HAESE MATHEMATICS

Mathematics

for the international student

Mathematics HL (Core)

Also suitable for HL & SL combined classes

third edition

David Martin
Robert Haese
Sandra Haese
Michael Haese
Mark Humphries

for use with

IB Diploma Programme

Roger Dixon
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WORKED SOLUTIONS

Haese Mathematics

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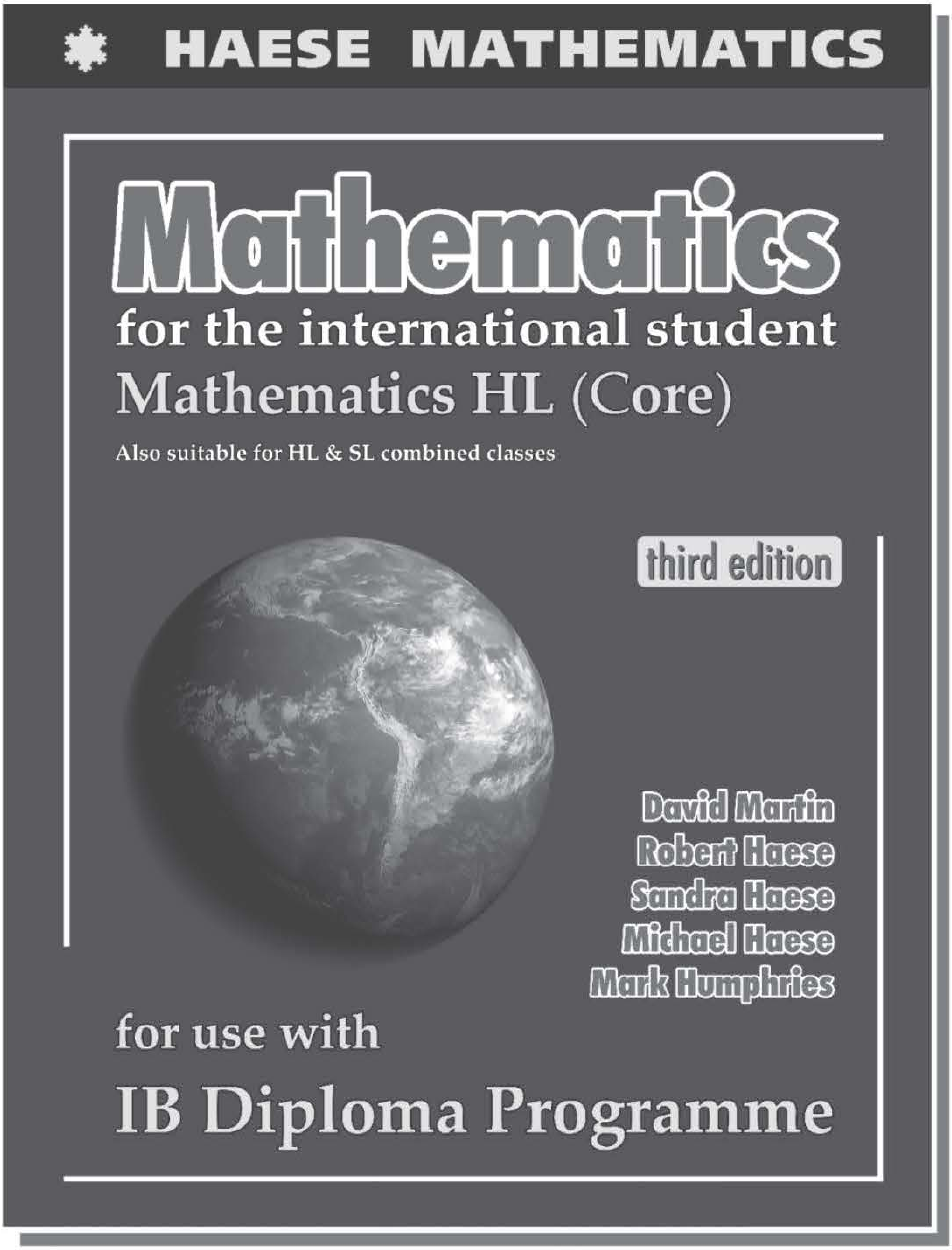

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Haese Mathematics

MATHEMATICS FOR THE INTERNATIONAL STUDENT
Mathematics HL (Core) third edition – WORKED SOLUTIONS
IB Diploma Programme

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FOREWORD

This book gives you fully worked solutions for every question (discussions, investigations and projects excepted) in each chapter of our textbook *Mathematics HL (Core) third edition* which is one of the textbooks in our series **Mathematics for the International Student** intended for use with IB Diploma and Middle Years courses.

Correct answers can sometimes be obtained by different methods. In this book, where applicable, each worked solution is modelled on the worked example in the textbook.

Be aware of the limitations of calculators and computer modelling packages. Understand that when your calculator gives an answer that is different from the answer you find in the book, you have not necessarily made a mistake, but the book may not be wrong either.

We have a list of errata for our books on our website. Please contact us if you notice any errors in this book.

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QUADRATICS

EXERCISE 1A.1

1 **a** $4x^2+7x=0$
$\therefore\ x(4x+7)=0$
$\therefore\ x=0$ or $4x+7=0$ {Null Factor law}
$\therefore\ x=0$ or $-\frac{7}{4}$

b $3x^2-7x=0$
$\therefore\ x(3x-7)=0$
$\therefore\ x=0$ or $3x-7=0$ {Null Factor law}
$\therefore\ x=0$ or $\frac{7}{3}$

c $2x^2-11x=0$
$\therefore\ x(2x-11)=0$
$\therefore\ x=0$ or $2x-11=0$ {Null Factor law}
$\therefore\ x=0$ or $\frac{11}{2}$

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$\therefore\ 6x^2-9x=0$
$\therefore\ 3x(2x-3)=0$
$\therefore\ x=0$ or $2x-3=0$ {Null Factor law}
$\therefore\ x=0$ or $\frac{3}{2}$

e $x^2-5x+6=0$
$\therefore\ (x-2)(x-3)=0$
$\therefore\ x-2=0$ or $x-3=0$ {Null Factor law}
$\therefore\ x=2$ or 3

f $x^2+21=10x$
$\therefore\ x^2-10x+21=0$
$\therefore\ (x-3)(x-7)=0$
$\therefore\ x-3=0$ or $x-7=0$ {Null Factor law}
$\therefore\ x=3$ or 7

g $9+x^2=6x$
$\therefore\ x^2-6x+9=0$
$\therefore\ (x-3)^2=0$
$\therefore\ x-3=0$
$\therefore\ x=3$

h $x^2+x=12$
$\therefore\ x^2+x-12=0$
$\therefore\ (x+4)(x-3)=0$
$\therefore\ x+4=0$ or $x-3=0$ {Null Factor law}
$\therefore\ x=-4$ or 3

i $x^2+8x=33$
$\therefore\ x^2+8x-33=0$
$\therefore\ (x+11)(x-3)=0$
$\therefore\ x+11=0$ or $x-3=0$ {Null Factor law}
$\therefore\ x=-11$ or 3

2 **a** $9x^2-12x+4=0$
$\therefore\ (3x-2)^2=0$
$\therefore\ x=\frac{2}{3}$

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c $3x^2=16x+12$
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f $3x^2+8x+4=0$
$\therefore\ (3x+2)(x+2)=0$
$\therefore\ x=-\frac{2}{3}$ or -2

g $3x^2=10x+8$
$\therefore\ 3x^2-10x-8=0$
$\therefore\ (3x+2)(x-4)=0$
$\therefore\ x=-\frac{2}{3}$ or 4

h $4x^2+4x=3$
$\therefore\ 4x^2+4x-3=0$
$\therefore\ (2x+3)(2x-1)=0$
$\therefore\ x=-\frac{3}{2}$ or $\frac{1}{2}$

i $4x^2=11x+3$
$\therefore\ 4x^2-11x-3=0$
$\therefore\ (4x+1)(x-3)=0$
$\therefore\ x=-\frac{1}{4}$ or 3

3 **a** $(x+1)^2=2x^2-5x+11$
$\therefore\ x^2+2x+1=2x^2-5x+11$
$\therefore\ x^2-7x+10=0$
$\therefore\ (x-2)(x-5)=0$
$\therefore\ x=2$ or 5

b $5-4x^2=3(2x+1)+2$
$\therefore\ 5-4x^2=6x+3+2$
$\therefore\ 4x^2+6x=0$
$\therefore\ 2x(2x+3)=0$
$\therefore\ x=0$ or $-\frac{3}{2}$

c $2x-\dfrac{1}{x}=-1$
$\therefore\ 2x^2-1=-x$
$\therefore\ 2x^2+x-1=0$
$\therefore\ (2x-1)(x+1)=0$
$\therefore\ x=\frac{1}{2}$ or -1

d $\dfrac{x+3}{1-x}=-\dfrac{9}{x}$
$\therefore\ x(x+3)=-9(1-x)$
$\therefore\ x^2+3x=-9+9x$
$\therefore\ x^2-6x+9=0$
$\therefore\ (x-3)^2=0$
$\therefore\ x=3$

EXERCISE 1A.2

1 **a** $(x+5)^2 = 2$
$\therefore\ x+5 = \pm\sqrt{2}$
$\therefore\ x = -5 \pm \sqrt{2}$

b $(x+6)^2 = -11$
has no real solutions as $(x+6)^2$ cannot be negative

c $(x-4)^2 = 8$
$\therefore\ x-4 = \pm\sqrt{8}$
$\therefore\ x = 4 \pm 2\sqrt{2}$

d $3(x-2)^2 = 18$
$\therefore\ (x-2)^2 = 6$
$\therefore\ x-2 = \pm\sqrt{6}$
$\therefore\ x = 2 \pm \sqrt{6}$

e $(2x+1)^2 = 3$
$\therefore\ 2x+1 = \pm\sqrt{3}$
$\therefore\ 2x = -1 \pm \sqrt{3}$
$\therefore\ x = -\frac{1}{2} \pm \frac{1}{2}\sqrt{3}$

f $(1-3x)^2 - 7 = 0$
$\therefore\ (1-3x)^2 = 7$
$\therefore\ 1-3x = \pm\sqrt{7}$
$\therefore\ 3x = 1 \pm \sqrt{7}$
$\therefore\ x = \frac{1}{3} \pm \frac{\sqrt{7}}{3}$

2 **a**
$$\begin{aligned} x^2 - 4x + 1 &= 0 \\ \therefore\ x^2 - 4x &= -1 \\ \therefore\ x^2 - 4x + (-2)^2 &= -1 + (-2)^2 \\ \therefore\ (x-2)^2 &= 3 \\ \therefore\ x-2 &= \pm\sqrt{3} \\ \therefore\ x &= 2 \pm \sqrt{3} \end{aligned}$$

b
$$\begin{aligned} x^2 + 6x + 2 &= 0 \\ \therefore\ x^2 + 6x &= -2 \\ \therefore\ x^2 + 6x + 3^2 &= -2 + 3^2 \\ \therefore\ (x+3)^2 &= 7 \\ \therefore\ x+3 &= \pm\sqrt{7} \\ \therefore\ x &= -3 \pm \sqrt{7} \end{aligned}$$

c
$$\begin{aligned} x^2 - 14x + 46 &= 0 \\ \therefore\ x^2 - 14x &= -46 \\ \therefore\ x^2 - 14x + (-7)^2 &= -46 + (-7)^2 \\ \therefore\ (x-7)^2 &= 3 \\ \therefore\ x-7 &= \pm\sqrt{3} \\ \therefore\ x &= 7 \pm \sqrt{3} \end{aligned}$$

d
$$\begin{aligned} x^2 &= 4x + 3 \\ \therefore\ x^2 - 4x &= 3 \\ \therefore\ x^2 - 4x + (-2)^2 &= 3 + (-2)^2 \\ \therefore\ (x-2)^2 &= 7 \\ \therefore\ x-2 &= \pm\sqrt{7} \\ \therefore\ x &= 2 \pm \sqrt{7} \end{aligned}$$

e
$$\begin{aligned} x^2 + 6x + 7 &= 0 \\ \therefore\ x^2 + 6x &= -7 \\ \therefore\ x^2 + 6x + 3^2 &= -7 + 3^2 \\ \therefore\ (x+3)^2 &= 2 \\ \therefore\ x+3 &= \pm\sqrt{2} \\ \therefore\ x &= -3 \pm \sqrt{2} \end{aligned}$$

f
$$\begin{aligned} x^2 &= 2x + 6 \\ \therefore\ x^2 - 2x &= 6 \\ \therefore\ x^2 - 2x + (-1)^2 &= 6 + (-1)^2 \\ \therefore\ (x-1)^2 &= 7 \\ \therefore\ x-1 &= \pm\sqrt{7} \\ \therefore\ x &= 1 \pm \sqrt{7} \end{aligned}$$

g
$$\begin{aligned} x^2 + 6x &= 2 \\ \therefore\ x^2 + 6x + 3^2 &= 2 + 3^2 \\ \therefore\ (x+3)^2 &= 11 \\ \therefore\ x+3 &= \pm\sqrt{11} \\ \therefore\ x &= -3 \pm \sqrt{11} \end{aligned}$$

h
$$\begin{aligned} x^2 + 10 &= 8x \\ \therefore\ x^2 - 8x &= -10 \\ \therefore\ x^2 - 8x + (-4)^2 &= -10 + (-4)^2 \\ \therefore\ (x-4)^2 &= 6 \\ \therefore\ x-4 &= \pm\sqrt{6} \\ \therefore\ x &= 4 \pm \sqrt{6} \end{aligned}$$

i
$$\begin{aligned} x^2 + 6x &= -11 \\ \therefore\ x^2 + 6x + 3^2 &= -11 + 3^2 \\ \therefore\ (x+3)^2 &= -2 \end{aligned}$$
$\therefore\ x$ has no real solutions, since the perfect square cannot be negative.

3 **a**
$$\begin{aligned} 2x^2 + 4x + 1 &= 0 \\ \therefore\ x^2 + 2x + \tfrac{1}{2} &= 0 \\ \therefore\ x^2 + 2x &= -\tfrac{1}{2} \\ \therefore\ x^2 + 2x + 1^2 &= -\tfrac{1}{2} + 1^2 \\ \therefore\ (x+1)^2 &= \tfrac{1}{2} \\ \therefore\ x+1 &= \pm\tfrac{1}{\sqrt{2}} \\ \therefore\ x &= -1 \pm \tfrac{1}{\sqrt{2}} \end{aligned}$$

b
$$\begin{aligned} 2x^2 - 10x + 3 &= 0 \\ \therefore\ x^2 - 5x + \tfrac{3}{2} &= 0 \\ \therefore\ x^2 - 5x &= -\tfrac{3}{2} \\ \therefore\ x^2 - 5x + (-\tfrac{5}{2})^2 &= -\tfrac{3}{2} + (-\tfrac{5}{2})^2 \\ \therefore\ (x - \tfrac{5}{2})^2 &= -\tfrac{3}{2} + \tfrac{25}{4} \\ \therefore\ (x - \tfrac{5}{2})^2 &= \tfrac{19}{4} \\ \therefore\ x - \tfrac{5}{2} &= \pm\tfrac{\sqrt{19}}{2} \\ \therefore\ x &= \tfrac{5}{2} \pm \tfrac{\sqrt{19}}{2} \end{aligned}$$

c

$$\begin{aligned} 3x^2+12x+5&=0\\ \therefore\ x^2+4x+\tfrac{5}{3}&=0\\ \therefore\ x^2+4x&=-\tfrac{5}{3}\\ \therefore\ x^2+4x+2^2&=-\tfrac{5}{3}+2^2\\ \therefore\ (x+2)^2&=\tfrac{7}{3}\\ \therefore\ x+2&=\pm\sqrt{\tfrac{7}{3}}\\ \therefore\ x&=-2\pm\sqrt{\tfrac{7}{3}} \end{aligned}$$

d

$$\begin{aligned} 3x^2&=6x+4\\ \therefore\ x^2&=2x+\tfrac{4}{3}\\ \therefore\ x^2-2x&=\tfrac{4}{3}\\ \therefore\ x^2-2x+(-1)^2&=\tfrac{4}{3}+(-1)^2\\ \therefore\ (x-1)^2&=\tfrac{7}{3}\\ \therefore\ x-1&=\pm\sqrt{\tfrac{7}{3}}\\ \therefore\ x&=1\pm\sqrt{\tfrac{7}{3}} \end{aligned}$$

e

$$\begin{aligned} 5x^2-15x+2&=0\\ \therefore\ x^2-3x+\tfrac{2}{5}&=0\\ \therefore\ x^2-3x&=-\tfrac{2}{5}\\ \therefore\ x^2-3x+(-\tfrac{3}{2})^2&=-\tfrac{2}{5}+(-\tfrac{3}{2})^2\\ \therefore\ (x-\tfrac{3}{2})^2&=-\tfrac{2}{5}+\tfrac{9}{4}=\tfrac{37}{20}\\ \therefore\ x-\tfrac{3}{2}&=\pm\sqrt{\tfrac{37}{20}}\\ \therefore\ x&=\tfrac{3}{2}\pm\sqrt{\tfrac{37}{20}} \end{aligned}$$

f

$$\begin{aligned} 4x^2+4x&=5\\ \therefore\ x^2+x&=\tfrac{5}{4}\\ \therefore\ x^2+x+(\tfrac{1}{2})^2&=\tfrac{5}{4}+(\tfrac{1}{2})^2\\ \therefore\ (x+\tfrac{1}{2})^2&=\tfrac{6}{4}\\ \therefore\ x+\tfrac{1}{2}&=\pm\tfrac{\sqrt{6}}{2}\\ \therefore\ x&=-\tfrac{1}{2}\pm\tfrac{\sqrt{6}}{2} \end{aligned}$$

EXERCISE 1A.3

1 **a** $x^2-4x-3=0$
has $a=1,\ b=-4,\ c=-3$

$$\begin{aligned} \therefore\ x&=\frac{-(-4)\pm\sqrt{(-4)^2-4(1)(-3)}}{2(1)}\\ &=\frac{4\pm\sqrt{28}}{2}\\ &=\frac{4\pm2\sqrt{7}}{2}\\ &=2\pm\sqrt{7} \end{aligned}$$

b $x^2+6x+7=0$
has $a=1,\ b=6,\ c=7$

$$\begin{aligned} \therefore\ x&=\frac{-6\pm\sqrt{6^2-4(1)(7)}}{2(1)}\\ &=\frac{-6\pm\sqrt{8}}{2}\\ &=\frac{-6\pm2\sqrt{2}}{2}\\ &=-3\pm\sqrt{2} \end{aligned}$$

c

$$\begin{aligned} x^2+1&=4x\\ \therefore\ x^2-4x+1&=0 \end{aligned}$$

which has $a=1,\ b=-4,\ c=1$

$$\begin{aligned} \therefore\ x&=\frac{-(-4)\pm\sqrt{(-4)^2-4(1)(1)}}{2(1)}\\ &=\frac{4\pm\sqrt{12}}{2}\\ &=\frac{4\pm2\sqrt{3}}{2}\\ &=2\pm\sqrt{3} \end{aligned}$$

d

$$\begin{aligned} x^2+4x&=1\\ \therefore\ x^2+4x-1&=0 \end{aligned}$$

which has $a=1,\ b=4,\ c=-1$

$$\begin{aligned} \therefore\ x&=\frac{-4\pm\sqrt{4^2-4(1)(-1)}}{2(1)}\\ &=\frac{-4\pm\sqrt{20}}{2}\\ &=\frac{-4\pm2\sqrt{5}}{2}\\ &=-2\pm\sqrt{5} \end{aligned}$$

e $x^2 - 4x + 2 = 0$
has $a = 1, \quad b = -4, \quad c = 2$

$$\therefore \quad x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(2)}}{2(1)}$$
$$= \frac{4 \pm \sqrt{8}}{2}$$
$$= \frac{4 \pm 2\sqrt{2}}{2}$$
$$= 2 \pm \sqrt{2}$$

f $2x^2 - 2x - 3 = 0$
has $a = 2, \quad b = -2, \quad c = -3$

$$\therefore \quad x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(2)(-3)}}{2(2)}$$
$$= \frac{2 \pm \sqrt{28}}{4}$$
$$= \frac{2 \pm 2\sqrt{7}}{4}$$
$$= \tfrac{1}{2} \pm \tfrac{\sqrt{7}}{2}$$

g
$$(3x+1)^2 = -2x$$
$$\therefore \quad 9x^2 + 6x + 1 = -2x$$
$$\therefore \quad 9x^2 + 8x + 1 = 0$$
which has $a = 9, \quad b = 8, \quad c = 1$

$$\therefore \quad x = \frac{-8 \pm \sqrt{8^2 - 4(9)(1)}}{2(9)}$$
$$= \frac{-8 \pm \sqrt{28}}{18}$$
$$= \frac{-8 \pm 2\sqrt{7}}{18} \quad \text{or} \quad -\tfrac{4}{9} \pm \tfrac{\sqrt{7}}{9}$$

h
$$(x+3)(2x+1) = 9$$
$$\therefore \quad 2x^2 + x + 6x + 3 = 9$$
$$\therefore \quad 2x^2 + 7x - 6 = 0$$
which has $a = 2, \quad b = 7, \quad c = -6$

$$\therefore \quad x = \frac{-7 \pm \sqrt{7^2 - 4(2)(-6)}}{2(2)}$$
$$= \frac{-7 \pm \sqrt{49 + 48}}{4}$$
$$= -\tfrac{7}{4} \pm \tfrac{\sqrt{97}}{4}$$

i $x^2 - 2\sqrt{2}x + 2 = 0$
has $a = 1, \quad b = -2\sqrt{2}, \quad c = 2$

$$\therefore \quad x = \frac{-(-2\sqrt{2}) \pm \sqrt{(-2\sqrt{2})^2 - 4(1)(2)}}{2(1)}$$
$$= \frac{2\sqrt{2} \pm \sqrt{8 - 8}}{2}$$
$$= \frac{2\sqrt{2} \pm 0}{2}$$
$$= \sqrt{2}$$

2 **a**
$$(x+2)(x-1) = 2 - 3x$$
$$\therefore \quad x^2 - x + 2x - 2 = 2 - 3x$$
$$\therefore \quad x^2 + 4x - 4 = 0$$
which has $a = 1, \quad b = 4, \quad c = -4$

$$\therefore \quad x = \frac{-4 \pm \sqrt{4^2 - 4(1)(-4)}}{2(1)}$$
$$= \frac{-4 \pm \sqrt{32}}{2}$$
$$= \frac{-4 \pm 4\sqrt{2}}{2}$$
$$= -2 \pm 2\sqrt{2}$$

b
$$(2x+1)^2 = 3 - x$$
$$\therefore \quad 4x^2 + 4x + 1 = 3 - x$$
$$\therefore \quad 4x^2 + 5x - 2 = 0$$
which has $a = 4, \quad b = 5, \quad c = -2$

$$\therefore \quad x = \frac{-5 \pm \sqrt{5^2 - 4(4)(-2)}}{2(4)}$$
$$= \frac{-5 \pm \sqrt{25 + 32}}{8}$$
$$= -\tfrac{5}{8} \pm \tfrac{\sqrt{57}}{8}$$

c
$$(x-2)^2 = 1 + x$$
$$\therefore\ x^2 - 4x + 4 = 1 + x$$
$$\therefore\ x^2 - 5x + 3 = 0$$
which has $a = 1,\ \ b = -5,\ \ c = 3$
$$\therefore\ x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(3)}}{2(1)}$$
$$= \frac{5 \pm \sqrt{25-12}}{2}$$
$$= \tfrac{5}{2} \pm \tfrac{\sqrt{13}}{2}$$

d
$$\frac{x-1}{2-x} = 2x + 1$$
$$\therefore\ x - 1 = (2x+1)(2-x)$$
$$\therefore\ x - 1 = 4x - 2x^2 + 2 - x$$
$$\therefore\ 2x^2 - 2x - 3 = 0$$
which has $a = 2,\ \ b = -2,\ \ c = -3$
$$\therefore\ x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(2)(-3)}}{2(2)}$$
$$= \frac{2 \pm \sqrt{28}}{4}$$
$$= \frac{2 \pm 2\sqrt{7}}{4} \text{ or } \tfrac{1}{2} \pm \tfrac{\sqrt{7}}{2}$$

e
$$x - \frac{1}{x} = 1$$
$$\therefore\ x^2 - 1 = x$$
$$\therefore\ x^2 - x - 1 = 0$$
which has $a = 1,\ \ b = -1,\ \ c = -1$
$$\therefore\ x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)}$$
$$= \frac{1 \pm \sqrt{1+4}}{2}$$
$$= \tfrac{1}{2} \pm \tfrac{\sqrt{5}}{2}$$

f
$$2x - \frac{1}{x} = 3$$
$$\therefore\ 2x^2 - 1 = 3x$$
$$\therefore\ 2x^2 - 3x - 1 = 0$$
which has $a = 2,\ \ b = -3,\ \ c = -1$
$$\therefore\ x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(-1)}}{2(2)}$$
$$= \frac{3 \pm \sqrt{9+8}}{4}$$
$$= \tfrac{3}{4} \pm \tfrac{\sqrt{17}}{4}$$

EXERCISE 1B

1 **a** $x^2 + 7x - 3 = 0$
has $a = 1,\ \ b = 7,\ \ c = -3$
$$\therefore\ \Delta = b^2 - 4ac$$
$$= 7^2 - 4(1)(-3)$$
$$= 61$$
Since $\Delta > 0$, there are two distinct real solutions.

b $x^2 - 3x + 2 = 0$
has $a = 1,\ \ b = -3,\ \ c = 2$
$$\therefore\ \Delta = b^2 - 4ac$$
$$= (-3)^2 - 4(1)(2)$$
$$= 1$$
Since $\Delta > 0$ and Δ is a square, there are two distinct rational solutions.

c $3x^2 + 2x - 1 = 0$
has $a = 3,\ \ b = 2,\ \ c = -1$
$$\therefore\ \Delta = b^2 - 4ac$$
$$= 2^2 - 4(3)(-1)$$
$$= 16$$
Since $\Delta > 0$ and Δ is a square, there are two distinct rational solutions.

d $5x^2 + 4x - 3 = 0$
has $a = 5,\ \ b = 4,\ \ c = -3$
$$\therefore\ \Delta = b^2 - 4ac$$
$$= 4^2 - 4(5)(-3)$$
$$= 76$$
Since $\Delta > 0$, there are two distinct real solutions.

e $x^2 + x + 5 = 0$
has $a = 1,\ \ b = 1,\ \ c = 5$
$$\therefore\ \Delta = b^2 - 4ac$$
$$= 1^2 - 4(1)(5)$$
$$= -19$$
Since $\Delta < 0$, there are no real solutions.

f $16x^2 - 8x + 1 = 0$
has $a = 16,\ \ b = -8,\ \ c = 1$
$$\therefore\ \Delta = b^2 - 4ac$$
$$= (-8)^2 - 4(16)(1)$$
$$= 0$$
$\therefore$ there is one repeated real solution.

2 **a** $6x^2 - 5x - 6 = 0$
has $a = 6,\ b = -5,\ c = -6$
$\therefore\ \Delta = b^2 - 4ac$
$= (-5)^2 - 4(6)(-6)$
$= 169$
$\therefore\ \sqrt{\Delta} = 13$, so the equation has rational roots.

b $2x^2 - 7x - 5 = 0$
has $a = 2,\ b = -7,\ c = -5$
$\therefore\ \Delta = b^2 - 4ac$
$= (-7)^2 - 4(2)(-5)$
$= 89$
$\therefore\ \sqrt{\Delta} = \sqrt{89}$, so the equation does not have rational roots.

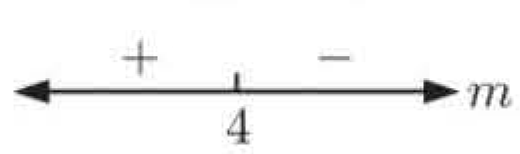
c $3x^2 + 4x + 1 = 0$
has $a = 3,\ b = 4,\ c = 1$
$\therefore\ \Delta = b^2 - 4ac$
$= 4^2 - 4(3)(1)$
$= 4$
$\therefore\ \sqrt{\Delta} = 2$, so the equation has rational roots.

d $6x^2 - 47x - 8 = 0$
has $a = 6,\ b = -47,\ c = -8$
$\therefore\ \Delta = b^2 - 4ac$
$= (-47)^2 - 4(6)(-8)$
$= 2401$
$\therefore\ \sqrt{\Delta} = 49$, so the equation has rational roots.

e $4x^2 - 3x + 2 = 0$
has $a = 4,\ b = -3,\ c = 2$
$\therefore\ \Delta = b^2 - 4ac$
$= (-3)^2 - 4(4)2$
$= -23$
Since $\Delta < 0$, the equation does not have rational roots.

f $8x^2 + 2x - 3 = 0$
has $a = 8,\ b = 2,\ c = -3$
$\therefore\ \Delta = b^2 - 4ac$
$= 2^2 - 4(8)(-3)$
$= 100$
$\therefore\ \sqrt{\Delta} = 10$, so the equation has rational roots.

3 **a** For $x^2 + 4x + m = 0$,
$a = 1,\ b = 4,\ c = m$
So, $\Delta = b^2 - 4ac$
$= 4^2 - 4(1)(m)$
$= 16 - 4m$
which has sign diagram

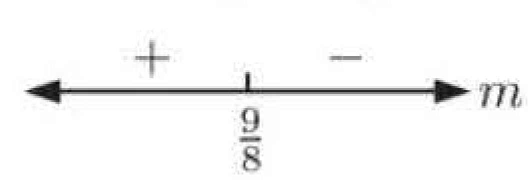


i For a repeated root, $\Delta = 0$
$\therefore\ m = 4$

ii For two distinct real roots, $\Delta > 0$
$\therefore\ m < 4$

iii For no real roots, $\Delta < 0$
$\therefore\ m > 4$

b For $mx^2 + 3x + 2 = 0$,
$a = m,\ b = 3,\ c = 2$
So, $\Delta = b^2 - 4ac$
$= 3^2 - 4(m)(2)$
$= 9 - 8m$
which has sign diagram

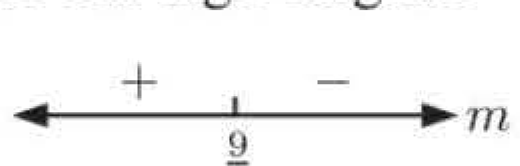


i For a repeated root, $\Delta = 0$
$\therefore\ m = \frac{9}{8}$

ii For two distinct real roots, $\Delta > 0$
$\therefore\ m < \frac{9}{8}$

iii For no real roots, $\Delta < 0$
$\therefore\ m > \frac{9}{8}$

c For $mx^2 - 3x + 1 = 0$,
$a = m,\ b = -3,\ c = 1$
So, $\Delta = b^2 - 4ac$
$= (-3)^2 - 4(m)(1)$
$= 9 - 4m$
which has sign diagram



i For a repeated root, $\Delta = 0$ $\quad \therefore\ m = \frac{9}{4}$

ii For two distinct real roots, $\Delta > 0$ $\quad \therefore\ m < \frac{9}{4}$

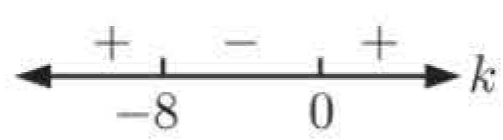
iii For no real roots, $\Delta < 0$ $\quad \therefore\ m > \frac{9}{4}$

4 **a** For $2x^2 + kx - k = 0$,

$a = 2, \quad b = k, \quad c = -k$

So, $\Delta = b^2 - 4ac$
$= k^2 - 4(2)(-k)$
$= k^2 + 8k$
$= k(k+8)$

which has sign diagram



i For two distinct real roots, $\Delta > 0$
$\therefore \ k < -8$ or $k > 0$

ii For two real roots, $\Delta \geqslant 0$
$\therefore \ k \leqslant -8$ or $k \geqslant 0$

iii For a repeated root, $\Delta = 0$
$\therefore \ k = -8$ or 0

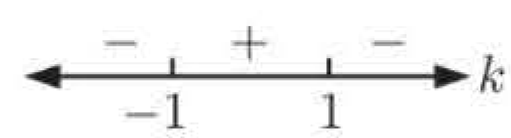
iv For no real roots, $\Delta < 0$
$\therefore \ -8 < k < 0$

b For $kx^2 - 2x + k = 0$,

$a = k, \quad b = -2, \quad c = k$

So, $\Delta = b^2 - 4ac$
$= (-2)^2 - 4(k)(k)$
$= 4 - 4k^2$
$= 4(1+k)(1-k)$

which has sign diagram



i For two distinct real roots, $\Delta > 0$
$\therefore \ -1 < k < 1$

ii For two real roots, $\Delta \geqslant 0$
$\therefore \ -1 \leqslant k \leqslant 1$

iii For a repeated root, $\Delta = 0$
$\therefore \ k = -1$ or 1

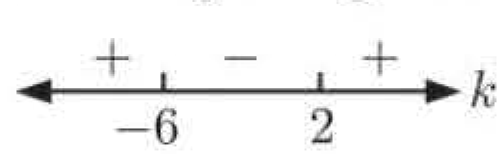
iv For no real roots, $\Delta < 0$
$\therefore \ k < -1$ or $k > 1$

c For $x^2 + (k+2)x + 4 = 0$,

$a = 1, \quad b = k+2, \quad c = 4$

So, $\Delta = b^2 - 4ac$
$= (k+2)^2 - 4(1)(4)$
$= k^2 + 4k + 4 - 16$
$= k^2 + 4k - 12$
$= (k+6)(k-2)$

which has sign diagram



i For two distinct real roots, $\Delta > 0$
$\therefore \ k < -6$ or $k > 2$

ii For two real roots, $\Delta \geqslant 0$
$\therefore \ k \leqslant -6$ or $k \geqslant 2$

iii For a repeated root, $\Delta = 0$
$\therefore \ k = -6$ or 2

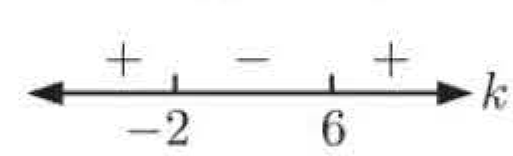
iv For no real roots, $\Delta < 0$
$\therefore \ -6 < k < 2$

d For $2x^2 + (k-2)x + 2 = 0$,

$a = 2, \quad b = k-2, \quad c = 2$

So, $\Delta = b^2 - 4ac$
$= (k-2)^2 - 4(2)(2)$
$= k^2 - 4k + 4 - 16$
$= k^2 - 4k - 12$
$= (k-6)(k+2)$

which has sign diagram



i For two distinct real roots, $\Delta > 0$
$\therefore \ k < -2$ or $k > 6$

ii For two real roots, $\Delta \geqslant 0$
$\therefore \ k \leqslant -2$ or $k \geqslant 6$

iii For a repeated root, $\Delta = 0$
$\therefore \ k = -2$ or 6

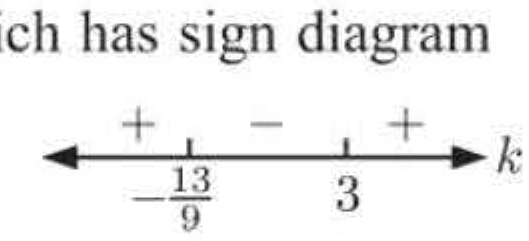
iv For no real roots, $\Delta < 0$
$\therefore \ -2 < k < 6$

e For $x^2 + (3k-1)x + (2k+10) = 0$,

$a = 1, \quad b = 3k-1, \quad c = 2k+10$

So, $\Delta = b^2 - 4ac$
$= (3k-1)^2 - 4(1)(2k+10)$
$= 9k^2 - 6k + 1 - 8k - 40$
$= 9k^2 - 14k - 39$
$= (9k+13)(k-3)$

which has sign diagram



i For two distinct real roots, $\Delta > 0$
$\therefore \ k < -\frac{13}{9}$ or $k > 3$

ii For two real roots, $\Delta \geqslant 0$
$\therefore \ k \leqslant -\frac{13}{9}$ or $k \geqslant 3$

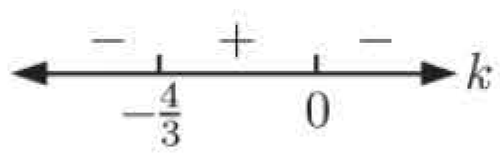
iii For a repeated root, $\Delta = 0$
$\therefore \ k = -\frac{13}{9}$ or 3

iv For no real roots, $\Delta < 0$
$\therefore \ -\frac{13}{9} < k < 3$

f For $(k+1)x^2 + kx + k = 0$,
$a = k+1, \quad b = k, \quad c = k$

So, $\Delta = b^2 - 4ac$
$= k^2 - 4(k+1)(k)$
$= k^2 - 4k^2 - 4k$
$= -3k^2 - 4k$
$= -k(3k+4)$

which has sign diagram



i For two distinct real roots, $\Delta > 0$
$\therefore\ -\frac{4}{3} < k < 0$

ii For two real roots, $\Delta \geqslant 0$
$\therefore\ -\frac{4}{3} \leqslant k \leqslant 0$

iii For a repeated root, $\Delta = 0$
$\therefore\ k = -\frac{4}{3}$ or 0

iv For no real roots, $\Delta < 0$
$\therefore\ k < -\frac{4}{3}$ or $k > 0$

EXERCISE 1C

1 **a** For $3x^2 - 2x + 7 = 0$:

sum of roots $= -\dfrac{b}{a} = -\dfrac{(-2)}{3} = \frac{2}{3}$

product of roots $= \dfrac{c}{a} = \frac{7}{3}$

b For $x^2 + 11x - 13 = 0$:

sum of roots $= -\dfrac{b}{a} = -\frac{11}{1} = -11$

product of roots $= \dfrac{c}{a} = \frac{-13}{1} = -13$

c For $5x^2 - 6x - 14 = 0$:

sum of roots $= -\dfrac{b}{a} = -\dfrac{(-6)}{5} = \frac{6}{5}$

product of roots $= \dfrac{c}{a} = -\frac{14}{5}$

2 For $kx^2 - (1+k)x + (3k+2) = 0, \qquad a = k, \quad b = -(1+k), \quad c = 3k+2$

$\therefore$ sum of roots $= -\dfrac{b}{a} = -\dfrac{-(1+k)}{k} = \dfrac{k+1}{k}$, product of roots $= \dfrac{c}{a} = \dfrac{3k+2}{k}$

Now, sum of the roots is twice their product

$$\therefore\ \frac{k+1}{k} = 2\left(\frac{3k+2}{k}\right)$$
$$\therefore\ k+1 = 2(3k+2)$$
$$\therefore\ k+1 = 6k+4$$
$$\therefore\ -3 = 5k$$
$$\therefore\ k = -\tfrac{3}{5}$$

Substituting $k = -\frac{3}{5}$ into the equation gives

$$-\tfrac{3}{5}x^2 - (1 - \tfrac{3}{5})x + (-\tfrac{9}{5} + 2) = 0$$
$$\therefore\ -\tfrac{3}{5}x^2 - \tfrac{2}{5}x + \tfrac{1}{5} = 0$$
$$\therefore\ -\tfrac{1}{5}(3x^2 + 2x - 1) = 0$$
$$\therefore\ -\tfrac{1}{5}(3x-1)(x+1) = 0$$
$$\therefore\ x = \tfrac{1}{3} \text{ or } -1$$

$\therefore$ the roots of the equation are $\frac{1}{3}$ and -1

3 For $ax^2 - 6x + a - 2 = 0, \qquad$ "a" $= a, \quad b = -6, \quad c = a-2$

a sum of roots $= -\dfrac{b}{a}$

$$\therefore\ \alpha + 2\alpha = -\frac{(-6)}{a}$$
$$\therefore\ 3\alpha = \frac{6}{a} \quad \{\text{or } \alpha = \frac{2}{a} \quad \text{.... (1)}\}$$

product of roots $= \dfrac{c}{a}$

$$\therefore\ \alpha \times 2\alpha = \frac{a-2}{a}$$
$$\therefore\ 2\alpha^2 = \frac{a-2}{a} \quad \text{.... (2)}$$

b Substituting (1) into (2) gives

$$2\left(\frac{2}{a}\right)^2 = \frac{a-2}{a}$$
$$\therefore\ 2\left(\frac{4}{a^2}\right) = \frac{a-2}{a}$$
$$\therefore\ \frac{8}{a} = a - 2$$
$$\therefore\ 8 = a^2 - 2a$$
$$\therefore\ a^2 - 2a - 8 = 0$$
$$\therefore\ (a-4)(a+2) = 0$$
$$\therefore\ a = 4 \text{ or } -2$$

If $a = 4$, then $\alpha = \frac{2}{4}$ {using (1)}
$= \frac{1}{2}$
and $2\alpha = 1$

If $a = -2$, then $\alpha = \frac{2}{-2}$ {using (1)}
$= -1$
and $2\alpha = -2$

$\therefore$ $a = 4$ and the roots are $\frac{1}{2}$ and 1 or $a = -2$ and the roots are -1 and -2.

4 For $kx^2 + (k-8)x + (1-k) = 0$, $a = k$, $b = k-8$, $c = 1-k$

Let the roots of the equation be m and $m+2$.

sum of roots $= -\dfrac{b}{a}$

$\therefore$ $m + (m+2) = -\dfrac{k-8}{k}$

$\therefore$ $2m + 2 = \dfrac{8-k}{k}$

$\therefore$ $2m = \dfrac{8-3k}{k}$

$\therefore$ $m = \dfrac{8-3k}{2k}$ .... (1)

and product of roots $= \dfrac{c}{a}$

$\therefore$ $m(m+2) = \dfrac{1-k}{k}$ .... (2)

Substituting (1) into (2) gives

$$\left(\frac{8-3k}{2k}\right)\left(\frac{8-3k}{2k}+2\right) = \frac{1-k}{k}$$

$$\therefore \left(\frac{8-3k}{2k}\right)\left(\frac{8+k}{2k}\right) = \frac{1-k}{k}$$

$$\therefore \frac{64-16k-3k^2}{4k^2} = \frac{1-k}{k}$$

$$\therefore \frac{64-16k-3k^2}{4k} = 1-k$$

$\therefore$ $64 - 16k - 3k^2 = 4k - 4k^2$

$\therefore$ $k^2 - 20k + 64 = 0$

$\therefore$ $(k-4)(k-16) = 0$

$\therefore$ $k = 4$ or 16

Using (1), if $k = 4$ then $m = \dfrac{8-3(4)}{2(4)} = \dfrac{-4}{8} = -\frac{1}{2}$, and $m + 2 = \frac{3}{2}$

and if $k = 16$ then $m = \dfrac{8-3(16)}{2(16)} = \dfrac{-40}{32} = -\frac{5}{4}$, and $m + 2 = \frac{3}{4}$

$\therefore$ $k = 4$ and the roots are $-\frac{1}{2}$ and $\frac{3}{2}$ or $k = 16$ and the roots are $-\frac{5}{4}$ and $\frac{3}{4}$.

5 The roots of $x^2 - 6x + 7 = 0$ are α and β.

sum of roots $= -\frac{-6}{1}$ and product of roots $= \frac{7}{1}$

$\therefore$ $\alpha + \beta = 6$ $\therefore$ $\alpha\beta = 7$

Now consider a quadratic equation with roots $\alpha + \dfrac{1}{\beta}$ and $\beta + \dfrac{1}{\alpha}$

sum of roots $= \alpha + \dfrac{1}{\beta} + \beta + \dfrac{1}{\alpha}$

$= (\alpha + \beta) + \left(\dfrac{1}{\alpha} + \dfrac{1}{\beta}\right)$

$= (\alpha + \beta) + \dfrac{\alpha + \beta}{\alpha\beta}$

$= 6 + \frac{6}{7}$

$= \frac{48}{7}$

product of roots $= \left(\alpha + \dfrac{1}{\beta}\right)\left(\beta + \dfrac{1}{\alpha}\right)$

$= \alpha\beta + 1 + 1 + \dfrac{1}{\alpha\beta}$

$= 7 + 2 + \frac{1}{7}$

$= 9 + \frac{1}{7}$

$= \frac{64}{7}$

$\therefore$ a quadratic equation $ax^2 + bx + c = 0$ with roots $\alpha + \dfrac{1}{\beta}$ and $\beta + \dfrac{1}{\alpha}$ has $-\dfrac{b}{a} = \frac{48}{7}$ and $\dfrac{c}{a} = \frac{64}{7}$.

The simplest solution to this is $a = 7$, $b = -48$, $c = 64$.

$\therefore$ the simplest quadratic equation with roots $\alpha + \dfrac{1}{\beta}$ and $\beta + \dfrac{1}{\alpha}$ is $7x^2 - 48x + 64 = 0$.

6 The roots of $2x^2 - 3x - 5 = 0$ are p and q.

$\therefore$ sum of roots $= -\frac{-3}{2}$ and product of roots $= \frac{-5}{2}$

$\therefore$ $p + q = \frac{3}{2}$ $\qquad$ $\therefore$ $pq = -\frac{5}{2}$

Now consider a quadratic equation with roots $p^2 + q$ and $q^2 + p$.

$$\begin{aligned}\text{sum of roots} &= p^2 + q + q^2 + p \\ &= (p^2 + q^2) + (p + q) \\ &= [(p+q)^2 - 2pq] + (p+q) \\ &= (\tfrac{3}{2})^2 - 2(-\tfrac{5}{2}) + \tfrac{3}{2} \\ &= \tfrac{9}{4} + 5 + \tfrac{3}{2} \\ &= \tfrac{35}{4}\end{aligned}$$

$$\begin{aligned}\text{product of roots} &= (p^2 + q)(q^2 + p) \\ &= p^2q^2 + p^3 + q^3 + qp \\ &= (pq)^2 + [(p+q)^3 - 3p^2q - 3pq^2] + pq \\ &= (pq)^2 + [(p+q)^3 - 3pq(p+q)] + pq \\ &= (-\tfrac{5}{2})^2 + (\tfrac{3}{2})^3 - 3(-\tfrac{5}{2})(\tfrac{3}{2}) + (-\tfrac{5}{2}) \\ &= \tfrac{25}{4} + \tfrac{27}{8} + \tfrac{45}{4} - \tfrac{5}{2} \\ &= \tfrac{147}{8}\end{aligned}$$

$\therefore$ a quadratic equation $ax^2 + bx + c = 0$ with roots $p^2 + q$ and $q^2 + p$

has $-\dfrac{b}{a} = \frac{35}{4}$ and $\dfrac{c}{a} = \frac{147}{8}$.

$\therefore$ $b = -\frac{35}{4}a$ and $c = \frac{147}{8}a$

$\therefore$ the quadratic equation is $ax^2 - \frac{35}{4}ax + \frac{147}{8}a = 0, \quad a \neq 0$

$\therefore$ $a(x^2 - \frac{35}{4}x + \frac{147}{8}) = 0, \quad a \neq 0$

$\therefore$ $a(8x^2 - 70x + 147) = 0, \quad a \neq 0$

7 $kx^2 + (k+2)x - 3 = 0$ will have roots which are real and positive if:

(1) $\Delta \geqslant 0$ $\qquad$ (2) sum of roots > 0 $\qquad$ (3) product of roots > 0

(1) $\Delta \geqslant 0$

$\therefore$ $(k+2)^2 - 4(k)(-3) \geqslant 0$

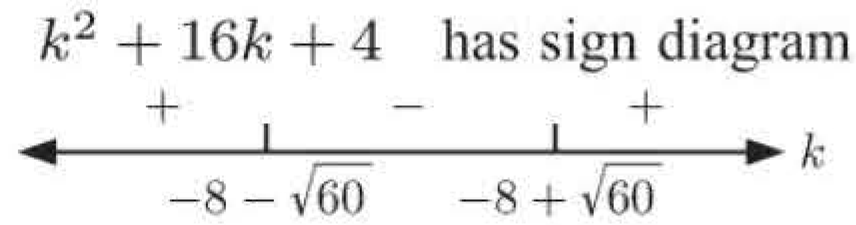
$\therefore$ $k^2 + 4k + 4 + 12k \geqslant 0$

$\therefore$ $k^2 + 16k + 4 \geqslant 0$

Now $k^2 + 16k + 4 = 0$ when

$$\begin{aligned}k &= \frac{-16 \pm \sqrt{16^2 - 4(1)(4)}}{2(1)} \\ &= \frac{-16 \pm \sqrt{240}}{2} \\ &= \frac{-16 \pm 2\sqrt{60}}{2} \\ &= -8 \pm \sqrt{60}\end{aligned}$$

$\therefore$ $k^2 + 16k + 4$ has sign diagram



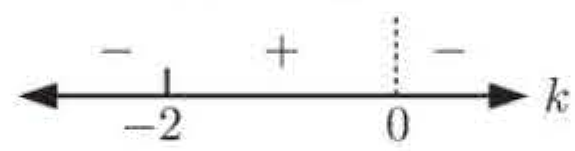
$\therefore$ $\Delta \geqslant 0$ when

$k \leqslant -8 - \sqrt{60}$ or $k \geqslant -8 + \sqrt{60}$

(2) sum of roots > 0

$\therefore$ $-\dfrac{k+2}{k} > 0$

$-\dfrac{k+2}{k}$ has sign diagram

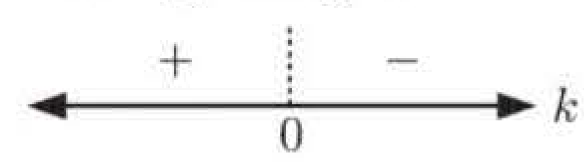


$\therefore$ sum of roots > 0 when $-2 < k < 0$

(3) product of roots > 0

$\therefore$ $\dfrac{-3}{k} > 0$

$-\dfrac{3}{k}$ has sign diagram

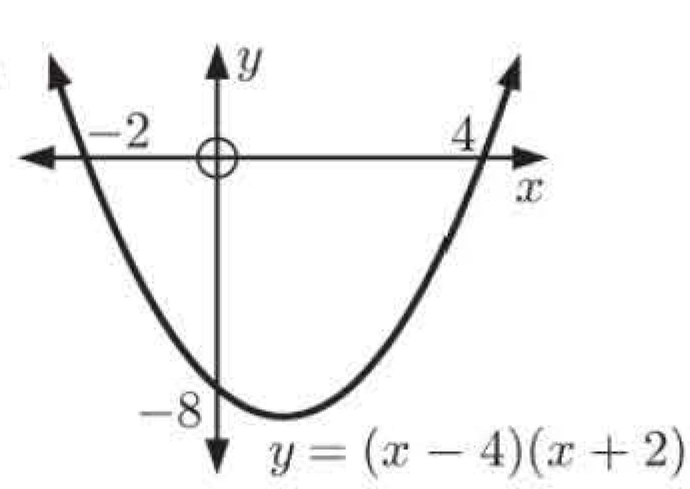


$\therefore$ product of roots > 0 when $k < 0$

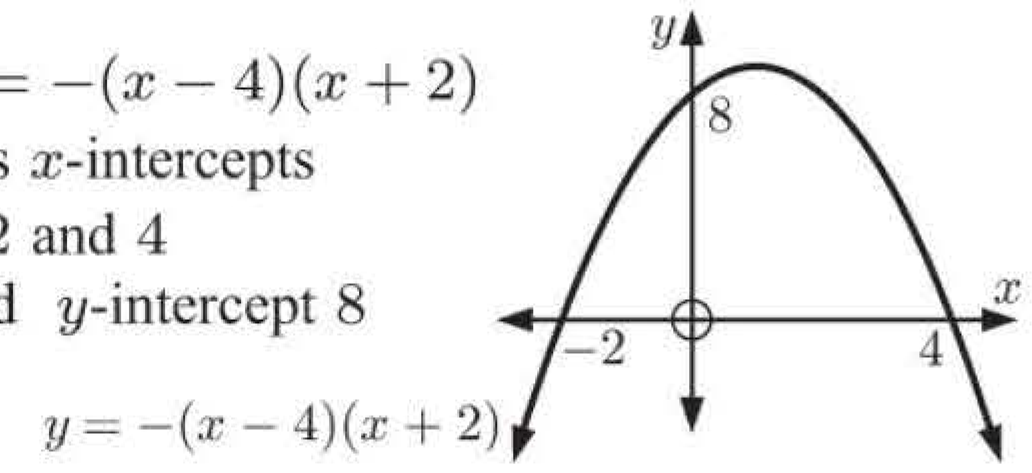
$\therefore$ the only values of k which satisfy all three conditions are $-8 + \sqrt{60} \leqslant k < 0$.

EXERCISE 1D.1

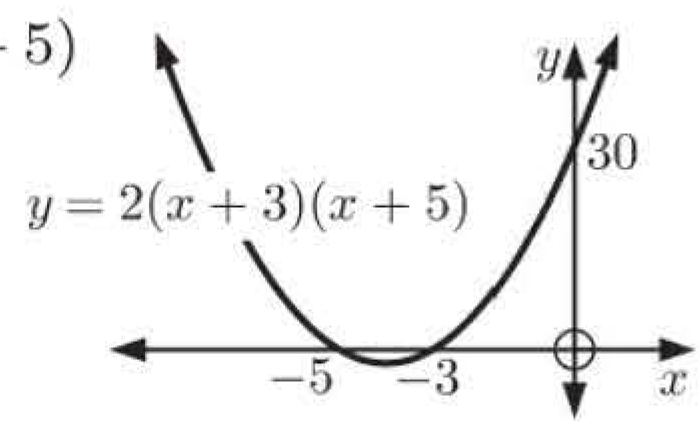
1 **a** $y = (x-4)(x+2)$ has x-intercepts -2 and 4 and y-intercept -8



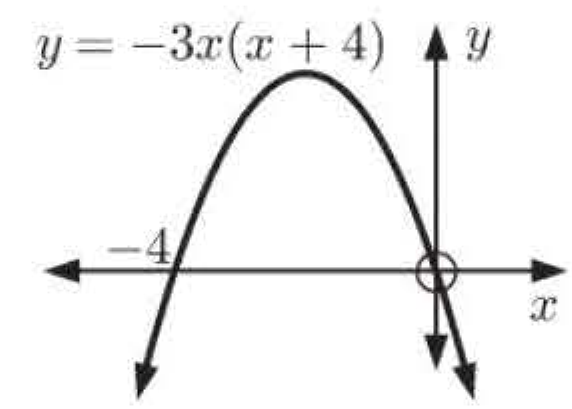
b $y = -(x-4)(x+2)$ has x-intercepts -2 and 4 and y-intercept 8



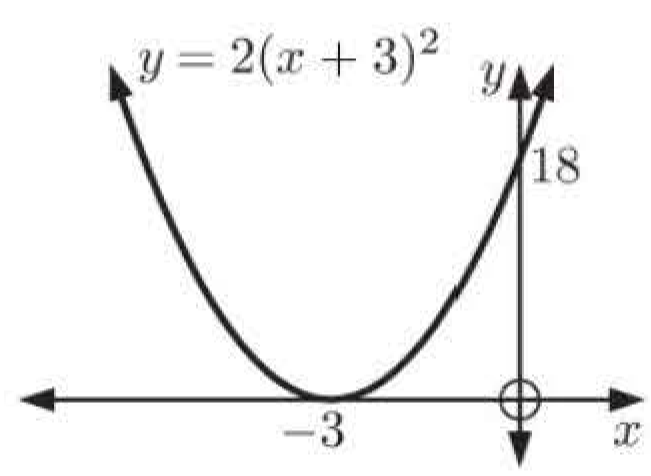
c $y = 2(x+3)(x+5)$ has x-intercepts -5 and -3 and y-intercept 30



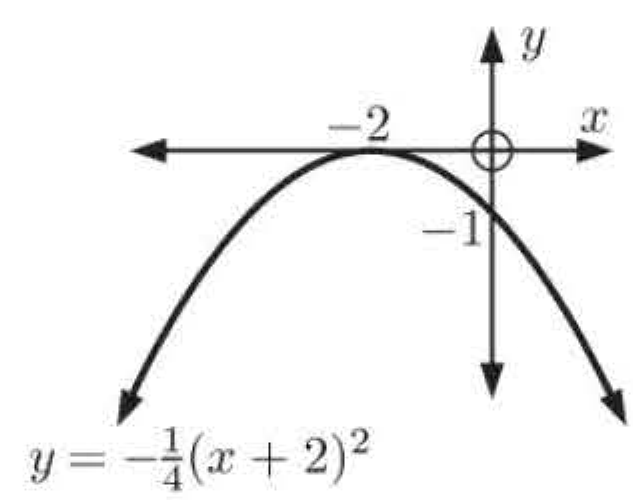
d $y = -3x(x+4)$ has x-intercepts 0 and -4 and y-intercept 0



e $y = 2(x+3)^2$ has x-intercept -3 and y-intercept 18



f $y = -\frac{1}{4}(x+2)^2$ has x-intercept -2 and y-intercept -1



2 **a** The average of the x-intercepts is 1, so the axis of symmetry is $x = 1$.

b The average of the x-intercepts is 1, so the axis of symmetry is $x = 1$.

c The average of the x-intercepts is -4, so the axis of symmetry is $x = -4$.

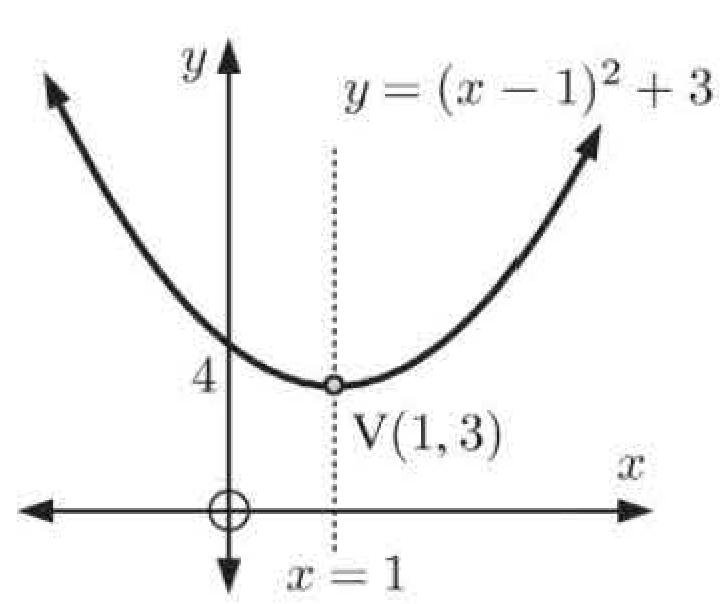
d The average of the x-intercepts is -2, so the axis of symmetry is $x = -2$.

e The only x-intercept is -3, so the axis of symmetry is $x = -3$.

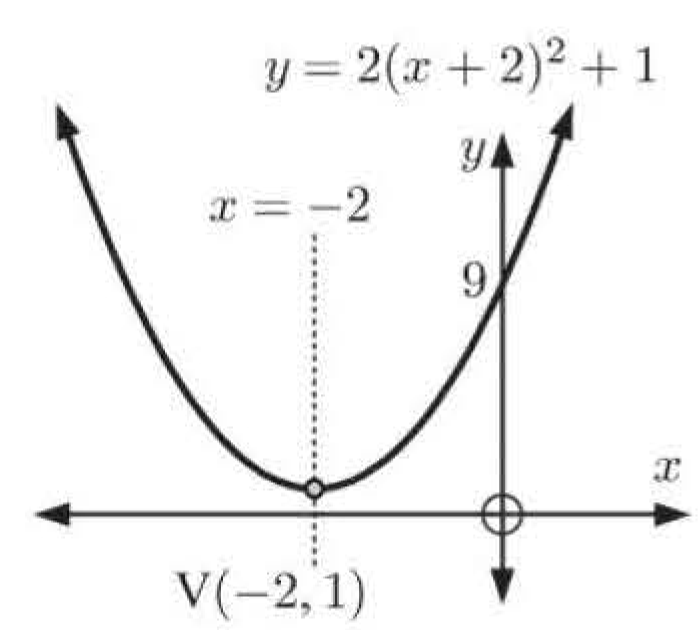
f The only x-intercept is -2, so the axis of symmetry is $x = -2$.

3 **a** C **b** E **c** B **d** F **e** G **f** H **g** A **h** D

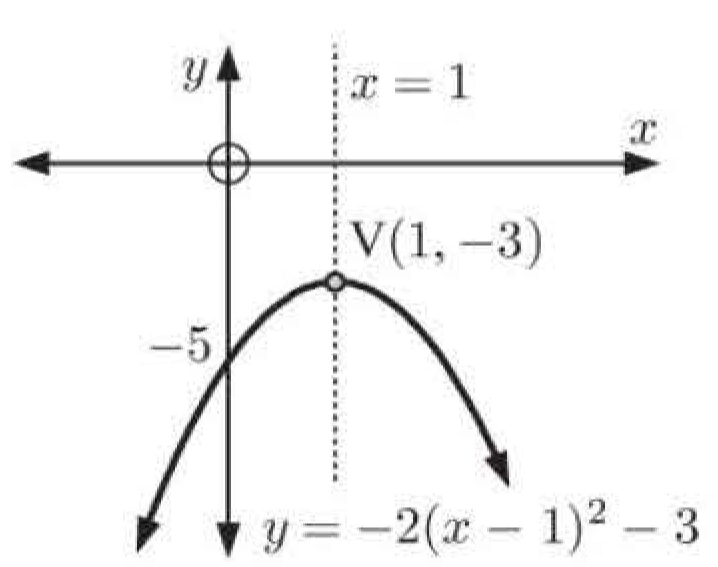
4 **a** The vertex is $(1, 3)$. The axis of symmetry is $x = 1$. The y-intercept is 4.



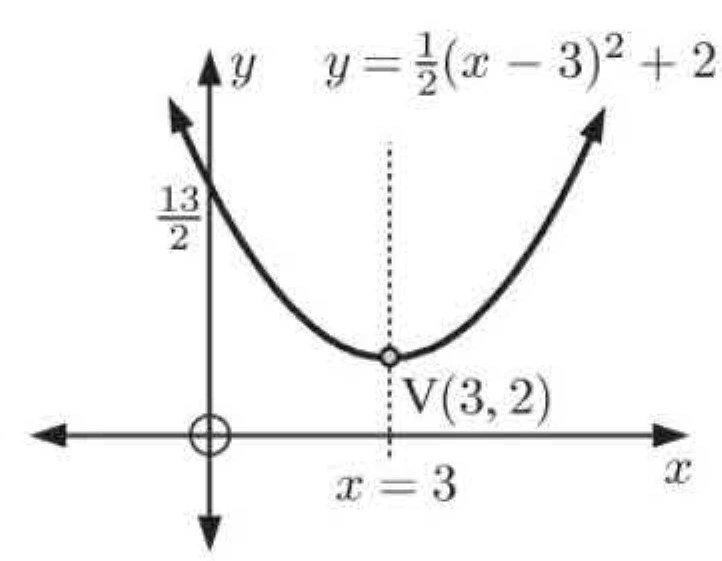
b The vertex is $(-2, 1)$. The axis of symmetry is $x = -2$. The y-intercept is 9.



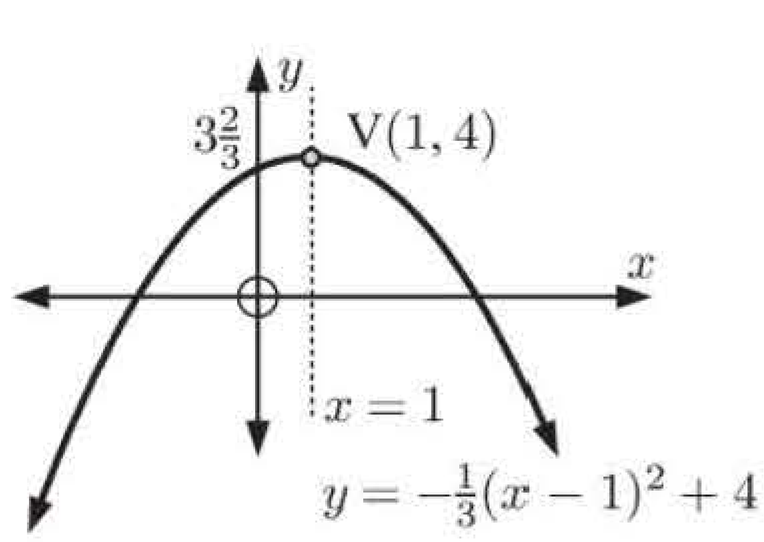
c The vertex is $(1, -3)$. The axis of symmetry is $x = 1$. The y-intercept is -5.



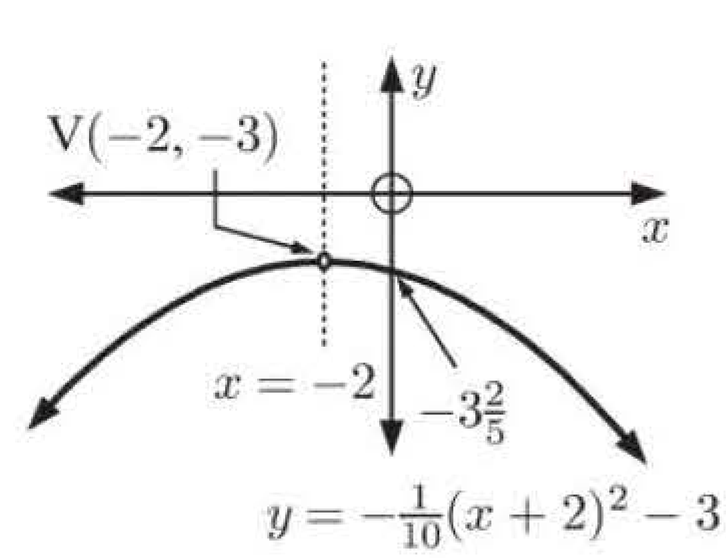
d The vertex is $(3, 2)$. The axis of symmetry is $x = 3$. The y-intercept is $6\frac{1}{2}$.



e The vertex is $(1, 4)$. The axis of symmetry is $x = 1$. The y-intercept is $3\frac{2}{3}$.



f The vertex is $(-2, -3)$. The axis of symmetry is $x = -2$. The y-intercept is $-3\frac{2}{5}$.



5 **a** G **b** A **c** E **d** B **e** I **f** C **g** D **h** F **i** H

6 **a** $y = x^2 - 4x + 2$
has $a = 1, \ b = -4, \ c = 2$
$\therefore \ -\dfrac{b}{2a} = -\dfrac{(-4)}{2(1)} = 2$
$\therefore$ the axis of symmetry is $x = 2$.
When $x = 2$,
$y = 2^2 - 4 \times 2 + 2 = -2$
$\therefore$ the vertex is at $(2, -2)$.

b $y = 2x^2 + 4$
has $a = 2, \ b = 0, \ c = 4$
$\therefore \ -\dfrac{b}{2a} = -\dfrac{0}{2(2)} = 0$
$\therefore$ the axis of symmetry is $x = 0$.
When $x = 0, \ y = 4$
$\therefore$ the vertex is at $(0, 4)$.

c $y = -3x^2 + 1$
has $a = -3, \ b = 0, \ c = 1$
$\therefore \ -\dfrac{b}{2a} = -\dfrac{0}{2(-3)} = 0$
$\therefore$ the axis of symmetry is $x = 0$.
When $x = 0, \ y = 1$
$\therefore$ the vertex is at $(0, 1)$.

d $y = -x^2 - 4x - 9$
has $a = -1, \ b = -4, \ c = -9$
$\therefore \ -\dfrac{b}{2a} = -\dfrac{(-4)}{2(-1)} = -2$
$\therefore$ the axis of symmetry is $x = -2$.
When $x = -2, \ y = -(-2)^2 - 4(-2) - 9$
$= -4 + 8 - 9$
$= -5$
$\therefore$ the vertex is at $(-2, -5)$.

e $y = 2x^2 + 6x - 1$
has $a = 2, \ b = 6, \ c = -1$
$\therefore \ -\dfrac{b}{2a} = -\dfrac{6}{2(2)} = -\dfrac{3}{2}$
$\therefore$ the axis of symmetry is $x = -\frac{3}{2}$.
When $x = -\frac{3}{2}, \ y = 2(-\frac{3}{2})^2 + 6(-\frac{3}{2}) - 1$
$= \frac{9}{2} - 9 - 1$
$= -\frac{11}{2}$
$\therefore$ the vertex is at $(-\frac{3}{2}, -\frac{11}{2})$.

f $y = -\frac{1}{2}x^2 + x - 5$
has $a = -\frac{1}{2}, \ b = 1, \ c = -5$
$\therefore \ -\dfrac{b}{2a} = -\dfrac{1}{2(-\frac{1}{2})} = 1$
$\therefore$ the axis of symmetry is $x = 1$.
When $x = 1$,
$y = -\frac{1}{2}(1)^2 + 1 - 5 = -\frac{9}{2}$
$\therefore$ the vertex is at $(1, -\frac{9}{2})$.

7 **a** When $y = 0, \ x^2 - 9 = 0$
$\therefore \ (x + 3)(x - 3) = 0$
$\therefore \ x = \pm 3$
$\therefore$ the x-intercepts are ± 3

b When $y = 0, \ 2x^2 - 6 = 0$
$\therefore \ x^2 - 3 = 0$
$\therefore \ (x + \sqrt{3})(x - \sqrt{3}) = 0$
$\therefore \ x = \pm\sqrt{3}$
$\therefore$ the x-intercepts are $\pm\sqrt{3}$

c When $y = 0, \ x^2 + x - 12 = 0$
$\therefore \ (x + 4)(x - 3) = 0$
$\therefore \ x = -4$ or 3
$\therefore$ the x-intercepts are -4 and 3

d When $y = 0, \ 4x - x^2 = 0$
$\therefore \ x(4 - x) = 0$
$\therefore \ x = 0$ or 4
$\therefore$ the x-intercepts are 0 and 4

e When $y = 0, \ -x^2 - 6x - 8 = 0$
$\therefore \ x^2 + 6x + 8 = 0$
$\therefore \ (x + 4)(x + 2) = 0$
$\therefore \ x = -4$ or -2
$\therefore$ the x-intercepts are -4 and -2

f When $y = 0, \ -2x^2 - 4x - 2 = 0$
$\therefore \ x^2 + 2x + 1 = 0$
$\therefore \ (x + 1)^2 = 0$
$\therefore \ x = -1$
$\therefore$ the x-intercept is -1 (touching)

g When $y = 0, \ 4x^2 - 24x + 36 = 0$
$\therefore \ x^2 - 6x + 9 = 0$
$\therefore \ (x - 3)^2 = 0$
$\therefore \ x = 3$
$\therefore$ the x-intercept is 3 (touching)

h When $y = 0$, $x^2 - 4x + 1 = 0$
$a = 1$, $b = -4$, and $c = 1$

$$\therefore \quad x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(1)}}{2(1)}$$
$$= \frac{4 \pm \sqrt{12}}{2}$$
$$= \frac{4 \pm 2\sqrt{3}}{2}$$
$$= 2 \pm \sqrt{3}$$

$\therefore$ the x-intercepts are $2 \pm \sqrt{3}$

i When $y = 0$, $x^2 + 8x + 11 = 0$
$a = 1$, $b = 8$, and $c = 11$

$$\therefore \quad x = \frac{-8 \pm \sqrt{8^2 - 4(1)(11)}}{2(1)}$$
$$= \frac{-8 \pm \sqrt{20}}{2}$$
$$= \frac{-8 \pm 2\sqrt{5}}{2}$$
$$= -4 \pm \sqrt{5}$$

$\therefore$ the x-intercepts are $-4 \pm \sqrt{5}$

8 **a** **i** $y = x^2 - 2x + 5$
has $a = 1$, $b = -2$, $c = 5$

$$\therefore \quad -\frac{b}{2a} = -\frac{(-2)}{2(1)} = 1$$

$\therefore$ the axis of symmetry is $x = 1$

ii When $x = 1$,
$$y = 1^2 - 2(1) + 5$$
$$= 1 - 2 + 5$$
$$= 4$$

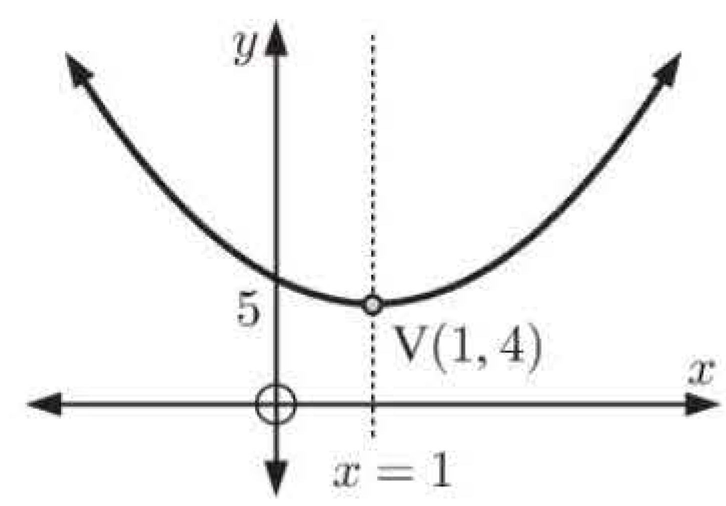
$\therefore$ the vertex is at $(1,\ 4)$

iii When $x = 0$, $y = 5$,
so the y-intercept is 5
When $y = 0$, $x^2 - 2x + 5 = 0$

$$\therefore \quad x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(5)}}{2(1)}$$
$$= \frac{2 \pm \sqrt{4 - 20}}{2}$$

This has no real solutions,
so there are no x-intercepts.

iv



b **i** $y = 2x^2 - 5x + 2$
has $a = 2$, $b = -5$, $c = 2$

$$\therefore \quad -\frac{b}{2a} = -\frac{(-5)}{2(2)} = \frac{5}{4}$$

$\therefore$ the axis of symmetry is $x = \frac{5}{4}$

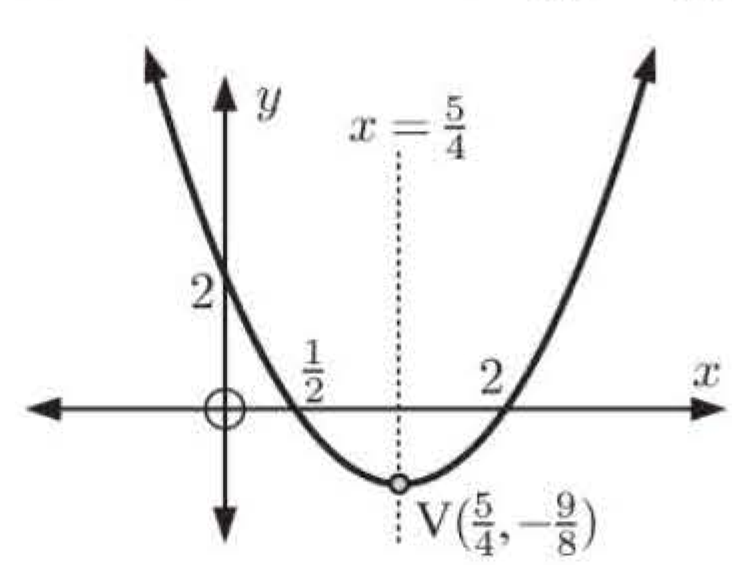
ii When $x = \frac{5}{4}$,
$$y = 2(\tfrac{5}{4})^2 - 5(\tfrac{5}{4}) + 2$$
$$= \tfrac{25}{8} - \tfrac{25}{4} + 2$$
$$= -\tfrac{9}{8}$$

$\therefore$ the vertex is at $(\frac{5}{4}, -\frac{9}{8})$

iii When $x = 0$, $y = 2$,
so the y-intercept is 2.
When $y = 0$, $2x^2 - 5x + 2 = 0$
$$\therefore \quad (2x - 1)(x - 2) = 0$$
$$\therefore \quad x = \tfrac{1}{2} \text{ or } 2$$

$\therefore$ the x-intercepts are $\frac{1}{2}$ and 2

iv



c **i** $y = -x^2 + 3x - 2$
has $a = -1$, $b = 3$, $c = -2$

$$\therefore \quad -\frac{b}{2a} = -\frac{3}{2(-1)} = \frac{3}{2}$$

$\therefore$ the axis of symmetry is $x = \frac{3}{2}$

ii When $x = \frac{3}{2}$, $y = -(\frac{3}{2})^2 + 3(\frac{3}{2}) - 2$
$$= -\tfrac{9}{4} + \tfrac{9}{2} - 2$$
$$= \tfrac{1}{4}$$

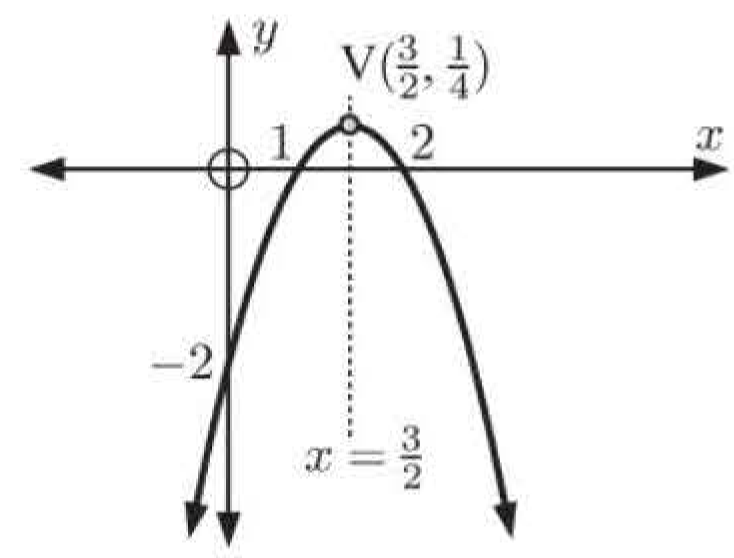
$\therefore$ the vertex is at $(\frac{3}{2}, \frac{1}{4})$

iii When $x = 0$, $y = -2$,
so the y-intercept is -2.
When $y = 0$, $-x^2 + 3x - 2 = 0$

$$\therefore \quad x^2 - 3x + 2 = 0$$
$$\therefore \quad (x-1)(x-2) = 0$$
$$\therefore \quad x = 1 \text{ or } 2$$

$\therefore$ the x-intercepts are 1 and 2

iv



d **i** $y = -2x^2 + x + 1$
has $a = -2$, $b = 1$, $c = 1$

$$\therefore \quad -\frac{b}{2a} = -\frac{1}{2(-2)} = \frac{1}{4}$$

$\therefore$ the axis of symmetry is $x = \frac{1}{4}$

ii When $x = \frac{1}{4}$,
$$y = -2(\tfrac{1}{4})^2 + \tfrac{1}{4} + 1$$
$$= -\tfrac{1}{8} + \tfrac{1}{4} + 1$$
$$= \tfrac{9}{8}$$

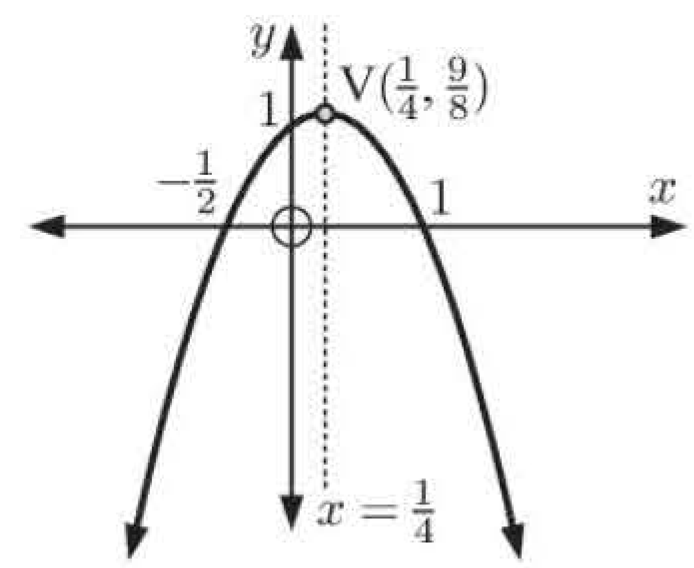
$\therefore$ the vertex is at $(\frac{1}{4}, \frac{9}{8})$

iii When $x = 0$, $y = 1$,
so the y-intercept is 1.
When $y = 0$, $-2x^2 + x + 1 = 0$

$$\therefore \quad 2x^2 - x - 1 = 0$$
$$\therefore \quad (2x+1)(x-1) = 0$$
$$\therefore \quad x = -\tfrac{1}{2} \text{ or } 1$$

$\therefore$ the x-intercepts are $-\frac{1}{2}$ and 1

iv



e **i** $y = 6x - x^2$
has $a = -1$, $b = 6$, $c = 0$

$$\therefore \quad -\frac{b}{2a} = -\frac{6}{2(-1)} = 3$$

$\therefore$ the axis of symmetry is $x = 3$

ii When $x = 3$,
$$y = 6 \times 3 - 3^2$$
$$= 9$$

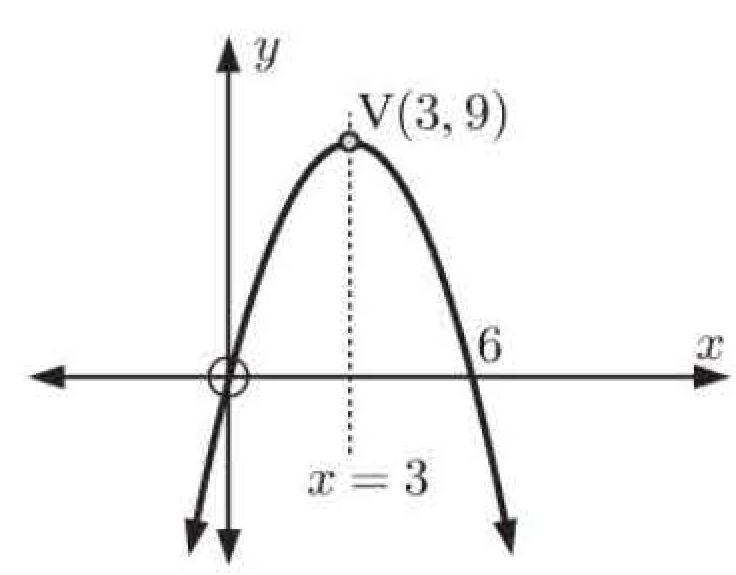
$\therefore$ the vertex is at $(3, 9)$

iii When $x = 0$, $y = 0$,
so the y-intercept is 0.
When $y = 0$, $6x - x^2 = 0$

$$\therefore \quad x(6-x) = 0$$
$$\therefore \quad x = 0 \text{ or } 6$$

$\therefore$ the x-intercepts are 0 and 6

iv



f **i** $y = -\frac{1}{4}x^2 + 2x + 1$
has $a = -\frac{1}{4}$, $b = 2$, $c = 1$

$$\therefore \quad -\frac{b}{2a} = -\frac{2}{2(-\frac{1}{4})} = 4$$

$\therefore$ the axis of symmetry is $x = 4$

ii When $x = 4$,
$$y = -\tfrac{1}{4}(4)^2 + 2(4) + 1$$
$$= -4 + 8 + 1$$
$$= 5$$

$\therefore$ the vertex is at $(4, 5)$

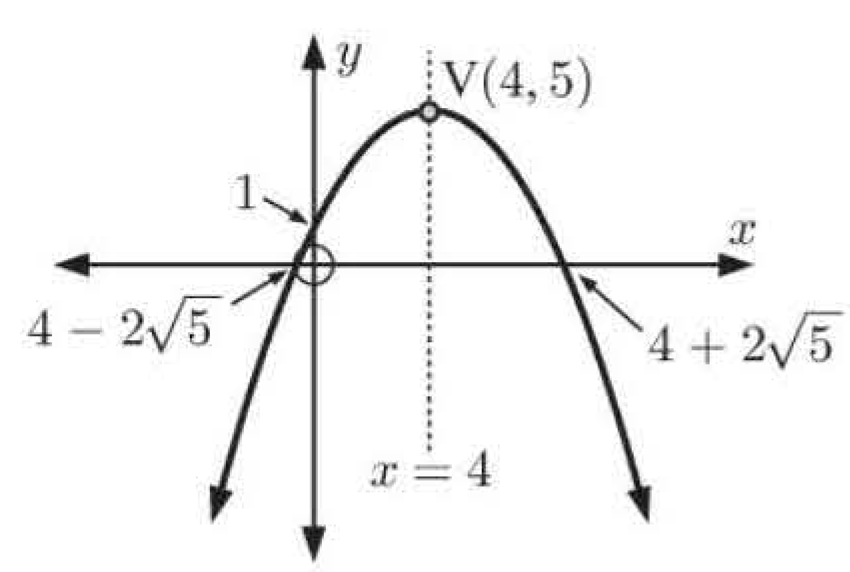
iii When $x = 0$, $y = 1$,
so the y-intercept is 1.
When $y = 0$, $-\frac{1}{4}x^2 + 2x + 1 = 0$

$$\therefore \quad x^2 - 8x - 4 = 0$$

$$\therefore \quad x = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(1)(-4)}}{2(1)}$$
$$= \frac{8 \pm \sqrt{80}}{2}$$
$$= \frac{8 \pm 4\sqrt{5}}{2}$$
$$= 4 \pm 2\sqrt{5}$$

$\therefore$ the x-intercepts are $4 \pm 2\sqrt{5}$.

iv



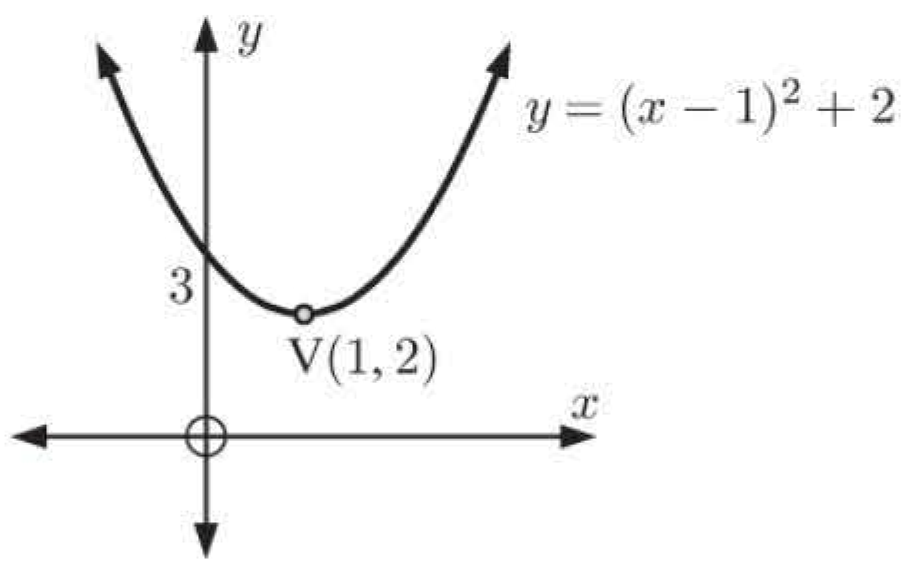
EXERCISE 1D.2

1 **a** $y = x^2 - 2x + 3$

$\therefore\ y = x^2 - 2x + 1^2 + 3 - 1^2$

$\therefore\ y = (x-1)^2 + 2$

$\therefore$ vertex is $(1, 2)$, y-intercept is 3

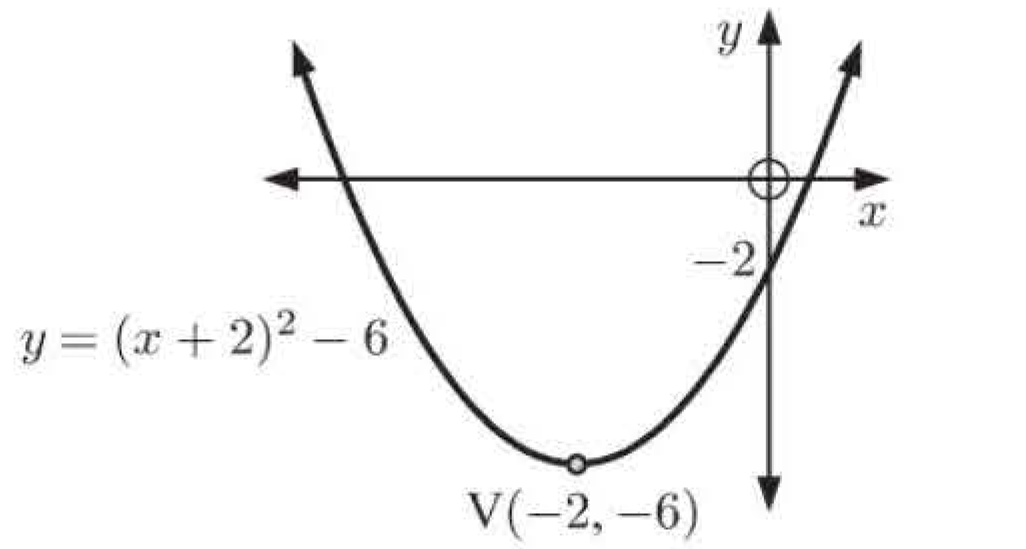


b $y = x^2 + 4x - 2$

$\therefore\ y = x^2 + 4x + 2^2 - 2 - 2^2$

$\therefore\ y = (x+2)^2 - 6$

$\therefore$ vertex is $(-2, -6)$, y-intercept is -2

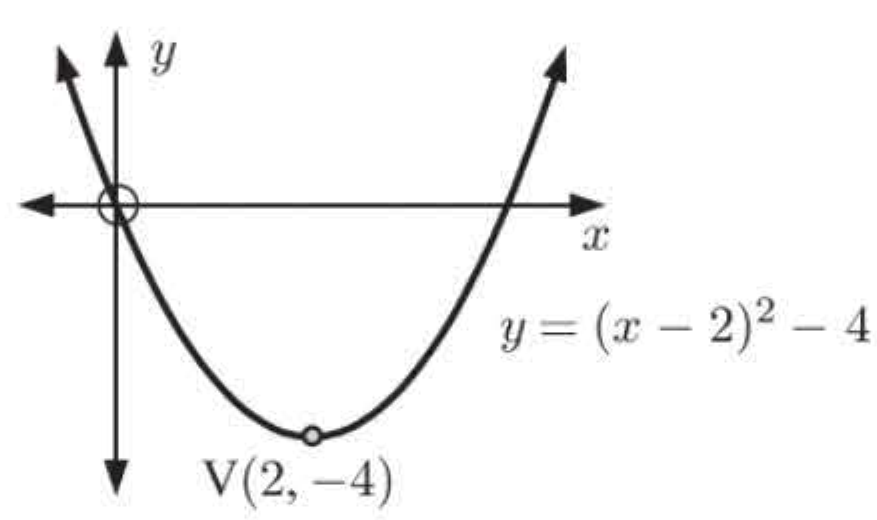


c $y = x^2 - 4x$

$\therefore\ y = x^2 - 4x + 2^2 - 2^2$

$\therefore\ y = (x-2)^2 - 4$

$\therefore$ vertex is $(2, -4)$, y-intercept is 0

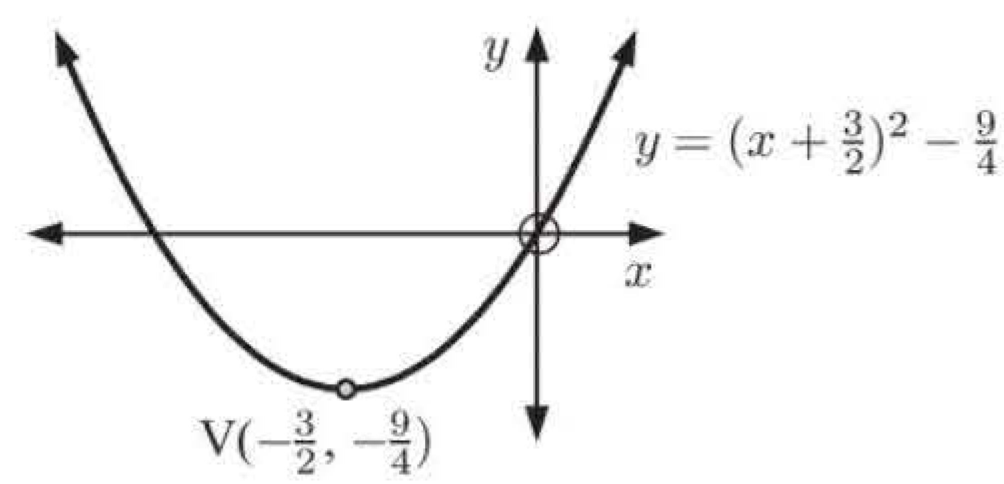


d $y = x^2 + 3x$

$\therefore\ y = x^2 + 3x + (\frac{3}{2})^2 - (\frac{3}{2})^2$

$\therefore\ y = (x + \frac{3}{2})^2 - \frac{9}{4}$

$\therefore$ vertex is $(-\frac{3}{2}, -\frac{9}{4})$, y-intercept is 0

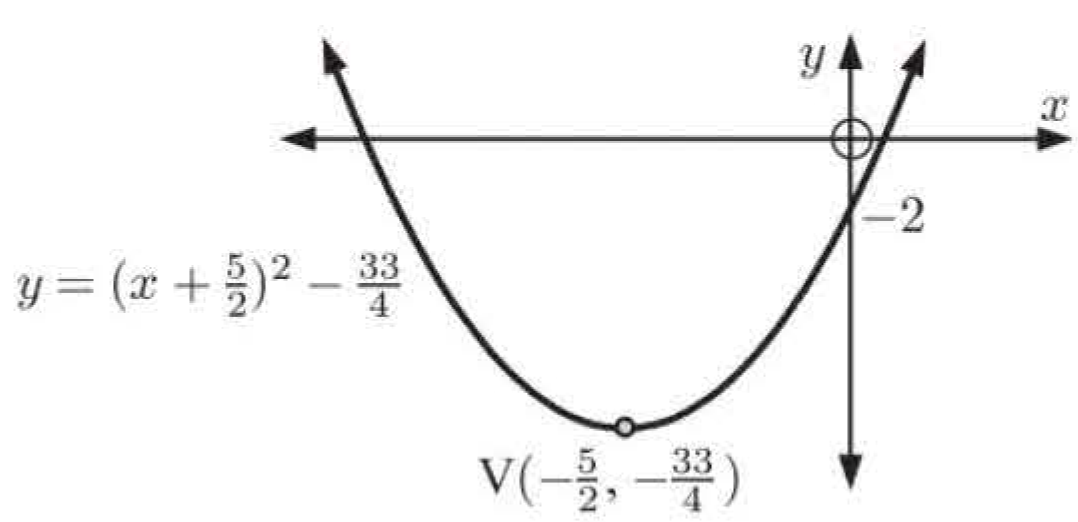


e $y = x^2 + 5x - 2$

$\therefore\ y = x^2 + 5x + (\frac{5}{2})^2 - 2 - (\frac{5}{2})^2$

$\therefore\ y = (x + \frac{5}{2})^2 - \frac{33}{4}$

$\therefore$ vertex is $(-\frac{5}{2}, -\frac{33}{4})$, y-intercept is -2

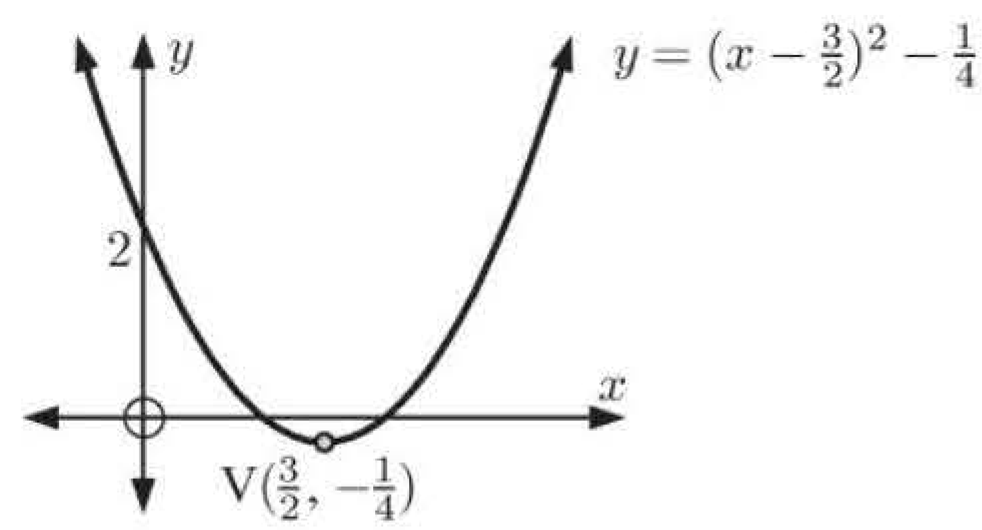


f $y = x^2 - 3x + 2$

$\therefore\ y = x^2 - 3x + (\frac{3}{2})^2 + 2 - (\frac{3}{2})^2$

$\therefore\ y = (x - \frac{3}{2})^2 - \frac{1}{4}$

$\therefore$ vertex is $(\frac{3}{2}, -\frac{1}{4})$, y-intercept is 2

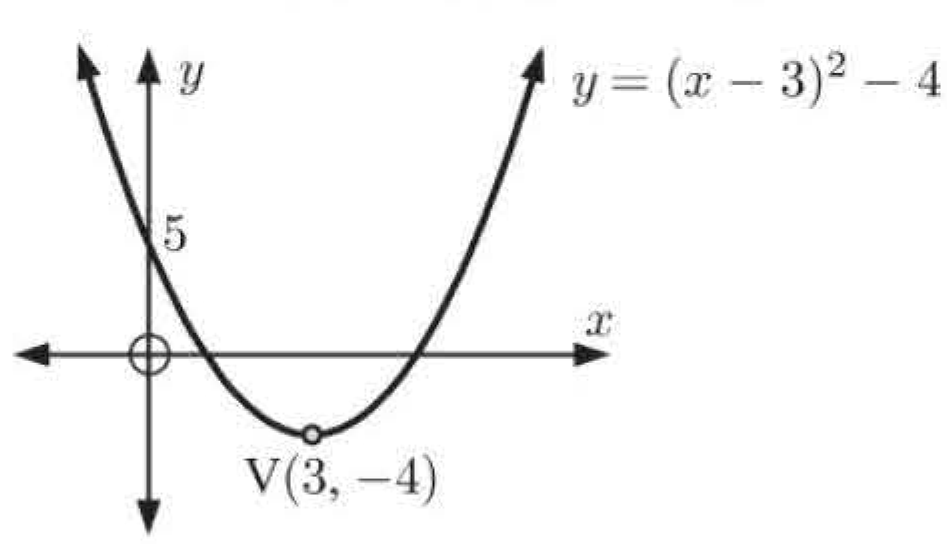


g $y = x^2 - 6x + 5$

$\therefore\ y = x^2 - 6x + 3^2 + 5 - 3^2$

$\therefore\ y = (x-3)^2 - 4$

$\therefore$ vertex is $(3, -4)$, y-intercept is 5

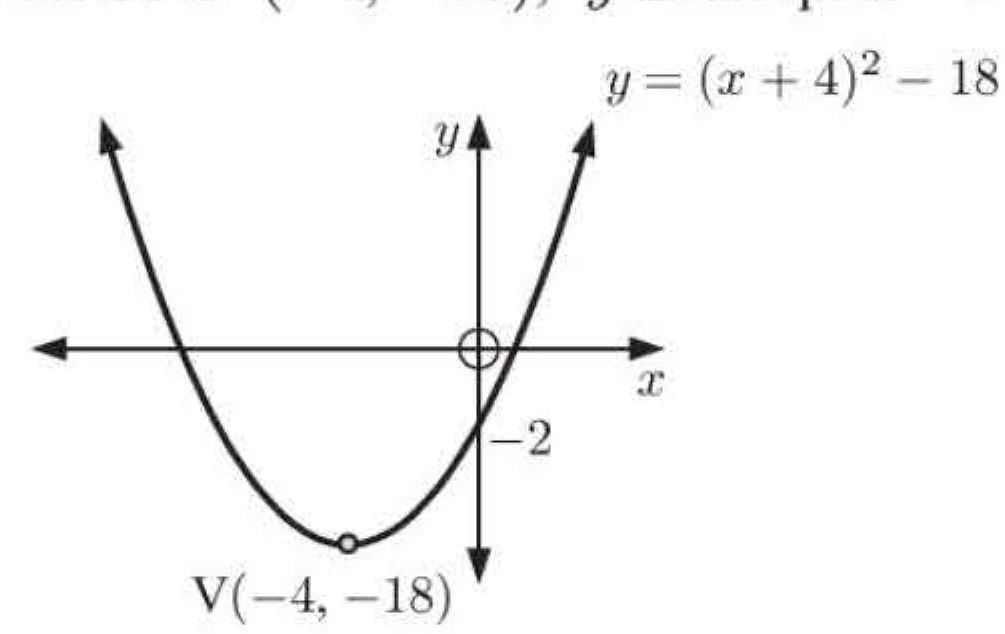


h $y = x^2 + 8x - 2$

$\therefore\ y = x^2 + 8x + 4^2 - 2 - 4^2$

$\therefore\ y = (x+4)^2 - 18$

$\therefore$ vertex is $(-4, -18)$, y-intercept is -2

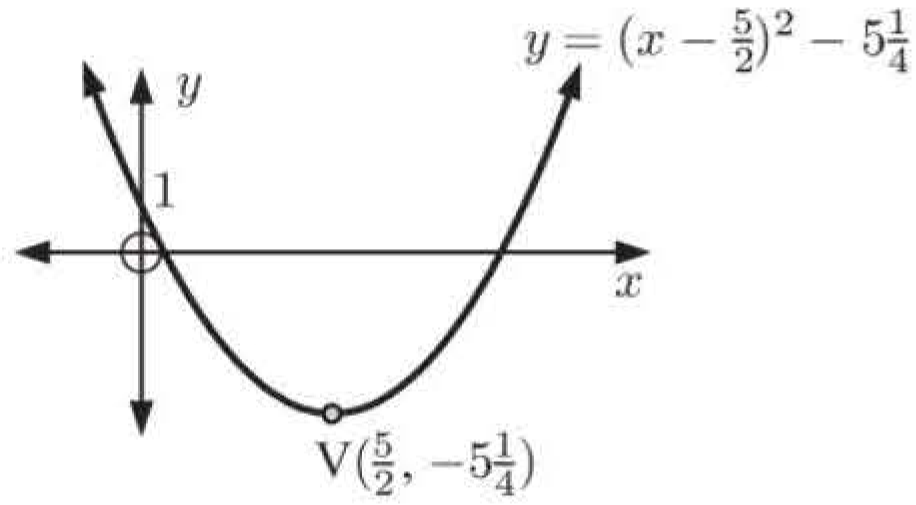


i $y = x^2 - 5x + 1$

$\therefore\ y = x^2 - 5x + (\frac{5}{2})^2 + 1 - (\frac{5}{2})^2$

$\therefore\ y = (x - \frac{5}{2})^2 - \frac{21}{4}$

$\therefore$ vertex is $(\frac{5}{2}, -5\frac{1}{4})$, y-intercept is 1



2 **a** **i** $y = 2x^2 + 4x + 5$

$= 2[x^2 + 2x + \frac{5}{2}]$

$= 2[x^2 + 2x + 1^2 - 1^2 + \frac{5}{2}]$

$= 2[(x+1)^2 + \frac{3}{2}]$

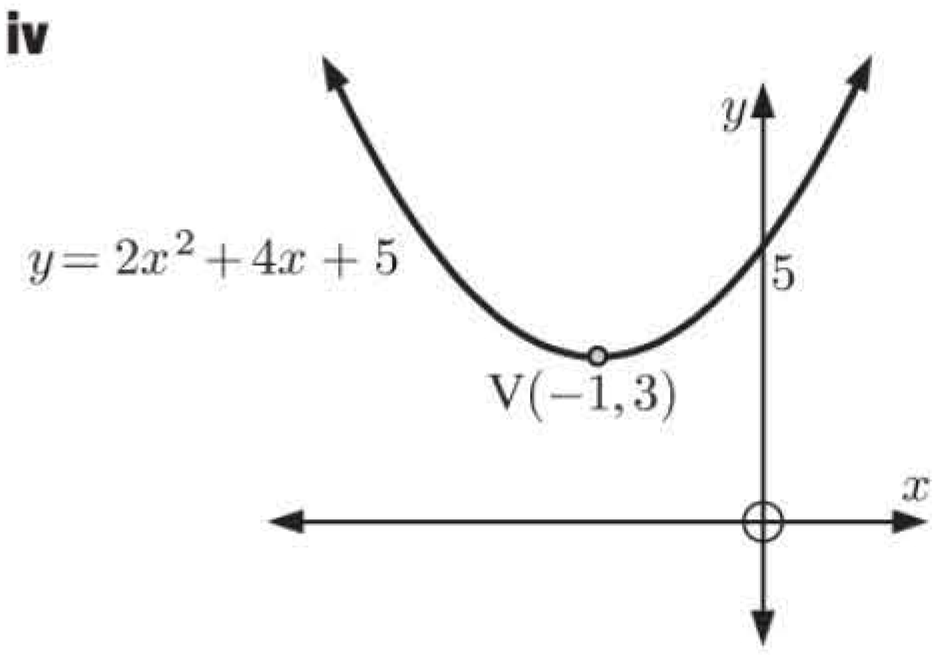
$= 2(x+1)^2 + 3$

ii The vertex is $(-1, 3)$.

iii When $x = 0$, $y = 5$

$\therefore$ the y-intercept is 5

iv



b **i** $y = 2x^2 - 8x + 3$

$= 2[x^2 - 4x + \frac{3}{2}]$

$= 2[x^2 - 4x + 2^2 - 2^2 + \frac{3}{2}]$

$= 2[(x-2)^2 - \frac{5}{2}]$

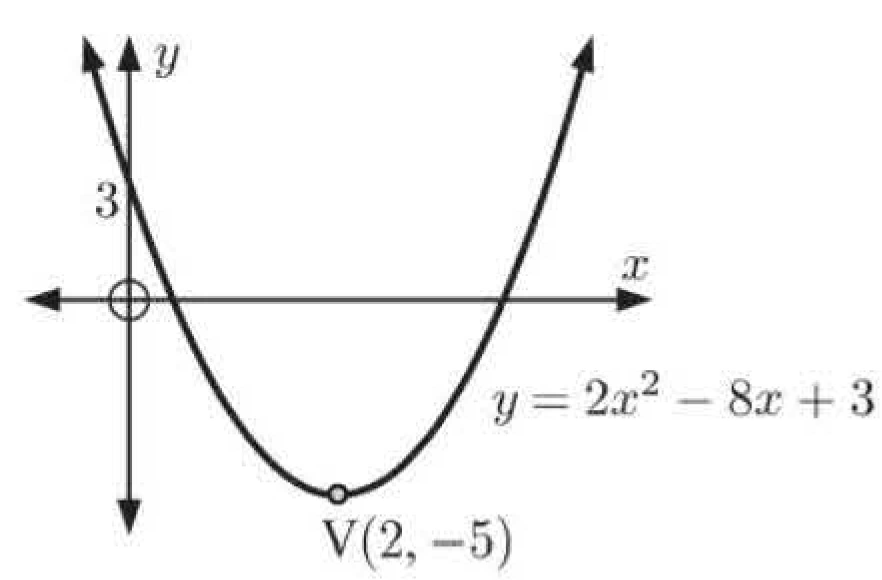
$= 2(x-2)^2 - 5$

ii The vertex is $(2, -5)$.

iii When $x = 0$, $y = 3$

$\therefore$ the y-intercept is 3

iv



c **i** $y = 2x^2 - 6x + 1$

$= 2[x^2 - 3x + \frac{1}{2}]$

$= 2[x^2 - 3x + (\frac{3}{2})^2 - (\frac{3}{2})^2 + \frac{1}{2}]$

$= 2[(x - \frac{3}{2})^2 - \frac{7}{4}]$

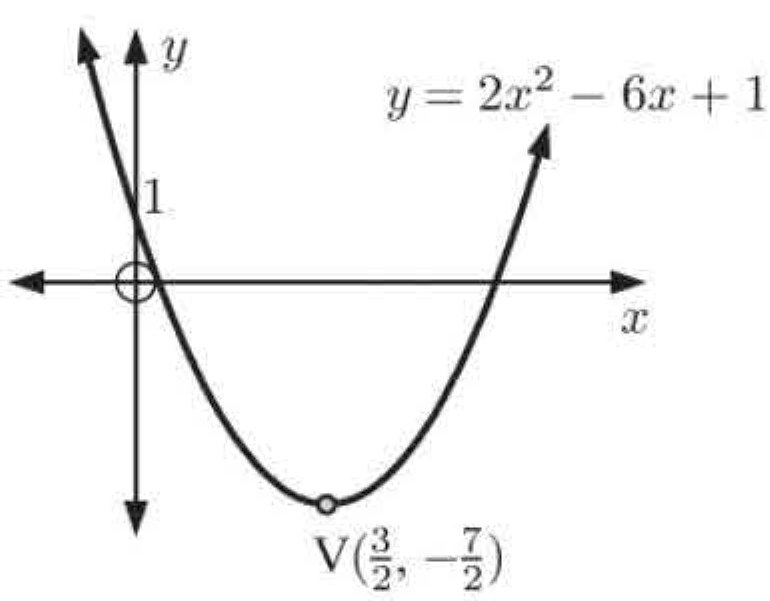
$= 2(x - \frac{3}{2})^2 - \frac{7}{2}$

ii The vertex is $(\frac{3}{2}, -\frac{7}{2})$.

iii When $x = 0$, $y = 1$

$\therefore$ the y-intercept is 1

iv



d **i** $y = 3x^2 - 6x + 5$

$= 3[x^2 - 2x + \frac{5}{3}]$

$= 3[x^2 - 2x + 1^2 - 1^2 + \frac{5}{3}]$

$= 3[(x-1)^2 + \frac{2}{3}]$

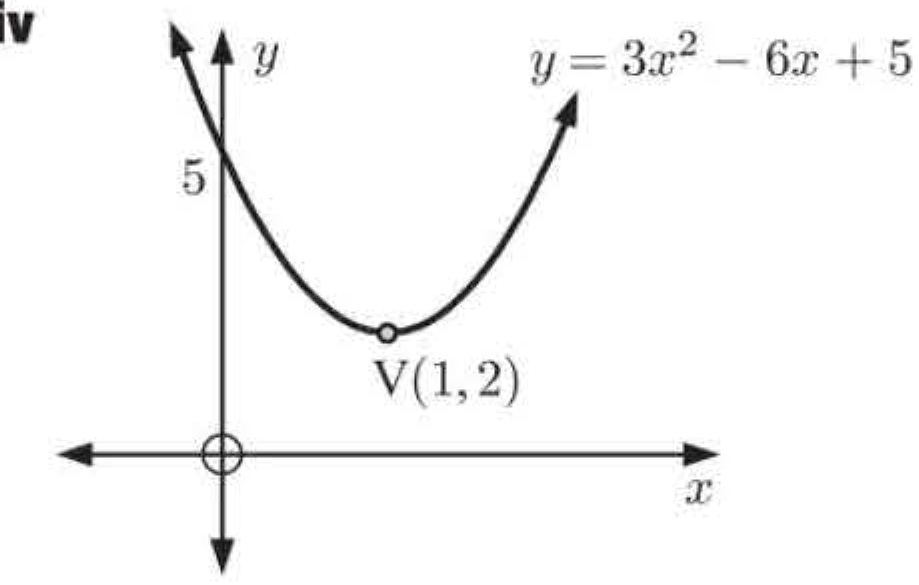
$= 3(x-1)^2 + 2$

ii The vertex is $(1, 2)$.

iii When $x = 0$, $y = 5$

$\therefore$ the y-intercept is 5

iv



e **i** $y = -x^2 + 4x + 2$

$= -[x^2 - 4x - 2]$

$= -[x^2 - 4x + 2^2 - 2^2 - 2]$

$= -[(x-2)^2 - 6]$

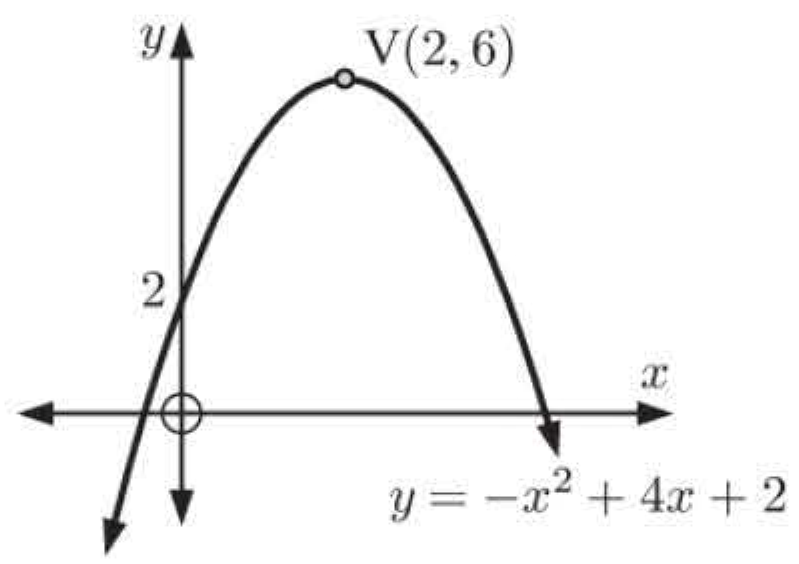
$= -(x-2)^2 + 6$

ii The vertex is $(2, 6)$.

iii When $x = 0$, $y = 2$

$\therefore$ the y-intercept is 2

iv



f **i** $y = -2x^2 - 5x + 3$

$= -2[x^2 + \frac{5}{2}x - \frac{3}{2}]$

$= -2[x^2 + \frac{5}{2}x + (\frac{5}{4})^2 - (\frac{5}{4})^2 - \frac{3}{2}]$

$= -2[(x + \frac{5}{4})^2 - \frac{25}{16} - \frac{24}{16}]$

$= -2[(x + \frac{5}{4})^2 - \frac{49}{16}]$

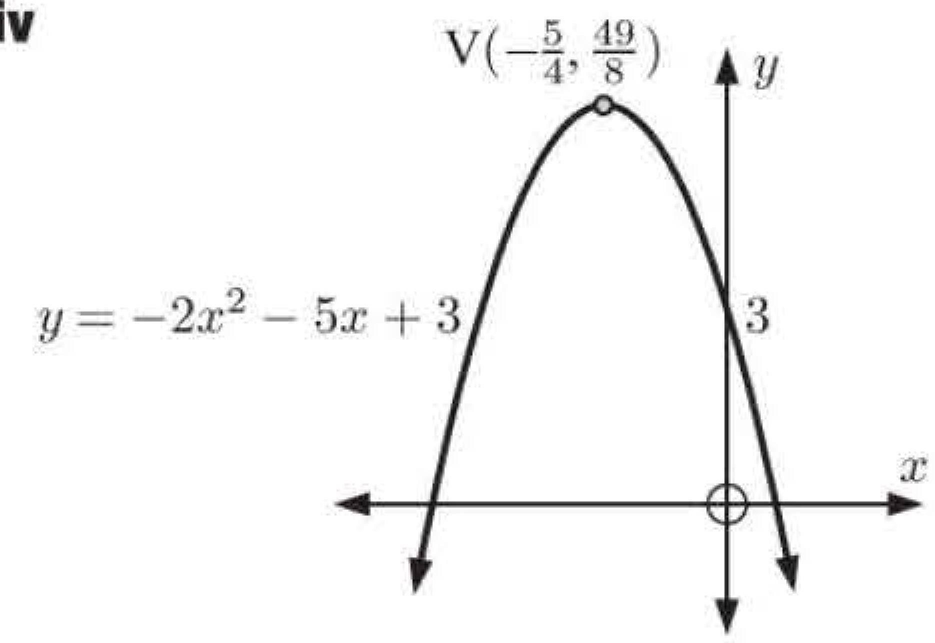
$= -2(x + \frac{5}{4})^2 + \frac{49}{8}$

ii The vertex is $(-\frac{5}{4}, \frac{49}{8})$.

iii When $x = 0$, $y = 3$

$\therefore$ the y-intercept is 3

iv



EXERCISE 1D.3

1 **a** $y = x^2 + x - 2$

has $a = 1$, $b = 1$, $c = -2$

$\therefore$ $\Delta = b^2 - 4ac$

$= 1^2 - 4(1)(-2)$

$= 9 > 0$

$\therefore$ the graph cuts the x-axis twice, and since $a > 0$, the graph is concave up.

b $y = x^2 + 7x - 2$

has $a = 1$, $b = 7$, $c = -2$

$\therefore$ $\Delta = b^2 - 4ac$

$= 7^2 - 4(1)(-2)$

$= 57 > 0$

$\therefore$ the graph cuts the x-axis twice, and since $a > 0$, the graph is concave up.

c $y = x^2 + 8x + 16$

has $a = 1$, $b = 8$, $c = 16$

$\therefore$ $\Delta = b^2 - 4ac$

$= 8^2 - 4(1)(16)$

$= 0$

$\therefore$ the graph touches the x-axis, and since $a > 0$, the graph is concave up.

d $y = x^2 + 4\sqrt{2}x + 8$

has $a = 1$, $b = 4\sqrt{2}$, $c = 8$

$\therefore$ $\Delta = b^2 - 4ac$

$= (4\sqrt{2})^2 - 4(1)(8)$

$= 0$

$\therefore$ the graph touches the x-axis, and since $a > 0$, the graph is concave up.

e $y = -x^2 + x + 6$

has $a = -1$, $b = 1$, $c = 6$

$\therefore$ $\Delta = b^2 - 4ac$

$= 1^2 - 4(-1)(6)$

$= 25 > 0$

$\therefore$ the graph cuts the x-axis twice, and since $a < 0$, the graph is concave down.

f $y = 9x^2 + 6x + 1$

has $a = 9$, $b = 6$, $c = 1$

$\therefore$ $\Delta = b^2 - 4ac$

$= 6^2 - 4(9)(1)$

$= 0$

$\therefore$ the graph touches the x-axis, and since $a > 0$, the graph is concave up.

2 **a** $x^2 - 3x + 6$

has $a = 1$, $b = -3$, $c = 6$

$\therefore$ $\Delta = b^2 - 4ac$

$= (-3)^2 - 4(1)(6)$

$= -15$

Since $a > 0$ and $\Delta < 0$,

$x^2 - 3x + 6$ is positive definite.

$\therefore$ $x^2 - 3x + 6 > 0$ for all x.

b $4x - x^2 - 6$

has $a = -1$, $b = 4$, $c = -6$

$\therefore$ $\Delta = b^2 - 4ac$

$= 4^2 - 4(-1)(-6)$

$= -8$

Since $a < 0$ and $\Delta < 0$,

$4x - x^2 - 6$ is negative definite.

$\therefore$ $4x - x^2 - 6 < 0$ for all x.

c $2x^2 - 4x + 7$ has $a = 2,\ b = -4,\ c = 7$

$\therefore\ \Delta = b^2 - 4ac$
$= (-4)^2 - 4(2)(7)$
$= -40$

Since $a > 0$ and $\Delta < 0$, $2x^2 - 4x + 7$ is positive definite.

d $-2x^2 + 3x - 4$ has $a = -2,\ b = 3,\ c = -4$

$\therefore\ \Delta = b^2 - 4ac$
$= 3^2 - 4(-2)(-4)$
$= -23$

Since $a < 0$ and $\Delta < 0$, $-2x^2 + 3x - 4$ is negative definite.

3 $3x^2 + kx - 1$ has $a = 3,\ b = k,\ c = -1$

$\therefore\ \Delta = b^2 - 4ac$
$= k^2 - 4(3)(-1)$
$= k^2 + 12$

$\therefore\ \Delta > 0$ for all k
{as $k^2 \geqslant 0$ for all k}

$\therefore\ 3x^2 + kx - 1$ has two real distinct roots for all k.
$\therefore$ it can never be positive definite.

4 $2x^2 + kx + 2$ has $a = 2,\ b = k,\ c = 2$

$\therefore\ \Delta = b^2 - 4ac$
$= k^2 - 4(2)(2)$
$= k^2 - 16$

Now $2x^2 + kx + 2$ has $a > 0$.
$\therefore$ it is positive definite provided $k^2 - 16 < 0$
$\therefore\ k^2 < 16$
$\therefore\ -4 < k < 4$

EXERCISE 1E

1 **a** The x-intercepts are 1 and 2.
$\therefore\ y = a(x-1)(x-2)$ for some $a \neq 0$.
But the y-intercept is 4.
$\therefore\ a(-1)(-2) = 4$
$\therefore\ 2a = 4$
$\therefore\ a = 2$
$\therefore\ y = 2(x-1)(x-2)$

b The graph touches the x-axis when $x = 2$.
$\therefore\ y = a(x-2)^2$ for some $a \neq 0$.
But the y-intercept is 8.
$\therefore\ a(-2)^2 = 8$
$\therefore\ 4a = 8$
$\therefore\ a = 2$
$\therefore\ y = 2(x-2)^2$

c The x-intercepts are 1 and 3.
$\therefore\ y = a(x-1)(x-3)$ for some $a \neq 0$.
But the y-intercept is 3.
$\therefore\ a(-1)(-3) = 3$
$\therefore\ 3a = 3$
$\therefore\ a = 1$
$\therefore\ y = (x-1)(x-3)$

d The x-intercepts are -1 and 3.
$\therefore\ y = a(x+1)(x-3)$ for some $a \neq 0$.
But the y-intercept is 3.
$\therefore\ a(1)(-3) = 3$
$\therefore\ -3a = 3$
$\therefore\ a = -1$
$\therefore\ y = -(x+1)(x-3)$

e The graph touches the x-axis when $x = 1$.
$\therefore\ y = a(x-1)^2$ for some $a \neq 0$.
But the y-intercept is -3.
$\therefore\ a(-1)^2 = -3$
$\therefore\ a = -3$
$\therefore\ y = -3(x-1)^2$

f The x-intercepts are -2 and 3.
$\therefore\ y = a(x+2)(x-3)$ for some $a \neq 0$.
But the y-intercept is 12.
$\therefore\ a(2)(-3) = 12$
$\therefore\ -6a = 12$
$\therefore\ a = -2$
$\therefore\ y = -2(x+2)(x-3)$

2 **a** As the axis of symmetry is $x = 3$, the other x-intercept is 4.
$\therefore\ y = a(x-2)(x-4)$ for some $a \neq 0$.
But the y-intercept $= 12$
$\therefore\ a(-2)(-4) = 12$
$\therefore\ 8a = 12$
$\therefore\ a = \frac{12}{8} = \frac{3}{2}$
$\therefore\ y = \frac{3}{2}(x-2)(x-4)$

b As the axis of symmetry is $x = -1$, the other x-intercept is 2.
$\therefore\ y = a(x+4)(x-2)$ for some $a \neq 0$.
But the y-intercept $= 4$
$\therefore\ a(4)(-2) = 4$
$\therefore\ -8a = 4$
$\therefore\ a = -\frac{1}{2}$
$\therefore\ y = -\frac{1}{2}(x+4)(x-2)$

c The graph touches the x-axis at $x=-3$,
$\therefore\ y=a(x+3)^2$ for some $a\neq 0$.
But the y-intercept is -12, so $a(3)^2=-12$

$$\therefore\ 9a=-12$$
$$\therefore\ a=-\tfrac{12}{9}=-\tfrac{4}{3}$$
$$\therefore\ y=-\tfrac{4}{3}(x+3)^2$$

3 **a** Since the x-intercepts are 5 and 1, the equation is $y=a(x-5)(x-1)$ for some $a\neq 0$.
But when $x=2$, $y=-9$

$$\therefore\ -9=a(2-5)(2-1)$$
$$\therefore\ -9=a(-3)(1)$$
$$\therefore\ -3a=-9$$
$$\therefore\ a=3$$

$\therefore$ the equation is $y=3(x-5)(x-1)$

$$\therefore\ y=3(x^2-6x+5)$$
$$\therefore\ y=3x^2-18x+15$$

b Since the x-intercepts are 2 and $-\frac{1}{2}$, the equation is $y=a(x-2)(x+\frac{1}{2})$ for some $a\neq 0$.
But when $x=3$, $y=-14$

$$\therefore\ -14=a(3-2)(3+\tfrac{1}{2})$$
$$\therefore\ -14=a(1)(\tfrac{7}{2})$$
$$\therefore\ \tfrac{7}{2}a=-14$$
$$\therefore\ a=-4$$

$\therefore$ the equation is $y=-4(x-2)(x+\frac{1}{2})$

$$\therefore\ y=-4(x^2-\tfrac{3}{2}x-1)$$
$$\therefore\ y=-4x^2+6x+4$$

c Since the graph touches the x-axis at 3, its equation is $y=a(x-3)^2$, for some $a\neq 0$.
But when $x=-2$, $y=-25$

$$\therefore\ -25=a(-2-3)^2$$
$$\therefore\ -25=25a$$
$$\therefore\ a=-1$$

$\therefore$ the equation is $y=-(x-3)^2$

$$\therefore\ y=-(x^2-6x+9)$$
$$\therefore\ y=-x^2+6x-9$$

d Since the graph touches the x-axis at -2, its equation is $y=a(x+2)^2$, for some $a\neq 0$.
But when $x=-1$, $y=4$

$$\therefore\ 4=a(-1+2)^2$$
$$\therefore\ 4=a$$

$\therefore$ the equation is $y=4(x+2)^2$

$$\therefore\ y=4(x^2+4x+4)$$
$$\therefore\ y=4x^2+16x+16$$

e Since the graph cuts the x-axis at 3 and has axis of symmetry $x=2$, it must also cut the x-axis at 1.
$\therefore$ the x-intercepts are 3 and 1, and the equation is $y=a(x-3)(x-1)$ for some $a\neq 0$.
But when $x=5$, $y=12$

$$\therefore\ 12=a(5-3)(5-1)$$
$$\therefore\ 12=a(2)(4)$$
$$\therefore\ 8a=12$$
$$\therefore\ a=\tfrac{12}{8}=\tfrac{3}{2}$$

$\therefore$ the equation is $y=\frac{3}{2}(x-3)(x-1)$

$$\therefore\ y=\tfrac{3}{2}(x^2-4x+3)$$
$$\therefore\ y=\tfrac{3}{2}x^2-6x+\tfrac{9}{2}$$

f Since the graph cuts the x-axis at 5 and has axis of symmetry $x=1$, it must also cut the x-axis at -3.
$\therefore$ the x-intercepts are 5 and -3, and the equation is $y=a(x-5)(x+3)$ for some $a\neq 0$.
But when $x=2$, $y=5$

$$\therefore\ 5=a(2-5)(2+3)$$
$$\therefore\ 5=a(-3)(5)$$
$$\therefore\ -15a=5$$
$$\therefore\ a=-\tfrac{5}{15}=-\tfrac{1}{3}$$

$\therefore$ the equation is $y=-\frac{1}{3}(x-5)(x+3)$

$$\therefore\ y=-\tfrac{1}{3}(x^2-2x-15)$$
$$\therefore\ y=-\tfrac{1}{3}x^2+\tfrac{2}{3}x+5$$

4 **a** The vertex is $(2, 4)$,
so the quadratic has equation
$y = a(x-2)^2 + 4$ for some $a \neq 0$.
But the graph passes through the origin
$\therefore \; 0 = a(0-2)^2 + 4$
$\therefore \; 4a + 4 = 0$
$\therefore \; a = -1$
$\therefore$ the equation is $y = -(x-2)^2 + 4$

b The vertex is $(2, -1)$,
so the quadratic has equation
$y = a(x-2)^2 - 1$ for some $a \neq 0$.
But the graph passes through $(0, 7)$
$\therefore \; 7 = a(0-2)^2 - 1$
$\therefore \; 7 = 4a - 1$
$\therefore \; a = 2$
$\therefore$ the equation is $y = 2(x-2)^2 - 1$

c The vertex is $(3, 8)$,
so the quadratic has equation
$y = a(x-3)^2 + 8$ for some $a \neq 0$.
But the graph passes through $(1, 0)$
$\therefore \; 0 = a(1-3)^2 + 8$
$\therefore \; 0 = 4a + 8$
$\therefore \; a = -2$
$\therefore$ the equation is $y = -2(x-3)^2 + 8$

d The vertex is $(4, -6)$,
so the quadratic has equation
$y = a(x-4)^2 - 6$ for some $a \neq 0$.
But the graph passes through $(7, 0)$
$\therefore \; 0 = a(7-4)^2 - 6$
$\therefore \; 9a - 6 = 0$
$\therefore \; a = \frac{2}{3}$
$\therefore$ the equation is $y = \frac{2}{3}(x-4)^2 - 6$

e The vertex is $(2, 3)$,
so the quadratic has equation
$y = a(x-2)^2 + 3$ for some $a \neq 0$.
But the graph passes through $(3, 1)$
$\therefore \; 1 = a(3-2)^2 + 3$
$\therefore \; 1 = a + 3$
$\therefore \; a = -2$
$\therefore$ the equation is $y = -2(x-2)^2 + 3$

f The vertex is $(\frac{1}{2}, -\frac{3}{2})$,
so the quadratic has equation
$y = a(x-\frac{1}{2})^2 - \frac{3}{2}$ for some $a \neq 0$.
But the graph passes through $(\frac{3}{2}, \frac{1}{2})$
$\therefore \; \frac{1}{2} = a(\frac{3}{2} - \frac{1}{2})^2 - \frac{3}{2}$
$\therefore \; \frac{1}{2} = a - \frac{3}{2}$
$\therefore \; a = 2$
$\therefore$ the equation is $y = 2(x-\frac{1}{2})^2 - \frac{3}{2}$

EXERCISE 1F

1 **a** $y = x^2 - 2x + 8$ meets $y = x + 6$
when $x^2 - 2x + 8 = x + 6$
$\therefore \; x^2 - 3x + 2 = 0$
$\therefore \; (x-1)(x-2) = 0$
$\therefore \; x = 1$ or 2
Substituting into $y = x + 6$,
when $x = 1$, $y = 7$
and when $x = 2$, $y = 8$
$\therefore$ the graphs intersect at $(1, 7)$ and $(2, 8)$.

b $y = -x^2 + 3x + 9$ meets $y = 2x - 3$
when $-x^2 + 3x + 9 = 2x - 3$
$\therefore \; x^2 - x - 12 = 0$
$\therefore \; (x-4)(x+3) = 0$
$\therefore \; x = 4$ or -3
Substituting into $y = 2x - 3$,
when $x = -3$, $y = 2(-3) - 3 = -9$
and when $x = 4$, $y = 2(4) - 3 = 5$
$\therefore$ the graphs intersect at $(-3, -9)$ and $(4, 5)$.

c $y = x^2 - 4x + 3$ meets $y = 2x - 6$
when $x^2 - 4x + 3 = 2x - 6$
$\therefore \; x^2 - 6x + 9 = 0$
$\therefore \; (x-3)^2 = 0$
$\therefore \; x = 3$
Substituting into $y = 2x - 6$,
when $x = 3$, $y = 0$
$\therefore$ the graphs touch at $(3, 0)$.

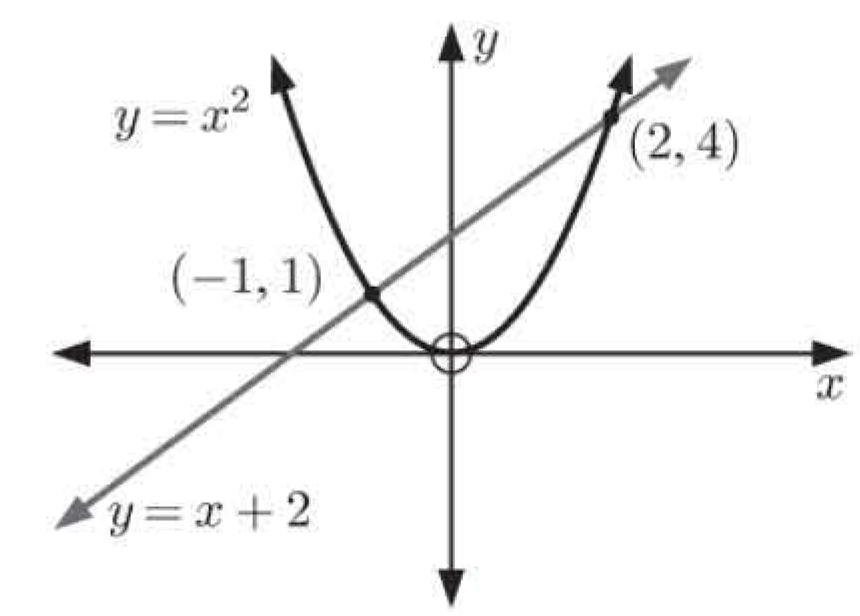
d $y = -x^2 + 4x - 7$ meets $y = 5x - 4$
when $-x^2 + 4x - 7 = 5x - 4$
$\therefore \; x^2 + x + 3 = 0$
which has $a = 1$, $b = 1$, $c = 3$

$$\therefore \; x = \frac{-1 \pm \sqrt{1^2 - 4(1)(3)}}{2(1)}$$

$$\therefore \; x = \frac{-1 \pm \sqrt{-11}}{2}$$

$\therefore$ there are no real solutions
$\therefore$ the graphs do not intersect.

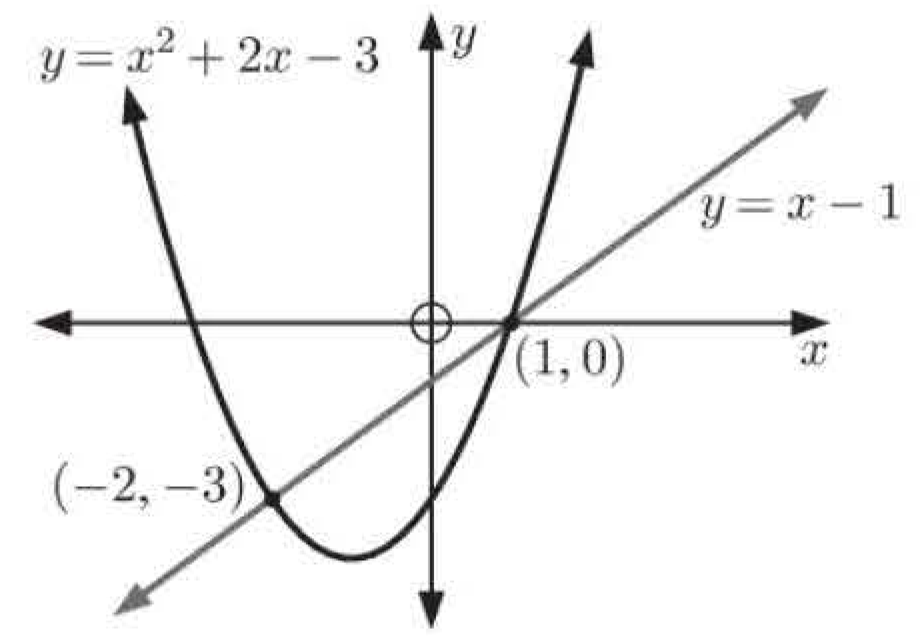
2 **a** **i**



$\therefore$ $y=x^2$ meets $y=x+2$ at the points $(-1,\ 1)$ and $(2,\ 4)$.

ii Using the graph in **a i**, $x^2 > x+2$ when $x < -1$ or $x > 2$.

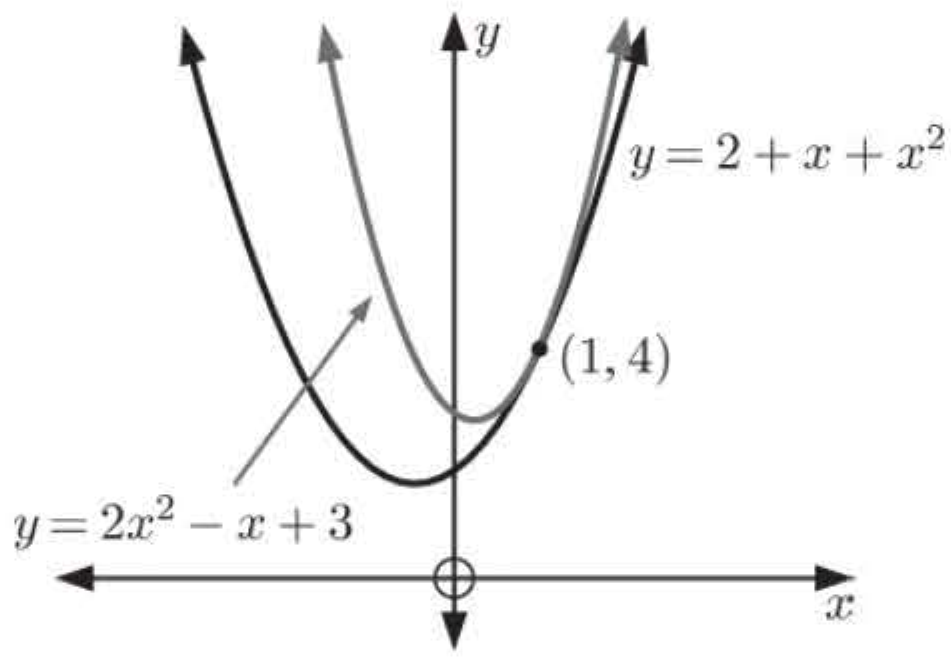
b **i**



$\therefore$ $y=x^2+2x-3$ meets $y=x-1$ at the points $(-2,\ -3)$ and $(1,\ 0)$.

ii Using the graph in **b i**, $x^2+2x-3 > x-1$ when $x < -2$ or $x > 1$.

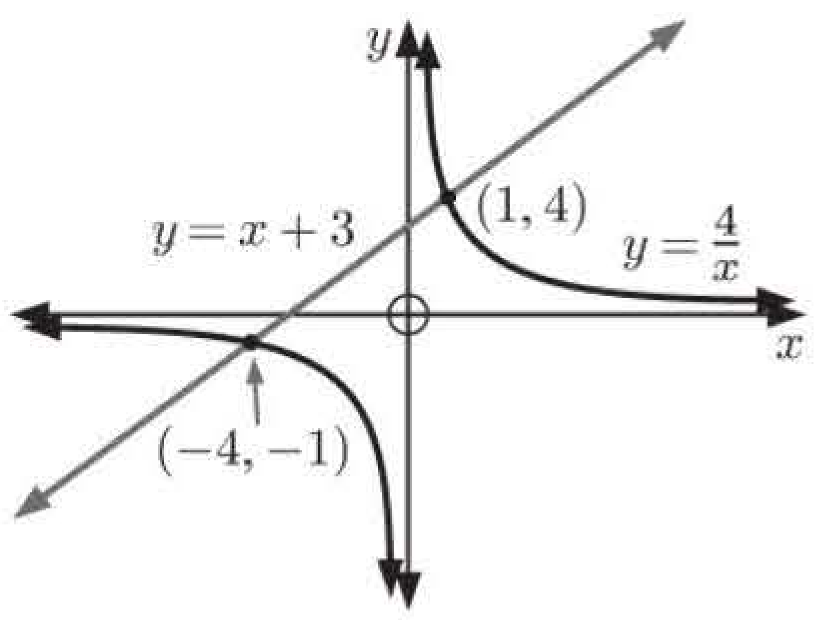
c **i**



$\therefore$ $y=2x^2-x+3$ touches $y=2+x+x^2$ at the point $(1,\ 4)$.

ii Using the graph in **c i**, $2x^2-x+3 > 2+x+x^2$ when $x \neq 1$.

d **i**



$\therefore$ $y=\dfrac{4}{x}$ meets $y=x+3$ at the points $(-4,\ -1)$ and $(1,\ 4)$.

ii Using the graph in **d i**, $\dfrac{4}{x} > x+3$ when $x < -4$ or $0 < x < 1$.

3 $y=3x+c$ is a tangent to $y=x^2-5x+7$ if they meet at exactly one point (touch).
$y=x^2-5x+7$ meets $y=3x+c$ when $x^2-5x+7=3x+c$
$$\therefore\ x^2-8x+7-c=0$$
The graphs meet exactly once when this equation has a repeated root $\therefore\ \Delta=0$
$$\therefore\ (-8)^2-4(1)(7-c)=0$$
$$\therefore\ 64-28+4c=0$$
$$\therefore\ 4c=-36$$
$$\therefore\ c=-9$$

4 $y=mx-2$ is a tangent to $y=x^2-4x+2$ if they meet at exactly one point (touch).
$y=x^2-4x+2$ meets $y=mx-2$ when $x^2-4x+2=mx-2$
$$\therefore\ x^2-(m+4)x+4=0$$
The graphs meet exactly once when this equation has a repeated root $\therefore\ \Delta=0$
$$\therefore\ (-(m+4))^2-4(1)(4)=0$$
$$\therefore\ m^2+8m+16-16=0$$
$$\therefore\ m(m+8)=0$$
$$\therefore\ m=0 \text{ or } -8$$

5 Lines with y-intercept 1 have the form $y=mx+1$.
$y=mx+1$ is a tangent to $y=3x^2+5x+4$ if they meet at exactly one point (touch).
$y=3x^2+5x+4$ meets $y=mx+1$ when $3x^2+5x+4=mx+1$
$$\therefore\ 3x^2+(5-m)x+3=0$$

The graphs meet exactly once when this equation has a repeated root $\therefore\ \Delta = 0$

$$\therefore\ (5-m)^2 - 4(3)(3) = 0$$
$$\therefore\ 25 - 10m + m^2 - 36 = 0$$
$$\therefore\ m^2 - 10m - 11 = 0$$
$$\therefore\ (m+1)(m-11) = 0$$
$$\therefore\ m = -1 \text{ or } 11$$

$\therefore$ the required lines have gradient -1 or 11.

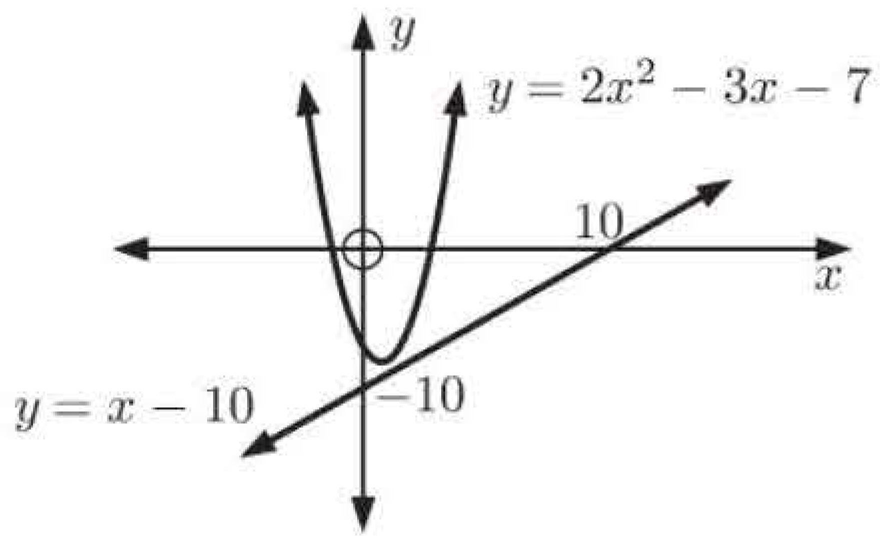
6 **a** $y = x + c$ meets $y = 2x^2 - 3x - 7$ when $2x^2 - 3x - 7 = x + c$

$\therefore\ 2x^2 - 4x - 7 - c = 0$

The graphs will never meet if this equation has no real roots $\therefore\ \Delta < 0$

$$\therefore\ (-4)^2 - 4(2)(-7-c) < 0$$
$$\therefore\ 16 + 56 + 8c < 0$$
$$\therefore\ 8c < -72$$
$$\therefore\ c < -9$$

b Choose c such that $c < -9$, for example $c = -10$:



EXERCISE 1G

1 Let the smaller of the integers be x.
The other integer is $(x + 12)$.
$\therefore$ the sum of their squares is

$$x^2 + (x+12)^2 = 74$$
$$\therefore\ x^2 + x^2 + 24x + 144 = 74$$
$$\therefore\ 2x^2 + 24x + 70 = 0$$
$$\therefore\ x^2 + 12x + 35 = 0$$
$$\therefore\ (x+7)(x+5) = 0$$
$$\therefore\ x = -7 \text{ or } -5$$

So, the integers are -7 and 5, or -5 and 7.

2 Let the number be x, so its reciprocal is $\dfrac{1}{x}$.

They have sum $x + \dfrac{1}{x} = 5\frac{1}{5}$

$$\therefore\ x^2 + 1 = \tfrac{26}{5}x$$
$$\therefore\ x^2 - \tfrac{26}{5}x + 1 = 0$$
$$\therefore\ 5x^2 - 26x + 5 = 0$$
$$\therefore\ (5x-1)(x-5) = 0$$
$$\therefore\ x = \tfrac{1}{5} \text{ or } 5$$

So, the number is either $\frac{1}{5}$ or 5.

3 Let the number be x so its square is x^2.

$\therefore$ the sum is $x + x^2 = 210$

$$\therefore\ x^2 + x - 210 = 0$$
$$\therefore\ (x+15)(x-14) = 0$$
$$\therefore\ x = -15 \text{ or } 14$$

But x is a natural number, so $x > 0$,
$\therefore$ the number is 14.

4 Suppose the numbers are x and $(x+2)$.

Then $x(x+2) = 255$

$$\therefore\ x^2 + 2x - 255 = 0$$
$$\therefore\ (x+17)(x-15) = 0$$
$$\therefore\ x = -17 \text{ or } 15$$

$\therefore$ the numbers are -17 and -15, or 15 and 17.

5 If the width of the rectangle is w cm, then its length is $(w+4)$ cm.

$\therefore$ the area is $w(w+4) = 26$

$$\therefore\ w^2 + 4w - 26 = 0$$

which has $a = 1$, $b = 4$, $c = -26$

$$\therefore\ w = \frac{-4 \pm \sqrt{4^2 - 4(1)(-26)}}{2(1)}$$

$$\therefore\ w = \frac{-4 \pm \sqrt{120}}{2} = -2 \pm \sqrt{30}$$

But $w > 0$, so $w = -2 + \sqrt{30}$
$$\approx 3.477$$

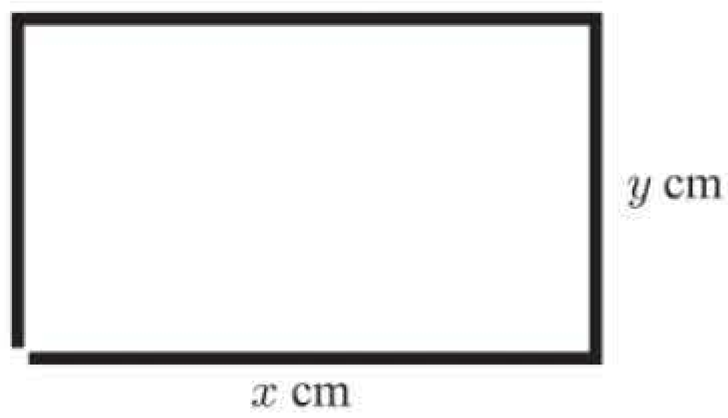
So, the width is approximately 3.48 cm.

6 **a** The base has sides of length x cm, so the areas of the top and bottom surfaces are both x^2 cm^2.
The box has height $(x+1)$ cm, so the area of each of the side faces is $x(x+1)$ cm^2.
$\therefore$ the total surface area is
$$\begin{aligned} A &= 2x^2 + 4x(x+1) \\ &= 2x^2 + 4x^2 + 4x \\ &= (6x^2 + 4x) \text{ cm}^2 \end{aligned}$$

b
$$\begin{aligned} 6x^2 + 4x &= 240 \\ \therefore\ 3x^2 + 2x - 120 &= 0 \\ \therefore\ (3x+20)(x-6) &= 0 \\ \therefore\ x &= -\tfrac{20}{3} \text{ or } 6 \end{aligned}$$
but $x > 0$, so $x = 6$
$\therefore$ the box is $6 \text{ cm} \times 6 \text{ cm} \times 7 \text{ cm}$.

7 Suppose the tinplate was $x \text{ cm} \times x \text{ cm}$.
When $3 \text{ cm} \times 3 \text{ cm}$ squares are cut from the corners, the base of the open box formed is $(x-6) \text{ cm} \times (x-6) \text{ cm}$.
The open box has height 3 cm, so its volume is $3 \times (x-6) \times (x-6) = 80$
$$\begin{aligned} \therefore\ 3(x^2 - 12x + 36) &= 80 \\ 3x^2 - 36x + 108 &= 80 \\ \therefore\ 3x^2 - 36x + 28 &= 0 \end{aligned}$$
which has $a = 3$, $b = -36$, $c = 28$
$$\therefore\ x = \frac{-(-36) \pm \sqrt{(-36)^2 - 4(3)(28)}}{2(3)}$$
$$= \frac{36 \pm \sqrt{960}}{6} \quad \text{and since } x > 6,$$
$$x = 6 + \frac{\sqrt{960}}{6} \approx 11.16$$
$\therefore$ the original piece of tinplate was about 11.2 cm square.

8



Suppose one side of the rectangle has length x cm and the other has length y cm.
The perimeter is $(2x + 2y)$ cm,
so $2x + 2y = 20$
$\therefore\ 2y = 20 - 2x$
$\therefore\ y = 10 - x$
The area of the rectangle is therefore $x(10-x)$ cm^2.

If the area is 30 cm^2, then
$$\begin{aligned} x(10-x) &= 30 \\ \therefore\ 10x - x^2 &= 30 \\ \therefore\ x^2 - 10x + 30 &= 0 \end{aligned}$$
which has $a = 1$, $b = -10$, $c = 30$
$$\therefore\ x = \frac{-(-10) \pm \sqrt{(-10)^2 - 4(1)(30)}}{2(1)}$$
$$= \frac{10 \pm \sqrt{100 - 120}}{2}$$
$$= \frac{10 \pm \sqrt{-20}}{2}$$
$\therefore$ x has no real solutions, so it is not possible.

9 The smaller rectangle is similar to the original rectangle.
$$\therefore\ \frac{\text{AB}}{\text{AD}} = \frac{\text{BC}}{\text{BY}}$$
Suppose $\text{AB} = x$ units, and $\text{AD} = \text{BC}$
$= 1$ unit
$$\therefore\ \frac{x}{1} = \frac{1}{x-1}$$
$\therefore\ x(x-1) = 1$
$\therefore\ x^2 - x - 1 = 0$
which has $a = 1$, $b = -1$, $c = -1$
$$\therefore\ x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)}$$
$$= \frac{1 \pm \sqrt{1+4}}{2}$$
$$= \frac{1 \pm \sqrt{5}}{2}$$
$$\therefore\ x = \frac{1+\sqrt{5}}{2}, \quad \text{since } x > 0$$
But $x = \dfrac{\text{AB}}{\text{AD}}$, which is the golden ratio
$\therefore$ the golden ratio is $\dfrac{1+\sqrt{5}}{2}$.

10 Suppose the express train travels at x km h^{-1}. We know speed $= \dfrac{\text{distance}}{\text{time}}$, so time $= \dfrac{\text{distance}}{\text{speed}}$.
$\therefore$ it takes the express train $\dfrac{160}{x}$ hours and the normal train $\dfrac{160}{x-10}$ hours.

$$\therefore \quad \frac{160}{x} + \tfrac{1}{2} = \frac{160}{x-10}$$

$$\therefore \quad 160(x-10) + \tfrac{1}{2}x(x-10) = 160x$$

$$\therefore \quad 160x - 1600 + \tfrac{1}{2}x^2 - 5x = 160x$$

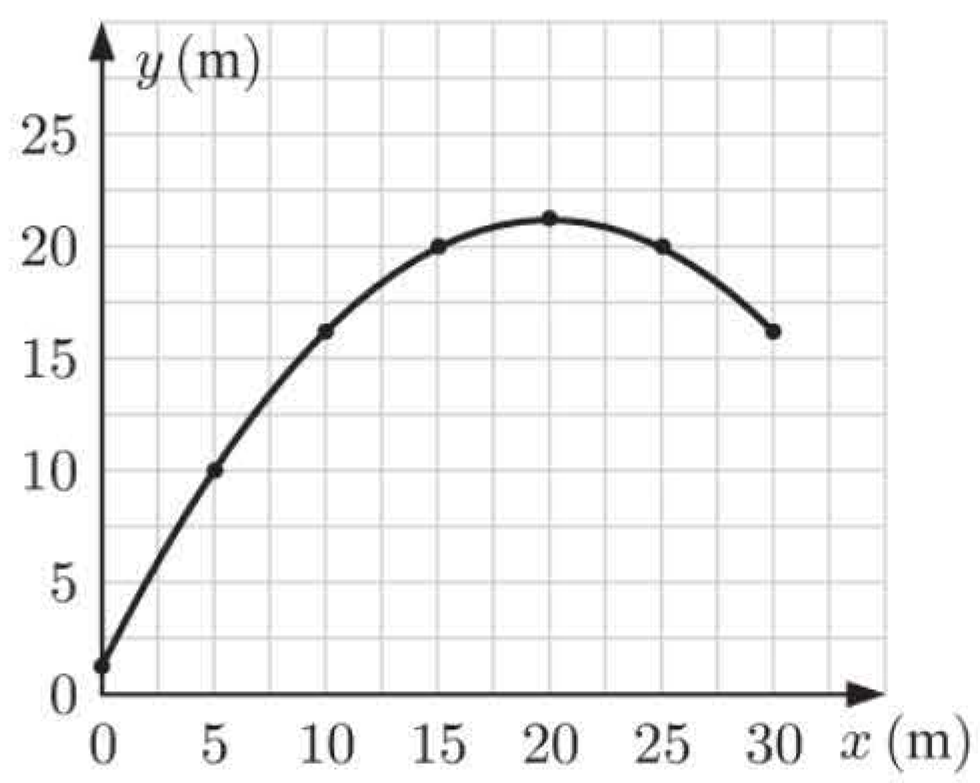
$$\therefore \quad x^2 - 10x - 3200 = 0 \quad \text{which has} \quad a = 1, \quad b = -10, \quad c = -3200$$

$$\therefore \quad x = \frac{-(-10) \pm \sqrt{(-10)^2 - 4(1)(-3200)}}{2(1)} = \frac{10 \pm \sqrt{12\,900}}{2}$$

But $x > 0$, so $x = \dfrac{10 + \sqrt{12\,900}}{2} \approx 61.8\ \text{km h}^{-1}$

$\therefore$ the express train travels on average at about $61.8\ \text{km h}^{-1}$.

11 **a**



b The graph is a parabola.

c The maximum height reached by the ball is 21.25 m.

d Let $f(x) = ax^2 + bx + c$ be the form of the formula for height given horizontal distance x.

$$f(0) = 1.25$$
$$\therefore \quad a(0)^2 + b(0) + c = 1.25$$
$$\therefore \quad c = 1.25$$
$$\text{Also} \quad f(5) = 10$$
$$\therefore \quad a(5)^2 + b(5) + 1.25 = 10$$
$$\therefore \quad 25a + 5b = 8.75 \quad (1)$$
$$\text{And} \quad f(15) = 20$$
$$\therefore \quad a(15)^2 + b(15) + 1.25 = 20$$
$$\therefore \quad 225a + 15b = 18.75$$
$$\therefore \quad 75a + 5b = 6.25 \quad \{\div 3\} \quad (2)$$

Solving simultaneously (or using technology):

$$\begin{array}{rl} 75a + 5b = 6.25 & (2) \\ -25a - 5b = -8.75 & -(1) \\ \hline 50a \qquad = -2.5 & \end{array}$$
$$\therefore \quad a = -0.05$$
$$\therefore \quad -1.25 + 5b = 8.75 \quad \{\text{substitute in (1)}\}$$
$$\therefore \quad 5b = 10$$
$$\therefore \quad b = 2$$

So, $f(x) = -0.05x^2 + 2x + 1.25$

e The ball bounces when it hits the ground

$$\therefore \quad f(x) = 0$$
$$\therefore \quad -0.05x^2 + 2x + 1.25 = 0$$

Using technology, $x = -0.616$ or $x = 40.6$

But $x \geqslant 0$ $\therefore$ ball bounces at 40.6 m.

But Badrani is 40 m from Abiola, so the ball will reach Badrani before it bounces.

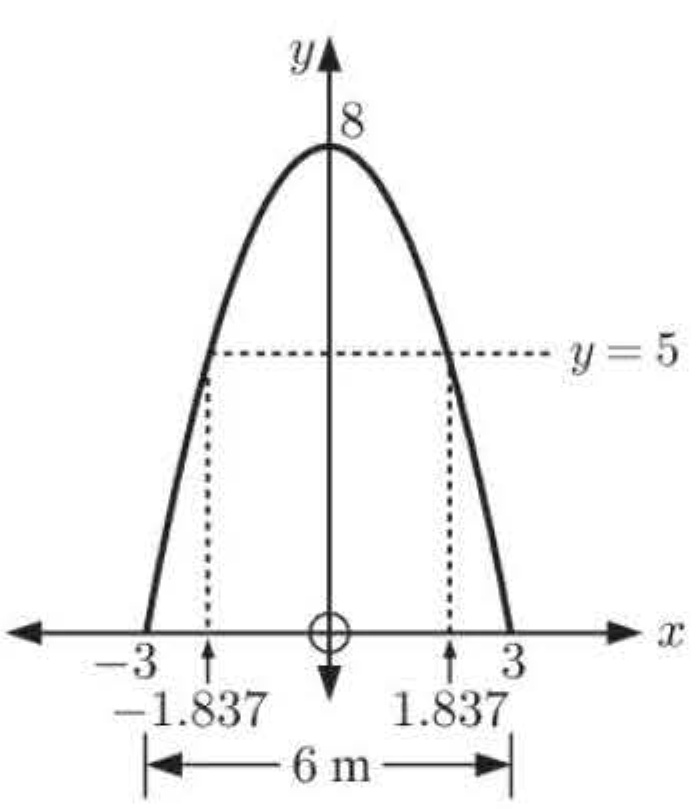
12 **a** The parabola has vertex $(0, 8)$, so it has equation

$$y = a(x-0)^2 + 8$$
$$\therefore \quad y = ax^2 + 8$$

When $x = 3$, $y = 0$, so

$$0 = a(3^2) + 8$$
$$\therefore \quad 9a = -8$$
$$\therefore \quad a = -\tfrac{8}{9}$$

$\therefore$ the equation of the parabola is $y = -\frac{8}{9}x^2 + 8$.



b The truck is 4 m wide, so we use the equation in **a** to find the width of the tunnel 5 m above ground level.

When $y = 5$, $-\frac{8}{9}x^2 + 8 = 5$

$\therefore\ -\frac{8}{9}x^2 = -3$

$\therefore\ x^2 = \frac{27}{8}$

$\therefore\ x = \pm\sqrt{\frac{27}{8}}$

$\therefore\ x \approx \pm 1.837$

So, the tunnel is $2 \times 1.837 \approx 3.67$ m wide, 5 m above ground level. But the truck is 4 m wide.

$\therefore$ the truck will not fit through the tunnel.

EXERCISE 1H

1 **a** For $y = x^2 - 2x$,
$a = 1,\ b = -2,\ c = 0.$

As $a > 0$, the shape is $\cup$

$\therefore$ the minimum value occurs when

$$x = \frac{-b}{2a} = \frac{2}{2} = 1$$

and $y = 1^2 - 2(1) = -1$

$\therefore$ the minimum value of $y = x^2 - 2x$ is -1, occurring when $x = 1$.

b For $y = 4x^2 - x + 5$,
$a = 4,\ b = -1,\ c = 5.$

As $a > 0$, the shape is $\cup$

$\therefore$ the minimum value occurs when

$$x = \frac{-b}{2a} = \frac{1}{8}$$

and $y = 4(\frac{1}{8})^2 - \frac{1}{8} + 5$

$= \frac{1}{16} - \frac{1}{8} + 5$

$= 4\frac{15}{16}$

$\therefore$ the minimum value of $y = 4x^2 - x + 5$ is $4\frac{15}{16}$, occuring when $x = \frac{1}{8}$.

c For $y = 7x - 2x^2$,
$a = -2,\ b = 7,\ c = 0.$

As $a < 0$, the shape is $\cap$

$\therefore$ the maximum value occurs when

$$x = \frac{-b}{2a} = \frac{-7}{-4} = \frac{7}{4}$$

and $y = 7(\frac{7}{4}) - 2(\frac{7}{4})^2$

$= \frac{49}{4} - \frac{49}{8}$

$= \frac{49}{8}$ or $6\frac{1}{8}$

$\therefore$ the maximum value of $y = 7x - 2x^2$ is $6\frac{1}{8}$, occuring when $x = \frac{7}{4}$.

2 **a** For $P = -3x^2 + 240x - 800$,
$a = -3,\ b = 240,\ c = -800.$

As $a < 0$, the shape is $\cap$

$\therefore$ the maximum profit occurs when

$$x = \frac{-b}{2a} = \frac{-240}{-6} = 40$$

$\therefore$ 40 refrigerators should be made each day to maximise the total profit.

b $P = -3(40)^2 + 240(40) - 800$

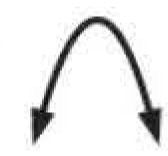
$= 4000$

$\therefore$ the maximum profit is \$4000.

3 **a** Let the other side be y m long.
The perimeter is 200 m.

$\therefore \quad 2x + 2y = 200$
$\therefore \quad x + y = 100$
$\therefore \quad y = 100 - x$
$\therefore$ the area $A = xy$
$\therefore \quad A = x(100 - x)$
$\therefore \quad A = 100x - x^2$

b $A = 100x - x^2$ is a quadratic function with $a = -1, \ b = 100, \ c = 0.$

As $a < 0$, the shape is 

$\therefore$ the area is maximised when

$$x = \frac{-b}{2a} = \frac{-100}{-2} = 50$$

and $y = 100 - 50 = 50$

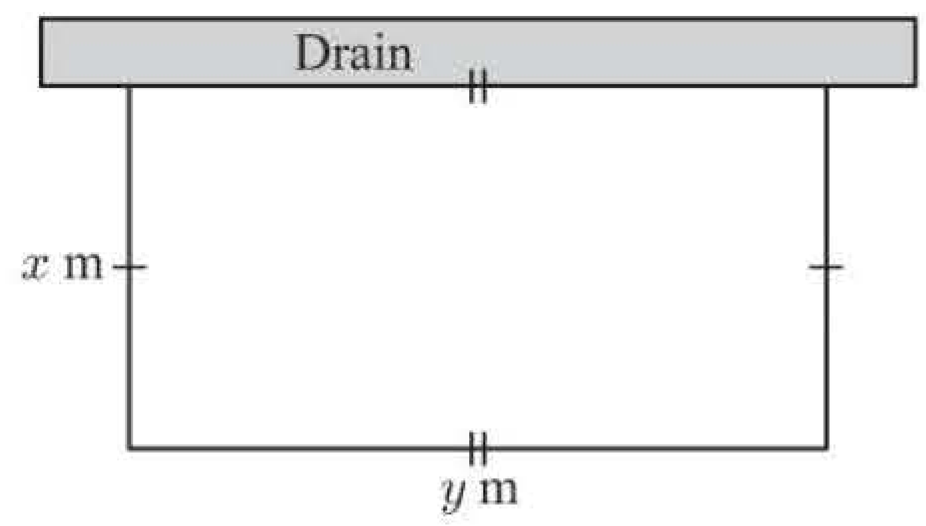
$\therefore$ the area of the rectangle is maximised when $x = y = 50$, which is when the rectangle is a square.

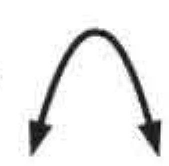
4 Let the dimensions of the paddock be x m $\times$ y m.
If 1000 m of fence is available, then

$2x + y = 1000$ {perimeter}
$\therefore \quad y = 1000 - 2x$ .... (1)

The area of the enclosure $A = xy$

Since $y = 1000 - 2x, \quad A = x(1000 - 2x)$
$= 1000x - 2x^2$
$\therefore \quad A = -2x^2 + 1000x$



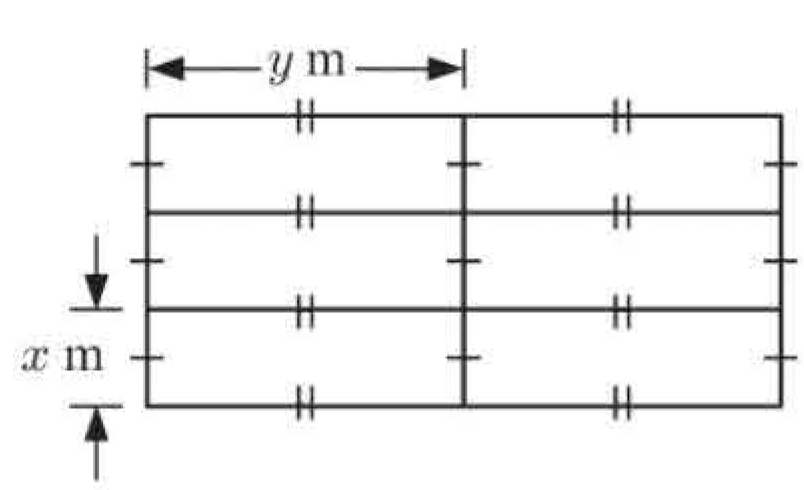
A is a quadratic and $a < 0$, so its shape is 

So, area is maximised when $x = \dfrac{-b}{2a} = \dfrac{-1000}{2 \times (-2)} = 250$

and when $x = 250, \ y = 1000 - 2(250) = 500$

$\therefore$ the paddock has a maximum area when the dimensions are 250 m $\times$ 500 m.

5 **a**



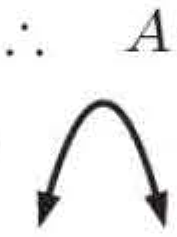
The length of fence required for this enclosure is $9x + 8y$. If 1800 m is available for this enclosure, then $9x + 8y = 1800$.

b If $9x + 8y = 1800$, then $y = \dfrac{1800 - 9x}{8}$.

The area of each pen is $A = xy$.

Substituting $y = \dfrac{1800 - 9x}{8}$ into A we get

$$A = x\left(\frac{1800 - 9x}{8}\right)$$
$$\therefore \quad A = \frac{1800x}{8} - \frac{9x^2}{8}$$
$$\therefore \quad A = -\tfrac{9}{8}x^2 + 225x \ \text{ m}^2$$

c The area is a quadratic function with $a < 0$, so its shape is 

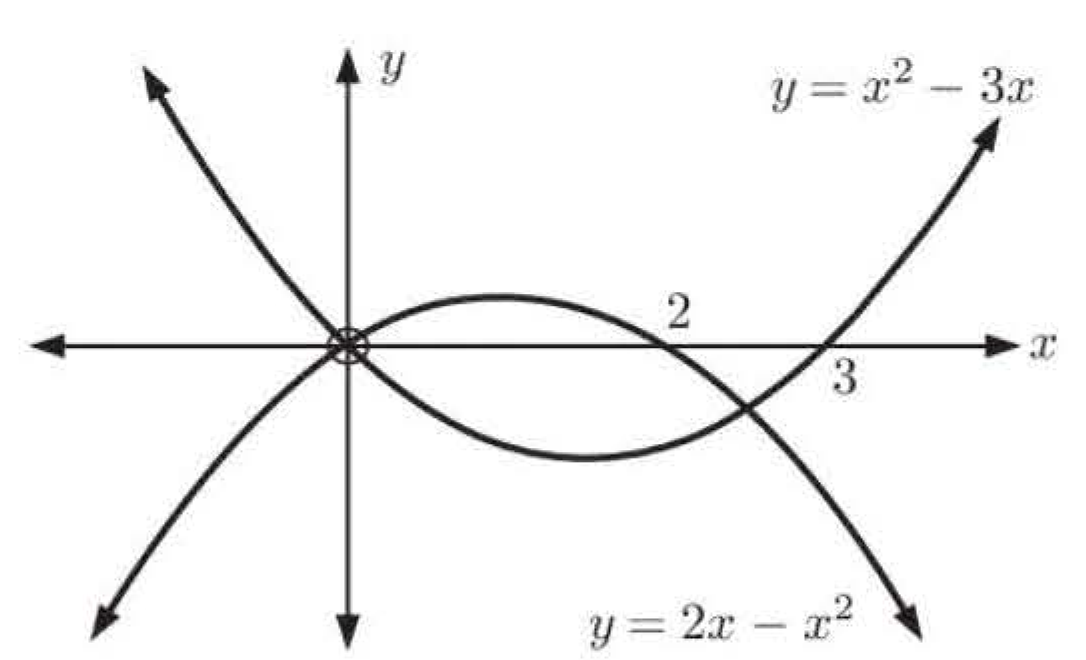
So, at $x = \dfrac{-b}{2a}$ we have a maximum

$\therefore \quad x = \dfrac{-225}{2 \times (-\frac{9}{8})} = 100$, and when $x = 100$, $y = \dfrac{1800 - 9(100)}{8} = 112.5$

Hence, the area is maximised when the dimensions are 100 m $\times$ 112.5 m.

6 **a** The graphs of $y = x^2 - 3x$ and $y = 2x - x^2$

meet where $x^2 - 3x = 2x - x^2$
$\therefore \quad 2x^2 - 5x = 0$
$\therefore \quad x(2x - 5) = 0$
$\therefore \quad x = 0$ or $2\frac{1}{2}$

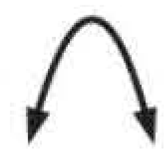


b The vertical separation between the curves is given by

$S = (2x - x^2) - (x^2 - 3x)$ $\quad \{y = 2x - x^2$ is above $y = x^2 - 3x$ for $0 \leqslant x \leqslant 2\frac{1}{2}\}$

$\therefore \quad S = 2x - x^2 - x^2 + 3x$

$\therefore \quad S = -2x^2 + 5x$

Thus S is a quadratic function with $a < 0$ so the shape is 

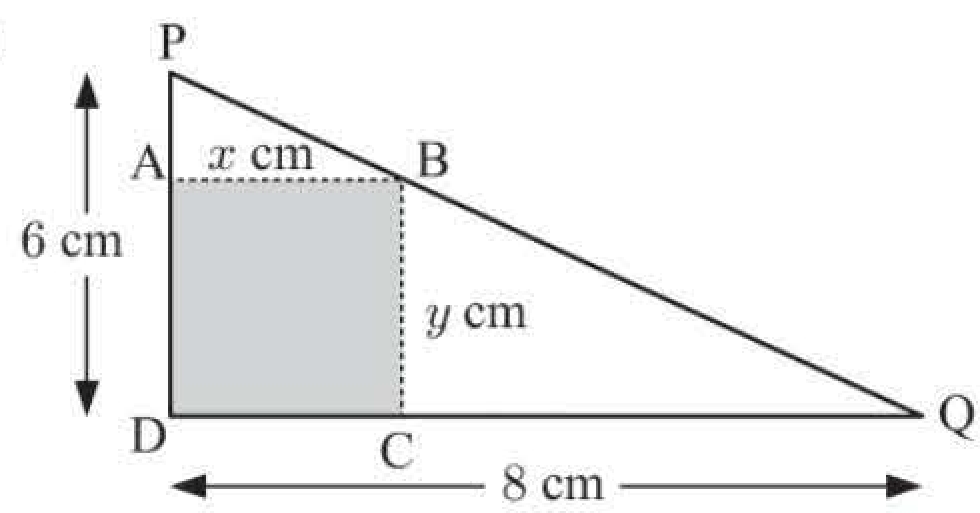
$\therefore$ the maximum separation occurs when $x = \dfrac{-b}{2a} = \dfrac{-5}{-4} = \frac{5}{4}$

and $S = -2(\frac{5}{4})^2 + 5(\frac{5}{4})$

$= -\frac{25}{8} + \frac{25}{4} = \frac{25}{8}$ or $3\frac{1}{8}$

$\therefore$ the maximum vertical separation between the curves for $0 \leqslant x \leqslant 2\frac{1}{2}$ is $3\frac{1}{8}$ units.

7 **a**



$\triangle$s PAB and PDQ are similar

$\{\widehat{APB}$ is common, $\widehat{ABP} = \widehat{DQP}$ as $[AB] \parallel [DQ]\}$

$\therefore \quad \dfrac{PA}{PD} = \dfrac{AB}{DQ}$

$\therefore \quad \dfrac{6-y}{6} = \dfrac{x}{8}$

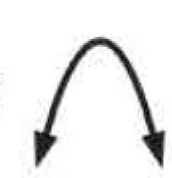
$\therefore \quad 6 - y = \frac{3}{4}x$

$\therefore \quad y = 6 - \frac{3}{4}x$

b Rectangle ABCD has area $A = xy$

$= x(6 - \frac{3}{4}x)$

$= -\frac{3}{4}x^2 + 6x$

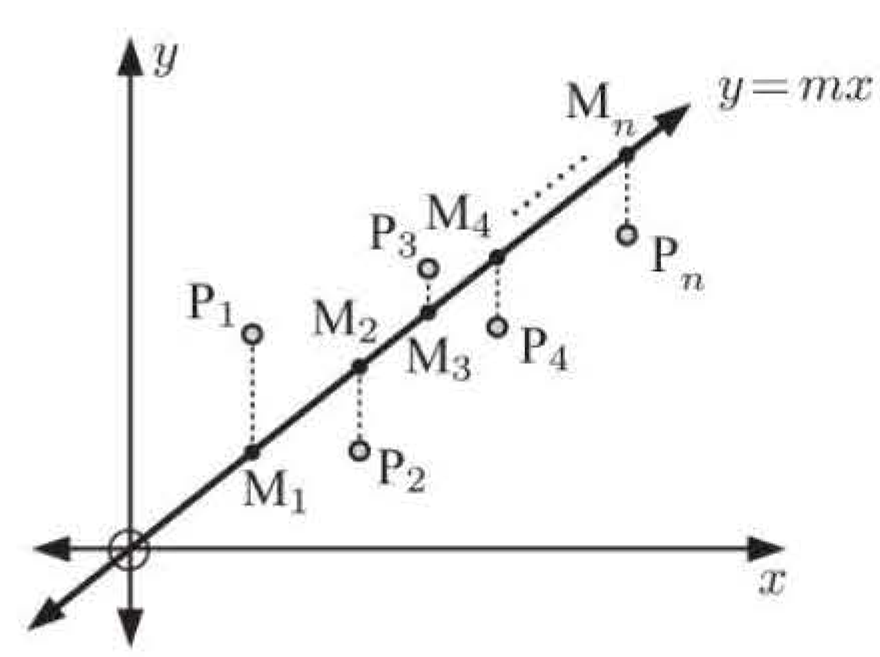
which is a quadratic with $a < 0$ $\therefore$ the shape is 

$\therefore$ the area is maximised when $x = \dfrac{-b}{2a} = \dfrac{-6}{-\frac{3}{2}} = 4$

and when $x = 4$, $y = 6 - \frac{3}{4}(4) = 3$

$\therefore$ the dimensions of rectangle ABCD of maximum area are 4 cm $\times$ 3 cm.

8 Let the 'line of best fit' through (0, 0) have slope m.



$\therefore$ the line has equation $y = mx$.

$\therefore$ for $P_1(a_1, b_1)$, the coordinates of M_1 are (a_1, ma_1).

$\therefore$ the distance between P_1 and M_1 is $b_1 - ma_1$.

In general, $P_iM_i = |b_i - ma_i|$, $i = 1, 2, ..., n$.

$\therefore \quad (P_1M_1)^2 + (P_2M_2)^2 + ... + (P_nM_n)^2$

$= |b_1 - ma_1|^2 + |b_2 - ma_2|^2 + ... + |b_n - ma_n|^2$

$= (b_1 - ma_1)^2 + (b_2 - ma_2)^2 + ... + (b_n - ma_n)^2 \quad \{|z|^2 = z^2\}$

$= b_1^{\,2} - 2b_1ma_1 + m^2a_1^{\,2} + b_2^{\,2} - 2b_2ma_2 + m^2a_2^{\,2} + ... + b_n^{\,2} - 2b_nma_n^{\,2} + m^2a_n^{\,2}$

$= m^2(a_1^{\,2} + a_2^{\,2} + ... + a_n^{\,2}) - m(2a_1b_1 + 2a_2b_2 + ... + 2a_nb_n) + (b_1^{\,2} + b_2^{\,2} + ... + b_n^{\,2})$

which is a quadratic in m, with $a = a_1^{\,2} + a_2^{\,2} + ... + a_n^{\,2}$, $b = -(2a_1b_1 + 2a_2b_2 + ... + 2a_nb_n)$, $c = b_1^{\,2} + b_2^{\,2} + ... + b_n^{\,2}$.

$a = a_1^{\,2} + a_2^{\,2} + ... + a_n^{\,2} > 0$, so the quadratic has shape 

$\therefore$ the sum $(P_1M_1)^2 + (P_2M_2)^2 + ... + (P_nM_n)^2$ is minimised when

$$m = \frac{-b}{2a} = \frac{2a_1b_1 + 2a_2b_2 + ... + 2a_nb_n}{2a_1^{\,2} + 2a_2^{\,2} + ... + 2a_n^{\,2}}$$

$$\therefore \quad m = \frac{a_1b_1 + a_2b_2 + ... + a_nb_n}{a_1^{\,2} + a_2^{\,2} + ... + a_n^{\,2}}$$

9 $y = (x-a-b)(x-a+b)(x+a-b)(x+a+b)$

$= [x-(a+b)][x+(a+b)][x-(a-b)][x+(a-b)]$ {rearranging}

$= [x^2-(a+b)^2][x^2-(a-b)^2]$

$= x^4 - x^2(a-b)^2 - x^2(a+b)^2 + (a+b)^2(a-b)^2$

$= x^4 - x^2[(a-b)^2+(a+b)^2] + [(a+b)(a-b)]^2$

$= x^4 - x^2(a^2 \cancel{-2ab} + b^2 + a^2 \cancel{+2ab} + b^2) + (a^2-b^2)^2$

$= x^4 - 2(a^2+b^2)x^2 + (a^2-b^2)^2$

which is a quadratic in x^2 with "a" $= 1$, "b" $= -2(a^2+b^2)$, "c" $= (a^2-b^2)^2$

"a" > 0, so the quadratic has shape 

$\therefore$ y is minimised when $x^2 = \dfrac{2(a^2+b^2)}{2} = a^2+b^2$

When $x^2 = a^2+b^2$, $y = (a^2+b^2)^2 - 2(a^2+b^2)(a^2+b^2) + (a^2-b^2)^2$

$= (a^2+b^2)^2 - 2(a^2+b^2)^2 + (a^2-b^2)^2$

$= (a^2-b^2)^2 - (a^2+b^2)^2$

$= a^4 - 2a^2b^2 + b^4 - a^4 - 2a^2b^2 - b^4$

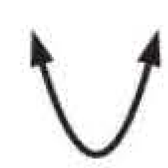
$= -4a^2b^2$

$\therefore$ the least value of y is $-4a^2b^2$.

10 $y = (a_1x-b_1)^2 + (a_2x-b_2)^2$

$= a_1^2x^2 - 2a_1b_1x + b_1^2 + a_2^2x^2 - 2a_2b_2x + b_2^2$

$= (a_1^2+a_2^2)x^2 - 2(a_1b_1+a_2b_2)x + (b_1^2+b_2^2)$

which is a quadratic in x with $a = a_1^2+a_2^2 > 0$, so it has shape 

$\therefore$ y is minimised when $x = \dfrac{-b}{2a} = \dfrac{2(a_1b_1+a_2b_2)}{2(a_1^2+a_2^2)} = \dfrac{a_1b_1+a_2b_2}{a_1^2+a_2^2}$

When $x = \dfrac{a_1b_1+a_2b_2}{a_1^2+a_2^2}$,

$$y = (a_1^2+a_2^2)\left(\frac{a_1b_1+a_2b_2}{a_1^2+a_2^2}\right)^2 - 2(a_1b_1+a_2b_2)\left(\frac{a_1b_1+a_2b_2}{a_1^2+a_2^2}\right) + b_1^2+b_2^2$$

$$= \frac{(a_1b_1+a_2b_2)^2}{a_1^2+a_2^2} - \frac{2(a_1b_1+a_2b_2)^2}{a_1^2+a_2^2} + b_1^2+b_2^2$$

$$= b_1^2+b_2^2 - \frac{(a_1b_1+a_2b_2)^2}{a_1^2+a_2^2}$$

But since $y = (a_1x-b_1)^2 + (a_2x-b_2)^2$, $y \geqslant 0$ for all x {sum of 2 squared terms}

$\therefore$ $b_1^2+b_2^2 - \dfrac{(a_1b_1+a_2b_2)^2}{a_1^2+a_2^2} \geqslant 0$

$\therefore$ $b_1^2+b_2^2 \geqslant \dfrac{(a_1b_1+a_2b_2)^2}{a_1^2+a_2^2}$

$\therefore$ $(a_1^2+a_2^2)(b_1^2+b_2^2) \geqslant (a_1b_1+a_2b_2)^2$ {$a_1^2+a_2^2 \geqslant 0$}

$\therefore$ $\sqrt{a_1^2+a_2^2}\sqrt{b_1^2+b_2^2} \geqslant \sqrt{(a_1b_1+a_2b_2)^2}$

$\therefore$ $|a_1b_1+a_2b_2| \leqslant \sqrt{a_1^2+a_2^2}\sqrt{b_1^2+b_2^2}$

11 Suppose one of the equations, say $x^2 + b_1x + c_1 = 0$ does not have two real roots.

$$\therefore \text{ its discriminant } \Delta < 0$$
$$\therefore b_1^2 - 4(1)(c_1) < 0$$
$$\therefore b_1^2 < 4c_1 \quad .... (1)$$
$$\therefore \left(\frac{2(c_1+c_2)}{b_2}\right)^2 < 4c_1 \qquad \{b_1b_2 = 2(c_1+c_2)\}$$
$$\therefore \frac{\cancel{4}(c_1+c_2)^2}{b_2^2} < \cancel{4}c_1$$
$$\therefore c_1^2 + 2c_1c_2 + c_2^2 < c_1b_2^2 \qquad \{b_2^2 > 0\}$$
$$\therefore c_1^2 - 2c_1c_2 + c_2^2 < c_1b_2^2 - 4c_1c_2 \qquad \{\text{subtract } 4c_1c_2 \text{ from both sides}\}$$
$$\therefore (c_1-c_2)^2 < c_1(b_2^2 - 4c_2)$$
$$\therefore c_1(b_2^2 - 4c_2) > (c_1-c_2)^2$$

Now $(c_1 - c_2)^2 \geqslant 0 \quad \therefore c_1(b_2^2 - 4c_2) > 0$

We know $c_1 > 0 \qquad \{4c_1 > b_1^2 > 0 \text{ using (1)}\}$

$$\therefore b_2^2 - 4c_2 > 0$$

$\therefore$ $x^2 + b_2x + c_2$ has two real roots $\{b_2^2 - 4c_2$ is the discriminant$\}$

$\therefore$ if one of the equations does not have two real roots, the other equation does have two real roots.

$\therefore$ at least one of the equations has two real roots.

REVIEW SET 1A

1 **a** The x-intercepts are -2 and 1.

b The axis of symmetry lies midway between the x-intercepts, so its equation is $x = -\frac{1}{2}$.

c When $x = 0$, $y = -2(2)(-1) = 4$

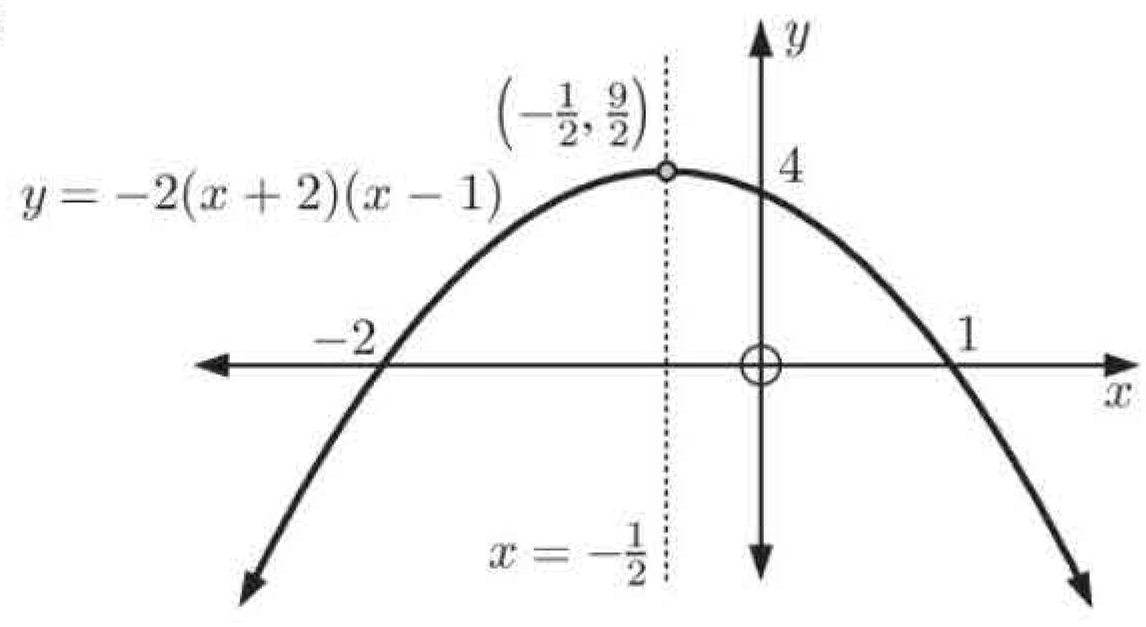
$\therefore$ the y-intercept is 4

d When $x = -\frac{1}{2}$, $y = -2(-\frac{1}{2}+2)(-\frac{1}{2}-1)$

$= -2(\frac{3}{2})(-\frac{3}{2}) = \frac{9}{2}$

$\therefore$ the vertex is $(-\frac{1}{2}, \frac{9}{2})$.

e



2 **a**
$$3x^2 - 12x = 0$$
$$\therefore 3x(x-4) = 0$$
$$\therefore x = 0 \text{ or } 4$$

b
$$3x^2 - x - 10 = 0$$
$$\therefore (3x+5)(x-2) = 0$$
$$\therefore x = -\tfrac{5}{3} \text{ or } 2$$

c
$$x^2 - 11x = 60$$
$$\therefore x^2 - 11x - 60 = 0$$
$$\therefore (x+4)(x-15) = 0$$
$$\therefore x = -4 \text{ or } 15$$

3 **a** $x^2 + 5x + 3 = 0$

has $a = 1$, $b = 5$, $c = 3$

$$\therefore x = \frac{-5 \pm \sqrt{5^2 - 4(1)(3)}}{2(1)}$$
$$\therefore x = -\tfrac{5}{2} \pm \tfrac{\sqrt{13}}{2}$$

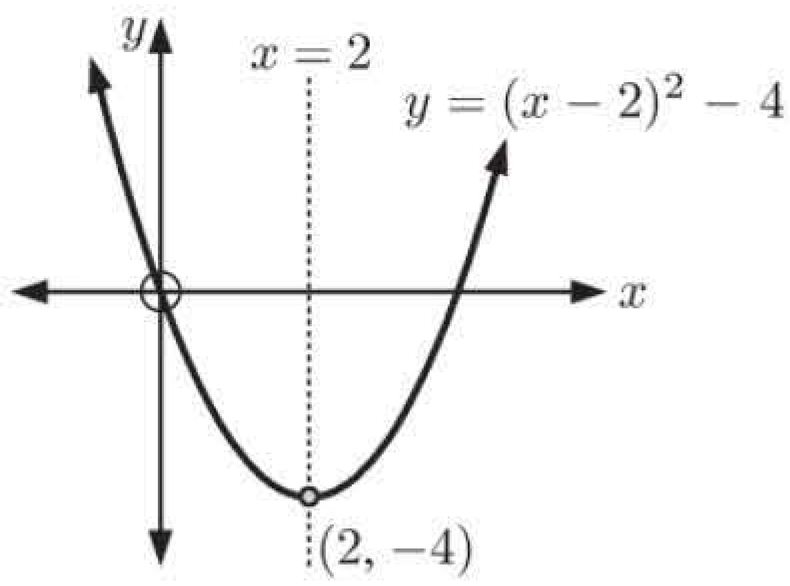
b $3x^2 + 11x - 2 = 0$

has $a = 3$, $b = 11$, $c = -2$

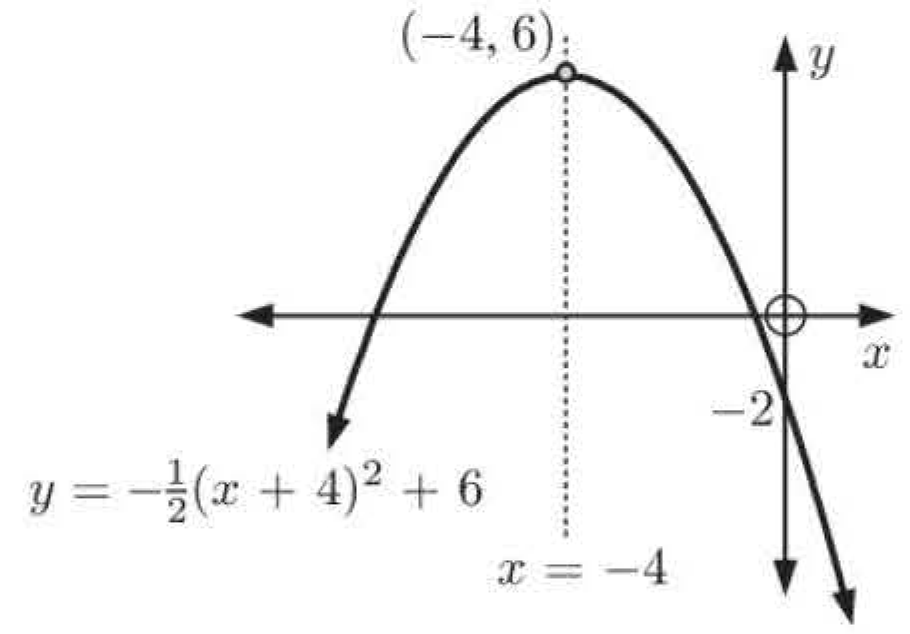
$$\therefore x = \frac{-11 \pm \sqrt{11^2 - 4(3)(-2)}}{2(3)}$$
$$\therefore x = -\tfrac{11}{6} \pm \tfrac{\sqrt{145}}{6}$$

4
$$x^2 + 7x - 4 = 0$$
$$\therefore x^2 + 7x + (\tfrac{7}{2})^2 - (\tfrac{7}{2})^2 - 4 = 0$$
$$\therefore (x + \tfrac{7}{2})^2 - \tfrac{49}{4} - 4 = 0$$
$$\therefore (x + \tfrac{7}{2})^2 = \tfrac{65}{4}$$
$$\therefore x + \tfrac{7}{2} = \pm\tfrac{\sqrt{65}}{2}$$
$$\therefore x = -\tfrac{7}{2} \pm \tfrac{\sqrt{65}}{2}$$

5 **a** $y = (x-2)^2 - 4$ has vertex $(2, -4)$ and axis of symmetry $x = 2$.
When $x = 0$, $y = (-2)^2 - 4 = 0$
so the y-intercept is 0.



b $y = -\frac{1}{2}(x+4)^2 + 6$ has vertex $(-4, 6)$ and axis of symmetry $x = -4$.
When $x = 0$, $y = -\frac{1}{2}(4)^2 + 6 = -2$
so the y-intercept is -2.



6 **a** The graph touches the x-axis at 4, so its vertex is $(4, 0)$.
$\therefore$ its equation is $y = a(x-4)^2$ for some $a \neq 0$.
The graph also passes through $(2, 12)$ $\therefore$ $a(2-4)^2 = 12$
$\therefore$ $4a = 12$
$\therefore$ $a = 3$
$\therefore$ the equation is $y = 3(x-4)^2$ which is $y = 3(x^2 - 8x + 16)$
or $y = 3x^2 - 24x + 48$

b The quadratic has vertex $(-4, 1)$, so its equation is $y = a(x+4)^2 + 1$ for some $a \neq 0$.
The graph also passes through $(1, 11)$ $\therefore$ $11 = a(1+4)^2 + 1$
$\therefore$ $25a = 10$
$\therefore$ $a = \frac{2}{5}$
$\therefore$ the equation is $y = \frac{2}{5}(x+4)^2 + 1$ which is $y = \frac{2}{5}(x^2 + 8x + 16) + 1$
or $y = \frac{2}{5}x^2 + \frac{16}{5}x + \frac{37}{5}$

7 $y = -2x^2 + 4x + 3$ has $a = -2$, $b = 4$, $c = 3$

Since $a < 0$, the graph has shape  and will have a maximum.

The axis of symmetry is $x = -\dfrac{b}{2a} = -\dfrac{4}{2(-2)} = 1$

When $x = 1$, $y = -2(1)^2 + 4(1) + 3$
$= 5$

$\therefore$ the maximum is 5, and this occurs when $x = 1$.

8 The roots of $2x^2 - 3x = 4$ or $2x^2 - 3x - 4 = 0$ are α and β.

sum of roots $= -\dfrac{b}{a}$ and product of roots $= \dfrac{c}{a}$

$\therefore$ $\alpha + \beta = -\dfrac{-3}{2}$ $\therefore$ $\alpha\beta = \dfrac{-4}{2}$

$\therefore$ $\alpha + \beta = \frac{3}{2}$ $\therefore$ $\alpha\beta = -2$

Now consider a quadratic equation with roots $\dfrac{1}{\alpha}$ and $\dfrac{1}{\beta}$.

sum of roots $= \dfrac{1}{\alpha} + \dfrac{1}{\beta} = \dfrac{\beta}{\alpha\beta} + \dfrac{\alpha}{\alpha\beta}$ and product of roots $= \dfrac{1}{\alpha} \times \dfrac{1}{\beta}$

$$= \frac{\alpha+\beta}{\alpha\beta} \qquad\qquad = \frac{1}{\alpha\beta}$$

$$= \frac{\frac{3}{2}}{-2} = -\tfrac{3}{4} \qquad\qquad = \frac{1}{-2} = -\tfrac{1}{2}$$

$\therefore$ a quadratic equation $ax^2 + bx + c = 0$ with roots $\dfrac{1}{\alpha}$ and $\dfrac{1}{\beta}$ has $-\dfrac{b}{a} = -\frac{3}{4}$ and $\dfrac{c}{a} = -\frac{1}{2}$

$\therefore$ $b = \frac{3}{4}a$ and $c = -\frac{1}{2}a$

The simplest solution to this is $a = 4$, $\therefore$ $b = 3$ and $c = -2$.

$\therefore$ the simplest quadratic equation with roots $\dfrac{1}{\alpha}$ and $\dfrac{1}{\beta}$ is $4x^2 + 3x - 2 = 0$.

9 **a**
$$\begin{aligned} x^2 + 10 &= 7x \\ \therefore\ x^2 - 7x + 10 &= 0 \\ \therefore\ (x-2)(x-5) &= 0 \\ \therefore\ x &= 2 \text{ or } 5 \end{aligned}$$

b
$$\begin{aligned} x + \frac{12}{x} &= 7 \\ \therefore\ x^2 + 12 &= 7x \\ \therefore\ x^2 - 7x + 12 &= 0 \\ \therefore\ (x-3)(x-4) &= 0 \\ \therefore\ x &= 3 \text{ or } 4 \end{aligned}$$

c
$$\begin{aligned} 2x^2 - 7x + 3 &= 0 \\ \therefore\ (2x-1)(x-3) &= 0 \\ \therefore\ x &= \tfrac{1}{2} \text{ or } 3 \end{aligned}$$

10 $y = x^2 - 3x$ meets $y = 3x^2 - 5x - 24$

when $x^2 - 3x = 3x^2 - 5x - 24$

$$\begin{aligned} \therefore\ 2x^2 - 2x - 24 &= 0 \\ \therefore\ x^2 - x - 12 &= 0 \\ \therefore\ (x-4)(x+3) &= 0 \\ \therefore\ x &= 4 \text{ or } -3 \end{aligned}$$

Substituting into $y = x^2 - 3x$,

when $x = 4$, $y = 4^2 - 3 \times 4 = 4$

and when $x = -3$, $y = (-3)^2 - 3(-3)$
$= 9 + 9 = 18$

$\therefore$ the graphs meet at $(4, 4)$ and $(-3, 18)$.

11 $y = -2x^2 + 5x + k$

has $a = -2$, $b = 5$, $c = k$.

$$\begin{aligned} \therefore\ \Delta &= b^2 - 4ac \\ &= 5^2 - 4(-2)k \\ &= 25 + 8k \end{aligned}$$

The graph does not cut the x-axis if $\Delta < 0$

$$\begin{aligned} \therefore\ 25 + 8k &< 0 \\ \therefore\ 8k &< -25 \\ \therefore\ k &< -\tfrac{25}{8} \end{aligned}$$

So, $k < -3\frac{1}{8}$

12 $2x^2 - 3x + m = 0$

has $a = 2$, $b = -3$, $c = m$

$$\begin{aligned} \therefore\ \Delta &= b^2 - 4ac \\ &= (-3)^2 - 4(2)m \\ &= 9 - 8m \end{aligned}$$

a There is a repeated root if $\Delta = 0$

$$\begin{aligned} \therefore\ 9 - 8m &= 0 \\ \therefore\ m &= \tfrac{9}{8} \end{aligned}$$

b There are two distinct real roots if $\Delta > 0$

$$\begin{aligned} \therefore\ 9 - 8m &> 0 \\ \therefore\ 8m &< 9 \\ \therefore\ m &< \tfrac{9}{8} \end{aligned}$$

c There are no real roots if $\Delta < 0$

$$\begin{aligned} \therefore\ 9 - 8m &< 0 \\ \therefore\ 8m &> 9 \\ \therefore\ m &> \tfrac{9}{8} \end{aligned}$$

13 Let the number be x, so its reciprocal is $\dfrac{1}{x}$.

$$\begin{aligned} \therefore\ x + \frac{1}{x} &= 2\tfrac{1}{30} = \tfrac{61}{30} \\ \therefore\ x^2 + 1 &= \tfrac{61}{30}x \\ \therefore\ 30x^2 + 30 &= 61x \\ \therefore\ 30x^2 - 61x + 30 &= 0 \\ \therefore\ (6x-5)(5x-6) &= 0 \\ \therefore\ x &= \tfrac{5}{6} \text{ or } \tfrac{6}{5} \end{aligned}$$

$\therefore$ the number is $\frac{5}{6}$ or $\frac{6}{5}$

14 Let the line with y-intercept $(0, 10)$ have equation $y = mx + 10$.

$y = 3x^2 + 7x - 2$ meets this line when $3x^2 + 7x - 2 = mx + 10$

$$\therefore \quad 3x^2 + (7 - m)x - 12 = 0$$

For $y = mx + 10$ to be tangential to $y = 3x^2 + 7x - 2$, this equation must have exactly one solution, so there is a repeated root.

$$\therefore \quad \Delta = 0$$
$$\therefore \quad (7-m)^2 - 4(3)(-12) = 0$$
$$\therefore \quad 49 - 14m + m^2 + 144 = 0$$
$$\therefore \quad m^2 - 14m + 193 = 0$$
$$\therefore \quad m = \frac{14 \pm \sqrt{(-14)^2 - 4(1)(193)}}{2}$$
$$\therefore \quad m = \frac{14 \pm \sqrt{-576}}{2} \quad \text{which has no real solutions}$$

$\therefore$ no line with y-intercept $(0, 10)$ can be tangential to $y = 3x^2 + 7x - 2$.

15 $kx^2 + (1 - 3k)x + (k - 6) = 0$ has $a = k$, $b = 1 - 3k$ and $c = k - 6$.

Let the roots be α and $-\dfrac{1}{\alpha}$.

$\therefore$ product of roots $= -1$

$$\therefore \quad \frac{c}{a} = \frac{k-6}{k} = -1$$
$$\therefore \quad k - 6 = -k$$
$$\therefore \quad 2k = 6$$
$$\therefore \quad k = 3$$

For $k = 3$ the equation is

$$3x^2 + (1 - 3(3))x + (3 - 6) = 0$$
$$\therefore \quad 3x^2 - 8x - 3 = 0$$
$$\therefore \quad (3x+1)(x-3) = 0$$
$$\therefore \quad x = -\tfrac{1}{3} \text{ or } 3$$

$\therefore$ $k = 3$, and the two roots of the equation are $-\frac{1}{3}$ and 3.

REVIEW SET 1B

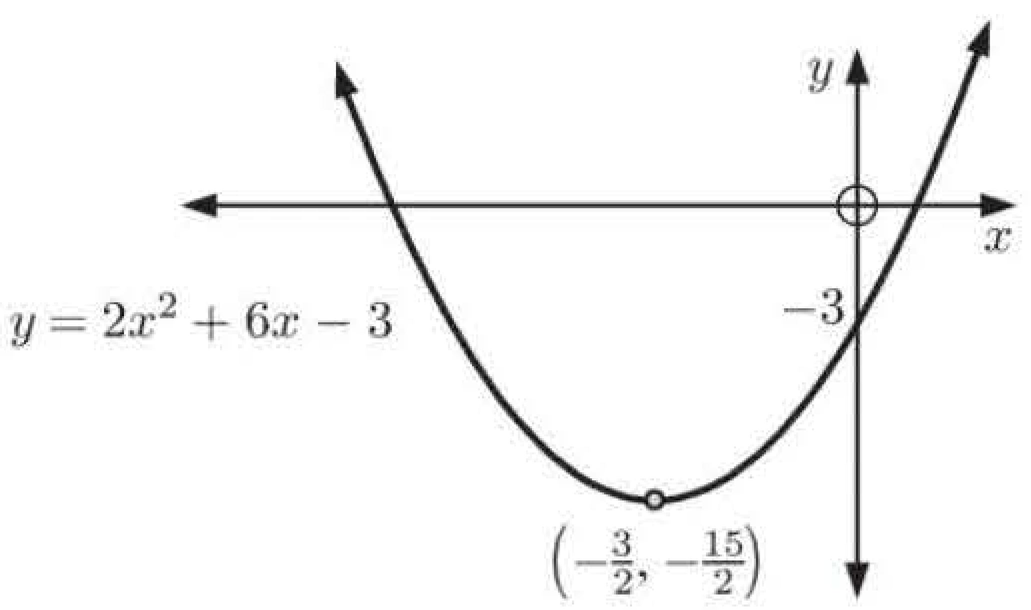
1 **a**
$$\begin{aligned} y &= 2x^2 + 6x - 3 \\ &= 2[x^2 + 3x - \tfrac{3}{2}] \\ &= 2[x^2 + 3x + (\tfrac{3}{2})^2 - (\tfrac{3}{2})^2 - \tfrac{3}{2}] \\ &= 2[(x + \tfrac{3}{2})^2 - \tfrac{9}{4} - \tfrac{3}{2}] \\ &= 2[(x + \tfrac{3}{2})^2 - \tfrac{15}{4}] \\ &= 2(x + \tfrac{3}{2})^2 - \tfrac{15}{2} \end{aligned}$$

b The vertex is $(-\frac{3}{2}, -\frac{15}{2})$.

c When $x = 0$, $y = -3$

$\therefore$ the y-intercept is -3.

d



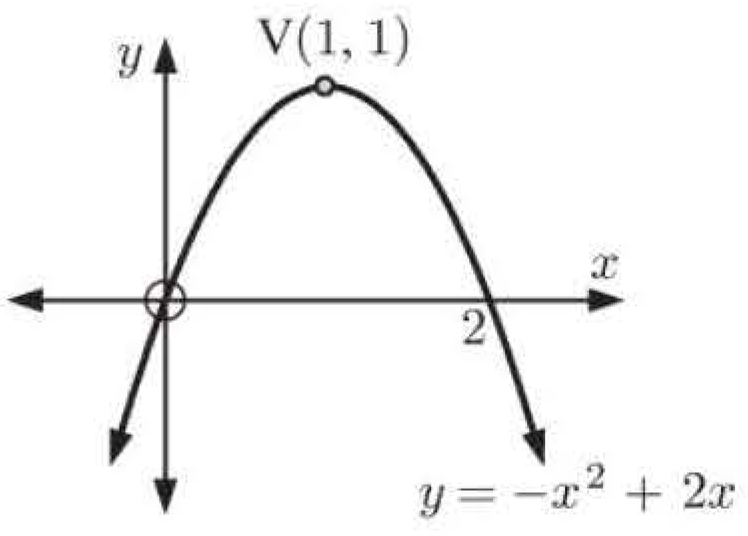
2 **a** Using technology, $x \approx 0.586$ or 3.414 **b** Using technology, $x \approx -0.186$ or 2.686

3 $y = -x^2 + 2x = x(2 - x)$

$\therefore$ the graph has x-intercepts 0 and 2, and y-intercept 0

Its axis of symmetry is midway between the x-intercepts, at $x = 1$, and when $x = 1$, $y = -1^2 + 2 = 1$

$\therefore$ the vertex is $(1, 1)$.



4 $y=-3x^2+8x+7$ has $a=-3$, $b=8$, $c=7$

The axis of symmetry is $x=-\dfrac{b}{2a}=-\dfrac{8}{2(-3)}=\frac{4}{3}$

When $x=\frac{4}{3}$, $y=-3(\frac{4}{3})^2+8(\frac{4}{3})+7$

$=-\frac{16}{3}+\frac{32}{3}+7=\frac{37}{3}$

$\therefore$ the axis of symmetry is $x=\frac{4}{3}$ and the vertex is $(\frac{4}{3}, \frac{37}{3})$ or $(\frac{4}{3}, 12\frac{1}{3})$.

5 **a** $2x^2-5x-7=0$

has $a=2$, $b=-5$, $c=-7$

$\therefore \quad \Delta=b^2-4ac$

$=(-5)^2-4(2)(-7)$

$=25+56=81$

$\therefore \quad \Delta>0$ and $\sqrt{\Delta}=9$

$\therefore$ there are two distinct real rational roots

b $3x^2-24x+48=0$

has $a=3$, $b=-24$, $c=48$

$\therefore \quad \Delta=b^2-4ac$

$=(-24)^2-4(3)(48)$

$=576-576$

$=0$

$\therefore$ there is a repeated real root

6 **a** $y=3x+c$ intersects the parabola $y=x^2+x-5$ when $x^2+x-5=3x+c$

$\therefore \quad x^2-2x-5-c=0$

The graphs meet in two distinct points when this equation has two distinct real roots.

$\therefore \quad \Delta>0$

$\therefore \quad (-2)^2-4(1)(-5-c)>0$

$\therefore \quad 4+20+4c>0$

$\therefore \quad 4c>-24$

$\therefore \quad c>-6$

b Choose c such that $c>-6$, for example, $c=-2$.

The graphs meet where $x^2+x-5=3x-2$

$\therefore \quad x^2-2x-3=0$

$\therefore \quad (x+1)(x-3)=0$

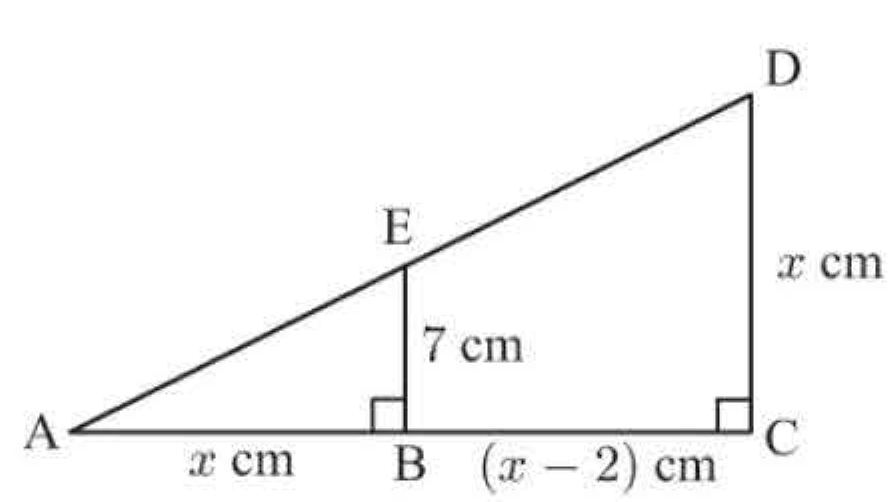
$\therefore \quad x=-1$ or 3

Using the line $y=3x-2$, when $x=-1$, $y=3(-1)-2=-5$

and when $x=3$, $y=3(3)-2=7$

$\therefore$ the points of intersection are $(-1, -5)$ and $(3, 7)$.

7 Suppose [AB] is x cm in length. Then, using the information given, we can label the diagram:



Now by similar triangles, $\dfrac{\text{BE}}{\text{AB}}=\dfrac{\text{CD}}{\text{AC}}$

$\therefore \quad \dfrac{7}{x}=\dfrac{x}{x+(x-2)}$

$\therefore \quad \dfrac{7}{x}=\dfrac{x}{2x-2}$

$\therefore \quad 7(2x-2)=x^2$

$\therefore \quad 14x-14=x^2$

$\therefore \quad x^2-14x+14=0$

which has $a=1$, $b=-14$ and $c=14$

$$\therefore \quad x=\frac{-(-14)\pm\sqrt{(-14)^2-4(1)(14)}}{2(1)}=\frac{14\pm\sqrt{140}}{2}$$

Now $x-2>0$, so $x=\dfrac{14+\sqrt{140}}{2}\approx 12.92$ cm

$\therefore$ [AB] is approximately 12.9 cm long.

8 **a** The total length of wire for the fence is 60 m.

$\therefore$ AB + BC + CD = 60

Since the enclosure is rectangular,

$$\text{CD} = \text{AB}$$
$$\therefore \quad 2\text{AB} + x = 60$$
$$\therefore \quad 2\text{AB} = 60 - x$$
$$\therefore \quad \text{AB} = 30 - \tfrac{1}{2}x$$

$\therefore$ the area of the rectangle is

$$A = x\left(30 - \tfrac{1}{2}x\right) = \left(30x - \tfrac{1}{2}x^2\right) \text{ m}^2$$

b $A = 30x - \frac{1}{2}x^2$

has $a = -\frac{1}{2}$ and $b = 30$.

Since $a < 0$, A has a maximum at the axis of symmetry, and this is at

$$x = -\frac{b}{2a} = -\frac{30}{2(-\frac{1}{2})} = 30$$

When $x = 30$, $\text{AB} = 30 - \frac{1}{2} \times 30 = 15$ m

$\therefore$ the enclosure is 15 m by 30 m.

9 **a** $y = 2x^2 + 4x - 1$

has $a = 2$, $b = 4$, $c = -1$

The axis of symmetry is $x = -\dfrac{b}{2a}$

$$\therefore \quad x = -\frac{4}{2 \times 2}$$
$$\therefore \quad x = -1$$

b When $x = -1$, $y = 2(-1)^2 + 4(-1) - 1 = 2 - 4 - 1 = -3$

$\therefore$ the vertex is $(-1, -3)$

c When $x = 0$, $y = -1$,

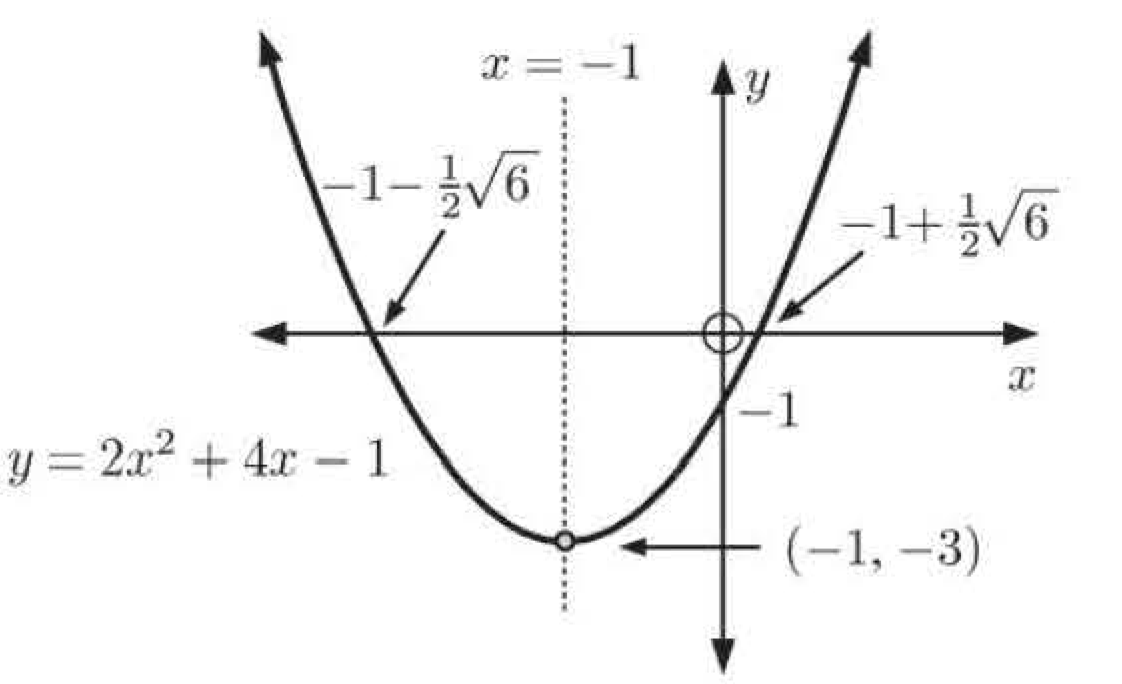
so the y-intercept is -1.

When $y = 0$, $2x^2 + 4x - 1 = 0$

$$\therefore \quad x = \frac{-4 \pm \sqrt{4^2 - 4(2)(-1)}}{2(2)} = \frac{-4 \pm \sqrt{24}}{4} = \frac{-4 \pm 2\sqrt{6}}{4} = -1 \pm \tfrac{1}{2}\sqrt{6}$$

$\therefore$ the x-intercepts are $-1 \pm \frac{1}{2}\sqrt{6}$.

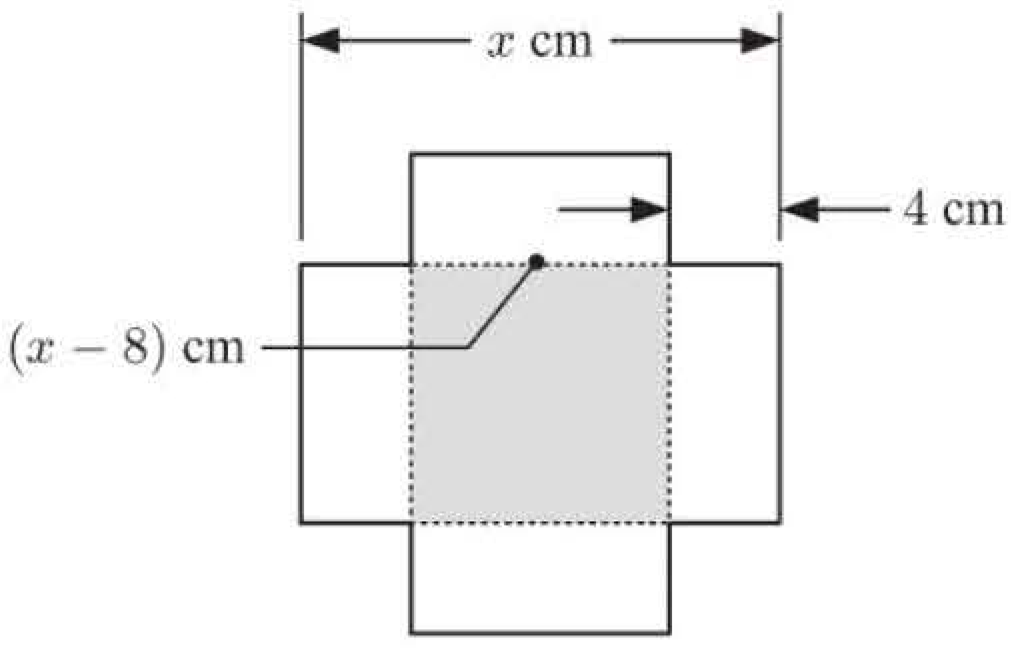
d



10 Since the container has a square base, the original tinplate must have been square.

Suppose its side was x cm long, so the base of the container is $(x - 8)$ cm by $(x - 8)$ cm.

The height of the container is 4 cm, so its capacity is $4(x-8)(x-8)$ cm^3.



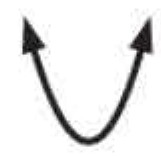
$$\therefore \quad 4(x-8)^2 = 120$$
$$\therefore \quad (x-8)^2 = 30$$
$$\therefore \quad x - 8 = \pm\sqrt{30}$$
$$\therefore \quad x = 8 \pm \sqrt{30}$$

Clearly, $x > 8$, so $x = 8 + \sqrt{30} \approx 13.48$

$\therefore$ the tinplate was about 13.5 cm by 13.5 cm.

11 **a**
$$-x^2 - 5x + 3 = x^2 + 3x + 11$$
$$\therefore \quad 2x^2 + 8x + 8 = 0$$
$$\therefore \quad x^2 + 4x + 4 = 0$$
$$\therefore \quad (x+2)^2 = 0$$
$$\therefore \quad x = -2$$

b From **a**, $x^2+3x+11=-x^2-5x+3$ has only one solution.
$\therefore$ the lines touch but do not cross.
So $y=x^2+3x+11$ is either above or below $y=-x^2-5x+3$ for all $x \neq -2$.
If $x=1$, $(1)^2+3(1)+11=15$ and $-(1)^2-5(1)+3=-3$
$\therefore$ $x^2+3x+11>-x^2-5x+3$ for $x \neq -2$.

12 **a** $y=3x^2+4x+7$
has $a=3$, $b=4$, and $c=7$
Since $a>0$,
the graph has shape 

and so has a minimum.
This occurs on the axis of symmetry

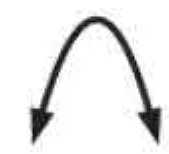
$$x=-\frac{b}{2a}$$

$$\therefore\ x=-\frac{4}{2(3)}=-\tfrac{2}{3}$$

When $x=-\frac{2}{3}$,

$$\begin{aligned} y&=3(-\tfrac{2}{3})^2+4(-\tfrac{2}{3})+7 \\ &=\tfrac{4}{3}-\tfrac{8}{3}+7 \\ &=\tfrac{17}{3}=5\tfrac{2}{3} \end{aligned}$$

$\therefore$ the minimum is $5\frac{2}{3}$ when $x=-\frac{2}{3}$

b $y=-2x^2-5x+2$
has $a=-2$, $b=-5$, and $c=2$
Since $a<0$,
the graph has shape 

and so has a maximum.
This occurs on the axis of symmetry

$$x=-\frac{b}{2a}$$

$$\therefore\ x=-\frac{-5}{2(-2)}=-\tfrac{5}{4}$$

When $x=-\frac{5}{4}$,

$$\begin{aligned} y&=-2(-\tfrac{5}{4})^2-5(-\tfrac{5}{4})+2 \\ &=-\tfrac{25}{8}+\tfrac{25}{4}+2 \\ &=\frac{-25+50+16}{8} \\ &=\tfrac{41}{8}=5\tfrac{1}{8} \end{aligned}$$

$\therefore$ the maximum is $5\frac{1}{8}$ when $x=-\frac{5}{4}$

13 **a** The total length of fencing is $(8x+9y)$ m

$$\therefore\ 8x+9y=600$$
$$\therefore\ 9y=600-8x$$
$$\therefore\ y=\frac{600-8x}{9}$$

The area of each pen is

$$\begin{aligned} A&=xy \\ &=x\left(\frac{600-8x}{9}\right)\ \text{m}^2 \end{aligned}$$

b

$$\begin{aligned} A&=x\left(\frac{600-8x}{9}\right) \\ &=\tfrac{600}{9}x-\tfrac{8}{9}x^2 \end{aligned}$$

which has $a=-\frac{8}{9}$, $b=\frac{600}{9}$
Since $a<0$, A is maximised at the axis of symmetry, which is $x=-\dfrac{b}{2a}$

$$\therefore\ x=-\frac{\frac{600}{9}}{2(-\frac{8}{9})}=\frac{600}{16}$$
$$\therefore\ x=\tfrac{75}{2}$$

When $x=\frac{75}{2}$, $y=\dfrac{600-8(\frac{75}{2})}{9}=33\frac{1}{3}$

$\therefore$ for maximum area, each pen should be $37\frac{1}{2}$ m $\times$ $33\frac{1}{3}$ m.

c The maximum area of each pen is

$$\begin{aligned} &37\tfrac{1}{2}\times 33\tfrac{1}{3} \\ &=\tfrac{75}{2}\times\tfrac{100}{3} \\ &=1250\ \text{m}^2 \end{aligned}$$

14 **a** $9x^2-kx+4$ touches the x-axis if

$$\Delta=0$$
$$\therefore\ (-k)^2-4(9)(4)=0$$
$$\therefore\ k^2-144=0$$
$$\therefore\ k=\pm 12$$

b The functions intersect when

$$9x^2-12x+4=9x^2+12x+4$$
$$\therefore\ 24x=0$$
$$\therefore\ x=0$$

$f(0)=9(0)^2-12(0)+4=4$
The two functions intersect at $(0,\,4)$.

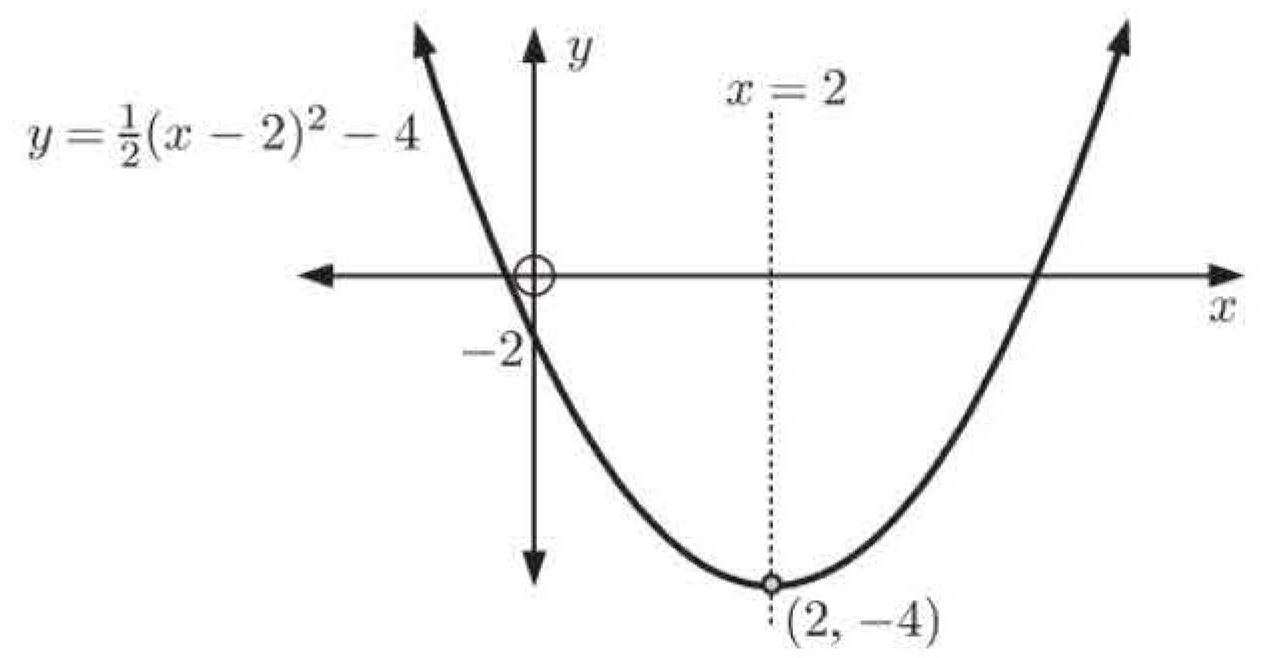
REVIEW SET 1C

1 **a** The axis of symmetry is $x = 2$.

b When $x = 2$, $y = \frac{1}{2}(2-2)^2 - 4$
$= -4$
$\therefore$ the vertex is $(2, -4)$

c When $x = 0$, $y = \frac{1}{2}(-2)^2 - 4$
$= -2$
$\therefore$ the y-intercept is -2

d



2 **a** $x^2 - 5x - 3 = 0$
has $a = 1$, $b = -5$, $c = -3$

$$\therefore \quad x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(-3)}}{2(1)}$$
$$= \tfrac{5}{2} \pm \tfrac{\sqrt{37}}{2}$$

b $2x^2 - 7x - 3 = 0$
has $a = 2$, $b = -7$, $c = -3$

$$\therefore \quad x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(2)(-3)}}{2(2)}$$
$$= \tfrac{7}{4} \pm \tfrac{\sqrt{73}}{4}$$

3 **a** $x^2 - 7x + 3 = 0$
has $a = 1$, $b = -7$, $c = 3$

$$\therefore \quad x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(1)(3)}}{2(1)}$$
$$= \frac{7 \pm \sqrt{49 - 12}}{2}$$
$$= \frac{7 \pm \sqrt{37}}{2}$$
$$\therefore \quad x = \tfrac{7}{2} \pm \tfrac{\sqrt{37}}{2}$$

b $2x^2 - 5x + 4 = 0$
has $a = 2$, $b = -5$, $c = 4$

$$\therefore \quad x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(2)(4)}}{2(2)}$$
$$= \frac{5 \pm \sqrt{25 - 32}}{4}$$
$$\therefore \quad x = \frac{5 \pm \sqrt{-7}}{4}$$
$\therefore$ x has no real solutions.

4 **a** The graph has vertex $(2, -20)$, so its equation is
$y = a(x-2)^2 - 20$ for some $a \neq 0$.
Now an x-intercept is 5
$\therefore$ $a(5-2)^2 - 20 = 0$
$\therefore$ $9a = 20$
and so $a = \frac{20}{9}$

So, the equation is $y = \frac{20}{9}(x-2)^2 - 20$.

b Since one x-intercept is 7 and the axis of symmetry is $x = 4$, the other x-intercept is $x = 1$.
$\therefore$ the graph has equation
$y = a(x-7)(x-1)$ for some $a \neq 0$.
The y-intercept is -2
$\therefore$ $a(-7)(-1) = -2$
$\therefore$ $a = -\frac{2}{7}$
$\therefore$ the equation is $y = -\frac{2}{7}(x-7)(x-1)$.

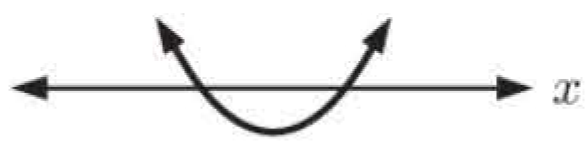
c The graph has vertex $(-3, 0)$, so its equation is $y = a(x+3)^2$ for some $a \neq 0$.
The y-intercept is 2
$\therefore$ $a(3)^2 = 2$
$\therefore$ $9a = 2$
and so $a = \frac{2}{9}$

So, the equation is $y = \frac{2}{9}(x+3)^2$.

5 **a** $y = 2x^2 + 3x - 7$
has $a = 2, \quad b = 3, \quad c = -7$

$$\begin{aligned}\therefore \quad \Delta &= b^2 - 4ac \\ &= 3^2 - 4(2)(-7) \\ &= 65\end{aligned}$$

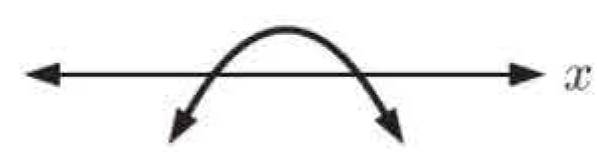
Since $\Delta > 0$, the graph cuts the x-axis twice.
Note that since $a > 0$, the graph is



b $y = -3x^2 - 7x + 4$
has $a = -3, \quad b = -7, \quad c = 4$

$$\begin{aligned}\therefore \quad \Delta &= b^2 - 4ac \\ &= (-7)^2 - 4(-3)4 \\ &= 97\end{aligned}$$

Since $\Delta > 0$, the graph cuts the x-axis twice.
Note that since $a < 0$, the graph is



6 **a** $y = -2x^2 + 3x + 2$
has $a = -2, \quad b = 3, \quad c = 2$

$$\begin{aligned}\therefore \quad \Delta &= b^2 - 4ac \\ &= 3^2 - 4(-2)(2) \\ &= 25\end{aligned}$$

Since $\Delta > 0$, the function is neither positive definite nor negative definite.

b $y = 3x^2 + x + 11$
has $a = 3, \quad b = 1, \quad c = 11$

$$\begin{aligned}\therefore \quad \Delta &= b^2 - 4ac \\ &= 1^2 - 4(3)(11) \\ &= -131\end{aligned}$$

$\therefore$ $\Delta < 0$, and since $a > 0$, the function is positive definite.

7 The quadratic has vertex $(2, 25)$.
$\therefore$ its equation is $y = a(x-2)^2 + 25$
The y-intercept is 1, so $a(-2)^2 + 25 = 1$

$$\begin{aligned}\therefore \quad 4a &= -24 \\ \therefore \quad a &= -6\end{aligned}$$

$\therefore$ the equation is $y = -6(x-2)^2 + 25$

8 Let the line with gradient -3 and y-intercept c have equation $y = -3x + c$.
$y = -3x + c$ is tangential to $y = 2x^2 - 5x + 1$ if they meet at exactly one point.
$y = 2x^2 - 5x + 1$ meets $y = -3x + c$ when $2x^2 - 5x + 1 = -3x + c$

$$\therefore \quad 2x^2 - 2x + 1 - c = 0$$

The graphs meet exactly once when this equation has a repeated root $\therefore \quad \Delta = 0$

$$\begin{aligned}\therefore \quad (-2)^2 - 4(2)(1-c) &= 0 \\ \therefore \quad 4 - 8 + 8c &= 0 \\ \therefore \quad 8c &= 4 \\ \therefore \quad c &= \tfrac{1}{2}\end{aligned}$$

$\therefore$ the y-intercept of the line is $\frac{1}{2}$.

9 $y = x^2 - 2x + k$
has $a = 1, \quad b = -2, \quad c = k$

$$\begin{aligned}\therefore \quad \Delta &= b^2 - 4ac \\ &= (-2)^2 - 4(1)k \\ &= 4 - 4k\end{aligned}$$

The graph cuts the x-axis twice if $\Delta > 0$

$$\begin{aligned}\therefore \quad 4 - 4k &> 0 \\ \therefore \quad 4k &< 4 \\ \therefore \quad k &< 1\end{aligned}$$

10 The x-intercepts are 3 and -2, so the equation is $y = a(x-3)(x+2)$ for some $a \neq 0$.
But the y-intercept is 24 $\therefore \quad a(-3)(2) = 24$

$$\begin{aligned}\therefore \quad -6a &= 24 \\ \therefore \quad a &= -4\end{aligned}$$

$\therefore$ the equation is $y = -4(x-3)(x+2)$

$$\begin{aligned}\therefore \quad y &= -4(x^2 - x - 6) \\ \therefore \quad y &= -4x^2 + 4x + 24\end{aligned}$$

11 $y = mx - 10$ is a tangent to $y = 3x^2 + 7x + 2$ if they meet at exactly one point (touch).

$y = 3x^2 + 7x + 2$ meets $y = mx - 10$ when $3x^2 + 7x + 2 = mx - 10$

$$\therefore \quad 3x^2 + (7 - m)x + 12 = 0$$

The graphs meet exactly once when this equation has a repeated root $\therefore \quad \Delta = 0$

$$\therefore \quad (7 - m)^2 - 4(3)(12) = 0$$
$$\therefore \quad 49 - 14m + m^2 - 144 = 0$$
$$\therefore \quad m^2 - 14m - 95 = 0$$
$$\therefore \quad (m + 5)(m - 19) = 0$$
$$\therefore \quad m = -5 \text{ or } 19$$

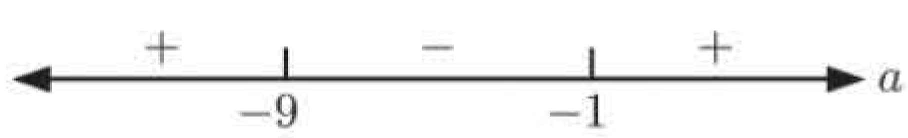
12 $ax^2 + [3 - a]x - 4 = 0$ will have roots which are real and positive if:

(1) $\Delta \geqslant 0$ (2) sum of roots > 0 (3) product of roots > 0

(1) $\Delta \geqslant 0$

$$\therefore \quad (3 - a)^2 - 4(a)(-4) \geqslant 0$$
$$\therefore \quad 9 - 6a + a^2 + 16a \geqslant 0$$
$$\therefore \quad a^2 + 10a + 9 \geqslant 0$$
$$\therefore \quad (a + 9)(a + 1) \geqslant 0$$

$(a + 9)(a + 1)$ has sign diagram



$\therefore \quad \Delta \geqslant 0$ when $a \leqslant -9$ or $a \geqslant -1$

(2) sum of roots > 0

$$\therefore \quad -\frac{3 - a}{a} > 0$$

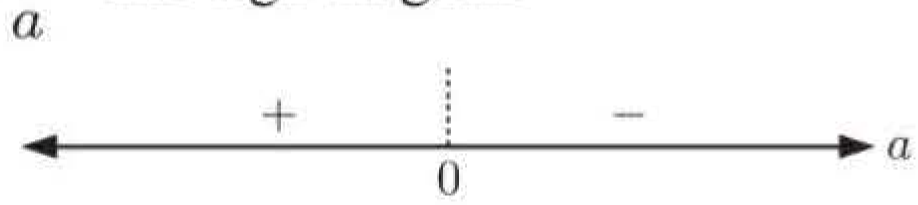
$-\dfrac{3 - a}{a}$ has sign diagram

$\therefore$ sum of roots > 0 when $a < 0$ or $a > 3$

(3) product of roots > 0

$$\therefore \quad \frac{-4}{a} > 0$$

$-\dfrac{4}{a}$ has sign diagram



$\therefore$ product of roots > 0 when $a < 0$

$\therefore$ the only values of a which satisfy all three conditions are $a \leqslant -9$ and $-1 \leqslant a < 0$.

13 **a** **i** The quadratic has x-intercepts ± 3, so its equation is $y = a(x + 3)(x - 3)$ for some $a \neq 0$.

Its y-intercept is -27, so

$$a(3)(-3) = -27$$
$$\therefore \quad -9a = -27$$
$$\therefore \quad a = 3$$

$\therefore$ the equation is $y = 3(x + 3)(x - 3)$ or $y = 3x^2 - 27$

ii The straight line has gradient $\dfrac{0 - (-27)}{3 - 0} = 9$ and its y-intercept is -27.

So, its equations is $y = 9x - 27$.

b From the graph, the straight line is above the curve when $0 < x < 3$.

14 $y = -5x + k$ meets $y = x^2 - 3x + c$ when $x^2 - 3x + c = -5x + k$

$$\therefore \quad x^2 + 2x + c - k = 0$$

For $y = -5x + k$ to be a tangent to $y = x^2 - 3x + c$, this equation must have exactly one solution, so there is a repeated root.

$$\therefore \quad \Delta = 0$$
$$\therefore \quad 2^2 - 4(1)(c - k) = 0$$
$$\therefore \quad 4 = 4(c - k)$$
$$\therefore \quad c - k = 1$$

15 The roots of $4x^2 - 3x - 3 = 0$ are p and q.

$\therefore$ sum of roots $= -\dfrac{-3}{4}$ and $\therefore$ product of roots $= \dfrac{-3}{4}$

$\therefore\ p + q = \frac{3}{4}$ $\qquad\therefore\ pq = -\frac{3}{4}$

Now consider a quadratic equation with roots p^3 and q^3.

$$\begin{aligned}\text{sum of roots} &= p^3 + q^3\\ &= (p+q)^3 - 3p^2q - 3pq^2\\ &= (p+q)^3 - 3pq(p+q)\\ &= (\tfrac{3}{4})^3 - 3(-\tfrac{3}{4})(\tfrac{3}{4})\\ &= \tfrac{27}{64} + \tfrac{27}{16}\\ &= \tfrac{135}{64}\end{aligned}$$

$$\begin{aligned}\text{product of roots} &= p^3 \times q^3\\ &= (pq)^3\\ &= (-\tfrac{3}{4})^3\\ &= -\tfrac{27}{64}\end{aligned}$$

$\therefore$ a quadratic equation $ax^2 + bx + c = 0$ with roots p^3 and q^3 has $-\dfrac{b}{a} = \frac{135}{64}$ and $\dfrac{c}{a} = -\frac{27}{64}$

$\therefore\ b = -\frac{135}{64}a$ and $c = -\frac{27}{64}a$

$\therefore$ the quadratic equation is $ax^2 - \frac{135}{64}ax - \frac{27}{64}a = 0,\quad a \neq 0$

$\therefore\ a(x^2 - \frac{135}{64}x - \frac{27}{64}) = 0,\quad a \neq 0$

$\therefore\ a(64x^2 - 135x - 27) = 0,\quad a \neq 0$

Chapter 2
FUNCTIONS

EXERCISE 2A

1 **a** $\{(1, 3), (2, 4), (3, 5), (4, 6)\}$ is a function since no two ordered pairs have the same x-coordinate.

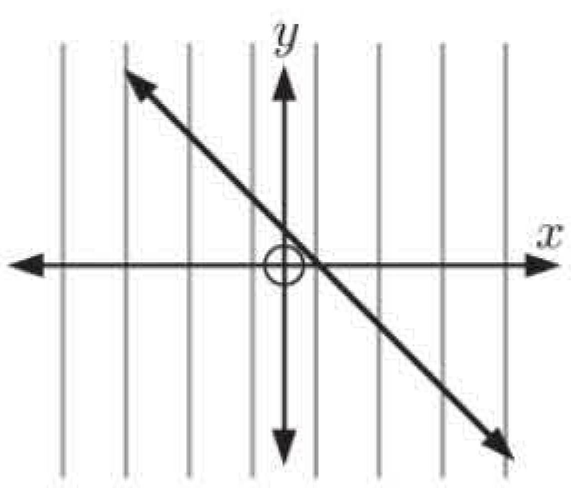
b $\{(1, 3), (3, 2), (1, 7), (-1, 4)\}$ is not a function since two of the ordered pairs, $(1, 3)$ and $(1, 7)$, have the same x-coordinate 1.

c $\{(2, -1), (2, 0), (2, 3), (2, 11)\}$ is not a function since each ordered pair has the same x-coordinate 2.

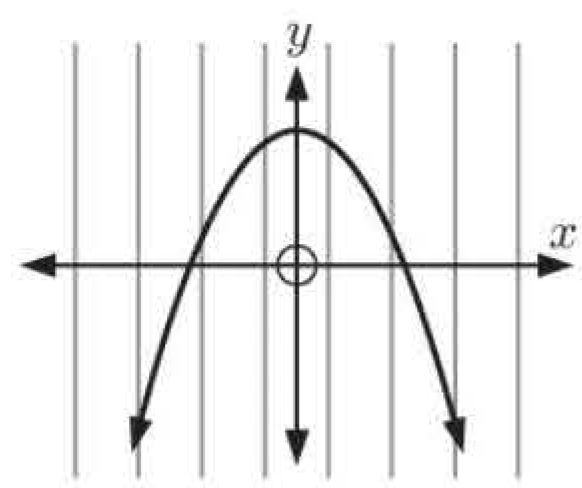
d $\{(7, 6), (5, 6), (3, 6), (-4, 6)\}$ is a function since no two ordered pairs have the same x-coordinate.

e $\{(0, 0), (1, 0), (3, 0), (5, 0)\}$ is a function since no two ordered pairs have the same x-coordinate.

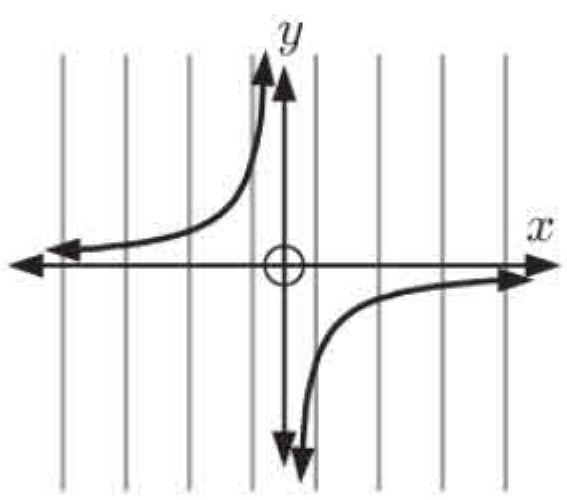
f $\{(0, 0), (0, -2), (0, 2), (0, 4)\}$ is not a function since each ordered pair has the same x-coordinate 0.

2 **a**


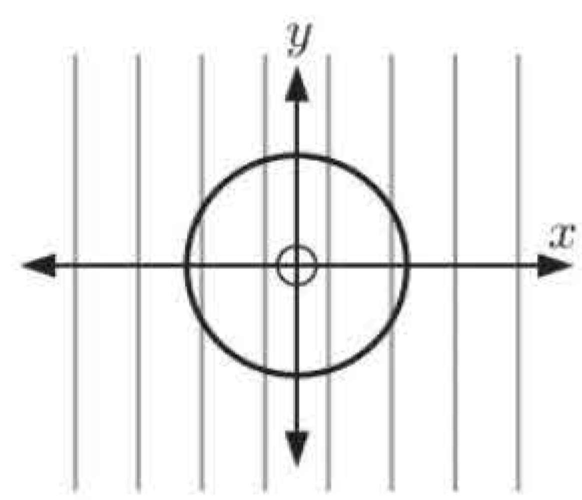
Each line cuts the graph no more than once, so it is a function.

b


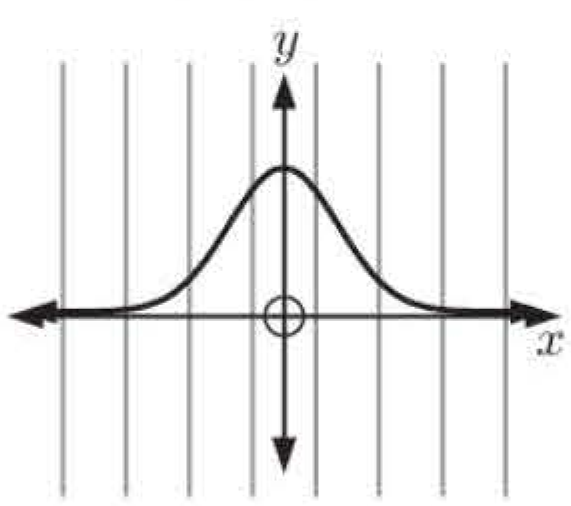
Each line cuts the graph no more than once, so it is a function.

c


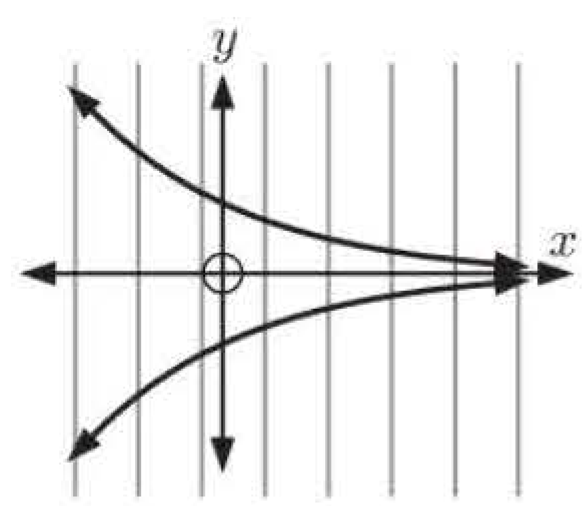
Each line cuts the graph no more than once, so it is a function.

d


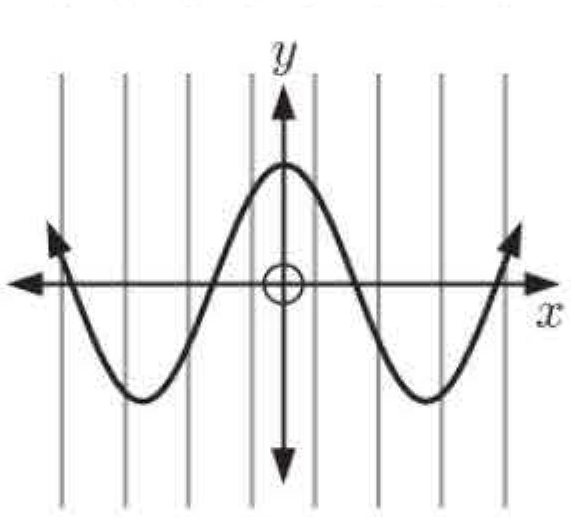
Some lines cut the graph more than once, so it is not a function.

e


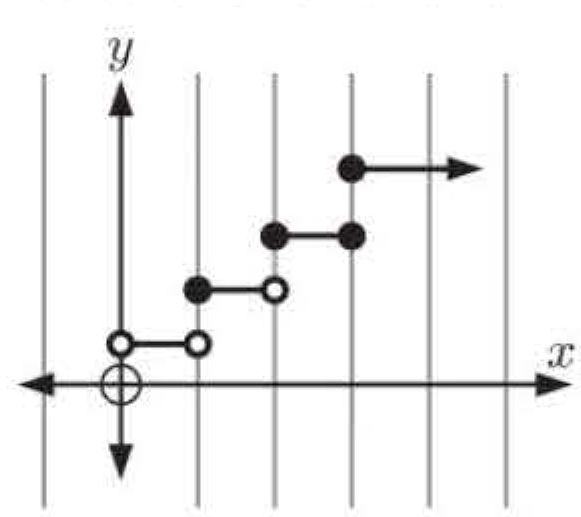
Each line cuts the graph no more than once, so it is a function.

f


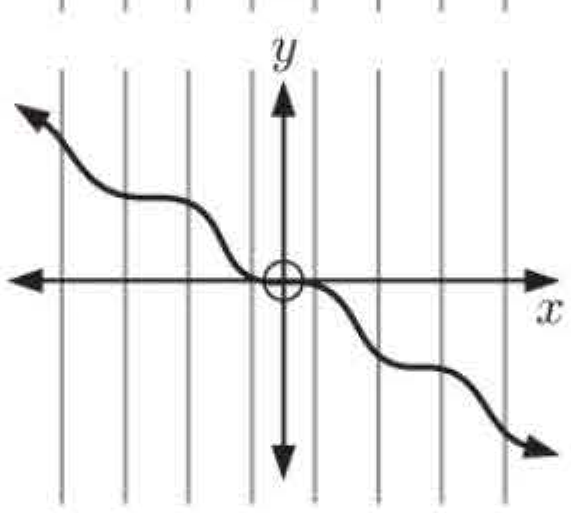
The lines cut the graph more than once, so it is not a function.

g


Each line cuts the graph no more than once, so it is a function.

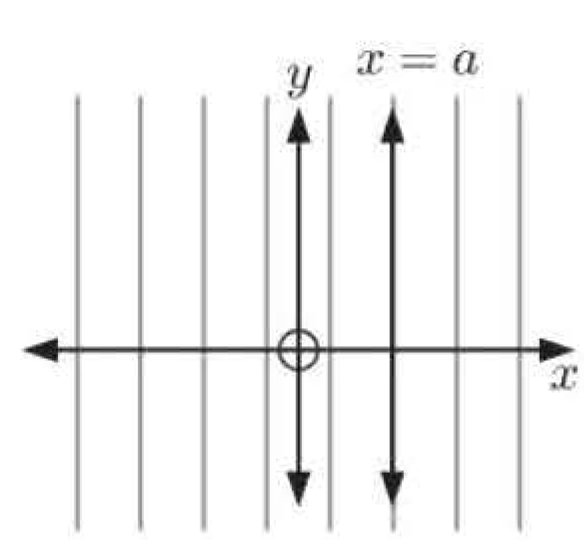
h


One line cuts the graph more than once, so it is not a function.

i


Each line cuts the graph no more than once, so it is a function.

3 The graph of a straight line is not a function if the graph is a vertical line. So, it is not a function if it has the form $x = a$ for some constant a.
The vertical line through $x = a$ cuts the graph at every point, so it is not a function.



4 $x^2 + y^2 = 9$ is the equation of a circle, centre $(0, 0)$ and radius 3.

Now $x^2 + y^2 = 9$

$\therefore\ y^2 = 9 - x^2$

$\therefore\ y = \pm\sqrt{9 - x^2}$

For any value of x where $-3 < x < 3$, y has two real values. Hence $x^2 + y^2 = 9$ is not a function.

EXERCISE 2B

1 **a** $f(0) = 3(0) + 2 = 2$ **b** $f(2) = 3(2) + 2 = 8$ **c** $f(-1) = 3(-1) + 2 = -1$

d $f(-5) = 3(-5) + 2 = -13$ **e** $f(-\frac{1}{3}) = 3(-\frac{1}{3}) + 2 = 1$

2 **a** $f(0) = 3(0) - 0^2 + 2$
$= 2$

b $f(3) = 3(3) - 3^2 + 2$
$= 9 - 9 + 2$
$= 2$

c $f(-3) = 3(-3) - (-3)^2 + 2$
$= -9 - 9 + 2$
$= -16$

d $f(-7) = 3(-7) - (-7)^2 + 2$
$= -21 - 49 + 2$
$= -68$

e $f(\frac{3}{2}) = 3(\frac{3}{2}) - (\frac{3}{2})^2 + 2$
$= \frac{9}{2} - \frac{9}{4} + 2$
$= \frac{17}{4}$

3 **a** $g(1) = 1 - \frac{4}{1} = -3$ **b** $g(4) = 4 - \frac{4}{4} = 3$ **c** $g(-1) = -1 - \frac{4}{(-1)} = 3$

d $g(-4) = -4 - \frac{4}{(-4)} = -3$ **e** $g(-\frac{1}{2}) = -\frac{1}{2} - \dfrac{4}{(-\frac{1}{2})} = -\frac{1}{2} + 8 = \frac{15}{2}$

4 **a** $f(a) = 7 - 3a$

b $f(-a) = 7 - 3(-a)$
$= 7 + 3a$

c $f(a + 3) = 7 - 3(a + 3)$
$= 7 - 3a - 9$
$= -3a - 2$

d $f(b - 1) = 7 - 3(b - 1)$
$= 7 - 3b + 3$
$= 10 - 3b$

e $f(x + 2) = 7 - 3(x + 2)$
$= 7 - 3x - 6$
$= 1 - 3x$

f $f(x + h) = 7 - 3(x + h)$
$= 7 - 3x - 3h$

5 **a** $F(x + 4)$
$= 2(x + 4)^2 + 3(x + 4) - 1$
$= 2(x^2 + 8x + 16) + 3x + 12 - 1$
$= 2x^2 + 16x + 32 + 3x + 11$
$= 2x^2 + 19x + 43$

b $F(2 - x)$
$= 2(2 - x)^2 + 3(2 - x) - 1$
$= 2(4 - 4x + x^2) + 6 - 3x - 1$
$= 8 - 8x + 2x^2 + 5 - 3x$
$= 2x^2 - 11x + 13$

c $F(-x)$
$= 2(-x)^2 + 3(-x) - 1$
$= 2x^2 - 3x - 1$

d $F(x^2)$
$= 2(x^2)^2 + 3(x^2) - 1$
$= 2x^4 + 3x^2 - 1$

e $F(x^2 - 1)$
$= 2(x^2 - 1)^2 + 3(x^2 - 1) - 1$
$= 2(x^4 - 2x^2 + 1) + 3x^2 - 3 - 1$
$= 2x^4 - 4x^2 + 2 + 3x^2 - 4$
$= 2x^4 - x^2 - 2$

f $F(x + h)$
$= 2(x + h)^2 + 3(x + h) - 1$
$= 2(x^2 + 2xh + h^2) + 3x + 3h - 1$
$= 2x^2 + 4xh + 2h^2 + 3x + 3h - 1$
$= 2x^2 + (4h + 3)x + 2h^2 + 3h - 1$

6 **a** **i** $G(2) = \dfrac{2(2)+3}{2-4} = \frac{7}{-2} = -\frac{7}{2}$

ii $G(0) = \dfrac{2(0)+3}{0-4} = \frac{3}{-4} = -\frac{3}{4}$

iii $G(-\frac{1}{2}) = \dfrac{2(-\frac{1}{2})+3}{-\frac{1}{2}-4} = \dfrac{-1+3}{(-\frac{9}{2})} = \dfrac{2}{(-\frac{9}{2})} = -\frac{4}{9}$

b $G(x) = \dfrac{2x+3}{x-4}$ is undefined when $x - 4 = 0$
$\therefore \quad x = 4$
So, when $x = 4$, $G(x)$ does not exist.

c $G(x+2) = \dfrac{2(x+2)+3}{(x+2)-4} = \dfrac{2x+4+3}{x+2-4} = \dfrac{2x+7}{x-2}$

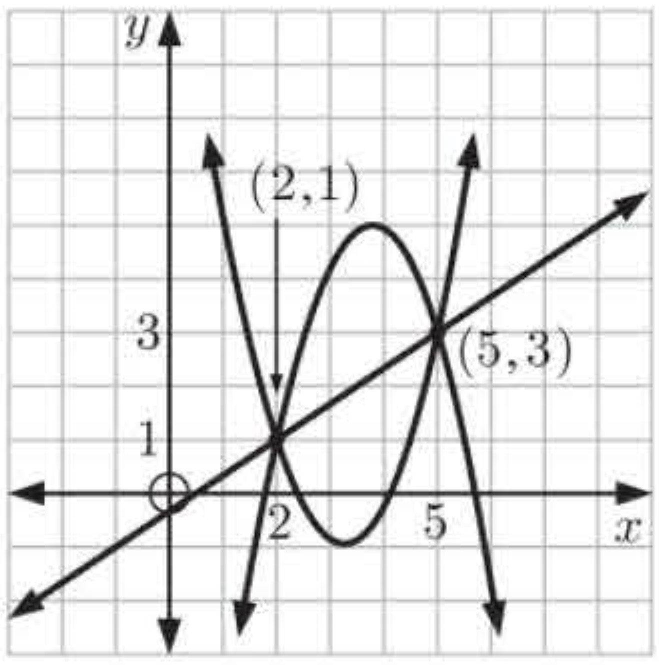
d $G(x) = -3$, so $\dfrac{2x+3}{x-4} = -3$ $\quad \therefore \quad 2x+3 = -3(x-4)$
$\therefore \quad 2x+3 = -3x+12$
$\therefore \quad 5x = 9$ and so $x = \frac{9}{5}$

7 f is the function which converts x into $f(x)$ whereas $f(x)$ is the value of the function at any value of x.

8 **a** $V(4) = 9650 - 860(4)$
$= 9650 - 3440$
$= 6210$
The value of the photocopier 4 years after purchase is 6210 euros.

b If $V(t) = 5780$,
then $9650 - 860t = 5780$
$\therefore \quad 860t = 3870$
$\therefore \quad t = 4.5$
The value of the photocopier is 5780 euros after $4\frac{1}{2}$ years.

c Original purchase price is when $t = 0$,
$V(0) = 9650 - 860(0)$
$= 9650$
The original purchase price was 9650 euros.

9 

First sketch the linear function which passes through the two points (2, 1) and (5, 3).
Then sketch two quadratic functions which also pass through the two points.

10 $f(x) = ax + b$ where $f(2) = 1$ and $f(-3) = 11$

So, $a(2) + b = 1$ and $a(-3) + b = 11$
$\therefore \quad 2a + b = 1$ $\qquad \therefore \quad -3a + b = 11$
$\therefore \quad b = 1 - 2a$ .... (1) $\qquad \therefore \quad b = 11 + 3a$ .... (2)

Solving (1) and (2) simultaneously, $1 - 2a = 11 + 3a$
$\therefore \quad 5a = -10$
$\therefore \quad a = -2$

Substituting $a = -2$ into (1) gives $b = 1 - 2(-2) = 5$. So, $a = -2$, $b = 5$.
Hence $f(x) = -2x + 5$

11 $f(x) = ax + \dfrac{b}{x}$ where $f(1) = 1$ and $f(2) = 5$

So, $a(1) + \dfrac{b}{1} = 1$ and $a(2) + \dfrac{b}{2} = 5$

$\therefore\ a + b = 1$

$\therefore\ a = 1 - b$ .... (1)

$\therefore\ 2a + \dfrac{b}{2} = 5$ .... (2)

Substituting (1) into (2), $2(1-b) + \dfrac{b}{2} = 5$

$\therefore\ 2 - 2b + \dfrac{b}{2} = 5$

$\therefore\ -\dfrac{3b}{2} = 3$

$\therefore\ b = -2$

Substituting $b = -2$ into (1) gives $a = 1 - (-2) = 3$.

So, $a = 3,\ b = -2$.

12 $T(x) = ax^2 + bx + c$ where $T(0) = -4$, $T(1) = -2$, and $T(2) = 6$

So, $a(0)^2 + b(0) + c = -4$

$\therefore\ c = -4$

Also, $a(1)^2 + b(1) + c = -2$ and $a(2)^2 + b(2) + c = 6$

$\therefore\ a + b + c = -2$ and $\therefore\ 4a + 2b + c = 6$

Substituting $c = -4$ into both equations gives

$a + b + (-4) = -2$ and $4a + 2b + (-4) = 6$

$\therefore\ a + b = 2$

$\therefore\ a = 2 - b$ .... (1)

$\therefore\ 4a + 2b = 10$ .... (2)

Substituting (1) into (2) gives $4(2-b) + 2b = 10$ $\therefore\ 8 - 4b + 2b = 10$

$\therefore\ -2b = 2$

$\therefore\ b = -1$

Substituting $b = -1$ into (1) gives $a = 2 - (-1) = 3$.

$\therefore\ a = 3,\ b = -1$, and $c = -4$. So, $T(x) = 3x^2 - x - 4$.

EXERCISE 2C

1 **a** Domain is $\{x \mid x \geqslant -1\}$
Range is $\{y \mid y \leqslant 3\}$

b Domain is $\{x \mid -1 < x \leqslant 5\}$
Range is $\{y \mid 1 < y \leqslant 3\}$

c Domain is $\{x \mid x \neq 2\}$
Range is $\{y \mid y \neq -1\}$

d Domain is $\{x \mid x \in \mathbb{R}\}$
Range is $\{y \mid 0 < y \leqslant 2\}$

e Domain is $\{x \mid x \in \mathbb{R}\}$
Range is $\{y \mid y \geqslant -1\}$

f Domain is $\{x \mid x \in \mathbb{R}\}$
Range is $\{y \mid y \leqslant 6\frac{1}{4}\}$ *or* $\{y \mid y \leqslant \frac{25}{4}\}$

g Domain is $\{x \mid x \geqslant -4\}$
Range is $\{y \mid y \geqslant -3\}$

h Domain is $\{x \mid x \in \mathbb{R}\}$
Range is $\{y \mid y > -2\}$

i Domain is $\{x \mid x \neq \pm 2\}$
Range is $\{y \mid y \leqslant -1 \text{ or } y > 0\}$

2 **a** $f(x)$ is defined when $x + 6 \geqslant 0$
$\therefore\ f(x)$ is defined for $x \geqslant -6$
$\therefore$ the domain is $\{x \mid x \geqslant -6\}$.

b $f(x)$ is defined when $x^2 \neq 0$
$\therefore\ f(x)$ is defined for $x \neq 0$
$\therefore$ the domain is $\{x \mid x \neq 0\}$.

c $f(x)$ is defined when $3 - 2x > 0$
$\therefore\ f(x)$ is defined for $x < \frac{3}{2}$
$\therefore$ the domain is $\{x \mid x < \frac{3}{2}\}$.

3 **a** $y = 2x - 1$ can take any x-value and any y-value.

$\therefore$ the domain is $\{x \mid x \in \mathbb{R}\}$ and the range is $\{y \mid y \in \mathbb{R}\}$.

b $y = 3$ can take any value of x, but the only permissible value for y is 3.

$\therefore$ the domain is $\{x \mid x \in \mathbb{R}\}$ and the range is $\{3\}$.

c $y = \sqrt{x}$ is defined when $x \geqslant 0$, and a square root cannot be negative.

$\therefore$ the domain is $\{x \mid x \geqslant 0\}$ and the range is $\{y \mid y \geqslant 0\}$.

d $y = \dfrac{1}{x+1}$ is defined when $x + 1 \neq 0$, or when $x \neq -1$.

$y = \dfrac{1}{x+1}$ cannot be 0 for any value of x.

$\therefore$ the domain is $\{x \mid x \neq -1\}$ and the range is $\{y \mid y \neq 0\}$.

e $y = -\dfrac{1}{\sqrt{x}}$ is defined when $x > 0$.

If x is always positive, then $y = -\dfrac{1}{\sqrt{x}}$ is always negative.

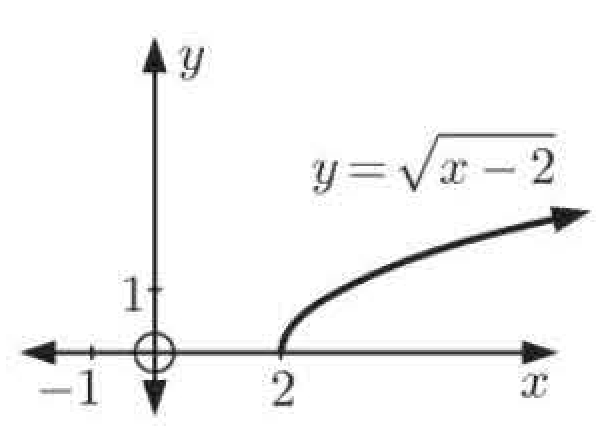
$\therefore$ the domain is $\{x \mid x > 0\}$ and the range is $\{y \mid y < 0\}$.

f $y = \dfrac{1}{3-x}$ is defined when $3 - x \neq 0$, or when $x \neq 3$.

$y = \dfrac{1}{3-x}$ cannot be 0 for any value of x.

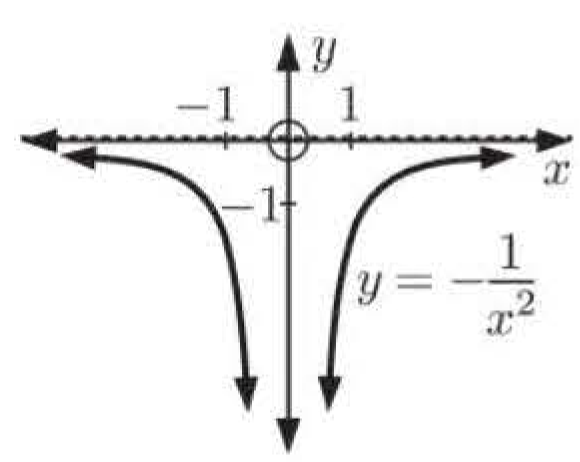
$\therefore$ the domain is $\{x \mid x \neq 3\}$ and the range is $\{y \mid y \neq 0\}$.

4 **a**



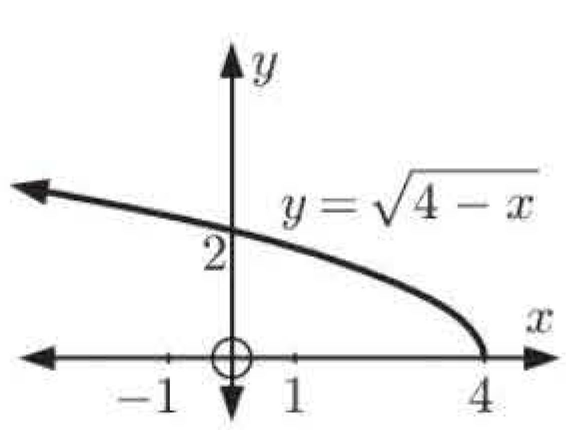
Domain is $\{x \mid x \geqslant 2\}$
Range is $\{y \mid y \geqslant 0\}$

b



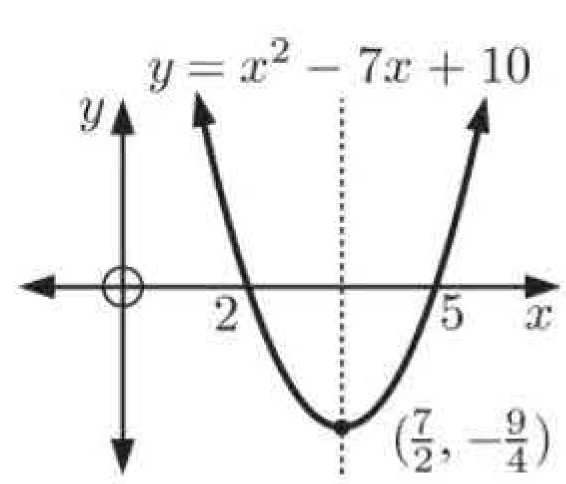
Domain is $\{x \mid x \neq 0\}$
Range is $\{y \mid y < 0\}$

c



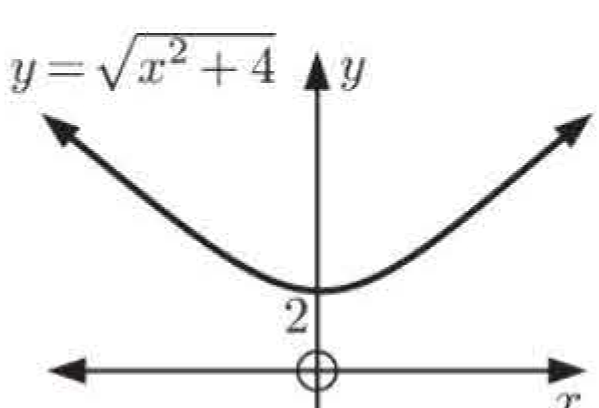
Domain is $\{x \mid x \leqslant 4\}$
Range is $\{y \mid y \geqslant 0\}$

d



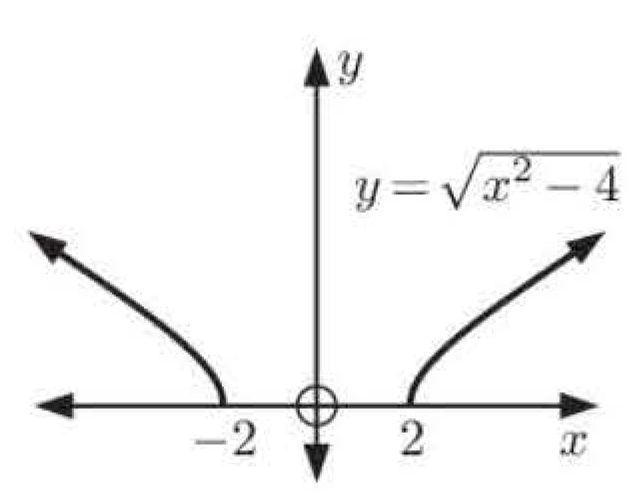
Domain is $\{x \mid x \in \mathbb{R}\}$
Range is $\{y \mid y \geqslant -2\frac{1}{4}\}$

e



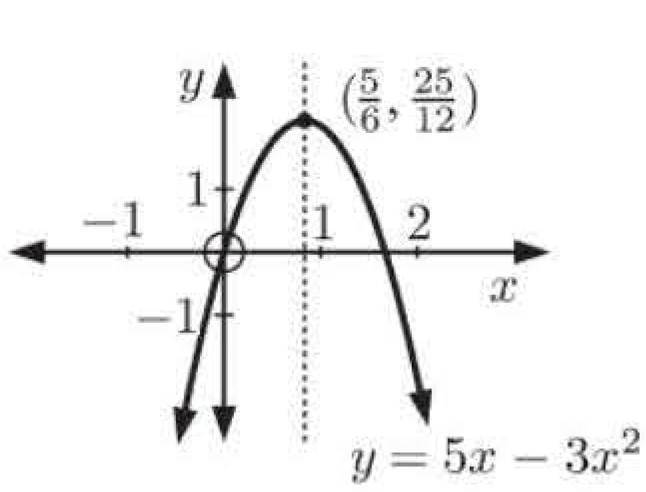
Domain is $\{x \mid x \in \mathbb{R}\}$
Range is $\{y \mid y \geqslant 2\}$

f



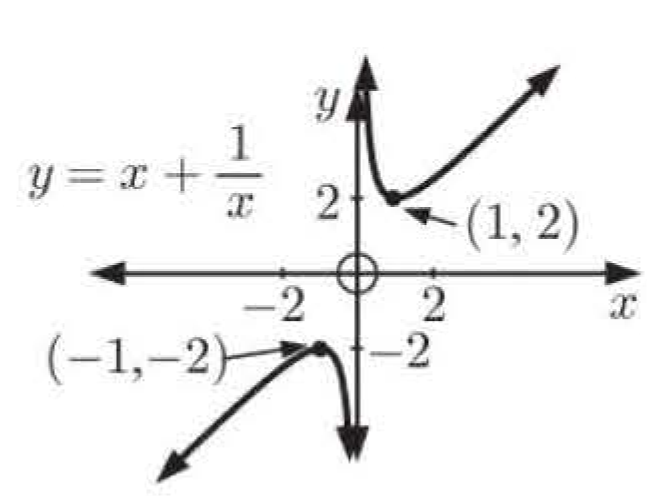
Domain is $\{x \mid x \leqslant -2$ or $x \geqslant 2\}$
Range is $\{y \mid y \geqslant 0\}$

g



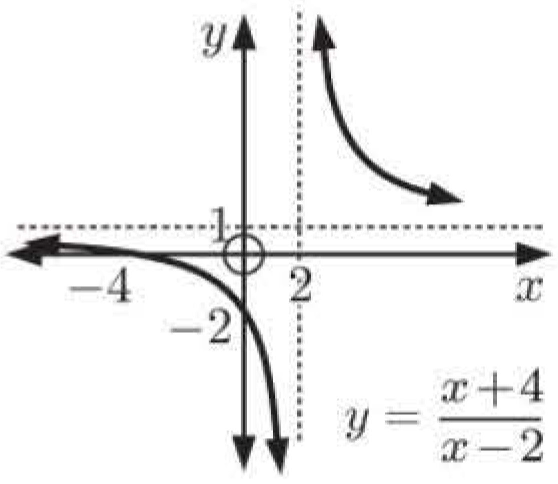
Domain is $\{x \mid x \in \mathbb{R}\}$
Range is $\{y \mid y \leqslant \frac{25}{12}\}$

h



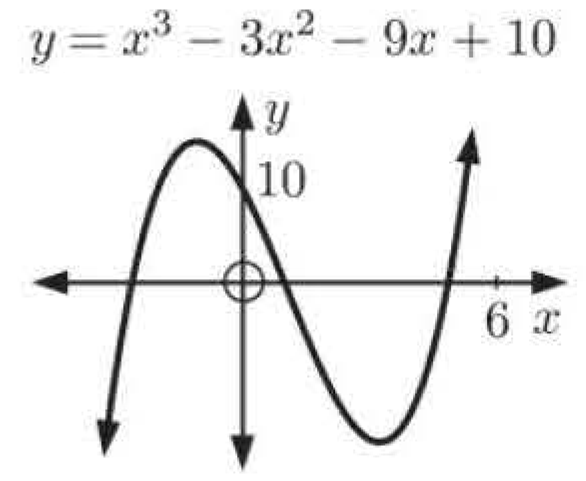
Domain is $\{x \mid x \neq 0\}$
Range is $\{y \mid y \leqslant -2$ or $y \geqslant 2\}$

i



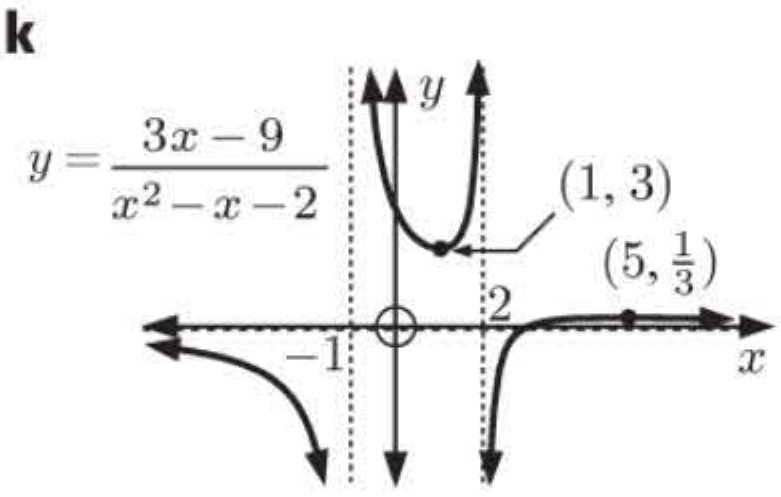
Domain is $\{x \mid x \neq 2\}$
Range is $\{y \mid y \neq 1\}$

j



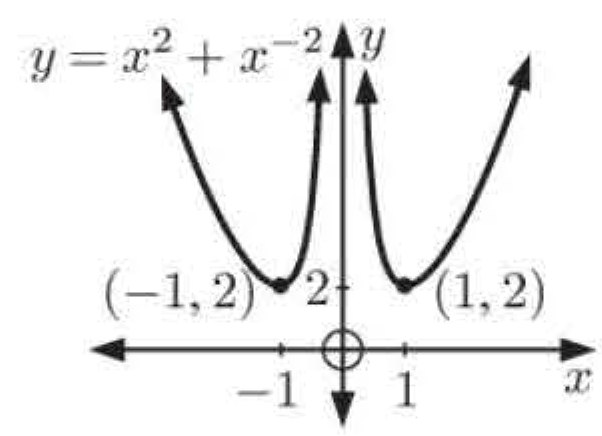
Domain is $\{x \mid x \in \mathbb{R}\}$
Range is $\{y \mid y \in \mathbb{R}\}$

k



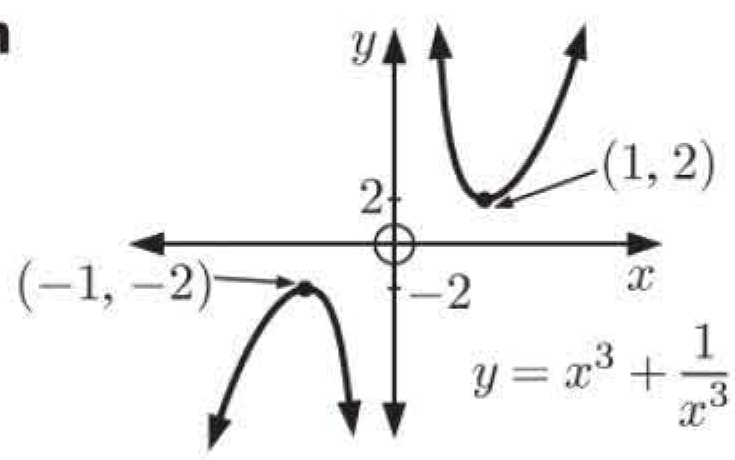
Domain is $\{x \mid x \neq -1$ and $x \neq 2\}$
Range is $\{y \mid y \leqslant \frac{1}{3}$ or $y \geqslant 3\}$

l



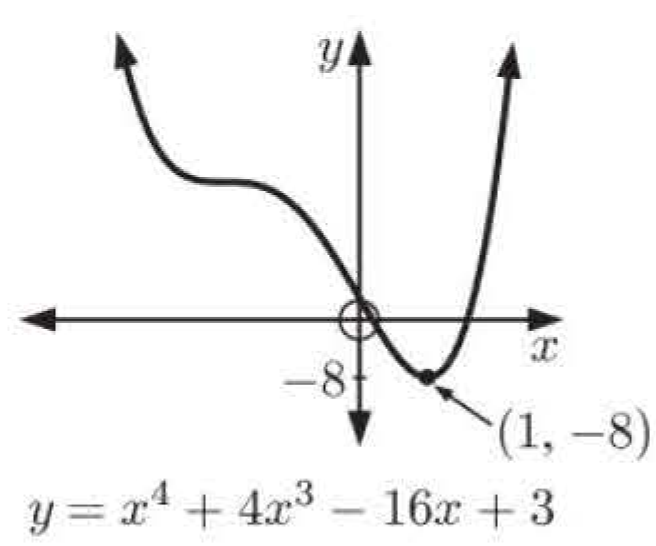
Domain is $\{x \mid x \neq 0\}$
Range is $\{y \mid y \geqslant 2\}$

m



Domain is $\{x \mid x \neq 0\}$
Range is $\{y \mid y \leqslant -2$ or $y \geqslant 2\}$

n



Domain is $\{x \mid x \in \mathbb{R}\}$
Range is $\{y \mid y \geqslant -8\}$

5 **a** The permissible values for x are 1, 2 and 3, so the domain is $\{1, 2, 3\}$.
The permissible values for y are 3, 5 and 7, so the range is $\{3, 5, 7\}$.

b The permissible values for x are -1, 0 and 2, so the domain is $\{-1, 0, 2\}$.
The permissible values for y are 3 and 5, so the range is $\{3, 5\}$.

c The permissible values for x are -3, -2, -1 and 3, so the domain is $\{-3, -2, -1, 3\}$.
The only permissible value for y is 1, so the range is $\{1\}$.

d The only solutions (x, y) to $x^2 + y^2 = 4$, where $x \in \mathbb{Z}$ and $y \geqslant 0$, are $(-2, 0)$, $(-1, \sqrt{3})$, $(0, 2)$, $(1, \sqrt{3})$ and $(2, 0)$.
$\therefore$ the domain is $\{-2, -1, 0, 1, 2\}$ and the range is $\{0, \sqrt{3}, 2\}$.

EXERCISE 2D

1 **a**
$$\begin{aligned}&(f \circ g)(x)\\&= f(g(x))\\&= f(1-x)\\&= 2(1-x)+3\\&= 2-2x+3\\&= 5-2x\end{aligned}$$

b
$$\begin{aligned}&(g \circ f)(x)\\&= g(f(x))\\&= g(2x+3)\\&= 1-(2x+3)\\&= 1-2x-3\\&= -2x-2\end{aligned}$$

c
$$\begin{aligned}&(f \circ g)(-3)\\&= 5-2(-3) \quad \{\text{from } \mathbf{a}\}\\&= 11\end{aligned}$$

2 **a**
$$\begin{aligned}&(g \circ g)(x)\\&= g(g(x))\\&= g(5x-7)\\&= 5(5x-7)-7\\&= 25x-35-7\\&= 25x-42\end{aligned}$$

b
$$\begin{aligned}(f \circ g)(1) &= f(g(1))\\ \text{Now } g(1) &= 5(1)-7\\ &= -2\\ \therefore\ (f \circ g)(1) &= f(-2)\\ &= \sqrt{6-(-2)}\\ &= \sqrt{8}\end{aligned}$$

c
$$\begin{aligned}(g \circ f)(6) &= g(f(6))\\ \text{Now } f(6) &= \sqrt{6-6}\\ &= 0\\ \therefore\ (g \circ f)(6) &= g(0)\\ &= 5(0)-7\\ &= -7\end{aligned}$$

3
$$\begin{aligned}(f \circ g)(x) &= f(g(x))\\ &= f(2-x)\\ &= (2-x)^2\end{aligned}$$

Domain is $\{x \mid x \in \mathbb{R}\}$
Range is $\{y \mid y \geqslant 0\}$

$$\begin{aligned}(g \circ f)(x) &= g(f(x))\\ &= g(x^2)\\ &= 2-x^2\end{aligned}$$

Domain is $\{x \mid x \in \mathbb{R}\}$
Range is $\{y \mid y \leqslant 2\}$

4 **a** **i** $(f \circ g)(x)$
$= f(g(x))$
$= f(3-x)$
$= (3-x)^2 + 1$
$= 9 - 6x + x^2 + 1$
$= x^2 - 6x + 10$

ii $(g \circ f)(x)$
$= g(f(x))$
$= g(x^2+1)$
$= 3 - (x^2+1)$
$= 3 - x^2 - 1$
$= 2 - x^2$

b $(g \circ f)(x) = f(x)$
$\therefore\ 2 - x^2 = f(x)$ {from **a ii**}
$\therefore\ 2 - x^2 = x^2 + 1$
$\therefore\ 2x^2 = 1$
$\therefore\ x^2 = \frac{1}{2}$
$\therefore\ x = \pm\frac{1}{\sqrt{2}}$

5 $(f \circ g)(0) = f(g(0)) = f(1) = 0$
$(f \circ g)(1) = f(g(1)) = f(2) = 1$
$(f \circ g)(2) = f(g(2)) = f(3) = 2$
$(f \circ g)(3) = f(g(3)) = f(0) = 3$
$\therefore\ f \circ g = \{(0, 0), (1, 1), (2, 2), (3, 3)\}$

6 **a** $(f \circ g)(2) = f(g(2)) = f(2) = 7$
$(f \circ g)(5) = f(g(5)) = f(0) = 2$
$(f \circ g)(7) = f(g(7)) = f(1) = 5$
$(f \circ g)(9) = f(g(9)) = f(3) = 9$
$\therefore\ f \circ g = \{(2, 7), (5, 2), (7, 5), (9, 9)\}$

b $(g \circ f)(0) = g(f(0)) = f(2) = 2$
$(g \circ f)(1) = g(f(1)) = f(5) = 0$
$(g \circ f)(2) = g(f(2)) = f(7) = 1$
$(g \circ f)(3) = g(f(3)) = f(9) = 3$
$\therefore\ g \circ f = \{(0, 2), (1, 0), (2, 1), (3, 3)\}$

7 **a**
$$\begin{aligned}&(f \circ g)(x)\\ &= f(g(x))\\ &= f\left(\frac{x+1}{x-1}\right)\\ &= \frac{\left(\frac{x+1}{x-1}\right)+3}{\left(\frac{x+1}{x-1}\right)+2} \times \frac{(x-1)}{(x-1)}\\ &= \frac{x+1+3(x-1)}{x+1+2(x-1)}\\ &= \frac{4x-2}{3x-1},\ x \neq 1\end{aligned}$$
$(f \circ g)(x)$ is undefined when $3x - 1 = 0$, which is when $x = \frac{1}{3}$
$\therefore$ domain is $\{x \mid x \neq \frac{1}{3} \text{ or } 1\}$

b
$$\begin{aligned}&(g \circ f)(x)\\ &= g(f(x))\\ &= g\left(\frac{x+3}{x+2}\right)\\ &= \frac{\left(\frac{x+3}{x+2}\right)+1}{\left(\frac{x+3}{x+2}\right)-1} \times \frac{(x+2)}{(x+2)}\\ &= \frac{x+3+(x+2)}{x+3-(x+2)}\\ &= \frac{2x+5}{1}\\ &= 2x+5,\ x \neq -2\end{aligned}$$
$\therefore$ domain is $\{x \mid x \neq -2\}$

c
$$\begin{aligned}&(g \circ g)(x)\\ &= g(g(x))\\ &= g\left(\frac{x+1}{x-1}\right)\\ &= \frac{\left(\frac{x+1}{x-1}\right)+1}{\left(\frac{x+1}{x-1}\right)-1} \times \frac{(x-1)}{(x-1)}\\ &= \frac{x+1+(x-1)}{x+1-(x-1)}\\ &= \frac{2x}{2}\\ &= x,\ x \neq 1\end{aligned}$$
$\therefore$ domain is $\{x \mid x \neq 1\}$

8 **a** $(f \circ g)(x) = f(g(x))$
$= f(x^2)$
$= \sqrt{1-x^2}$

b $(f \circ g)(x) = \sqrt{1-x^2}$ is defined when $1 - x^2 \geqslant 0$
$\therefore\ x^2 \leqslant 1$
$\therefore\ -1 \leqslant x \leqslant 1$
$(f \circ g)(x) = \sqrt{1-x^2}$ is always positive, and always $\leqslant 1$.
$\therefore$ the domain is $\{x \mid -1 \leqslant x \leqslant 1\}$
and the range is $\{y \mid 0 \leqslant y \leqslant 1\}$

9 **a** $ax + b = cx + d$ is true for all x {given}
When $x = 0$, $a(0) + b = c(0) + d$
$\therefore\ b = d$ .... $(*)$

When $x = 1$, $a(1) + b = c(1) + d$
$\therefore\ a + b = c + d$
But from $(*)$, $b = d$, so $a + d = c + d$
$\therefore\ a = c$

b
$$\begin{aligned}
&(f\circ g)(x)=x \quad \text{for all } x \quad \{\text{given}\}\\
\therefore\ & f(g(x))=x\\
\therefore\ & f(ax+b)=x\\
\therefore\ & 2(ax+b)+3=x\\
\therefore\ & 2ax+2b+3=x \quad \text{for all } x\\
\therefore\ & 2a=1 \quad \text{and} \quad 2b+3=0 \quad \{\text{using } \mathbf{a}\}\\
\therefore\ & a=\tfrac{1}{2} \quad \text{and} \quad 2b=-3
\end{aligned}$$
So, $a=\frac{1}{2}$ and $b=-\frac{3}{2}$ as required.

c
$$\begin{aligned}
\text{If}\ & (g\circ f)(x)=x\\
\text{then}\ & g(f(x))=x\\
\therefore\ & g(2x+3)=x\\
\therefore\ & a(2x+3)+b=x\\
\therefore\ & 2ax+3a+b=x\\
\therefore\ & 2a=1 \quad \text{and} \quad 3a+b=0 \quad \{\text{using } \mathbf{a}\}\\
\therefore\ & a=\tfrac{1}{2} \quad \text{and} \quad b=-3a
\end{aligned}$$
So, $a=\frac{1}{2}$ and $b=-\frac{3}{2}$

$\therefore$ the result in **b** is also true if $(g\circ f)(x)=x$ for all x.

EXERCISE 2E

1
$$\begin{aligned}
f(x)&=\frac{1}{x^2}+2\\
\therefore\ f(-x)&=\frac{1}{(-x)^2}+2\\
&=\frac{1}{x^2}+2\\
&=f(x)
\end{aligned}$$
$\therefore\ f(x)=\dfrac{1}{x^2}+2$ is an even function.

2
$$\begin{aligned}
f(x)&=x^3-3x\\
\therefore\ f(-x)&=(-x)^3-3(-x)\\
&=(-1)^3x^3+3x\\
&=-x^3+3x\\
&=-(x^3-3x)\\
&=-f(x)
\end{aligned}$$
$\therefore\ f(x)=x^3-3x$ is an odd function.

3 **a**
$$\begin{aligned}
f(x)&=5x\\
\therefore\ f(-x)&=5(-x)\\
&=-5x\\
&=-f(x)
\end{aligned}$$
$\therefore\ f(x)=5x$ is an odd function.

b
$$\begin{aligned}
f(x)&=-4x+3\\
\therefore\ f(-x)&=-4(-x)+3\\
&=4x+3
\end{aligned}$$
which is neither $f(x)$ or $-f(x)$.

$\therefore\ f(x)=-4x+3$ is neither even nor odd.

c
$$\begin{aligned}
f(x)&=\frac{3}{x^2-4}\\
\therefore\ f(-x)&=\frac{3}{(-x)^2-4}\\
&=\frac{3}{x^2-4}\\
&=f(x)
\end{aligned}$$
$\therefore\ f(x)=\dfrac{3}{x^2-4}$ is an even function.

d
$$\begin{aligned}
f(x)&=2x^3-\frac{5}{x}\\
\therefore\ f(-x)&=2(-x)^3-\frac{5}{-x}\\
&=2(-1)^3x^3+\frac{5}{x}\\
&=-2x^3+\frac{5}{x}\\
&=-\left(2x^3-\frac{5}{x}\right)\\
&=-f(x)
\end{aligned}$$
$\therefore\ f(x)=2x^3-\dfrac{5}{x}$ is an odd function.

e
$$\begin{aligned}
f(x)&=x^2+\frac{7}{x^2}-3\\
\therefore\ f(-x)&=(-x)^2+\frac{7}{(-x)^2}-3\\
&=x^2+\frac{7}{x^2}-3\\
&=f(x)
\end{aligned}$$
$\therefore\ f(x)=x^2+\dfrac{7}{x^2}-3$ is an even function.

f
$$\begin{aligned}
f(x)&=\sqrt{x}\\
\therefore\ f(-x)&=\sqrt{-x}
\end{aligned}$$
which is neither $f(x)$ nor $-f(x)=-\sqrt{x}$.

$\therefore\ f(x)=\sqrt{x}$ is neither even nor odd.

4 $f(x)=(2x+3)(x+a)$ is an even function

$$\therefore\ f(-x)=f(x)$$
$$\therefore\ (2(-x)+3)(-x+a)=(2x+3)(x+a)$$
$$\therefore\ (-2x+3)(-x+a)=(2x+3)(x+a)$$
$$\therefore\ \cancel{2x^2}-2ax-3x+\cancel{3a}=\cancel{2x^2}+2ax+3x+\cancel{3a}$$
$$\therefore\ -2a-3=2a+3 \qquad \{\text{equating coefficients of } x\}$$
$$\therefore\ 4a=-6$$
$$\therefore\ a=\frac{-6}{4}=-\tfrac{3}{2}$$

5 $g(x)=(x+1)\left(\dfrac{1}{x}+b\right)$ is an odd function

$$\therefore\ g(-x)=-g(x)$$
$$\therefore\ (-x+1)\left(\frac{1}{-x}+b\right)=-(x+1)\left(\frac{1}{x}+b\right)$$
$$\therefore\ \frac{x}{x}-bx-\frac{1}{x}+b=-\left(\frac{x}{x}+bx+\frac{1}{x}+b\right)$$
$$\therefore\ 1\cancel{-bx}\cancel{-\frac{1}{x}}+b=-1\cancel{-bx}\cancel{-\frac{1}{x}}-b$$
$$\therefore\ 1+b=-1-b$$
$$\therefore\ 2b=-2$$
$$\therefore\ b=-1$$

6 **a** $f(x)=ax^2+bx+c,\ a\neq 0$ is an even function

$$\therefore\ f(-x)=f(x)$$
$$\therefore\ a(-x)^2+b(-x)+c=ax^2+bx+c$$
$$\therefore\ \cancel{ax^2}-bx+\cancel{c}=\cancel{ax^2}+bx+\cancel{c}$$
$$\therefore\ -b=b \qquad \{\text{equating coefficients of } x\}$$
$$\therefore\ b=0$$

b If $g(x)=ax^3+bx^2+cx+d,\ a\neq 0$ is an odd function, then $g(-x)=-g(x)$

$$\therefore\ a(-x)^3+b(-x)^2+c(-x)+d=-(ax^3+bx^2+cx+d)$$
$$\therefore\ a(-1)^3x^3+bx^2-cx+d=-ax^3-bx^2-cx-d$$
$$\therefore\ -ax^3+bx^2-cx+d=-ax^3-bx^2-cx-d$$

Equating coefficients of x^3: $-a=-a$ $\therefore\ a\in\mathbb{R},\ a\neq 0$

Equating coefficients of x^2: $b=-b$ $\therefore\ b=0$

Equating coefficients of x: $-c=-c$ $\therefore\ c\in\mathbb{R}$

Equating constant terms: $d=-d$ $\therefore\ d=0$

So, the cubic function $g(x)=ax^3+bx^2+cx+d,\ a\neq 0$ is an odd function when $b=0,\ d=0$.

c If $h(x)=ax^4+bx^3+cx^2+dx+e,\ a\neq 0$ is an even function, then $h(-x)=h(x)$

$$\therefore\ a(-x)^4+b(-x)^3+c(-x)^2+d(-x)+e=ax^4+bx^3+cx^2+dx+e$$
$$\therefore\ ax^4-bx^3+cx^2-dx+e=ax^4+bx^3+cx^2+dx+e$$

Equating coefficients of x^4: $a=a$ $\therefore\ a\in\mathbb{R},\ a\neq 0$

Equating coefficients of x^3: $-b=b$ $\therefore\ b=0$

Equating coefficients of x^2: $c=c$ $\therefore\ c\in\mathbb{R}$

Equating coefficients of x: $-d=d$ $\therefore\ d=0$

Equating constant terms: $e=e$ $\therefore\ e\in\mathbb{R}$

So, the quartic function $h(x)=ax^4+bx^3+cx^2+dx+e,\ a\neq 0$ is an even function when $b=0,\ d=0$.

7 **a** Suppose $h(x)=f(x)+g(x)$ where $f(x)$ and $g(x)$ are even functions.

Now $h(-x) = f(-x)+g(-x)$
$= f(x)+g(x)$ $\{f(x)$ and $g(x)$ are even functions$\}$
$= h(x)$ for all x

$\therefore$ $h(x)$ is even.

Thus the sum of two even functions is an even function.

b Suppose $h(x)=f(x)-g(x)$ where $f(x)$ and $g(x)$ are odd functions.

Now $h(-x) = f(-x)-g(-x)$
$= -f(x) - -g(x)$ $\{f(x)$ and $g(x)$ are odd functions$\}$
$= -(f(x)-g(x))$
$= -h(x)$

$\therefore$ $h(x)$ is odd.

Thus the difference between two odd functions is an odd function.

c Suppose $h(x)=f(x)g(x)$ where $f(x)$ and $g(x)$ are odd functions.

Now $h(-x) = f(-x)g(-x)$
$= -f(x)\times -g(x)$ $\{f(x)$ and $g(x)$ are odd functions$\}$
$= f(x)g(x)$
$= h(x)$

$\therefore$ $h(x)$ is even.

Thus the product of two odd functions is an even function.

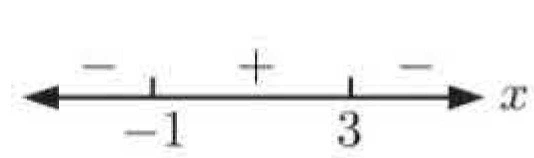
8 $f(x)$ is an even function and $g(x)$ is an odd function.

$(f\circ g)(-x) = f(g(-x))$
$= f(-g(x))$ $\{g(x)$ is an odd function$\}$
$= f(g(x))$ $\{f(x)$ is an even function$\}$
$= (f\circ g)(x)$

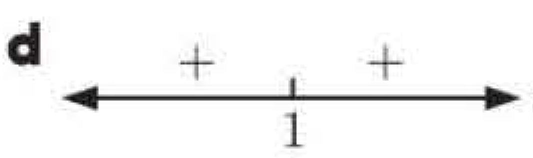
So $(f\circ g)(x)$ is an even function.

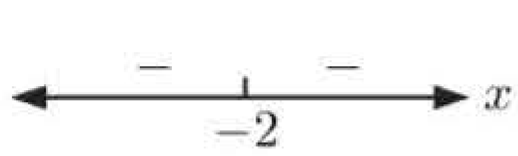
EXERCISE 2F

1 **a**


b


c


d


e


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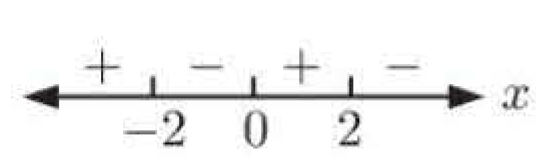
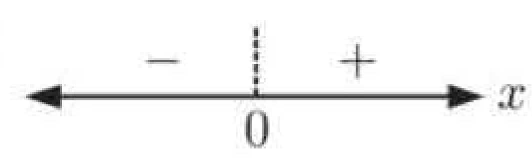
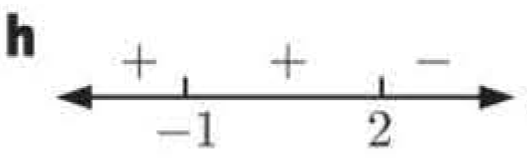
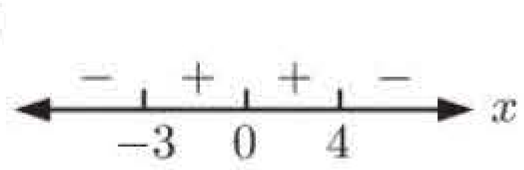


g


h


i


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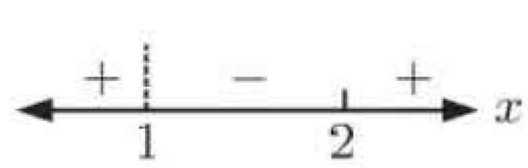

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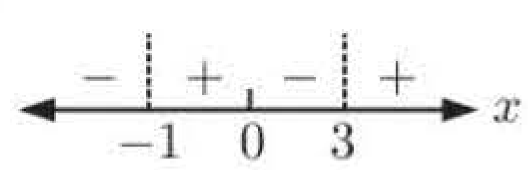
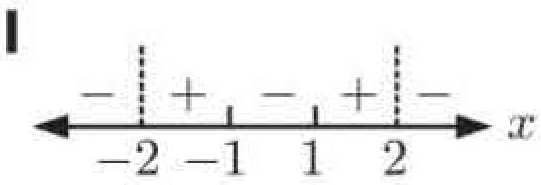
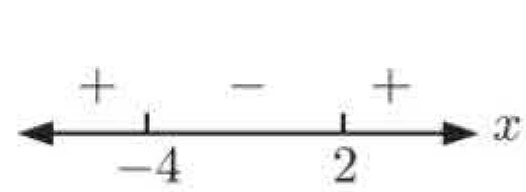
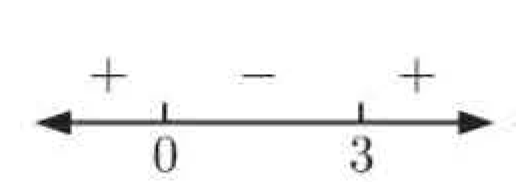


l


2 **a** $y=(x+4)(x-2)$ is zero when $x=-4$ or 2.
When $x=0$, $y=(4)(-2)=-8<0$.
The factors are distinct and linear, so the signs alternate.

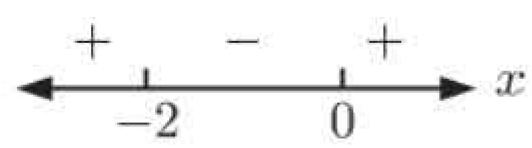
$\therefore$ sign diagram is:


b $y=x(x-3)$ is zero when $x=0$ or 3.
When $x=10$, $y=10(7)=70>0$.
The factors are distinct and linear, so the signs alternate.

$\therefore$ sign diagram is:


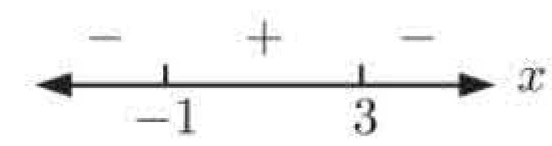
c $y = x(x+2)$ is zero when $x = -2$ or 0.
When $x = 10$, $y = 10(12) = 120 > 0$.
The factors are distinct and linear, so the signs alternate.

$\therefore$ sign diagram is:



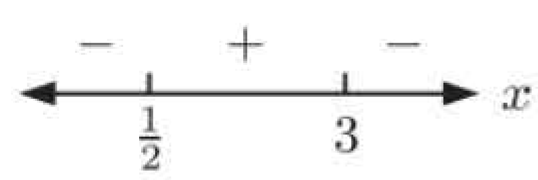
d $y = -(x+1)(x-3)$ is zero when $x = -1$ or 3.
When $x = 0$, $y = -(1)(-3) = 3 > 0$.
The factors are distinct and linear, so the signs alternate.

$\therefore$ sign diagram is:



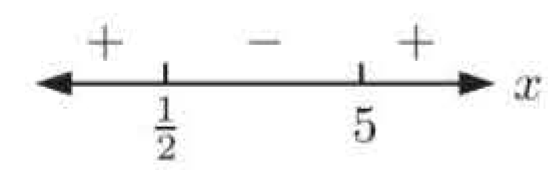
e $y = (2x-1)(3-x)$ is zero when $x = \frac{1}{2}$ or 3.
When $x = 0$, $y = (-1)(3) = -3 < 0$.
The factors are distinct and linear, so the signs alternate.

$\therefore$ sign diagram is:



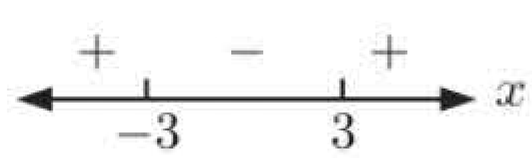
f $y = (5-x)(1-2x)$ is zero when $x = \frac{1}{2}$ or 5.
When $x = 0$, $y = (5)(1) = 5 > 0$.
The factors are distinct and linear, so the signs alternate.

$\therefore$ sign diagram is:



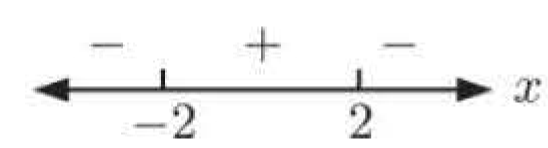
g $y = x^2 - 9 = (x+3)(x-3)$ is zero when $x = -3$ or 3.
When $x = 0$, $y = (3)(-3) = -9 < 0$.
The factors are distinct and linear, so the signs alternate.

$\therefore$ sign diagram is:



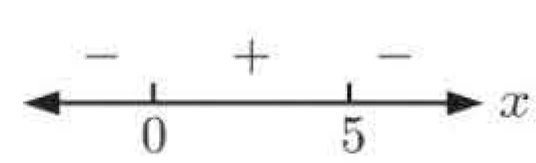
h $y = 4 - x^2 = (2+x)(2-x)$ is zero when $x = -2$ or 2.
When $x = 0$, $y = (2)(2) = 4 > 0$.
The factors are distinct and linear, so the signs alternate.

$\therefore$ sign diagram is:



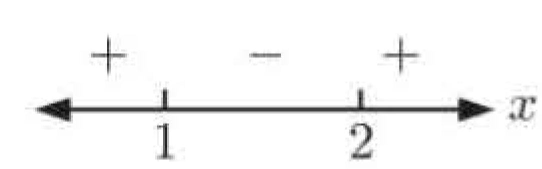
i $y = 5x - x^2 = x(5-x)$ is zero when $x = 0$ or 5.
When $x = 10$, $y = 10(-5) = -50 < 0$.
The factors are distinct and linear, so the signs alternate.

$\therefore$ sign diagram is:



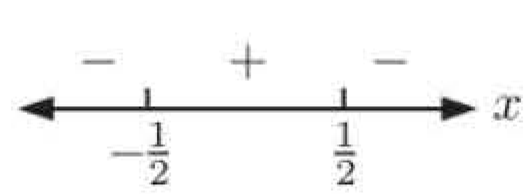
j $y = x^2 - 3x + 2 = (x-1)(x-2)$ is zero when $x = 1$ or 2.
When $x = 0$, $y = (-1)(-2) = 2 > 0$.
The factors are distinct and linear, so the signs alternate.

$\therefore$ sign diagram is:



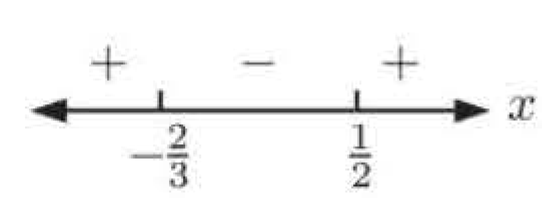
k $y = 2 - 8x^2 = 2(1+2x)(1-2x)$ is zero when $x = -\frac{1}{2}$ or $\frac{1}{2}$.
When $x = 0$, $y = 2(1)(1) = 2 > 0$.
The factors are distinct and linear, so the signs alternate.

$\therefore$ sign diagram is:



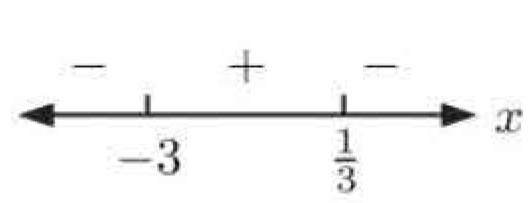
l $y = 6x^2 + x - 2 = (3x+2)(2x-1)$ is zero when $x = -\frac{2}{3}$ or $\frac{1}{2}$.
When $x = 0$, $y = (2)(-1) = -2 < 0$.
The factors are distinct and linear, so the signs alternate.

$\therefore$ sign diagram is:



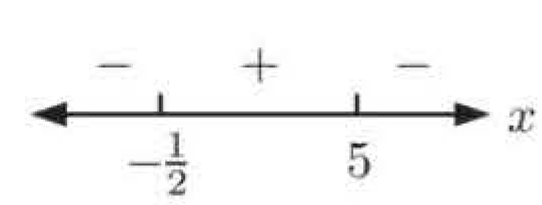
m $y = 6 - 16x - 6x^2 = 2(3+x)(1-3x)$ is zero when $x = -3$ or $\frac{1}{3}$.
When $x = 0$, $y = 2(3)(1) = 6 > 0$.
The factors are distinct and linear, so the signs alternate.

$\therefore$ sign diagram is:

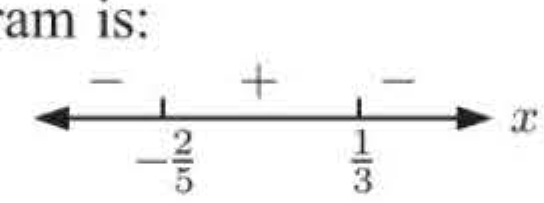


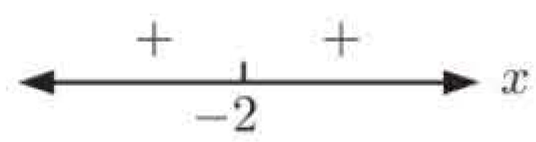
n $y = -2x^2 + 9x + 5 = (2x+1)(5-x)$ is zero when $x = -\frac{1}{2}$ or 5.
When $x = 0$, $y = (1)(5) = 5 > 0$.
The factors are distinct and linear, so the signs alternate.

$\therefore$ sign diagram is:

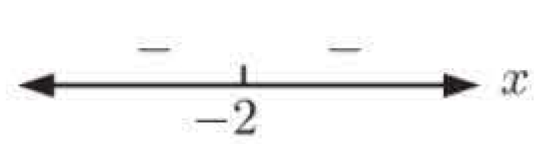


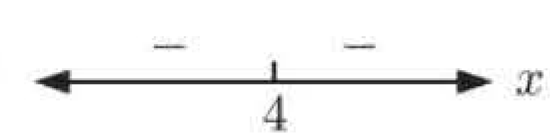
o $y = -15x^2 - x + 2 = (5x+2)(1-3x)$ is zero when $x = -\frac{2}{5}$ or $\frac{1}{3}$.
When $x = 0$, $y = (2)(1) = 2 > 0$.
The factors are distinct and linear, so the signs alternate. $\therefore$ sign diagram is:

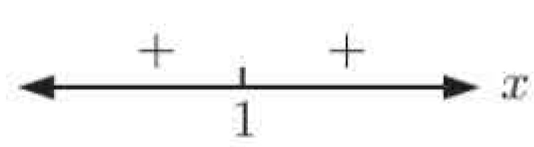


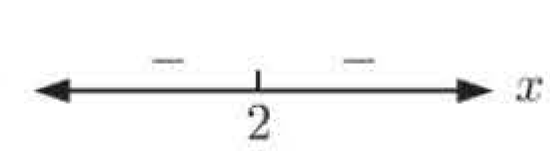
3 **a** $y=(x+2)^2$ is zero when $x=-2$.
When $x=0$, $y=2^2=4>0$.
The linear factor is squared, so the sign does not change.
$\therefore$ sign diagram is: 

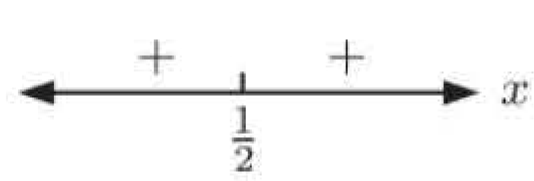
b $y=(x-3)^2$ is zero when $x=3$.
When $x=0$, $y=(-3)^2=9>0$.
The linear factor is squared, so the sign does not change.
$\therefore$ sign diagram is: 

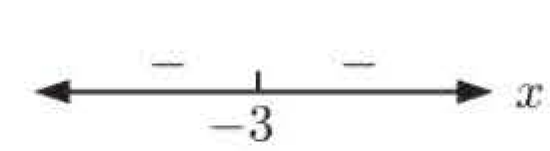
c $y=-(x+2)^2$ is zero when $x=-2$.
When $x=0$, $y=-(2^2)=-4<0$.
The linear factor is squared, so the sign does not change.
$\therefore$ sign diagram is: 

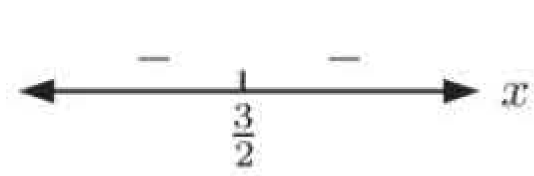
d $y=-(x-4)^2$ is zero when $x=4$.
When $x=0$, $y=-(-4)^2=-16<0$.
The linear factor is squared, so the sign does not change.
$\therefore$ sign diagram is: 

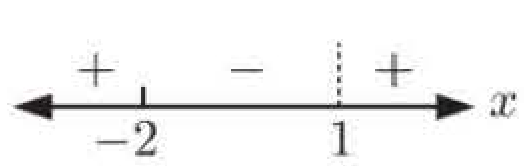
e $y=x^2-2x+1=(x-1)^2$ is zero when $x=1$.
When $x=0$, $y=(-1)^2=1>0$.
The linear factor is squared, so the sign does not change.
$\therefore$ sign diagram is: 

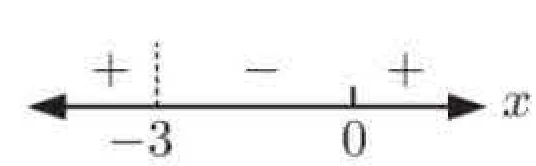
f $y=-x^2+4x-4=-(x-2)^2$ is zero when $x=2$.
When $x=0$, $y=-(-2)^2=-4<0$.
The linear factor is squared, so the sign does not change.
$\therefore$ sign diagram is: 

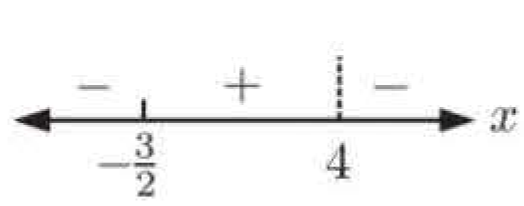
g $y=4x^2-4x+1=(2x-1)^2$ is zero when $x=\frac{1}{2}$.
When $x=0$, $y=(-1)^2=1>0$.
The linear factor is squared, so the sign does not change.
$\therefore$ sign diagram is: 

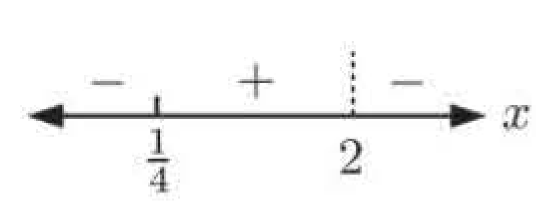
h $y=-x^2-6x-9=-(x+3)^2$ is zero when $x=-3$.
When $x=0$, $y=-(3^2)=-9<0$.
The linear factor is squared, so the sign does not change.
$\therefore$ sign diagram is: 

i $y=-4x^2+12x-9=-(2x-3)^2$ is zero when $x=\frac{3}{2}$.
When $x=0$, $y=-(-3)^2=-9<0$.
The linear factor is squared, so the sign does not change.
$\therefore$ sign diagram is: 

4 **a** $y=\dfrac{x+2}{x-1}$ is zero when $x=-2$ and undefined when $x=1$.
When $x=0$, $y=\frac{2}{-1}=-2<0$.
Since the factors are distinct and linear, the signs alternate.
$\therefore$ sign diagram is: 

b $y=\dfrac{x}{x+3}$ is zero when $x=0$ and undefined when $x=-3$.
When $x=10$, $y=\frac{10}{13}>0$.
Since the factors are distinct and linear, the signs alternate.
$\therefore$ sign diagram is: 

c $y=\dfrac{2x+3}{4-x}$ is zero when $x=-\frac{3}{2}$ and undefined when $x=4$.
When $x=0$, $y=\frac{3}{4}>0$.
Since the factors are distinct and linear, the signs alternate.
$\therefore$ sign diagram is: 

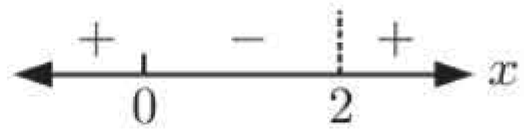
d $y=\dfrac{4x-1}{2-x}$ is zero when $x=\frac{1}{4}$ and undefined when $x=2$.
When $x=0$, $y=\frac{-1}{2}=-\frac{1}{2}<0$.
Since the factors are distinct and linear, the signs alternate.
$\therefore$ sign diagram is: 

e $y = \dfrac{3x}{x-2}$ is zero when $x = 0$ and undefined when $x = 2$.

When $x = 5$, $y = \frac{15}{3} = 5 > 0$.

Since the factors are distinct and linear, the signs alternate.

$\therefore$ sign diagram is:

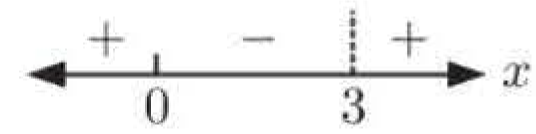


f $y = \dfrac{-8x}{3-x}$ is zero when $x = 0$ and undefined when $x = 3$.

When $x = 5$, $y = \frac{-40}{-2} = 20 > 0$.

Since the factors are distinct and linear, the signs alternate.

$\therefore$ sign diagram is:

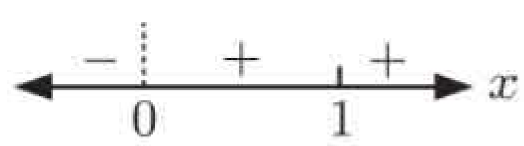


g $y = \dfrac{(x-1)^2}{x}$ is zero when $x = 1$ and undefined when $x = 0$.

When $x = 2$, $y = \frac{1^2}{2} = \frac{1}{2} > 0$.

Since the $(x-1)$ factor is squared, the sign does not change at $x = 1$.

$\therefore$ sign diagram is:

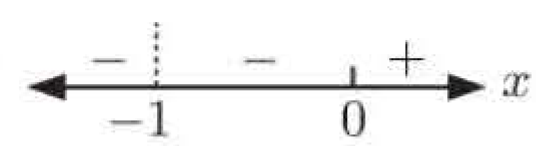


h $y = \dfrac{4x}{(x+1)^2}$ is zero when $x = 0$ and undefined when $x = -1$.

When $x = 1$, $y = \frac{4}{2^2} = 1 > 0$.

Since the $(x+1)$ factor is squared, the sign does not change at $x = -1$.

$\therefore$ sign diagram is:

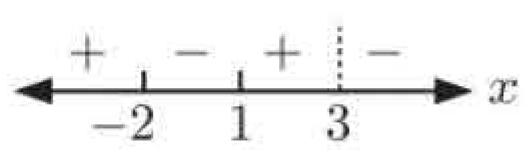


i $y = \dfrac{(x+2)(x-1)}{3-x}$ is zero when $x = -2$ or 1 and undefined when $x = 3$.

When $x = 0$, $y = \frac{(2)(-1)}{3} = -\frac{2}{3} < 0$.

Since the factors are distinct and linear, the signs alternate.

$\therefore$ sign diagram is:

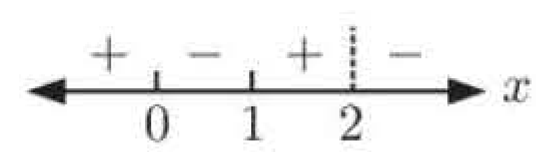


j $y = \dfrac{x(x-1)}{2-x}$ is zero when $x = 0$ or 1 and undefined when $x = 2$.

When $x = 3$, $y = \frac{3(2)}{-1} = -6 < 0$.

Since the factors are distinct and linear, the signs alternate.

$\therefore$ sign diagram is:

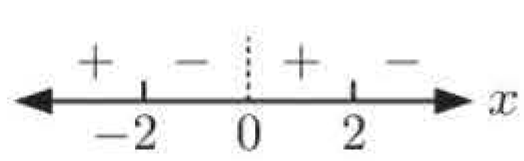


k $y = \dfrac{x^2-4}{-x} = \dfrac{(x-2)(x+2)}{-x}$ is zero when $x = \pm 2$ and undefined when $x = 0$.

When $x = 1$, $y = \dfrac{(-1)(3)}{-1} = 3 > 0$.

Since the factors are distinct and linear, the signs alternate.

$\therefore$ sign diagram is:

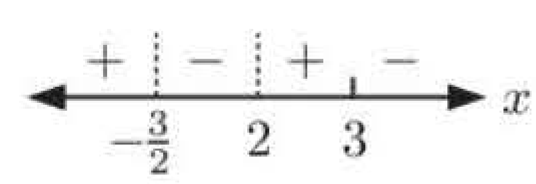


l $y = \dfrac{3-x}{2x^2-x-6} = \dfrac{3-x}{(2x+3)(x-2)}$ is zero when $x = 3$ and undefined when $x = -\frac{3}{2}$ or 2.

When $x = 0$, $y = \frac{3}{-6} = -\frac{1}{2} < 0$.

Since the factors are distinct and linear, the signs alternate.

$\therefore$ sign diagram is:

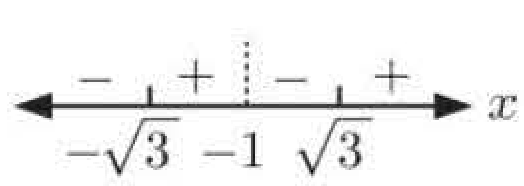


m $y = \dfrac{x^2-3}{x+1} = \dfrac{(x+\sqrt{3})(x-\sqrt{3})}{x+1}$ is zero when $x = \pm\sqrt{3}$ and undefined when $x = -1$.

When $x = 0$, $y = \frac{-3}{1} = -3 < 0$.

Since the factors are distinct and linear, the signs alternate.

$\therefore$ sign diagram is:

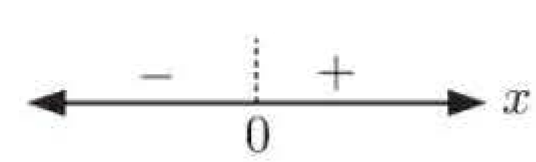


n $y = \dfrac{x^2+1}{x}$ is never zero (since $x^2 + 1 > 0$ for all real x), and undefined when $x = 0$.

When $x = 1$, $y = \frac{2}{1} = 2 > 0$.

Since the factor is distinct and linear, the sign alternates.

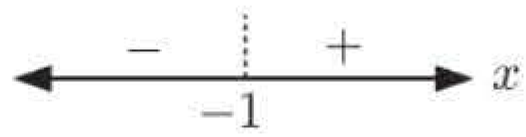
$\therefore$ sign diagram is:



o $y = \dfrac{x^2+2x+4}{x+1}$ is never zero (since $x^2+2x+4>0$ for all real x), and undefined when $x=-1$.

When $x=0$, $y=\frac{4}{1}=4>0$.

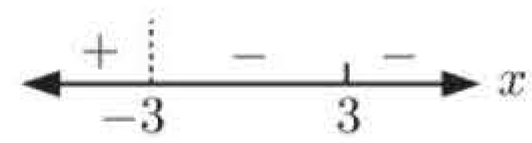
Since the factor is distinct and linear, the sign alternates.

$\therefore$ sign diagram is: 

p $y = \dfrac{-(x-3)^2(x^2+2)}{x+3}$ is zero when $x=3$ and undefined when $x=-3$.

When $x=0$, $y=\dfrac{-(-3)^2(2)}{3}=-6<0$.

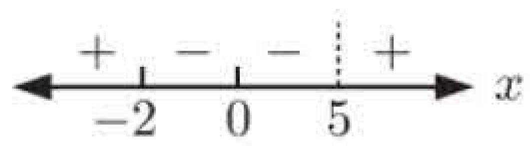
Since the $(x-3)$ factor is squared, the sign does not change at $x=3$.

$\therefore$ sign diagram is: 

q $y = \dfrac{-x^2(x+2)}{5-x}$ is zero when $x=0$ or -2 and undefined when $x=5$.

When $x=1$, $y=\dfrac{-1^2(3)}{4}=-\frac{3}{4}<0$.

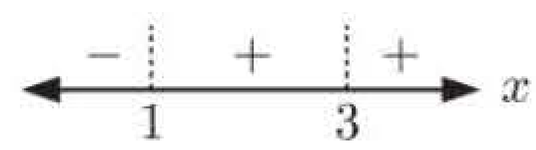
Since the x factor is squared, the sign does not change at $x=0$.

$\therefore$ sign diagram is: 

r $y = \dfrac{x^2+4}{(x-3)^2(x-1)}$ is never zero (since $x^2+4>0$ for all real x) and undefined when $x=3$ or 1.

When $x=0$, $y=\dfrac{4}{(-3)^2(-1)}=-\frac{4}{9}<0$.

Since the $(x-3)$ factor is squared, the sign does not change at $x=3$.

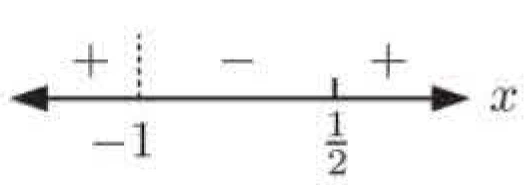
$\therefore$ sign diagram is: 

s
$$\begin{aligned} y &= \frac{x-5}{x+1}+3\frac{(x+1)}{(x+1)} \\ &= \frac{x-5+3x+3}{x+1} \\ &= \frac{4x-2}{x+1} \end{aligned}$$

which is zero when $x=\frac{1}{2}$ and undefined when $x=-1$.

When $x=0$, $y=\frac{-2}{1}=-2<0$.

Since the factors are distinct and linear, the signs alternate.

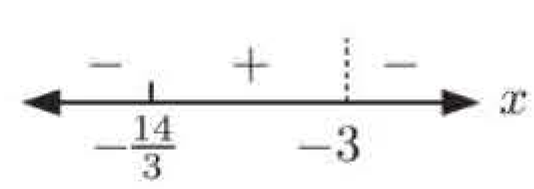
$\therefore$ sign diagram is: 

t
$$\begin{aligned} y &= \frac{x-2}{x+3}-4\frac{(x+3)}{(x+3)} \\ &= \frac{x-2-4x-12}{x+3} \\ &= \frac{-3x-14}{x+3} \end{aligned}$$

which is zero when $x=-\frac{14}{3}$ and undefined when $x=-3$.

When $x=0$, $y=\frac{-14}{3}<0$.

Since the factors are distinct and linear, the signs alternate.

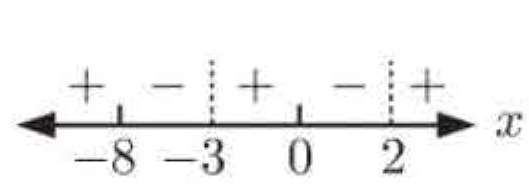
$\therefore$ sign diagram is: 

u
$$\begin{aligned} y &= \frac{3x+2}{x-2}-\frac{x-3}{x+3} \\ &= \frac{(3x+2)(x+3)-(x-3)(x-2)}{(x-2)(x+3)} \\ &= \frac{3x^2+11x+6-(x^2-5x+6)}{(x-2)(x+3)} \\ &= \frac{2x^2+16x}{(x-2)(x+3)} \\ &= \frac{2x(x+8)}{(x-2)(x+3)} \end{aligned}$$

which is zero when $x=0$ or -8 and undefined when $x=2$ or -3.

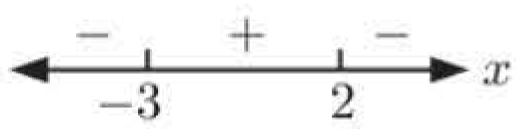
When $x=1$, $y=\dfrac{2(1)(9)}{(-1)(4)}=-\frac{18}{4}<0$.

Since the factors are distinct and linear, the signs alternate.

$\therefore$ sign diagram is: 

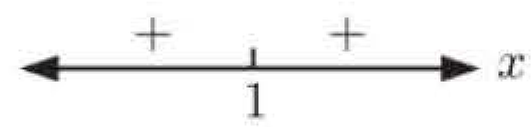
EXERCISE 2G

1 **a** Sign diagram of $(2-x)(x+3)$ is



$\therefore\ (2-x)(x+3) \geqslant 0$ when $x \in [-3, 2]$

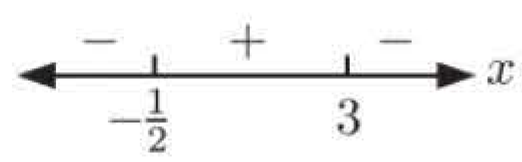
b Sign diagram of $(x-1)^2$ is



$\therefore\ (x-1)^2 < 0$ is never true

$\therefore$ no solutions

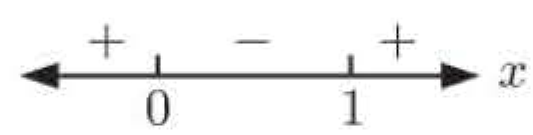
c Sign diagram of $(2x+1)(3-x)$ is



$\therefore\ (2x+1)(3-x) > 0$ when $x \in \,]-\frac{1}{2}, 3[$

d
$$x^2 \geqslant x$$
$$\therefore\ x^2 - x \geqslant 0$$
$$\therefore\ x(x-1) \geqslant 0$$

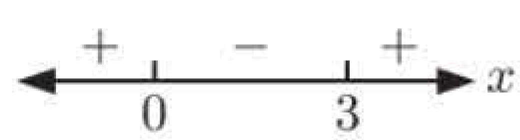
Sign diagram of LHS is



$\therefore\ x \in \,]-\infty, 0]$ or $[1, \infty[$

e
$$x^2 \geqslant 3x$$
$$\therefore\ x^2 - 3x \geqslant 0$$
$$\therefore\ x(x-3) \geqslant 0$$

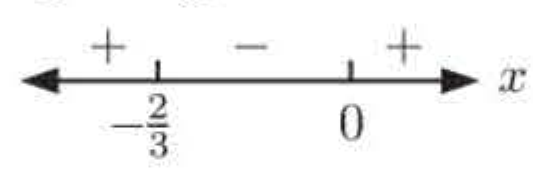
Sign diagram of LHS is



$\therefore$ LHS $\geqslant 0$ when $x \in \,]-\infty, 0]$
or $[3, \infty[$

f
$$3x^2 + 2x < 0$$
$$\therefore\ x(3x+2) < 0$$

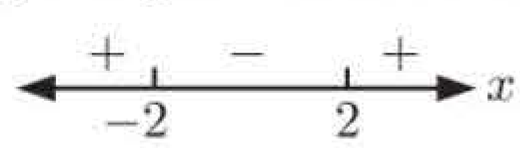
Sign diagram of LHS is



$\therefore$ LHS < 0 when $x \in \,]-\frac{2}{3}, 0[$

g
$$x^2 < 4$$
$$\therefore\ x^2 - 4 < 0$$
$$\therefore\ (x+2)(x-2) < 0$$

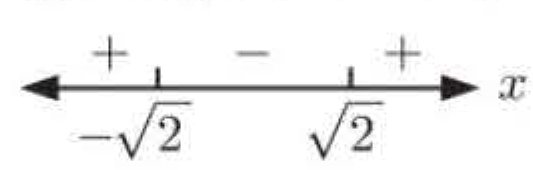
Sign diagram of LHS is



$\therefore$ LHS < 0 when $x \in \,]-2, 2[$

h
$$2x^2 \geqslant 4$$
$$\therefore\ 2x^2 - 4 \geqslant 0$$
$$\therefore\ 2(x+\sqrt{2})(x-\sqrt{2}) \geqslant 0$$

Sign diagram of LHS is

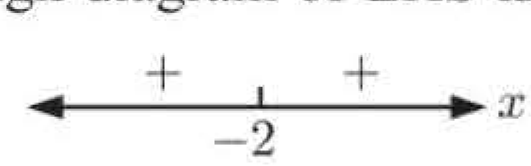


$\therefore$ LHS $\geqslant 0$ when $x \in \,]-\infty, -\sqrt{2}]$
or $[\sqrt{2}, \infty[$

i $x^2 + 4x + 4 > 0$

$\therefore\ (x+2)^2 > 0$

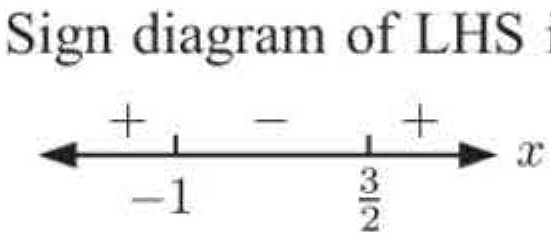
Sign diagram of LHS is



$\therefore$ LHS > 0 when $x \neq -2$

j
$$2x^2 \geqslant x + 3$$
$$\therefore\ 2x^2 - x - 3 \geqslant 0$$
$$\therefore\ (2x-3)(x+1) \geqslant 0$$

Sign diagram of LHS is

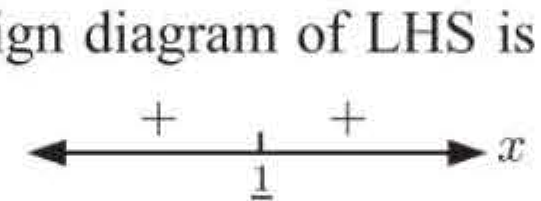


$\therefore$ LHS $\geqslant 0$ when $x \in \,]-\infty, -1]$
or $[\frac{3}{2}, \infty[$

k $4x^2 - 4x + 1 < 0$

$\therefore\ (2x-1)^2 < 0$

Sign diagram of LHS is

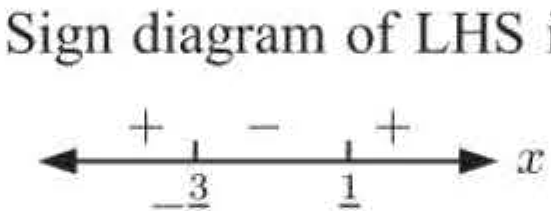


$\therefore$ LHS < 0 is never true

$\therefore$ no solutions

l
$$6x^2 + 7x < 3$$
$$\therefore\ 6x^2 + 7x - 3 < 0$$
$$\therefore\ (3x-1)(2x+3) < 0$$

Sign diagram of LHS is

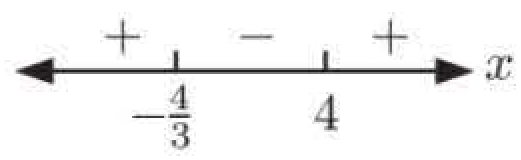


$\therefore$ LHS < 0 when $x \in \,]-\frac{3}{2}, \frac{1}{3}[$

m
$$3x^2 > 8(x+2)$$
$\therefore\ 3x^2 - 8x - 16 > 0$

$\therefore\ (3x+4)(x-4) > 0$

Sign diagram of LHS is

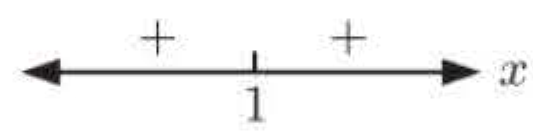


$\therefore$ LHS > 0 when $x \in\ ]-\infty, -\frac{4}{3}[$
or $]4, \infty[$

n
$$2x^2 - 4x + 2 > 0$$
$\therefore\ 2(x^2 - 2x + 1) > 0$

$\therefore\ 2(x-1)^2 > 0$

Sign diagram of LHS is

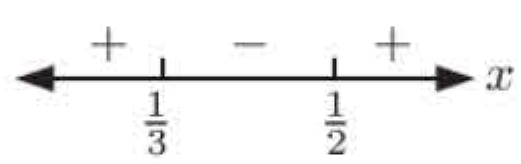


$\therefore$ LHS > 0 when $x \neq 1$

o
$$6x^2 + 1 \leqslant 5x$$
$\therefore\ 6x^2 - 5x + 1 \leqslant 0$

$\therefore\ (3x-1)(2x-1) \leqslant 0$

Sign diagram of LHS is



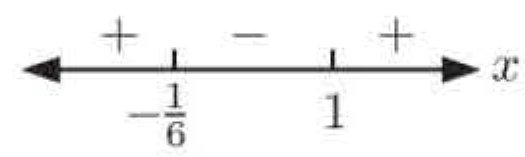
$\therefore$ LHS $\leqslant 0$ when $x \in [\frac{1}{3}, \frac{1}{2}]$

p
$$1 + 5x < 6x^2$$
$\therefore\ -6x^2 + 5x + 1 < 0$

$\therefore\ 6x^2 - 5x - 1 > 0$

$\therefore\ (6x+1)(x-1) > 0$

Sign diagram of LHS is

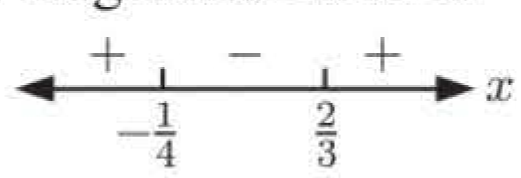


$\therefore$ LHS > 0 when $x \in\ ]-\infty, -\frac{1}{6}[$
or $]1, \infty[$

q
$$12x^2 \geqslant 5x + 2$$
$\therefore\ 12x^2 - 5x - 2 \geqslant 0$

$\therefore\ (3x-2)(4x+1) \geqslant 0$

Sign diagram of LHS is

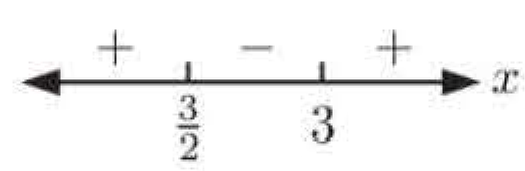


$\therefore$ LHS $\geqslant 0$ when $x \in\ ]-\infty, -\frac{1}{4}]$
or $[\frac{2}{3}, \infty[$

r
$$2x^2 + 9 > 9x$$
$\therefore\ 2x^2 - 9x + 9 > 0$

$\therefore\ (2x-3)(x-3) > 0$

Sign diagram of LHS is

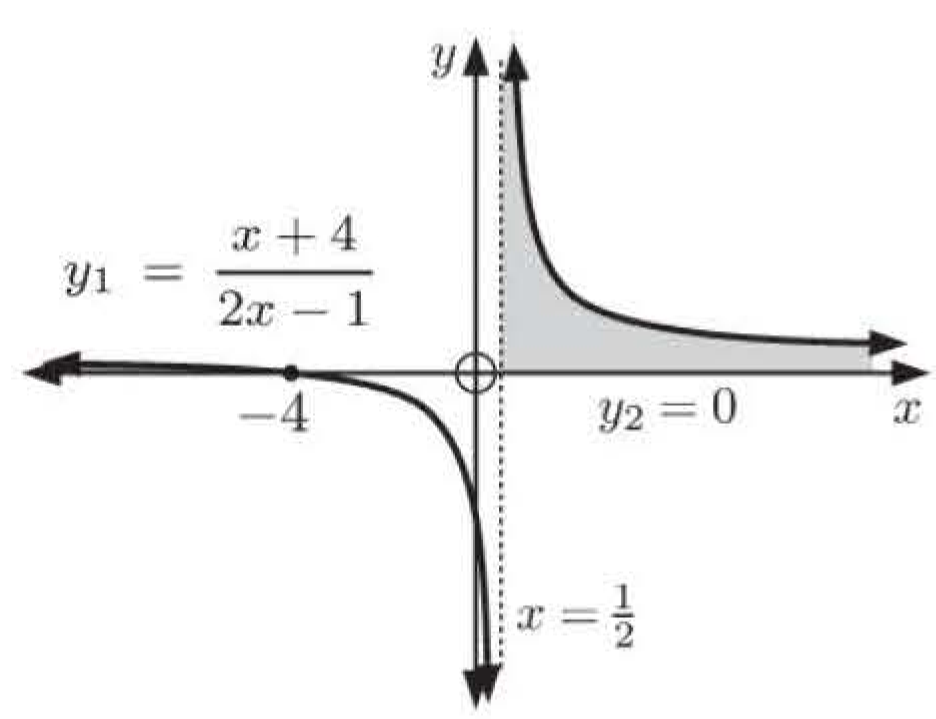


$\therefore$ LHS > 0 when $x \in\ ]-\infty, \frac{3}{2}[$
or $]3, \infty[$

2 **a** The graphs of $y_1 = \dfrac{x+4}{2x-1}$ and $y_2 = 0$ intersect at $x = -4$.

$y_1 = \dfrac{x+4}{2x-1}$ has vertical asymptote $x = \frac{1}{2}$.

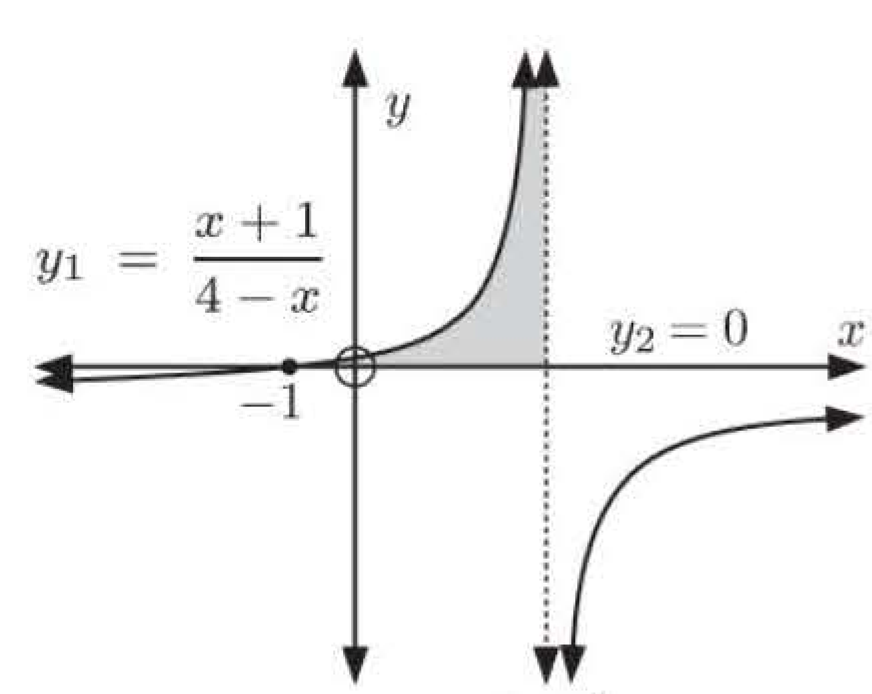
So, $\dfrac{x+4}{2x-1} > 0$ when $x < -4$ and $x > \frac{1}{2}$.



b The graphs of $y_1 = \dfrac{x+1}{4-x}$ and $y_2 = 0$ intersect at $x = -1$.

$y_1 = \dfrac{x+1}{4-x}$ has vertical asymptote $x = 4$.

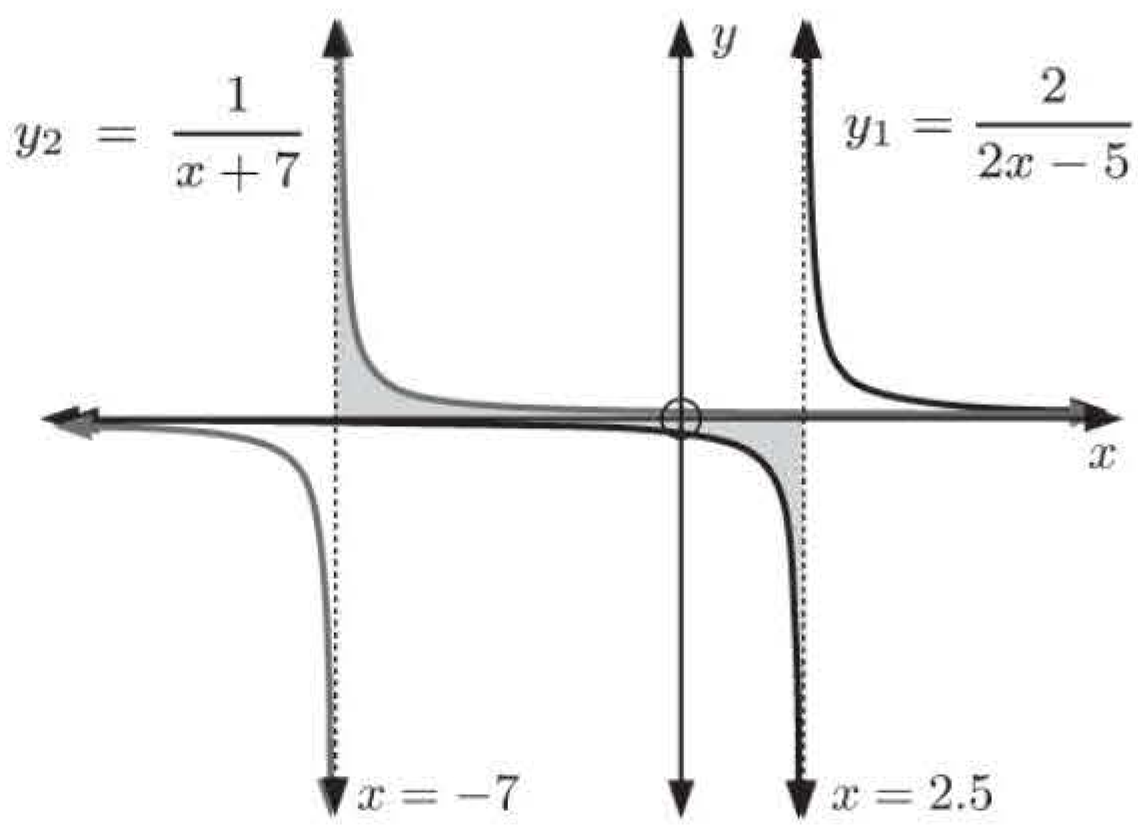
So, $\dfrac{x+1}{4-x} \geqslant 0$ when $-1 \leqslant x < 4$.



c The graphs of $y_1 = \dfrac{2}{2x-5}$ and $y_2 = \dfrac{1}{x+7}$ do not intersect.

$y_1 = \dfrac{2}{2x-5}$ has vertical asymptote $x = 2.5$ and $y_2 = \dfrac{1}{x+7}$ has vertical asymptote $x = -7$.

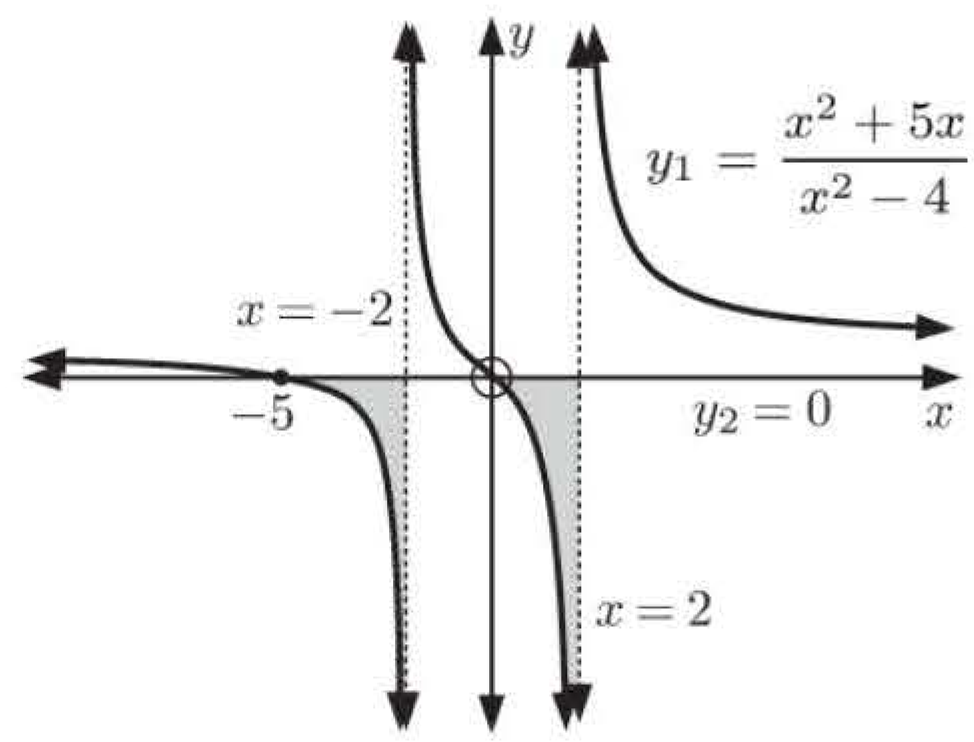
So, $\dfrac{2}{2x-5} < \dfrac{1}{x+7}$ when $-7 < x < 2.5$.



d The graphs of $y_1 = \dfrac{x^2+5x}{x^2-4}$ and $y_2 = 0$ intersect at $x = -5$ and $x = 0$.

$y_1 = \dfrac{x^2+5x}{x^2-4}$ has vertical asymptotes $x = -2$ and $x = 2$.

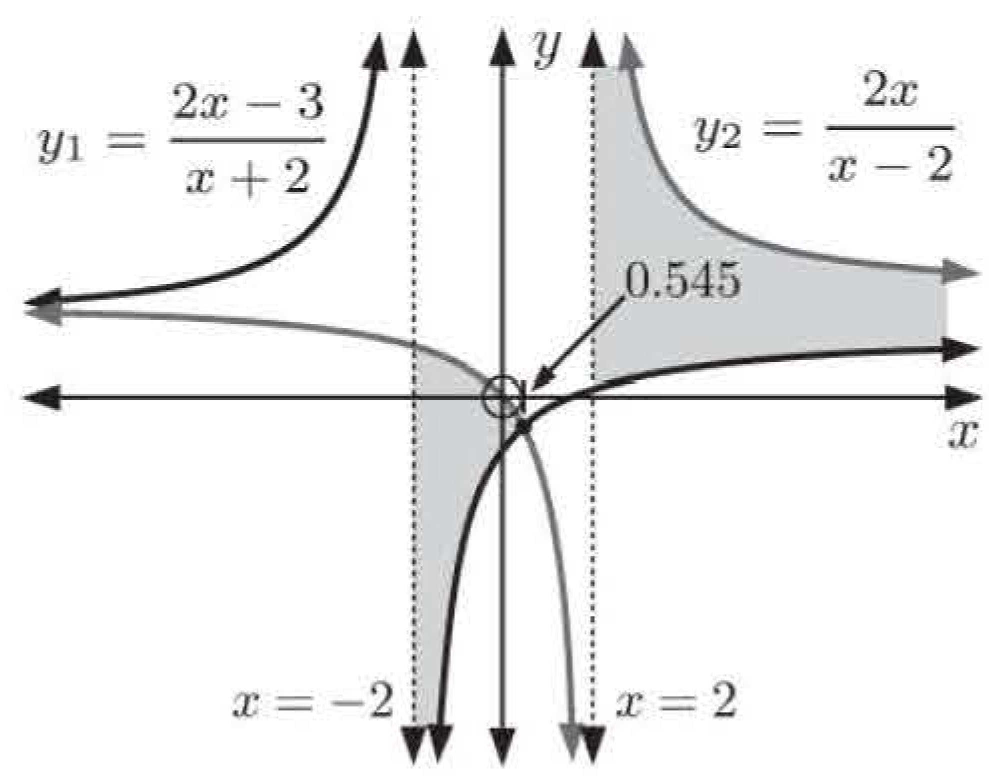
So, $\dfrac{x^2+5x}{x^2-4} \leqslant 0$ when $-5 \leqslant x < -2$ and $0 \leqslant x < 2$.



e The graphs of $y_1 = \dfrac{2x-3}{x+2}$ and $y_2 = \dfrac{2x}{x-2}$ intersect at $x \approx 0.545$.

$y_1 = \dfrac{2x-3}{x+2}$ has vertical asymptote $x = -2$ and $y_2 = \dfrac{2x}{x-2}$ has vertical asymptote $x = 2$.

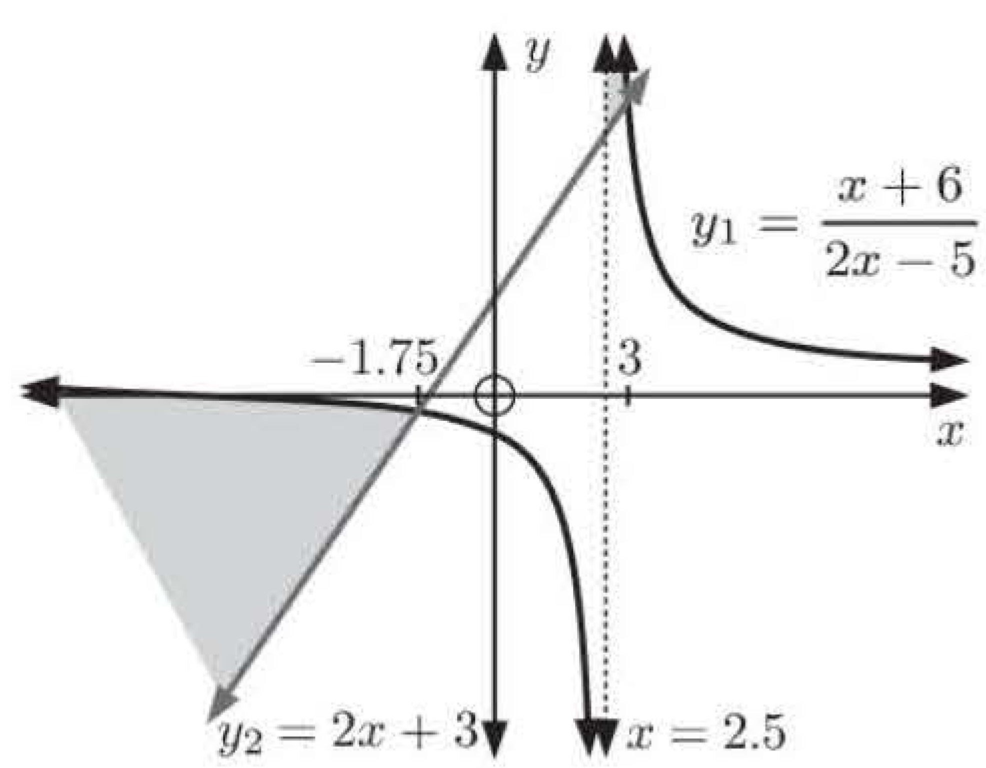
So, $\dfrac{2x-3}{x+2} < \dfrac{2x}{x-2}$ when $-2 < x < 0.545$ and $x > 2$.



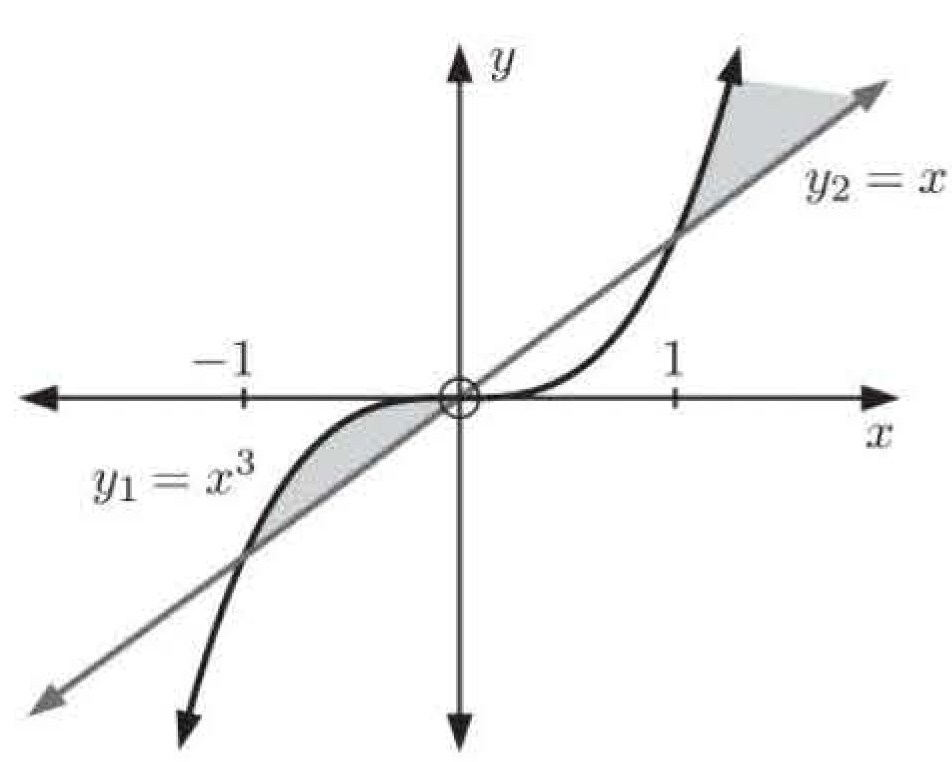
f The graphs of $y_1 = \dfrac{x+6}{2x-5}$ and $y_2 = 2x+3$ intersect at $x = -1.75$ and $x = 3$.

$y_1 = \dfrac{x+6}{2x-5}$ has vertical asymptote $x = 2.5$.

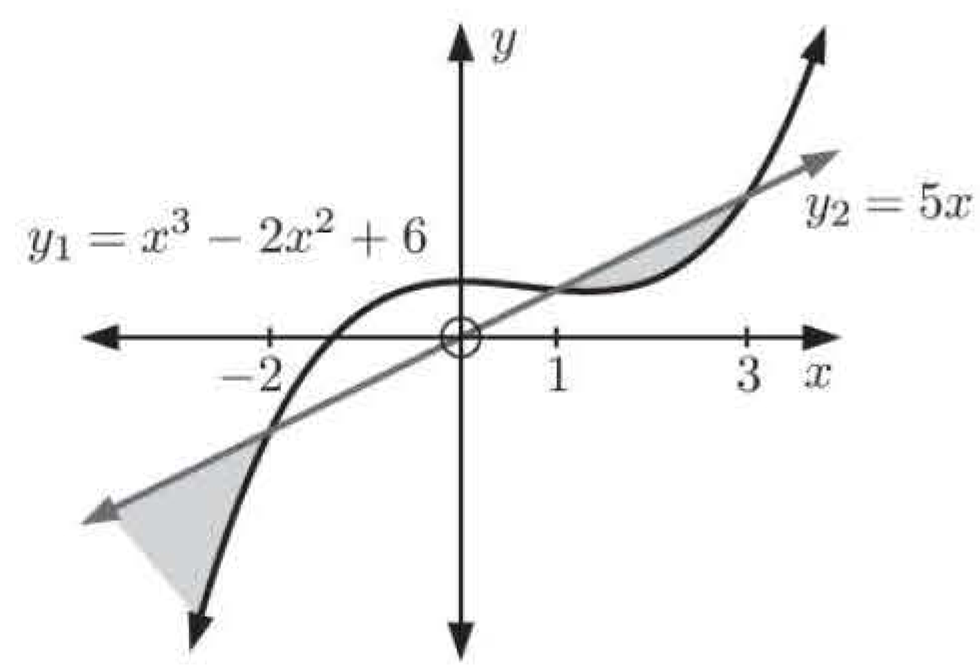
So, $\dfrac{x+6}{2x-5} > 2x+3$ when $x < -1.75$ and $2.5 < x < 3$.



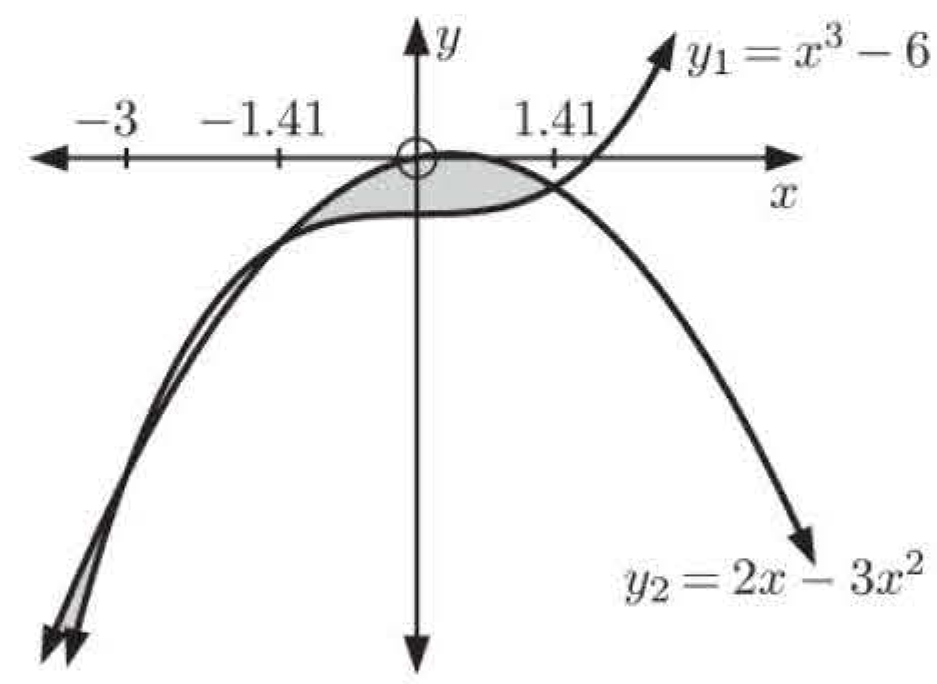
3 **a** The graphs of $y_1 = x^3$ and $y_2 = x$ intersect at $x = -1$, $x = 0$, and $x = 1$.
So, $x^3 \geqslant x$ when $-1 \leqslant x \leqslant 0$ and $x \geqslant 1$.



b The graphs of $y_1 = x^3 - 2x^2 + 6$ and $y_2 = 5x$ intersect at $x = -2$, $x = 1$, and $x = 3$.
So, $x^3 - 2x^2 + 6 < 5x$ when $x < -2$ and $1 < x < 3$.



c The graphs of $y_1 = x^3 - 6$ and $y_2 = 2x - 3x^2$ intersect at $x = -3$, $x \approx -1.41$, and $x \approx 1.41$.
So, $x^3 - 6 \leqslant 2x - 3x^2$ when $x \leqslant -3$ and $-1.41 \leqslant x \leqslant 1.41$.



EXERCISE 2H.1

1 **a** $|a| = |-2|$
$= 2$

b $|b| = |3|$
$= 3$

c $|a|\,|b| = |-2|\,|3|$
$= 2 \times 3$
$= 6$

d $|ab| = |-2 \times 3|$
$= |-6|$
$= 6$

e $|a - b| = |-2 - 3|$
$= |-5|$
$= 5$

f $|a| - |b| = |-2| - |3|$
$= 2 - 3$
$= -1$

g $|a + b| = |-2 + 3|$
$= |1|$
$= 1$

h $|a| + |b| = |-2| + |3|$
$= 2 + 3$
$= 5$

i $|a|^2 = |-2|^2$
$= 2^2$
$= 4$

j $a^2 = (-2)^2$
$= 4$

k $\left|\dfrac{c}{a}\right| = \left|\dfrac{-4}{-2}\right|$
$= |2|$
$= 2$

l $\dfrac{|c|}{|a|} = \dfrac{|-4|}{|-2|}$
$= \frac{4}{2}$
$= 2$

2 **a** $|5 - x| = |5 - (-3)|$
$= |8|$
$= 8$

b $|5| - |x| = |5| - |-3|$
$= 5 - 3$
$= 2$

c $\left|\dfrac{2x+1}{1-x}\right| = \left|\dfrac{2(-3)+1}{1-(-3)}\right|$
$= \left|\frac{-5}{4}\right|$
$= \frac{5}{4}$

d $\left|3 - 2x - x^2\right| = \left|3 - 2(-3) - (-3)^2\right|$
$= |0|$
$= 0$

3 **a**

a	b	$\|a\| + \|b\|$	$\|a\| - \|b\|$	$\|a + b\|$	$\|a - b\|$	$\|b - a\|$
6	2	8	4	8	4	4
6	-2	8	4	4	8	8
-6	2	8	4	4	8	8
-6	-2	8	4	8	4	4

b **i** False, $|a+b| \neq |a| + |b|$.
For example, $|6+(-2)| = 4$ but $|6| + |-2| = 8$.

ii False, $|a-b| \neq |a| - |b|$.
For example, $|6-(-2)| = 8$ but $|6| - |-2| = 4$.

c $$\begin{aligned} |a-b| &= \sqrt{(a-b)^2} \\ &= \sqrt{(b-a)^2} \\ &= |b-a| \end{aligned}$$

4 **a**

a	b	$\lvert ab\rvert$	$\lvert a\rvert \lvert b\rvert$	$\left\lvert \frac{a}{b} \right\rvert$	$\frac{\lvert a\rvert}{\lvert b\rvert}$
6	2	12	12	3	3
6	-2	12	12	3	3
-6	2	12	12	3	3
-6	-2	12	12	3	3

b **i** $$\begin{aligned} |ab| &= \sqrt{(ab)^2} \\ &= \sqrt{a^2b^2} \\ &= \sqrt{a^2}\sqrt{b^2} \\ &= |a|\,|b| \end{aligned}$$

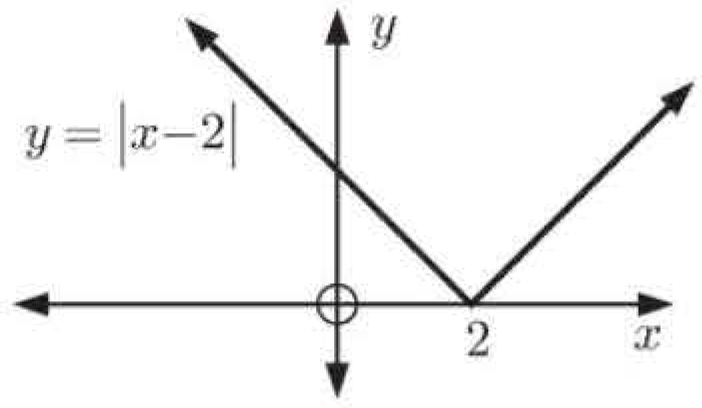
ii $$\begin{aligned} \left|\frac{a}{b}\right| &= \sqrt{\left(\frac{a}{b}\right)^2} \\ &= \sqrt{\frac{a^2}{b^2}} \\ &= \frac{\sqrt{a^2}}{\sqrt{b^2}} \\ &= \frac{|a|}{|b|} \end{aligned}$$

5 **a** $y = |x-2|$

When $x \geqslant 2$, $x - 2 \geqslant 0$,
so $y = x - 2$

When $x < 2$, $x - 2 < 0$,
so $y = -(x-2)$
$= 2 - x$

$$\therefore \quad y = \begin{cases} x-2 & \text{when } x \geqslant 2 \\ 2-x & \text{when } x < 2 \end{cases}$$

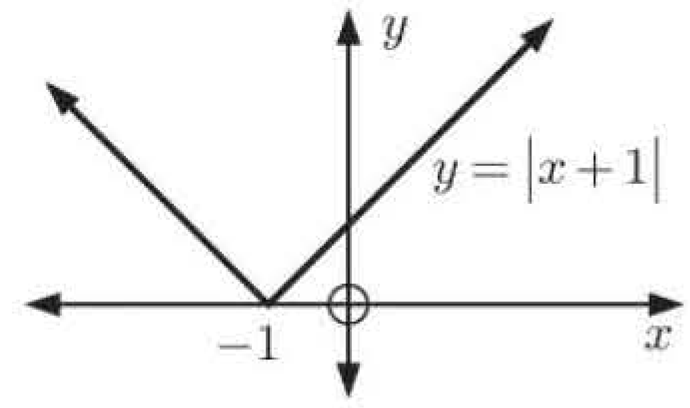


b $y = |x+1|$

When $x \geqslant -1$, $x + 1 \geqslant 0$,
so $y = x + 1$

When $x < -1$, $x + 1 < 0$,
so $y = -(x+1)$
$= -x - 1$

$$\therefore \quad y = \begin{cases} x+1 & \text{when } x \geqslant -1 \\ -x-1 & \text{when } x < -1 \end{cases}$$

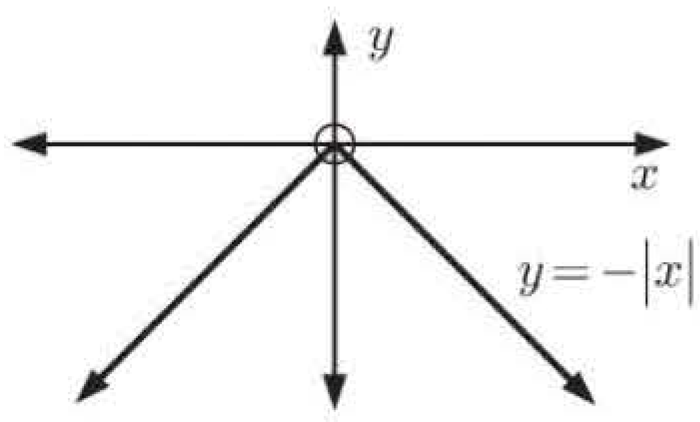


c $y = -|x|$

When $x \geqslant 0$, $y = -x$

When $x < 0$, $y = -(-x)$
$= x$

$$\therefore \quad y = \begin{cases} -x & \text{when } x \geqslant 0 \\ x & \text{when } x < 0 \end{cases}$$

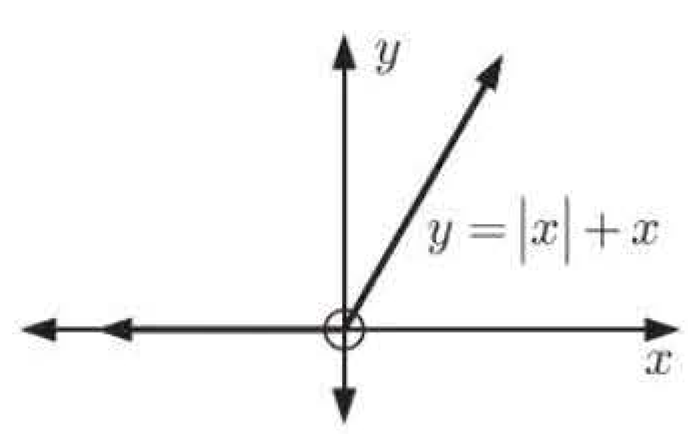


d $y = |x| + x$

When $x \geqslant 0$, $y = x + x$
$= 2x$

When $x < 0$, $y = -x + x$
$= 0$

$$\therefore \quad y = \begin{cases} 2x & \text{when } x \geqslant 0 \\ 0 & \text{when } x < 0 \end{cases}$$



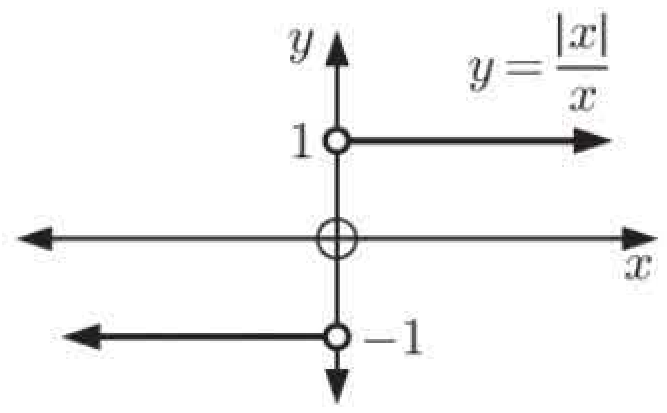
e $$y = \frac{|x|}{x}$$

When $x > 0$, $y = \dfrac{x}{x} = 1$

When $x < 0$, $y = \dfrac{-x}{x} = -1$

When $x = 0$, y is undefined.

$$\therefore \quad y = \begin{cases} 1 & \text{when } x > 0 \\ \text{undefined} & \text{when } x = 0 \\ -1 & \text{when } x < 0 \end{cases}$$

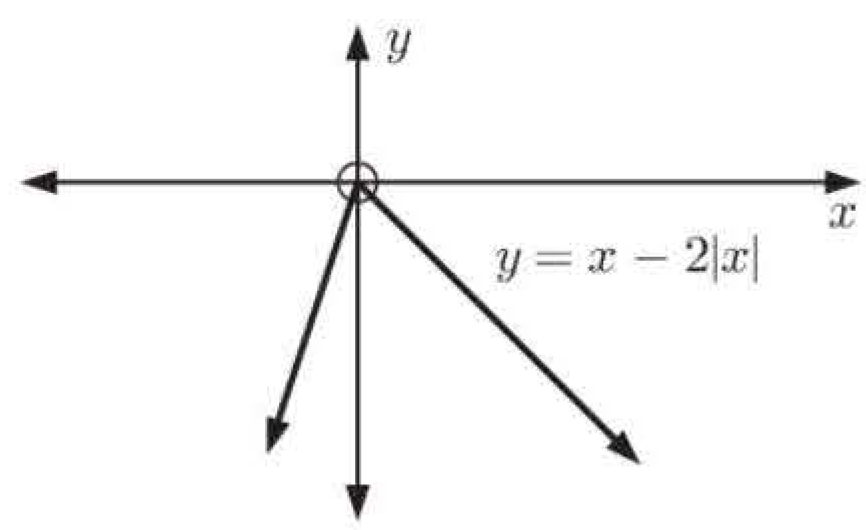


f $$y = x - 2|x|$$

When $x \geqslant 0$, $y = x - 2x$
$\qquad = -x$

When $x < 0$, $y = x - 2(-x)$
$\qquad = 3x$

$$\therefore \quad y = \begin{cases} -x & \text{when } x \geqslant 0 \\ 3x & \text{when } x < 0 \end{cases}$$



g $$y = |x| + |x-2|$$

When $x \geqslant 2$, $x - 2 \geqslant 0$ and $x \geqslant 0$

$\therefore \quad y = x + (x-2)$
$\qquad = 2x - 2$

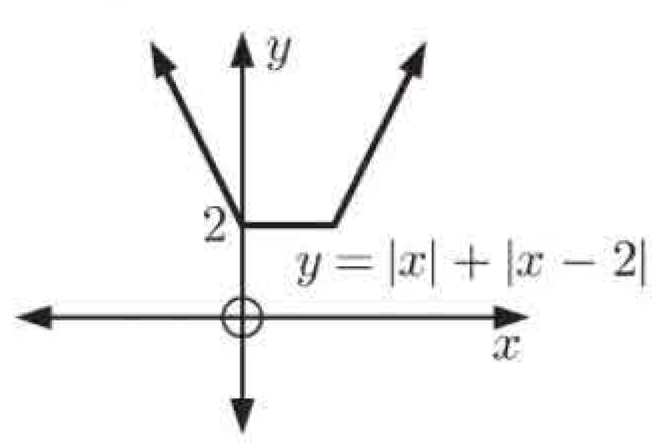
When $0 \leqslant x < 2$, $x \geqslant 0$ and $x - 2 < 0$

$\therefore \quad y = x - (x-2)$
$\qquad = 2$

When $x < 0$, $x - 2 < 0$

$\therefore \quad y = -x - (x-2)$
$\qquad = 2 - 2x$

$$\therefore \quad y = \begin{cases} 2x-2 & \text{when } x \geqslant 2 \\ 2 & \text{when } 0 \leqslant x < 2 \\ 2-2x & \text{when } x < 0 \end{cases}$$



h $$y = |x| - |x-1|$$

When $x \geqslant 1$, $x - 1 \geqslant 0$ and $x \geqslant 0$

$\therefore \quad y = x - (x-1)$
$\qquad = 1$

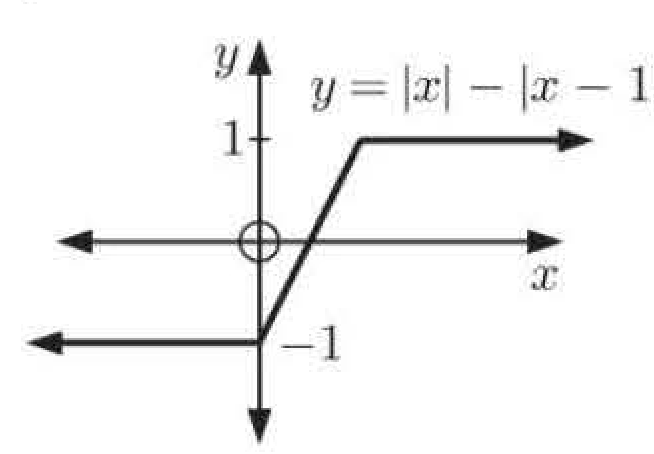
When $0 \leqslant x < 1$, $x \geqslant 0$ and $x - 1 < 0$

$\therefore \quad y = x + (x-1)$
$\qquad = 2x - 1$

When $x < 0$, $x - 1 < 0$

$\therefore \quad y = -x + (x-1)$
$\qquad = -1$

$$\therefore \quad y = \begin{cases} 1 & \text{when } x \geqslant 1 \\ 2x-1 & \text{when } 0 \leqslant x < 1 \\ -1 & \text{when } x < 0 \end{cases}$$



EXERCISE 2H.2

1 **a** $|x| = 3$
$\therefore \quad x = \pm 3$

b $|x| = -5$ has no solution as $|x|$ is never negative.

c $|x| = 0$
$\therefore \quad x = 0$

d $|x - 1| = 3$
$\therefore \quad x - 1 = \pm 3$
$\therefore \quad x = 1 + 3$ or $1 - 3$
$\therefore \quad x = 4$ or -2

e $|3 - x| = 4$
$\therefore \quad 3 - x = \pm 4$
$\therefore \quad x - 3 = \pm 4$
$\therefore \quad x = 3 + 4$ or $3 - 4$
$\therefore \quad x = 7$ or -1

f $|x + 5| = -1$ has no solution as $|x + 5|$ is never negative.

g $|3x - 2| = 1$
$\therefore \quad 3x - 2 = \pm 1$
$\therefore \quad 3x = 2 + 1$ or $2 - 1$
$\therefore \quad 3x = 3$ or 1
$\therefore \quad x = 1$ or $\frac{1}{3}$

h $|3 - 2x| = 3$
$\therefore \quad 3 - 2x = \pm 3$
$\therefore \quad -2x = -3 + 3$ or $-3 - 3$
$\therefore \quad -2x = 0$ or -6
$\therefore \quad x = 0$ or 3

i $|2 - 5x| = 12$
$\therefore \quad 2 - 5x = \pm 12$
$\therefore \quad -5x = -2 + 12$ or $-2 - 12$
$\therefore \quad -5x = 10$ or -14
$\therefore \quad x = -2$ or $\frac{14}{5}$

2 **a** $\left|\dfrac{x}{x-1}\right| = 3$

$\therefore \dfrac{x}{x-1} = \pm 3$

If $\dfrac{x}{x-1} = 3$

then $x = 3x - 3$

$\therefore -2x = -3$

$\therefore x = \frac{3}{2}$

If $\dfrac{x}{x-1} = -3$

then $x = -3x + 3$

$\therefore 4x = 3$

$\therefore x = \frac{3}{4}$

So, $x = \frac{3}{2}$ or $\frac{3}{4}$

b $\left|\dfrac{2x-1}{x+1}\right| = 5$

$\therefore \dfrac{2x-1}{x+1} = \pm 5$

If $\dfrac{2x-1}{x+1} = 5$

then $2x - 1 = 5x + 5$

$\therefore -3x = 6$

$\therefore x = -2$

If $\dfrac{2x-1}{x+1} = -5$

then $2x - 1 = -5x - 5$

$\therefore 7x = -4$

$\therefore x = -\frac{4}{7}$

So, $x = -2$ or $-\frac{4}{7}$

c $\left|\dfrac{x+3}{1-3x}\right| = \frac{1}{2}$

$\therefore \dfrac{x+3}{1-3x} = \pm\frac{1}{2}$

If $\dfrac{x+3}{1-3x} = \frac{1}{2}$

then $2(x+3) = 1 - 3x$

$\therefore 5x = -5$

$\therefore x = -1$

If $\dfrac{x+3}{1-3x} = -\frac{1}{2}$

then $2(x+3) = -(1-3x)$

$\therefore -x = -7$

$\therefore x = 7$

So, $x = -1$ or 7

3 **a** $|3x-1| = |x+2|$

$\therefore 3x - 1 = \pm(x+2)$

If $3x - 1 = x + 2$

then $2x = 3$

$\therefore x = \frac{3}{2}$

If $3x - 1 = -x - 2$

$\therefore 4x = -1$

$\therefore x = -\frac{1}{4}$

So, $x = \frac{3}{2}$ or $-\frac{1}{4}$

b $|2x+5| = |1-x|$

$\therefore 2x + 5 = \pm(1-x)$

If $2x + 5 = 1 - x$

then $3x = -4$

$\therefore x = -\frac{4}{3}$

If $2x + 5 = -(1-x)$

then $2x + 5 = -1 + x$

$\therefore x = -6$

So, $x = -\frac{4}{3}$ or -6

c $|x+1| = |2-x|$

$\therefore x + 1 = \pm(2-x)$

If $x + 1 = 2 - x$

then $2x = 1$

$\therefore x = \frac{1}{2}$

If $x + 1 = -(2-x)$

then $x + 1 = -2 + x$

$\therefore 1 = -2$

which is false

So, $x = \frac{1}{2}$ is the only solution.

d $|x| = |5-x|$

$\therefore x = \pm(5-x)$

If $x = 5 - x$

then $2x = 5$

$\therefore x = \frac{5}{2}$

If $x = -(5-x)$

then $x = -5 + x$

$\therefore 0 = -5$

which is false

So, $x = \frac{5}{2}$ is the only solution.

e $|1-4x| = 2\,|x-1|$

$\therefore 1 - 4x = \pm 2(x-1)$

If $1 - 4x = 2(x-1)$

then $1 - 4x = 2x - 2$

$\therefore -6x = -3$

$\therefore x = \frac{1}{2}$

If $1 - 4x = -2(x-1)$

then $1 - 4x = -2x + 2$

$\therefore -2x = 1$

$\therefore x = -\frac{1}{2}$

So, $x = \pm\frac{1}{2}$

f $|3x+2| = 2\,|2-x|$

$\therefore 3x + 2 = \pm 2(2-x)$

If $3x + 2 = 2(2-x)$

then $3x + 2 = 4 - 2x$

$\therefore 5x = 2$

$\therefore x = \frac{2}{5}$

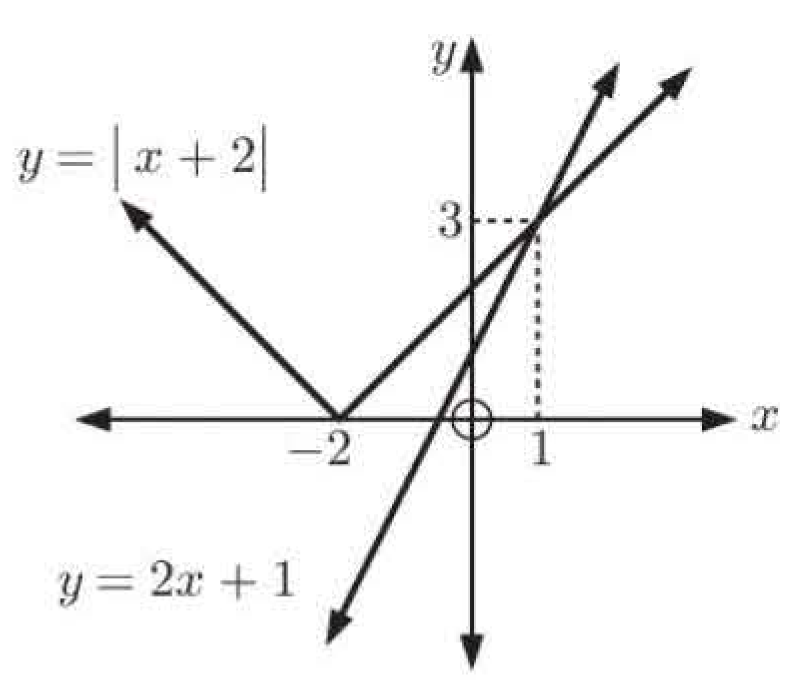
If $3x + 2 = -2(2-x)$

then $3x + 2 = -4 + 2x$

$\therefore x = -6$

So, $x = \frac{2}{5}$ or -6

4 **a** **i**



The lines $y = |x+2|$ and $y = 2x + 1$ intersect at $(1, 3)$.

$\therefore$ the solution is $x = 1$.

ii $|x+2| = 2x+1$

$\therefore\ x+2 = \pm(2x+1)$

If $x+2 = 2x+1$

then $-x = -1$

$\therefore\ x = 1$

If $x+2 = -(2x+1)$

then $x+2 = -2x-1$

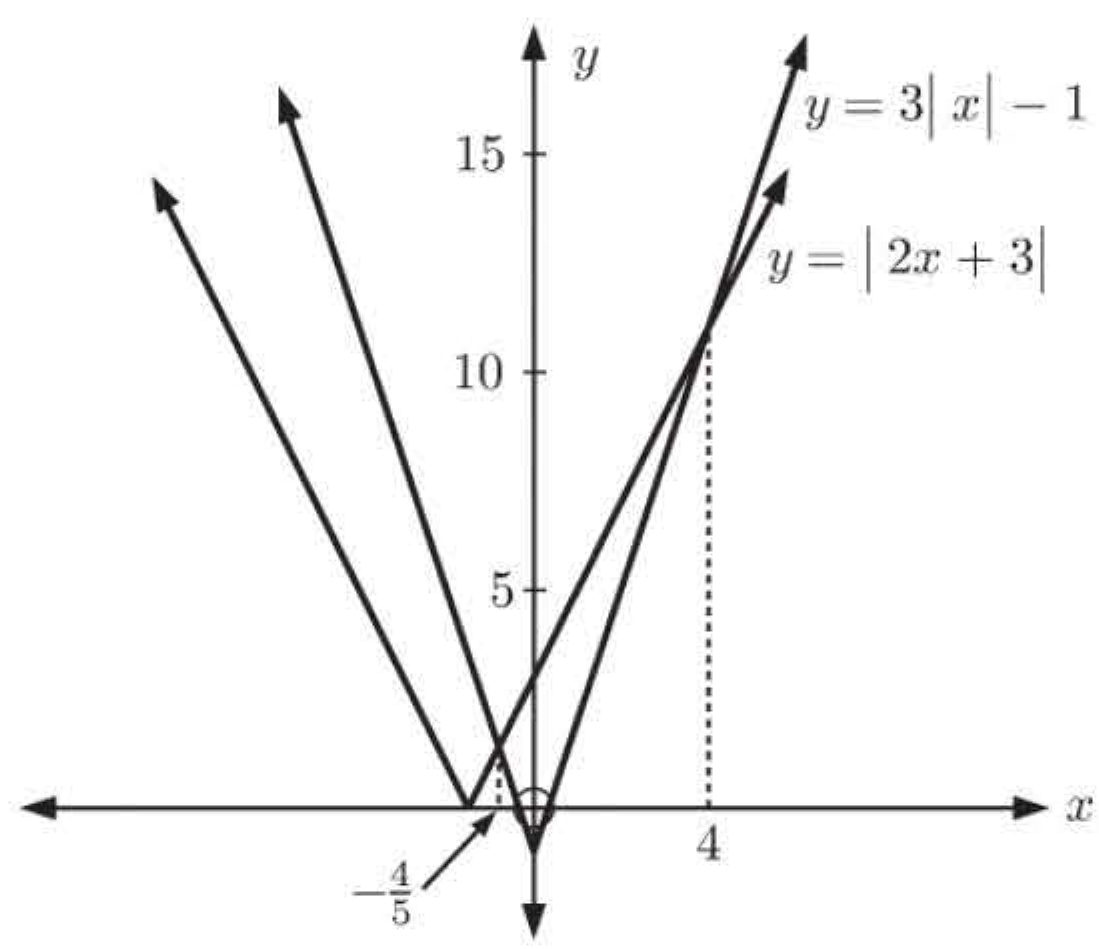
$\therefore\ 3x = -3$

$\therefore\ x = -1$

However, $x=-1$ is not a valid solution, because when $x=-1$, $2x+1<0$, and $|x+2|$ is never negative.

$\therefore\ x=1$ is the only solution.

b **i**



The lines $y=|2x+3|$ and $y=3|x|-1$ intersect at $(-\frac{4}{5}, \frac{7}{5})$ and $(4, 11)$.

$\therefore$ the solution is $x=-\frac{4}{5}$ or 4.

ii Let $y_1 = |2x+3|$, $y_2 = 3|x|-1$

When $x < -\frac{3}{2}$, $y_1 = -(2x+3)$

and $y_2 = 3(-x)-1$

$\therefore\ -(2x+3) = 3(-x)-1$

$\therefore\ -2x-3 = -3x-1$

$\therefore\ x = 2$

This is not in the domain $x<-\frac{3}{2}$, so is not a valid solution.

When $-\frac{3}{2} \leqslant x < 0$, $y_1 = 2x+3$

and $y_2 = 3(-x)-1$

$\therefore\ 2x+3 = 3(-x)-1$

$\therefore\ 2x+3 = -3x-1$

$\therefore\ 5x = -4$

$\therefore\ x = -\frac{4}{5}$

When $x \geqslant 0$, $y_1 = 2x+3$

and $y_2 = 3x-1$

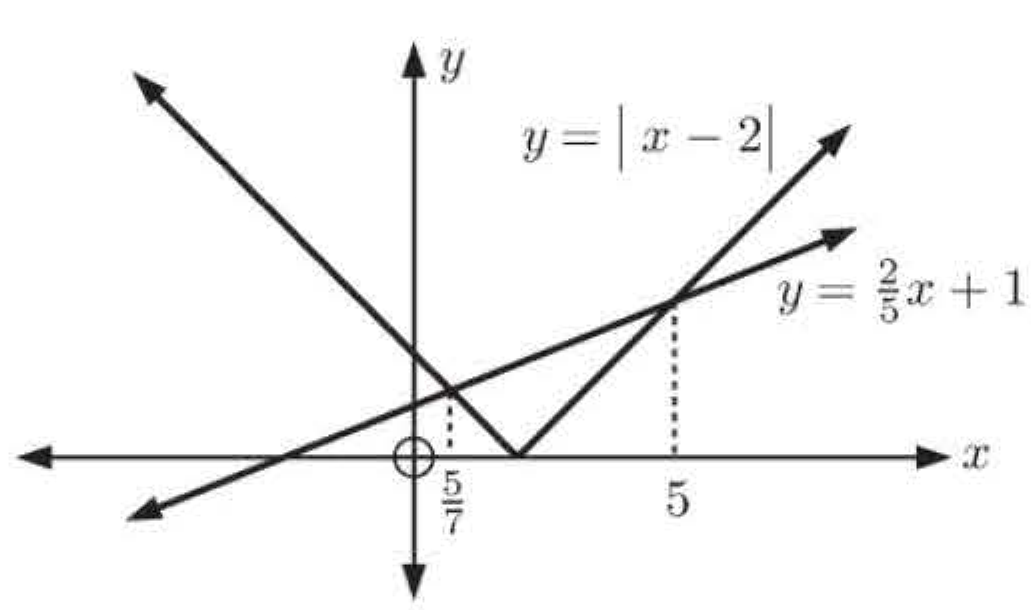
$\therefore\ 2x+3 = 3x-1$

$\therefore\ -x = -4$

$\therefore\ x = 4$

So, the solution is $x=-\frac{4}{5}$ or 4.

c **i**



The lines $y=|x-2|$ and $y=\frac{2}{5}x+1$ intersect at $(\frac{5}{7}, \frac{9}{7})$ and $(5, 3)$.

$\therefore$ the solution is $x=\frac{5}{7}$ or 5.

ii $|x-2| = \frac{2}{5}x+1$

$\therefore\ x-2 = \pm(\frac{2}{5}x+1)$

If $x-2 = \frac{2}{5}x+1$

then $\frac{3}{5}x = 3$

$\therefore\ x = 5$

If $x-2 = -(\frac{2}{5}x+1)$

then $x-2 = -\frac{2}{5}x-1$

$\therefore\ \frac{7}{5}x = 1$

$\therefore\ x = \frac{5}{7}$

So, the solution is $x=\frac{5}{7}$ or 5.

5 **a** $|x| < 4$

$\therefore\ -4 < x < 4$

$\therefore\ x \in\ ]-4, 4[$

b $|x| \geqslant 3$

$\therefore\ x \leqslant -3$ or $x \geqslant 3$

$\therefore\ x \in\ ]-\infty, -3]$ or $[3, \infty[$

c $|x+3| \leqslant 1$

$\therefore\ -1 \leqslant x+3 \leqslant 1$

$\therefore\ -4 \leqslant x \leqslant -2$

$\therefore\ x \in [-4, -2]$

d $|2x-1| < 3$

$\therefore\ -3 < 2x-1 < 3$

$\therefore\ -2 < 2x < 4$

$\therefore\ -1 < x < 2$

$\therefore\ x \in \,]-1, 2[$

e $|3-4x| > 2$

$\therefore\ |4x-3| > 2$

$\therefore\ 4x-3 < -2$ or $4x-3 > 2$

$\therefore\ 4x < 1$ $\quad\therefore\ 4x > 5$

$\therefore\ x < \frac{1}{4}$ $\quad\therefore\ x > \frac{5}{4}$

$\therefore\ x \in \,]-\infty, \frac{1}{4}[$ or $]\frac{5}{4}, \infty[$

f $|x| \geqslant |2-x|$

$\therefore\ x^2 \geqslant (2-x)^2$

$\therefore\ x^2 - (2-x)^2 \geqslant 0$

$\therefore\ [x+(2-x)][x-(2-x)] \geqslant 0$

$\therefore\ 2(2x-2) \geqslant 0$

$\therefore\ x \geqslant 1$

So, $x \in [1, \infty[$

g $3\,|x| \leqslant |1-2x|$

$\therefore\ 9x^2 \leqslant (1-2x)^2$

$\therefore\ 9x^2 - (1-2x)^2 \leqslant 0$

$\therefore\ [3x+(1-2x)][3x-(1-2x)] \leqslant 0$

$\therefore\ (x+1)(5x-1) \leqslant 0$

$\therefore\ -1 \leqslant x \leqslant \frac{1}{5}$

So, $x \in [-1, \frac{1}{5}]$

h $\left|\dfrac{x}{x-2}\right| \geqslant 3$

$$\therefore\ \left(\frac{x}{x-2}\right)^2 \geqslant 3^2$$

$$\therefore\ \left(\frac{x}{x-2}\right)^2 - 3^2 \geqslant 0$$

$$\therefore\ \left(\frac{x}{x-2}+3\right)\left(\frac{x}{x-2}-3\right) \geqslant 0$$

$$\therefore\ \left(\frac{x+3x-6}{x-2}\right)\left(\frac{x-3x+6}{x-2}\right) \geqslant 0$$

$$\therefore\ \frac{(4x-6)(-2x+6)}{(x-2)^2} \geqslant 0$$

$\therefore\ x \in [\frac{3}{2}, 3]$ but $x \neq 2$

$\therefore\ x \in [\frac{3}{2}, 2[$ or $]2, 3]$

i $\left|\dfrac{2x+3}{x-1}\right| \geqslant 2$

$$\therefore\ \left(\frac{2x+3}{x-1}\right)^2 \geqslant 2^2$$

$$\therefore\ \left(\frac{2x+3}{x-1}\right)^2 - 2^2 \geqslant 0$$

$$\therefore\ \left(\frac{2x+3}{x-1}+2\right)\left(\frac{2x+3}{x-1}-2\right) \geqslant 0$$

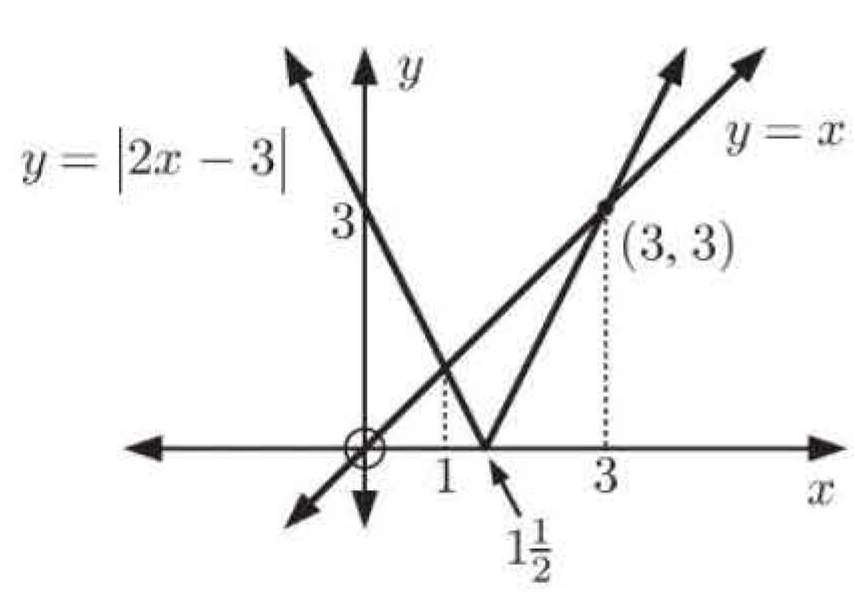
$$\therefore\ \left(\frac{2x+3+2x-2}{x-1}\right)\left(\frac{2x+3-2x+2}{x-1}\right) \geqslant 0$$

$$\therefore\ \frac{(4x+1)(5)}{(x-1)^2} \geqslant 0$$

$\therefore\ x \in [-\frac{1}{4}, \infty[$ but $x \neq 1$

$\therefore\ x \in [-\frac{1}{4}, 1[$ or $]1, \infty[$

j

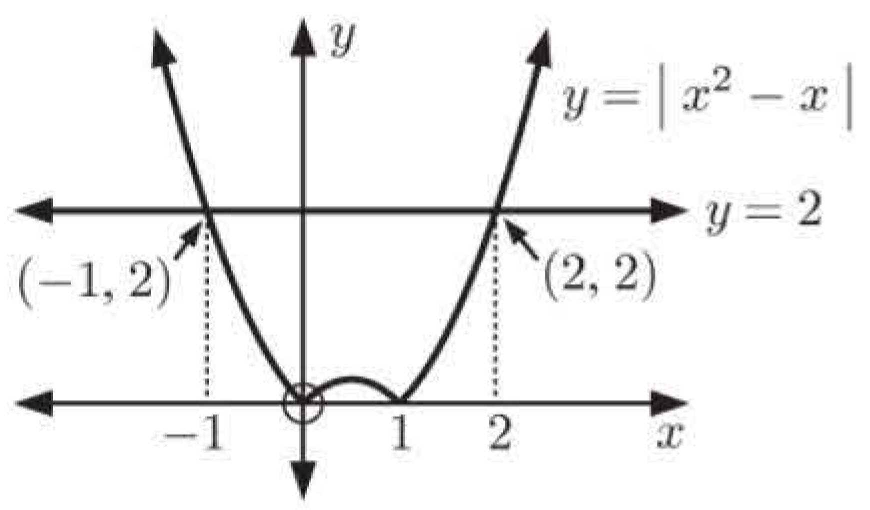


$|2x-3| < x$ when the modulus graph is below the line.

$\therefore\ 1 < x < 3$

$\therefore\ x \in \,]1, 3[$

k

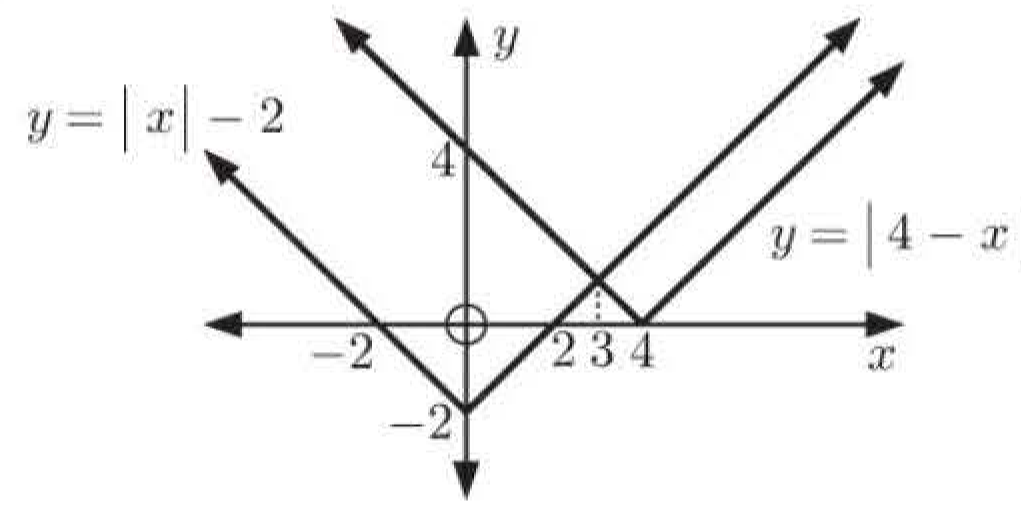


$|x^2 - x| > 2$ when the modulus graph is above the line.

$\therefore\ x < -1$ or $x > 2$

$\therefore\ x \in\ ]-\infty, -1[$ or $]2, \infty[$

l

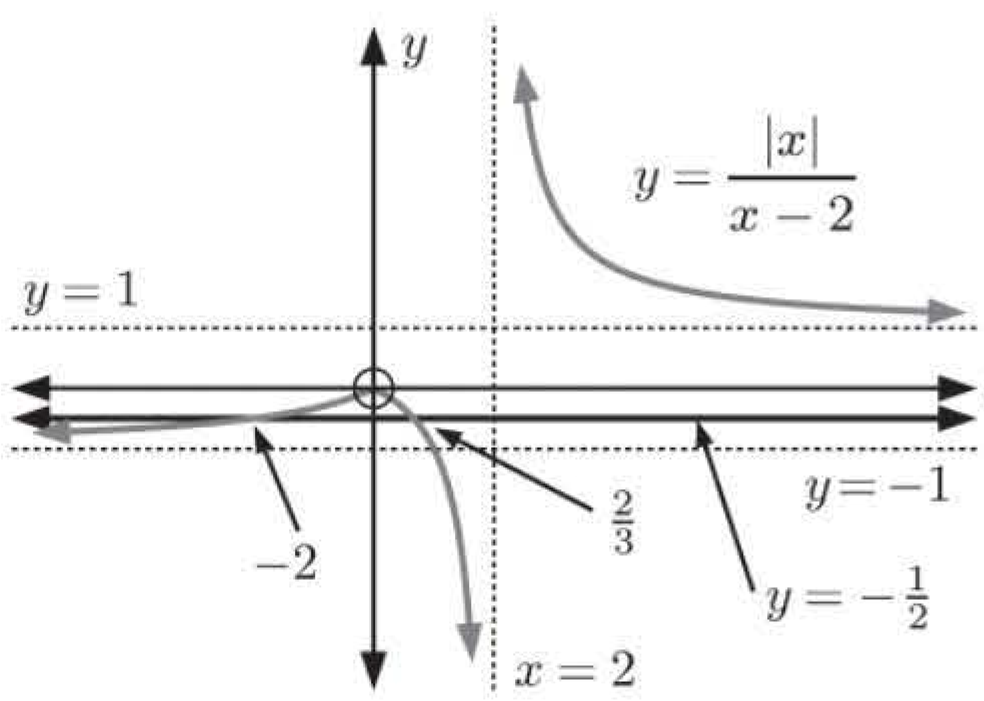


$|x| - 2 \geqslant |4 - x|$ when the graph of $y = |x| - 2$ is above or on the graph of $y = |4 - x|$.

$\therefore\ x \geqslant 3$

$\therefore\ x \in [3, \infty[$

6



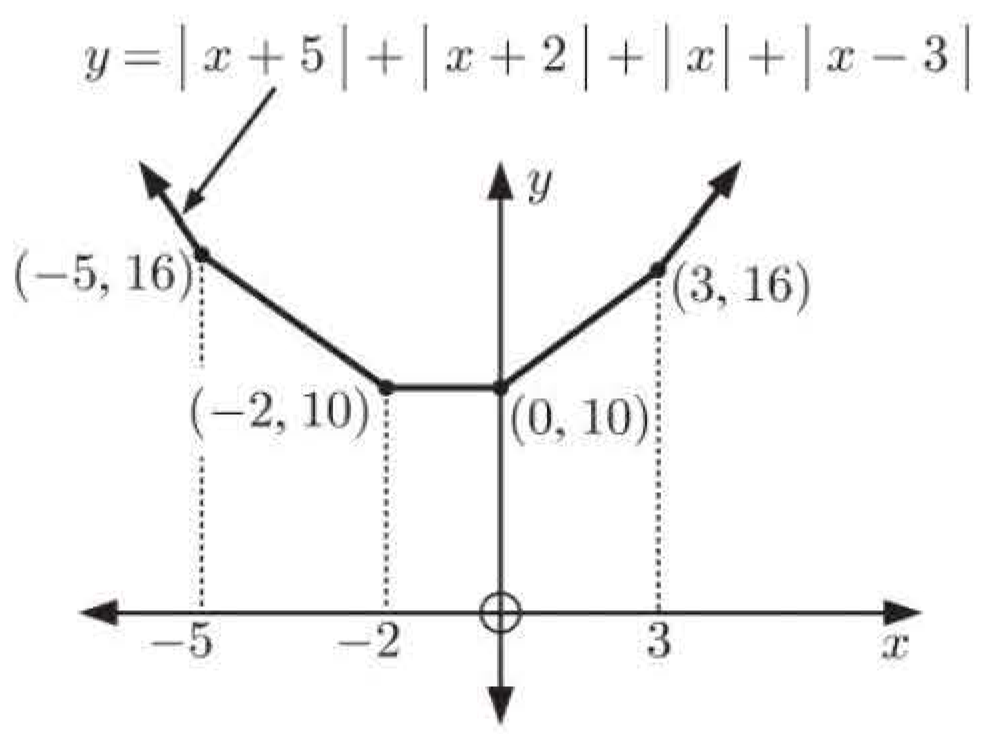
If $\dfrac{|x|}{x-2} \geqslant -\frac{1}{2}$ then the graph of $y = f(x)$ is above or on $y = -\frac{1}{2}$.

They intersect at -2 and $\frac{2}{3}$

$\therefore\ -2 \leqslant x \leqslant \frac{2}{3}$ or $x > 2$

$\therefore\ x \in [-2, \frac{2}{3}]$ or $]2, \infty[$

7 a



b **i** If x is a position along (AB) then:

$$\begin{aligned} \text{XP} &= |x - (-5)| = |x + 5| \\ \text{XQ} &= |x - (-2)| = |x + 2| \\ \text{XO} &= |x - 0| = |x| \\ \text{XR} &= |x - 3| \end{aligned}$$

The total length is

$$|x + 5| + |x + 2| + |x| + |x - 3|$$

ii The minimum length is 10 km when $-2 \leqslant x \leqslant 0$, so x can be anywhere between O and Q.

iii We need to graph

$$y = |x + 5| + |x + 2| + |x| + |x - 3| + |x - 7|$$

From technology, the minimum cable length is 17 km when $x = 0$, so x is at O.

8 a

$$\begin{aligned} |x+y|^2 &= (x+y)^2 \\ &= x^2 + 2xy + y^2 \\ \text{and}\quad (|x| + |y|)^2 &= |x|^2 + 2\,|x|\,|y| + |y|^2 \\ &= x^2 + 2\,|x|\,|y| + y^2 \end{aligned}$$

Now $xy \leqslant |x|\,|y|$

$\therefore\ x^2 + 2xy + y^2 \leqslant x^2 + 2\,|x|\,|y| + y^2$

$\therefore\ |x+y|^2 \leqslant (|x| + |y|)^2$

$\therefore\ |x + y| \leqslant |x| + |y|$ {both sides $\geqslant 0$}

$\therefore$ the statement is true for all x, y.

b

$$\begin{aligned} |x-y|^2 &= (x-y)^2 \\ &= x^2 - 2xy + y^2 \\ \text{and}\quad (|x| - |y|)^2 &= |x^2| - 2\,|x|\,|y| + |y|^2 \\ &= x^2 - 2\,|x|\,|y| + y^2 \end{aligned}$$

Now $xy \leqslant |x|\,|y|$

$\therefore\ -2xy \geqslant -2\,|x|\,|y|$

$\therefore\ x^2 - 2xy + y^2 \geqslant x^2 - 2\,|x|\,|y| + y^2$

$\therefore\ |x-y|^2 \geqslant (|x| - |y|)^2$

$\therefore\ |x - y| \geqslant |x| - |y|$ {both sides $\geqslant 0$}

$\therefore$ the statement is true for all x, y.

EXERCISE 2I

1 a i $f : x \mapsto \dfrac{3}{x-2}$ is undefined when $x = 2$, so $x = 2$ is a vertical asymptote.

As $|x| \to \infty$, $f(x) \to 0$, so $y = 0$ is a horizontal asymptote.

ii Domain is $\{x \mid x \neq 2\}$, Range is $\{y \mid y \neq 0\}$

iii $f(0) = \dfrac{3}{0-2} = -\frac{3}{2}$

So, the y-intercept is $-\frac{3}{2}$.

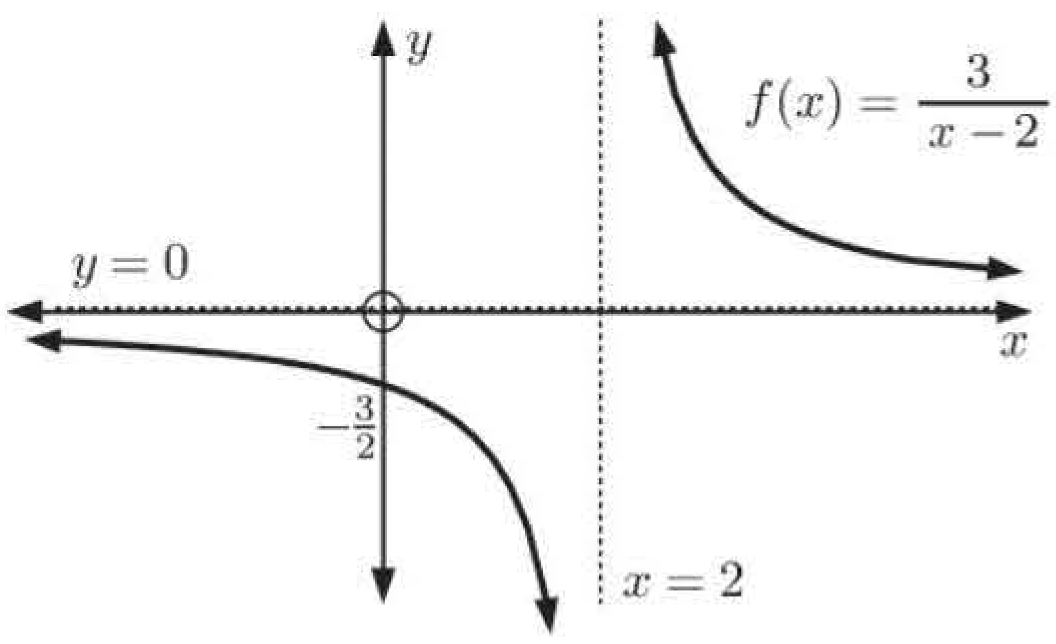
$f(x) = 0$ when $\dfrac{3}{x-2} = 0$,

which has no solutions.

$\therefore$ there is no x-intercept.

iv As $x \to 2^-$, $y \to -\infty$.
As $x \to 2^+$, $y \to \infty$.
As $x \to \infty$, $y \to 0^+$.
As $x \to -\infty$, $y \to 0^-$.

v



b **i** $f(x) = 2 - \dfrac{3}{x+1}$ is undefined when $x = -1$, so $x = -1$ is a vertical asymptote.

As $|x| \to \infty$, $\dfrac{3}{x+1} \to 0$, so $f(x) \to 2$ $\therefore$ $y = 2$ is a horizontal asymptote.

ii Domain is $\{x \mid x \neq -1\}$, Range is $\{y \mid y \neq 2\}$

iii $f(0) = 2 - \dfrac{3}{0+1} = -1$

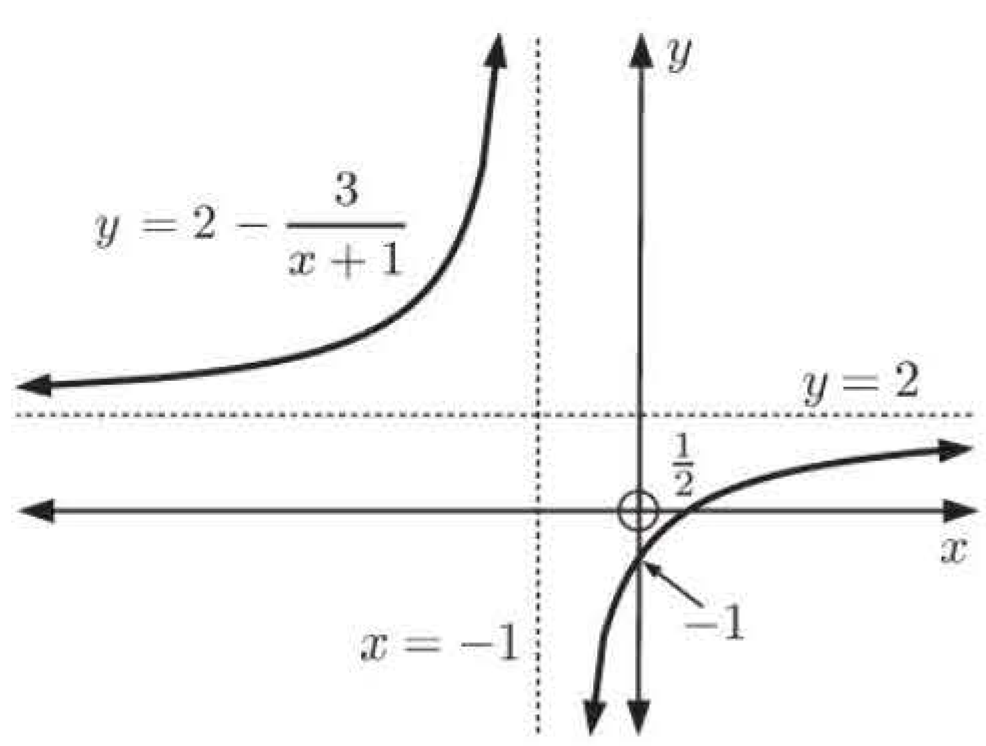
So, the y-intercept is -1.

$$f(x) = 0 \text{ when } 2 - \frac{3}{x+1} = 0$$
$$\therefore \quad \frac{3}{x+1} = 2$$
$$\therefore \quad x + 1 = \tfrac{3}{2}$$
$$\therefore \quad x = \tfrac{1}{2}$$

So, the x-intercept is $\frac{1}{2}$.

iv As $x \to -1^-$, $y \to \infty$.
As $x \to -1^+$, $y \to -\infty$.
As $x \to \infty$, $y \to 2^-$.
As $x \to -\infty$, $y \to 2^+$.

v



c **i** $f : x \mapsto \dfrac{x+3}{x-2}$ is undefined when $x = 2$, so $x = 2$ is a vertical asymptote.

Now $f(x) = \dfrac{x+3}{x-2} = \dfrac{1 + \frac{3}{x}}{1 - \frac{2}{x}}$

$\therefore$ as $|x| \to \infty$, $f(x) \to \frac{1}{1} = 1$, and so $y = 1$ is a horizontal asymptote.

ii Domain is $\{x \mid x \neq 2\}$, Range is $\{y \mid y \neq 1\}$

iii $f(0) = \dfrac{0+3}{0-2} = -\frac{3}{2}$

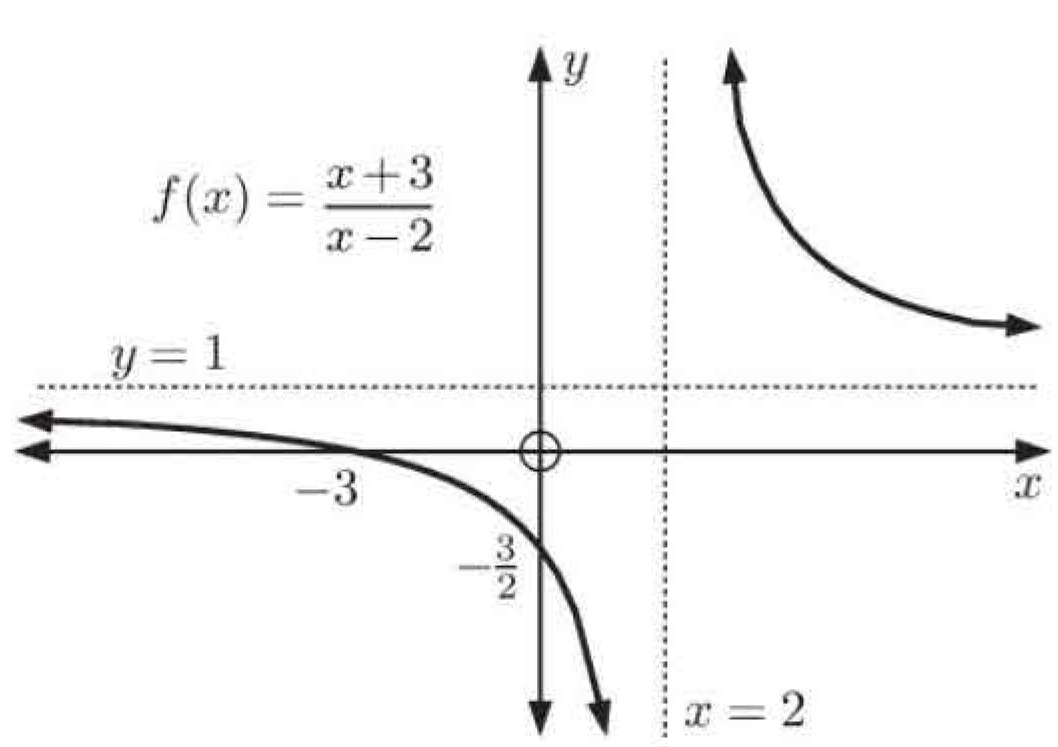
So, the y-intercept is $-\frac{3}{2}$.

$$f(x) = 0 \text{ when } \frac{x+3}{x-2} = 0$$
$$\therefore \quad x + 3 = 0$$
$$\therefore \quad x = -3$$

So, the x-intercept is -3.

iv As $x \to 2^-$, $y \to -\infty$.
As $x \to 2^+$, $y \to \infty$.
As $x \to \infty$, $y \to 1^+$.
As $x \to -\infty$, $y \to 1^-$.

v



d **i** $f(x) = \dfrac{3x-1}{x+2}$ is undefined when $x = -2$, so $x = -2$ is a vertical asymptote.

$f(x) = \dfrac{3x-1}{x+2} = \dfrac{3-\frac{1}{x}}{1+\frac{2}{x}}$

As $|x| \to \infty$, $f(x) \to \frac{3}{1} = 3$ and so $y = 3$ is a horizontal asymptote.

ii Domain is $\{x \mid x \neq -2\}$, Range is $\{y \mid y \neq 3\}$

iii $f(0) = \dfrac{3(0)-1}{0+2} = -\frac{1}{2}$

So, the y-intercept is $-\frac{1}{2}$.

$f(x) = 0$ when $\dfrac{3x-1}{x+2} = 0$

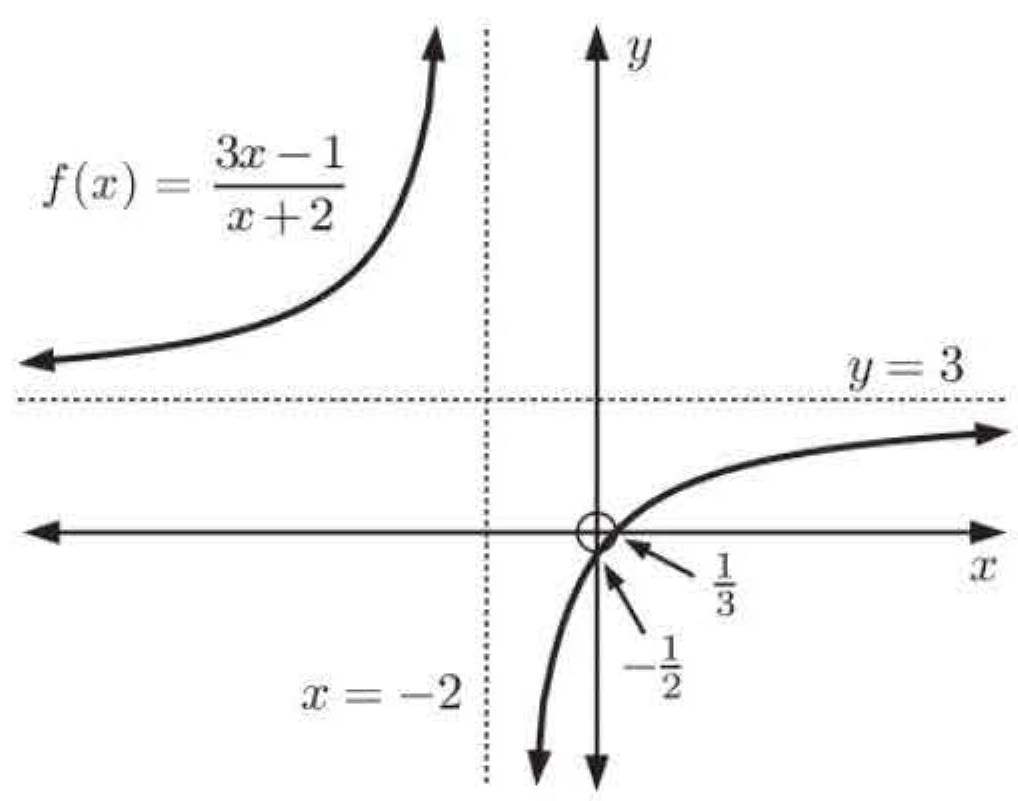
$\therefore \quad 3x - 1 = 0$

$\therefore \quad x = \frac{1}{3}$

So, the x-intercept is $\frac{1}{3}$.

iv As $x \to -2^-$, $y \to \infty$.
As $x \to -2^+$, $y \to -\infty$.
As $x \to \infty$, $y \to 3^-$.
As $x \to -\infty$, $y \to 3^+$.

v



2 **a** The function is defined when $cx + d \neq 0$, or when $x \neq -\dfrac{d}{c}$.

So, the domain is $\{x \mid x \neq -\dfrac{d}{c}\}$.

b The equation of the vertical asymptote is $x = -\dfrac{d}{c}$.

c To find the horizontal asymptote of $y = \dfrac{ax+b}{cx+d}$, we consider the function's behavior as $|x| \to \infty$.

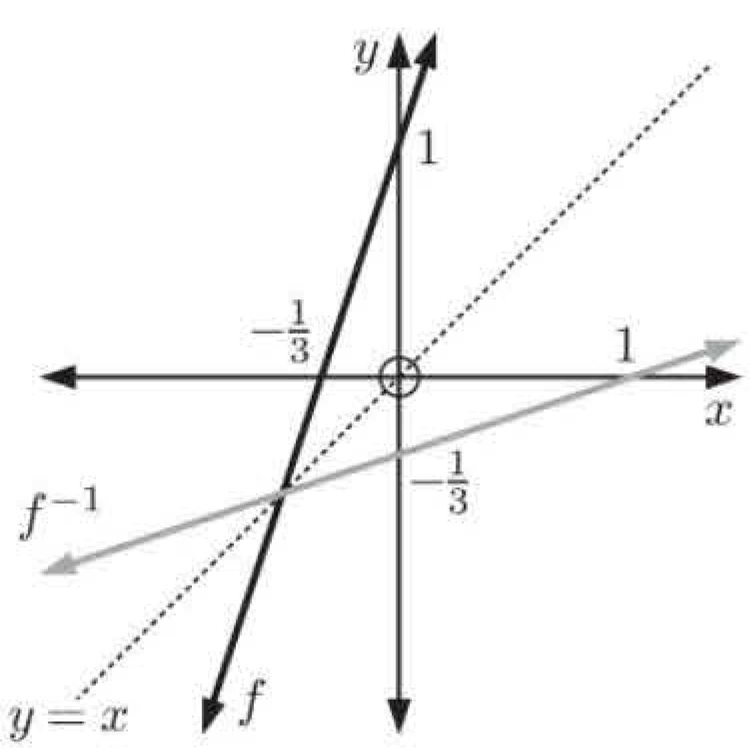
Now for $c \neq 0$,

$$\frac{ax+b}{cx+d} = \frac{acx+bc}{c(cx+d)} = \frac{a(cx+d)+bc-ad}{c(cx+d)} = \frac{a}{c} + \frac{bc-ad}{c(cx+d)} = \frac{a}{c} + \frac{b-\frac{ad}{c}}{cx+d}$$

$\therefore$ as $|x| \to \infty$, $y \to \dfrac{a}{c}$, and so $y = \dfrac{a}{c}$ is a horizontal asymptote.

EXERCISE 2J

1 **a** **i**



ii $f(x)$ passes through $(0, 1)$ and $(-\frac{1}{3}, 0)$

$\therefore$ $f^{-1}(x)$ passes through $(1, 0)$ and $(0, -\frac{1}{3})$

$f^{-1}(x)$ has gradient $\dfrac{-\frac{1}{3}-0}{0-1} = \dfrac{-\frac{1}{3}}{-1} = \frac{1}{3}$

So, its equation is $\dfrac{y-0}{x-1} = \frac{1}{3}$

which is $y = \dfrac{x-1}{3}$

So, $f^{-1}(x) = \dfrac{x-1}{3}$

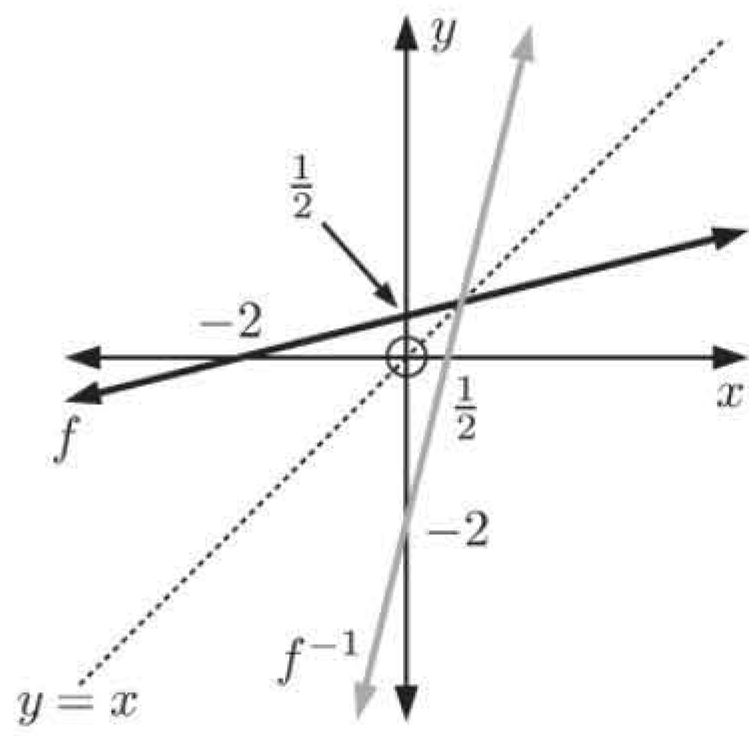
iii f is $y = 3x + 1$

so f^{-1} is $x = 3y + 1$

$\therefore \quad x - 1 = 3y$

$\therefore \quad y = \dfrac{x-1}{3}$. So, $f^{-1}(x) = \dfrac{x-1}{3}$

b **i**



ii $f(x)$ passes through $(0, \frac{1}{2})$ and $(-2, 0)$

$\therefore$ $f^{-1}(x)$ passes through $(\frac{1}{2}, 0)$ and $(0, -2)$

$f^{-1}(x)$ has gradient $\dfrac{-2-0}{0-\frac{1}{2}} = \dfrac{-2}{-\frac{1}{2}} = 4$

So, its equation is $\dfrac{y-0}{x-\frac{1}{2}} = 4$

which is $y = 4x - 2$

So, $f^{-1}(x) = 4x - 2$

iii f is $y = \dfrac{x+2}{4}$

so f^{-1} is $x = \dfrac{y+2}{4}$

$\therefore \quad 4x = y + 2$

$\therefore \quad y = 4x - 2.$ So, $f^{-1}(x) = 4x - 2$

2 **a** **i** f is $y = 2x + 5$

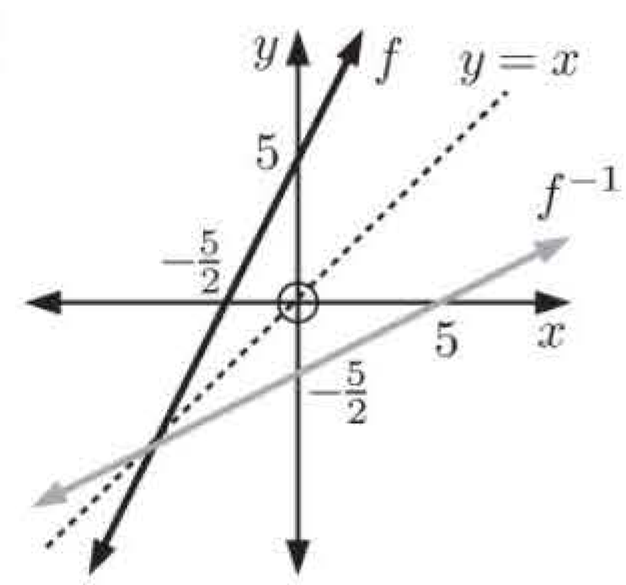
so f^{-1} is $x = 2y + 5$

$\therefore \quad x - 5 = 2y$

$\therefore \quad y = \dfrac{x-5}{2}$

So, $f^{-1}(x) = \dfrac{x-5}{2}$

ii



$f(x)$ passes through $(0, 5)$ and $(-\frac{5}{2}, 0)$

$\therefore$ $f^{-1}(x)$ passes through $(5, 0)$ and $(0, -\frac{5}{2})$.

iii $(f^{-1} \circ f)(x) = f^{-1}(2x+5)$
$$= \frac{2x+5-5}{2}$$
$$= \frac{2x}{2}$$
$$= x$$

and $(f \circ f^{-1})(x) = f(f^{-1}(x))$
$$= f\left(\frac{x-5}{2}\right)$$
$$= 2\left(\frac{x-5}{2}\right) + 5$$
$$= x - 5 + 5$$
$$= x$$

b **i** f is $y = \dfrac{3-2x}{4}$

so f^{-1} is $x = \dfrac{3-2y}{4}$

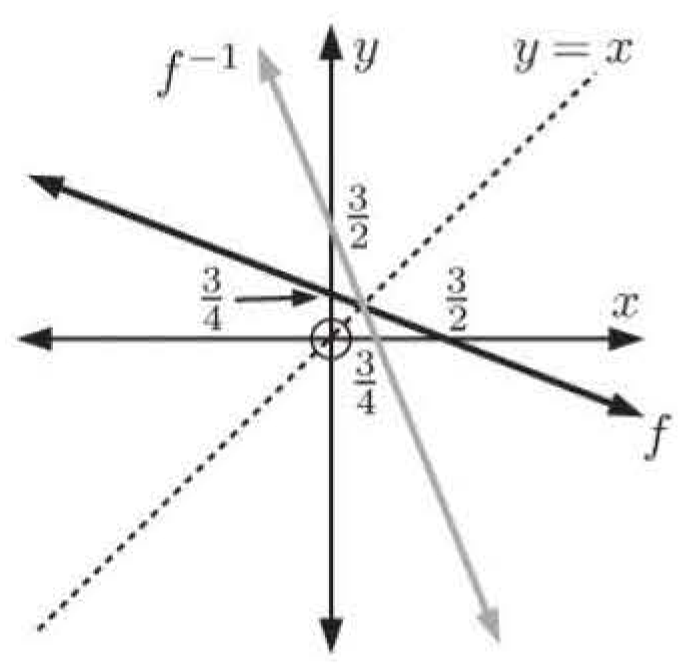
$\therefore \quad 4x = 3 - 2y$

$\therefore \quad 4x - 3 = -2y$

$\therefore \quad y = -2x + \frac{3}{2}$

So, $f^{-1}(x) = -2x + \frac{3}{2}$

ii



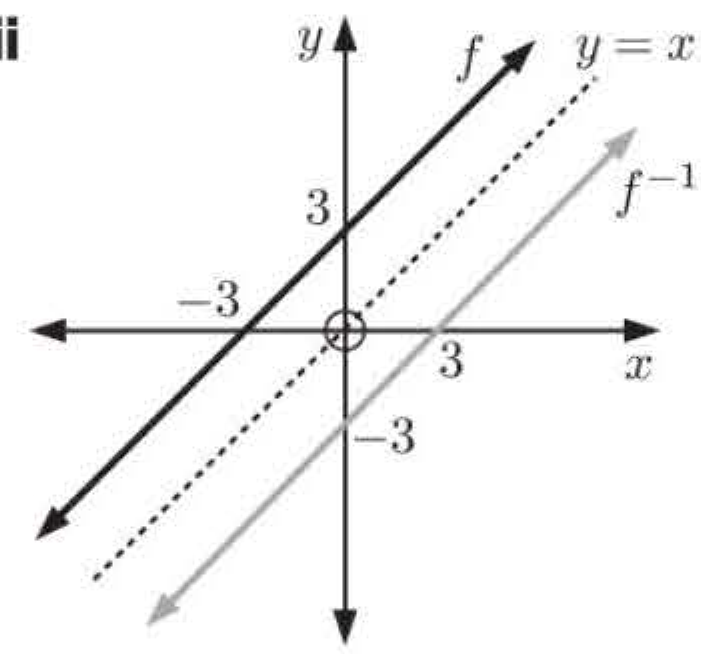
$f(x)$ passes through $(0, \frac{3}{4})$ and $(\frac{3}{2}, 0)$

$\therefore$ $f^{-1}(x)$ passes through $(\frac{3}{4}, 0)$ and $(0, \frac{3}{2})$.

iii $(f^{-1} \circ f)(x) = f^{-1}(f(x))$
$$= f^{-1}\left(\frac{3-2x}{4}\right)$$
$$= -2\left(\frac{3-2x}{4}\right) + \tfrac{3}{2}$$
$$= \frac{3-2x}{-2} + \tfrac{3}{2}$$
$$= -\tfrac{3}{2} + x + \tfrac{3}{2}$$
$$= x$$

and $(f \circ f^{-1})(x) = f(f^{-1}(x))$
$$= f\left(-2x + \tfrac{3}{2}\right)$$
$$= \frac{3 - 2(-2x + \frac{3}{2})}{4}$$
$$= \frac{3 + 4x - 3}{4}$$
$$= \frac{4x}{4}$$
$$= x$$

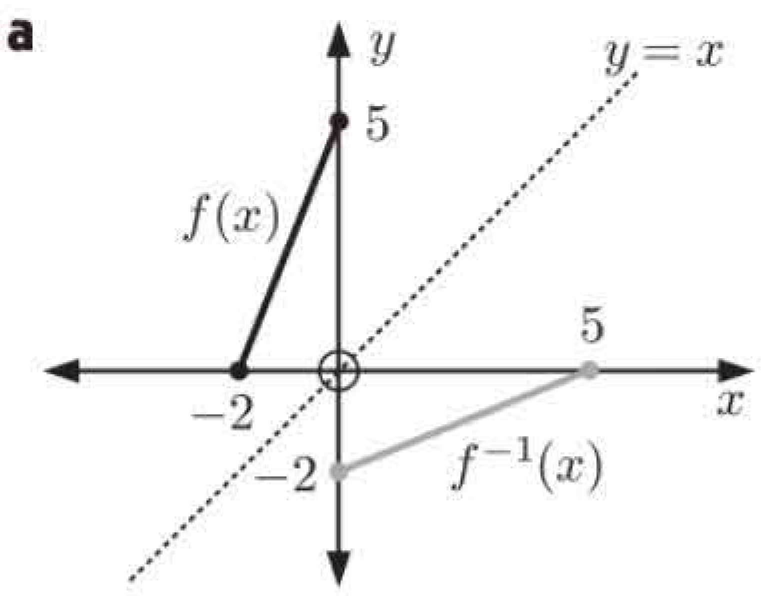
c **i** f is $y = x + 3$
so f^{-1} is $x = y + 3$
$\therefore \quad y = x - 3$
So, $f^{-1}(x) = x - 3$

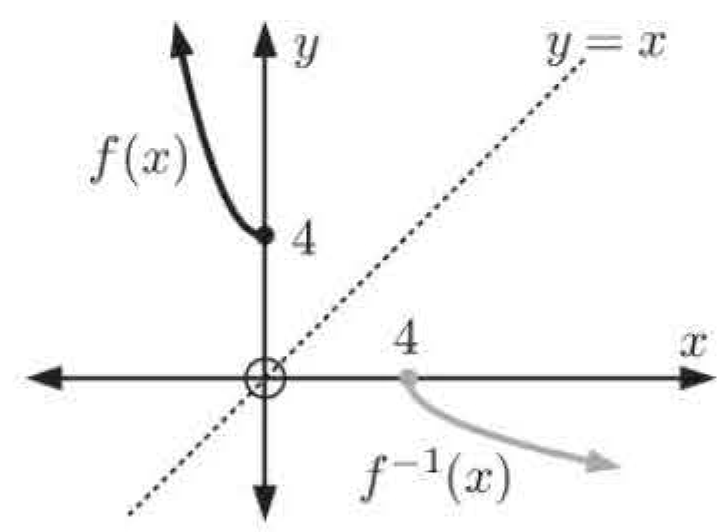
ii 

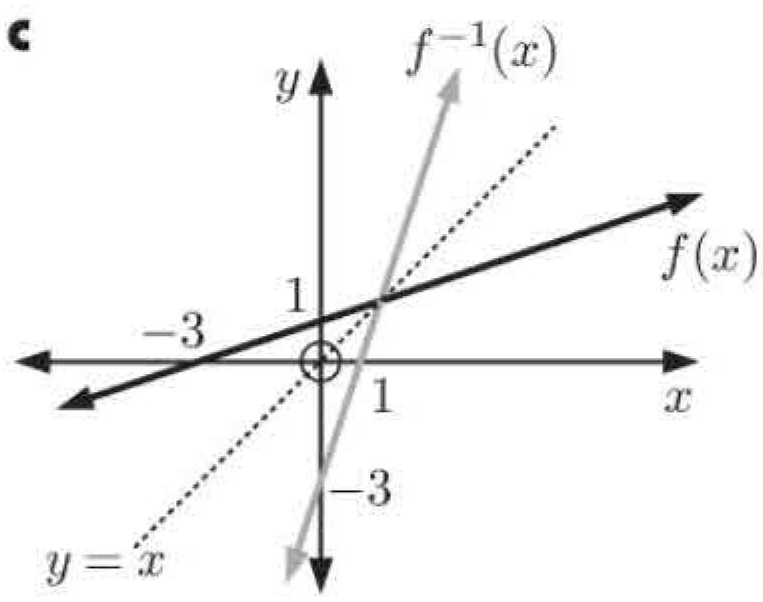
$f(x)$ passes through $(0, 3)$ and $(-3, 0)$
$\therefore$ $f^{-1}(x)$ passes through $(3, 0)$ and $(0, -3)$.

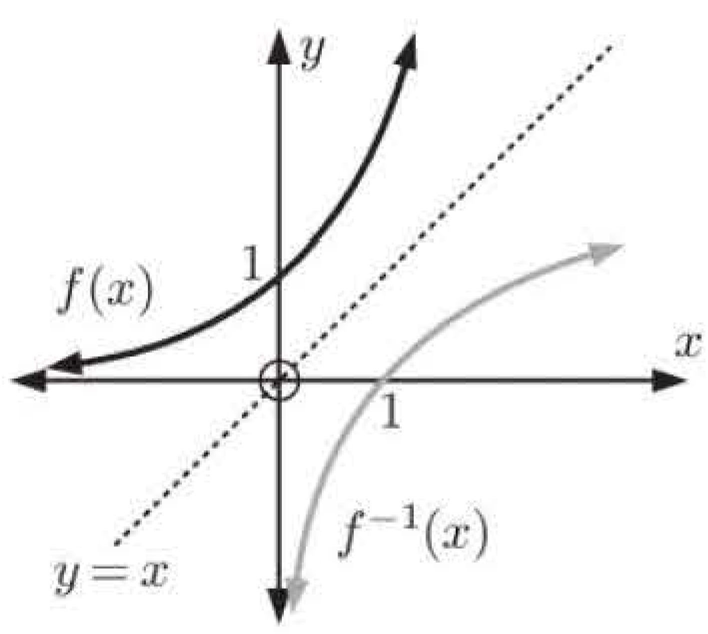
iii $(f^{-1} \circ f)(x) = f^{-1}(f(x))$
$= f^{-1}(x + 3)$
$= (x + 3) - 3$
$= x$

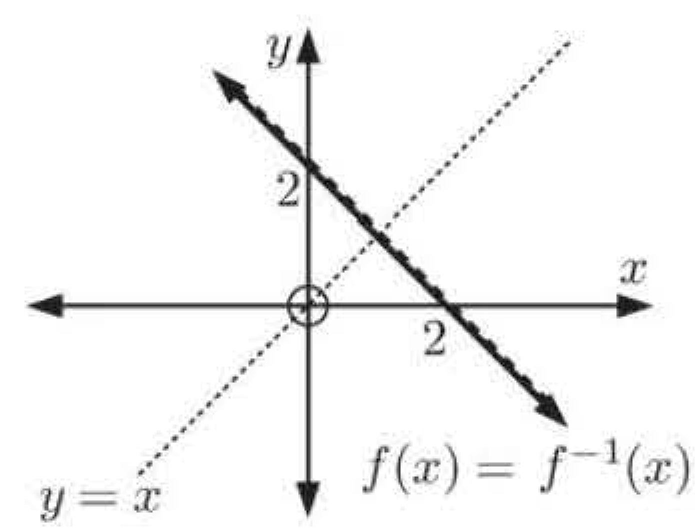
and $(f \circ f^{-1})(x) = f(f^{-1}(x))$
$= f(x - 3)$
$= (x - 3) + 3$
$= x$

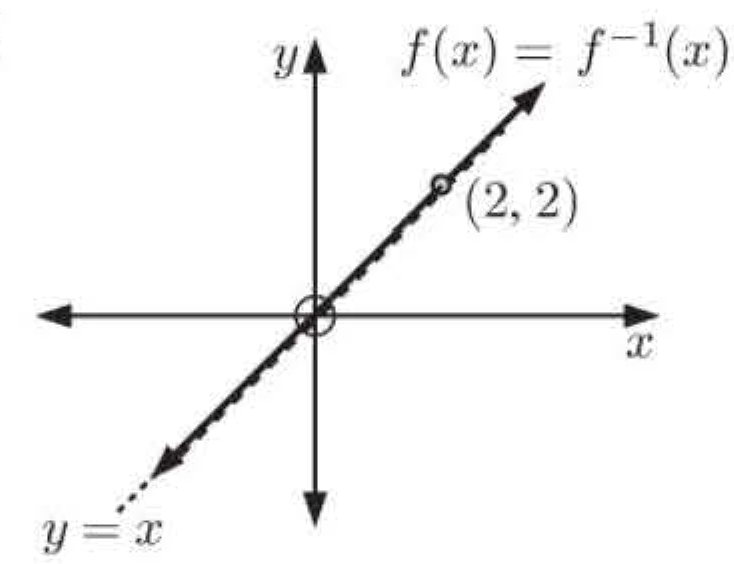
3 **a** 

b 

c 

d 

e 

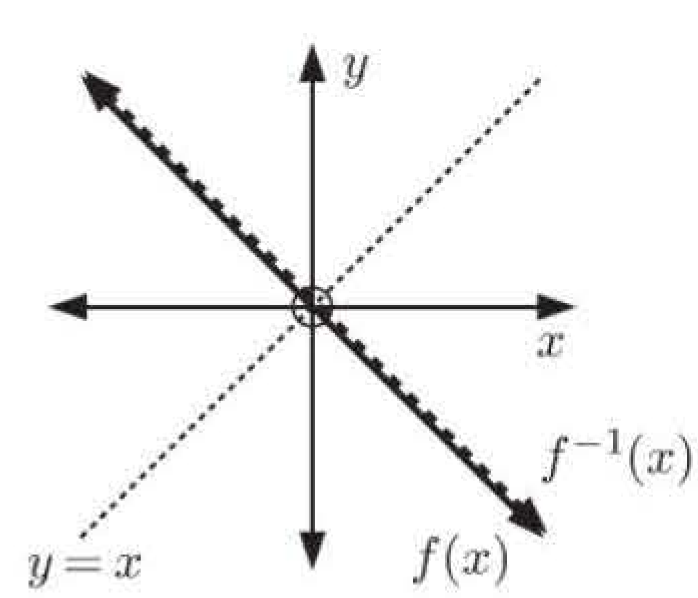
f 

4 **a** Domain of $f(x)$ is $\{x \mid -2 \leqslant x \leqslant 0\}$ **b** Range of $f(x)$ is $\{y \mid 0 \leqslant y \leqslant 5\}$

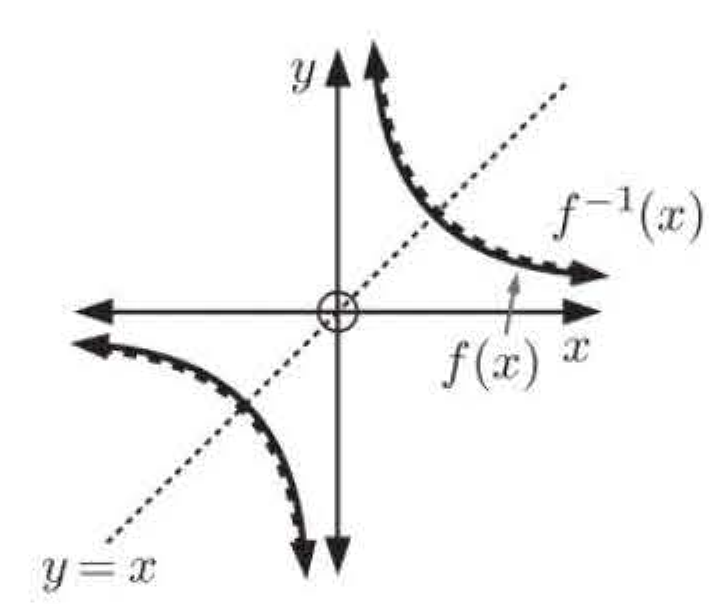
c Domain of $f^{-1}(x)$ is $\{x \mid 0 \leqslant x \leqslant 5\}$ **d** Range of $f^{-1}(x)$ is $\{y \mid -2 \leqslant y \leqslant 0\}$

5 **a** The functions in **3 e** and **3 f** are self-inverse functions.

b Any linear function of the form $y = a - x$ will be a self-inverse function, for example $y = -x$ (where $a = 0$):



c Any rational function of the form $y = \dfrac{a}{x}$ will be a self-inverse function, for example $y = \dfrac{2}{x}$ (where $a = 2$):



6 f is $y = 2x - 5$

$\therefore$ the inverse function is $x = 2y - 5$

$\therefore\ 2y = x + 5$

$\therefore\ y = \dfrac{x+5}{2}$

$\therefore\ f^{-1}(x) = \dfrac{x+5}{2}$

To find $(f^{-1})^{-1}(x)$, we need to find the inverse function for $y = \dfrac{x+5}{2}$

This is $x = \dfrac{y+5}{2}$

$\therefore\ 2x = y + 5$

$\therefore\ y = 2x - 5$

This is the original function $f(x)$.

So, $(f^{-1})^{-1}(x) = f(x)$.

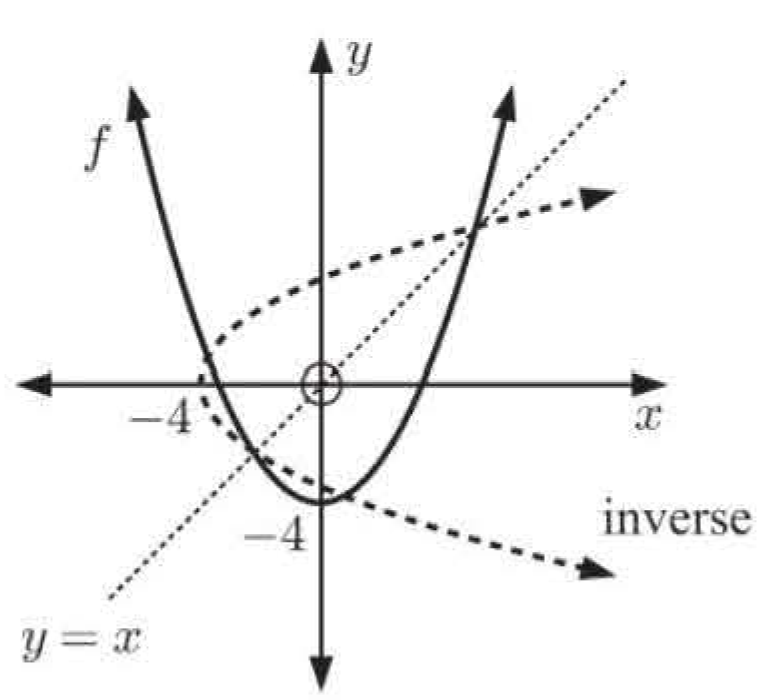
7 **a** For $\{(1, 2), (2, 4), (3, 5)\}$, there is at most one x-value corresponding to each y-value, so the function has an inverse. The inverse function is $\{(2, 1), (4, 2), (5, 3)\}$.

b For $\{(-1, 3), (0, 2), (1, 3)\}$, there are two x-values corresponding to the y-value of 3. So, the function does not have an inverse.

c For $\{(2, 1), (-1, 0), (0, 2), (1, 3)\}$, there is at most one x-value corresponding to each y-value, so the function has an inverse. The inverse function is $\{(0, -1), (1, 2), (2, 0), (3, 1)\}$.

d For $\{(-1, -1), (0, 0), (1, 1)\}$, there is at most one x-value corresponding to each y-value, so the function has an inverse. The inverse function is $\{(-1, -1), (0, 0), (1, 1)\}$.

8 **a**



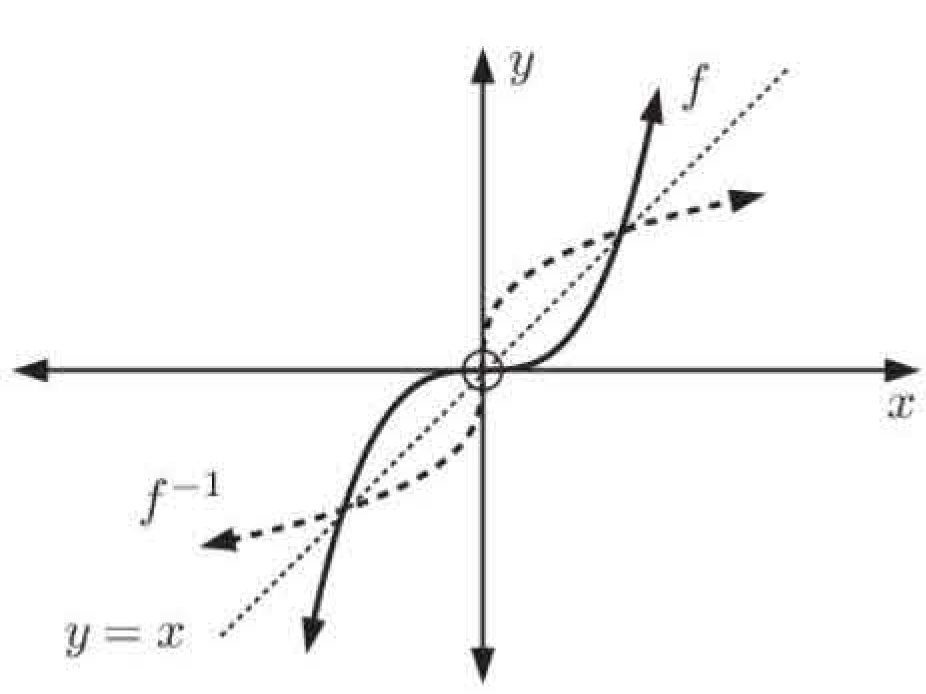
b Using the 'horizontal line test', f does not have an inverse function as a horizontal line through $y = x^2 - 4$ cuts it more than once.

c For $x \geqslant 0$, any horizontal line cuts it only once.

$\therefore$ f does have an inverse function for $x \geqslant 0$;

$f^{-1}(x) = \sqrt{x+4}$.

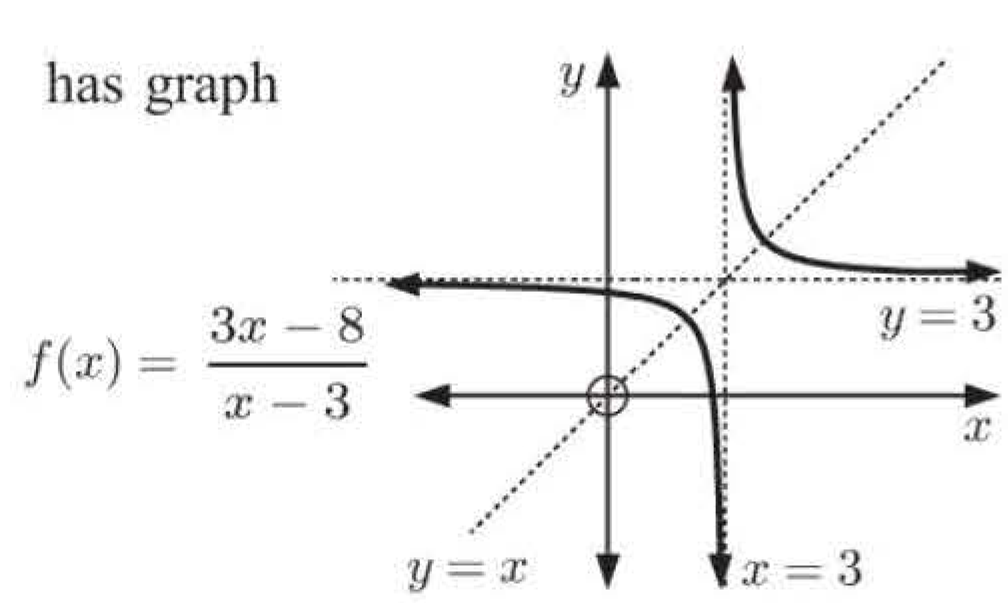
9



10 $f(x) = \dfrac{1}{x}$ has inverse function $x = \dfrac{1}{y}$ or $y = \dfrac{1}{x}$

So, $f^{-1}(x) = \dfrac{1}{x}$, which means $f(x)$ is a self-inverse function.

11 **a** $f(x) = \dfrac{3x-8}{x-3}$ has graph



The vertical line test shows it to be a function.

Symmetry about $y = x$ shows it is a self-inverse function.

b $f(x) = \dfrac{3x-8}{x-3}$ has inverse function $x = \dfrac{3y-8}{y-3}$

$$\therefore \quad x(y-3) = 3y-8$$
$$\therefore \quad xy - 3x = 3y - 8$$
$$\therefore \quad xy - 3y = 3x - 8$$
$$\therefore \quad y(x-3) = 3x-8$$
$$\therefore \quad y = \frac{3x-8}{x-3}$$
$$\therefore \quad f^{-1}(x) = \frac{3x-8}{x-3}$$

So, $f(x) = f^{-1}(x)$, which means $f(x)$ is a self-inverse function.

12 **a** If $y = f(x)$ has an inverse function, then the inverse function must also be a function. Thus, it must satisfy the 'vertical line test', i.e., no vertical line can cut it more than once. This condition for the inverse function cannot be satisfied if the original function does not satisfy the 'horizontal line test'. Thus, the 'horizontal line test' is a valid test for the existence of an inverse function.

b **i** This graph satisfies the 'horizontal line test' and therefore has an inverse function.

ii, iii These graphs both fail the 'horizontal line test' so neither of these have inverse functions.

c **ii** Domain $\{x \mid x \geqslant 1\}$ or $\{x \mid x \leqslant 1\}$ **iii** Domain $\{x \mid x \geqslant 1\}$ or $\{x \mid x \leqslant -2\}$

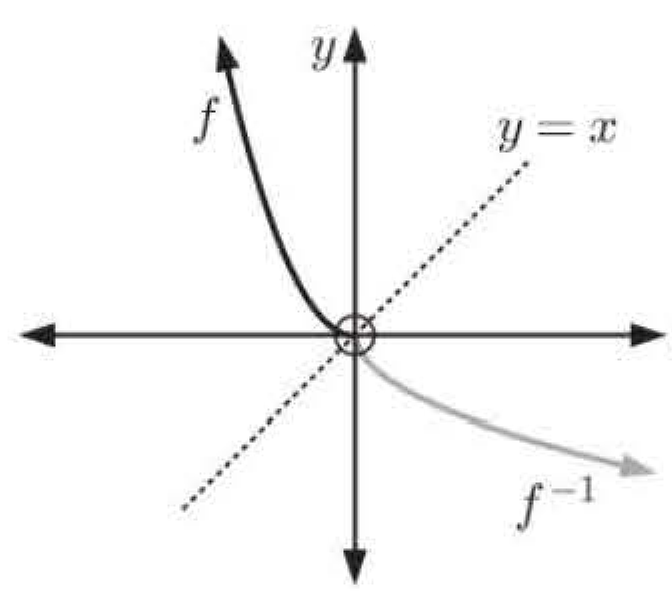
13 **a** f is $y = x^2, \ x \leqslant 0$

so f^{-1} is $x = y^2, \ y \leqslant 0$

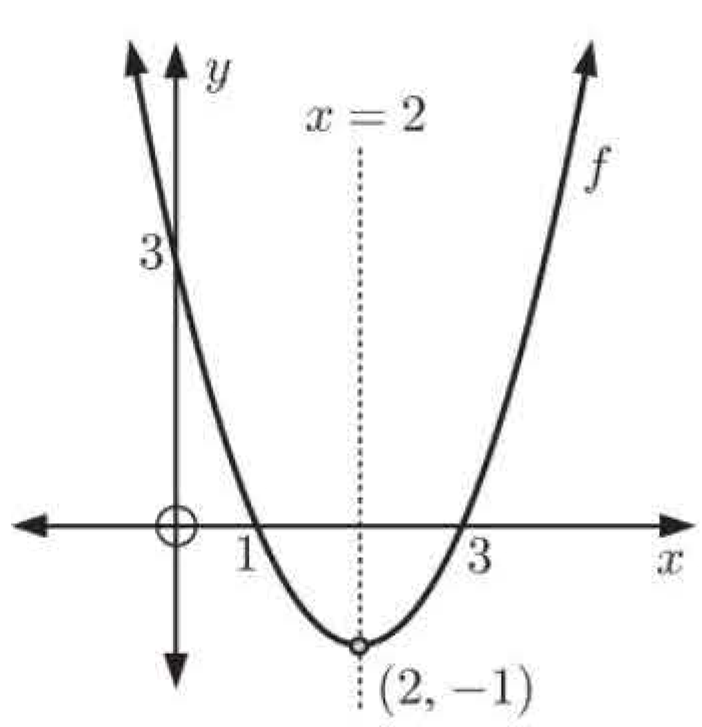
$\therefore \quad y = -\sqrt{x}$

So, $f^{-1}(x) = -\sqrt{x}$

b



14 **a**



$f : x \mapsto x^2 - 4x + 3$ satisfies the 'vertical line test' so is therefore a function. It does not however satisfy the horizontal line test as any horizontal line above the vertex cuts the graph twice. Therefore it does not have an inverse function.

b **i** For $g(x) = x^2 - 4x + 3$ where $x \geqslant 2$, all horizontal lines cut the graph no more than once. Therefore $g(x)$ has an inverse function for $x \geqslant 2$.

ii g is $y = x^2 - 4x + 3, \ x \geqslant 2$

so g^{-1} is $x = y^2 - 4y + 3, \ y \geqslant 2$

$\therefore \quad x = (y-2)^2 - 4 + 3$

$\therefore \quad x = (y-2)^2 - 1$

$\therefore \quad x + 1 = (y-2)^2$

$\therefore \quad y - 2 = \sqrt{x+1}, \ y \geqslant 2, \ x \geqslant -1$

$\therefore \quad y = 2 + \sqrt{1+x}, \ y \geqslant 2, \ x \geqslant -1$

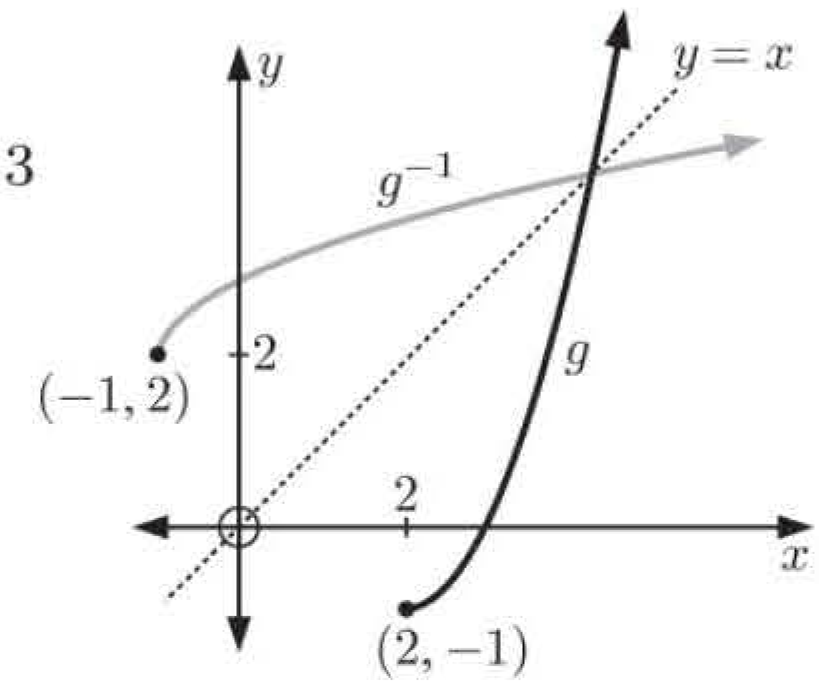
So, $g^{-1}(x) = 2 + \sqrt{1+x}$ as required.

iii **A** Domain of g is $\{x \mid x \geqslant 2\}$.
Range is $\{y \mid y \geqslant -1\}$

B Domain of g^{-1} is $\{x \mid x \geqslant -1\}$.
Range is $\{y \mid y \geqslant 2\}$

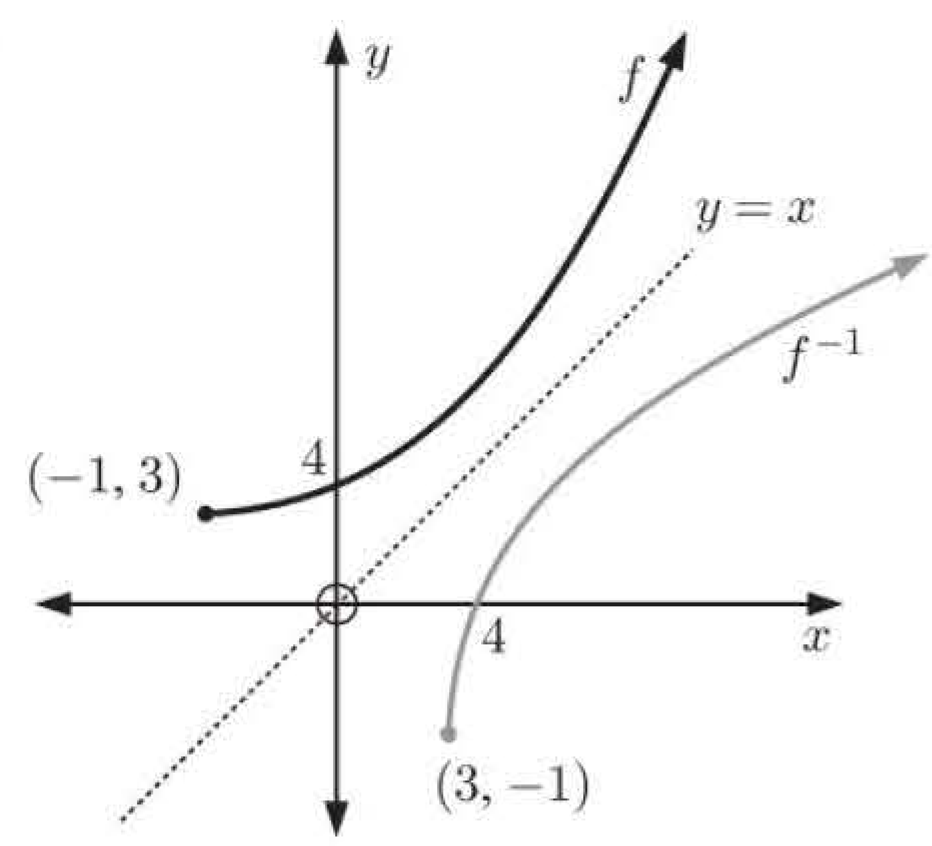
iv $(g \circ g^{-1})(x) = g(g^{-1}(x))$
$= \left(2+\sqrt{1+x}\right)^2 - 4\left(2+\sqrt{1+x}\right)+3$
$= 4+4\sqrt{1+x}+1+x-8-4\sqrt{1+x}+3$
$= x$

$(g^{-1} \circ g)(x) = g^{-1}(g(x))$
$= 2+\sqrt{1+x^2-4x+3}$
$= 2+\sqrt{(x-2)^2}$
$= 2+x-2$
$= x$



15 **a** f is $y=(x+1)^2+3, \quad x \geqslant -1$
so f^{-1} is $x=(y+1)^2+3, \quad y \geqslant -1$
$\therefore \quad x-3=(y+1)^2$
$\therefore \quad y+1=\sqrt{x-3}, \quad y \geqslant -1, \ x \geqslant 3$
$\therefore \quad y=\sqrt{x-3}-1, \quad y \geqslant -1, \ x \geqslant 3$

b



c **i** Domain $\{x \mid x \geqslant -1\}$. Range $\{y \mid y \geqslant 3\}$.
ii Domain $\{x \mid x \geqslant 3\}$. Range $\{y \mid y \geqslant -1\}$.

16 **a** g is $y=\dfrac{8-x}{2}$
so g^{-1} is $x=\dfrac{8-y}{2}$
$\therefore \quad 2x=8-y$
$\therefore \quad y=8-2x$
So, $g^{-1}(x)=8-2x$
$\therefore \quad g^{-1}(-1)=8-2(-1)=10$

b f is $y=2x+5$
so f^{-1} is $x=2y+5$
$\therefore \quad 2y=x-5$
$\therefore \quad y=\dfrac{x-5}{2}$
So, $f^{-1}(x)=\dfrac{x-5}{2}$
$\therefore \quad f^{-1}(-3)=\dfrac{-3-5}{2} = \dfrac{-8}{2} = -4$ and $g^{-1}(6)=8-2\times 6 = 8-12 = -4$
$\therefore \quad f^{-1}(-3)-g^{-1}(6) = -4-(-4) = -4+4 = 0$ as required

c $(f \circ g^{-1})(x)=9$
$\therefore \quad f(g^{-1}(x))=9$
$\therefore \quad f(8-2x)=9$
$\therefore \quad 2(8-2x)+5=9$
$\therefore \quad 16-4x+5=9$
$\therefore \quad -4x=-12$
$\therefore \quad x=3$

17 **a** **i** f is $y=5^x$
so $f(2)=5^2$
$=25$

ii g is $y=\sqrt{x}$ where $y \geqslant 0$
so g^{-1} is $x=\sqrt{y}$ where $x \geqslant 0$
$\therefore \quad y=x^2$
$\therefore \quad g^{-1}(x)=x^2, \ x \geqslant 0$
$\therefore \quad g^{-1}(4)=4^2$
$=16$

b $(g^{-1} \circ f)(x)=25$
$\therefore \quad g^{-1}(f(x))=25$
$\therefore \quad g^{-1}(5^x)=25$
$\therefore \quad (5^x)^2=25 \quad \{\text{as } g^{-1}(x)=x^2, \ x \geqslant 0\}$
$\therefore \quad 5^{2x}=5^2$
$\therefore \quad 2x=2$
$\therefore \quad x=1$

18 f is $y = 2x$
so f^{-1} is $x = 2y$
$\therefore\ y = \dfrac{x}{2}$
$\therefore\ f^{-1}(x) = \dfrac{x}{2}$

g is $y = 4x - 3$
so g^{-1} is $x = 4y - 3$
$\therefore\ 4y = x + 3$
$\therefore\ y = \dfrac{x+3}{4}$
$\therefore\ g^{-1}(x) = \dfrac{x+3}{4}$

$$\begin{aligned}(g \circ f)(x) &= g(f(x))\\ &= g(2x)\\ &= 4(2x) - 3\end{aligned}$$
$\therefore\ (g \circ f)(x) = 8x - 3$
$\therefore\ g \circ f$ is $y = 8x - 3$
so $(g \circ f)^{-1}$ is $x = 8y - 3$
$\therefore\ y = \dfrac{x+3}{8}$
So, $(g \circ f)^{-1}(x) = \dfrac{x+3}{8}$

Now
$$\begin{aligned}(f^{-1} \circ g^{-1})(x) &= f^{-1}(g^{-1}(x))\\ &= f^{-1}\left(\frac{x+3}{4}\right)\\ &= \frac{\left(\frac{x+3}{4}\right)}{2}\end{aligned}$$
$\therefore\ (f^{-1} \circ g^{-1})(x) = \dfrac{x+3}{8} = (g \circ f)^{-1}(x)$ as required

19 **a** f is $y = 2x$
so f^{-1} is $x = 2y$
$\therefore\ y = \dfrac{x}{2}$
so $f^{-1}(x) = \dfrac{x}{2} \neq 2x$
So, $f^{-1}(x) \neq f(x)$

b f is $y = x$
so f^{-1} is $x = y$
$\therefore\ y = x$
so $f^{-1}(x) = x$
So, $f^{-1}(x) = f(x)$

c f is $y = -x$
so f^{-1} is $x = -y$
$\therefore\ y = -x$
so $f^{-1}(x) = -x$
So, $f^{-1}(x) = f(x)$

d f is $y = \dfrac{2}{x}$
so f^{-1} is $x = \dfrac{2}{y}$
$\therefore\ y = \dfrac{2}{x}$
so $f^{-1}(x) = \dfrac{2}{x}$
So, $f^{-1}(x) = f(x)$

e f is $y = -\dfrac{6}{x}$
so f^{-1} is $x = -\dfrac{6}{y}$
$\therefore\ y = -\dfrac{6}{x}$
so $f^{-1}(x) = -\dfrac{6}{x}$
So, $f^{-1}(x) = f(x)$

So, $f^{-1}(x) = f(x)$ is true for parts **b**, **c**, **d**, and **e**.

20 **a** $f(x)$ passes through $A(x, f(x))$, so $f^{-1}(x)$ passes through $B(f(x), x)$.

b Substituting the coordinates of $B(f(x), x)$ into $y = f^{-1}(x)$ gives $f^{-1}(f(x)) = x$.

c B has coordinates $(x, f^{-1}(x))$ since it lies on $y = f^{-1}(x)$,
so A has coordinates $(f^{-1}(x), x)$ as $f(x)$ is the inverse of $f^{-1}(x)$.
Substituting the coordinates of $A(f^{-1}(x), x)$ into $y = f(x)$ gives $x = f(f^{-1}(x))$.
$\therefore\ f(f^{-1}(x)) = x$ as required.

EXERCISE 2K

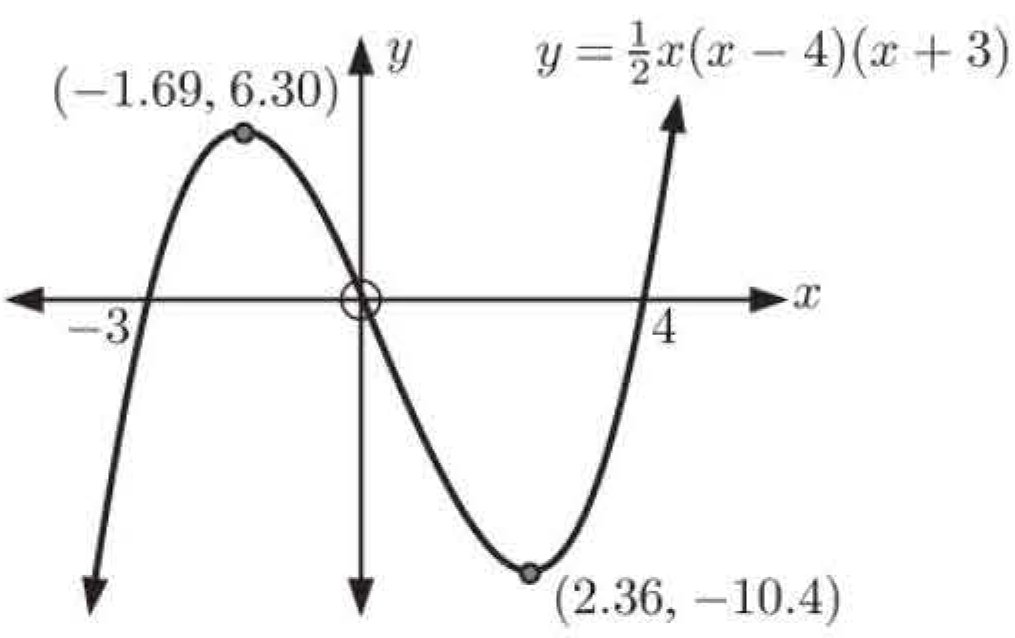
1 **a** **i** x-intercepts are -3, 0, and 4, y-intercept is 0 {using technology}

ii $(-1.69, 6.30)$ is a local maximum, $(2.36, -10.4)$ is a local minimum {using technology}

iii $y = \frac{1}{2}x(x-4)(x+3)$ is defined for all $x \in \mathbb{R}$.
$\therefore$ there are no vertical asymptotes.
As $x \to \infty$, $y \to \infty$
As $x \to -\infty$, $y \to -\infty$
$\therefore$ there are no horizontal asymptotes.

iv Domain is $\{x \mid x \in \mathbb{R}\}$.
Range is $\{y \mid y \in \mathbb{R}\}$.

v



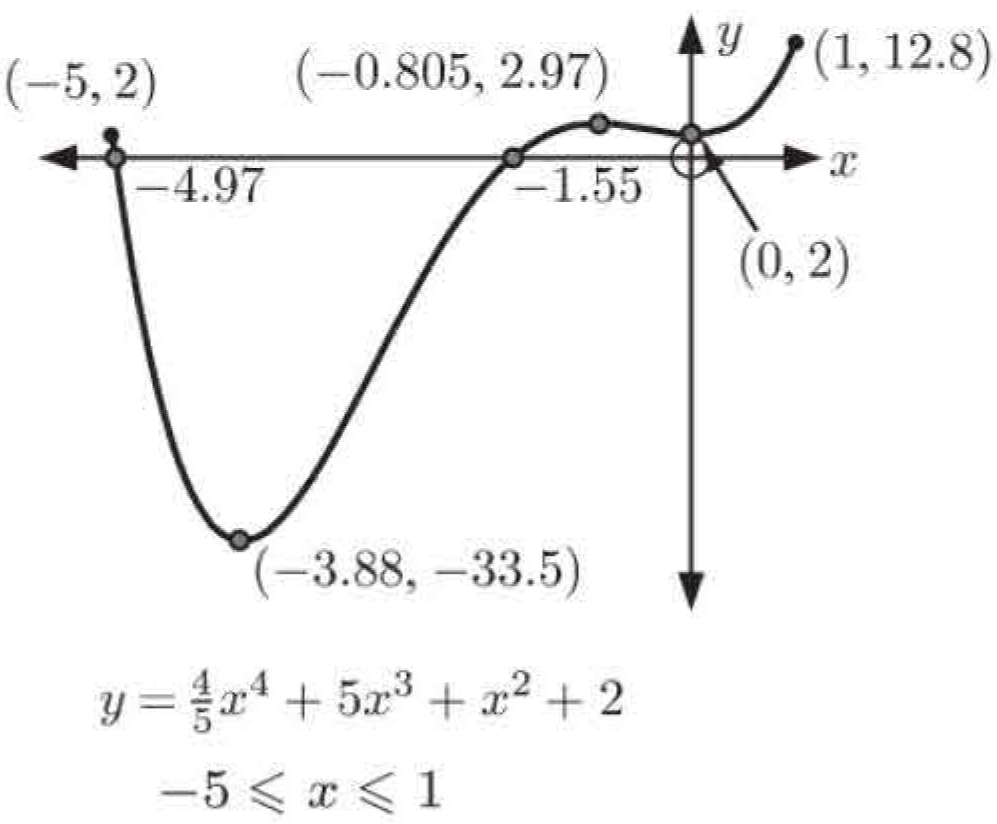
b **i** x-intercepts are ≈ -4.97 and ≈ -1.55, y-intercept is 2 {using technology}

ii $(-3.88, -33.5)$ and $(0, 2)$ are local minima,
$(-0.805, 2.97)$ is a local maximum {using technology}

iii $y = \frac{4}{5}x^4 + 5x^3 + 5x^2 + 2$ is defined for $-5 \leqslant x \leqslant 1$.
$\therefore$ there are no vertical asymptotes.
The function has no horizontal asymptotes.

iv We need to check the endpoints of the function.
When $x = -5$, $y = 2$.
When $x = 1$, $y = 12.8$.
So, the domain is $\{x \mid -5 \leqslant x \leqslant 1\}$,
and the range is $\{y \mid -33.5 \leqslant y \leqslant 12.8\}$.

v



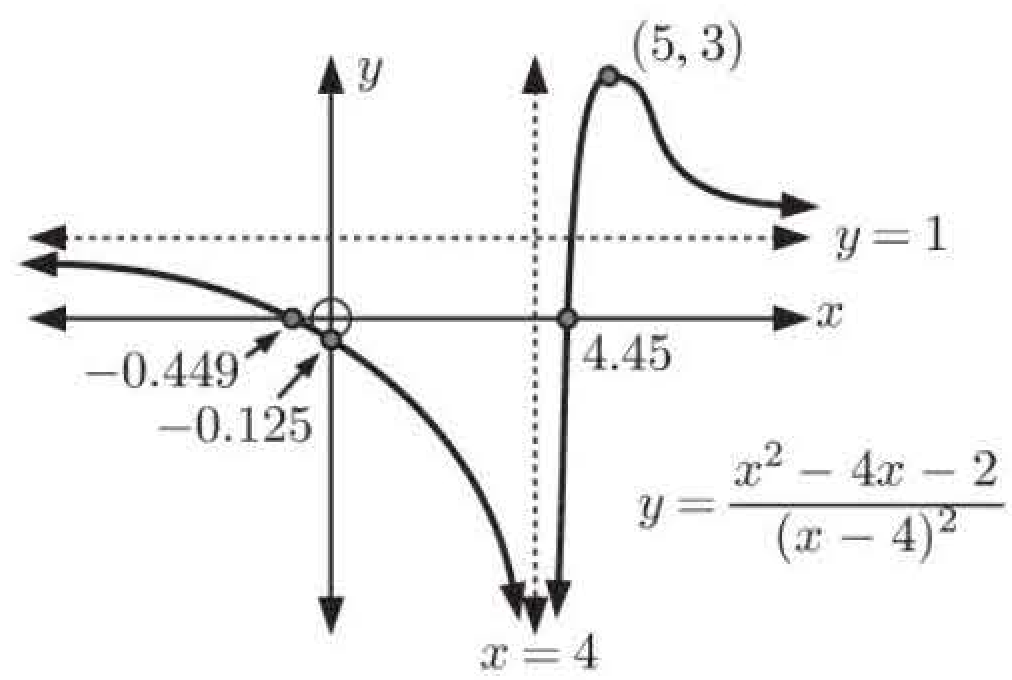
c **i** x-intercepts are ≈ -0.449 and ≈ 4.45, y-intercept is -0.125 {using technology}

ii $(5, 3)$ is a local maximum {using technology}

iii As $x \to 4^-$, $y \to -\infty$
As $x \to 4^+$, $y \to -\infty$
$\therefore$ the vertical asymptote is $x = 4$.
As $x \to \infty$, $y \to 1^+$
As $x \to -\infty$, $y \to 1^-$
$\therefore$ the horizontal asymptote is $y = 1$.

iv Domain is $\{x \mid x \neq 4\}$.
Range is $\{y \mid y \leqslant 3\}$.

v



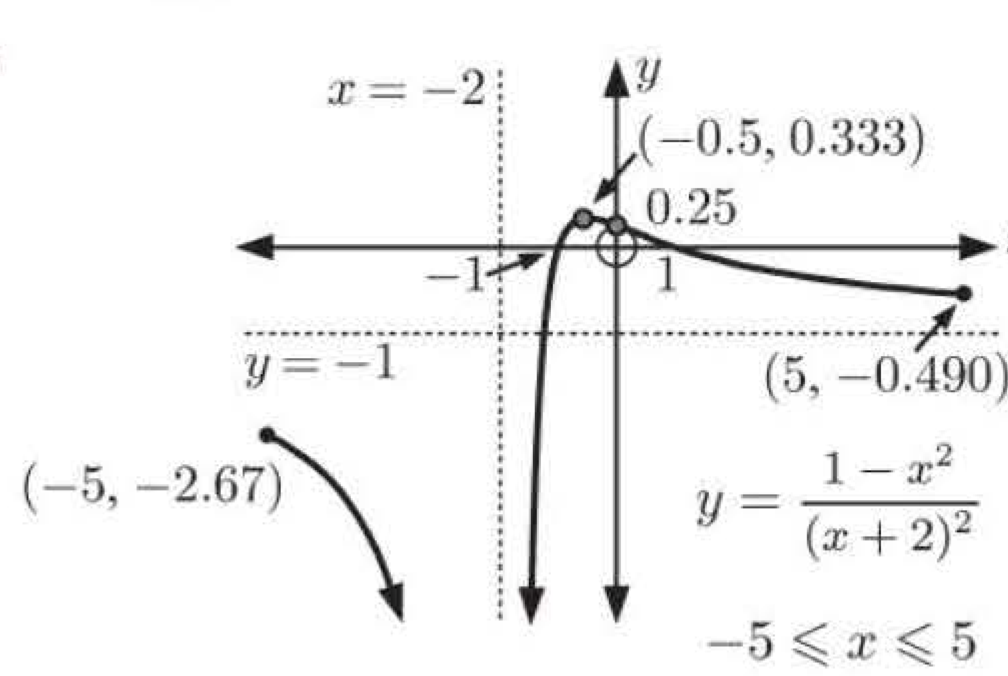
d **i** x-intercepts are -1 and 1, y-intercept is 0.25 {using technology}

ii $(-0.5, 0.333)$ is a local maximum {using technology}

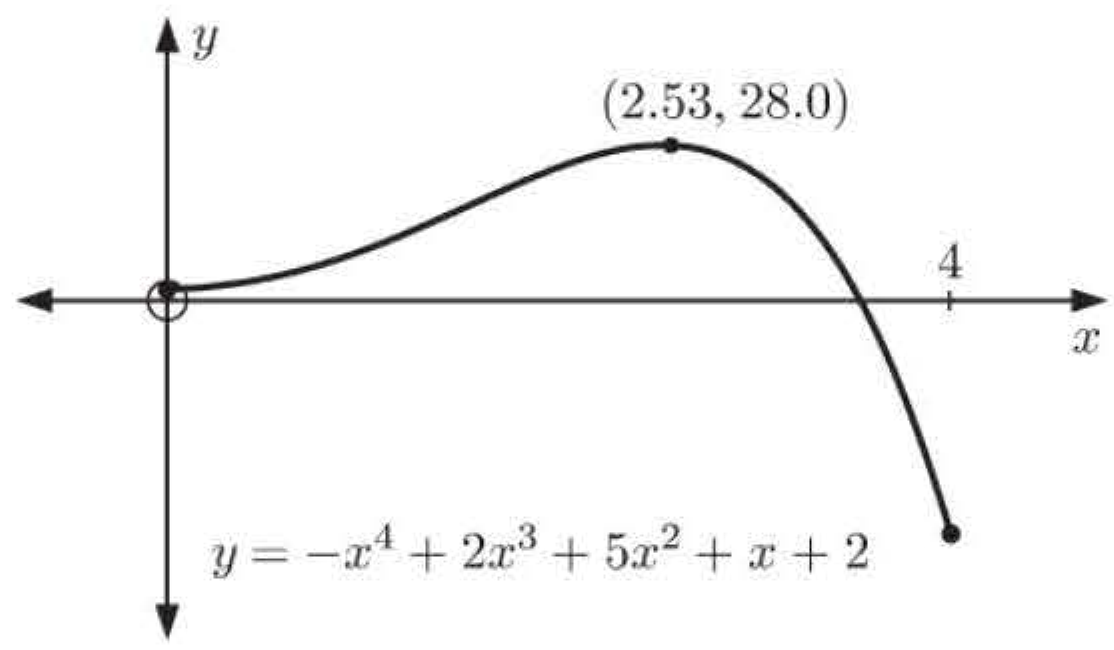
iii As $x \to -2^-$, $y \to -\infty$
As $x \to -2^+$, $y \to -\infty$
$\therefore$ the vertical asymptote is $x = -2$.
On $x \in \mathbb{R}$, $y = \dfrac{1-x^2}{(x+2)^2}$ has horizontal asymptote $y = -1$, but we are only considering the function on the domain $-5 \leqslant x \leqslant 5$.
Strictly speaking there is no horizontal asymptote on this domain.

iv Domain is $\{x \mid -5 \leqslant x \leqslant 5, \ x \neq 2\}$.
Range is $\{y \mid y \leqslant 0.333\}$.

v

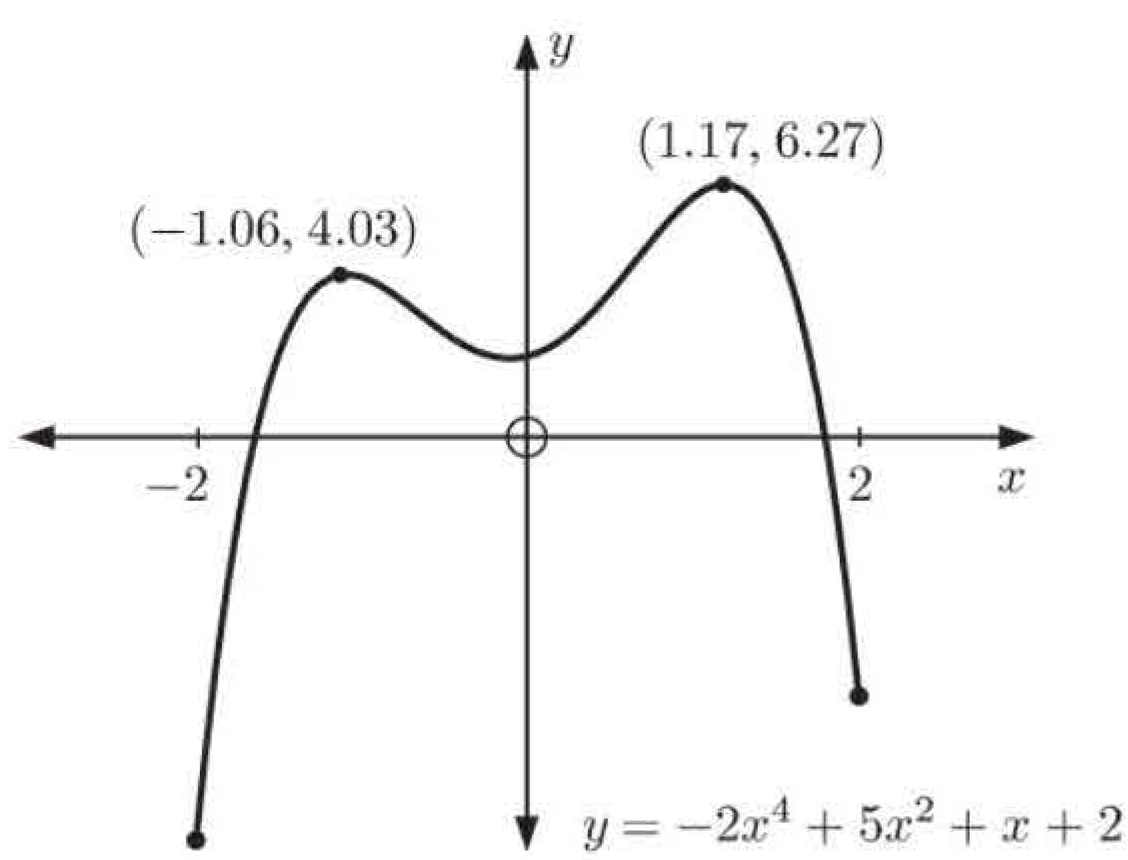


2 **a** Graphing $y = -x^4 + 2x^3 + 5x^2 + x + 2$ on $0 \leqslant x \leqslant 4$ using technology:



Clearly the maximum value occurs at the local maximum, which is $(2.53, 28.0)$, so the maximum value is 28.0.

b Graphing $y = -2x^4 + 5x^2 + x + 2$ on $-2 \leqslant x \leqslant 2$ using technology:



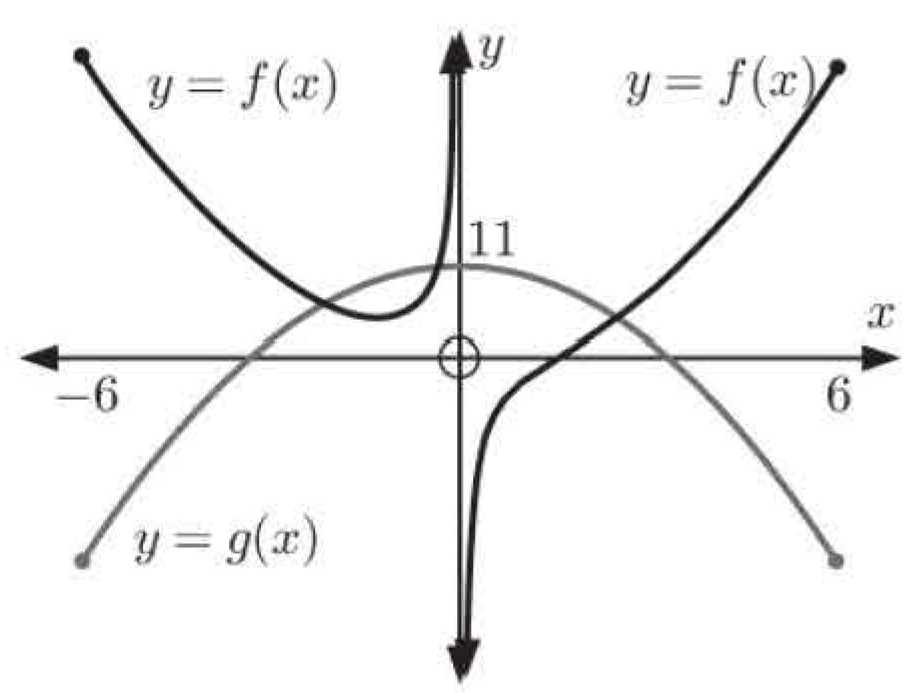
i The higher of the two maxima $(1.17, 6.27)$ is the global maximum on $-2 \leqslant x \leqslant 2$, so the maximum value is 6.27.

ii The global maximum on $-2 \leqslant x \leqslant 0$ occurs at the lower maximum $(-1.06, 4.03)$, so the maximum value on this interval is 4.03.

iii The higher maximum $(1.17, 6.27)$ is in $0 \leqslant x \leqslant 2$, so the maximum value on this interval is 6.27.

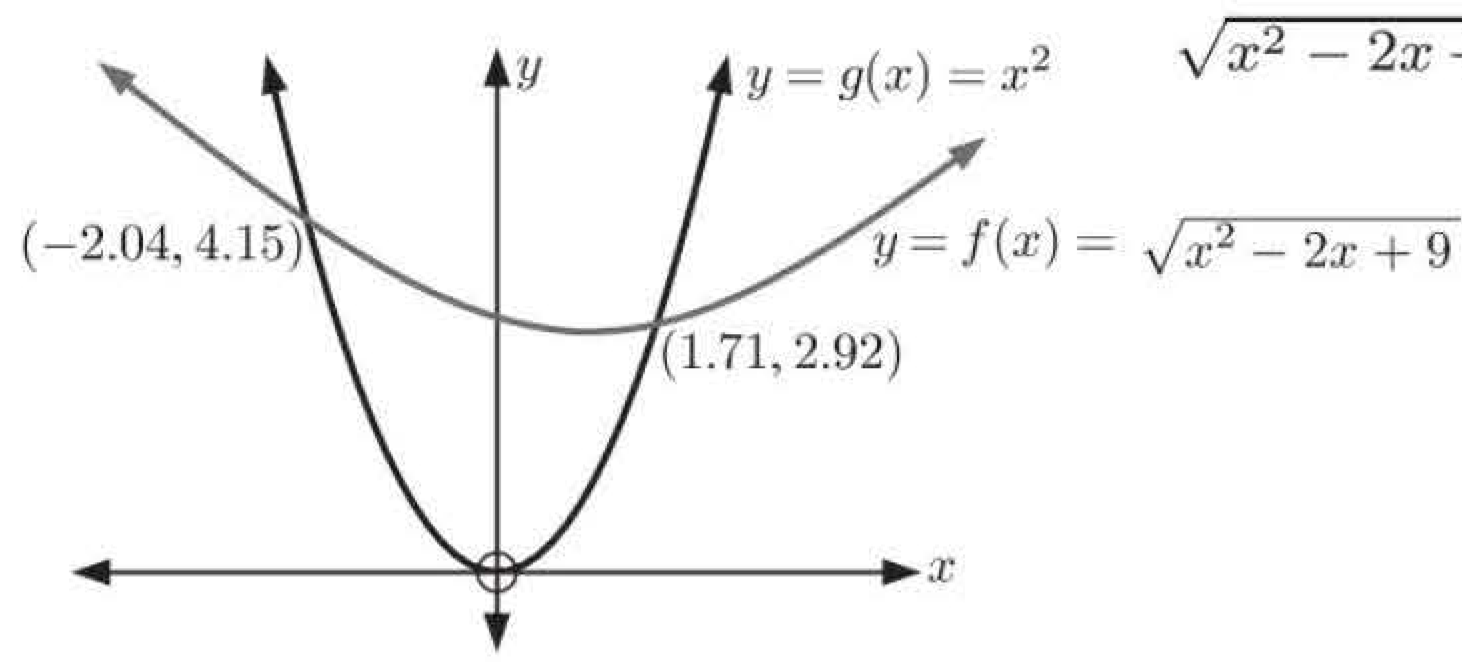
EXERCISE 2L

1 **a**



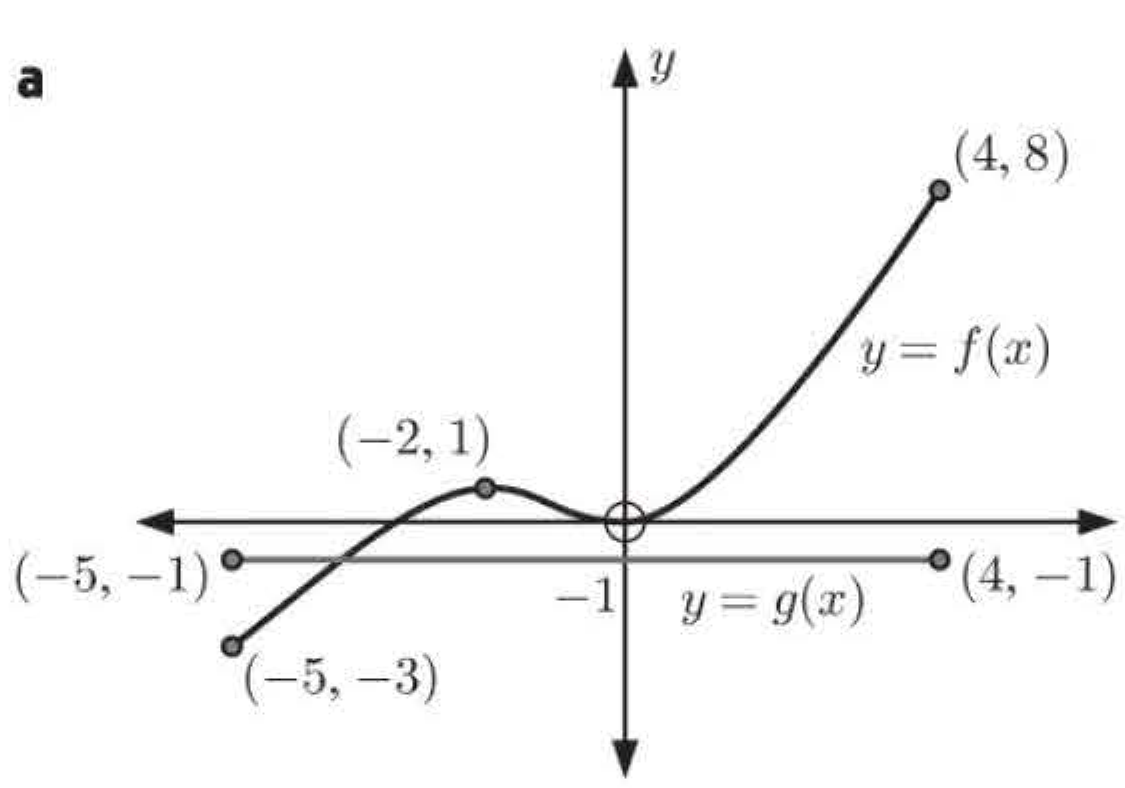
b Using technology, the other solutions are $x \approx -2.14$ and $x \approx -0.373$.

2



$\sqrt{x^2 - 2x + 9} = x^2$ when $x \approx -2.04$ or 1.71

3 **a**



b $f(x)$ and $g(x)$ intersect at one point, so $f(x) = g(x)$ has one solution on the domain $-5 \leqslant x \leqslant 4$.

c **i** $f(x) = h(x)$ on the domain $-5 \leqslant x \leqslant 4$ has three solutions when $0 < k < 1$.

ii $f(x) = h(x)$ on the domain $-5 \leqslant x \leqslant 4$ has two solutions when $k = 0$ or $k = 1$.

iii $f(x) = h(x)$ on the domain $-5 \leqslant x \leqslant 4$ has one solution when $-3 \leqslant k < 0$ or $1 < k \leqslant 8$.

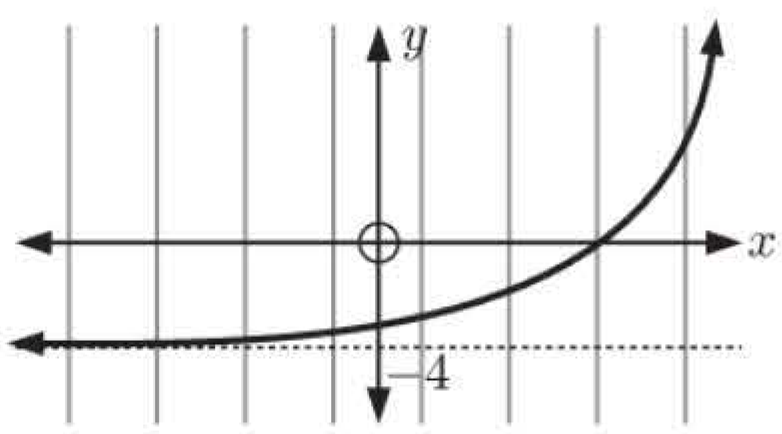
iv $f(x) = h(x)$ on the domain $-5 \leqslant x \leqslant 4$ has no solutions when $k < -3$ or $k > 8$.

REVIEW SET 2A

1 **a** **i** Domain is $\{x \mid x \in \mathbb{R}\}$.

ii Range is $\{y \mid y > -4\}$.

iii

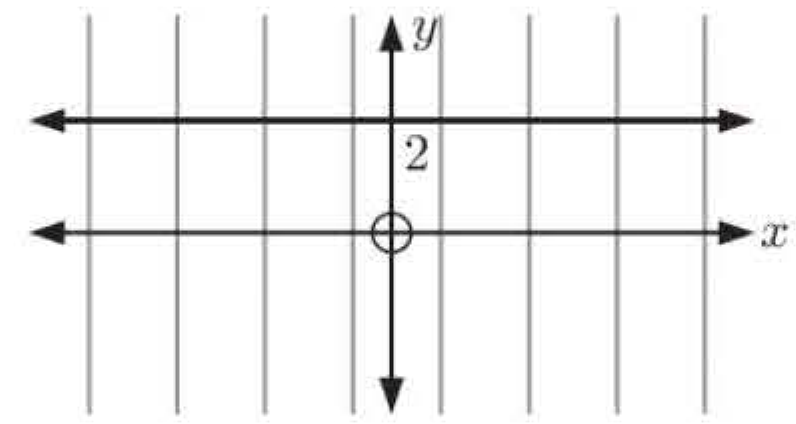


Each line cuts the graph no more than once, so the graph shows a function.

b **i** Domain is $\{x \mid x \in \mathbb{R}\}$.

ii Range is $\{2\}$.

iii

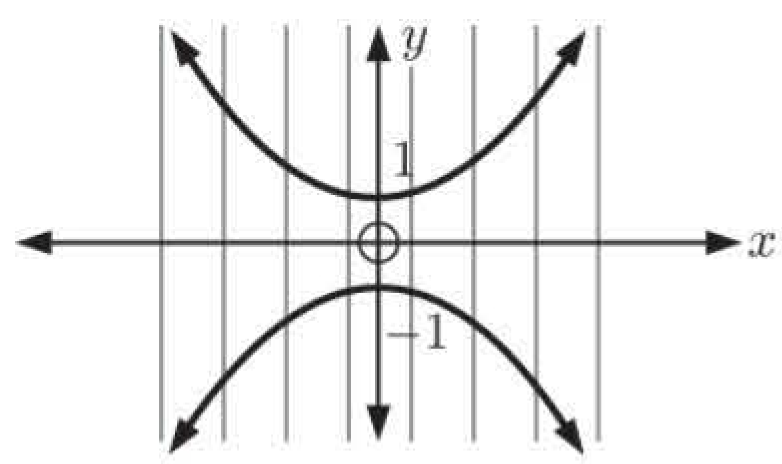


Each line cuts the graph no more than once, so the graph shows a function.

c **i** Domain is $\{x \mid x \in \mathbb{R}\}$.

ii Range is $\{y \mid y \leqslant -1 \text{ or } y \geqslant 1\}$.

iii

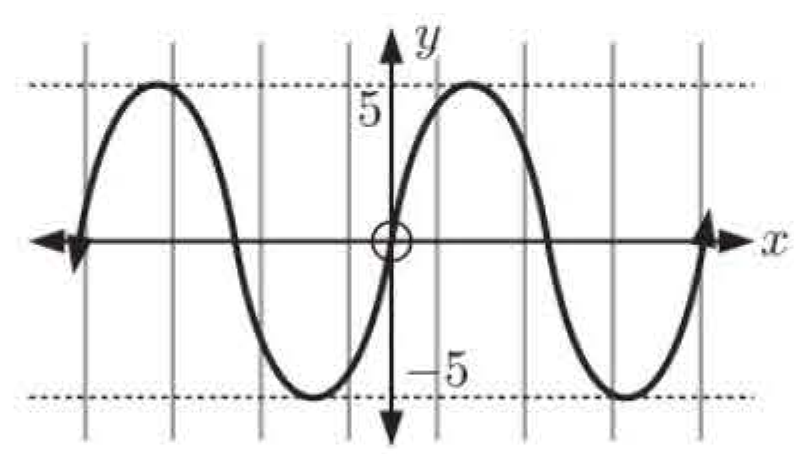


The lines cut the graph more than once, so the graph does not show a function.

d **i** Domain is $\{x \mid x \in \mathbb{R}\}$.

ii Range is $\{y \mid -5 \leqslant y \leqslant 5\}$.

iii



Each line cuts the graph no more than once, so the graph shows a function.

2 **a** $f(x) = 2x - x^2$
$$\begin{aligned} f(2) &= 2(2) - 2^2 \\ &= 0 \end{aligned}$$

b $$\begin{aligned} f(-3) &= 2(-3) - (-3)^2 \\ &= -6 - 9 \\ &= -15 \end{aligned}$$

c $$\begin{aligned} f(-\tfrac{1}{2}) &= 2(-\tfrac{1}{2}) - (-\tfrac{1}{2})^2 \\ &= -1 - \tfrac{1}{4} \\ &= -\tfrac{5}{4} \end{aligned}$$

3 $f(x) = ax + b$, where $f(1) = 7$ and $f(3) = -5$

When $f(1) = 7$,
$$\begin{aligned} 7 &= a(1) + b \\ \therefore\ 7 &= a + b \\ \therefore\ a &= 7 - b \quad \text{.... (1)} \end{aligned}$$

When $f(3) = -5$,
$$\begin{aligned} -5 &= a(3) + b \\ \therefore\ -5 &= 3a + b \\ \therefore\ -5 &= 3(7 - b) + b \quad \{\text{using (1)}\} \\ \therefore\ -5 &= 21 - 3b + b \\ \therefore\ 2b &= 26 \quad \text{and so} \quad b = 13 \end{aligned}$$

Substituting $b = 13$ into (1), $a = 7 - 13 = -6$

$\therefore\ a = -6$ and $b = 13$

4 **a** $$\begin{aligned} f(g(x)) &= \sqrt{1 - x^2} \\ &= f(1 - x^2) \end{aligned}$$

So, $f(x) = \sqrt{x}$
$g(x) = 1 - x^2$

Note: There are other possible functions f and g.

b $$\begin{aligned} g(f(x)) &= \left(\frac{x-2}{x+1}\right)^2 \\ &= g\left(\frac{x-2}{x+1}\right) \end{aligned}$$

So, $g(x) = x^2$
$f(x) = \dfrac{x-2}{x+1}$

5 **a** $|4x - 2| = |x + 7|$
$\therefore\ 4x - 2 = \pm(x + 7)$

If $4x - 2 = x + 7$
then $3x = 9$
$\therefore\ x = 3$

If $4x - 2 = -(x + 7)$
then $4x - 2 = -x - 7$
$\therefore\ 5x = -5$
$\therefore\ x = -1$

So, $x = -1$ or 3

b $x^2 + 6x > 16$

$\therefore\ x^2 + 6x - 16 > 0$

$\therefore\ (x+8)(x-2) > 0$

Sign diagram of LHS is: 

$\therefore\ x^2 + 6x - 16 > 0$ when $x \in\ ]-\infty, -8[$ or $]2, \infty[$

$x^2 + 6x > 16$ when $x \in\ ]-\infty, -8[$ or $]2, \infty[$

6 $g(x) = x^2 - 3x$

a
$$\begin{aligned} g(x+1) &= (x+1)^2 - 3(x+1) \\ &= x^2 + 2x + 1 - 3x - 3 \\ &= x^2 - x - 2 \end{aligned}$$

b
$$\begin{aligned} g(x^2-2) &= (x^2-2)^2 - 3(x^2-2) \\ &= x^4 - 4x^2 + 4 - 3x^2 + 6 \\ &= x^4 - 7x^2 + 10 \end{aligned}$$

7 **a** **i** Domain is $\{x \mid x \in \mathbb{R}\}$. Range is $\{y \mid y \geqslant -5\}$.

ii x-intercepts are -1 and 5, y-intercept is $-\frac{25}{9}$

iii The graph passes the 'vertical line test' so is therefore a function.

iv The graph does not pass the 'horizontal line test' so it does not have an inverse function.

b **i** Domain is $\{x \mid x \in \mathbb{R}\}$. Range is $\{y \mid y = 1 \text{ or } -3\}$.

ii There are no x-intercepts, y-intercept is 1.

iii The graph passes the 'vertical line test' so is therefore a function.

iv The graph does not pass the 'horizontal line test' so it does not have an inverse function.

8 **a**
$$\begin{aligned} f(x) &= -\frac{4}{x} \\ \therefore\ f(-x) &= -\frac{4}{-x} \\ &= \frac{4}{x} \\ &= -f(x) \end{aligned}$$

$\therefore\ f(x) = -\dfrac{4}{x}$ is an odd function.

b
$$\begin{aligned} f(x) &= \frac{2x-3}{x+1} \\ \therefore\ f(-x) &= \frac{2(-x)-3}{-x+1} \\ &= \frac{-2x-3}{-x+1} \end{aligned}$$

which is neither $f(x)$ or $-f(x)$.

$\therefore\ f(x) = \dfrac{2x-3}{x+1}$ is neither even nor odd.

c
$$\begin{aligned} f(x) &= \sqrt{x^2-5} \\ \therefore\ f(-x) &= \sqrt{(-x)^2-5} \\ &= \sqrt{x^2-5} \\ &= f(x) \end{aligned}$$

$\therefore\ f(x) = \sqrt{x^2-5}$ is an even function.

9 **a**
f is $y = 4x + 2$

$\therefore\ f^{-1}(x)$ is $x = 4y + 2$

$\therefore\ y = \dfrac{x-2}{4}$

$\therefore\ f^{-1}(x) = \dfrac{x-2}{4}$

b
f is $y = \dfrac{3-5x}{4}$

so $f^{-1}(x)$ is $x = \dfrac{3-5y}{4}$

$\therefore\ 4x = 3 - 5y$

$\therefore\ y = \dfrac{3-4x}{5}$

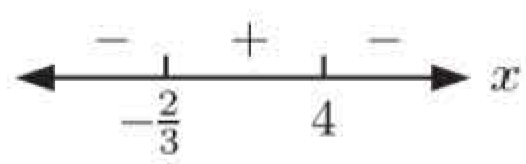
$\therefore\ f^{-1}(x) = \dfrac{3-4x}{5}$

10 **a** $y = (3x+2)(4-x)$ is zero when $x = -\frac{2}{3}$ or 4.

When $x = 0$, $y = (2)(4) = 8 > 0$.

Since the factors are distinct and linear, the signs alternate.

$\therefore$ sign diagram is

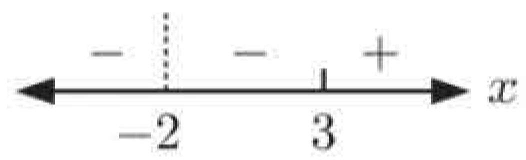


b $y = \dfrac{x-3}{x^2+4x+4} = \dfrac{x-3}{(x+2)^2}$ is zero when $x = 3$ and undefined when $x = -2$.

When $x = 0$, $y = \dfrac{-3}{2^2} = -\frac{3}{4} < 0$.

Since the $(x+2)$ factor is squared, the sign does not change at $x = -2$

$\therefore$ sign diagram is



11 $f(x) = ax + b$

Now $f(2) = 1$, so $a(2) + b = 1$

$\therefore\ b = 1 - 2a \quad .... (*)$

Now $f^{-1}(3) = 4$, so

$$\begin{aligned} f(4) &= 3 \\ \therefore\ a(4) + b &= 3 \\ \therefore\ 4a + (1-2a) &= 3 \qquad \{\text{from } (*)\} \\ \therefore\ 2a &= 2 \\ \therefore\ a &= 1 \end{aligned}$$

Substituting $a = 1$ into $(*)$, $b = 1 - 2(1) = -1$

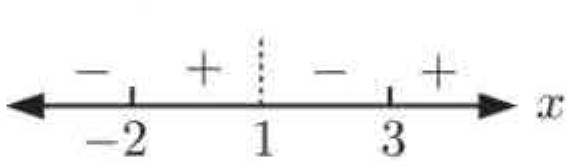
So, $a = 1$ and $b = -1$.

12 **a** $y = \dfrac{(x+2)(x-3)}{x-1}$ is zero when $x = -2$ or 3, and undefined when $x = 1$.

When $x = 0$, $y = \frac{(2)(-3)}{-1} = 6 > 0$.

Since the factors are distinct and linear, the signs alternate.

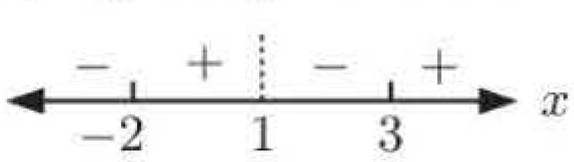
$\therefore$ the sign diagram is



b $\dfrac{x^2-x-6}{x-1} < 0$

$\therefore\ \dfrac{(x+2)(x-3)}{x-1} < 0$

From **a**, sign diagram of LHS is



$\therefore$ LHS < 0 when $x \in \,]-\infty, -2[$ or $]1, 3[$

13 **a**

$$\begin{aligned} f(x) &= x^2 \\ \therefore\ f(-3) &= (-3)^2 = 9 \\ g(x) &= 1 - 6x \\ \therefore\ g(-\tfrac{4}{3}) &= 1 - 6(-\tfrac{4}{3}) \\ &= 1 + 8 = 9 \end{aligned}$$

$\therefore\ f(-3) = g(-\frac{4}{3})$

b $(f \circ g)(-2) = f(g(-2))$

Now, $g(-2) = 1 - 6(-2)$
$= 13$

$$\begin{aligned} \therefore\ (f \circ g)(-2) &= f(13) \\ &= 13^2 \\ &= 169 \end{aligned}$$

c $f(5) = 5^2 = 25$

So, we need to find x such that $1 - 6x = 25$

$\therefore\ -6x = 24$

$\therefore\ x = -4$

14 f is $y = 3x + 6$

so $f^{-1}(x)$ is $x = 3y + 6$

$\therefore\ y = \dfrac{x-6}{3}$

$\therefore\ f^{-1}(x) = \dfrac{x-6}{3}$

h is $y = \dfrac{x}{3}$

so $h^{-1}(x)$ is $x = \dfrac{y}{3}$

$\therefore\ y = 3x$

$\therefore\ h^{-1}(x) = 3x$

Now $(f^{-1}\circ h^{-1})(x) = f^{-1}(h^{-1}(x))$
$$= f^{-1}(3x)$$
$$= \frac{3x-6}{3}$$
$$= x-2$$

$$(h\circ f)(x) = h(f(x))$$
$$= h(3x+6)$$
$$= \frac{3x+6}{3}$$
$\therefore\ h\circ f$ is $y = x+2$
so $(h\circ f)^{-1}(x)$ is $x = y+2$
$\therefore\ y = x-2$
$\therefore\ (h\circ f)^{-1}(x) = x-2$

$\therefore\ (f^{-1}\circ h^{-1})(x) = (h\circ f)^{-1}(x)$ as required.

15 **a** $h(x) = (x-4)^2+3,\quad x \geqslant 4$

$\therefore\ y = (x-4)^2+3,\quad x\geqslant 4$

$\therefore\ h^{-1}(x)$ is $x = (y-4)^2+3,\quad y\geqslant 4$

$\therefore\ x-3 = (y-4)^2$

$\therefore\ y-4 = \pm\sqrt{x-3}$

$\therefore\ y = 4\pm\sqrt{x-3}$

But $y\geqslant 4$, so $y = 4+\sqrt{x-3}$

$\therefore\ h^{-1}(x) = 4+\sqrt{x-3},\quad x\geqslant 3$

b
$$\begin{aligned}&(h\circ h^{-1})(x)\\ &= h\left(h^{-1}(x)\right)\\ &= h(4+\sqrt{x-3})\\ &= (4+\sqrt{x-3}-4)^2+3\\ &= \left(\sqrt{x-3}\right)^2+3\\ &= x-3+3\\ &= x\end{aligned}$$

$$\begin{aligned}&(h^{-1}\circ h)(x)\\ &= h^{-1}(h(x))\\ &= h^{-1}\left((x-4)^2+3\right)\\ &= 4+\sqrt{(x-4)^2+3-3}\\ &= 4+\sqrt{(x-4)^2}\\ &= 4+x-4 \qquad \{\text{as } x\geqslant 4\}\\ &= x\end{aligned}$$

$\therefore\ (h\circ h^{-1})(x) = (h^{-1}\circ h)(x) = x$

REVIEW SET 2B

1 **a** $y = (x-1)(x-5)$

$\therefore$ the x-intercepts are $x = 1$ and 5

The vertex is at $x = 3$, with $y = (3-1)(3-5) = 2\times(-2) = -4$

$\therefore$ the vertex is at $(3, -4)$

The domain is $\{x \mid x\in\mathbb{R}\}$. The range is $\{y \mid y\geqslant -4\}$.

b From the graph, the domain is $\{x \mid x\neq 0,\ x\neq 2\}$ and the range is $\{y \mid y\leqslant -1$ or $y>0\}$.

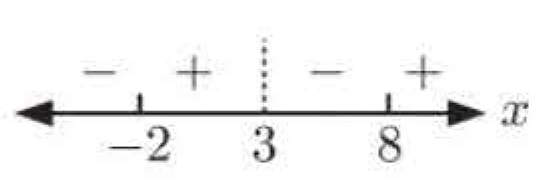
2 **a**
$$\begin{aligned}(f\circ g)(x) &= f(g(x))\\ &= f(x^2+2)\\ &= 2(x^2+2)-3\\ &= 2x^2+4-3\\ &= 2x^2+1\end{aligned}$$

b
$$\begin{aligned}(g\circ f)(x) &= g(f(x))\\ &= g(2x-3)\\ &= (2x-3)^2+2\\ &= 4x^2-12x+9+2\\ &= 4x^2-12x+11\end{aligned}$$

3 **a** $y = \dfrac{x^2-6x-16}{x-3} = \dfrac{(x+2)(x-8)}{x-3}$ is zero when $x = -2$ or 8 and undefined when $x = 3$.

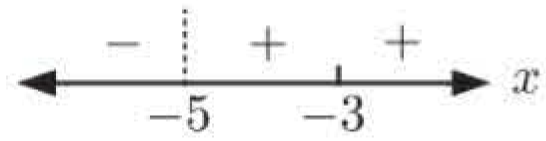
When $x = 0$, $y = \frac{-16}{-3} > 0$.

Since the factors are single, the signs alternate. So, the sign diagram is:

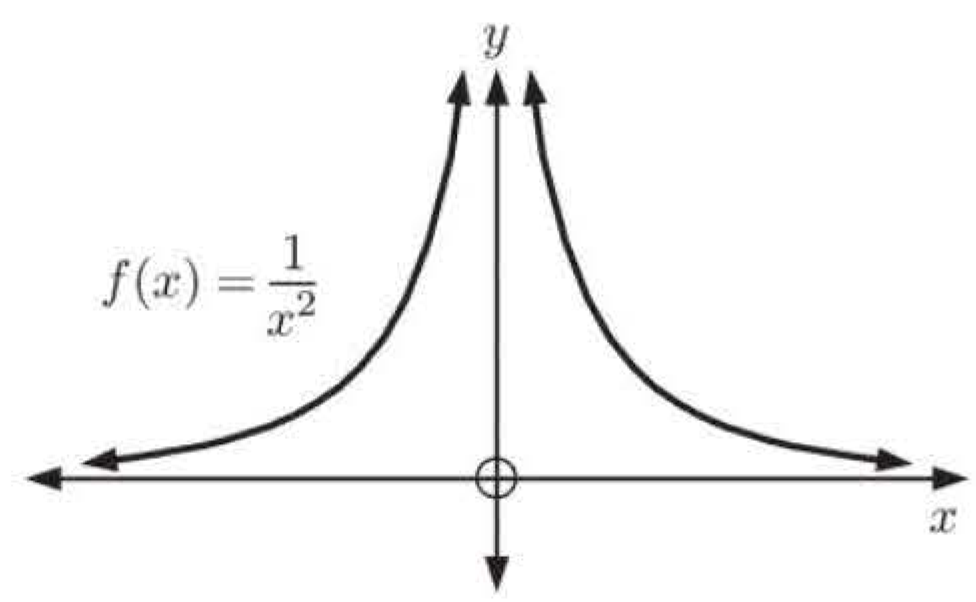


b $y = \dfrac{x+9}{x+5} + x\left(\dfrac{x+5}{x+5}\right) = \dfrac{x^2+6x+9}{x+5} = \dfrac{(x+3)^2}{x+5}$ is zero when $x = -3$ and undefined when $x = -5$.

When $x = 0$, $y = \dfrac{3^2}{5} > 0$. The $(x+3)$ factor is squared, so the sign does not change at $x = -3$.

So, the sign diagram is: 

4 **a** $f(x) = \dfrac{1}{x^2}$ is meaningless when $x = 0$.

b 

c Domain of $f(x)$ is $\{x \mid x \neq 0\}$.
Range of $f(x)$ is $\{y \mid y > 0\}$.

5 **a** $f(x) = \dfrac{ax+3}{x-b}$ has asymptotes $x = -1$, $y = 2$.

$f(x)$ is undefined when $x - b = 0$

$\therefore\ x = b$ is the vertical asymptote.

But $x = -1$ is the vertical asymptote, so $b = -1$.

So, $f(x) = \dfrac{ax+3}{x-(-1)} = \dfrac{ax+3}{x+1} = \dfrac{a+\frac{3}{x}}{1+\frac{1}{x}}$

As $|x| \to \infty$, $f(x) \to \dfrac{a}{1} = a$ so the horizontal asymptote is $y = a$.

But $y = 2$ is the horizontal asymptote, so $a = 2$.

$\therefore\ a = 2$ and $b = -1$.

b Domain of f is $\{x \mid x \neq -1\}$ and range of f is $\{y \mid y \neq 2\}$.

$\therefore$ domain of f^{-1} is $\{x \mid x \neq 2\}$ and range of f^{-1} is $\{y \mid y \neq -1\}$.

6 **a** Domain is $\{x \mid x > -3\}$.
Range is $\{y \mid -3 < y < 5\}$.

b Domain is $\{x \mid x \neq 1\}$.
Range is $\{y \mid y \leqslant -3 \text{ or } y \geqslant 5\}$.

7 **a** $\left|\dfrac{2x+1}{x-2}\right| = 3$

$\therefore\ \dfrac{2x+1}{x-2} = \pm 3$

If $\dfrac{2x+1}{x-2} = 3$

then $2x+1 = 3x-6$

$\therefore\ -x = -7$

$\therefore\ x = 7$

If $\dfrac{2x+1}{x-2} = -3$

then $2x+1 = -3x+6$

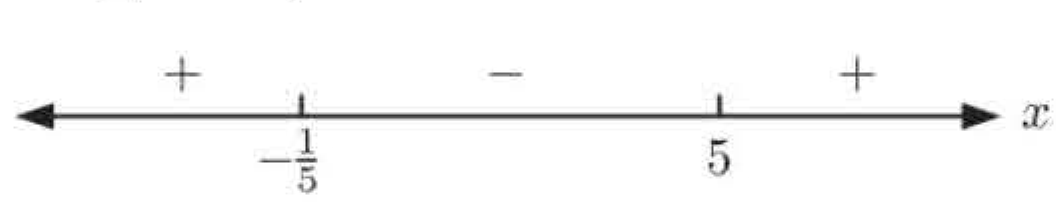
$\therefore\ 5x = 5$

$\therefore\ x = 1$

So, $x = 1$ or 7

b

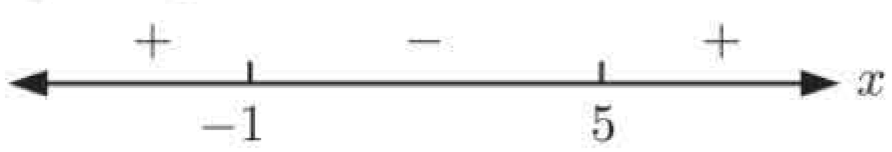
$$\begin{aligned}
|3x-2| &\geqslant |2x+3| \\
\therefore\ |3x-2|^2 &\geqslant |2x+3|^2 \\
\therefore\ (3x-2)^2 - (2x+3)^2 &\geqslant 0 \\
\therefore\ 9x^2 - 12x + 4 - (4x^2+12x+9) &\geqslant 0 \\
\therefore\ 5x^2 - 24x - 5 &\geqslant 0 \\
\therefore\ (5x+1)(x-5) &\geqslant 0
\end{aligned}$$

Sign diagram of LHS is 

$\therefore$ LHS $\geqslant 0$ when $x \in\ ]-\infty, -\frac{1}{5}]$ or $[5, \infty[$

8 **a**
$$x^2 - 5 \leqslant 4x$$
$$\therefore\ x^2 - 4x - 5 \leqslant 0$$
$$\therefore\ (x+1)(x-5) \leqslant 0$$

Sign diagram of LHS is

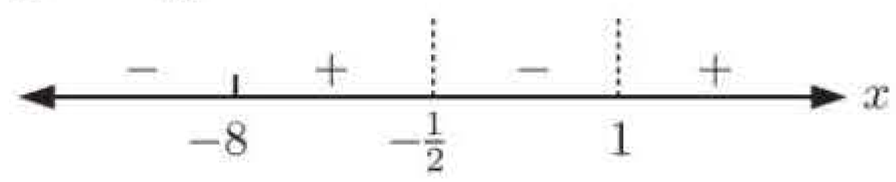


$\therefore\ x^2 - 4x - 5 \leqslant 0$ when $x \in [-1, 5]$

$\therefore\ x^2 - 5 \leqslant 4x$ when $x \in [-1, 5]$.

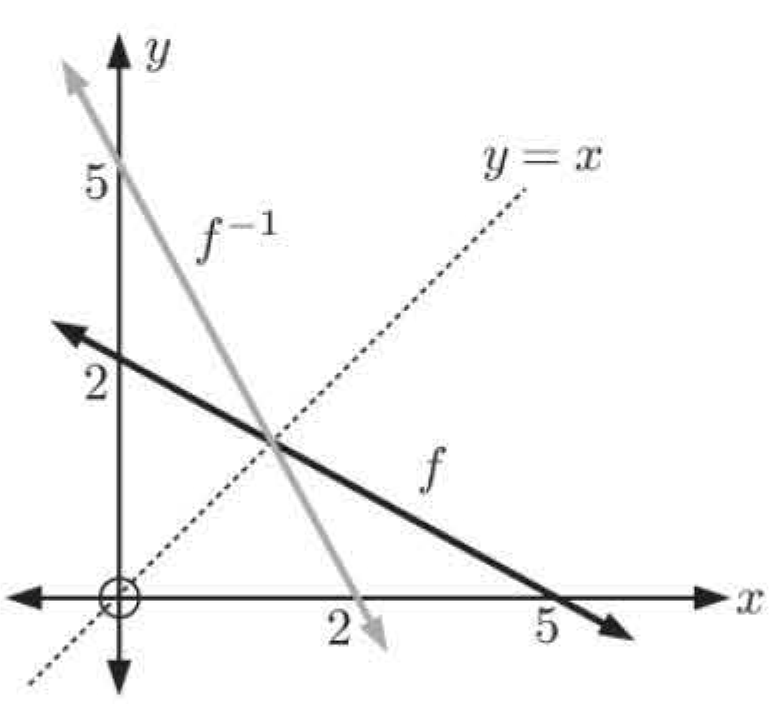
b
$$\frac{3}{x-1} > \frac{5}{2x+1}$$
$$\therefore\ \frac{3}{x-1} - \frac{5}{2x+1} > 0$$
$$\therefore\ \frac{3(2x+1) - 5(x-1)}{(x-1)(2x+1)} > 0$$
$$\therefore\ \frac{x+8}{(x-1)(2x+1)} > 0$$

Sign diagram of LHS is

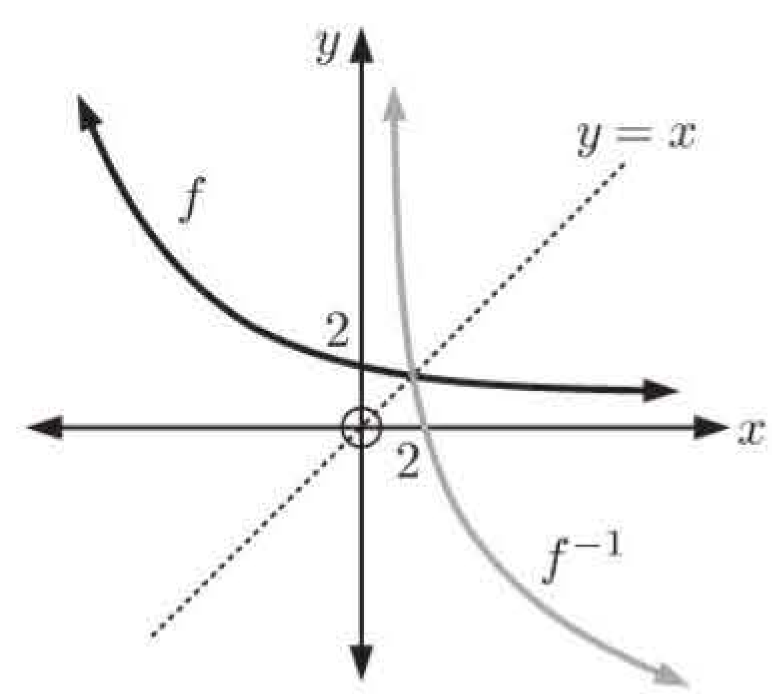


$\therefore$ LHS > 0 when $x \in \left]-8, -\frac{1}{2}\right[$ or $]1, \infty[$

9 **a**



b



10 **a** $f : x \mapsto \dfrac{4x+1}{2-x}$ is undefined when $x = 2$

$\therefore\ x = 2$ is a vertical asymptote.

Now $f(x) = \dfrac{4x+1}{2-x} = \dfrac{4 + \frac{1}{x}}{-1 + \frac{2}{x}}$

$\therefore$ as $|x| \to \infty$, $f(x) \to \frac{4}{-1} = -4$,

and so $y = -4$ is a horizontal asymptote.

b The domain is $\{x \mid x \neq 2\}$.

The range is $\{y \mid y \neq -4\}$.

c As $x \to 2^-$, $y \to \infty$.

As $x \to 2^+$, $y \to -\infty$.

As $x \to \infty$, $y \to -4^-$.

As $x \to -\infty$, $y \to -4^+$.

d $f(0) = \dfrac{4(0)+1}{2-0} = \frac{1}{2}$

So, the y-intercept is $\frac{1}{2}$.

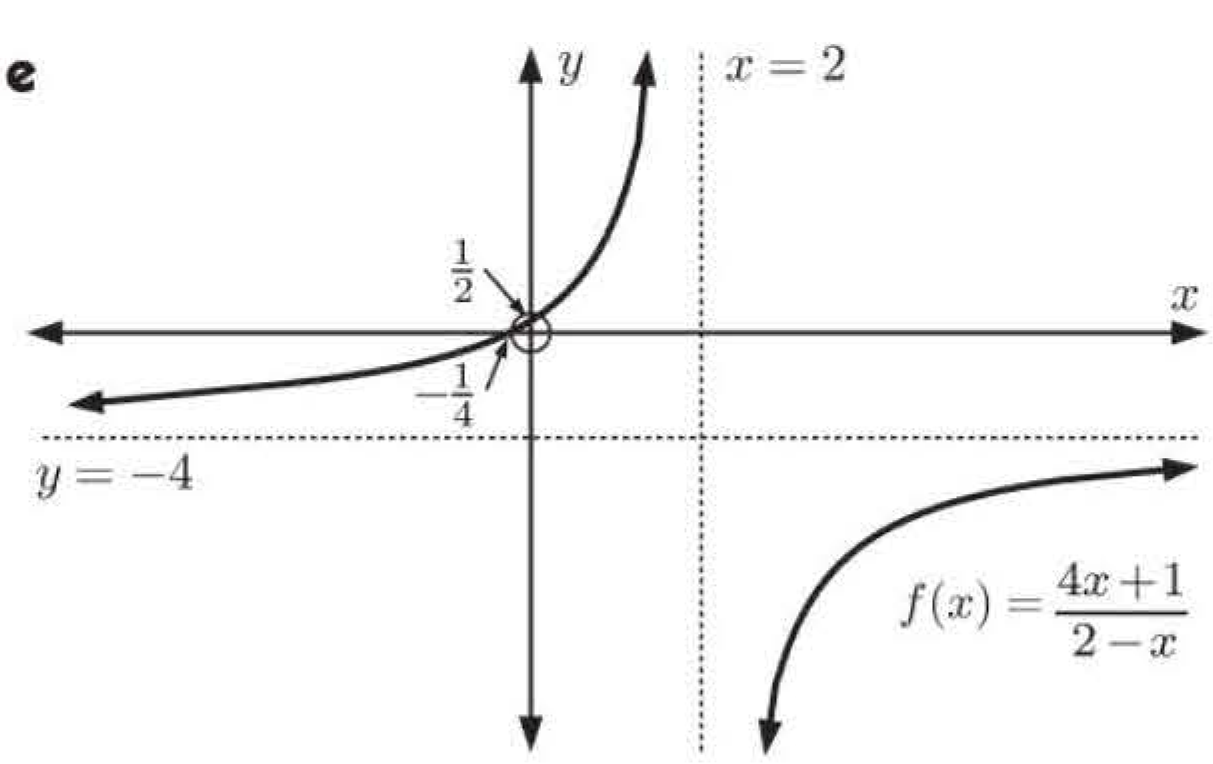
$f(x) = 0$ when $\dfrac{4x+1}{2-x} = 0$

$\therefore\ 4x + 1 = 0$

$\therefore\ x = -\frac{1}{4}$

So, the x-intercept is $-\frac{1}{4}$.

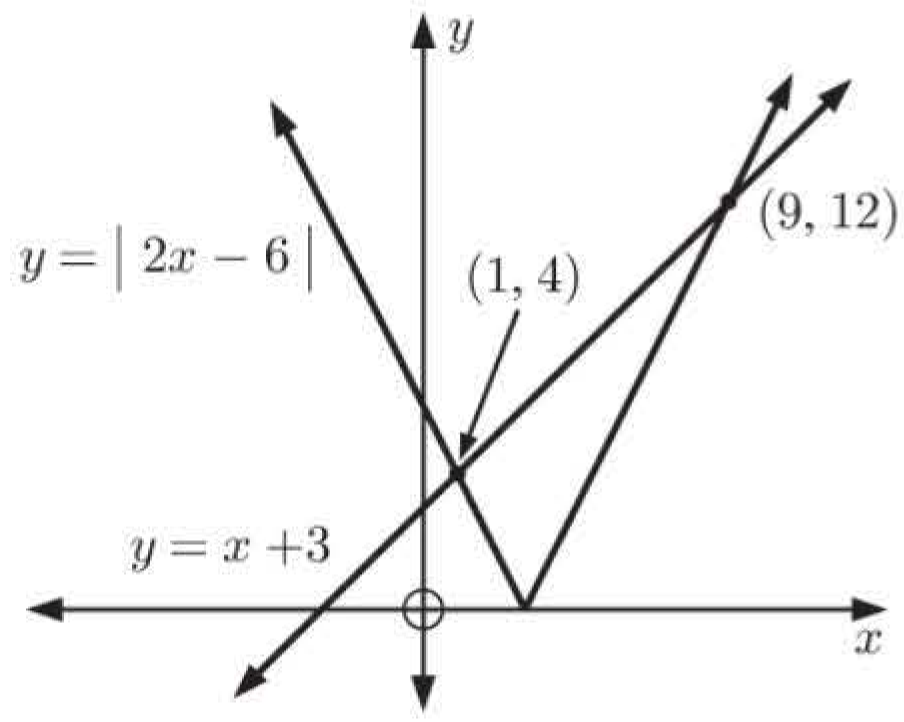
e



11
$$f(x) = (x-3)^2 + ax \quad \text{is an even function}$$
$$\therefore\ f(-x) = f(x)$$
$$\therefore\ (-x-3)^2 + a(-x) = (x-3)^2 + ax$$
$$\therefore\ \cancel{x^2} + 6x + \cancel{9} - ax = \cancel{x^2} - 6x + \cancel{9} + ax$$

Equating coefficients of x:
$$6 - a = -6 + a$$
$$\therefore\ 2a = 12$$
$$\therefore\ a = 6$$

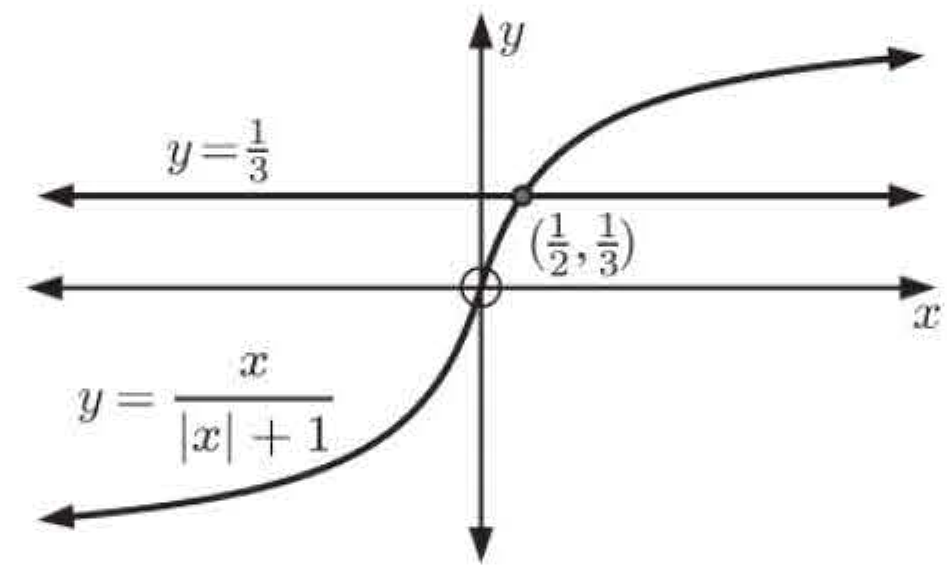
12 **a**



$y = |2x + 6|$ and $y = x + 3$ intersect at $x = 1$ and $x = 9$.

$\therefore$ from the graph, $|2x + 6| > x + 3$ when $x \in \,]-\infty, 1[$ or $]9, \infty[$

b



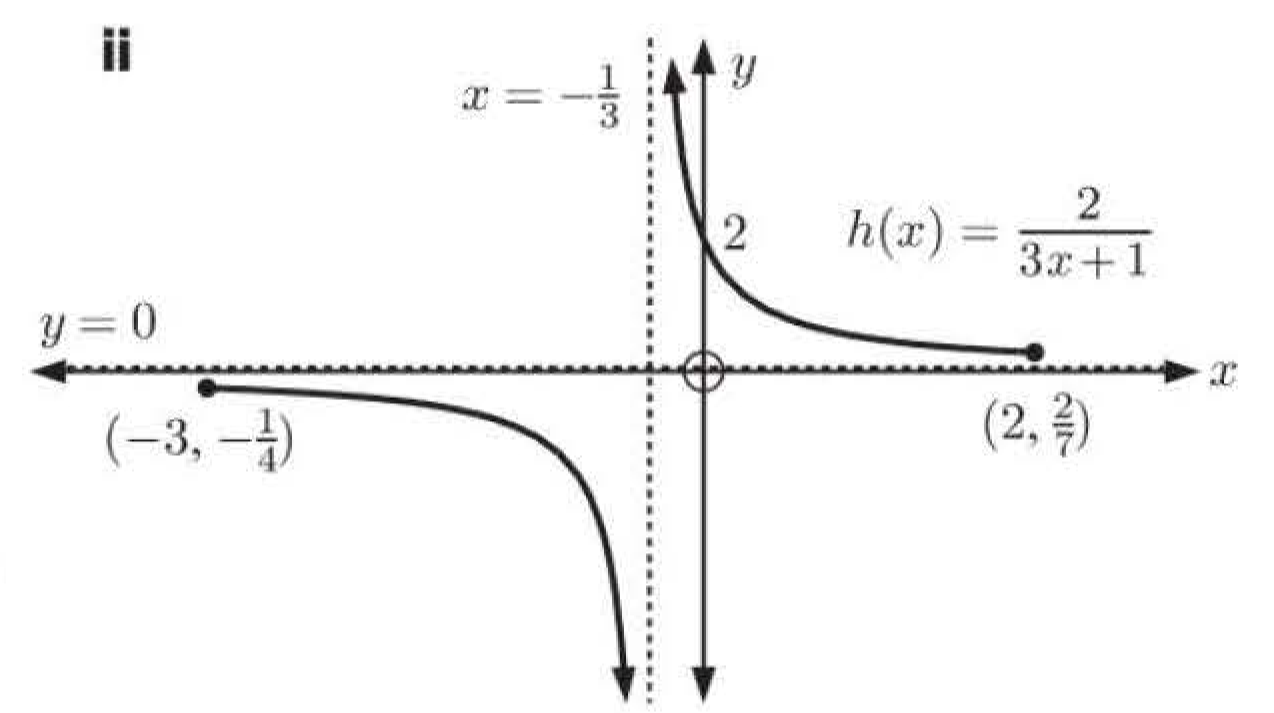
$$\therefore \quad \frac{x}{|x| + 1} \geqslant \frac{1}{3} \quad \text{for} \quad x \in [\tfrac{1}{2}, \infty[$$

13 **a**
$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g(3x + 1) \\ &= \frac{2}{3x + 1}\end{aligned}$$

b
$$\begin{aligned}(g \circ f)(x) &= -4 \\ \therefore \quad \frac{2}{3x + 1} &= -4 \\ \therefore \quad -4(3x + 1) &= 2 \\ \therefore \quad -12x - 4 &= 2 \\ \therefore \quad 12x &= -6 \\ \therefore \quad x &= -\tfrac{1}{2}\end{aligned}$$

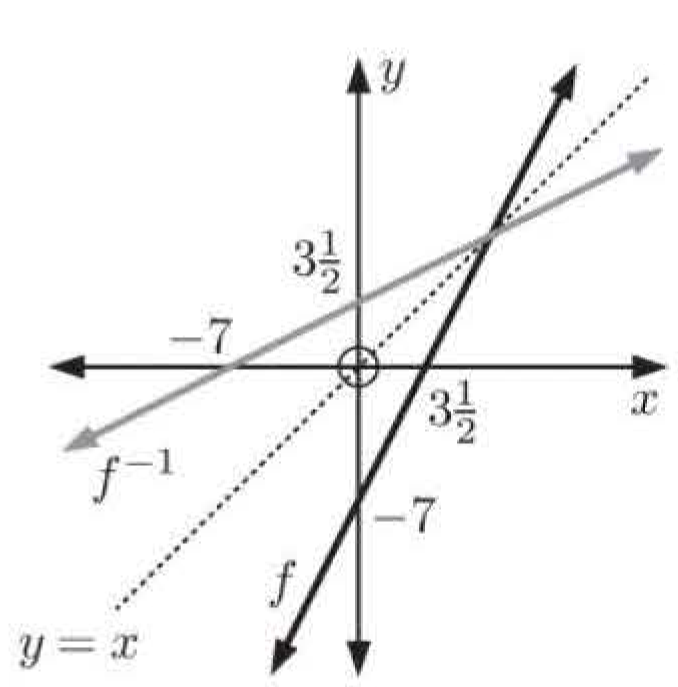
c **i** $h(x) = \dfrac{2}{3x + 1}$ is undefined when $3x + 1 = 0$ or $x = -\frac{1}{3}$.
So, $x = -\frac{1}{3}$ is a vertical asymptote.
As $|x| \to \infty$, $h(x) \to 0$
$\therefore$ $y = 0$ is a horizontal asymptote.

ii



iii Range of h is $\{y \mid y \leqslant -\frac{1}{4}$ or $y \geqslant \frac{2}{7}\}$.

14 **a**



b The function f is $y = 2x - 7$
so f^{-1} is $x = 2y - 7$
$$\therefore \quad y = \frac{x + 7}{2}$$
$$\text{So,} \quad f^{-1}(x) = \frac{x + 7}{2}$$

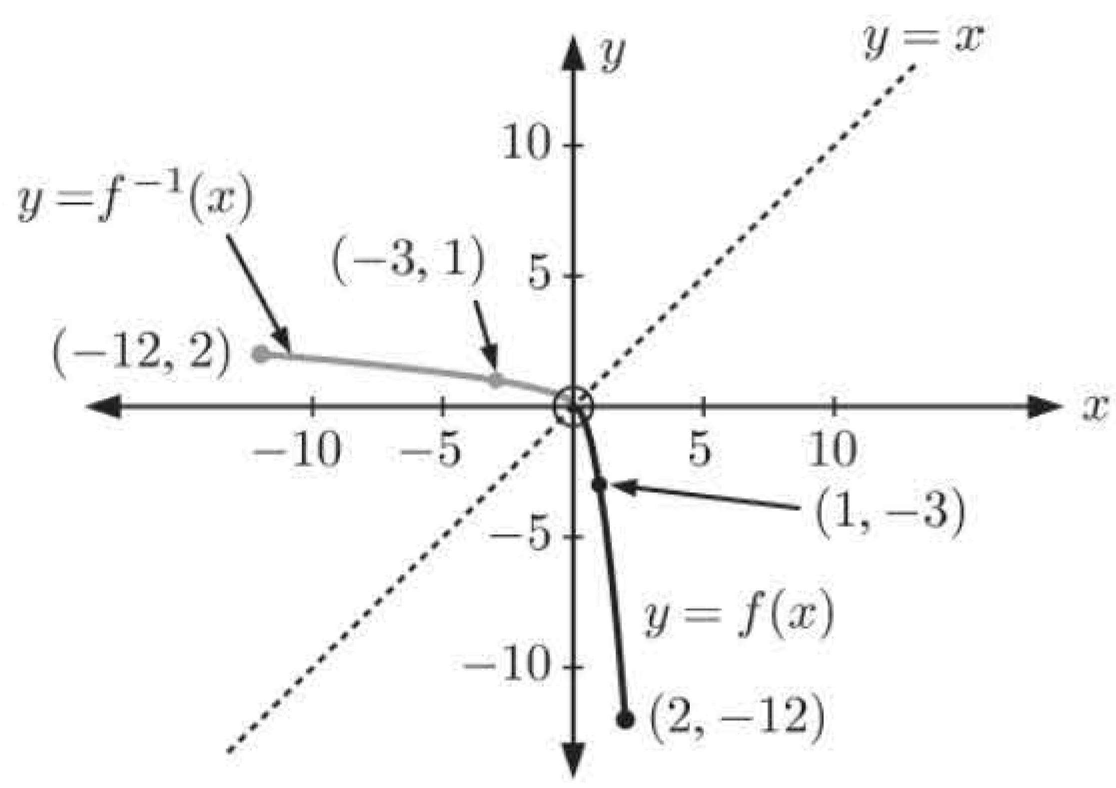
c $(f \circ f^{-1})(x)$ and $(f^{-1} \circ f)(x)$

$$\begin{aligned}&= f\left(f^{-1}(x)\right) \\ &= f\left(\frac{x + 7}{2}\right) \\ &= 2\left(\frac{x + 7}{2}\right) - 7 \\ &= x + 7 - 7 \\ &= x\end{aligned}$$

$$\begin{aligned}&= f^{-1}\left(f(x)\right) \\ &= f^{-1}(2x - 7) \\ &= \frac{2x - 7 + 7}{2} \\ &= \frac{2x}{2} \\ &= x\end{aligned}$$

So, $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$.

15 **a**



b Range of f^{-1} is $\{y \mid 0 \leqslant y \leqslant 2\}$.

c **i**

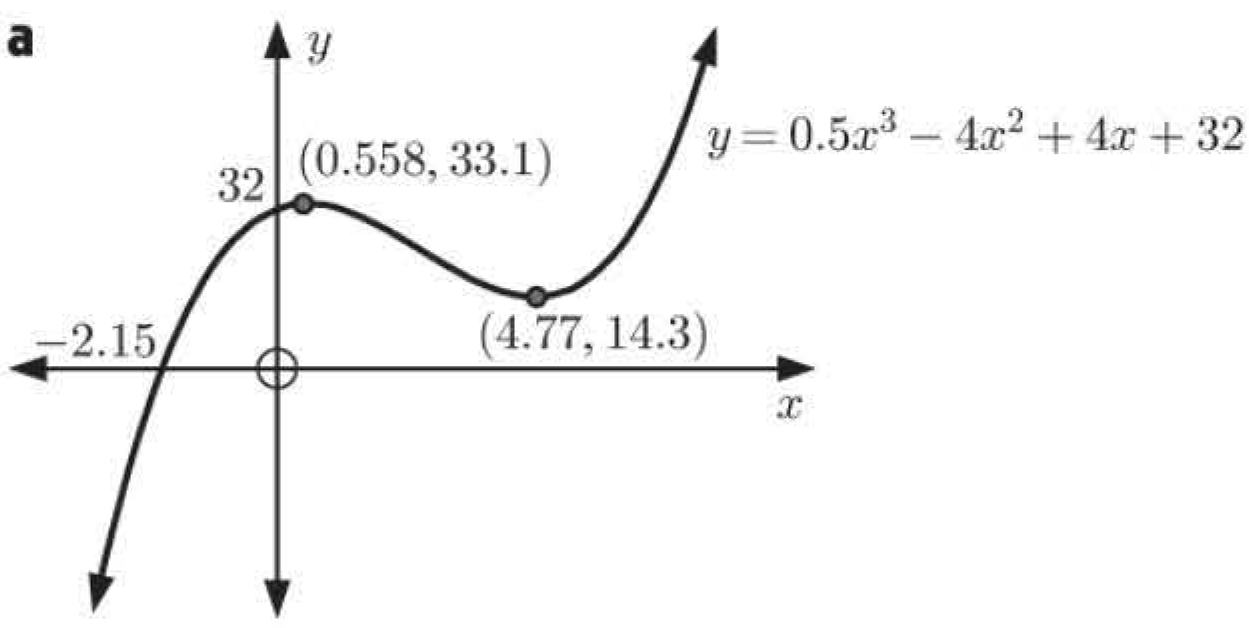
$$\begin{aligned} f(x) &= -10 \\ \therefore\ -3x^2 &= -10, \quad 0 \leqslant x \leqslant 2 \\ \therefore\ x^2 &= \tfrac{10}{3} \\ \therefore\ x &= \sqrt{\tfrac{10}{3}} \quad \{0 \leqslant x \leqslant 2\} \\ \therefore\ x &\approx 1.83 \end{aligned}$$

ii $f^{-1}(x) = 1$ so $f(1) = x$

The graph shows that $f(1) = -3$

$\therefore\ x = -3$

16 **a**



b $(0.558, 33.1)$ is a local maximum,

$(4.77, 14.3)$ is a local minimum

{using technology}

c Check endpoints:

$$\begin{aligned} f(0) &= 32 \\ f(6) &= 0.5(6)^3 - 4(6)^2 + 4(6) + 32 \\ &= 20 \end{aligned}$$

$\therefore$ for $f(x)$ on the interval $0 \leqslant x \leqslant 6$, the maximum value is 33.1 and the minimum value is 14.3.

REVIEW SET 2C

1 **a** Domain is $\{x \mid x \geqslant -2\}$. Range is $\{y \mid 1 \leqslant y < 3\}$.

b Domain is $\{x \mid x \in \mathbb{R}\}$. Range is $\{y \mid y = -1, 1, \text{ or } 2\}$.

2 **a**

$$\begin{aligned} f(x) &= x^2 + 3 \\ \therefore\ f(-3) &= (-3)^2 + 3 \\ &= 9 + 3 \\ &= 12 \end{aligned}$$

b

$$\begin{aligned} x^2 + 3 &= 4 \\ \therefore\ x^2 &= 1 \\ \therefore\ x &= \pm 1 \end{aligned}$$

3 **a** $f(x) = 10 + \dfrac{3}{2x-1}$

is undefined when $2x - 1 = 0$

$\therefore\ x = \frac{1}{2}$

b $f(x) = \sqrt{x+7}$

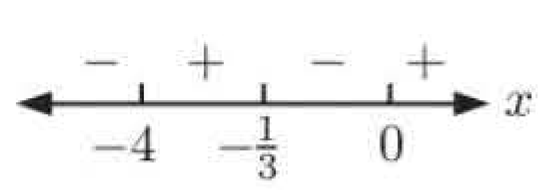
is undefined when $x + 7 < 0$

$\therefore\ x < -7$

4 **a** $f(x) = x(x+4)(3x+1)$ is zero when $x = 0,\ -4,$ and $-\frac{1}{3}$.

When $x = 10,\ y = 10(14)(31) = 4340 > 0$.

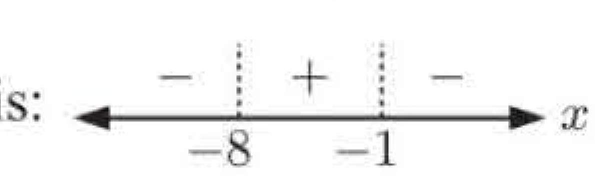
The factors are single, so the signs alternate.

$\therefore$ sign diagram is: 

b $f(x) = \dfrac{-11}{(x+1)(x+8)}$ is undefined when $x = -1$ or -8.

When $x = 0,\ y = \frac{-11}{(1)(8)} = -\frac{11}{8} < 0$.

The factors are single, so the signs alternate.

$\therefore$ sign diagram is: 

5 **a**

$$\begin{aligned} h(x) &= 7 - 3x \\ h(2x-1) &= 7 - 3(2x-1) \\ &= 7 - 6x + 3 \\ &= 10 - 6x \end{aligned}$$

b

$$\begin{aligned} h(2x-1) &= -2 \\ \therefore\ 10 - 6x &= -2 \quad \{\text{using } \mathbf{a}\} \\ \therefore\ -6x &= -12 \\ \therefore\ x &= 2 \end{aligned}$$

6 **a** **i** $(f \circ g)(x) = f(g(x))$
$= f(\sqrt{x})$
$= 1 - 2\sqrt{x}$

ii $(g \circ f)(x) = g(f(x))$
$= g(1 - 2x)$
$= \sqrt{1 - 2x}$

b **i** Domain is $\{x \mid x \geqslant 0\}$.
Range is $\{y \mid y \leqslant 1\}$.

ii Domain is $\{x \mid x \leqslant 0.5\}$.
Range is $\{y \mid y \geqslant 0\}$.

7 $f(x) = ax^2 + bx + c$, where $f(0) = 5$, $f(-2) = 21$, and $f(3) = -4$

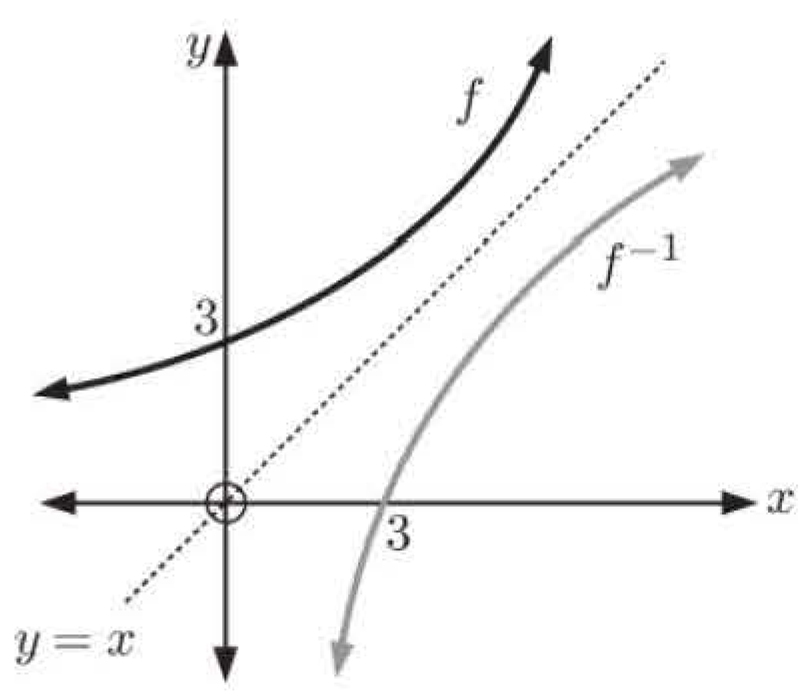
When $f(0) = 5$,
$5 = a(0)^2 + b(0) + c$
$\therefore\ 5 = c$
$\therefore\ c = 5$ .... (1)

When $f(-2) = 21$,
$21 = a(-2)^2 + b(-2) + c$
$= 4a - 2b + c$
$= 4a - 2b + 5$ {using (1)}
$\therefore\ 4a - 2b = 16$
$\therefore\ 2a - b = 8$ and so $b = 2a - 8$ .... (2)

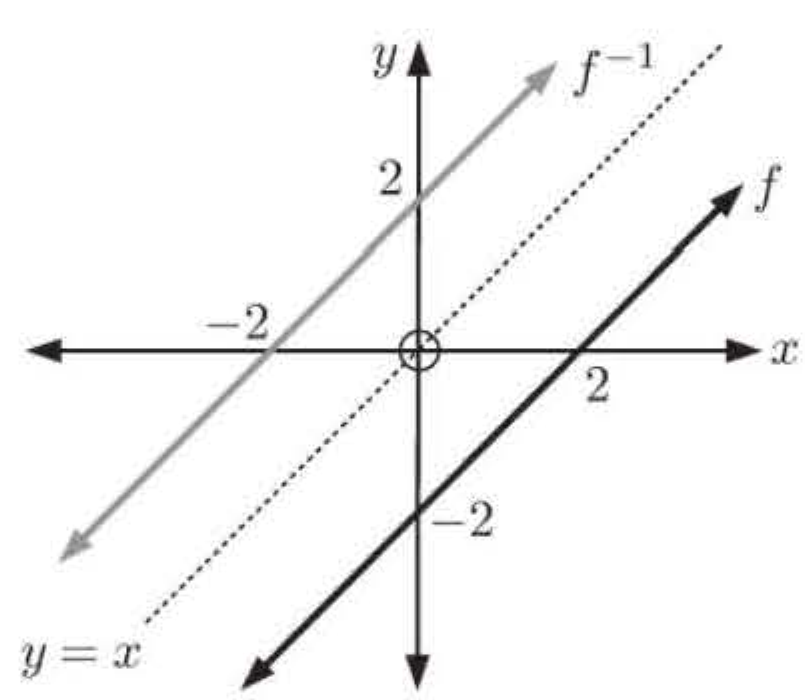
When $f(3) = -4$, $-4 = a(3)^2 + b(3) + c$
$\therefore\ -4 = 9a + 3b + c$
$\therefore\ -4 = 9a + 3b + 5$ {using (1)}
$\therefore\ -4 = 9a + 3(2a - 8) + 5$ {using (2)}
$\therefore\ -4 = 9a + 6a - 24 + 5$
$\therefore\ 15 = 15a$ and so $a = 1$

Now, substituting $a = 1$ into (2) gives $b = 2(1) - 8 = -6$
So, $a = 1$, $b = -6$, $c = 5$.

8 **a**



b



9
$$g(x) = |f(x)|$$
$$= \begin{cases} f(x) & \text{if } f(x) \geqslant 0 \\ -f(x) & \text{if } f(x) < 0 \end{cases}$$

Now $g(-x) = |f(-x)|$
$= |-f(x)|$ $\{f(x)$ is an odd function$\}$
$$= \begin{cases} f(x) & \text{if } f(x) \geqslant 0 \\ -f(x) & \text{if } f(x) < 0 \end{cases}$$
$= g(x)$

$\therefore$ $g(x)$ is an even function.

10 **a** f is $y = 7 - 4x$
$\therefore\ f^{-1}$ is $x = 7 - 4y$
$\therefore\ y = \dfrac{7 - x}{4}$
So, $f^{-1}(x) = \dfrac{7 - x}{4}$

b f is $y = \dfrac{3 + 2x}{5}$
$\therefore\ f^{-1}$ is $x = \dfrac{3 + 2y}{5}$
$\therefore\ 5x = 3 + 2y$
$\therefore\ y = \dfrac{5x - 3}{2}$
So, $f^{-1}(x) = \dfrac{5x - 3}{2}$

11

$$\begin{aligned} f \text{ is } \quad y &= 5x-2 \\ \therefore\ f^{-1} \text{ is } \quad x &= 5y-2 \\ \therefore\ y &= \frac{x+2}{5} \\ \therefore\ f^{-1}(x) &= \frac{x+2}{5} \end{aligned}$$

$$\begin{aligned} h \text{ is } \quad y &= \frac{3x}{4} \\ \therefore\ h^{-1} \text{ is } \quad x &= \frac{3y}{4} \\ \therefore\ y &= \frac{4x}{3} \\ \therefore\ h^{-1}(x) &= \frac{4x}{3} \end{aligned}$$

Now
$$\begin{aligned} (f^{-1} \circ h^{-1})(x) &= f^{-1}(h^{-1}(x)) \\ &= f^{-1}\left(\frac{4x}{3}\right) \\ &= \frac{\frac{4x}{3}+2}{5} \\ &= \frac{4x+6}{15} \end{aligned}$$

and
$$\begin{aligned} (h \circ f)(x) &= h(f(x)) \\ &= h(5x-2) \\ &= \frac{3(5x-2)}{4} \\ \text{So,}\quad y &= \frac{15x-6}{4} \\ \therefore\ (h\circ f)^{-1}(x) \quad \text{is} \quad x &= \frac{15y-6}{4} \\ \therefore\ 4x &= 15y-6 \\ \therefore\ y &= \frac{4x+6}{15} \\ \therefore\ (h \circ f)^{-1}(x) &= \frac{4x+6}{15} \end{aligned}$$

Hence, $(f^{-1} \circ h^{-1})(x) = (h \circ f)^{-1}(x)$ as required.

12 **a**

$$\begin{aligned} 2x^2 + x &\leqslant 10 \\ \therefore\ 2x^2+x-10 &\leqslant 0 \\ \therefore\ (2x+5)(x-2) &\leqslant 0 \end{aligned}$$

Sign diagram of LHS is

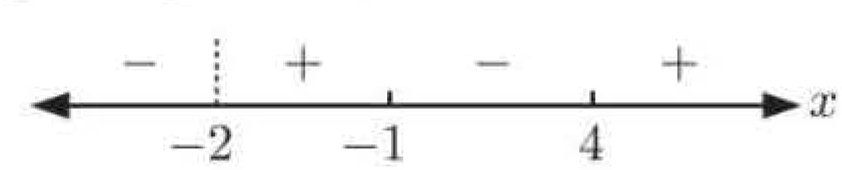
$\therefore$ LHS $\leqslant 0$ when $x \in [-\frac{5}{2}, 2]$

$\therefore$ $2x^2 + x \leqslant 10$ when $x \in [-\frac{5}{2}, 2]$

b

$$\begin{aligned} \frac{x^2-3x-4}{x+2} &> 0 \\ \therefore\ \frac{(x+1)(x-4)}{x+2} &> 0 \end{aligned}$$

Sign diagram of LHS is



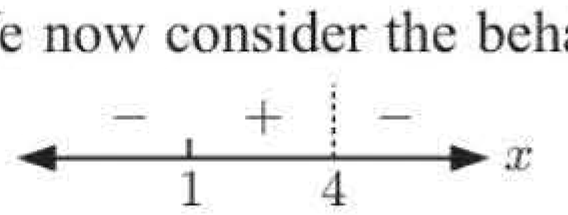
$\therefore$ LHS > 0 when $x \in \,]-2, -1[$ or $]4, \infty[$

13

$$\begin{aligned} f \text{ is } \quad y &= 2x+11 \\ \text{so}\quad f^{-1}(x) \quad \text{is} \quad x &= 2y+11 \\ \therefore\ y &= \frac{x-11}{2} \\ \therefore\ f^{-1}(x) &= \frac{x-11}{2} \end{aligned}$$

$$\begin{aligned} g(x) &= x^2 \\ (g \circ f^{-1})(x) &= g(f^{-1}(x)) \\ &= g\left(\frac{x-11}{2}\right) \\ &= \left(\frac{x-11}{2}\right)^2 \\ \therefore\ (g \circ f^{-1})(3) &= \left(\frac{3-11}{2}\right)^2 \\ &= (-4)^2 = 16 \end{aligned}$$

14 The domain is $\{x \mid x \neq 4\}$, so $x = 4$ is a vertical asymptote.
The range is $\{y \mid y \neq -1\}$, so $y = -1$ is a horizontal asymptote.
We now consider the behaviour of the function near the asymptotes, using the sign diagram



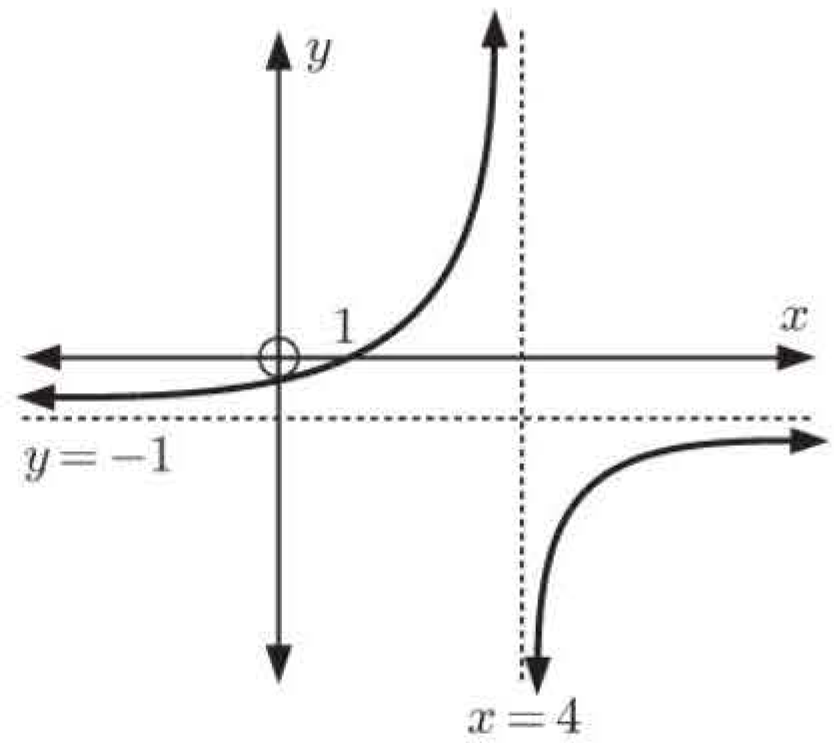
to help us.

As $x \to \infty$, $y \to -1$
As $x \to -\infty$, $y \to -1$

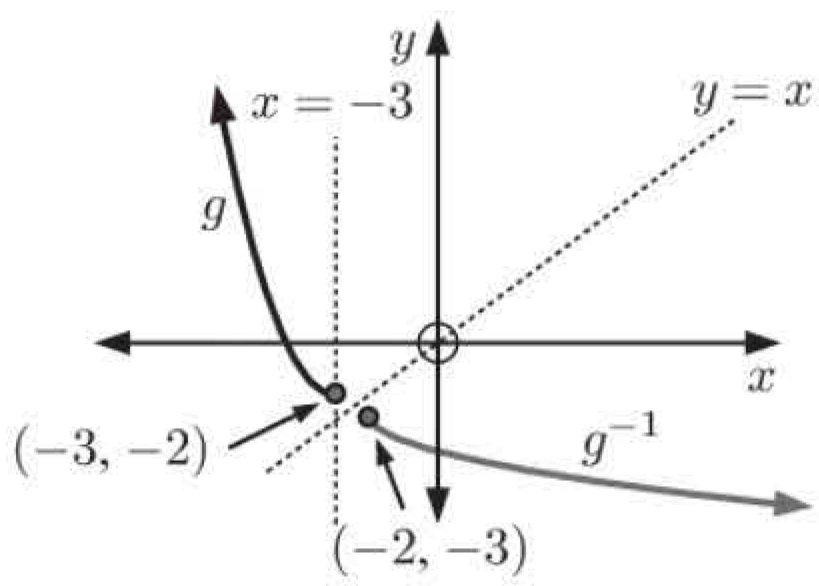
Note that we cannot tell whether the function tends to -1 from above or below.

As $x \to 4^-$, $y \to \infty$
As $x \to 4^+$, $y \to -\infty$

So, the function could be:
(**Note:** There may be other answers.)



15 **a**, **d**



b Any horizontal line through the graph cuts it no more than once. Therefore it has an inverse function.

c
$$\begin{aligned} g(x) &= x^2 + 6x + 7, \quad x \leqslant -3 \\ \therefore \quad y &= x^2 + 6x + 7, \quad x \leqslant -3 \\ \therefore \quad g^{-1}(x) \text{ is } \quad x &= y^2 + 6y + 7, \quad y \leqslant -3 \\ &= (y+3)^2 - 9 + 7 \\ \therefore \quad x + 2 &= (y+3)^2 \\ \therefore \quad y + 3 &= \pm\sqrt{x+2} \\ \therefore \quad y &= -3 \pm \sqrt{x+2} \end{aligned}$$

but $y \leqslant -3$, so $y = -3 - \sqrt{x+2}$

So, $g^{-1}(x) = -3 - \sqrt{x+2}$, $x \geqslant -2$

d The range of g is $\{y \mid y \geqslant -2\}$.

e The domain of g^{-1} is $\{x \mid x \geqslant -2\}$, and the range of g^{-1} is $\{y \mid y \leqslant -3\}$.

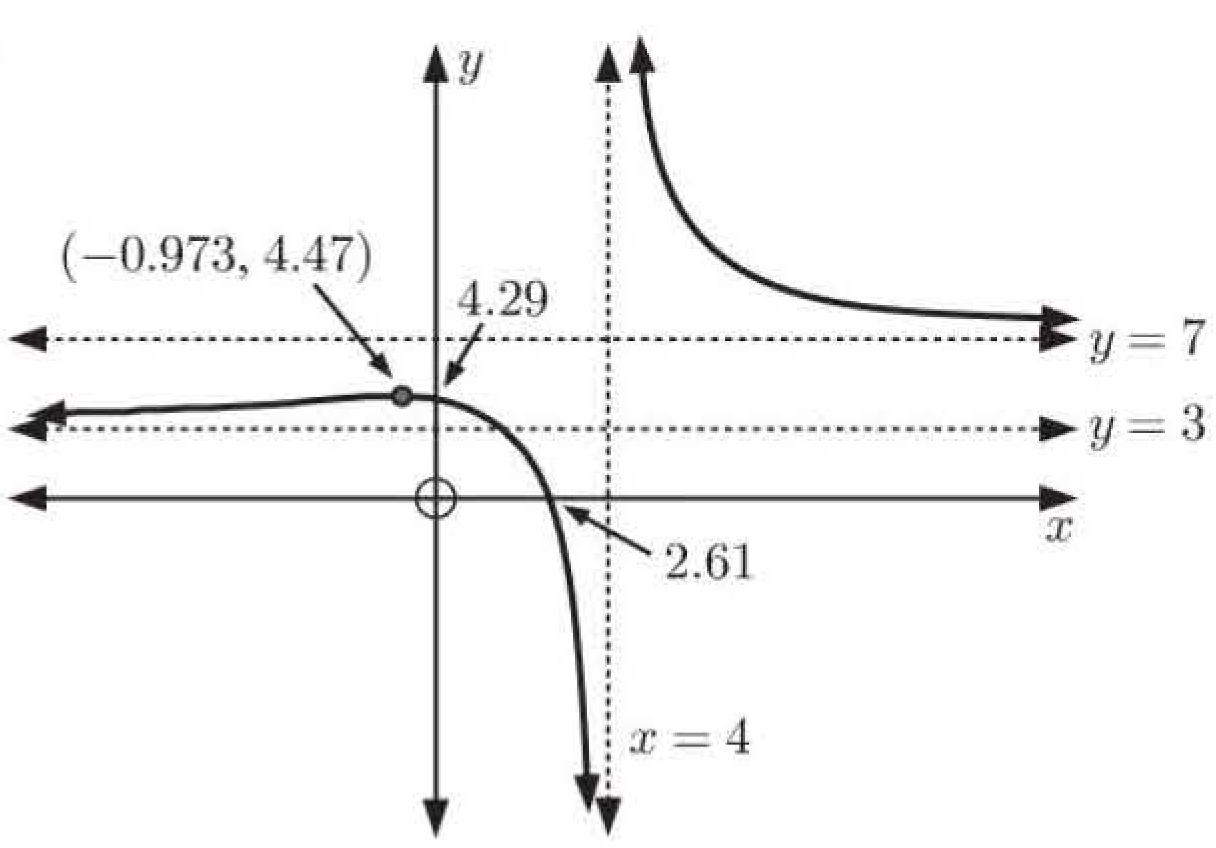
16 **a** x-intercept is 2.61, y-intercept is 4.29 {using technology}

b $(-0.973, 4.47)$ is a local maximum

c As $x \to 4^-$, $y \to -\infty$
As $x \to 4^+$, $y \to \infty$
$\therefore$ the vertical asymptote is $x = 4$.
As $x \to \infty$, $y \to 7^+$
As $x \to -\infty$, $y \to 3^+$
$\therefore$ the horizontal asymptotes are $y = 3$ and $y = 7$.

d Domain is $\{x \mid x \neq 4\}$.
Range is $\{y \mid y \leqslant 4.47, \ y > 7\}$.

e



Chapter 3
EXPONENTIALS

EXERCISE 3A

1 **a** $2^1 = 2,\ 2^2 = 4,\ 2^3 = 8,\ 2^4 = 16,\ 2^5 = 32,\ 2^6 = 64$

b $3^1 = 3,\ 3^2 = 9,\ 3^3 = 27,\ 3^4 = 81,\ 3^5 = 243,\ 3^6 = 729$

c $4^1 = 4,\ 4^2 = 16,\ 4^3 = 64,\ 4^4 = 256,\ 4^5 = 1024,\ 4^6 = 4096$

2 **a** $5^1 = 5,\ 5^2 = 25,\ 5^3 = 125,\ 5^4 = 625$ **b** $6^1 = 6,\ 6^2 = 36,\ 6^3 = 216,\ 6^4 = 1296$

c $7^1 = 7,\ 7^2 = 49,\ 7^3 = 343,\ 7^4 = 2401$

3 **a**
$$\begin{aligned}&(-1)^5\\&= (-1)\times(-1)\times(-1)\times(-1)\times(-1)\\&= 1\times1\times(-1)\\&= -1\end{aligned}$$

b
$$\begin{aligned}&(-1)^6\\&= (-1)^5\times(-1)\\&= (-1)\times(-1)\\&= 1\end{aligned}$$

c
$$\begin{aligned}&(-1)^{14}\\&= 1\end{aligned}$$

d
$$\begin{aligned}&(-1)^{19}\\&= -1\end{aligned}$$

e
$$\begin{aligned}&(-1)^{8}\\&= 1\end{aligned}$$

f
$$\begin{aligned}&-1^{8}\\&= -(1^8)\\&= -1\end{aligned}$$

g
$$\begin{aligned}&-(-1)^{8}\\&= -(1)\\&= -1\end{aligned}$$

h
$$\begin{aligned}&(-2)^5\\&= (-2)\times(-2)\times(-2)\times(-2)\times(-2)\\&= 4\times4\times(-2)\\&= -32\end{aligned}$$

i
$$\begin{aligned}&-2^5\\&= -(2^5)\\&= -32\end{aligned}$$

j
$$\begin{aligned}&-(-2)^6\\&= -(-2)^5\times(-2)\\&= 32\times(-2)\\&= -64\end{aligned}$$

k
$$\begin{aligned}&(-5)^4\\&= (-5)\times(-5)\times(-5)\times(-5)\\&= 25\times25\\&= 625\end{aligned}$$

l
$$\begin{aligned}&-(-5)^4\\&= -(-5)\times(-5)\times(-5)\times(-5)\\&= -25\times25\\&= -625\end{aligned}$$

4 **a** $4^7 = 16\,384$ **b** $7^4 = 2401$ **c** $-5^5 = -3125$ **d** $(-5)^5 = -3125$

e $8^6 = 262\,144$ **f** $(-8)^6 = 262\,144$ **g** $-8^6 = -262\,144$

h $2.13^9 \approx 902.436\,039\,6$ **i** $-2.13^9 \approx -902.436\,039\,6$ **j** $(-2.13)^9 \approx -902.436\,039\,6$

5 **a** $9^{-1} = 0.\overline{1}$ **b** $\dfrac{1}{9^1} = 0.\overline{1}$ **c** $6^{-2} = 0.02\overline{7}$ **d** $\dfrac{1}{6^2} = 0.02\overline{7}$

e $3^{-4} \approx 0.012\,345\,679$ **f** $\dfrac{1}{3^4} \approx 0.012\,345\,679$ **g** $17^0 = 1$ **h** $(0.366)^0 = 1$

We notice that $a^{-n} = \dfrac{1}{a^n}$ and $a^0 = 1$ for $a \neq 0$.

6 $3^{101} = \underbrace{3^4\times3^4\times3^4\times \ldots\ldots \times 3^4}_{\text{25 of these}}\times 3^1$

But $3^4 = 81$ which ends in a 1

$\therefore\ \underbrace{3^4\times3^4\times3^4\times \ldots\ldots \times 3^4}_{\text{25 of these}}$ ends in a 1

$\therefore\ 3^{101}$ ends in a 3

7 $7^1 = 7,\quad 7^2 = 49,\quad 7^3 = 343,\quad 7^4 = 2401,\quad 7^5 = 16\,807$

Now $7^{217} = \underbrace{7^4\times7^4\times7^4\times \ldots\ldots \times 7^4}_{\text{54 of these, so this part ends in a 1}}\times 7^1$

$\therefore\ 7^{217}$ ends in $1\times7 = 7$.

EXERCISE 3B

1 **a** $5^4 \times 5^7 = 5^{4+7}$
$= 5^{11}$

b $d^2 \times d^6 = d^{2+6}$
$= d^8$

c $\dfrac{k^8}{k^3} = k^{8-3}$
$= k^5$

d $\dfrac{7^5}{7^6} = 7^{5-6}$
$= 7^{-1}$
$= \frac{1}{7}$

e $(x^2)^5 = x^{2\times 5}$
$= x^{10}$

f $(3^4)^4 = 3^{4\times 4}$
$= 3^{16}$

g $\dfrac{p^3}{p^7} = p^{3-7}$
$= p^{-4}$ or $\dfrac{1}{p^4}$

h $n^3 \times n^9 = n^{3+9}$
$= n^{12}$

i $(5^t)^3 = 5^{t\times 3}$
$= 5^{3t}$

j $7^x \times 7^2 = 7^{x+2}$

k $\dfrac{10^3}{10^q} = 10^{3-q}$

l $(c^4)^m = c^{4\times m}$
$= c^{4m}$

2 **a** $4 = 2\times 2$
$= 2^2$

b $\dfrac{1}{4} = \dfrac{1}{2^2}$
$= 2^{-2}$

c $8 = 2\times 2\times 2$
$= 2^3$

d $\dfrac{1}{8} = \dfrac{1}{2^3}$
$= 2^{-3}$

e 32
$= 2\times 2\times 2\times 2\times 2$
$= 2^5$

f $\dfrac{1}{32} = \dfrac{1}{2^5}$
$= 2^{-5}$

g $2 = 2^1$

h $\dfrac{1}{2} = \dfrac{1}{2^1}$
$= 2^{-1}$

i $64 = 32\times 2$
$= 2^5\times 2^1$
$= 2^6$

j $\dfrac{1}{64} = \dfrac{1}{2^6}$
$= 2^{-6}$

k $128 = 64\times 2$
$= 2^6\times 2^1$
$= 2^7$

l $\dfrac{1}{128} = \dfrac{1}{2^7}$
$= 2^{-7}$

3 **a** $9 = 3\times 3$
$= 3^2$

b $\dfrac{1}{9} = \dfrac{1}{3^2}$
$= 3^{-2}$

c $27 = 3\times 3\times 3$
$= 3^3$

d $\dfrac{1}{27} = \dfrac{1}{3^3}$
$= 3^{-3}$

e $3 = 3^1$

f $\dfrac{1}{3} = \dfrac{1}{3^1}$
$= 3^{-1}$

g $81 = 3\times 3\times 3\times 3$
$= 3^4$

h $\dfrac{1}{81} = \dfrac{1}{3^4}$
$= 3^{-4}$

i $1 = 3^0$

j $243 = 81\times 3$
$= 3^4\times 3^1$
$= 3^5$

k $\dfrac{1}{243} = \dfrac{1}{3^5}$
$= 3^{-5}$

4 **a** $2\times 2^a = 2^1\times 2^a$
$= 2^{a+1}$

b $4\times 2^b = 2^2\times 2^b$
$= 2^{b+2}$

c $8\times 2^t = 2^3\times 2^t$
$= 2^{t+3}$

d $(2^{x+1})^2 = 2^{2(x+1)}$
$= 2^{2x+2}$

e $(2^{1-n})^{-1} = 2^{-(1-n)}$
$= 2^{n-1}$

f $\dfrac{2^c}{4} = \dfrac{2^c}{2^2} = 2^{c-2}$

g $\dfrac{2^m}{2^{-m}} = 2^{m-(-m)}$
$= 2^{2m}$

h $\dfrac{4}{2^{1-n}} = \dfrac{2^2}{2^{1-n}}$
$= 2^{2-(1-n)}$
$= 2^{n+1}$

i $\dfrac{2^{x+1}}{2^x} = 2^{x+1-x}$
$= 2^1$

j $\dfrac{4^x}{2^{1-x}} = \dfrac{(2^2)^x}{2^{1-x}}$
$= 2^{2x-(1-x)}$
$= 2^{3x-1}$

5 **a** $9 \times 3^p = 3^2 \times 3^p$
$= 3^{p+2}$

b $27^a = (3^3)^a$
$= 3^{3a}$

c $3 \times 9^n = 3^1 \times (3^2)^n$
$= 3^{2n+1}$

d $27 \times 3^d = 3^3 \times 3^d$
$= 3^{d+3}$

e $9 \times 27^t = 3^2 \times (3^3)^t$
$= 3^{3t+2}$

f $\dfrac{3^y}{3} = \dfrac{3^y}{3^1} = 3^{y-1}$

g $\dfrac{3}{3^y} = \dfrac{3^1}{3^y}$
$= 3^{1-y}$

h $\dfrac{9}{27^t} = \dfrac{3^2}{(3^3)^t}$
$= 3^{2-3t}$

i $\dfrac{9^a}{3^{1-a}} = \dfrac{(3^2)^a}{3^{1-a}}$
$= 3^{2a-(1-a)}$
$= 3^{3a-1}$

j $\dfrac{9^{n+1}}{3^{2n-1}} = \dfrac{(3^2)^{n+1}}{3^{2n-1}}$
$= 3^{2n+2-(2n-1)}$
$= 3^3$

6 **a** $(2a)^2 = 2^2 \times a^2$
$= 4a^2$

b $(3b)^3 = 3^3 \times b^3$
$= 27b^3$

c $(ab)^4 = a^4 \times b^4$
$= a^4b^4$

d $(pq)^3 = p^3 \times q^3$
$= p^3q^3$

e $\left(\dfrac{m}{n}\right)^2 = \dfrac{m^2}{n^2}$

f $\left(\dfrac{a}{3}\right)^3 = \dfrac{a^3}{3^3} = \dfrac{a^3}{27}$

g $\left(\dfrac{b}{c}\right)^4 = \dfrac{b^4}{c^4}$

h $\left(\dfrac{2a}{b}\right)^0 = 1,\ b \neq 0$

i $\left(\dfrac{m}{3n}\right)^4 = \dfrac{m^4}{3^4 \times n^4} = \dfrac{m^4}{81n^4}$

j $\left(\dfrac{xy}{2}\right)^3 = \dfrac{x^3y^3}{2^3} = \dfrac{x^3y^3}{8}$

7 **a** $(-2a)^2$
$= (-2)^2a^2$
$= 4a^2$

b $(-6b^2)^2$
$= (-6)^2b^4$
$= 36b^4$

c $(-2a)^3$
$= (-2)^3a^3$
$= -8a^3$

d $(-3m^2n^2)^3$
$= (-3)^3m^6n^6$
$= -27m^6n^6$

e $(-2ab^4)^4$
$= (-2)^4a^4b^{16}$
$= 16a^4b^{16}$

f $\left(\dfrac{-2a^2}{b^2}\right)^3$
$= \dfrac{(-2)^3a^6}{b^6}$
$= -\dfrac{8a^6}{b^6}$

g $\left(\dfrac{-4a^3}{b}\right)^2$
$= \dfrac{(-4)^2a^6}{b^2}$
$= \dfrac{16a^6}{b^2}$

h $\left(\dfrac{-3p^2}{q^3}\right)^2$
$= \dfrac{(-3)^2p^4}{q^6}$
$= \dfrac{9p^4}{q^6}$

8 **a** $ab^{-2} = \dfrac{a}{b^2}$

b $(ab)^{-2} = \dfrac{1}{(ab)^2}$
$= \dfrac{1}{a^2b^2}$

c $(2ab^{-1})^2 = 2^2a^2b^{-2}$
$= \dfrac{4a^2}{b^2}$

d $(3a^{-2}b)^2 = 3^2a^{-4}b^2$
$= \dfrac{9b^2}{a^4}$

e $\dfrac{a^2b^{-1}}{c^2} = \dfrac{a^2}{bc^2}$

f $\dfrac{a^2b^{-1}}{c^{-2}} = \dfrac{a^2c^2}{b}$

g $\dfrac{1}{a^{-3}} = a^3$

h $\dfrac{a^{-2}}{b^{-3}} = \dfrac{b^3}{a^2}$

i $\dfrac{2a^{-1}}{d^2} = \dfrac{2}{ad^2}$

j $\dfrac{12a}{m^{-3}} = 12am^3$

9 **a** $\dfrac{1}{a^n} = a^{-n}$

b $\dfrac{1}{b^{-n}} = b^n$

c $\dfrac{1}{3^{2-n}} = 3^{n-2}$

d $\dfrac{a^n}{b^{-m}} = a^nb^m$

e $\dfrac{a^{-n}}{a^{2+n}} = a^{-n-(2+n)}$
$= a^{-2n-2}$

10 **a** $(\frac{5}{3})^0 = 1$ **b** $(\frac{7}{4})^{-1} = \frac{4}{7}$ **c** $(\frac{1}{6})^{-1} = \frac{6}{1} = 6$ **d** $\dfrac{3^3}{3^0} = \frac{27}{1} = 27$

e $(\frac{4}{3})^{-2} = \dfrac{3^2}{4^2} = \frac{9}{16}$ **f** $2^1 + 2^{-1} = 2 + \frac{1}{2} = \frac{5}{2}$ **g** $(1\frac{2}{3})^{-3} = (\frac{5}{3})^{-3} = \dfrac{3^3}{5^3} = \frac{27}{125}$ **h** $5^2 + 5^1 + 5^{-1} = 25 + 5 + \frac{1}{5} = \frac{151}{5}$

11 **a** $5^3 = 21 + 23 + 25 + 27 + 29$ **b** $7^3 = 43 + 45 + 47 + 49 + 51 + 53 + 55$

c $12^3 = 133 + 135 + 137 + 139 + 141 + 143 + 145 + 147 + 149 + 151 + 153 + 155$

EXERCISE 3C

1 **a** $\sqrt[5]{2} = 2^{\frac{1}{5}}$ **b** $\dfrac{1}{\sqrt[5]{2}} = \dfrac{1}{2^{\frac{1}{5}}} = 2^{-\frac{1}{5}}$ **c** $2\sqrt{2} = 2^1 \times 2^{\frac{1}{2}} = 2^{\frac{3}{2}}$ **d** $4\sqrt{2} = 2^2 \times 2^{\frac{1}{2}} = 2^{\frac{5}{2}}$

e $\dfrac{1}{\sqrt[3]{2}} = \dfrac{1}{2^{\frac{1}{3}}} = 2^{-\frac{1}{3}}$ **f** $2 \times \sqrt[3]{2} = 2^1 \times 2^{\frac{1}{3}} = 2^{\frac{4}{3}}$ **g** $\dfrac{4}{\sqrt{2}} = \dfrac{2^2}{2^{\frac{1}{2}}} = 2^{\frac{3}{2}}$

h $(\sqrt{2})^3 = (2^{\frac{1}{2}})^3 = 2^{\frac{3}{2}}$ **i** $\dfrac{1}{\sqrt[3]{16}} = \dfrac{1}{\sqrt[3]{2^4}} = \dfrac{1}{2^{\frac{4}{3}}} = 2^{-\frac{4}{3}}$ **j** $\dfrac{1}{\sqrt{8}} = \dfrac{1}{\sqrt{2^3}} = \dfrac{1}{2^{\frac{3}{2}}} = 2^{-\frac{3}{2}}$

2 **a** $\sqrt[3]{3} = 3^{\frac{1}{3}}$ **b** $\dfrac{1}{\sqrt[3]{3}} = \dfrac{1}{3^{\frac{1}{3}}} = 3^{-\frac{1}{3}}$ **c** $\sqrt[4]{3} = 3^{\frac{1}{4}}$ **d** $3\sqrt{3} = 3^1 \times 3^{\frac{1}{2}} = 3^{\frac{3}{2}}$

e $\dfrac{1}{9\sqrt{3}} = \dfrac{1}{3^2 \times 3^{\frac{1}{2}}} = \dfrac{1}{3^{\frac{5}{2}}} = 3^{-\frac{5}{2}}$

3 **a** $\sqrt[3]{7} = 7^{\frac{1}{3}}$ **b** $\sqrt[4]{27} = \sqrt[4]{3^3} = 3^{\frac{3}{4}}$ **c** $\sqrt[5]{16} = \sqrt[5]{2^4} = 2^{\frac{4}{5}}$ **d** $\sqrt[3]{32} = \sqrt[3]{2^5} = 2^{\frac{5}{3}}$

e $\sqrt[7]{49} = \sqrt[7]{7^2} = 7^{\frac{2}{7}}$ **f** $\dfrac{1}{\sqrt[3]{7}} = \dfrac{1}{7^{\frac{1}{3}}} = 7^{-\frac{1}{3}}$ **g** $\dfrac{1}{\sqrt[4]{27}} = \dfrac{1}{3^{\frac{3}{4}}} = 3^{-\frac{3}{4}}$ **h** $\dfrac{1}{\sqrt[5]{16}} = \dfrac{1}{2^{\frac{4}{5}}} = 2^{-\frac{4}{5}}$

i $\dfrac{1}{\sqrt[3]{32}} = \dfrac{1}{2^{\frac{5}{3}}} = 2^{-\frac{5}{3}}$ **j** $\dfrac{1}{\sqrt[7]{49}} = \dfrac{1}{7^{\frac{2}{7}}} = 7^{-\frac{2}{7}}$

4 **a** $3^{\frac{3}{4}} \approx 2.28$ **b** $2^{\frac{7}{8}} \approx 1.83$ **c** $2^{-\frac{1}{3}} \approx 0.794$ **d** $4^{-\frac{3}{5}} \approx 0.435$

e $\sqrt[4]{8} \approx 1.68$ **f** $\sqrt[5]{27} \approx 1.93$ **g** $\dfrac{1}{\sqrt[3]{7}} \approx 0.523$

5 **a** $4^{\frac{3}{2}} = (2^2)^{\frac{3}{2}} = 2^3 = 8$ **b** $8^{\frac{5}{3}} = (2^3)^{\frac{5}{3}} = 2^5 = 32$ **c** $16^{\frac{3}{4}} = (2^4)^{\frac{3}{4}} = 2^3 = 8$ **d** $25^{\frac{3}{2}} = (5^2)^{\frac{3}{2}} = 5^3 = 125$

e $32^{\frac{2}{5}} = (2^5)^{\frac{2}{5}}$
$= 2^2$
$= 4$

f $4^{-\frac{1}{2}} = (2^2)^{-\frac{1}{2}}$
$= 2^{-1}$
$= \frac{1}{2}$

g $9^{-\frac{3}{2}} = (3^2)^{-\frac{3}{2}}$
$= 3^{-3}$
$= \frac{1}{27}$

h $8^{-\frac{4}{3}} = (2^3)^{-\frac{4}{3}}$
$= 2^{-4}$
$= \frac{1}{16}$

i $27^{-\frac{4}{3}} = (3^3)^{-\frac{4}{3}}$
$= 3^{-4}$
$= \frac{1}{81}$

j $125^{-\frac{2}{3}} = (5^3)^{-\frac{2}{3}}$
$= 5^{-2}$
$= \frac{1}{25}$

EXERCISE 3D.1

1 **a** $x^2(x^3 + 2x^2 + 1)$
$= x^2 \times x^3 + x^2 \times 2x^2 + x^2 \times 1$
$= x^5 + 2x^4 + x^2$

b $2^x(2^x + 1)$
$= 2^x \times 2^x + 2^x \times 1$
$= 2^{2x} + 2^x$
$= 4^x + 2^x$

c $x^{\frac{1}{2}}(x^{\frac{1}{2}} + x^{-\frac{1}{2}})$
$= x^{\frac{1}{2}} \times x^{\frac{1}{2}} + x^{\frac{1}{2}} \times x^{-\frac{1}{2}}$
$= x^1 + x^0$
$= x + 1$

d $7^x(7^x + 2)$
$= 7^x \times 7^x + 7^x \times 2$
$= 7^{2x} + 2(7^x)$
$= 49^x + 2(7^x)$

e $3^x(2 - 3^{-x})$
$= 3^x \times 2 - 3^x \times 3^{-x}$
$= 2(3^x) - 3^0$
$= 2(3^x) - 1$

f $x^{\frac{1}{2}}(x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + 3x^{-\frac{1}{2}})$
$= x^{\frac{1}{2}} \times x^{\frac{3}{2}} + x^{\frac{1}{2}} \times 2x^{\frac{1}{2}} + x^{\frac{1}{2}} \times 3x^{-\frac{1}{2}}$
$= x^2 + 2x^1 + 3x^0$
$= x^2 + 2x + 3$

g $2^{-x}(2^x + 5)$
$= 2^{-x} \times 2^x + 2^{-x} \times 5$
$= 2^0 + 5(2^{-x})$
$= 1 + 5(2^{-x})$

h $5^{-x}(5^{2x} + 5^x)$
$= 5^{-x} \times 5^{2x} + 5^{-x} \times 5^x$
$= 5^x + 5^0$
$= 5^x + 1$

i $x^{-\frac{1}{2}}(x^2 + x + x^{\frac{1}{2}})$
$= x^{-\frac{1}{2}} \times x^2 + x^{-\frac{1}{2}} \times x^1 + x^{-\frac{1}{2}} \times x^{\frac{1}{2}}$
$= x^{\frac{3}{2}} + x^{\frac{1}{2}} + x^0$
$= x^{\frac{3}{2}} + x^{\frac{1}{2}} + 1$

2 **a** $(2^x - 1)(2^x + 3)$
$= 2^x \times 2^x + 2^x \times 3 - 1 \times 2^x - 3$
$= 2^{2x} + 2(2^x) - 3$
$= 4^x + 2^{x+1} - 3$

b $(3^x + 2)(3^x + 5)$
$= 3^x \times 3^x + 3^x \times 5 + 2 \times 3^x + 10$
$= 3^{2x} + 7(3^x) + 10$
$= 9^x + 7(3^x) + 10$

c $(5^x - 2)(5^x - 4)$
$= 5^x \times 5^x - 5^x \times 4 - 2 \times 5^x + 8$
$= 5^{2x} - 6(5^x) + 8$
$= 25^x - 6(5^x) + 8$

d $(2^x + 3)^2$
$= (2^x)^2 + 2 \times 2^x \times 3 + 3^2$
$= 2^{2x} + 6(2^x) + 9$
$= 4^x + 6(2^x) + 9$

e $(3^x - 1)^2$
$= (3^x)^2 - 2 \times 3^x \times 1 + 1^2$
$= 3^{2x} - 2(3^x) + 1$
$= 9^x - 2(3^x) + 1$

f $(4^x + 7)^2$
$= (4^x)^2 + 2 \times 4^x \times 7 + 7^2$
$= 4^{2x} + 14(4^x) + 49$
$= 16^x + 14(4^x) + 49$

g $(x^{\frac{1}{2}} + 2)(x^{\frac{1}{2}} - 2)$
$= (x^{\frac{1}{2}})^2 - 2^2$
$= x - 4$

h $(2^x + 3)(2^x - 3)$
$= (2^x)^2 - 3^2$
$= 2^{2x} - 9$
$= 4^x - 9$

i $(x^{\frac{1}{2}}+x^{-\frac{1}{2}})(x^{\frac{1}{2}}-x^{-\frac{1}{2}})$
$= (x^{\frac{1}{2}})^2-(x^{-\frac{1}{2}})^2$
$= x^1-x^{-1}$
$= x-x^{-1}$

j $\left(x+\frac{2}{x}\right)^2$
$= x^2+2\times x\times\left(\frac{2}{x}\right)+\left(\frac{2}{x}\right)^2$
$= x^2+4+\frac{4}{x^2}$

k $(7^x-7^{-x})^2$
$= (7^x)^2-2\times 7^x\times 7^{-x}+(7^{-x})^2$
$= 7^{2x}-2\times 7^0+7^{-2x}$
$= 7^{2x}-2+7^{-2x}$

l $(5-2^{-x})^2$
$= 5^2-2\times 5\times 2^{-x}+(2^{-x})^2$
$= 25-10(2^{-x})+2^{-2x}$
$= 25-10(2^{-x})+4^{-x}$

EXERCISE 3D.2

1

a $5^{2x}+5^x$
$= 5^x\times 5^x+5^x$
$= 5^x(5^x+1)$

b $3^{n+2}+3^n$
$= 3^n\times 3^2+3^n$
$= 3^n(3^2+1)$
$= 10(3^n)$

c 7^n+7^{3n}
$= 7^n+7^n\times 7^{2n}$
$= 7^n(1+7^{2n})$

d $5^{n+1}-5$
$= 5\times 5^n-5$
$= 5(5^n-1)$

e $6^{n+2}-6$
$= 6\times 6^{n+1}-6$
$= 6(6^{n+1}-1)$

f $4^{n+2}-16$
$= 4^2\times 4^n-16$
$= 16\times 4^n-16$
$= 16(4^n-1)$

2

a 9^x-4
$= (3^x)^2-2^2$
$= (3^x+2)(3^x-2)$

b 4^x-25
$= (2^x)^2-5^2$
$= (2^x+5)(2^x-5)$

c $16-9^x$
$= 4^2-(3^x)^2$
$= (4+3^x)(4-3^x)$

d $25-4^x$
$= 5^2-(2^x)^2$
$= (5+2^x)(5-2^x)$

e 9^x-4^x
$= (3^x)^2-(2^x)^2$
$= (3^x+2^x)(3^x-2^x)$

f $4^x+6(2^x)+9$
$= (2^x)^2+6(2^x)+9$
$= (2^x+3)^2$
$\{a^2+6a+9=(a+3)^2\}$

g $9^x+10(3^x)+25$
$= (3^x)^2+10(3^x)+25$
$= (3^x+5)^2$
$\{a^2+10a+25=(a+5)^2\}$

h $4^x-14(2^x)+49$
$= (2^x)^2-14(2^x)+49$
$= (2^x-7)^2$
$\{a^2-14a+49=(a-7)^2\}$

i $25^x-4(5^x)+4$
$= (5^x)^2-4(5^x)+4$
$= (5^x-2)^2$
$\{a^2-4a+4=(a-2)^2\}$

3

a $4^x+9(2^x)+18$
$= (2^x)^2+9(2^x)+18$
$= (2^x+3)(2^x+6)$
$\{a^2+9a+18=(a+3)(a+6)\}$

b 4^x-2^x-20
$= (2^x)^2-2^x-20$
$= (2^x+4)(2^x-5)$
$\{a^2-a-20=(a+4)(a-5)\}$

c $9^x+9(3^x)+14$
$= (3^x)^2+9(3^x)+14$
$= (3^x+2)(3^x+7)$
$\{a^2+9a+14=(a+2)(a+7)\}$

d $9^x+4(3^x)-5$
$= (3^x)^2+4(3^x)-5$
$= (3^x+5)(3^x-1)$
$\{a^2+4a-5=(a+5)(a-1)\}$

e 25^x+5^x-2
$= (5^x)^2+5^x-2$
$= (5^x+2)(5^x-1)$
$\{a^2+a-2=(a+2)(a-1)\}$

f $49^x-7^{x+1}+12$
$= (7^x)^2-7(7^x)+12$
$= (7^x-4)(7^x-3)$
$\{a^2-7a+12=(a-4)(a-3)\}$

4 **a** $\dfrac{12^n}{6^n}=\left(\dfrac{12}{6}\right)^n$
$=2^n$

b $\dfrac{20^a}{2^a}=\left(\dfrac{20}{2}\right)^a$
$=10^a$

c $\dfrac{6^b}{2^b}=\left(\dfrac{6}{2}\right)^b$
$=3^b$

d $\dfrac{4^n}{20^n}=\left(\dfrac{4}{20}\right)^n$
$=\left(\frac{1}{5}\right)^n$
$=\dfrac{1}{5^n}$

e $\dfrac{35^x}{7^x}=\left(\dfrac{35}{7}\right)^x$
$=5^x$

f $\dfrac{6^a}{8^a}=\left(\dfrac{6}{8}\right)^a$
$=\left(\frac{3}{4}\right)^a$

g $\dfrac{5^{n+1}}{5^n}=\dfrac{5\times\cancel{5^n}}{\cancel{5^n}_1}$
$=5$

h $\dfrac{5^{n+1}}{5}=\dfrac{\cancel{5}\times 5^n}{\cancel{5}_1}$
$=5^n$

5 **a** $\dfrac{6^m+2^m}{2^m}$
$=\dfrac{2^m3^m+2^m}{2^m}$
$=\dfrac{\cancel{2^m}(3^m+1)}{\cancel{2^m}\,1}$
$=3^m+1$

b $\dfrac{2^n+12^n}{2^n}$
$=\dfrac{2^n+2^n6^n}{2^n}$
$=\dfrac{\cancel{2^n}(1+6^n)}{\cancel{2^n}\,1}$
$=1+6^n$

c $\dfrac{8^n+4^n}{2^n}$
$=\dfrac{2^n4^n+2^n2^n}{2^n}$
$=\dfrac{\cancel{2^n}(4^n+2^n)}{\cancel{2^n}\,1}$
$=4^n+2^n$

d $\dfrac{12^x-3^x}{3^x}$
$=\dfrac{3^x4^x-3^x}{3^x}$
$=\dfrac{\cancel{3^x}(4^x-1)}{\cancel{3^x}\,1}$
$=4^x-1$

e $\dfrac{6^n+12^n}{1+2^n}$
$=\dfrac{6^n+6^n2^n}{1+2^n}$
$=\dfrac{6^n\cancel{(1+2^n)}}{\cancel{1+2^n}\,1}$
$=6^n$

f $\dfrac{5^{n+1}-5^n}{4}$
$=\dfrac{5^n\times5-5^n}{4}$
$=\dfrac{5^n\cancel{(5-1)}}{\cancel{4}\,1}$
$=5^n$

g $\dfrac{5^{n+1}-5^n}{5^n}$
$=\dfrac{5^n\times5-5^n}{5^n}$
$=\dfrac{\cancel{5^n}(5-1)}{\cancel{5^n}\,1}$
$=4$

h $\dfrac{4^n-2^n}{2^n}$
$=\dfrac{2^n2^n-2^n}{2^n}$
$=\dfrac{\cancel{2^n}(2^n-1)}{\cancel{2^n}\,1}$
$=2^n-1$

i $\dfrac{2^n-2^{n-1}}{2^n}$
$=\dfrac{2^{n-1}\times2-2^{n-1}}{2^{n-1}\times2}$
$=\dfrac{\cancel{2^{n-1}}(2-1)}{\cancel{2^{n-1}}\times2}$
$=\frac{1}{2}$

6 **a** $2^n(n+1)+2^n(n-1)$
$=2^n(n+1+n-1)$
$=2^n(2n)$
$=n2^{n+1}$

b $3^n\left(\dfrac{n-1}{6}\right)-3^n\left(\dfrac{n+1}{6}\right)$
$=3^n\left(\dfrac{n-1}{6}-\dfrac{n+1}{6}\right)$
$=3^n(-\frac{1}{3})$
$=3^n\times-3^{-1}$
$=-3^{n-1}$

EXERCISE 3E

1 **a** $2^x=8$
$\therefore\ 2^x=2^3$
$\therefore\ x=3$

b $5^x=25$
$\therefore\ 5^x=5^2$
$\therefore\ x=2$

c $3^x=81$
$\therefore\ 3^x=3^4$
$\therefore\ x=4$

d $7^x=1$
$\therefore\ 7^x=7^0$
$\therefore\ x=0$

e $3^x = \frac{1}{3}$
$\therefore\ 3^x = 3^{-1}$
$\therefore\ x = -1$

f $2^x = \sqrt{2}$
$\therefore\ 2^x = 2^{\frac{1}{2}}$
$\therefore\ x = \frac{1}{2}$

g $5^x = \frac{1}{125}$
$\therefore\ 5^x = 5^{-3}$
$\therefore\ x = -3$

h $4^{x+1} = 64$
$\therefore\ 4^{x+1} = 4^3$
$\therefore\ x + 1 = 3$
$\therefore\ x = 2$

i $2^{x-2} = \frac{1}{32}$
$\therefore\ 2^{x-2} = 2^{-5}$
$\therefore\ x - 2 = -5$
$\therefore\ x = -3$

j $3^{x+1} = \frac{1}{27}$
$\therefore\ 3^{x+1} = 3^{-3}$
$\therefore\ x + 1 = -3$
$\therefore\ x = -4$

k $7^{x+1} = 343$
$\therefore\ 7^{x+1} = 7^3$
$\therefore\ x + 1 = 3$
$\therefore\ x = 2$

l $5^{1-2x} = \frac{1}{5}$
$\therefore\ 5^{1-2x} = 5^{-1}$
$\therefore\ 1 - 2x = -1$
$\therefore\ -2x = -2$
$\therefore\ x = 1$

2 **a** $8^x = 32$
$\therefore\ 2^{3x} = 2^5$
$\therefore\ 3x = 5$
$\therefore\ x = \frac{5}{3}$

b $4^x = \frac{1}{8}$
$\therefore\ 2^{2x} = 2^{-3}$
$\therefore\ 2x = -3$
$\therefore\ x = -\frac{3}{2}$

c $9^x = \frac{1}{27}$
$\therefore\ 3^{2x} = 3^{-3}$
$\therefore\ 2x = -3$
$\therefore\ x = -\frac{3}{2}$

d $25^x = \frac{1}{5}$
$\therefore\ 5^{2x} = 5^{-1}$
$\therefore\ 2x = -1$
$\therefore\ x = -\frac{1}{2}$

e $27^x = \frac{1}{9}$
$\therefore\ 3^{3x} = 3^{-2}$
$\therefore\ 3x = -2$
$\therefore\ x = -\frac{2}{3}$

f $16^x = \frac{1}{32}$
$\therefore\ 2^{4x} = 2^{-5}$
$\therefore\ 4x = -5$
$\therefore\ x = -\frac{5}{4}$

g $4^{x+2} = 128$
$\therefore\ 2^{2(x+2)} = 2^7$
$\therefore\ 2x + 4 = 7$
$\therefore\ 2x = 3$
$\therefore\ x = \frac{3}{2}$

h $25^{1-x} = \frac{1}{125}$
$\therefore\ 5^{2(1-x)} = 5^{-3}$
$\therefore\ 2 - 2x = -3$
$\therefore\ -2x = -5$
$\therefore\ x = \frac{5}{2}$

i $4^{4x-1} = \frac{1}{2}$
$\therefore\ 2^{2(4x-1)} = 2^{-1}$
$\therefore\ 8x - 2 = -1$
$\therefore\ 8x = 1$
$\therefore\ x = \frac{1}{8}$

j $9^{x-3} = 27$
$\therefore\ 3^{2(x-3)} = 3^3$
$\therefore\ 2x - 6 = 3$
$\therefore\ 2x = 9$
$\therefore\ x = \frac{9}{2}$

k $\left(\frac{1}{2}\right)^{x+1} = 8$
$\therefore\ \left(2^{-1}\right)^{x+1} = 2^3$
$\therefore\ -x - 1 = 3$
$\therefore\ -x = 4$
$\therefore\ x = -4$

l $\left(\frac{1}{3}\right)^{x+2} = 9$
$\therefore\ \left(3^{-1}\right)^{x+2} = 3^2$
$\therefore\ -x - 2 = 2$
$\therefore\ -x = 4$
$\therefore\ x = -4$

m $81^x = 27^{-x}$
$\therefore\ 3^{4x} = 3^{-3x}$
$\therefore\ 4x = -3x$
$\therefore\ 7x = 0$
$\therefore\ x = 0$

n $\left(\frac{1}{4}\right)^{1-x} = 32$
$\therefore\ \left(2^{-2}\right)^{1-x} = 2^5$
$\therefore\ -2 + 2x = 5$
$\therefore\ 2x = 7$
$\therefore\ x = \frac{7}{2}$

o $\left(\frac{1}{7}\right)^x = 49$
$\therefore\ 7^{-x} = 7^2$
$\therefore\ -x = 2$
$\therefore\ x = -2$

p $\left(\frac{1}{3}\right)^{x+1} = 243$
$\therefore\ \left(3^{-1}\right)^{x+1} = 3^5$
$\therefore\ -x - 1 = 5$
$\therefore\ -x = 6$
$\therefore\ x = -6$

3 **a** $4^{2x+1} = 8^{1-x}$
$\therefore\ \left(2^2\right)^{2x+1} = \left(2^3\right)^{1-x}$
$\therefore\ 4x + 2 = 3 - 3x$
$\therefore\ 7x = 1$
$\therefore\ x = \frac{1}{7}$

b $9^{2-x} = \left(\frac{1}{3}\right)^{2x+1}$
$\therefore\ \left(3^2\right)^{2-x} = \left(3^{-1}\right)^{2x+1}$
$\therefore\ 4 - 2x = -2x - 1$
$\therefore\ 4 = -1$
This is clearly false, so no solutions exist.

c $2^x \times 8^{1-x} = \frac{1}{4}$
$\therefore\ 2^x \times \left(2^3\right)^{1-x} = 2^{-2}$
$\therefore\ x + 3 - 3x = -2$
$\therefore\ -2x = -5$
$\therefore\ x = \frac{5}{2}$
(or $2\frac{1}{2}$)

4 **a** $3 \times 2^x = 24$
$\therefore\ 2^x = 8$
$\therefore\ 2^x = 2^3$
$\therefore\ x = 3$

b $7 \times 2^x = 56$
$\therefore\ 2^x = 8$
$\therefore\ 2^x = 2^3$
$\therefore\ x = 3$

c $3 \times 2^{x+1} = 24$
$\therefore\ 2^{x+1} = 8$
$\therefore\ 2^{x+1} = 2^3$
$\therefore\ x+1 = 3$
$\therefore\ x = 2$

d $12 \times 3^{-x} = \frac{4}{3}$
$\therefore\ 3^{-x} = \frac{4}{3} \div 12$
$\therefore\ 3^{-x} = \frac{4}{3} \times \frac{1}{12}$
$\therefore\ 3^{-x} = \frac{1}{9}$
$\therefore\ 3^{-x} = 3^{-2}$
$\therefore\ x = 2$

e $4 \times \left(\frac{1}{3}\right)^x = 36$
$\therefore\ \left(\frac{1}{3}\right)^x = 9$
$\therefore\ \left(3^{-1}\right)^x = 3^2$
$\therefore\ 3^{-x} = 3^2$
$\therefore\ -x = 2$
$\therefore\ x = -2$

f $5 \times \left(\frac{1}{2}\right)^x = 20$
$\therefore\ \left(\frac{1}{2}\right)^x = 4$
$\therefore\ \left(2^{-1}\right)^x = 2^2$
$\therefore\ -x = 2$
$\therefore\ x = -2$

5 **a** $4^x - 6(2^x) + 8 = 0$
$\therefore\ (2^x)^2 - 6(2^x) + 8 = 0$
$\therefore\ (2^x - 2)(2^x - 4) = 0 \qquad \{a^2 - 6a + 8 = (a-2)(a-4)\}$
$\therefore\ 2^x = 2 \text{ or } 4$
$\therefore\ 2^x = 2^1 \text{ or } 2^2$
$\therefore\ x = 1 \text{ or } 2$

b $4^x - 2^x - 2 = 0$
$\therefore\ (2^x)^2 - 2^x - 2 = 0$
$\therefore\ (2^x - 2)(2^x + 1) = 0 \qquad \{a^2 - a - 2 = (a-2)(a+1)\}$
$\therefore\ 2^x = 2 \text{ or } -1$
$\therefore\ 2^x = 2^1 \qquad \{\text{since } 2^x \text{ cannot be negative}\}$
$\therefore\ x = 1$

c $9^x - 12(3^x) + 27 = 0$
$\therefore\ (3^x)^2 - 12(3^x) + 27 = 0$
$\therefore\ (3^x - 3)(3^x - 9) = 0 \qquad \{a^2 - 12a + 27 = (a-3)(a-9)\}$
$\therefore\ 3^x = 3 \text{ or } 9$
$\therefore\ 3^x = 3^1 \text{ or } 3^2$
$\therefore\ x = 1 \text{ or } 2$

d $9^x = 3^x + 6$
$\therefore\ (3^x)^2 - 3^x - 6 = 0$
$\therefore\ (3^x - 3)(3^x + 2) = 0 \qquad \{a^2 - a - 6 = (a-3)(a+2)\}$
$\therefore\ 3^x = 3 \text{ or } -2$
$\therefore\ 3^x = 3^1 \qquad \{\text{since } 3^x \text{ cannot be negative}\}$
$\therefore\ x = 1$

e $25^x - 23(5^x) - 50 = 0$
$\therefore\ (5^x)^2 - 23(5^x) - 50 = 0$
$\therefore\ (5^x - 25)(5^x + 2) = 0 \qquad \{a^2 - 23a - 50 = (a-25)(a+2)\}$
$\therefore\ 5^x = 25 \text{ or } -2$
$\therefore\ 5^x = 5^2 \qquad \{\text{since } 5^x \text{ cannot be negative}\}$
$\therefore\ x = 2$

f $49^x + 1 = 2(7^x)$
$\therefore\ (7^x)^2 - 2(7^x) + 1 = 0$
$\therefore\ (7^x - 1)^2 = 0 \qquad \{a^2 - 2a + 1 = (a-1)^2\}$
$\therefore\ 7^x = 1$
$\therefore\ 7^x = 7^0$
$\therefore\ x = 0$

EXERCISE 3F

1 **a** When $x = \frac{1}{2}$, $y = 2^{\frac{1}{2}}$

From point A, $y \approx 1.4$

$\therefore \quad 2^{\frac{1}{2}} \approx 1.4$

b When $x = 0.8$, $y = 2^{0.8}$

From point B, $y \approx 1.7$

$\therefore \quad 2^{0.8} \approx 1.7$

c When $x = 1.5$, $y = 2^{1.5}$

From point C, $y \approx 2.8$

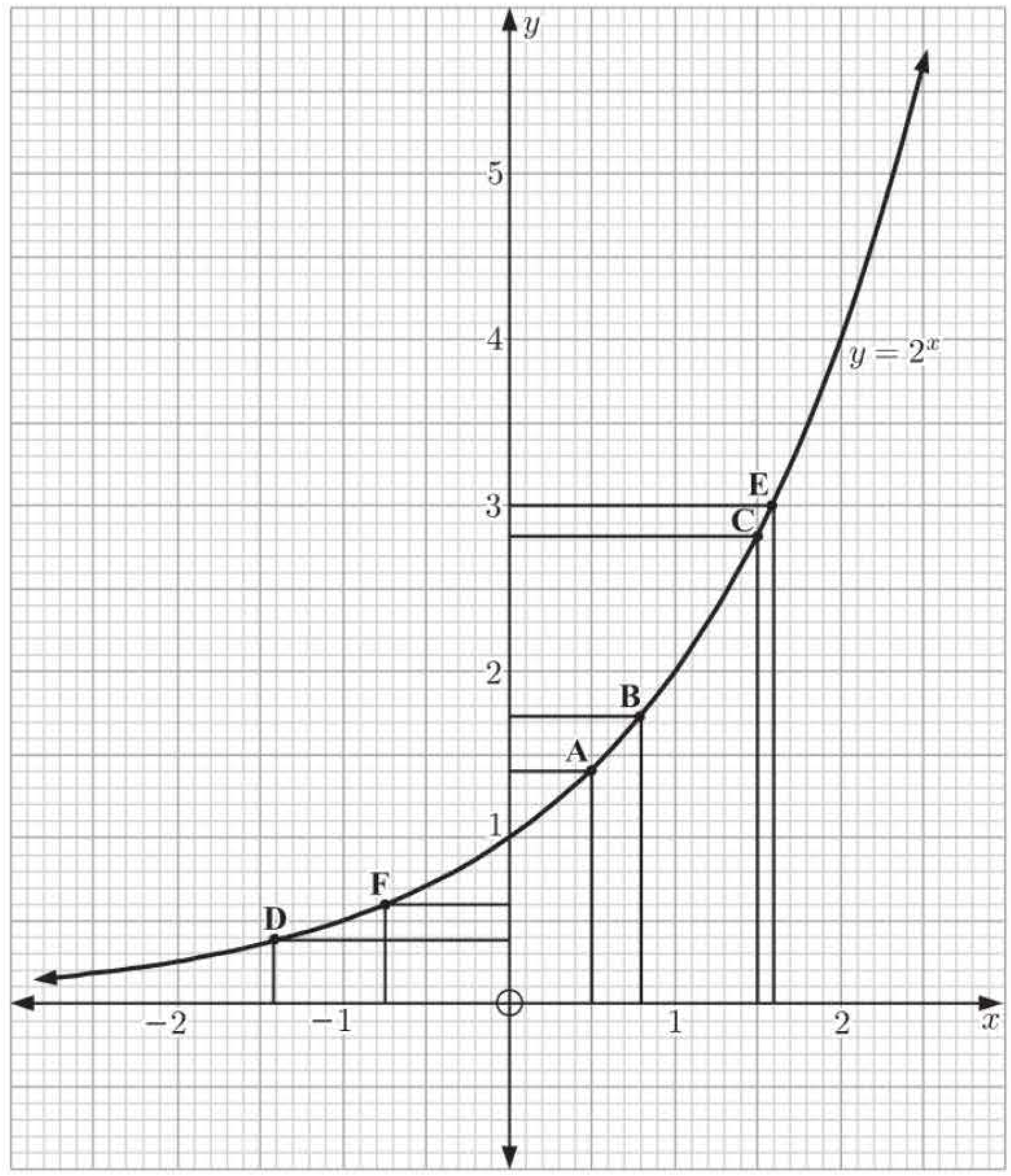
$\therefore \quad 2^{1.5} \approx 2.8$

d When $x = -\sqrt{2}$, $y = 2^{-\sqrt{2}}$

Using **a** we know $x \approx -1.4$

From point D, $y \approx 0.4$

$\therefore \quad 2^{-\sqrt{2}} \approx 0.4$



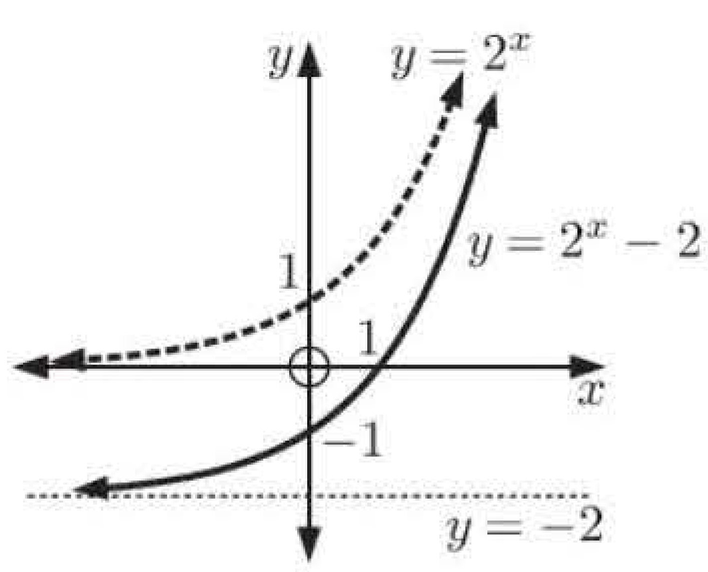
2 **a** When $2^x = 3$, $x \approx 1.6$ from point E. **b** When $2^x = 0.6$, $x \approx -0.7$ from point F.

3 The graph of $y = 2^x$ has a horizontal asymptote of $y = 0$.

$\therefore$ there is no value of x such that $2^x = 0$.

$\therefore$ $2^x = 0$ has no solutions.

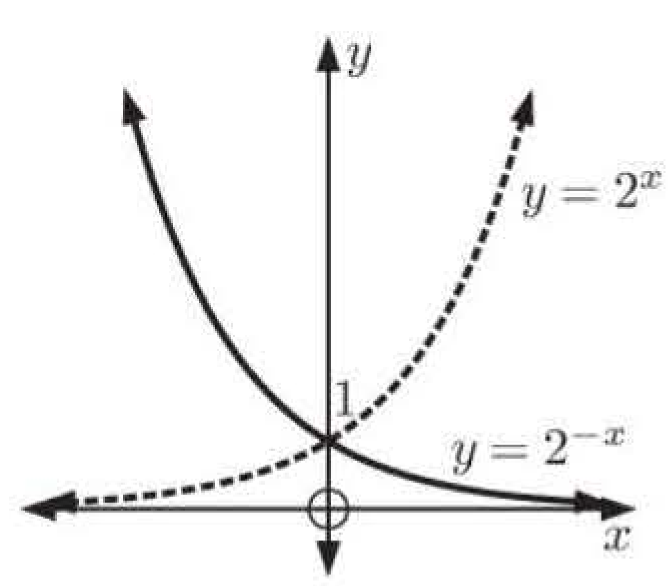
4 **a**



a vertical translation of 2 units downwards

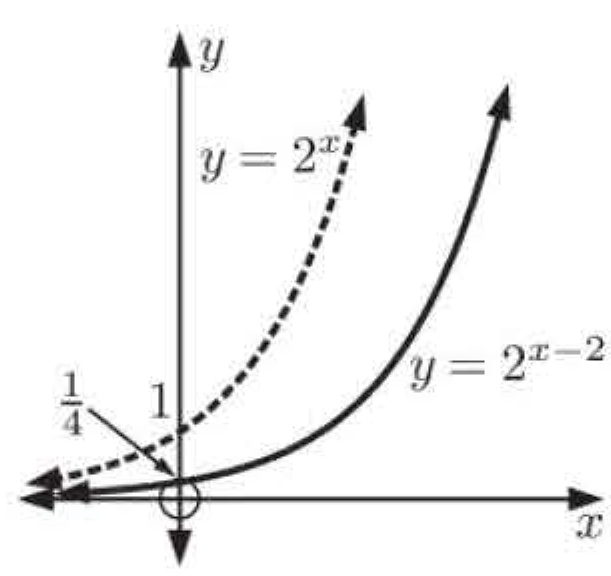
$y = -2$ is the H.A.

b



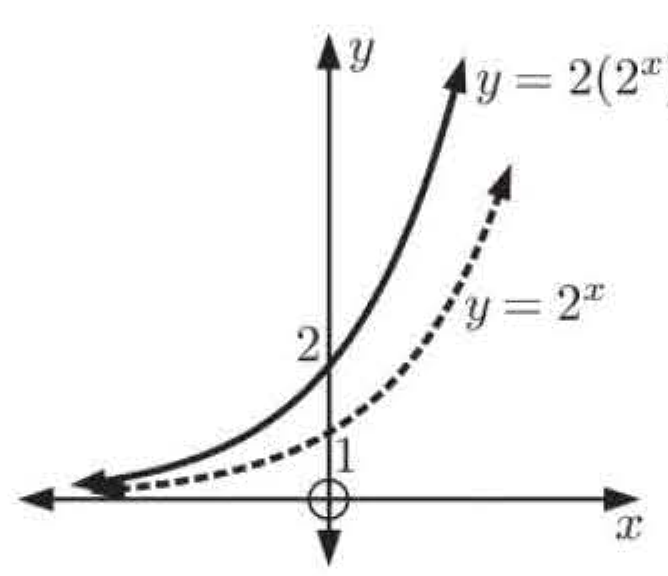
a reflection in the y-axis

c



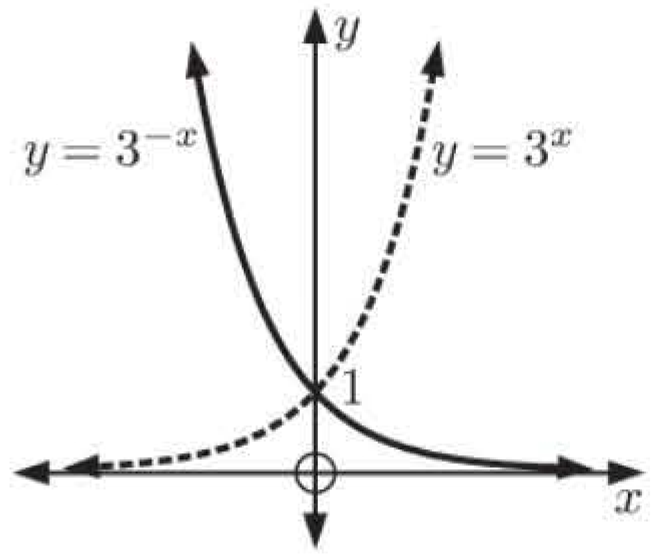
a horizontal translation of 2 units right

d



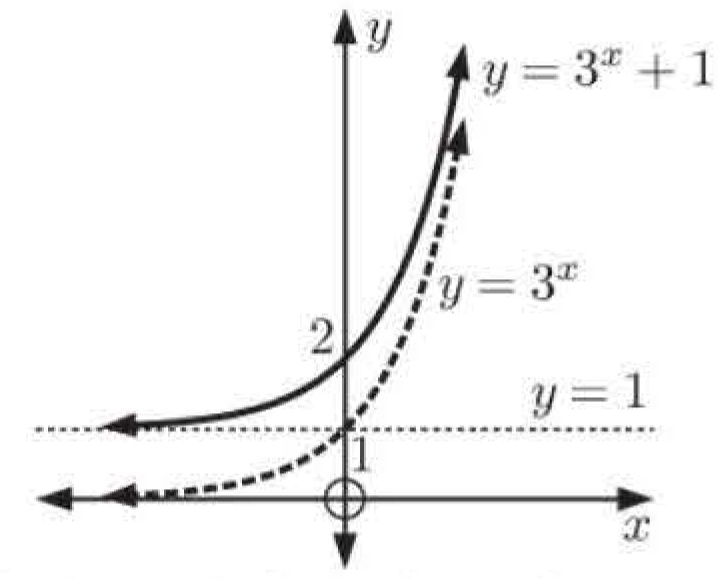
a vertical stretch of factor 2

5 **a**



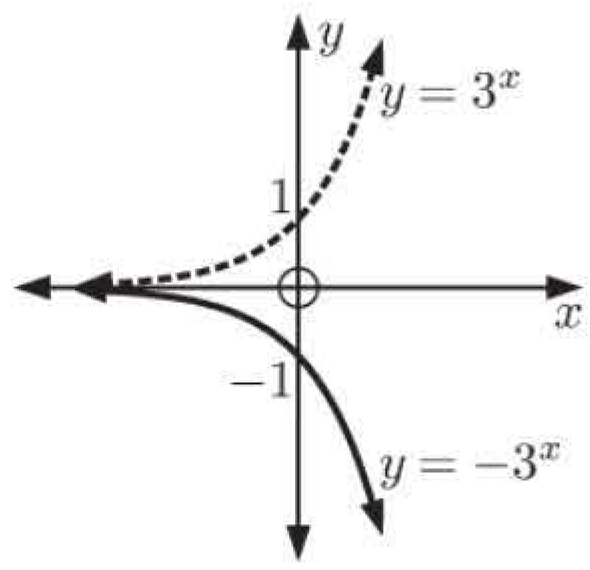
a reflection in the y-axis

b



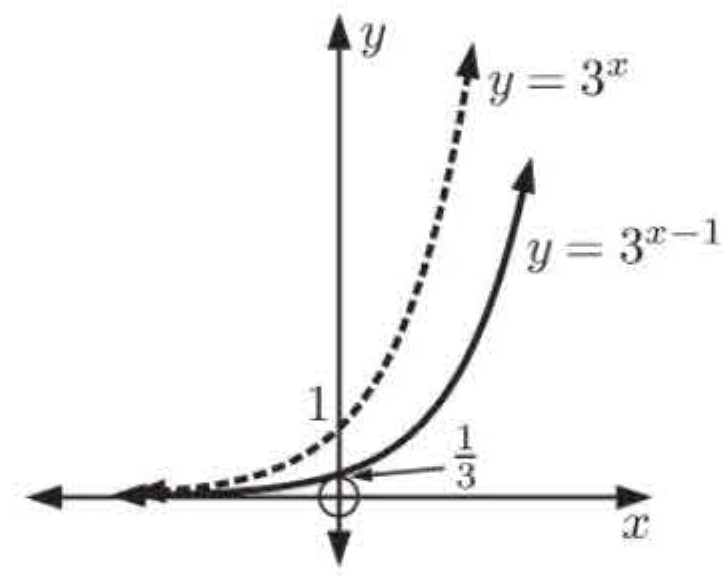
a vertical translation of 1 unit upwards
$y=1$ is the H.A.

c



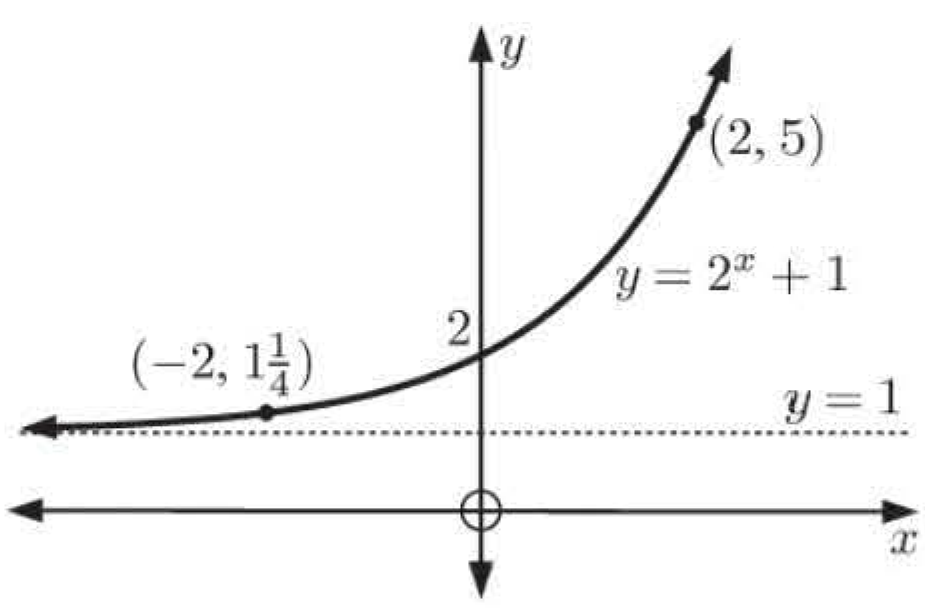
a reflection in the x-axis

d



a horizontal translation of 1 unit right

6 **a** **i**



a vertical translation of 1 unit upwards
When $x=2$, $y=4+1=5$
When $x=-2$, $y=\frac{1}{4}+1=1\frac{1}{4}$

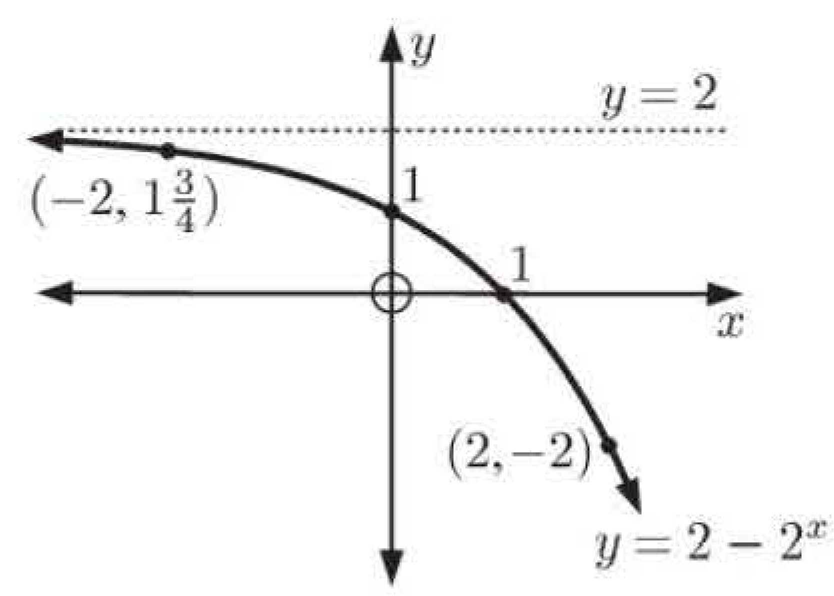
ii Domain $=\{x \mid x \in \mathbb{R}\}$,
Range $=\{y \mid y > 1\}$

iii Using technology, when
$x=\sqrt{2}$, $y \approx 3.67$

iv As $x \to \infty$, $y \to \infty$
As $x \to -\infty$, $y \to 1^+$

v The horizontal asymptote is $y=1$.

b **i**



When $x=0$, $y=2-2^0=2-1=1$
$\therefore$ the y-intercept is 1
When $x=1$, $y=2-2=0$
When $x=2$, $y=2-4=-2$
When $x=-2$, $y=2-\frac{1}{4}=1\frac{3}{4}$

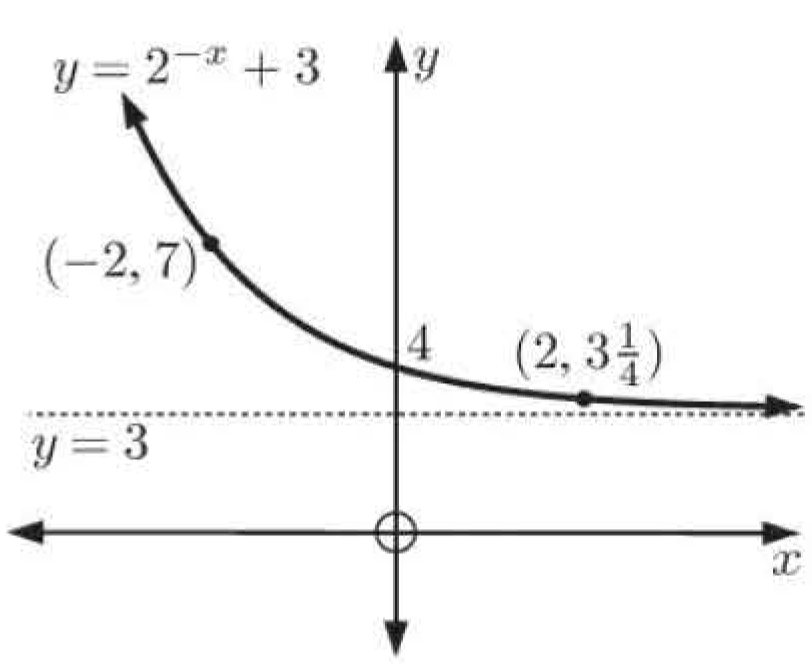
ii Domain $=\{x \mid x \in \mathbb{R}\}$,
Range $=\{y \mid y < 2\}$

iii Using technology, when
$x=\sqrt{2}$, $y \approx -0.665$

iv As $x \to \infty$, $y \to -\infty$
As $x \to -\infty$, $y \to 2^-$

v The horizontal asymptote is $y=2$.

c **i**



When $x=0$, $y=1+3=4$
When $x=2$, $y=\frac{1}{4}+3=3\frac{1}{4}$
When $x=-2$, $y=2^2+3=7$

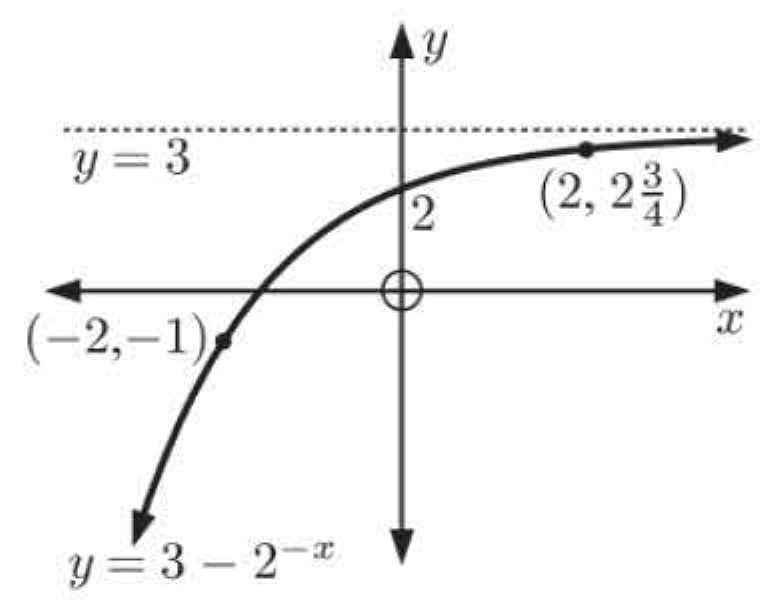
ii Domain $=\{x \mid x \in \mathbb{R}\}$,
Range $=\{y \mid y > 3\}$

iii Using technology, when
$x=\sqrt{2}$, $y \approx 3.38$

iv As $x \to \infty$, $y \to 3^+$
As $x \to -\infty$, $y \to \infty$

v The horizontal asymptote is $y=3$.

d **i**



When $x = 0,\quad y = 3 - 1 = 2$

When $x = 2,\quad y = 3 - \frac{1}{4} = 2\frac{3}{4}$

When $x = -2,\quad y = 3 - 4 = -1$

ii Domain $= \{x \mid x \in \mathbb{R}\}$,
Range $= \{y \mid y < 3\}$

iii Using technology, when
$x = \sqrt{2},\quad y \approx 2.62$

iv As $x \to \infty,\quad y \to 3^-$
As $x \to -\infty,\quad y \to -\infty$

v The horizontal asymptote is $y = 3$.

EXERCISE 3G.1

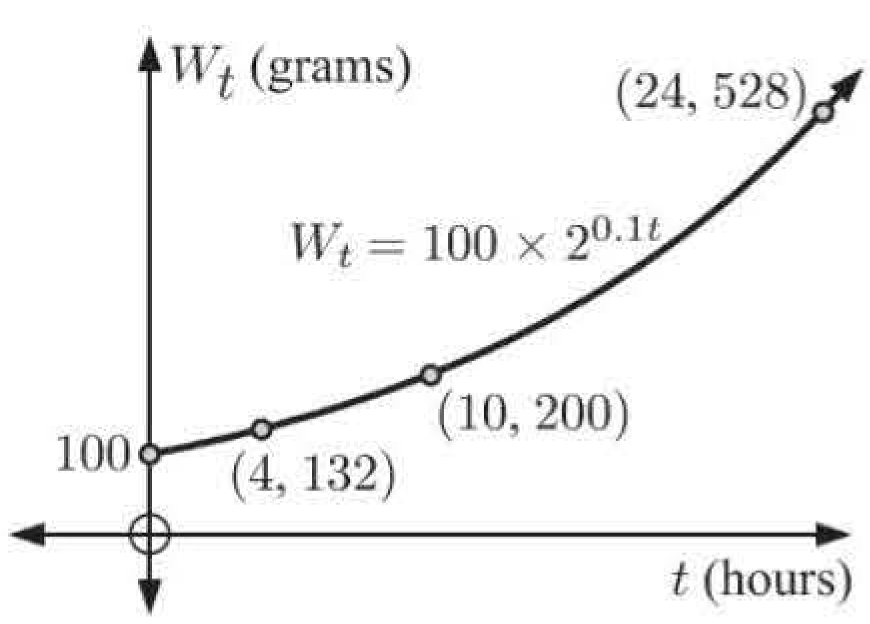
1 **a** When $t = 0$, $W_0 = 100$ grams = the initial weight

b **i** When $t = 4$,
$W_4 = 100 \times 2^{0.1\times 4}$
$= 100 \times 2^{0.4}$
≈ 132 grams

ii When $t = 10$,
$W_{10} = 100 \times 2^1$
$= 200$ grams

iii When $t = 24$, $W_{24} = 100 \times 2^{0.1\times 24}$
$= 100 \times 2^{2.4}$
≈ 528 grams

c



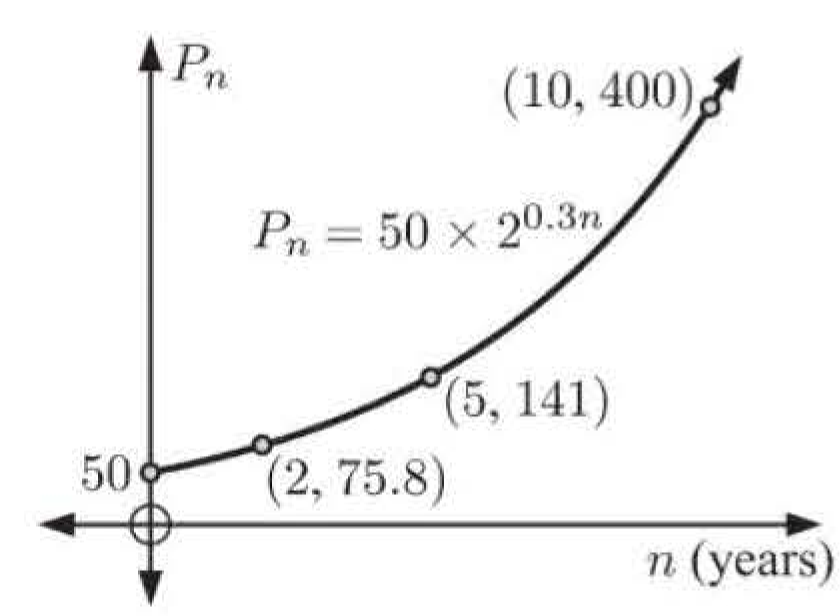
2 **a** $P_0 = 50$ (the initial population)

b **i** When $n = 2$,
$P_2 = 50 \times 2^{0.3\times 2}$
$= 50 \times 2^{0.6}$
≈ 75.785
So, the expected population is 76 possums.

ii When $n = 5$,
$P_5 = 50 \times 2^{0.3\times 5}$
$= 50 \times 2^{1.5}$
≈ 141.421
So, the expected population is 141 possums.

iii When $n = 10$, $P_{10} = 50 \times 2^{0.3\times 10}$
$= 50 \times 2^3 = 400$
So, the expected population is 400 possums.

c



3 **a** $B_0 = 6$ pairs $= 12$ bears

b In 2018, $t = 20$
$\therefore\ B_{20} = 12 \times 2^{0.18\times 20}$
$= 12 \times 2^{3.6}$
≈ 145.509
≈ 146 bears

c In 2008, $t = 10$

$$\therefore\ \%\text{ increase} = \left(\frac{B_{20} - B_{10}}{B_{10}}\right) \times 100\%$$
$$= \left(\frac{12 \times 2^{3.6} - 12 \times 2^{1.8}}{12 \times 2^{1.8}}\right) \times 100\%$$
$$= \left(\frac{2^{3.6} - 2^{1.8}}{2^{1.8}}\right) \times 100\%$$
$$\approx 248\%$$

4 **a** **i** When $t = 0$, $V_0 = V_0 \times 2^0$
$= V_0$
So, the speed is V_0.

ii When $t = 20$, $V_{20} = V_0 \times 2^{0.05\times 20}$
$= V_0 \times 2^1$
$= 2V_0$
So, the speed is $2V_0$.

b V_0 becomes $2V_0$. So, there was a 100% increase in speed.

c $$\left(\frac{V_{50}-V_{20}}{V_{20}}\right)\times 100\% = \left(\frac{V_0\times 2^{2.5}-V_0\times 2^1}{V_0\times 2^1}\right)\times 100\%$$
$$= \left(\frac{2^{2.5}-2^1}{2^1}\right)\times 100\%$$
$$\approx 183\%$$

This expression is the percentage increase in speed from the speed at 20°C to the speed at 50°C. ($V_{50}-V_{20}$ is the increase in speed.)

EXERCISE 3G.2

1 $W(t) = 250\times(0.998)^t$ grams

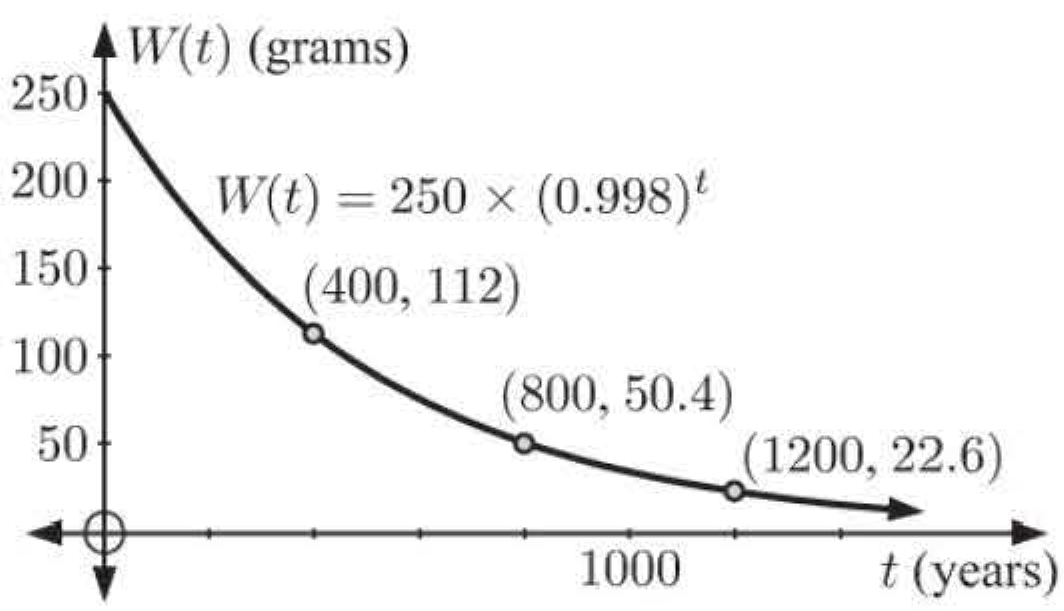
a $W(0) = 250\times(0.998)^0$
$= 250\times 1 = 250$ grams $\quad\therefore$ 250 g of radioactive substance was put aside.

b **i** When $t = 400$,
$W(400)$
$= 250\times(0.998)^{400}$
≈ 112 grams

ii When $t = 800$,
$W(800)$
$= 250\times(0.998)^{800}$
≈ 50.4 grams

iii When $t = 1200$,
$W(1200)$
$= 250\times(0.998)^{1200}$
≈ 22.6 grams

c



d When $W(t) = 125$
$250\times(0.998)^t = 125$
$\therefore\ (0.998)^t = 0.5$
$\therefore\ t\approx 346.2$ {using technology}

It takes approximately 346 years.

2 $T(t) = 100\times 2^{-0.02t}$

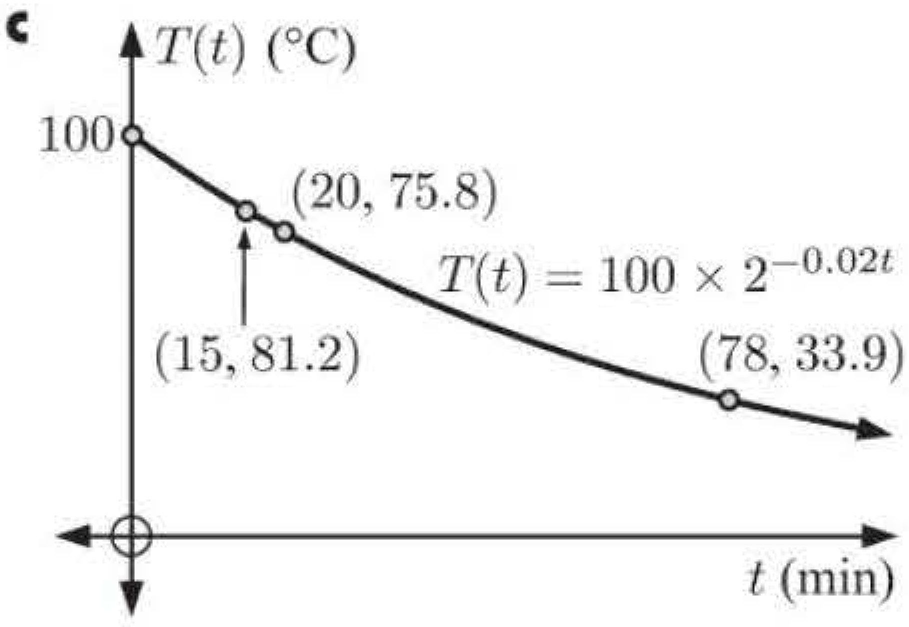
a $T(0) = 100\times 2^0$
$= 100\times 1$
$= 100$°C

b **i** $T(15) = 100\times 2^{-0.02\times 15}$
$= 100\times 2^{-0.3}$
≈ 81.2°C

ii $T(20) = 100\times 2^{-0.02\times 20}$
$= 100\times 2^{-0.4}$
≈ 75.8°C

iii $T(78) = 100\times 2^{-0.02\times 78}$
$= 100\times 2^{-1.56}$
≈ 33.9°C

c



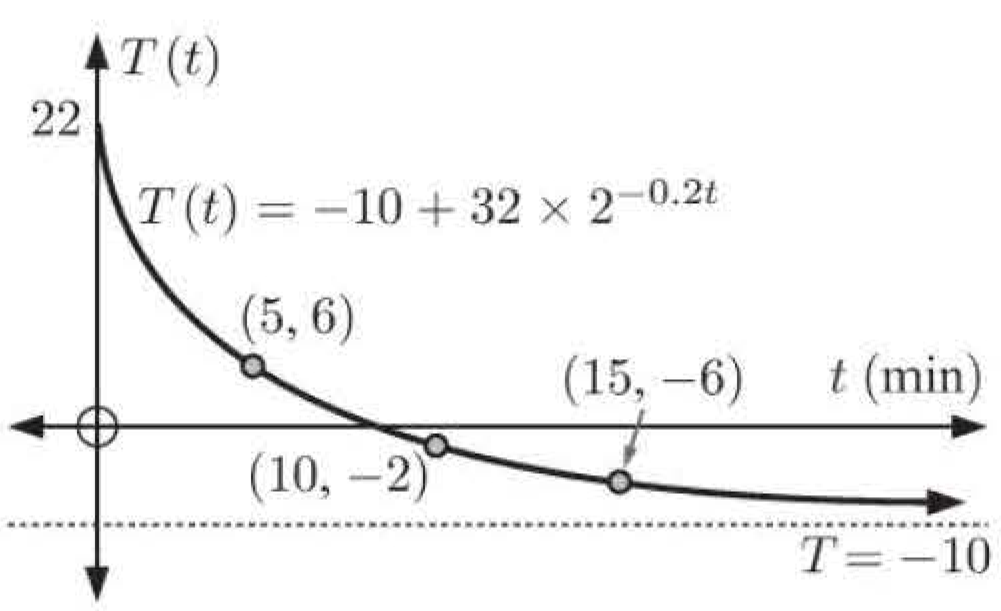
3 **a** **i** $T(0) = -10+32\times 2^0$
$= -10+32\times 1 = 22$°C

ii $T(5) = -10+32\times 2^{-0.2\times 5}$
$= -10+32\times 2^{-1} = 6$°C

iii $T(10) = -10+32\times 2^{-0.2\times 10}$
$= -10+32\times 2^{-2} = -2$°C

iv $T(15) = -10+32\times 2^{-0.2\times 15}$
$= -10+32\times 2^{-3} = -6$°C

b



c $32\times 2^{-0.2t}$ is always > 0 since 2^t is always > 0
$\therefore\ -10+32\times 2^{-0.2t}$ is always > -10
$\therefore$ the temperature of the packet of peas will never reach -10°C.

4 $W_t = 1000 \times 2^{-0.03t}$

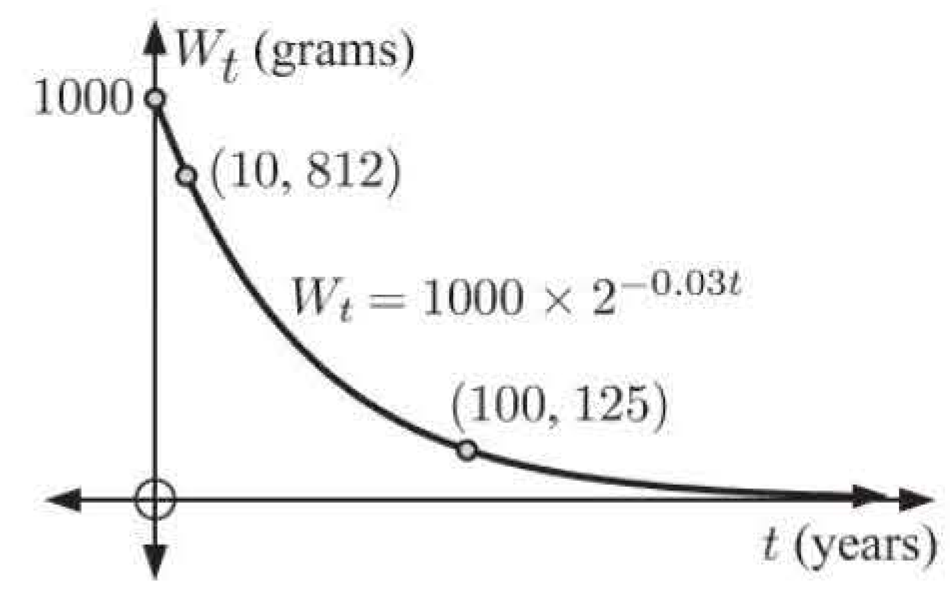
a $W_0 = 1000 \times 2^0$
$= 1000 \times 1$
$= 1000$ g

b **i** W_{10}
$= 1000 \times 2^{-0.3}$
≈ 812 g

ii W_{100}
$= 1000 \times 2^{-3}$
$= 125$ g

iii W_{1000}
$= 1000 \times 2^{-30}$
$\approx 9.31 \times 10^{-7}$ g

c



d When $W_t = 10$, $1000 \times 2^{-0.03t} = 10$
$\therefore\ (2^{-0.03})^t = 0.01$
$\therefore\ t \approx 221.46$ {using technology}

There is 10 g of the substance remaining after approximately 221 years.

e Initial weight $= W_0 = 1000$ g

Amount remaining after t years $= W_t = 1000 \times 2^{-0.03t}$

Amount that has decayed after t years $= W_0 - W_t$
$= 1000 - 1000 \times 2^{-0.03t}$
$= 1000(1 - 2^{-0.03t})$ g

5 **a** When $t = 0$, $W_0 = W_0 2^0$
$= W_0$ grams

$\therefore$ the original weight was W_0 grams.

b $\%$ change $= \left(\dfrac{W_{1000} - W_0}{W_0}\right) \times 100\%$
$= \left(\dfrac{W_0 \times 2^{-0.2} - W_0}{W_0}\right) \times 100\%$
$= (2^{-0.2} - 1) \times 100\%$
$\approx -12.9\%$

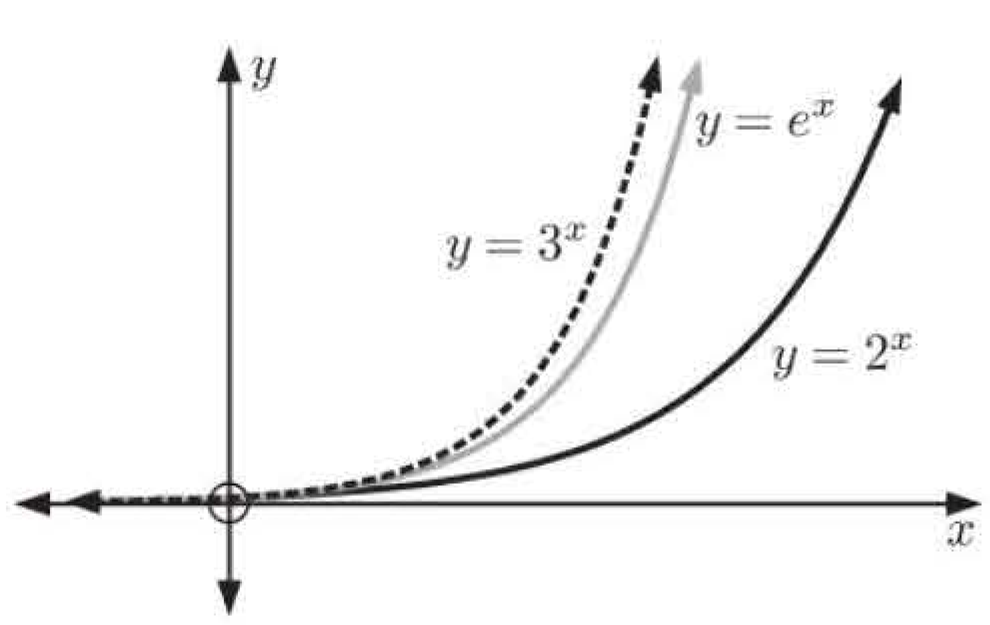
The weight loss was about 12.9%.

c $W_0 \times 2^{-0.0002t} = \frac{1}{512} W_0$
$\therefore\ (2^{-0.0002})^t = \frac{1}{512}$
$\therefore\ t = 45\,000$ {using technology}

It would take 45 000 years.

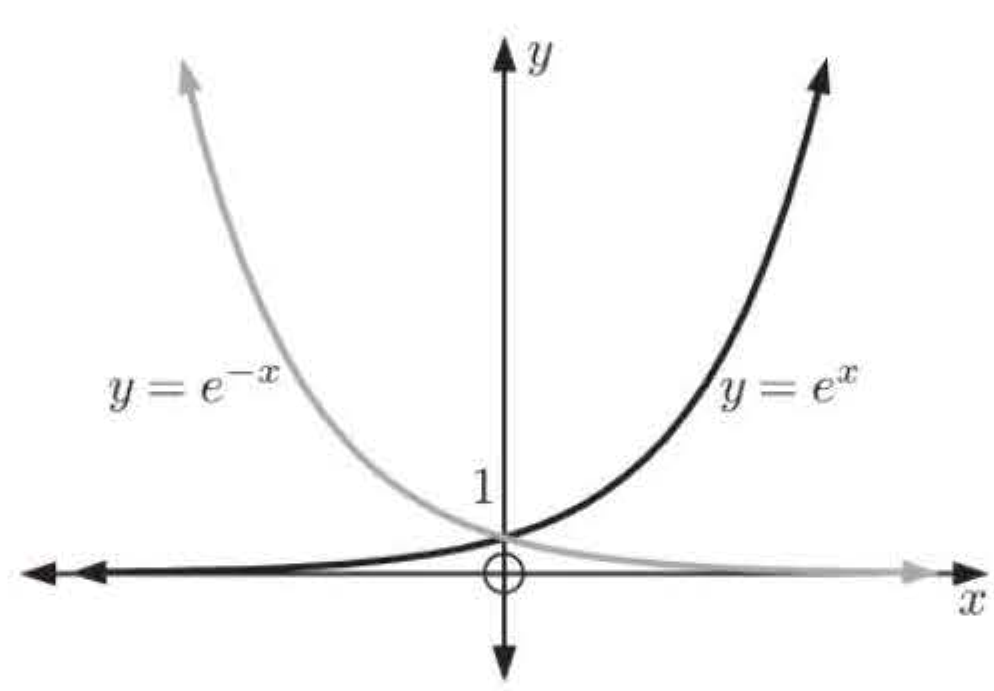
EXERCISE 3H

1



The graph of $y = e^x$ lies between $y = 2^x$ and $y = 3^x$.

2



One is the other reflected in the y-axis.

3 When $x = 0$, $y = ae^0 = a \times 1 = a$ $\therefore$ the y-intercept is a.

4 **a** The graph of $y = e^x$ is entirely above the x-axis.
$y > 0$ for all x
$\therefore\ e^x > 0$ for all x
$\therefore\ 2e^x > 0$ for all x
$\therefore\ y = 2e^x$ cannot be negative.

b **i** When $x = -20$, $y = 2e^{-20} \approx 4.12 \times 10^{-9}$
$\approx 0.000\,000\,004\,12$

ii When $x = 20$, $y = 2e^{20} \approx 9.70 \times 10^8$
$\approx 970\,000\,000$

5 **a** $e^2 \approx 7.39$ **b** $e^3 \approx 20.1$ **c** $e^{0.7} \approx 2.01$ **d** $\sqrt{e} \approx 1.65$ **e** $e^{-1} \approx 0.368$

6 **a** $\sqrt{e} = e^{\frac{1}{2}}$

b $\dfrac{1}{\sqrt{e}} = \dfrac{1}{e^{\frac{1}{2}}} = e^{-\frac{1}{2}}$

c $\dfrac{1}{e^2} = e^{-2}$

d $e\sqrt{e} = e^1 e^{\frac{1}{2}} = e^{\frac{3}{2}}$

7 **a** $\left(e^{0.36}\right)^{\frac{t}{2}} = e^{0.36 \times \frac{t}{2}} = e^{0.18t}$

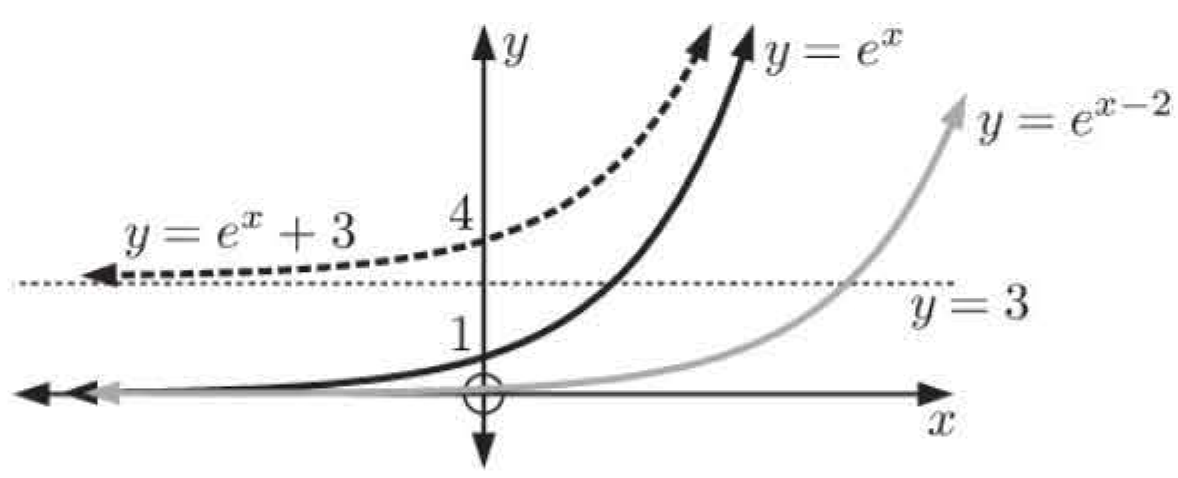
b $\left(e^{0.064}\right)^{\frac{t}{16}} = e^{0.064 \times \frac{t}{16}} = e^{0.004t}$

c $\left(e^{-0.04}\right)^{\frac{t}{8}} = e^{-0.04 \times \frac{t}{8}} = e^{-0.005t}$

d $\left(e^{-0.836}\right)^{\frac{t}{5}} = e^{-0.836 \times \frac{t}{5}} \approx e^{-0.167t}$

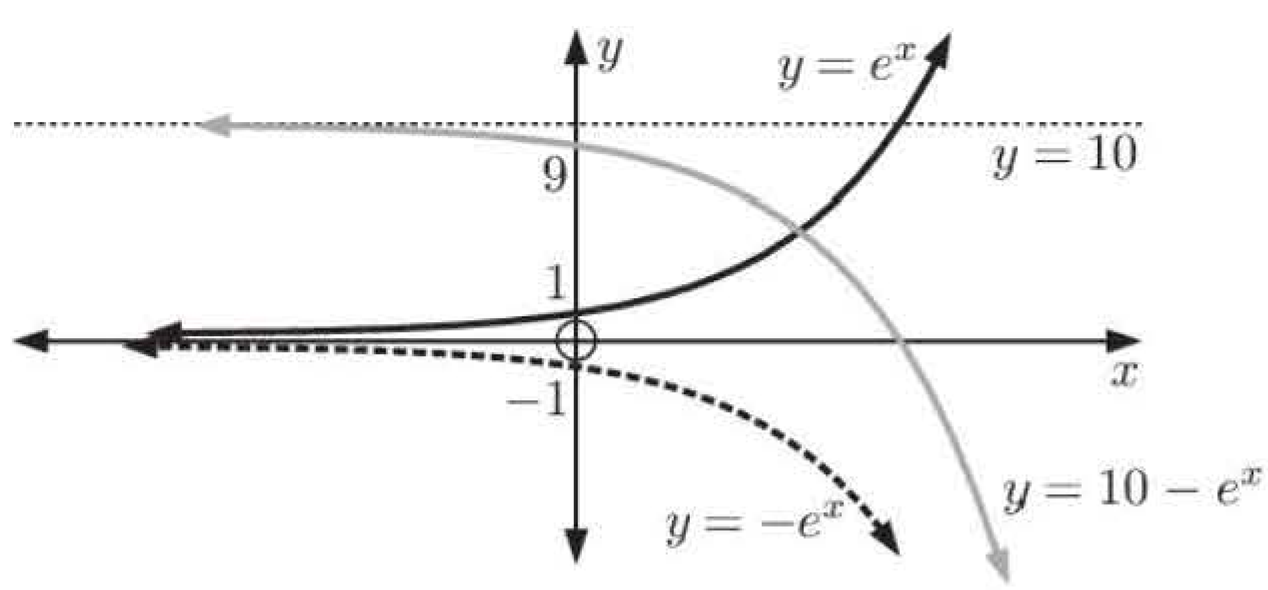
8 **a** ≈ 10.074 **b** $\approx 0.099\,261$ **c** ≈ 125.09 **d** $\approx 0.007\,994\,5$
e ≈ 41.914 **f** ≈ 42.429 **g** ≈ 3540.3 **h** $\approx 0.006\,342\,4$

9



Domain of f, g, and h is $\{x \mid x \in \mathbb{R}\}$
Range of f is $\{y \mid y > 0\}$
Range of g is $\{y \mid y > 0\}$
Range of h is $\{y \mid y > 3\}$

10



Domain of f, g, and h is $\{x \mid x \in \mathbb{R}\}$
Range of f is $\{y \mid y > 0\}$
Range of g is $\{y \mid y < 0\}$
Range of h is $\{y \mid y < 10\}$

11 **a**
$$\begin{aligned}&(e^x+1)^2\\ &= (e^x)^2 + 2 \times e^x \times 1 + 1^2\\ &= e^{2x} + 2e^x + 1\end{aligned}$$

b
$$\begin{aligned}&(1+e^x)(1-e^x)\\ &= 1^2 - (e^x)^2\\ &= 1 - e^{2x}\end{aligned}$$

c
$$\begin{aligned}&e^x(e^{-x} - 3)\\ &= e^x \times e^{-x} - e^x \times 3\\ &= e^0 - 3e^x\\ &= 1 - 3e^x\end{aligned}$$

12 $W(t) = 2e^{\frac{t}{2}}$ grams

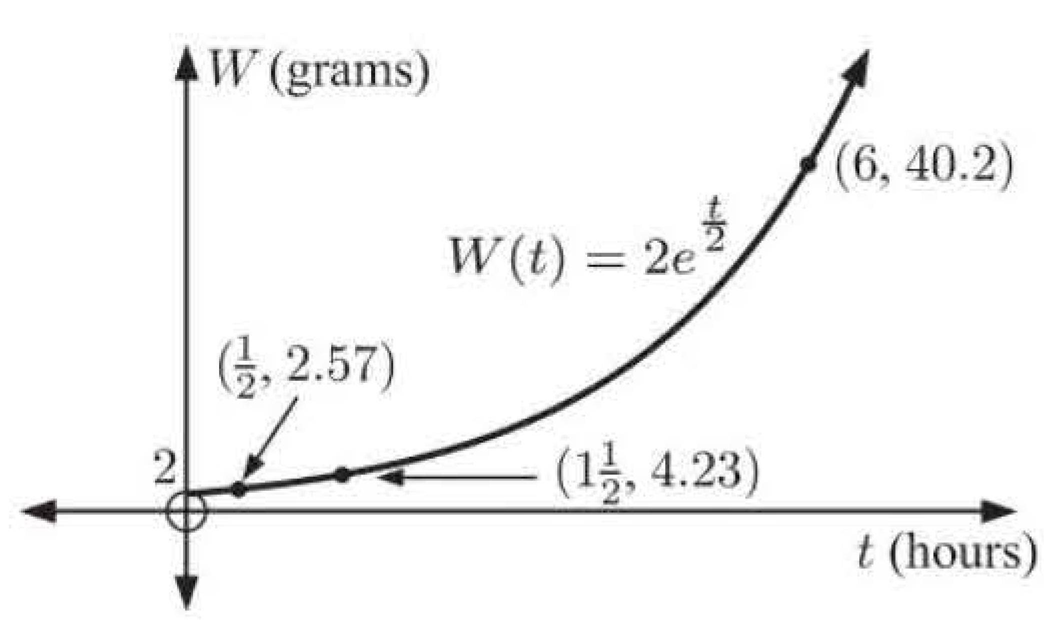
a **i** $W(0) = 2e^0 = 2 \times 1 = 2$ g

ii $W(\frac{1}{2}) = 2e^{\frac{1}{4}} \approx 2.57$ g

iii $W(1\frac{1}{2}) = 2e^{\frac{3}{4}} \approx 4.23$ g

iv $W(6) = 2e^3 \approx 40.2$ g

b



13 **a**
$$\begin{aligned}e^x &= \sqrt{e}\\ \therefore\ e^x &= e^{\frac{1}{2}}\\ \therefore\ x &= \tfrac{1}{2}\end{aligned}$$

b
$$\begin{aligned}e^{\frac{1}{2}x} &= \frac{1}{e^2}\\ \therefore\ e^{\frac{1}{2}x} &= e^{-2}\\ \therefore\ \tfrac{1}{2}x &= -2\\ \therefore\ x &= -4\end{aligned}$$

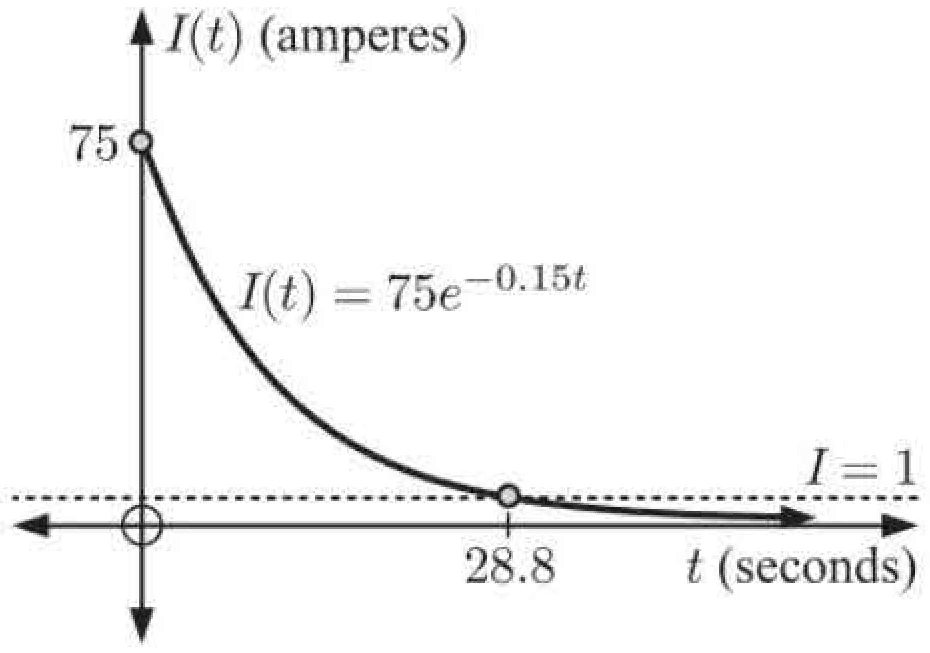
14 $I(t) = 75e^{-0.15t}$

a **i** $I(1) = 75e^{-0.15}$
≈ 64.6 amps

ii $I(10) = 75e^{-1.5}$
≈ 16.7 amps

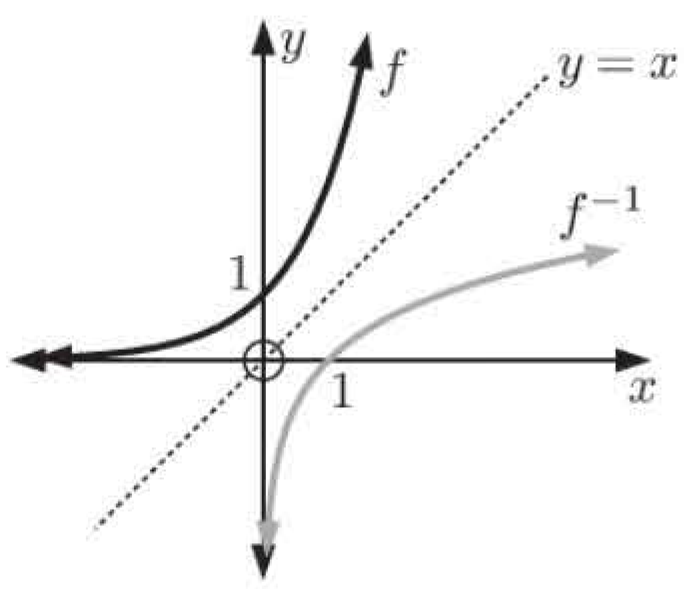
c We need to solve $75e^{-0.15t} = 1$.
Using technology, $t \approx 28.8$ s

b



15 $f(x) = e^x$

a



b Domain of f^{-1} is $\{x \mid x > 0\}$
Range of f^{-1} is $\{y \mid y \in \mathbb{R}\}$

REVIEW SET 3A

1 **a** $-(-1)^{10}$
$= -1$

b $-(-3)^3$
$= -(-27)$
$= 27$

c $3^0 - 3^{-1}$
$= 1 - \frac{1}{3}$
$= \frac{2}{3}$

2 **a** $a^4b^5 \times a^2b^2$
$= a^{4+2} \times b^{5+2}$
$= a^6b^7$

b $6xy^5 \div 9x^2y^5$
$= \frac{6}{9}x^{1-2}y^{5-5}$
$= \frac{2}{3}x^{-1}y^0$
$= \dfrac{2}{3x}$

c $\dfrac{5(x^2y)^2}{(5x^2)^2}$
$= \dfrac{5 \times x^4y^2}{25x^4}$
$= \frac{1}{5}x^0y^2$
$= \dfrac{y^2}{5}$

3 **a** **i** $f(4) = 3^4$
$= 81$

ii $f(-1) = 3^{-1}$
$= \frac{1}{3}$

b $f(x+2) = kf(x)$
$\therefore\ 3^{x+2} = k \times 3^x$
$\therefore\ 3^2 \times 3^x = k \times 3^x$
$\therefore\ 3^2 = k$
$\therefore\ k = 9$

4 **a** $x^{-2} \times x^{-3}$
$= x^{-2+(-3)}$
$= x^{-5}$
$= \dfrac{1}{x^5}$

b $2(ab)^{-2}$
$= 2 \times \dfrac{1}{(ab)^2}$
$= \dfrac{2}{a^2b^2}$

c $2ab^{-2}$
$= 2a \times \left(\dfrac{1}{b^2}\right)$
$= \dfrac{2a}{b^2}$

5 **a** $\dfrac{27}{9^a} = \dfrac{3^3}{(3^2)^a}$
$= 3^{3-2a}$

b $\left(\sqrt{3}\right)^{1-x} \times 9^{1-2x} = (3^{\frac{1}{2}})^{1-x} \times (3^2)^{1-2x}$
$= 3^{\frac{1}{2}-\frac{1}{2}x+2-4x}$
$= 3^{\frac{5}{2}-\frac{9}{2}x}$

6 **a** $8^{\frac{2}{3}} = \left(2^3\right)^{\frac{2}{3}} = 2^2 = 4$

b $27^{-\frac{2}{3}} = \left(3^3\right)^{-\frac{2}{3}} = 3^{-2} = \dfrac{1}{3^2} = \frac{1}{9}$

7 **a**
$$\begin{aligned} & mn^{-2} \\ &= m \times \frac{1}{n^2} \\ &= \frac{m}{n^2} \end{aligned}$$

b
$$\begin{aligned} & (mn)^{-3} \\ &= \frac{1}{(mn)^3} \\ &= \frac{1}{m^3 n^3} \end{aligned}$$

c
$$\begin{aligned} & \frac{m^2 n^{-1}}{p^{-2}} \\ &= m^2 \left(\frac{1}{n}\right) p^2 \\ &= \frac{m^2 p^2}{n} \end{aligned}$$

d
$$\begin{aligned} & (4m^{-1}n)^2 \\ &= 4^2 m^{-2} n^2 \\ &= \frac{16n^2}{m^2} \end{aligned}$$

8 **a**
$$\begin{aligned} & (3 - e^x)^2 \\ &= 3^2 - 2 \times 3 \times e^x + (e^x)^2 \\ &= 9 - 6e^x + e^{2x} \end{aligned}$$

b
$$\begin{aligned} & (\sqrt{x} + 2)(\sqrt{x} - 2) \\ &= (\sqrt{x})^2 - 2^2 \\ &= x - 4 \end{aligned}$$

c
$$\begin{aligned} & 2^{-x}(2^{2x} + 2^x) \\ &= 2^{-x+2x} + 2^{-x+x} \\ &= 2^x + 2^0 \\ &= 2^x + 1 \end{aligned}$$

9 **a**
$$\begin{aligned} 2^{x-3} &= \tfrac{1}{32} \\ \therefore \quad 2^{x-3} &= 2^{-5} \\ \therefore \quad x - 3 &= -5 \\ \therefore \quad x &= -2 \end{aligned}$$

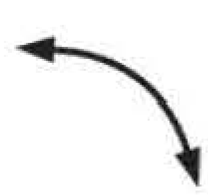
b
$$\begin{aligned} 9^x &= 27^{2-2x} \\ \therefore \quad (3^2)^x &= (3^3)^{2-2x} \\ \therefore \quad 2x &= 6 - 6x \\ \therefore \quad 8x &= 6 \\ \therefore \quad x &= \tfrac{6}{8} = \tfrac{3}{4} \end{aligned}$$

c
$$\begin{aligned} e^{2x} &= \frac{1}{\sqrt{e}} \\ \therefore \quad e^{2x} &= e^{-\frac{1}{2}} \\ \therefore \quad 2x &= -\tfrac{1}{2} \\ \therefore \quad x &= -\tfrac{1}{4} \end{aligned}$$

10 Use the general exponential function $y = a \times b^{x-c} + d$.

a $y = -e^x$

$\left.\begin{array}{ll} a = -1 & \therefore\ a < 0 \\ b = e & \therefore\ b > 1 \end{array}\right\}$ function is decreasing



When $x = 0$, $y = -e^0 = -1$

$\therefore$ y-intercept is $y = -1$.

$\therefore$ the graph is **C**.

b $y = 3 \times 2^x$

$\left.\begin{array}{ll} a = 3 & \therefore\ a > 0 \\ b = 2 & \therefore\ b > 1 \end{array}\right\}$ function is increasing

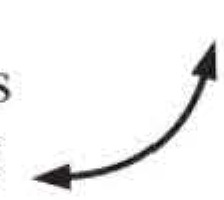
When $x = 0$, $y = 3 \times 2^0 = 3$

$\therefore$ y-intercept is $y = 3$.

$\therefore$ the graph is **E**.

c $y = e^x + 1$

$\left.\begin{array}{ll} a = 1 & \therefore\ a > 0 \\ b = e & \therefore\ b > 1 \end{array}\right\}$ function is increasing



When $x = 0$, $y = e^0 + 1 = 2$

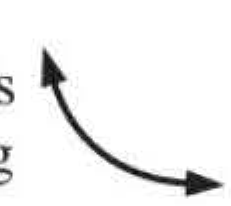
$\therefore$ y-intercept is $y = 2$.

$d = 1$ $\therefore$ $y = 1$ is a horizontal asymptote.

$\therefore$ the graph is **A**.

d $y = 3^{-x} = \dfrac{1}{3^x} = \left(\frac{1}{3}\right)^x$

$\left.\begin{array}{ll} a = 1 & \therefore\ a > 0 \\ b = \frac{1}{3} & \therefore\ 0 < b < 1 \end{array}\right\}$ function is decreasing



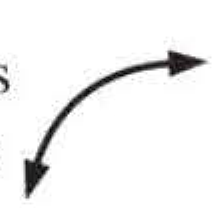
When $x = 0$, $y = 3^0 = 1$

$\therefore$ y-intercept is $y = 1$.

$\therefore$ the graph is **B**.

e $y = -e^{-x} = -\dfrac{1}{e^x} = -\left(\dfrac{1}{e}\right)^x$

$\left.\begin{array}{ll} a = -1 & \therefore\ a < 0 \\ b = \dfrac{1}{e} & \therefore\ 0 < b < 1 \end{array}\right\}$ function is increasing



When $x = 0$, $y = -e^0 = -1$

$\therefore$ y-intercept is $y = -1$.

$\therefore$ the graph is **D**.

11 $y = a^x$

a $a^{2x} = (a^x)^2 = y^2$

b $a^{-x} = (a^x)^{-1} = y^{-1}$

c $\dfrac{1}{\sqrt{a^x}} = \dfrac{1}{\sqrt{y}} = y^{-\frac{1}{2}}$

REVIEW SET 3B

1 **a** 4×2^n
$= 2^2 \times 2^n$
$= 2^{n+2}$

b $7^{-1} - 7^0$
$= \frac{1}{7} - 1$
$= -\frac{6}{7}$

c $\left(\frac{2}{3}\right)^{-3}$
$= \left(\frac{3}{2}\right)^3$
$= \frac{27}{8}$
$= 3\frac{3}{8}$

d $\left(\dfrac{2a^{-1}}{b^2}\right)^2$
$= \dfrac{2^2a^{-2}}{b^4}$
$= \dfrac{4}{a^2b^4}$

2 **a** $3^{\frac{3}{4}} \approx 2.28$ **b** $27^{-\frac{1}{5}} \approx 0.517$ **c** $\sqrt[4]{100} \approx 3.16$

3 $f(x) = 3 \times 2^x$

a $f(0) = 3 \times 2^0$
$= 3 \times 1$
$= 3$

b $f(3) = 3 \times 2^3$
$= 3 \times 8$
$= 24$

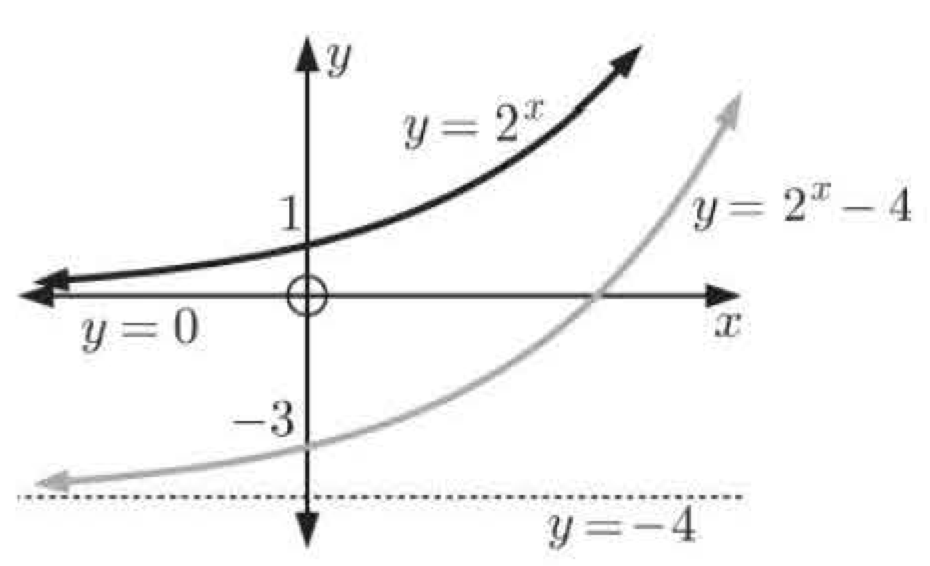
c $f(-2) = 3 \times 2^{-2}$
$= 3 \times \dfrac{1}{2^2} = \frac{3}{4}$

4 $f(x) = 2^{-x} + 1$

a $f(\frac{1}{2}) = 2^{-\frac{1}{2}} + 1$
$= \dfrac{1}{\sqrt{2}} + 1$
≈ 1.71

b $f(a) = 3$
$\therefore\ 2^{-a} + 1 = 3$
$\therefore\ 2^{-a} = 2$
$\therefore\ 2^{-a} = 2^1$
$\therefore\ -a = 1$
$\therefore\ a = -1$

5



$y = 2^x$ has y-intercept 1 and horizontal asymptote $y = 0$
$y = 2^x - 4$ has y-intercept -3 and horizontal asymptote $y = -4$

6 $T = 80 \times (0.913)^t$ °C

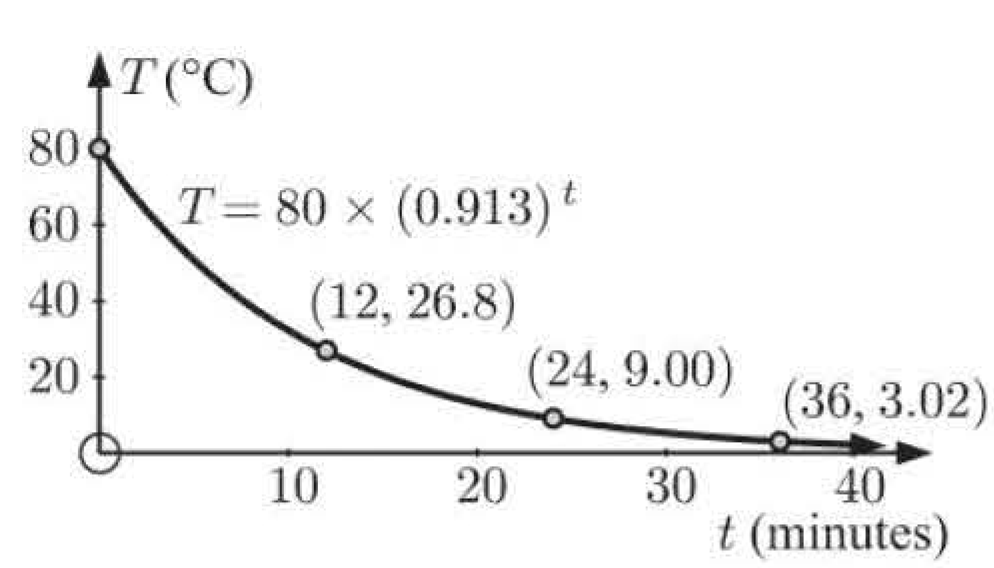
a When $t = 0$, $T = 80 \times (0.913)^0$
$= 80 \times 1$
$= 80$ $\therefore$ the initial temperature was 80°C.

b **i** When $t = 12$,
$T = 80 \times (0.913)^{12}$
≈ 26.8°C

ii When $t = 24$,
$T = 80 \times (0.913)^{24}$
≈ 9.00°C

iii When $t = 36$,
$T = 80 \times (0.913)^{36}$
≈ 3.02°C

c

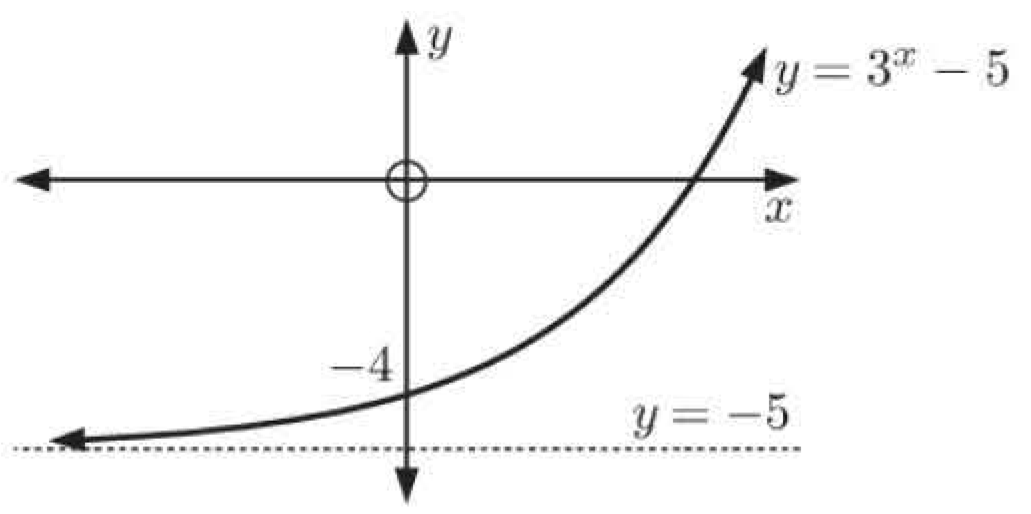


d When $T = 25$
$80 \times (0.913)^t = 25$
$\therefore\ 0.913^t = 0.3125$
$\therefore\ t \approx 12.8$ min {using technology}

7 **a** When $x = 0$, $y = 3^0 - 5 = 1 - 5 = -4$
When $x = 1$, $y = 3^1 - 5 = 3 - 5 = -2$
When $x = 2$, $y = 3^2 - 5 = 9 - 5 = 4$
When $x = -1$, $y = 3^{-1} - 5 = \frac{1}{3} - 5 = -4\frac{2}{3}$
When $x = -2$, $y = 3^{-2} - 5 = \frac{1}{9} - 5 = -4\frac{8}{9}$

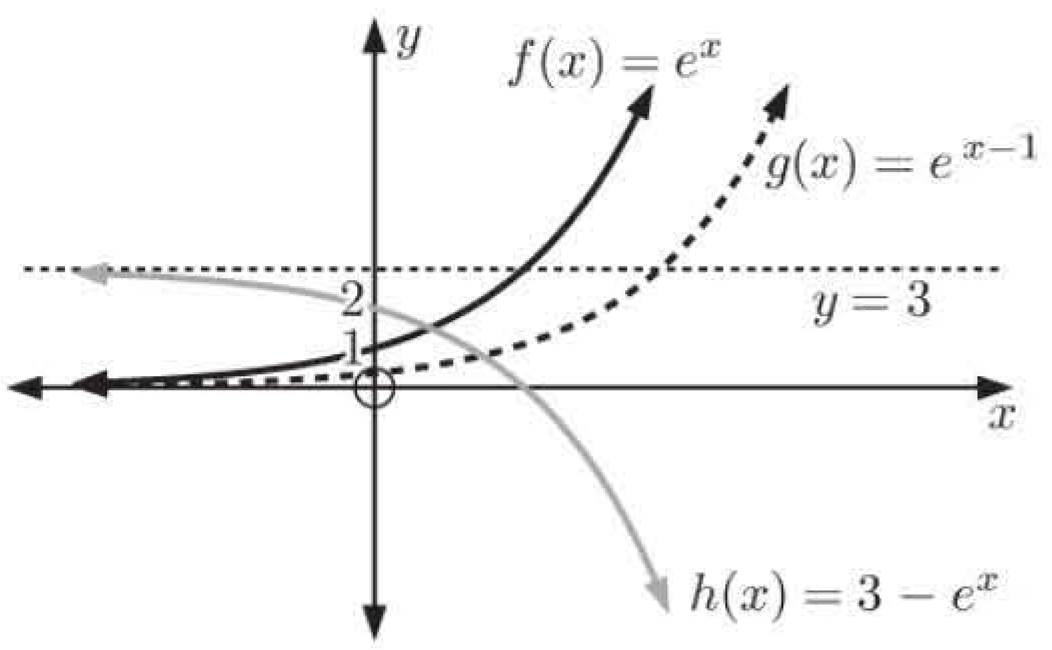
b As $x \to \infty$, $3^x \to \infty$ and so $y \to \infty$
As $x \to -\infty$, $3^x \to 0$ and so $y \to -5^+$

c



d $y = -5$ is the horizontal asymptote.

8 **a**

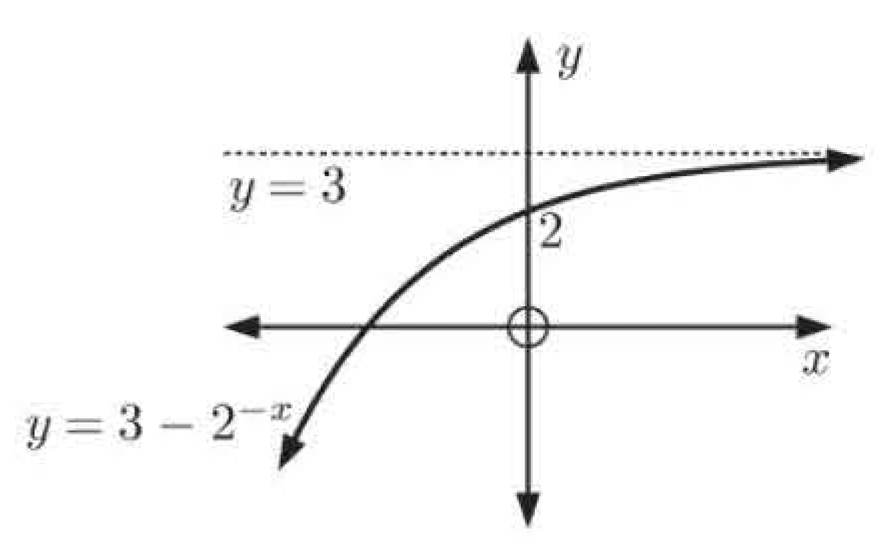


b Domain of f, g, and h is $\{x \mid x \in \mathbb{R}\}$
Range of f is $\{y \mid y > 0\}$
Range of g is $\{y \mid y > 0\}$
Range of h is $\{y \mid y < 3\}$

9 **a** When $x = 0$, $y = 3 - 2^0 = 3 - 1 = 2$
When $x = 1$, $y = 3 - 2^{-1} = 3 - \frac{1}{2} = 2\frac{1}{2}$
When $x = 2$, $y = 3 - 2^{-2} = 3 - \frac{1}{4} = 2\frac{3}{4}$
When $x = -1$, $y = 3 - 2^1 = 3 - 2 = 1$
When $x = -2$, $y = 3 - 2^2 = 3 - 4 = -1$

b As $x \to \infty$, $2^{-x} \to 0$, $\therefore\ y \to 3^-$
As $x \to -\infty$, $2^{-x} \to \infty$, $\therefore\ y \to -\infty$

c



d horizontal asymptote is $y = 3$

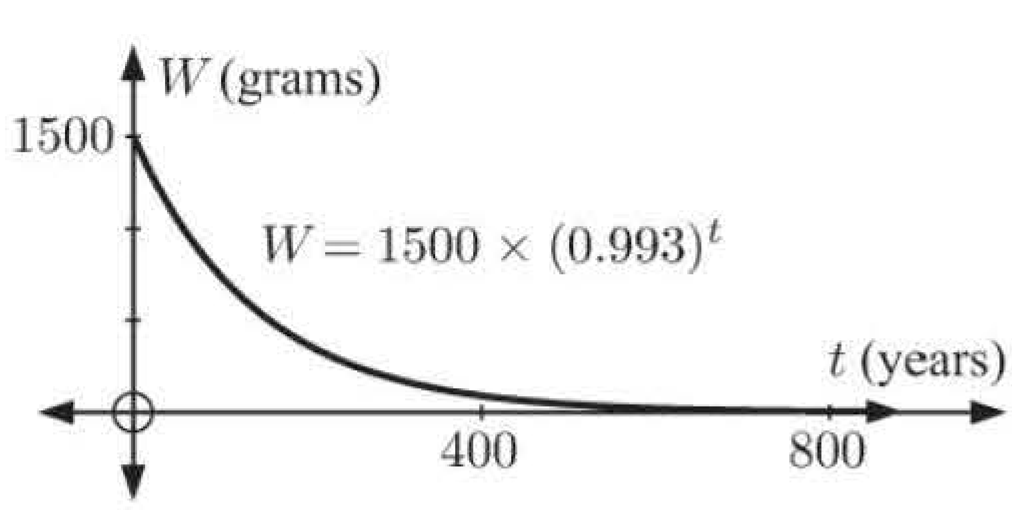
10 $W = 1500 \times (0.993)^t$ grams

a When $t = 0$,
$W = 1500 \times (0.993)^0$
$= 1500 \times 1$
$= 1500$ grams

b **i** When $t = 400$,
$W = 1500 \times (0.993)^{400}$
≈ 90.3 grams

ii When $t = 800$,
$W = 1500 \times (0.993)^{800}$
≈ 5.44 grams

c



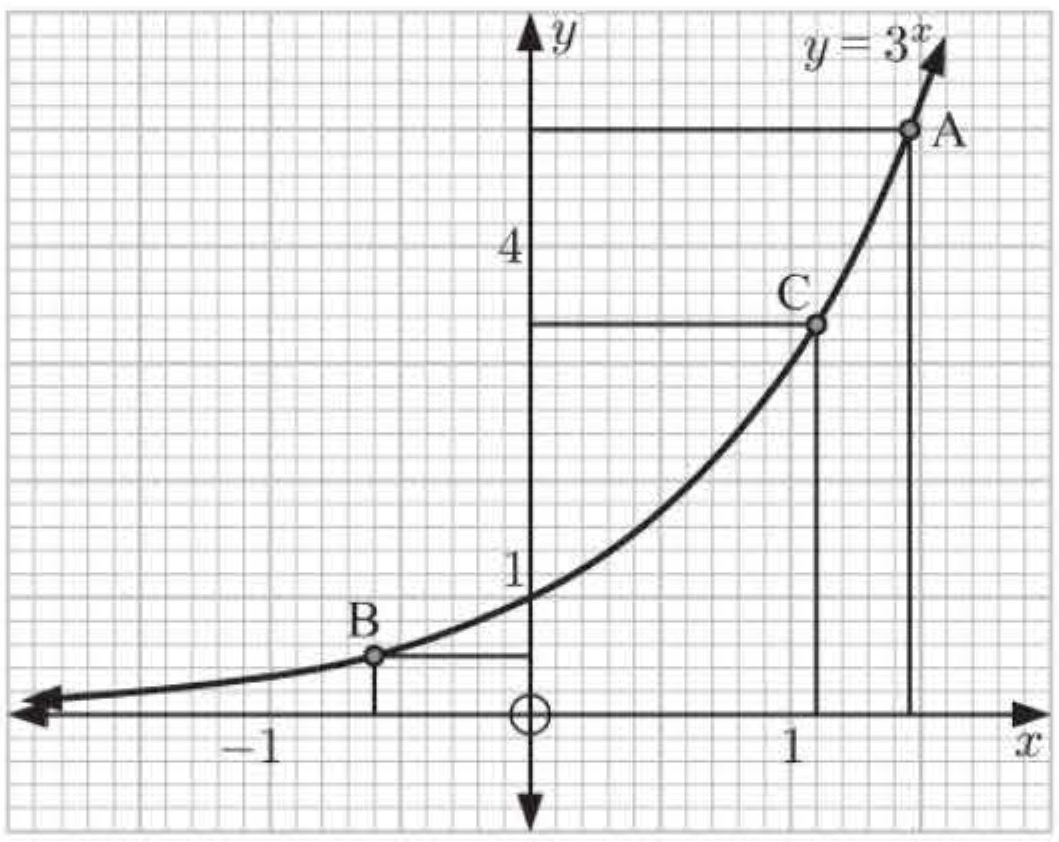
d When $W = 100$,
$1500 \times (0.993)^t = 100$
$\therefore\ (0.993)^t \approx 0.0667$
$\therefore\ t \approx 385.5$ {using technology}
So, it will take about 386 years.

REVIEW SET 3C

1 **a** When $y = 3^x = 5$,
$x \approx 1.5$ from point A.

b When $y = 3^x = \frac{1}{2}$,
$x \approx -0.6$ from point B.

c $6 \times 3^x = 20$
$\therefore\ 3^x = \frac{20}{6} = 3\frac{1}{3}$
When $y = 3^x = 3\frac{1}{3}$,
$x \approx 1.1$ from point C.



2 **a** $(a^7)^3$
$= a^{7\times 3}$
$= a^{21}$

b $pq^2 \times p^3q^4$
$= p^{1+3}q^{2+4}$
$= p^4q^6$

c $\dfrac{8ab^5}{2a^4b^4}$
$= \frac{8}{2}a^{1-4}b^{5-4}$
$= 4a^{-3}b^1$
$= \dfrac{4b}{a^3}$

3 **a** 2×2^{-4}
$= 2^1 \times 2^{-4}$
$= 2^{1+(-4)}$
$= 2^{-3}$

b $16 \div 2^{-3}$
$= 2^4 \div 2^{-3}$
$= 2^{4-(-3)}$
$= 2^7$

c 8^4
$= (2^3)^4$
$= 2^{12}$

4 **a** $b^{-3} = \dfrac{1}{b^3}$

b $(ab)^{-1}$
$= a^{-1}b^{-1}$
$= \dfrac{1}{ab}$

c ab^{-1}
$= a \times \dfrac{1}{b}$
$= \dfrac{a}{b}$

5 $\dfrac{2^{x+1}}{2^{1-x}} = 2^{x+1-(1-x)}$
$= 2^{x+1-1+x}$
$= 2^{2x}$

6 **a** $1 = 5^0$

b $5\sqrt{5}$
$= 5^1 \times 5^{\frac{1}{2}}$
$= 5^{\frac{3}{2}}$

c $\dfrac{1}{\sqrt[4]{5}}$
$= \dfrac{1}{5^{\frac{1}{4}}}$
$= 5^{-\frac{1}{4}}$

d 25^{a+3}
$= (5^2)^{a+3}$
$= 5^{2a+6}$

7 **a** $e^x(e^{-x} + e^x)$
$= e^0 + e^{2x}$
$= 1 + e^{2x}$

b $(2^x + 5)^2$
$= (2^x)^2 + 2 \times 2^x \times 5 + 5^2$
$= 2^{2x} + 5 \times 2^{x+1} + 25$
$= 4^x + 5 \times 2^{x+1} + 25$
$\{or\ \ 2^{2x} + 10(2^x) + 25\}$

c $(x^{\frac{1}{2}} - 7)(x^{\frac{1}{2}} + 7)$
$= (x^{\frac{1}{2}})^2 - 7^2$
$= x^1 - 49$
$= x - 49$

8 **a** $6 \times 2^x = 192$
$\therefore \; 2^x = 32$
$\therefore \; 2^x = 2^5$
$\therefore \; x = 5$

b $4 \times (\frac{1}{3})^x = 324$
$\therefore \; (\frac{1}{3})^x = 81$
$\therefore \; (3^{-1})^x = 3^4$
$\therefore \; 3^{-x} = 3^4$
$\therefore \; x = -4$

9 The point $(1, \sqrt{8})$ lies on the graph of $y = 2^{kx}$.
$\therefore \; 2^{k \times 1} = \sqrt{8}$
$\therefore \; 2^k = \sqrt{2^3}$
$\therefore \; 2^k = 2^{\frac{3}{2}}$
$\therefore \; k = \frac{3}{2}$

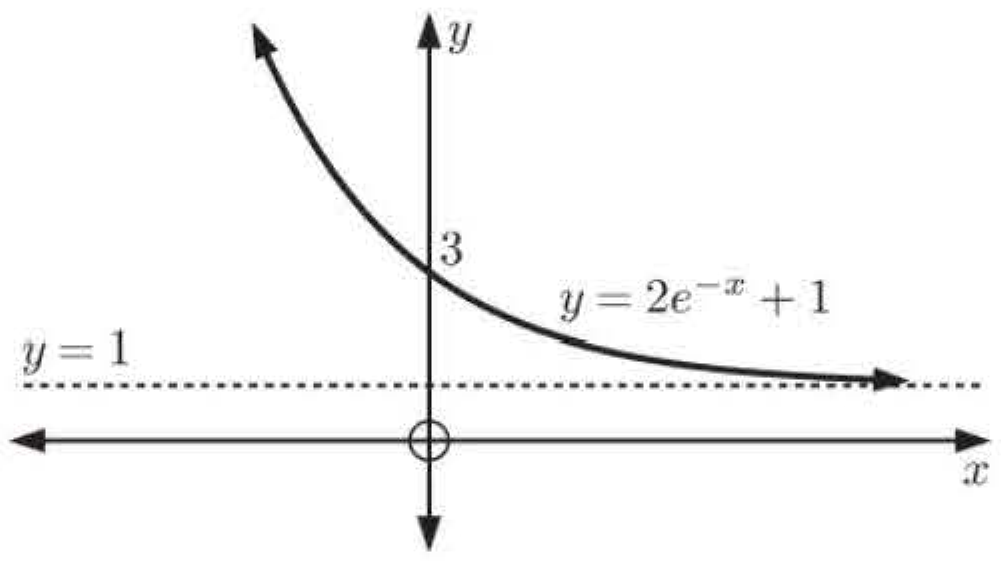
10 **a** $2^{x+1} = 32$
$\therefore \; 2^{x+1} = 2^5$
$\therefore \; x + 1 = 5$
$\therefore \; x = 4$

b $4^{x+1} = \left(\frac{1}{8}\right)^x$
$\therefore \; (2^2)^{x+1} = (2^{-3})^x$
$\therefore \; 2x + 2 = -3x$
$\therefore \; 5x = -2$
$\therefore \; x = -\frac{2}{5}$

11 **a** When $x = 0$, $y = 2e^{-0} + 1 = 3$
When $x = 1$, $y = 2e^{-1} + 1 \approx 1.74$
When $x = 2$, $y = 2e^{-2} + 1 \approx 1.27$
When $x = -1$, $y = 2e^{1} + 1 \approx 6.44$
When $x = -2$, $y = 2e^{2} + 1 \approx 15.8$

b As $x \to \infty$, $y \to 1^+$
As $x \to -\infty$, $y \to \infty$

c



d $y = 1$ is a horizontal asymptote.

Chapter 4
LOGARITHMS

EXERCISE 4A

1 **a** $\log 10\,000 = \log_{10} 10^4 = 4$

b $\log 0.001 = \log_{10} 10^{-3} = -3$

c $\log 10 = \log_{10} 10^1 = 1$

d $\log 1 = \log_{10} 10^0 = 0$

e $\log \sqrt{10} = \log_{10} 10^{\frac{1}{2}} = \frac{1}{2}$

f $\log \sqrt[3]{10} = \log_{10} 10^{\frac{1}{3}} = \frac{1}{3}$

g $\log\left(\frac{1}{\sqrt[4]{10}}\right) = \log_{10} 10^{-\frac{1}{4}} = -\frac{1}{4}$

h $\log 10\sqrt{10} = \log_{10} 10^{\frac{3}{2}} = \frac{3}{2} = 1\frac{1}{2}$

i $\log \sqrt[3]{100} = \log_{10}(10^2)^{\frac{1}{3}} = \log_{10} 10^{\frac{2}{3}} = \frac{2}{3}$

j $\log\left(\frac{100}{\sqrt{10}}\right) = \log_{10}\left(\frac{10^2}{10^{\frac{1}{2}}}\right) = \log_{10} 10^{\frac{3}{2}} = \frac{3}{2} = 1\frac{1}{2}$

k $\log\left(10 \times \sqrt[3]{10}\right) = \log_{10}(10^1 \times 10^{\frac{1}{3}}) = \log_{10} 10^{\frac{4}{3}} = \frac{4}{3} = 1\frac{1}{3}$

l $\log 1000\sqrt{10} = \log_{10}(10^3 \times 10^{\frac{1}{2}}) = \log_{10} 10^{\frac{7}{2}} = \frac{7}{2} = 3\frac{1}{2}$

2 **a** $\log 10^n = \log_{10} 10^n = n$

b $\log(10^a \times 100) = \log_{10}\left(10^a \times 10^2\right) = \log_{10}\left(10^{a+2}\right) = a + 2$

c $\log\left(\frac{10}{10^m}\right) = \log_{10}(10^{1-m}) = 1 - m$

d $\log\left(\frac{10^a}{10^b}\right) = \log_{10}(10^{a-b}) = a - b$

3 **a** $6 = 10^{\log 6} \approx 10^{0.7782}$

b $60 = 10^{\log 60} \approx 10^{1.7782}$

c $6000 = 10^{\log 6000} \approx 10^{3.7782}$

d $0.6 = 10^{\log(0.6)} \approx 10^{-0.2218}$

e $0.006 = 10^{\log(0.006)} \approx 10^{-2.2218}$

f $15 = 10^{\log 15} \approx 10^{1.1761}$

g $1500 = 10^{\log 1500} \approx 10^{3.1761}$

h $1.5 = 10^{\log 1.5} \approx 10^{0.1761}$

i $0.15 = 10^{\log(0.15)} \approx 10^{-0.8239}$

j $0.000\,15 = 10^{\log(0.000\,15)} \approx 10^{-3.8239}$

4 **a** **i** $\log 3 \approx 0.477$

ii $\log 300 \approx 2.477$

b $\log 300 = \log(3 \times 10^2) \approx \log(10^{0.477} \times 10^2) \approx \log 10^{2.477} \approx 2.477 \approx \log 3 + 2$

5 **a** **i** $\log 5 \approx 0.699$

ii $\log 0.05 \approx -1.301$

b $\log 0.05 = \log(5 \times 10^{-2}) \approx \log(10^{0.699} \times 10^{-2}) \approx \log 10^{(0.699-2)} \approx 0.699 - 2 \approx \log 5 - 2$

6 **a** $\log x = 2$ $\therefore\ x = 10^2$ $\therefore\ x = 100$

b $\log x = 1$ $\therefore\ x = 10^1$ $\therefore\ x = 10$

c $\log x = 0$ $\therefore\ x = 10^0$ $\therefore\ x = 1$

d $\log x = -1$ $\therefore\ x = 10^{-1}$ $\therefore\ x = \frac{1}{10}$

e $\log x = \frac{1}{2}$ $\therefore\ x = 10^{\frac{1}{2}}$ $(= \sqrt{10})$

f $\log x = -\frac{1}{2}$ $\therefore\ x = 10^{-\frac{1}{2}}$ $\left(= \dfrac{1}{10^{\frac{1}{2}}} = \dfrac{1}{\sqrt{10}}\right)$

g $\log x = 4$ $\therefore\ x = 10^4$ $\therefore\ x = 10\,000$

h $\log x = -5$ $\therefore\ x = 10^{-5}$ $\therefore\ x = 0.000\,01$

i $\log x \approx 0.8351$ $\therefore\ x \approx 10^{0.8351}$ $\therefore\ x \approx 6.84$

j $\log x \approx 2.1457$ $\therefore\ x \approx 10^{2.1457}$ $\therefore\ x \approx 140$

k $\log x \approx -1.378$ $\therefore\ x \approx 10^{-1.378}$ $\therefore\ x \approx 0.0419$

l $\log x \approx -3.1997$ $\therefore\ x \approx 10^{-3.1997}$ $\therefore\ x \approx 0.000\,631$

EXERCISE 4B

1 **a** $10^2 = 100$ **b** $10^4 = 10\,000$ **c** $10^{-1} = 0.1$

d $10^{\frac{1}{2}} = \sqrt{10}$ **e** $2^3 = 8$ **f** $3^2 = 9$

g $2^{-2} = \frac{1}{4}$ **h** $3^{1.5} = \sqrt{27}$ **i** $5^{-\frac{1}{2}} = \frac{1}{\sqrt{5}}$

2 **a** $\log_2 4 = 2$ **b** $\log_4 64 = 3$ **c** $\log_5 25 = 2$

d $\log_7 49 = 2$ **e** $\log_2 64 = 6$ **f** $\log_2\left(\frac{1}{8}\right) = -3$

g $\log_{10}(0.01) = -2$ **h** $\log_2\left(\frac{1}{2}\right) = -1$ **i** $\log_3\left(\frac{1}{27}\right) = -3$

3 **a** $\log_{10} 100\,000$ $= \log_{10} 10^5$ $= 5$

b $\log_{10}(0.01)$ $= \log_{10} 10^{-2}$ $= -2$

c $\log_3 \sqrt{3}$ $= \log_3 3^{\frac{1}{2}}$ $= \frac{1}{2}$

d $\log_2 8$ $= \log_2 2^3$ $= 3$

e $\log_2 64$ $= \log_2 2^6$ $= 6$

f $\log_2 128$ $= \log_2 2^7$ $= 7$

g $\log_5 25$ $= \log_5 5^2$ $= 2$

h $\log_5 125$ $= \log_5 5^3$ $= 3$

i $\log_2(0.125)$ $= \log_2\left(\frac{1}{8}\right)$ $= \log_2\left(2^{-3}\right)$ $= -3$

j $\log_9 3$ $= \log_9 9^{\frac{1}{2}}$ $= \frac{1}{2}$

k $\log_4 16$ $= \log_4 4^2$ $= 2$

l $\log_{36} 6$ $= \log_{36} 36^{\frac{1}{2}}$ $= \frac{1}{2}$

m $\log_3 243$ $= \log_3 3^5$ $= 5$

n $\log_2 \sqrt[3]{2}$ $= \log_2 2^{\frac{1}{3}}$ $= \frac{1}{3}$

o $\log_a a^n$ $= n,\ a > 0$

p $\log_8 2$ $= \log_8 8^{\frac{1}{3}}$ $= \frac{1}{3}$

q $\log_t\left(\frac{1}{t}\right)$ $= \log_t t^{-1}$ $= -1,\ t > 0$

r $\log_6 6\sqrt{6}$ $= \log_6(6^1 \times 6^{\frac{1}{2}})$ $= \log_6 6^{\frac{3}{2}}$ $= \frac{3}{2}$

s $\log_4 1$ $= \log_4 4^0$ $= 0$

t $\log_9 9$ $= \log_9 9^1$ $= 1$

4 **a** $\log_{10} 152 \approx 2.18$ **b** $\log_{10} 25 \approx 1.40$ **c** $\log_{10} 74 \approx 1.87$ **d** $\log_{10} 0.8 \approx -0.0969$

5 **a** $\log_2 x = 3$
$\therefore \quad x = 2^3$
$\therefore \quad x = 8$

b $\log_4 x = \frac{1}{2}$
$\therefore \quad x = 4^{\frac{1}{2}}$
$\therefore \quad x = 2$

c $\log_x 81 = 4$
$\therefore \quad 81 = x^4$
$\therefore \quad x = \pm\sqrt[4]{81}$
$\therefore \quad x = \pm 3$
$\therefore \quad x = 3 \;\{\text{as } x > 0\}$

d $\log_2(x-6) = 3$
$\therefore \quad x - 6 = 2^3$
$\therefore \quad x - 6 = 8$
$\therefore \quad x = 14$

6 **a** $\log_4 16$
$= \log_4 4^2$
$= 2$

b $\log_2 4$
$= \log_2 2^2$
$= 2$

c $\log_3(\frac{1}{3})$
$= \log_3 3^{-1}$
$= -1$

d $\log_{10} \sqrt[4]{1000}$
$= \log_{10}(10^3)^{\frac{1}{4}}$
$= \log_{10} 10^{\frac{3}{4}}$
$= \frac{3}{4}$

e $\log_7\left(\frac{1}{\sqrt{7}}\right)$
$= \log_7 7^{-\frac{1}{2}}$
$= -\frac{1}{2}$

f $\log_5(25\sqrt{5})$
$= \log_5(5^2 5^{\frac{1}{2}})$
$= \log_5 5^{\frac{5}{2}}$
$= \frac{5}{2}$

g $\log_3\left(\frac{1}{\sqrt{27}}\right)$
$= \log_3\left(\dfrac{1}{(3^3)^{\frac{1}{2}}}\right)$
$= \log_3 3^{-\frac{3}{2}}$
$= -\frac{3}{2}$

h $\log_4\left(\frac{1}{2\sqrt{2}}\right)$
$= \log_4\left(2^{-\frac{3}{2}}\right)$
$= \log_4\left((2^2)^{-\frac{3}{4}}\right)$
$= \log_4 4^{-\frac{3}{4}}$
$= -\frac{3}{4}$

i $\log_x x^2$
$= 2, \;\; x > 0$

j $\log_x \sqrt{x}$
$= \log_x x^{\frac{1}{2}}$
$= \frac{1}{2}, \;\; x > 0$

k $\log_m m^3$
$= 3, \;\; m > 0$

l $\log_x (x\sqrt{x})$
$= \log_x(x^1 \times x^{\frac{1}{2}})$
$= \log_x x^{\frac{3}{2}}$
$= \frac{3}{2}, \;\; x > 0$

m $\log_n\left(\frac{1}{n}\right)$
$= \log_n n^{-1}$
$= -1, \;\; n > 0$

n $\log_a\left(\frac{1}{a^2}\right)$
$= \log_a a^{-2}$
$= -2, \;\; a > 0$

o $\log_a\left(\frac{1}{\sqrt{a}}\right)$
$= \log_a a^{-\frac{1}{2}}$
$= -\frac{1}{2}, \;\; a > 0$

p $\log_m \sqrt{m^5}$
$= \log_m \left(m^5\right)^{\frac{1}{2}}$
$= \log_m m^{\frac{5}{2}}$
$= \frac{5}{2}, \;\; m > 0$

EXERCISE 4C.1

1 **a** $\log 8 + \log 2$
$= \log(8 \times 2)$
$= \log 16$

b $\log 4 + \log 5$
$= \log(4 \times 5)$
$= \log 20$

c $\log 40 - \log 5$
$= \log\left(\frac{40}{5}\right)$
$= \log 8$

d $\log p - \log m$
$= \log\left(\frac{p}{m}\right)$

e $\log_4 8 - \log_4 2$
$= \log_4\left(\frac{8}{2}\right)$
$= \log_4 4$
$= 1$

f $\log 5 + \log(0.4)$
$= \log(5 \times 0.4)$
$= \log 2$

g $\log 2 + \log 3 + \log 4$
$= \log(2 \times 3 \times 4)$
$= \log 24$

h $1 + \log_2 3$
$= \log_2 2^1 + \log_2 3$
$= \log_2(2 \times 3)$
$= \log_2 6$

i $\log 4 - 1$
$= \log 4 - \log 10^1$
$= \log\left(\frac{4}{10}\right)$
$= \log 0.4$

j $\log 5+\log 4-\log 2 = \log\left(\frac{5\times 4}{2}\right) = \log 10 = 1$

k $2+\log 2 = \log 10^2+\log 2 = \log(100\times 2) = \log 200$

l $t+\log w = \log 10^t+\log w = \log\left(10^t\times w\right)$

m $\log_m 40-2 = \log_m 40-\log_m m^2 = \log_m\left(\frac{40}{m^2}\right)$

n $\log_3 6-\log_3 2-\log_3 3 = \log_3(6\div 2\div 3) = \log_3 1 = 0$

o $\log 50-4 = \log 50-\log 10^4 = \log\left(\frac{50}{10^4}\right) = \log 0.005$

p $3-\log_5 50 = \log_5 5^3-\log_5 50 = \log_5\left(\frac{125}{50}\right) = \log_5\left(\frac{5}{2}\right)$

q $\log_5 100-\log_5 4 = \log_5\left(\frac{100}{4}\right) = \log_5 25 = \log_5 5^2 = 2$

r $\log\left(\frac{4}{3}\right)+\log 3+\log 7 = \log\left(\frac{4}{3}\times 3\times 7\right) = \log 28$

2 **a** $5\log 2+\log 3 = \log 2^5+\log 3 = \log(2^5\times 3) = \log 96$

b $2\log 3+3\log 2 = \log 3^2+\log 2^3 = \log(9\times 8) = \log 72$

c $3\log 4-\log 8 = \log 4^3-\log 8 = \log\left(\frac{64}{8}\right) = \log 8$

d $2\log_3 5-3\log_3 2 = \log_3 5^2-\log_3 2^3 = \log_3\left(\frac{25}{8}\right)$

e $\frac{1}{2}\log_6 4+\log_6 3 = \log_6 4^{\frac{1}{2}}+\log_6 3 = \log_6(2\times 3) = \log_6 6 = 1$

f $\frac{1}{3}\log\left(\frac{1}{8}\right) = \log\left(\frac{1}{8}\right)^{\frac{1}{3}} = \log\left(2^{-3}\right)^{\frac{1}{3}} = \log 2^{-1} = \log\left(\frac{1}{2}\right)$ or $-\log 2$

g $3-\log 2-2\log 5 = \log 10^3-\log 2-\log 5^2 = \log(1000\div 2\div 25) = \log 20$

h $1-3\log 2+\log 20 = \log 10^1-\log 2^3+\log 20 = \log(10\div 8\times 20) = \log 25$

i $2-\frac{1}{2}\log_n 4-\log_n 5 = \log_n n^2-\log_n 4^{\frac{1}{2}}-\log_n 5 = \log_n(n^2\div 2\div 5) = \log_n\left(\frac{n^2}{10}\right)$

3 **a** $\dfrac{\log 4}{\log 2} = \dfrac{\log 2^2}{\log 2} = \dfrac{2\log 2}{\log 2} = 2$

b $\dfrac{\log_5 27}{\log_5 9} = \dfrac{\log_5 3^3}{\log_5 3^2} = \dfrac{3\log_5 3}{2\log_5 3} = \frac{3}{2}$

c $\dfrac{\log 8}{\log 2} = \dfrac{\log 2^3}{\log 2} = \dfrac{3\log 2}{\log 2} = 3$

d $\dfrac{\log 3}{\log 9} = \dfrac{\log 3}{\log 3^2} = \dfrac{\log 3}{2\log 3} = \frac{1}{2}$

e $\dfrac{\log_3 25}{\log_3(0.2)} = \dfrac{\log_3 5^2}{\log_3 5^{-1}} = \dfrac{2\log_3 5}{-1\log_3 5} = -2$

f $\dfrac{\log_4 8}{\log_4(0.25)} = \dfrac{\log_4 2^3}{\log_4 2^{-2}}$ $\{0.25=\frac{1}{4}=\frac{1}{2^2}\}$ $= \dfrac{3\log_4 2}{-2\log_4 2} = -\frac{3}{2}$

4 **a** $\log 9 = \log 3^2$
$= 2\log 3$

b $\log\sqrt{2} = \log 2^{\frac{1}{2}}$
$= \frac{1}{2}\log 2$

c $\log\left(\frac{1}{8}\right) = \log\left(\frac{1}{2^3}\right)$
$= \log 2^{-3}$
$= -3\log 2$

d $\log\left(\frac{1}{5}\right) = \log 5^{-1}$
$= -1\log 5$
$= -\log 5$

e $\log 5 = \log\left(\frac{10}{2}\right)$
$= \log 10^1 - \log 2$
$= 1 - \log 2$

f $\log 5000 = \log\left(\frac{10\,000}{2}\right)$
$= \log 10^4 - \log 2$
$= 4 - \log 2$

5 **a** $\log_b 6$
$= \log_b(2\times 3)$
$= \log_b 2 + \log_b 3$
$= p + q$

b $\log_b 45$
$= \log_b(3^2 5)$
$= 2\log_b 3 + \log_b 5$
$= 2q + r$

c $\log_b 108$
$= \log_b(2^2 3^3)$
$= 2\log_b 2 + 3\log_b 3$
$= 2p + 3q$

d $\log_b\left(\frac{5\sqrt{3}}{2}\right)$
$= \log_b(5\times 3^{\frac{1}{2}}) - \log_b 2$
$= \log_b 5 + \frac{1}{2}\log_b 3 - \log_b 2$
$= r + \frac{1}{2}q - p$

e $\log_b\left(\frac{5}{32}\right)$
$= \log_b 5 - \log_b 2^5$
$= \log_b 5 - 5\log_b 2$
$= r - 5p$

f $\log_b(0.\overline{2})$
$= \log_b(\frac{2}{9})$
$= \log_b 2 - \log_b 3^2$
$= p - 2q$

6 **a** $\log_2(PR)$
$= \log_2 P + \log_2 R$
$= x + z$

b $\log_2(RQ^2)$
$= \log_2 R + \log_2 Q^2$
$= \log_2 R + 2\log_2 Q$
$= z + 2y$

c $\log_2\left(\frac{PR}{Q}\right)$
$= \log_2(PR) - \log_2 Q$
$= \log_2 P + \log_2 R - \log_2 Q$
$= x + z - y$

d $\log_2\left(P^2\sqrt{Q}\right)$
$= \log_2 P^2 + \log_2 Q^{\frac{1}{2}}$
$= 2\log_2 P + \frac{1}{2}\log_2 Q$
$= 2x + \frac{1}{2}y$

e $\log_2\left(\frac{Q^3}{\sqrt{R}}\right)$
$= \log_2 Q^3 - \log_2 R^{\frac{1}{2}}$
$= 3\log_2 Q - \frac{1}{2}\log_2 R$
$= 3y - \frac{1}{2}z$

f $\log_2\left(\frac{R^2\sqrt{Q}}{P^3}\right)$
$= \log_2 R^2 + \log_2 Q^{\frac{1}{2}} - \log_2 P^3$
$= 2\log_2 R + \frac{1}{2}\log_2 Q - 3\log_2 P$
$= 2z + \frac{1}{2}y - 3x$

7 **a** $\log_t N^2 = 1.72$
$\therefore\ 2\log_t N = 1.72$
$\therefore\ \log_t N = 1.72 \div 2$
$= 0.86$

b $\log_t(MN)$
$= \log_t M + \log_t N$
$= 1.29 + 0.86$
$= 2.15$

c $\log_t\left(\frac{N^2}{\sqrt{M}}\right)$
$= \log_t N^2 - \log_t M^{\frac{1}{2}}$
$= 1.72 - \frac{1}{2}\log_t M$
$= 1.72 - \frac{1}{2}(1.29)$
$= 1.075$

EXERCISE 4C.2

1 **a** $y = 2^x$
$\therefore\ \log y = \log 2^x$
$\therefore\ \log y = x\log 2$

b $y = 20b^3$
$\therefore\ \log y = \log(20b^3)$
$\therefore\ \log y = \log 20 + \log b^3$
$\therefore\ \log y \approx 1.30 + 3\log b$

c $M = ad^4$
$\therefore\ \log M = \log(ad^4)$
$\therefore\ \log M = \log a + \log d^4$
$\therefore\ \log M = \log a + 4\log d$

d $T = 5\sqrt{d} = 5d^{\frac{1}{2}}$
$\therefore\ \log T = \log(5d^{\frac{1}{2}})$
$\therefore\ \log T = \log 5 + \log d^{\frac{1}{2}}$
$\therefore\ \log T \approx 0.699 + \frac{1}{2}\log d$

e $R = b\sqrt{l} = bl^{\frac{1}{2}}$
$\therefore\ \log R = \log(bl^{\frac{1}{2}})$
$\therefore\ \log R = \log b + \log l^{\frac{1}{2}}$
$\therefore\ \log R = \log b + \frac{1}{2}\log l$

f $Q = \frac{a}{b^n}$
$\therefore\ \log Q = \log\left(\frac{a}{b^n}\right)$
$\therefore\ \log Q = \log a - \log b^n$
$\therefore\ \log Q = \log a - n\log b$

g
$$y = ab^x$$
$$\therefore \ \log y = \log(ab^x)$$
$$\therefore \ \log y = \log a + \log b^x$$
$$\therefore \ \log y = \log a + x\log b$$

h
$$F = \frac{20}{\sqrt{n}} = \frac{20}{n^{\frac{1}{2}}}$$
$$\therefore \ \log F = \log\left(\frac{20}{n^{\frac{1}{2}}}\right)$$
$$\therefore \ \log F = \log 20 - \log n^{\frac{1}{2}}$$
$$\therefore \ \log F \approx 1.30 - \tfrac{1}{2}\log n$$

i
$$L = \frac{ab}{c}$$
$$\therefore \ \log L = \log\left(\frac{ab}{c}\right)$$
$$\therefore \ \log L = \log ab - \log c$$
$$\therefore \ \log L = \log a + \log b - \log c$$

j
$$N = \sqrt{\frac{a}{b}}$$
$$\therefore \ N = \left(\frac{a}{b}\right)^{\frac{1}{2}}$$
$$\therefore \ \log N = \log\left(\frac{a}{b}\right)^{\frac{1}{2}}$$
$$\therefore \ \log N = \tfrac{1}{2}\log\left(\frac{a}{b}\right)$$
$$\therefore \ \log N = \tfrac{1}{2}\log a - \tfrac{1}{2}\log b$$

k
$$S = 200 \times 2^t$$
$$\therefore \ \log S = \log(200 \times 2^t)$$
$$\therefore \ \log S = \log 200 + \log 2^t$$
$$\therefore \ \log S = \log 200 + t\log 2$$
$$\therefore \ \log S \approx 2.30 + t\log 2$$

l
$$y = \frac{a^m}{b^n}$$
$$\therefore \ \log y = \log\left(\frac{a^m}{b^n}\right)$$
$$\therefore \ \log y = \log a^m - \log b^n$$
$$\therefore \ \log y = m\log a - n\log b$$

2 **a**
$$\log D = \log e + \log 2$$
$$= \log(e \times 2)$$
$$\therefore \ D = 2e$$

b
$$\log_a F = \log_a 5 - \log_a t$$
$$= \log_a\left(\frac{5}{t}\right)$$
$$\therefore \ F = \frac{5}{t}$$

c
$$\log P = \tfrac{1}{2}\log x$$
$$= \log x^{\frac{1}{2}}$$
$$\therefore \ P = \sqrt{x}$$

d
$$\log_n M = 2\log_n b + \log_n c$$
$$= \log_n b^2 + \log_n c$$
$$= \log_n(b^2c)$$
$$\therefore \ M = b^2c$$

e
$$\log B = 3\log m - 2\log n$$
$$= \log m^3 - \log n^2$$
$$= \log\left(\frac{m^3}{n^2}\right)$$
$$\therefore \ B = \frac{m^3}{n^2}$$

f
$$\log N = -\tfrac{1}{3}\log p$$
$$= \log p^{-\frac{1}{3}}$$
$$= \log\left(\frac{1}{\sqrt[3]{p}}\right)$$
$$\therefore \ N = \frac{1}{\sqrt[3]{p}}$$

g
$$\log P = 3\log x + 1$$
$$= \log x^3 + \log 10^1$$
$$= \log(10x^3)$$
$$\therefore \ P = 10x^3$$

h
$$\log_a Q = 2 - \log_a x$$
$$= \log_a a^2 - \log_a x$$
$$= \log_a\left(\frac{a^2}{x}\right)$$
$$\therefore \ Q = \frac{a^2}{x}$$

3 **a**
$$y = 3 \times 2^x$$
$$\therefore \ \log_2 y = \log_2(3 \times 2^x)$$
$$\therefore \ \log_2 y = \log_2 3 + \log_2 2^x$$
$$\therefore \ \log_2 y = \log_2 3 + x$$

b
$$\log_2 y = \log_2 3 + x \quad \{\text{from } \mathbf{a}\}$$
$$\therefore \ x = \log_2 y - \log_2 3$$
$$\therefore \ x = \log_2\left(\frac{y}{3}\right)$$

c **i** When $y = 3$,
$$x = \log_2\left(\tfrac{3}{3}\right)$$
$$\therefore \ x = \log_2 1$$
$$\therefore \ x = 0$$

ii When $y = 12$,
$$x = \log_2\left(\tfrac{12}{3}\right)$$
$$\therefore \ x = \log_2 4$$
$$\therefore \ x = \log_2 2^2$$
$$\therefore \ x = 2$$

iii When $y = 30$,
$$x = \log_2\left(\tfrac{30}{3}\right)$$
$$\therefore \ x = \log_2 10$$
$$\therefore \ x \approx 3.32$$

4 **a** $\log_3 27 + \log_3\left(\frac{1}{3}\right) = \log_3 x$
$\therefore\ \log_3\left(27 \times \frac{1}{3}\right) = \log_3 x$
$\therefore\ \log_3 9 = \log_3 x$
$\therefore\ x = 9$

b $\log_5 x = \log_5 8 - \log_5(6-x)$
$\therefore\ \log_5 x = \log_5\left(\dfrac{8}{6-x}\right)$
$\therefore\ x = \dfrac{8}{6-x}$
$\therefore\ 6x - x^2 = 8$
$\therefore\ x^2 - 6x + 8 = 0$
$\therefore\ (x-2)(x-4) = 0$
$\therefore\ x = 2$ or 4

Note: $x > 0$ and $6 - x > 0$ so $0 < x < 6$

c $\log_5 125 - \log_5 \sqrt{5} = \log_5 x$
$\therefore\ \log_5\left(\frac{125}{\sqrt{5}}\right) = \log_5 x$
$\therefore\ x = \frac{125}{\sqrt{5}} = 25\sqrt{5}$

d $\log_{20} x = 1 + \log_{20} 10$
$\therefore\ \log_{20} x = \log_{20} 20^1 + \log_{20} 10$
$= \log_{20} 200$
$\therefore\ x = 200$

e $\log x + \log(x+1) = \log 30$
$\therefore\ \log[x(x+1)] = \log 30$
$\therefore\ x^2 + x = 30$
$\therefore\ x^2 + x - 30 = 0$
$\therefore\ (x+6)(x-5) = 0$
$\therefore\ x = -6$ or 5
but $x > 0$ for $\log x$ to exist
$\therefore\ x = 5$

f $\log(x+2) - \log(x-2) = \log 5$
$\therefore\ \log\left(\dfrac{x+2}{x-2}\right) = \log 5$
$\therefore\ \dfrac{x+2}{x-2} = 5$
$\therefore\ x + 2 = 5x - 10$
$\therefore\ -4x = -12$
$\therefore\ x = 3$

Note: $x + 2 > 0$ and $x - 2 > 0$
$\therefore\ x > 2$ ✓

EXERCISE 4D.1

1 **a** $\ln e^2$
$= 2 \quad \{\ln e^x = x\}$

b $\ln e^3$
$= 3$

c $\ln \sqrt{e}$
$= \ln e^{\frac{1}{2}}$
$= \frac{1}{2}$

d $\ln 1$
$= \ln e^0$
$= 0$

e $\ln\left(\dfrac{1}{e}\right)$
$= \ln e^{-1}$
$= -1$

f $\ln \sqrt[3]{e}$
$= \ln e^{\frac{1}{3}}$
$= \frac{1}{3}$

g $\ln\left(\dfrac{1}{e^2}\right)$
$= \ln e^{-2}$
$= -2$

h $\ln\left(\dfrac{1}{\sqrt{e}}\right)$
$= \ln e^{-\frac{1}{2}}$
$= -\frac{1}{2}$

2 **a** $e^{\ln 3}$
$= 3$
{using $e^{\ln x} = x$}

b $e^{2\ln 3}$
$= 3^2$
{using $e^{x \ln a} = a^x$}
$= 9$

c $e^{-\ln 5}$
$= e^{-1 \ln 5}$
$= 5^{-1}$
$= \frac{1}{5}$

d $e^{-2\ln 2}$
$= 2^{-2}$
$= \dfrac{1}{2^2}$
$= \frac{1}{4}$

3 $\ln x$ exists only when $x > 0$.
$\therefore\ \ln(-2)$ and $\ln(0)$ do not exist.

Note: If $\ln(-2) = a$ then $-2 = e^a$
and $e^a = -2$ has no solutions as $e^a > 0$ for all a.

4 **a** $\ln e^a$
$= a$

b $\ln(e \times e^a)$
$= \ln e^{1+a}$
$= a + 1$

c $\ln(e^a \times e^b)$
$= \ln(e^{a+b})$
$= a + b$

d $\ln(e^a)^b$
$= \ln e^{ab}$
$= ab$

e $\ln\left(\dfrac{e^a}{e^b}\right)$
$= \ln(e^{a-b})$
$= a - b$

5 **a** $6 = e^{1.7918}$ **b** $60 = e^{4.0943}$ **c** $6000 = e^{8.6995}$ **d** $0.6 = e^{-0.5108}$
e $0.006 = e^{-5.1160}$ **f** $15 = e^{2.7081}$ **g** $1500 = e^{7.3132}$ **h** $1.5 = e^{0.4055}$
i $0.15 = e^{-1.8971}$ **j** $0.000\,15 = e^{-8.8049}$

6 **a** $\ln x = 3$
$\therefore\ x = e^3$
$\therefore\ x \approx 20.1$

b $\ln x = 1$
$\therefore\ x = e^1$
$\therefore\ x = e \approx 2.72$

c $\ln x = 0$
$\therefore\ x = e^0$
$\therefore\ x = 1$

d $\ln x = -1$
$\therefore\ x = e^{-1}$
$\therefore\ x \approx 0.368$

e $\ln x = -5$
$\therefore\ x = e^{-5}$
$\therefore\ x \approx 0.006\,74$

f $\ln x \approx 0.835$
$\therefore\ x \approx e^{0.835}$
$\therefore\ x \approx 2.30$

g $\ln x \approx 2.145$
$\therefore\ x \approx e^{2.145}$
$\therefore\ x \approx 8.54$

h $\ln x \approx -3.2971$
$\therefore\ x \approx e^{-3.2971}$
$\therefore\ x \approx 0.0370$

EXERCISE 4D.2

1 **a** $\ln 15 + \ln 3$
$= \ln(15 \times 3)$
$= \ln 45$

b $\ln 15 - \ln 3$
$= \ln(\frac{15}{3})$
$= \ln 5$

c $\ln 20 - \ln 5$
$= \ln(\frac{20}{5})$
$= \ln 4$

d $\ln 4 + \ln 6$
$= \ln(4 \times 6)$
$= \ln 24$

e $\ln 5 + \ln(0.2)$
$= \ln(5 \times 0.2)$
$= \ln 1$
$= 0$

f $\ln 2 + \ln 3 + \ln 5$
$= \ln(2 \times 3 \times 5)$
$= \ln 30$

g $1 + \ln 4$
$= \ln e^1 + \ln 4$
$= \ln(e \times 4)$
$= \ln(4e)$

h $\ln 6 - 1$
$= \ln 6 - \ln e^1$
$= \ln\left(\dfrac{6}{e}\right)$

i $\ln 5 + \ln 8 - \ln 2$
$= \ln(5 \times 8 \div 2)$
$= \ln 20$

j $2 + \ln 4$
$= \ln e^2 + \ln 4$
$= \ln(e^2 \times 4)$
$= \ln(4e^2)$

k $\ln 20 - 2$
$= \ln 20 - \ln e^2$
$= \ln\left(\dfrac{20}{e^2}\right)$

l $\ln 12 - \ln 4 - \ln 3$
$= \ln(12 \div 4 \div 3)$
$= \ln 1$
$= 0$

2 **a** $5\ln 3 + \ln 4$
$= \ln(3^5) + \ln 4$
$= \ln(243 \times 4)$
$= \ln 972$

b $3\ln 2 + 2\ln 5$
$= \ln(2^3) + \ln(5^2)$
$= \ln(8 \times 25)$
$= \ln 200$

c $3\ln 2 - \ln 8$
$= \ln(2^3) - \ln 8$
$= \ln(\frac{8}{8})$
$= \ln 1 = 0$

d $3\ln 4 - 2\ln 2$
$= \ln(4^3) - \ln(2^2)$
$= \ln(\frac{64}{4})$
$= \ln 16$

e $\frac{1}{3}\ln 8 + \ln 3$
$= \ln(8^{\frac{1}{3}}) + \ln 3$
$= \ln(2 \times 3)$
$= \ln 6$

f $\frac{1}{3}\ln(\frac{1}{27})$
$= \ln\left((\frac{1}{27})^{\frac{1}{3}}\right)$
$= \ln\left(\dfrac{1}{27^{\frac{1}{3}}}\right)$
$= \ln(\frac{1}{3})$

g $-\ln 2$
$= \ln(2^{-1})$
$= \ln(\frac{1}{2})$

h $-\ln(\frac{1}{2})$
$= \ln((\frac{1}{2})^{-1})$
$= \ln 2$

i $-2\ln(\frac{1}{4})$
$= \ln((\frac{1}{4})^{-2})$
$= \ln(4^2)$
$= \ln 16$

3 **a** $\ln 27 = \ln 3^3 = 3\ln 3$

b $\ln\sqrt{3} = \ln 3^{\frac{1}{2}} = \frac{1}{2}\ln 3$

c $\ln(\frac{1}{16}) = \ln\left(\frac{1}{2^4}\right) = \ln(2^{-4}) = -4\ln 2$

d $\ln(\frac{1}{6}) = \ln 6^{-1} = -1\ln 6 = -\ln 6$

e $\ln\left(\frac{1}{\sqrt{2}}\right) = \ln 2^{-\frac{1}{2}} = -\frac{1}{2}\ln 2$

f $\ln\left(\frac{e}{5}\right) = \ln e^1 - \ln 5 = 1 - \ln 5$

4 **a** $\ln\sqrt[3]{5} = \ln 5^{\frac{1}{3}} = \frac{1}{3}\ln 5$

b $\ln(\frac{1}{32}) = \ln 2^{-5} = -5\ln 2$

c $\ln\left(\frac{1}{\sqrt[5]{2}}\right) = \ln\left(\frac{1}{2^{\frac{1}{5}}}\right) = \ln 2^{-\frac{1}{5}} = -\frac{1}{5}\ln 2$

d $\ln\left(\frac{e^2}{8}\right) = \ln e^2 - \ln 8 = 2 - \ln 2^3 = 2 - 3\ln 2$

5 **a**
$$\begin{aligned} \ln D &= \ln x + 1 \\ \therefore\ \ln D - \ln x &= 1 \\ \therefore\ \ln\left(\frac{D}{x}\right) &= 1 \\ \therefore\ \frac{D}{x} &= e^1 \\ \therefore\ D &= ex \end{aligned}$$

b
$$\begin{aligned} \ln F &= -\ln p + 2 \\ \therefore\ \ln F + \ln p &= 2 \\ \therefore\ \ln(Fp) &= 2 \\ \therefore\ Fp &= e^2 \\ \therefore\ F &= \frac{e^2}{p} \end{aligned}$$

c
$$\begin{aligned} \ln P &= \tfrac{1}{2}\ln x \\ \therefore\ \ln P &= \ln x^{\frac{1}{2}} \\ \therefore\ P &= \sqrt{x} \end{aligned}$$

d
$$\begin{aligned} \ln M &= 2\ln y + 3 \\ \therefore\ \ln M - 2\ln y &= 3 \\ \therefore\ \ln\left(\frac{M}{y^2}\right) &= 3 \\ \therefore\ \frac{M}{y^2} &= e^3 \\ \therefore\ M &= e^3y^2 \end{aligned}$$

e
$$\begin{aligned} \ln B &= 3\ln t - 1 \\ \therefore\ \ln B - \ln t^3 &= -1 \\ \therefore\ \ln\left(\frac{B}{t^3}\right) &= -1 \\ \therefore\ \frac{B}{t^3} &= e^{-1} \\ \therefore\ B &= \frac{t^3}{e} \end{aligned}$$

f
$$\begin{aligned} \ln N &= -\tfrac{1}{3}\ln g \\ \therefore\ \ln N &= \ln g^{-\frac{1}{3}} \\ \therefore\ N &= g^{-\frac{1}{3}} \\ \therefore\ N &= \frac{1}{\sqrt[3]{g}} \end{aligned}$$

g
$$\begin{aligned} \ln Q &\approx 3\ln x + 2.159 \\ \therefore\ \ln Q - 3\ln x &\approx 2.159 \\ \therefore\ \ln\left(\frac{Q}{x^3}\right) &\approx 2.159 \\ \therefore\ \frac{Q}{x^3} &\approx e^{2.159} \\ \therefore\ \frac{Q}{x^3} &\approx 8.66 \\ \therefore\ Q &\approx 8.66x^3 \end{aligned}$$

h
$$\begin{aligned} \ln D &\approx 0.4\ln n - 0.6582 \\ \therefore\ \ln D - \ln n^{0.4} &\approx -0.6582 \\ \therefore\ \ln\left(\frac{D}{n^{0.4}}\right) &\approx -0.6582 \\ \therefore\ \frac{D}{n^{0.4}} &\approx e^{-0.6582} \\ \therefore\ \frac{D}{n^{0.4}} &\approx 0.518 \\ \therefore\ D &\approx 0.518n^{0.4} \end{aligned}$$

EXERCISE 4E

1 **a**
$$\begin{aligned} 2^x &= 10 \\ \therefore\ \log 2^x &= \log 10 \\ \therefore\ x\log 2 &= \log 10^1 \\ \therefore\ x &= \frac{1}{\log 2} \end{aligned}$$

b
$$\begin{aligned} 3^x &= 20 \\ \therefore\ \log 3^x &= \log 20 \\ \therefore\ x\log 3 &= \log 20 \\ \therefore\ x &= \frac{\log 20}{\log 3} \end{aligned}$$

c
$$\begin{aligned} 4^x &= 100 \\ \therefore\ \log 4^x &= \log 100 \\ \therefore\ x\log 4 &= \log 10^2 \\ \therefore\ x &= \frac{2}{\log 4} \end{aligned}$$

d $\left(\frac{1}{2}\right)^x = 0.0625$
$\therefore \log(\frac{1}{2})^x = \log(\frac{1}{16})$
$\therefore x\log(2^{-1}) = \log(2^{-4})$
$\therefore x = \dfrac{-4\log 2}{-\log 2}$
$\therefore x = 4$

e $\left(\frac{3}{4}\right)^x = 0.1$
$\therefore \log(\frac{3}{4})^x = \log 10^{-1}$
$\therefore x\log(\frac{3}{4}) = -1$
$\therefore x = -\dfrac{1}{\log(\frac{3}{4})}$

f $10^x = 0.000\,01$
$\therefore \log 10^x = \log 0.000\,01$
$\therefore \log 10^x = \log 10^{-5}$
$\therefore x\log 10 = -5\log 10$
$\therefore x = -5$

2 **a** $e^x = 10$
$\therefore x = \ln 10$

b $e^x = 1000$
$\therefore x = \ln 1000$

c $2e^x = 0.3$
$\therefore e^x = 0.15$
$\therefore x = \ln 0.15$

d $e^{\frac{x}{2}} = 5$
$\therefore \dfrac{x}{2} = \ln 5$
$\therefore x = 2\ln 5$

e $e^{2x} = 18$
$\therefore 2x = \ln 18$
$\therefore x = \frac{1}{2}\ln 18$

f $e^{-\frac{x}{2}} = 1$
$\therefore -\dfrac{x}{2} = \ln 1$
$\therefore -\dfrac{x}{2} = 0$
$\therefore x = 0$

3 **a** $R = 200 \times 2^{0.25t}$
$\therefore 2^{0.25t} = \dfrac{R}{200}$
$\therefore \log 2^{0.25t} = \log\left(\dfrac{R}{200}\right)$
$\therefore 0.25t\log 2 = \log R - \log 200$
$\therefore t = \dfrac{\log R - \log 200}{0.25\log 2}$

b **i** When $R = 600$,
$t = \dfrac{\log 600 - \log 200}{0.25\log 2}$
$\therefore t \approx 6.34$

ii When $R = 1425$,
$t = \dfrac{\log 1425 - \log 200}{0.25\log 2}$
$\therefore t \approx 11.3$

4 **a** $M = 20 \times 5^{-0.02x}$
$\therefore 5^{-0.02x} = \dfrac{M}{20}$
$\therefore \log 5^{-0.02x} = \log\left(\dfrac{M}{20}\right)$
$\therefore -0.02x\log 5 = \log M - \log 20$
$\therefore x = \dfrac{\log M - \log 20}{-0.02\log 5}$

b **i** When $M = 100$,
$x = \dfrac{\log 100 - \log 20}{-0.02\log 5}$
$\therefore x = -50$

ii When $M = 232$,
$x = \dfrac{\log 232 - \log 20}{-0.02\log 5}$
$\therefore x \approx -76.1$

5 **a** $4 \times 2^{-x} = 0.12$
$\therefore 2^{-x} = 0.03$
$\therefore \log 2^{-x} = \log(0.03)$
$\therefore -x\log 2 = \log(0.03)$
$\therefore x = -\dfrac{\log(0.03)}{\log 2}$

b $300 \times 5^{0.1x} = 1000$
$\therefore 5^{0.1x} = \frac{10}{3}$
$\therefore \log 5^{0.1x} = \log\left(\frac{10}{3}\right)$
$\therefore 0.1x\log 5 = \log\left(\frac{10}{3}\right)$
$\therefore x\log 5 = 10\log\left(\frac{10}{3}\right)$
$\therefore x = \dfrac{10\log\left(\frac{10}{3}\right)}{\log 5}$

c $32 \times 3^{-0.25x} = 4$

$\therefore\ 3^{-0.25x} = \frac{1}{8}$

$\therefore\ \log 3^{-0.25x} = \log\left(\frac{1}{8}\right)$

$\therefore\ -0.25x \log 3 = \log\left(\frac{1}{8}\right)$

$\therefore\ x \log 3 = -4 \log\left(\frac{1}{8}\right)$

$\therefore\ x = \dfrac{-4\log\left(\frac{1}{8}\right)}{\log 3}$

d $20 \times e^{2x} = 840$

$\therefore\ e^{2x} = 42$

$\therefore\ \ln e^{2x} = \ln 42$

$\therefore\ 2x = \ln 42$

$\therefore\ x = \frac{1}{2}\ln 42$

e $50 \times e^{-0.03x} = 0.05$

$\therefore\ e^{-0.03x} = 0.001$

$\therefore\ \ln e^{-0.03x} = \ln(0.001)$

$\therefore\ -0.03x = \ln(0.001)$

$\therefore\ -\frac{3}{100}x = \ln(0.001)$

$\therefore\ x = -\frac{100}{3}\ln(0.001)$

f $41e^{0.3x} - 27 = 0$

$\therefore\ 41e^{0.3x} = 27$

$\therefore\ e^{0.3x} = \frac{27}{41}$

$\therefore\ \ln e^{0.3x} = \ln\left(\frac{27}{41}\right)$

$\therefore\ 0.3x = \ln\left(\frac{27}{41}\right)$

$\therefore\ \frac{3}{10}x = \ln\left(\frac{27}{41}\right)$

$\therefore\ x = \frac{10}{3}\ln\left(\frac{27}{41}\right)$

6 **a** $e^{2x} = 2e^x$

$\therefore\ e^x(e^x - 2) = 0$

$\therefore\ e^x = 2 \quad \{\text{as } e^x > 0\}$

$\therefore\ x = \ln 2$

b $e^x = e^{-x}$

$\therefore\ x = -x$

$\therefore\ 2x = 0$

$\therefore\ x = 0$

c $e^{2x} - 5e^x + 6 = 0$

$\therefore\ (e^x - 3)(e^x - 2) = 0$

$\therefore\ e^x = 3 \text{ or } 2$

$\therefore\ x = \ln 3 \text{ or } \ln 2$

d $e^x + 2 = 3e^{-x}$

$\therefore\ e^{2x} + 2e^x = 3 \quad \{\times e^x\}$

$\therefore\ e^{2x} + 2e^x - 3 = 0$

$\therefore\ (e^x + 3)(e^x - 1) = 0$

$\therefore\ e^x = -3 \text{ or } 1$

$\therefore\ e^x = 1 \quad \{\text{as } e^x > 0\}$

$\therefore\ x = \ln 1$

$\therefore\ x = 0$

e $1 + 12e^{-x} = e^x$

$\therefore\ e^x + 12 = e^{2x} \quad \{\times e^x\}$

$\therefore\ e^{2x} - e^x - 12 = 0$

$\therefore\ (e^x - 4)(e^x + 3) = 0$

$\therefore\ e^x = 4 \text{ or } -3$

$\therefore\ e^x = 4 \quad \{\text{as } e^x > 0\}$

$\therefore\ x = \ln 4$

f $e^x + e^{-x} = 3$

$\therefore\ e^{2x} + 1 = 3e^x \quad \{\times e^x\}$

$\therefore\ e^{2x} - 3e^x + 1 = 0$

$\therefore\ e^x = \dfrac{3 \pm \sqrt{9-4}}{2}$

$\therefore\ e^x = \dfrac{3 \pm \sqrt{5}}{2}$

$\therefore\ x = \ln\left(\frac{3+\sqrt{5}}{2}\right) \text{ or } \ln\left(\frac{3-\sqrt{5}}{2}\right)$

$\approx 0.962 \text{ or } -0.962$

7 **a** $y = e^x$ and $y = e^{2x} - 6$

meet when $e^x = e^{2x} - 6$

$\therefore\ e^{2x} - e^x - 6 = 0$

$\therefore\ (e^x - 3)(e^x + 2) = 0$

$\therefore\ e^x = 3 \text{ or } -2$

$\therefore\ e^x = 3 \quad \{\text{as } e^x > 0\}$

$\therefore\ x = \ln 3$ and $y = e^x = 3$

$\therefore$ they meet at $(\ln 3, 3)$.

b $y = 2e^x + 1$ and $y = 7 - e^x$

meet when $2e^x + 1 = 7 - e^x$

$\therefore\ 3e^x = 6$

$\therefore\ e^x = 2$

$\therefore\ x = \ln 2$ and $y = 7 - e^x = 5$

$\therefore$ they meet at $(\ln 2, 5)$.

c $y = 3 - e^x$ and $y = 5e^{-x} - 3$ meet when $3 - e^x = 5e^{-x} - 3$

$\therefore\ 3e^x - e^{2x} = 5 - 3e^x \quad \{\times e^x\}$

$\therefore\ e^{2x} - 6e^x + 5 = 0$

$\therefore\ (e^x - 5)(e^x - 1) = 0$

$\therefore\ e^x = 1 \text{ or } 5$

$\therefore\ x = 0 \text{ or } \ln 5$

When $x = 0$, $y = 3 - e^0 = 3 - 1 = 2$

When $x = \ln 5$, $y = 3 - e^{\ln 5} = 3 - 5 = -2$

$\therefore$ they meet at $(0, 2)$ and $(\ln 5, -2)$.

EXERCISE 4F

1 **a** $\log_3 12 = \dfrac{\log_{10} 12}{\log_{10} 3} \approx 2.26$

b $\log_{\frac{1}{2}} 1250 = \dfrac{\log_{10} 1250}{\log_{10}(0.5)} \approx -10.3$

c $\log_3(0.067) = \dfrac{\log_{10}(0.067)}{\log_{10} 3} \approx -2.46$

d $\log_{0.4}(0.006\,984) = \dfrac{\log_{10}(0.006\,984)}{\log_{10}(0.4)} \approx 5.42$

2 **a** $2^x = 0.051$

$\therefore\ x = \log_2(0.051)$

$\therefore\ x = \dfrac{\ln(0.051)}{\ln 2}$

$\therefore\ x \approx -4.29$

b $4^x = 213.8$

$\therefore\ x = \log_4 213.8$

$\therefore\ x = \dfrac{\ln(213.8)}{\ln 4}$

$\therefore\ x \approx 3.87$

c $3^{2x+1} = 4.069$

$\therefore\ 2x + 1 = \log_3(4.069)$

$\therefore\ 2x + 1 = \dfrac{\ln(4.069)}{\ln 3}$

$\therefore\ 2x + 1 \approx 1.2774$

$\therefore\ 2x \approx 0.2774$

$\therefore\ x \approx 0.139$

3 **a** $25^x - 3(5^x) = 0$

$\therefore\ 5^{2x} - 3(5^x) = 0$

$\therefore\ 5^x(5^x - 3) = 0$

$\therefore\ 5^x = 3$ {as $5^x > 0$ for all x}

$\therefore\ x = \log_5 3$

$\therefore\ x = \dfrac{\log 3}{\log 5}$

b $8(9^x) - 3^x = 0$

$\therefore\ 8 \times 3^{2x} - 3^x = 0$

$\therefore\ 3^x(8 \times 3^x - 1) = 0$

$\therefore\ 8 \times 3^x - 1 = 0$ {as $3^x > 0$ for all x}

$\therefore\ 3^x = \frac{1}{8}$

$\therefore\ x = \log_3(\frac{1}{8})$

$\therefore\ x = \dfrac{\log(\frac{1}{8})}{\log 3}$

c $2^x - 2(4^x) = 0$

$\therefore\ 2^x - 2 \times 2^{2x} = 0$

$\therefore\ 2^x(1 - 2 \times 2^x) = 0$

$\therefore\ 1 - 2 \times 2^x = 0$ {as $2^x > 0$ for all x}

$\therefore\ 2^x = \frac{1}{2}$

$\therefore\ x = \log_2(\frac{1}{2})$

$\therefore\ x = \dfrac{\log(\frac{1}{2})}{\log 2}$

$\therefore\ x = -1$

4 **a** $\log_4 x^3 + \log_2 \sqrt{x} = 8$

$\therefore\ \dfrac{\log x^3}{\log 4} + \dfrac{\log x^{\frac{1}{2}}}{\log 2} = 8$

$\therefore\ \dfrac{3\log x}{2\log 2} + \dfrac{\frac{1}{2}\log x}{\log 2} = 8$

$\therefore\ \dfrac{3\log x}{2\log 2} + \dfrac{\log x}{2\log 2} = 8$

$\therefore\ \dfrac{4\log x}{2\log 2} = 8$

$\therefore\ \log x = 4\log 2$

$\therefore\ \log x = \log 2^4$

$\therefore\ x = 16$

b $\log_{16} x^5 = \log_{64} 125 - \log_4 \sqrt{x}$

$\therefore\ \dfrac{\log x^5}{\log 16} = \dfrac{\log 125}{\log 64} - \dfrac{\log x^{\frac{1}{2}}}{\log 4}$

$\therefore\ \dfrac{5\log x}{4\log 2} = \dfrac{\log 125}{6\log 2} - \dfrac{\frac{1}{2}\log x}{2\log 2}$

$\therefore\ \dfrac{15\log x}{12\log 2} = \dfrac{2\log 125}{12\log 2} - \dfrac{3\log x}{12\log 2}$

$\therefore\ 15\log x = 2\log 125 - 3\log x$

$\therefore\ 18\log x = 2\log 125$

$\therefore\ \log x = \frac{1}{9}\log 5^3$

$\therefore\ \log x = \log(5^3)^{\frac{1}{9}}$

$\therefore\ x = 5^{\frac{1}{3}} \approx 1.71$

5
$$4^x \times 5^{4x+3} = 10^{2x+3}$$
$$\therefore \log(4^x \times 5^{4x+3}) = \log 10^{2x+3}$$
$$\therefore x\log 4 + (4x+3)\log 5 = 2x+3$$
$$\therefore x\log 4 + 4x\log 5 + 3\log 5 = 2x+3$$
$$\therefore x[\log 4 + 4\log 5 - 2] = 3 - 3\log 5$$
$$\therefore x = \frac{3-3\log 5}{\log 4 + 4\log 5 - 2}$$
$$\therefore x = \frac{\log 10^3 - \log 5^3}{\log 4 + \log 5^4 - \log 10^2}$$
$$\therefore x = \frac{\log(\frac{1000}{125})}{\log\left(\frac{4\times 5^4}{10^2}\right)}$$
$$\therefore x = \frac{\log 8}{\log 25} \quad \text{or} \quad \log_{25} 8$$

6
$$\log_9 x + \log_{27} x = p$$
$$\therefore \frac{\log x}{\log 9} + \frac{\log x}{\log 27} = p$$
$$\therefore \frac{\log x}{2\log 3} + \frac{\log x}{3\log 3} = p$$
$$\therefore \frac{3\log x}{6\log 3} + \frac{2\log x}{6\log 3} = p$$
$$\therefore \frac{5\log x}{6\log 3} = p$$
$$\therefore \tfrac{5}{6}\log_3 x = p$$
$$\therefore \log_3 x = \tfrac{6}{5}p$$

Now, $\log_{81} x = \dfrac{\log x}{\log 81}$
$$= \frac{\log x}{4\log 3}$$
$$= \tfrac{1}{4}\log_3 x$$
$$= \tfrac{1}{4}\times\tfrac{6}{5}p$$
$$= \tfrac{3}{10}p$$

So, $\log_3 x + \log_{81} x = \tfrac{6}{5}p + \tfrac{3}{10}p$
$$= \tfrac{15}{10}p$$
$$= \tfrac{3}{2}p$$

7
$$\frac{4}{\log_5 4} + \frac{3}{\log_7 8} = \frac{4}{\frac{\log 4}{\log 5}} + \frac{3}{\frac{\log 8}{\log 7}}$$
$$= \frac{4\log 5}{\log 4} + \frac{3\log 7}{\log 8}$$
$$= \frac{4\log 5}{2\log 2} + \frac{3\log 7}{3\log 2}$$
$$= \frac{2\log 5}{\log 2} + \frac{\log 7}{\log 2}$$
$$= \log_2 25 + \log_2 7$$
$$= \log_2(25\times 7)$$
$$= \log_2 175$$

So, $2^{\frac{4}{\log_5 4} + \frac{3}{\log_7 8}} = 2^{\log_2 175} = 175$

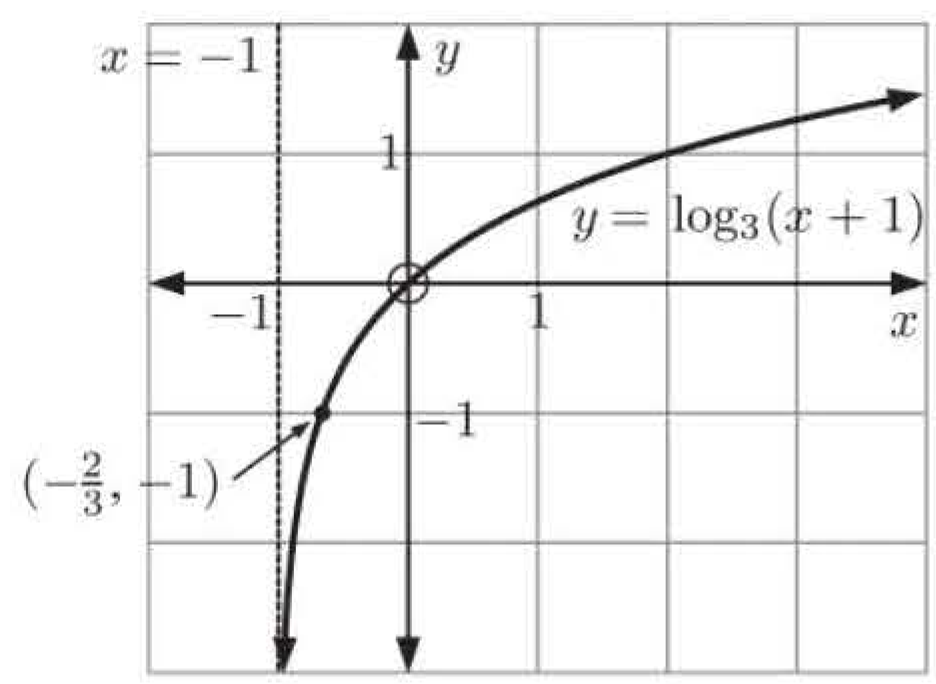
EXERCISE 4G

1 **a** $f(x) = \log_3(x+1)$

i We require $x + 1 > 0 \quad \therefore x > -1$
So, the domain is $\{x \mid x > -1\}$ and the range is $\{y \mid y \in \mathbb{R}\}$.

ii As $x \to -1^+$, $y \to -\infty$,
so $x = -1$ is a vertical asymptote.
As $x \to \infty$, $y \to \infty$.
When $x = 0$, $y = \log_3 1 = 0$
So, the y-intercept is 0.
When $y = 0$, $\log_3(x+1) = 0$
$$\therefore x + 1 = 3^0$$
$$\therefore x + 1 = 1$$
$$\therefore x = 0$$
So, the x-intercept is 0.

iii We graph using $y = \dfrac{\log(x+1)}{\log 3}$



iv If $f(x) = -1$

then $\log_3(x+1) = -1$

$\therefore \quad x + 1 = 3^{-1}$

$\therefore \quad x = \frac{1}{3} - 1$

$\therefore \quad x = -\frac{2}{3}$

which checks with the graph

v f is defined by $y = \log_3(x+1)$

$\therefore$ f^{-1} is defined by $x = \log_3(y+1)$

$\therefore \quad y + 1 = 3^x$

$\therefore \quad y = 3^x - 1$

$\therefore \quad f^{-1}(x) = 3^x - 1$

Horizontal asymptote is $y = -1$.
Domain is $x \in \mathbb{R}$.
Range is $\{y \mid y > -1\}$.

b $f(x) = 1 - \log_3(x+1)$

i We require $x + 1 > 0 \quad \therefore \quad x > -1$
So, the domain is $\{x \mid x > -1\}$ and the range is $\{y \mid y \in \mathbb{R}\}$.

ii As $x \to -1^+$, $y \to \infty$,
so $x = -1$ is a vertical asymptote.
As $x \to \infty$, $y \to -\infty$.
When $x = 0$, $y = 1 - \log_3 1$
$= 1 - 0 = 1$
So, the y-intercept is 1.
When $y = 0$, $1 - \log_3(x+1) = 0$

$\therefore \quad \log_3(x+1) = 1$

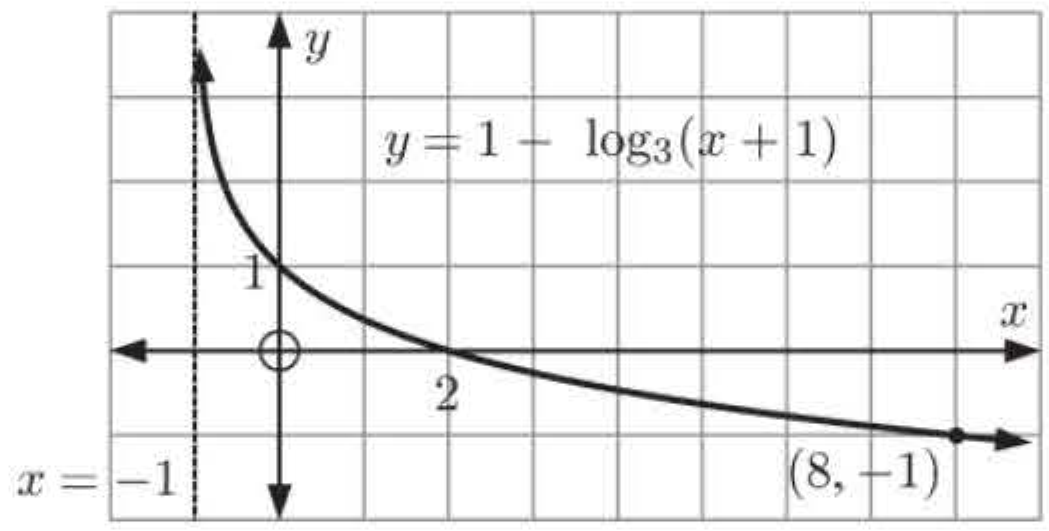
$\therefore \quad x + 1 = 3^1$

$= 3$

$\therefore \quad x = 2$

So, the x-intercept is 2.

iii We graph using $y = 1 - \dfrac{\log(x+1)}{\log 3}$



iv If $f(x) = -1$

then $1 - \log_3(x+1) = -1$

$\therefore \quad \log_3(x+1) = 2$

$\therefore \quad x + 1 = 3^2$

$\therefore \quad x = 8$

which checks with the graph

v f is defined by $y = 1 - \log_3(x+1)$

$\therefore$ f^{-1} is defined by $x = 1 - \log_3(y+1)$

$\therefore \quad \log_3(y+1) = 1 - x$

$\therefore \quad y + 1 = 3^{1-x}$

$\therefore \quad y = 3^{1-x} - 1$

$\therefore \quad f^{-1}(x) = 3^{1-x} - 1$

Horizontal asymptote is $y = -1$.
Domain is $x \in \mathbb{R}$.
Range is $\{y \mid y > -1\}$.

c $f(x) = \log_5(x-2) - 2$

i We require $x - 2 > 0 \quad \therefore \quad x > 2$.
So, the domain is $\{x \mid x > 2\}$ and the range is $\{y \mid y \in \mathbb{R}\}$.

ii As $x \to 2^+$, $y \to -\infty$,
so $x = 2$ is a vertical asymptote.
As $x \to \infty$, $y \to \infty$.
When $x = 0$, y is undefined.
$\therefore$ there is no y-intercept.
When $y = 0$, $\log_5(x-2) = 2$

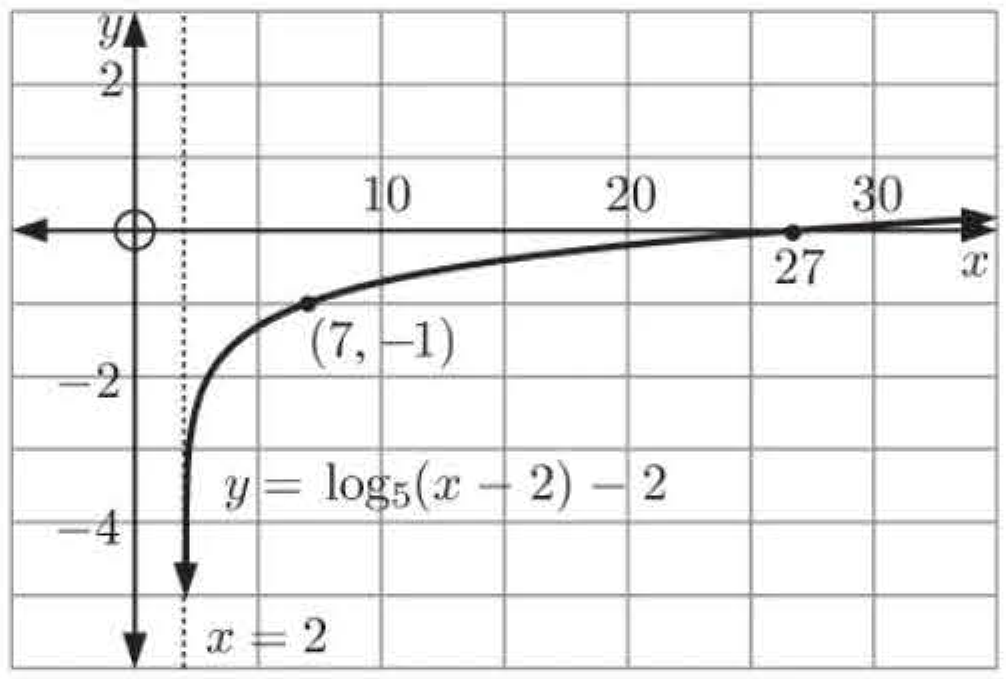
$\therefore \quad x - 2 = 5^2$

$= 25$

$\therefore \quad x = 27$

So, the x-intercept is 27.

iii We graph using $y = \dfrac{\log(x-2)}{\log 5} - 2$



iv If $f(x) = -1$

then $\log_5(x-2) - 2 = -1$

$\therefore \log_5(x-2) = 1$

$\therefore x - 2 = 5^1$

$\therefore x = 5 + 2$

$\therefore x = 7$

which checks with the graph

v f is defined by $y = \log_5(x-2) - 2$

$\therefore f^{-1}$ is defined by $x = \log_5(y-2) - 2$

$\therefore x + 2 = \log_5(y-2)$

$\therefore y - 2 = 5^{x+2}$

$\therefore y = 5^{x+2} + 2$

$\therefore f^{-1}(x) = 5^{x+2} + 2$

Horizontal asymptote is $y = 2$.
Domain is $x \in \mathbb{R}$.
Range is $\{y \mid y > 2\}$.

d $f(x) = 1 - \log_5(x-2)$

i We require $x - 2 > 0$ $\therefore x > 2$.
So, the domain is $\{x \mid x > 2\}$ and the range is $\{y \mid y \in \mathbb{R}\}$.

ii As $x \to 2^+$, $y \to \infty$,
so $x = 2$ is a vertical asymptote.
As $x \to \infty$, $y \to -\infty$.
When $x = 0$, y is undefined.
$\therefore$ there is no y-intercept.
When $y = 0$, $1 - \log_5(x-2) = 0$

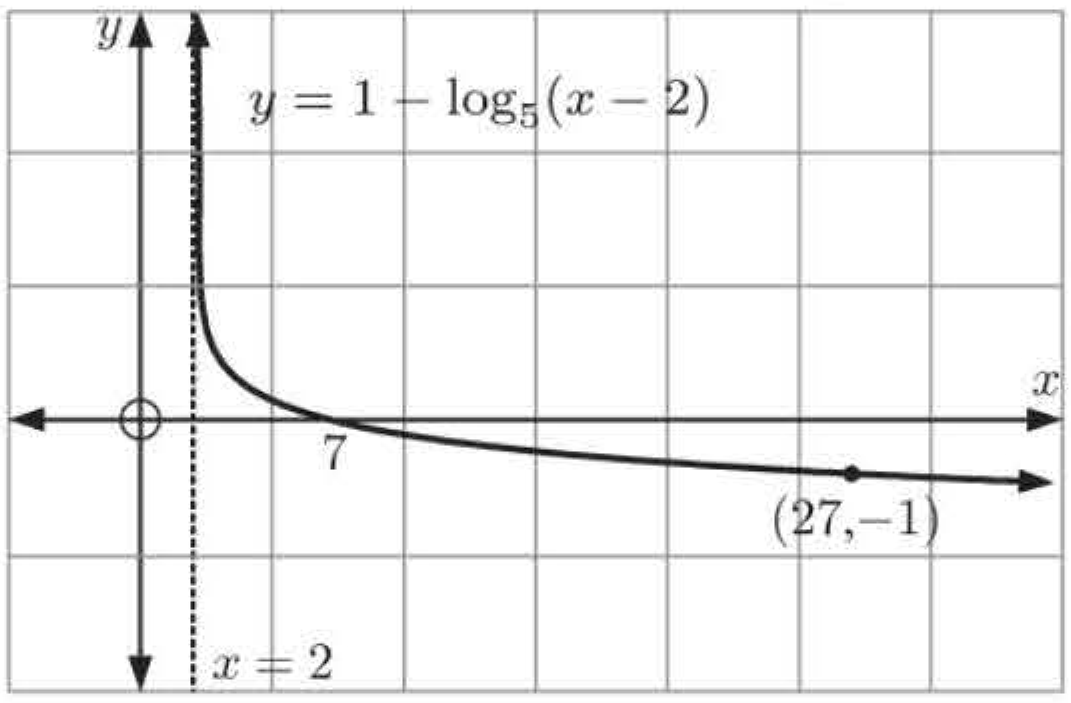
$\therefore \log_5(x-2) = 1$

$\therefore x - 2 = 5^1$

$\therefore x = 7$

So, x-intercept is 7.

iii We graph using $y = 1 - \dfrac{\log(x-2)}{\log 5}$



iv If $f(x) = -1$

then $1 - \log_5(x-2) = -1$

$\therefore \log_5(x-2) = 2$

$\therefore x - 2 = 5^2$

$\therefore x = 27$

which checks with the graph

v f is defined by $y = 1 - \log_5(x-2)$

$\therefore f^{-1}$ is defined by $x = 1 - \log_5(y-2)$

$\therefore \log_5(y-2) = 1 - x$

$\therefore y - 2 = 5^{1-x}$

$\therefore y = 5^{1-x} + 2$

$\therefore f^{-1}(x) = 5^{1-x} + 2$

Horizontal asymptote is $y = 2$.
Domain is $x \in \mathbb{R}$.
Range is $\{y \mid y > 2\}$.

e $f(x) = 1 - 2\log_2 x$

i We require $x > 0$.
So, the domain is $\{x \mid x > 0\}$ and the range is $\{y \mid y \in \mathbb{R}\}$.

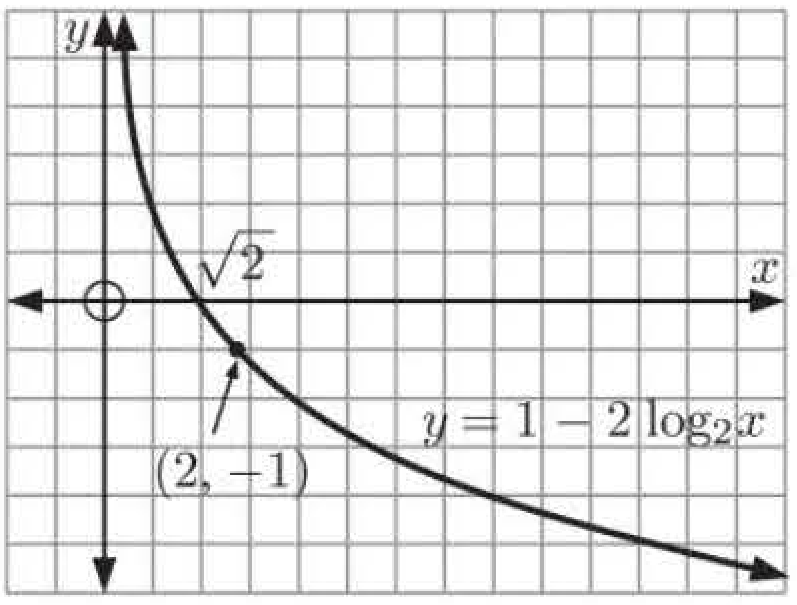
ii As $x \to 0^+$, $y \to \infty$,
so $x = 0$ is a vertical asymptote.
As $x \to \infty$, $y \to -\infty$.
When $x = 0$, y is undefined.
$\therefore$ there is no y-intercept.
When $y = 0$, $\log_2 x = \frac{1}{2}$

$\therefore x = 2^{\frac{1}{2}}$

$\therefore x = \sqrt{2}$

$\therefore$ x-intercept is $\sqrt{2} \approx 1.41$.

iii We graph using $y = 1 - \dfrac{2\log x}{\log 2}$



iv If $f(x) = -1$

then $1 - 2\log_2 x = -1$

$\therefore\ -2\log_2 x = -2$

$\therefore\ \log_2 x = 1$

$\therefore\ x = 2^1$

$\therefore\ x = 2$

which checks with the graph

v f is defined by $y = 1 - 2\log_2 x$

$\therefore\ f^{-1}$ is defined by $x = 1 - 2\log_2 y$

$\therefore\ 2\log_2 y = 1 - x$

$\therefore\ \log_2 y = \dfrac{1-x}{2}$

$\therefore\ y = 2^{\frac{1-x}{2}}$

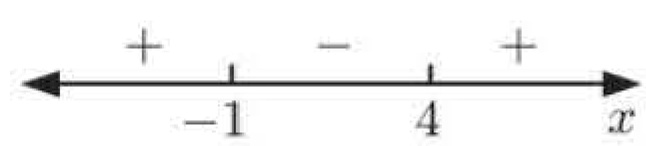
$\therefore\ f^{-1}(x) = 2^{\frac{1-x}{2}}$

Horizontal asymptote is $y = 0$.

Domain is $x \in \mathbb{R}$. Range is $\{y \mid y > 0\}$.

f $f(x) = \log_2(x^2 - 3x - 4)$

i We require $x^2 - 3x - 4 > 0$ $\therefore$ $(x+1)(x-4) > 0$.

Sign diagram of LHS: 

$\therefore$ LHS > 0 when $x < -1$ or $x > 4$.

So, the domain is $\{x \mid x < -1 \text{ or } x > 4\}$ and the range is $\{y \mid y \in \mathbb{R}\}$.

ii As $x \to -1^-$, $y \to -\infty$,

As $x \to 4^+$, $y \to -\infty$

so $x = -1$ and $x = 4$ are vertical asymptotes.

When $x = 0$, y is undefined.

$\therefore$ there is no y-intercept.

When $y = 0$,

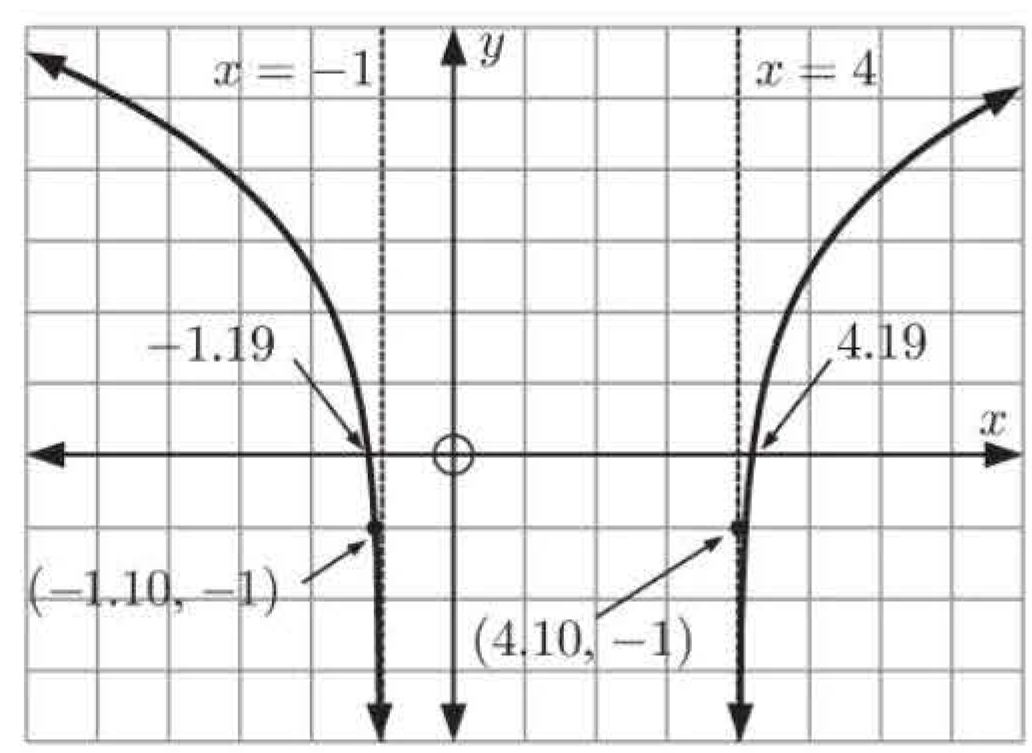
$\log_2(x^2 - 3x - 4) = 0$

$\therefore\ x^2 - 3x - 4 = 2^0 = 1$

$\therefore\ x^2 - 3x - 5 = 0$

$\therefore\ x \approx -1.19$ or 4.19 {using technology}

iii We graph using $y = \dfrac{\log(x^2 - 3x - 4)}{\log 2}$



iv If $f(x) = -1$

then $\log_2(x^2 - 3x - 4) = -1$

$\therefore\ x^2 - 3x - 4 = 2^{-1}$

$\therefore\ x^2 - 3x - \frac{9}{2} = 0$

$\therefore\ 2x^2 - 6x - 9 = 0$

$\therefore\ x \approx -1.10$ or 4.10 {using technology}

which checks with the graph

v If f is defined by $y = \log_2(x^2 - 3x - 4)$, $x > 4$

then f^{-1} is defined by $x = \log_2(y^2 - 3y - 4)$, $y > 4$

$\therefore\ y^2 - 3y - 4 = 2^x$, $y > 4$

$\therefore\ y^2 - 3y - 4 - 2^x = 0$, $y > 4$

$$\therefore\ y = \frac{3 \pm \sqrt{9 + 4(4 + 2^x)}}{2}, \quad y > 4$$

$$\therefore\ y = \frac{3 + \sqrt{25 + 2^{x+2}}}{2} \quad \text{as } y > 4$$

$$\therefore\ f^{-1}(x) = \frac{3 + \sqrt{25 + 2^{x+2}}}{2}$$

If f is defined by $y = \log_2(x^2 - 3x - 4)$, $x < -1$

then f^{-1} is defined by $x = \log_2(y^2 - 3y - 4)$, $y < -1$

and by the same working, $y = \dfrac{3 - \sqrt{25 + 2^{x+2}}}{2}$ as $y < -1$

$$\therefore \quad f^{-1}(x) = \frac{3 - \sqrt{25 + 2^{x+2}}}{2}$$

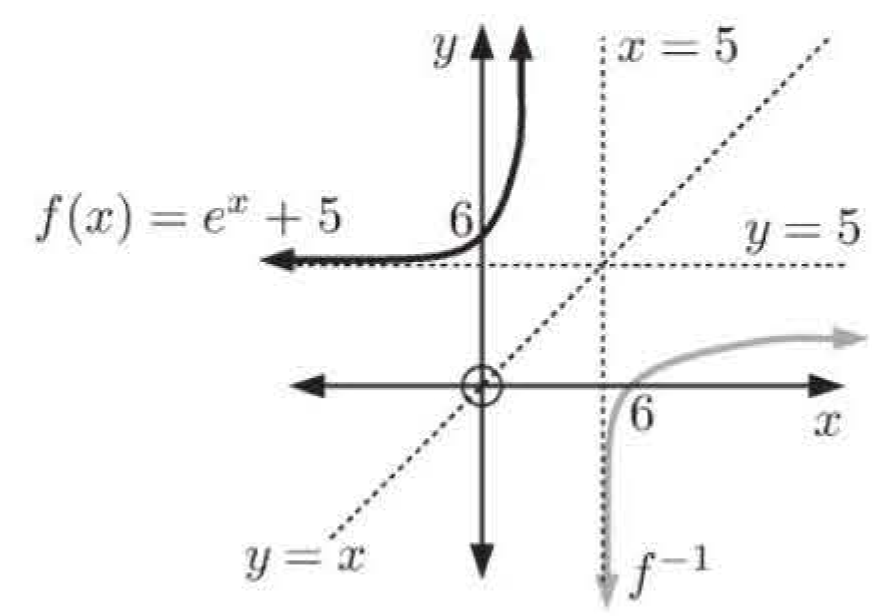
2 **a** **i**

$$f(x) = e^x + 5$$
or $y = e^x + 5$

has inverse function

$$x = e^y + 5$$
$$\therefore \quad x - 5 = e^y$$
$$\therefore \quad y = \ln(x - 5)$$
$$\therefore \quad f^{-1}(x) = \ln(x - 5)$$

ii



iii Domain of f is $\{x \mid x \in \mathbb{R}\}$, range is $\{y \mid y > 5\}$.
Domain of f^{-1} is $\{x \mid x > 5\}$, range is $\{y \mid y \in \mathbb{R}\}$.

iv f has a H.A. $y = 5$. f^{-1} has a V.A. $x = 5$.
When $x = 0$, $y = e^0 + 5$
$\therefore \; y = 6$
$\therefore$ y-intercept of f is 6.
When $y = 0$, $e^x + 5$ is undefined
$\therefore$ f has no x-intercept.
$\therefore$ the x-intercept of f^{-1} is 6, and f^{-1} has no y-intercept.

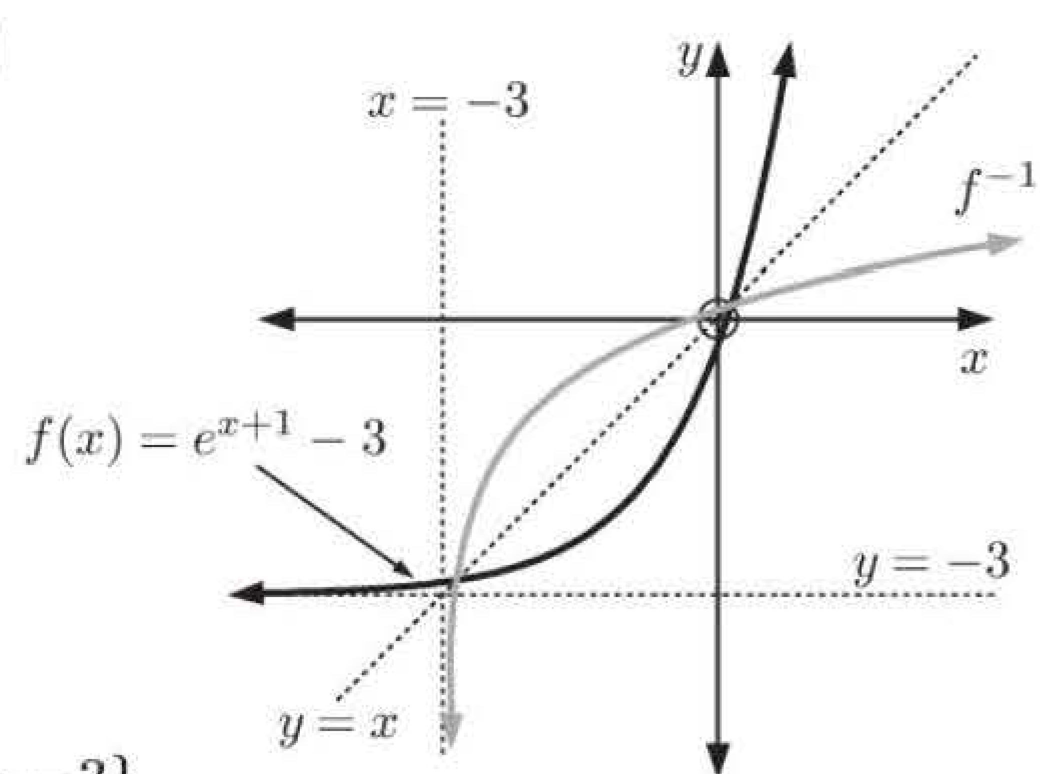
b **i**

$$f(x) = e^{x+1} - 3$$
or $y = e^{x+1} - 3$

has inverse function

$$x = e^{y+1} - 3$$
$$\therefore \quad x + 3 = e^{y+1}$$
$$\therefore \quad y + 1 = \ln(x + 3)$$
$$\therefore \quad f^{-1}(x) = \ln(x + 3) - 1$$

ii



iii Domain of f is $\{x \mid x \in \mathbb{R}\}$, range is $\{y \mid y > -3\}$.
Domain of f^{-1} is $\{x \mid x > -3\}$, range is $\{y \mid y \in \mathbb{R}\}$.

iv f has a H.A. $y = -3$. f^{-1} has a V.A. $x = -3$.
When $x = 0$, $y = e^{0+1} - 3 = e - 3$
$\therefore$ the y-intercept of f is $e - 3 \approx -0.282$
When $y = 0$, $e^{x+1} - 3 = 0$
$\therefore \; e^{x+1} = 3$
$\therefore \; x + 1 = \ln 3$
$\therefore \; x = \ln 3 - 1$
$\therefore$ the x-intercept of f is $\ln 3 - 1 \approx 0.0986$
$\therefore$ the x-intercept of f^{-1} is $e - 3 \approx -0.282$
and the y-intercept of f^{-1} is $\ln 3 - 1 \approx 0.0986$

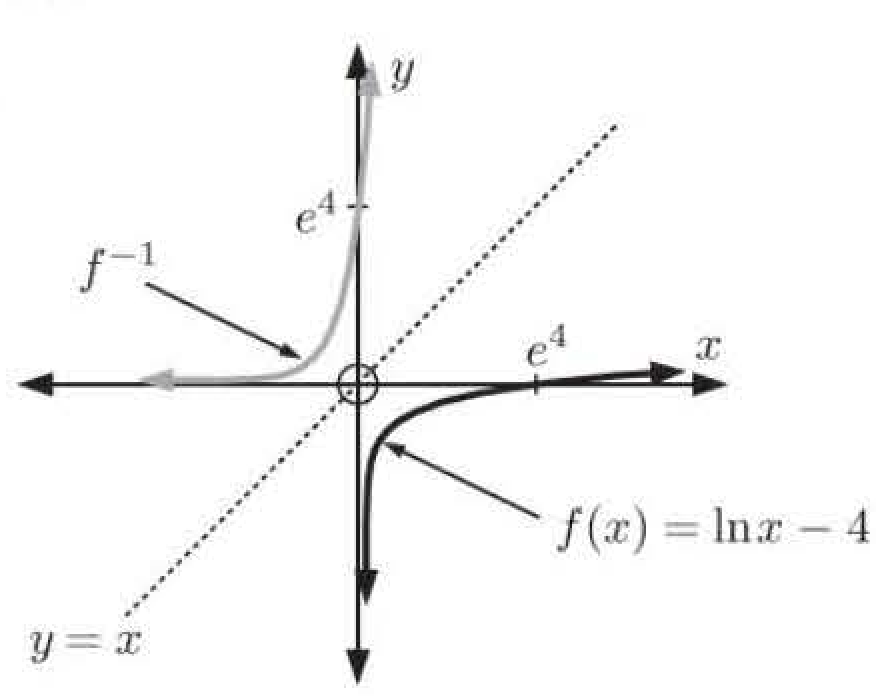
c **i**

$$f(x) = \ln x - 4, \; x > 0$$
$$\therefore \quad y = \ln x - 4$$

and has inverse function

$$x = \ln y - 4$$
$$\therefore \quad x + 4 = \ln y$$
$$\therefore \quad y = e^{x+4}$$
$$\therefore \quad f^{-1}(x) = e^{x+4}$$

ii



iii Domain of f is $\{x \mid x > 0\}$, range is $\{y \mid y \in \mathbb{R}\}$.
Domain of f^{-1} is $\{x \mid x \in \mathbb{R}\}$, range is $\{y \mid y > 0\}$.

iv f has a V.A. $x = 0$. f^{-1} has a H.A. $y = 0$.
When $x = 0$, $\ln x - 4$ is undefined.
$\therefore$ f has no y-intercept.
When $y = 0$, $\ln x - 4 = 0$

$$\therefore \ln x = 4$$
$$\therefore x = e^4 \approx 54.6$$

$\therefore$ the x-intercept of f is e^4.
$\therefore$ f^{-1} has no x-intercept and the y-intercept of f^{-1} is e^4.

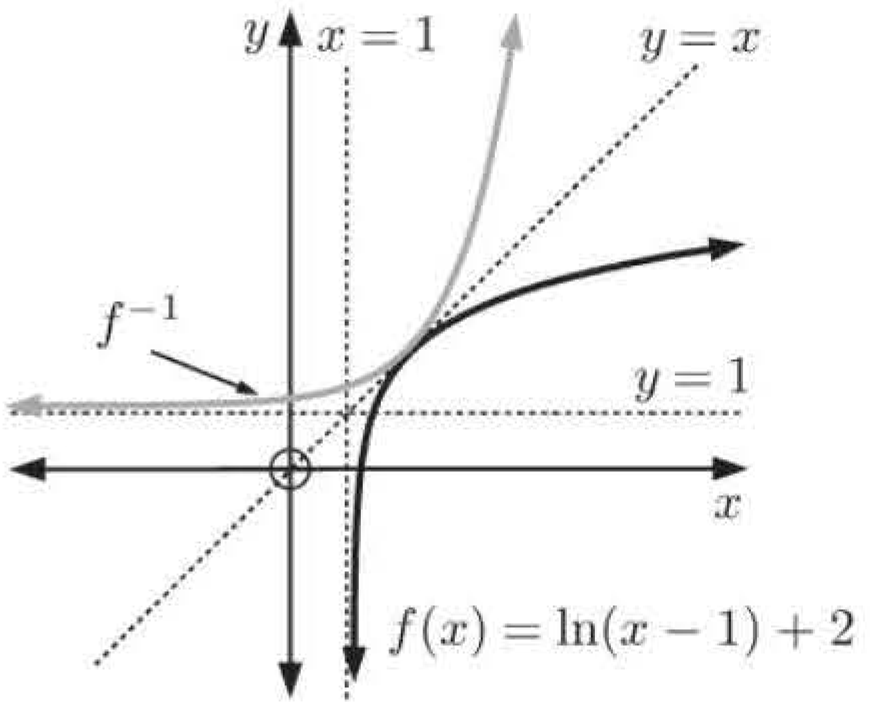
d **i**

$$f(x) = \ln(x-1) + 2, \; x > 1$$
$$\therefore y = \ln(x-1) + 2$$

and has inverse function

$$x = \ln(y-1) + 2$$
$$\therefore \ln(y-1) = x - 2$$
$$\therefore y - 1 = e^{x-2}$$
$$\therefore y = e^{x-2} + 1$$
$$\therefore f^{-1}(x) = e^{x-2} + 1$$

ii



iii Domain of f is $\{x \mid x > 1\}$, range is $\{y \mid y \in \mathbb{R}\}$.
Domain of f^{-1} is $\{x \mid x \in \mathbb{R}\}$, range is $\{y \mid y > 1\}$.

iv f has a V.A. $x = 1$. f^{-1} has a H.A. $y = 1$.
When $x = 0$, $\ln(x-1) + 2$ is undefined.
$\therefore$ f has no y-intercept.
When $y = 0$, $\ln(x-1) + 2 = 0$

$$\therefore \ln(x-1) = -2$$
$$\therefore x - 1 = e^{-2}$$
$$\therefore x = 1 + e^{-2}$$

$\therefore$ the x-intercept of f is $1 + e^{-2}$.
$\therefore$ f^{-1} has no x-intercept and the y-intercept of f^{-1} is $1 + e^{-2}$.

3 **a** f is $y = e^{2x}$
so the inverse function f^{-1} is

$$x = e^{2y}$$
$$\therefore 2y = \ln x$$
$$\therefore y = \tfrac{1}{2}\ln x$$
$$\therefore f^{-1}(x) = \tfrac{1}{2}\ln x$$
$$\therefore (f^{-1} \circ g)(x) = f^{-1}(g(x)) = f^{-1}(2x-1) = \tfrac{1}{2}\ln(2x-1)$$

b

$$(g \circ f)(x) = g(f(x)) = g(e^{2x}) = 2(e^{2x}) - 1$$

So, $y = 2e^{2x} - 1$ which has inverse

$$x = 2e^{2y} - 1$$
$$\therefore x + 1 = 2e^{2y}$$
$$\therefore \tfrac{1}{2}(x+1) = e^{2y}$$
$$\therefore 2y = \ln\left(\frac{x+1}{2}\right)$$
$$\therefore y = \tfrac{1}{2}\ln\left(\frac{x+1}{2}\right)$$
$$\therefore (g \circ f)^{-1}(x) = \tfrac{1}{2}\ln\left(\frac{x+1}{2}\right)$$

4 **a** $y = \ln x$ cuts the x-axis when $y = 0$

$\therefore \ln x = 0$

$\therefore x = e^0 = 1$

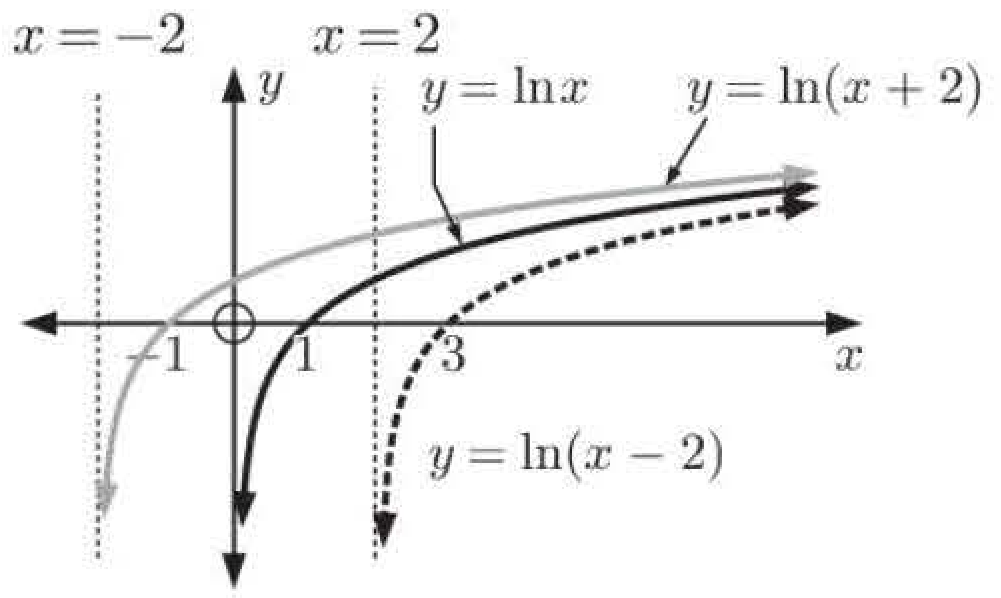
So, graph A is that of $y = \ln x$.

Note: x-intercept of $y = \ln(x-2)$ is when $x - 2 = e^0 = 1$

$\therefore x = 3$

b The x-intercept of $y = \ln(x+2)$ occurs when $x + 2 = e^0 = 1$

$\therefore x = -1$



c $y = \ln x$ has a V.A. of $x = 0$.
$y = \ln(x-2)$ has a V.A. of $x = 2$.
$y = \ln(x+2)$ has a V.A. of $x = -2$.

5 Since $y = \ln(x^2)$, $y = 2\ln x$ {log law}

$\therefore$ the new y-values are twice the old y-values.

$\therefore$ Kelly is correct.

(Note that $y = \ln x^2$ is also defined for $x < 0$. However, we are only concerned with $y = \ln x^2$ for $x > 0$.)

6 **a** $f(x) = e^{x+3} + 2$ or $y = e^{x+3} + 2$ has inverse function

$x = e^{y+3} + 2$

$\therefore x - 2 = e^{y+3}$

$\therefore \ln(x-2) = y + 3$

$\therefore y = \ln(x-2) - 3$

So, $f^{-1}(x) = \ln(x-2) - 3$

b **i** $f(x) < 2.1$ when $e^{x+3} + 2 < 2.1$

$\therefore e^{x+3} < 0.1$

$\therefore x + 3 < \ln(0.1)$

$\therefore x < \ln(0.1) - 3$

$\therefore x < -5.30$

ii Similarly, $f(x) < 2.01$ when

$x < \ln(0.01) - 3$

$\therefore x < -7.61$

iii $f(x) < 2.001$ when

$x < \ln(0.001) - 3$

$\therefore x < -9.91$

iv $f(x) < 2.0001$ when

$x < \ln(0.0001) - 3$

$\therefore x < -12.2$

We conjecture that the H.A. is $y = 2$.

c As $x \to \infty$, $y \to \infty$

As $x \to -\infty$, $e^{x+3} \to 0$ $\therefore y \to 2$

$\therefore$ H.A. is $y = 2$.

d f has a H.A. $y = 2$ and range $\{y \mid y > 2\}$

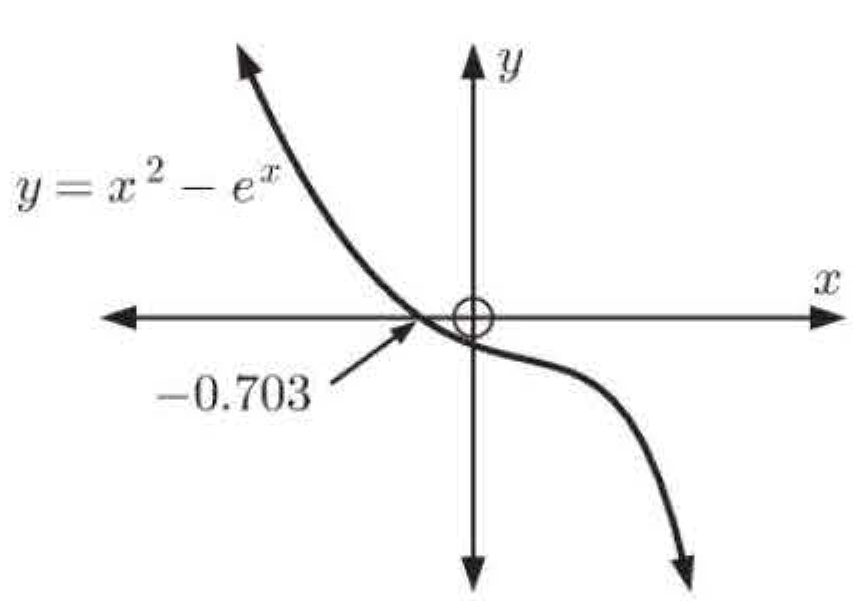
$\therefore f^{-1}$ has a V.A. $x = 2$ and domain $\{x \mid x > 2\}$

7 **a** $x^2 > e^x \Rightarrow x^2 - e^x > 0$

$\therefore$ to solve $x^2 > e^x$, we find where the graph of $y = x^2 - e^x$ is above the x-axis.

The graph cuts the x-axis when $x \approx -0.703$

$\therefore x^2 > e^x$ when $x < -0.703$

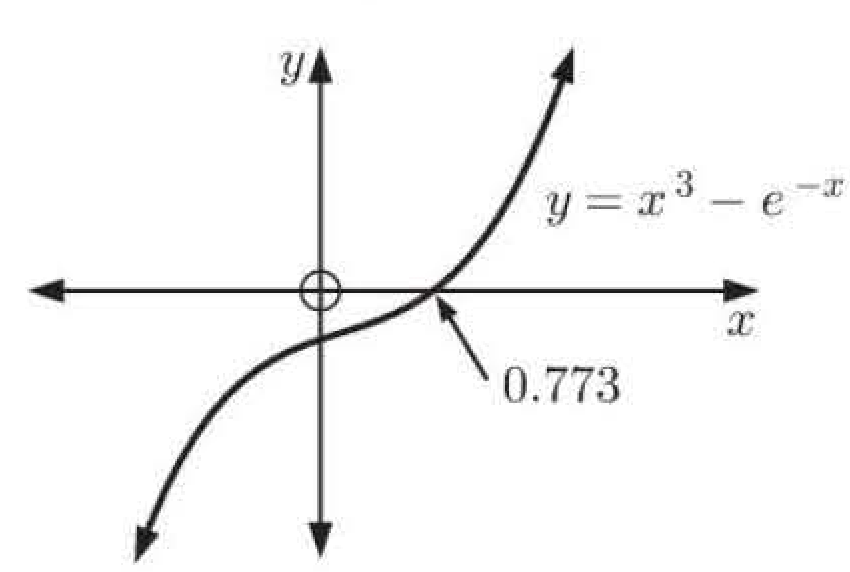


b $x^3 < e^{-x} \Rightarrow x^3 - e^{-x} < 0$

$\therefore$ to solve $x^3 < e^{-x}$, we find where the graph of $y = x^3 - e^{-x}$ is below the x-axis.

The graph cuts the x-axis when $x \approx 0.773$

$\therefore x^3 < e^{-x}$ when $x < 0.773$

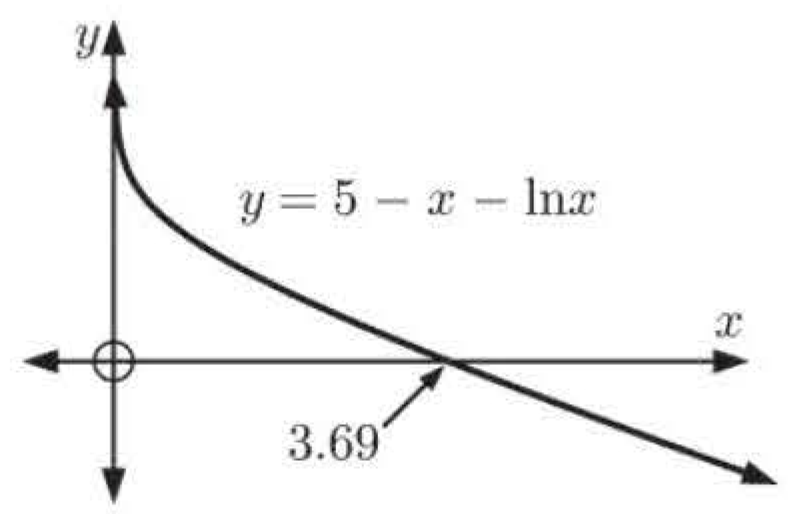


c $5 - x > \ln x \quad \Rightarrow \quad 5 - x - \ln x > 0$

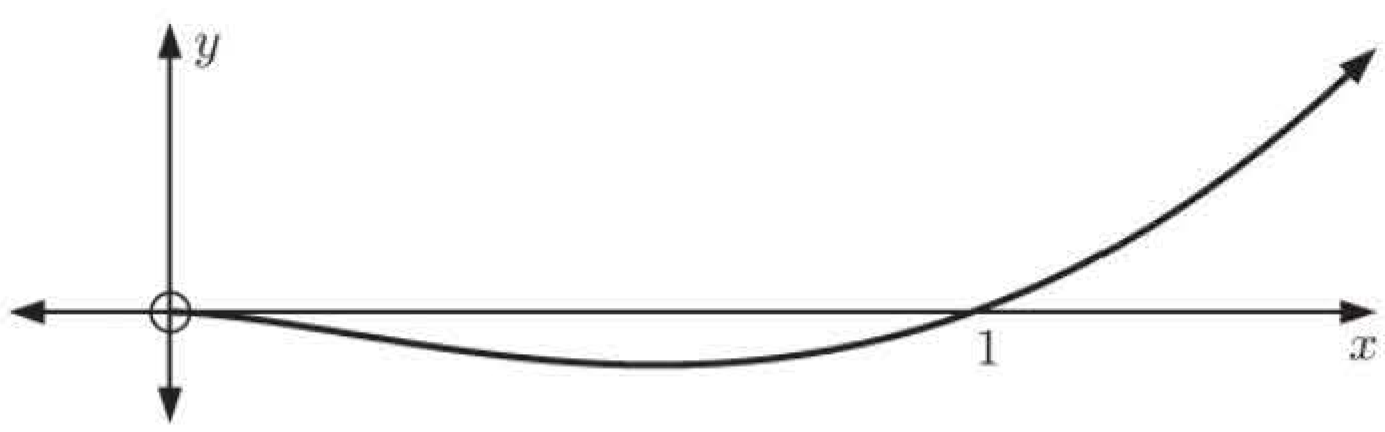
$\therefore$ to solve $5 - x > \ln x$, we find where the graph of $y = 5 - x - \ln x$ is above the x-axis.

The graph cuts the x-axis when $x \approx 3.69$ and $\ln x$ is only defined for $x > 0$

$\therefore \quad 5 - x > \ln x \quad$ when $\quad 0 < x < 3.69$



8 f is defined when $\ln x$ is defined. This is when $x > 0$.
So, the domain is $x \in \,]0, \infty[$



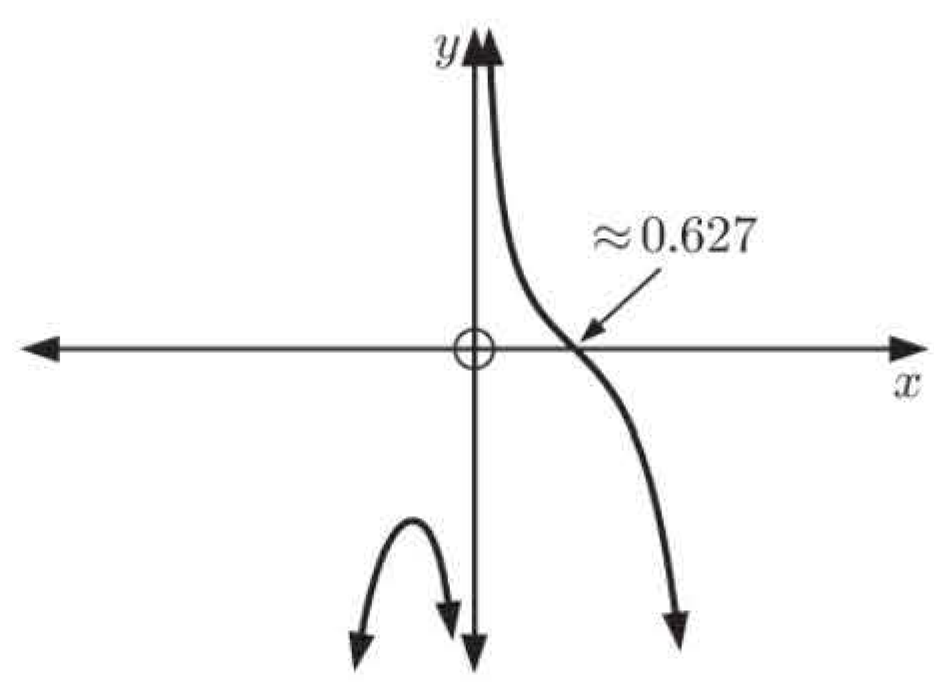
If $f(x) \leqslant 0$ then $x^2 \ln x \leqslant 0$

$\therefore$ the graph of $y = x^2 \ln x$ is on or below $y = 0$.

$\therefore \quad 0 < x \leqslant 1 \quad \therefore \quad x \in \,]0, 1]$

9 $f(x) = \dfrac{2}{x} - e^{2x^2 - x + 1}$

a



b domain is $\{x \mid x \in \mathbb{R}, \ x \neq 0\}$
range is $\{y \mid y \in \mathbb{R}\}$

c If $e^{2x^2 - x + 1} > \dfrac{2}{x}$

then $\dfrac{2}{x} - e^{2x^2 - x + 1} < 0$

So, we want x such that $f(x) < 0$.
This is for $x < 0$ or $x > 0.627$

$\therefore \quad x \in \,]-\infty, 0[$ or $x \in \,]0.627, \infty[$

EXERCISE 4H

1 $W_t = 20 \times 2^{0.15t}$ grams

a When $W_t = 30$,

$$20 \times 2^{0.15t} = 30$$
$$\therefore \quad 2^{0.15t} = 1.5$$
$$\therefore \quad \log 2^{0.15t} = \log(1.5)$$
$$\therefore \quad 0.15t \log 2 = \log(1.5)$$
$$\therefore \quad t = \frac{\log(1.5)}{0.15 \times \log 2}$$
$$\therefore \quad t \approx 3.90 \text{ hours}$$

$\therefore$ it takes about 3.90 hours to reach 30 g.

b When $W_t = 100$,

$$20 \times 2^{0.15t} = 100$$
$$\therefore \quad 2^{0.15t} = 5$$
$$\therefore \quad \log 2^{0.15t} = \log 5$$
$$\therefore \quad 0.15t \log 2 = \log 5$$
$$\therefore \quad t = \frac{\log 5}{0.15 \times \log 2}$$
$$\therefore \quad t \approx 15.5 \text{ hours}$$

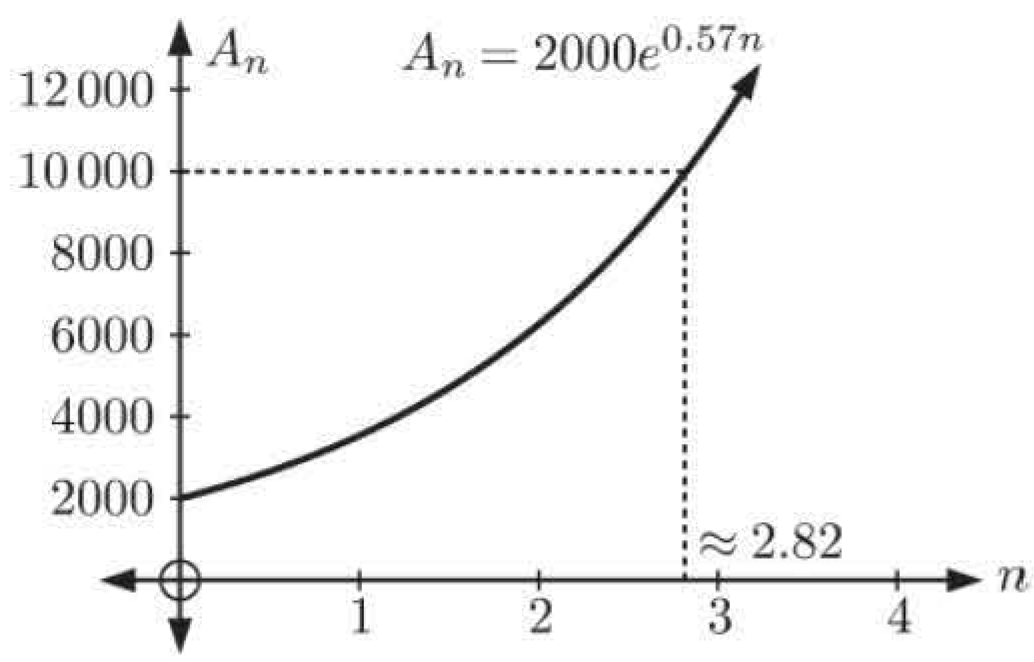
$\therefore$ it takes about 15.5 hours to reach 100 g.

2 When $M_t = 50$, $\quad 25 \times e^{0.1t} = 50$

$$\therefore \quad e^{0.1t} = 2$$
$$\therefore \quad \ln e^{0.1t} = \ln 2$$
$$\therefore \quad 0.1t = \ln 2$$
$$\therefore \quad t = 10 \ln 2$$

$\therefore$ it takes $10 \ln 2$ hours to reach 50 g.

3 **a**



b When $A_n = 10\,000$, $t \approx 2.82$
$\therefore$ we estimate that it will take 2.82 weeks for the infested area to reach 10 000 ha.

c **i** When $A_n = 10\,000$,
$$\begin{aligned} 2000 \times e^{0.57n} &= 10\,000 \\ \therefore\ e^{0.57n} &= 5 \\ \therefore\ \ln e^{0.57n} &= \ln 5 \\ \therefore\ 0.57n &= \ln 5 \\ \therefore\ n &= \frac{\ln 5}{0.57} \\ \therefore\ n &\approx 2.82 \end{aligned}$$
$\therefore$ it takes about 2.82 weeks for the infested area to reach 10 000 hectares.

4 $r = 107.5\% = 1.075$, $u_1 = 160\,000$, $u_{n+1} = 250\,000$
$$\begin{aligned} u_{n+1} &= u_1 \times r^n \\ \therefore\ 250\,000 &= 160\,000 \times (1.075)^n \\ \therefore\ (1.075)^n &= \tfrac{25}{16} \\ \therefore\ \log(1.075)^n &= \log(\tfrac{25}{16}) \\ \therefore\ n\log(1.075) &= \log(\tfrac{25}{16}) \\ \therefore\ n &= \frac{\log(\frac{25}{16})}{\log(1.075)} \approx 6.1709 \end{aligned}$$
$\therefore$ it would take 6.17 years or 6 years 62 days.

5 $u_1 = 10\,000$, $u_{n+1} = 15\,000$, $r = 104.8\% = 1.048$
$$\begin{aligned} u_{n+1} &= u_1 \times r^n \\ \therefore\ 15\,000 &= 10\,000 \times (1.048)^n \\ \therefore\ (1.048)^n &= 1.5 \\ \therefore\ \log(1.048)^n &= \log(1.5) \\ \therefore\ n\log(1.048) &= \log(1.5) \\ \therefore\ n &= \frac{\log(1.5)}{\log(1.048)} \\ \therefore\ n &\approx 8.648 \end{aligned}$$
$\therefore$ it would take 9 years. {interest compounded annually}

6 **a** 8.4% p.a. compounded monthly is $\dfrac{8.4\%}{12} = 0.7\%$ a month $= 0.007$
So $r = 1 + 0.007$
$\therefore\ r = 1.007$

b $u_1 = 15\,000$ and $u_{n+1} = 25\,000$
$$\begin{aligned} u_{n+1} &= u_1 \times r^n \\ \therefore\ 25\,000 &= 15\,000 \times (1.007)^n \\ \therefore\ (1.007)^n &= \tfrac{25}{15} = \tfrac{5}{3} \\ \therefore\ \log(1.007)^n &= \log(\tfrac{5}{3}) \\ \therefore\ n\log(1.007) &= \log(\tfrac{5}{3}) \\ \therefore\ n &= \frac{\log(\frac{5}{3})}{\log(1.007)} \approx 73.23 \end{aligned}$$
$\therefore$ he will withdraw the money after 74 months.

7 $M_t = 1000e^{-0.04t}$ $\quad \therefore\ M_0 = 1000e^0 = 1000$ g

a For M_t to halve, $M_t = 500$
$$\begin{aligned} \therefore\ 1000e^{-0.04t} &= 500 \\ \therefore\ e^{-0.04t} &= 0.5 \\ \therefore\ -0.04t &= \ln(0.5) \\ \therefore\ t &= \frac{\ln(0.5)}{-0.04} \\ \therefore\ t &\approx 17.3 \text{ years} \end{aligned}$$

b For $M_t = 25$ g,
$$\begin{aligned} \therefore\ 1000e^{-0.04t} &= 25 \\ \therefore\ e^{-0.04t} &= 0.025 \\ \therefore\ -0.04t &= \ln(0.025) \\ \therefore\ t &= \frac{\ln(0.025)}{-0.04} \\ \therefore\ t &\approx 92.2 \text{ years} \end{aligned}$$

c For $M_t = 1\%$ of M_0

$$\therefore \quad 1000e^{-0.04t} = 0.01 \times 1000$$
$$\therefore \quad e^{-0.04t} = 0.01$$
$$\therefore \quad -0.04t = \ln(0.01)$$
$$\therefore \quad t = \frac{\ln(0.01)}{-0.04}$$
$$\therefore \quad t \approx 115 \text{ years}$$

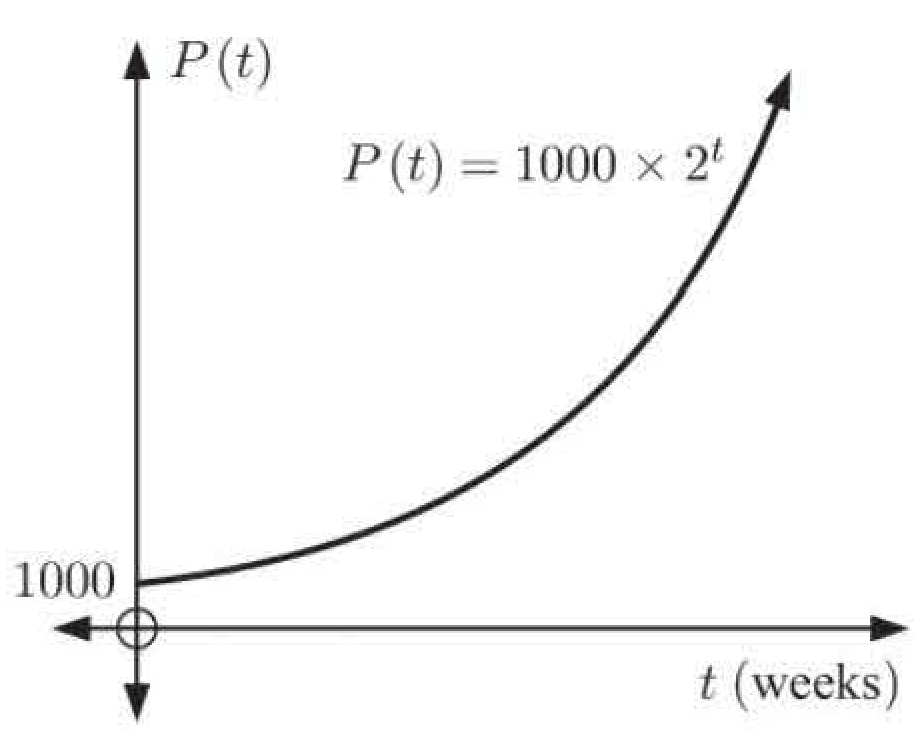
8 $V = 50(1 - e^{-0.2t})\ \text{m s}^{-1}$

So, when $V = 40$, $50(1 - e^{-0.2t}) = 40$

$$\therefore \quad 1 - e^{-0.2t} = 0.8$$
$$\therefore \quad e^{-0.2t} = 0.2$$
$$\therefore \quad -0.2t = \ln(0.2)$$
$$\therefore \quad -\tfrac{1}{5}t = \ln(\tfrac{1}{5})$$
$$\therefore \quad -\tfrac{1}{5}t = \ln(5^{-1})$$
$$\therefore \quad -\tfrac{1}{5}t = -\ln 5$$
$$\therefore \quad t = 5\ln 5$$

$\therefore$ it will take $5\ln 5$ seconds for the man's speed to reach $40\ \text{m s}^{-1}$.

9 **a**



b When $P(t) = 20\,000$,

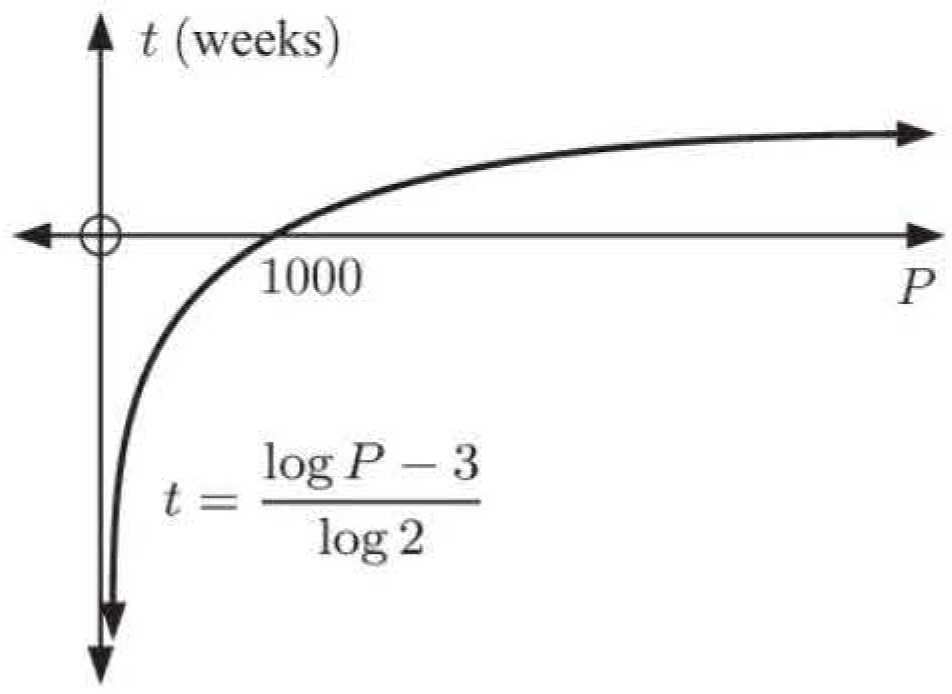
$$1000 \times 2^t = 20\,000$$
$$\therefore \quad 2^t = 20$$
$$\therefore \quad \log 2^t = \log 20$$
$$\therefore \quad t\log 2 = \log 20$$
$$\therefore \quad t = \frac{\log 20}{\log 2} \approx 4.32$$

$\therefore$ it will take 4.32 weeks for the population to reach 20 000 mice.

c

$$P = 1000 \times 2^t$$
$$\therefore \quad 2^t = \frac{P}{1000}$$
$$\therefore \quad \log 2^t = \log\left(\frac{P}{1000}\right)$$
$$\therefore \quad t\log 2 = \log P - \log 1000$$
$$\therefore \quad t = \frac{\log P - 3}{\log 2}$$

d



10 $T = 4 + 96 \times e^{-0.03t}\ ^\circ\text{C}$

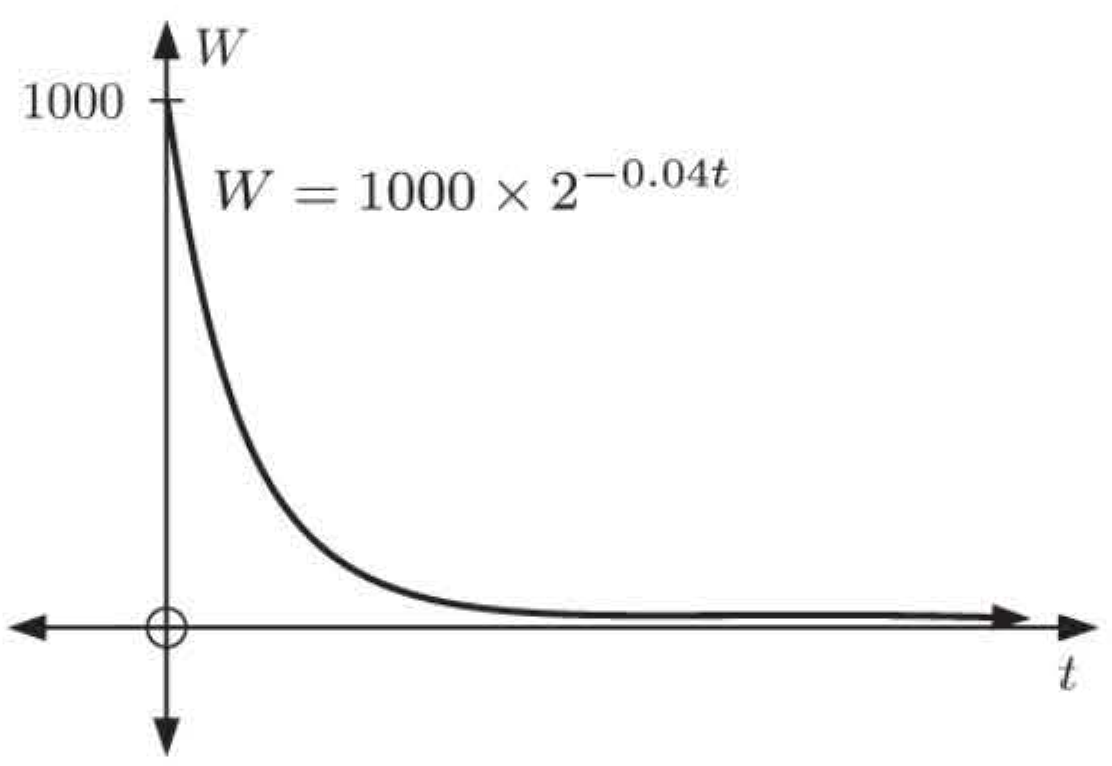
a When $T = 25$,

$$4 + 96 \times e^{-0.03t} = 25$$
$$\therefore \quad 96 \times e^{-0.03t} = 21$$
$$\therefore \quad e^{-0.03t} = \tfrac{21}{96}$$
$$\therefore \quad -0.03t = \ln(\tfrac{21}{96})$$
$$\therefore \quad t = \frac{\ln(\frac{21}{96})}{-0.03}$$
$$\therefore \quad t \approx 50.7 \text{ minutes}$$

b When $T = 5$,

$$4 + 96 \times e^{-0.03t} = 5$$
$$\therefore \quad 96 \times e^{-0.03t} = 1$$
$$\therefore \quad e^{-0.03t} = \tfrac{1}{96}$$
$$\therefore \quad -0.03t = \ln(\tfrac{1}{96})$$
$$\therefore \quad t = \frac{\ln(\frac{1}{96})}{-0.03}$$
$$\therefore \quad t \approx 152 \text{ minutes}$$

11 **a**



b $$W = 1000 \times 2^{-0.04t}$$
$$\therefore\ 2^{-0.04t} = \frac{W}{1000}$$
$$\therefore\ \log 2^{-0.04t} = \log\left(\frac{W}{1000}\right)$$
$$\therefore\ -0.04t \log 2 = \log W - \log 1000$$
$$\therefore\ 0.04t \log 2 = 3 - \log W$$
$$\therefore\ t = \frac{3 - \log W}{0.04 \log 2}$$

c **i** When $W = 20$,
$$t = \frac{3 - \log 20}{0.04 \log 2} \approx 141$$
$\therefore$ it will take about 141 years for the weight to reach 20 grams.

ii When $W = 0.001$,
$$t = \frac{3 - \log(0.001)}{0.04 \log 2} \approx 498$$
$\therefore$ it will take about 498 years for the weight to reach 0.001 grams.

12 $W = W_0 \times 2^{-0.0002t}$ grams

a When W is 25% of original,
$$W = \tfrac{1}{4} \text{ of } W_0$$
$$\therefore\ W_0 \times 2^{-0.0002t} = \tfrac{1}{4} \times W_0$$
$$\therefore\ 2^{-0.0002t} = 2^{-2}$$
$$\therefore\ 0.0002t = 2$$
$$\therefore\ t = \frac{2}{0.0002}$$
$$\therefore\ t = 10\,000$$
$\therefore$ it would take 10 000 years.

b When W is 0.1% of original,
$$W = \tfrac{0.1}{100} \text{ of } W_0$$
$$\therefore\ W_0 \times 2^{-0.0002t} = \tfrac{1}{1000} \times W_0$$
$$\therefore\ \log 2^{-0.0002t} = \log(0.001)$$
$$\therefore\ -0.0002t \log 2 = \log(0.001)$$
$$\therefore\ t = \frac{\log(0.001)}{-0.0002 \times \log 2}$$
$$\therefore\ t \approx 49\,829$$
$\therefore$ it would take about 49 800 years.

13 $I = I_0 \times 2^{-0.02t}$ amps

When I is 10% of its original value,
$$I = 10\% \text{ of } I_0$$
$$\therefore\ I_0 \times 2^{-0.02t} = 0.1 \times I_0$$
$$\therefore\ 2^{-0.02t} = 0.1$$
$$\therefore\ \log 2^{-0.02t} = \log(0.1)$$
$$\therefore\ -0.02t \log 2 = \log(0.1)$$
$$\therefore\ -\tfrac{1}{50} t \log 2 = -1$$
$$\therefore\ t = \frac{50}{\log 2} \text{ seconds}$$

14 $V = 60(1 - 2^{-0.2t})$ m s^{-1}

When $V = 50$, $60(1 - 2^{-0.2t}) = 50$
$$\therefore\ 1 - 2^{-0.2t} = 0.8\overline{3}$$
$$\therefore\ 2^{-0.2t} = 0.1\overline{6}$$
$$\therefore\ \log 2^{-0.2t} = \log 0.1\overline{6}$$
$$\therefore\ -0.2t \log 2 = \log 0.1\overline{6}$$
$$\therefore\ t = \frac{\log 0.1\overline{6}}{-0.2 \times \log 2}$$
$$\therefore\ t \approx 12.9 \text{ seconds}$$

REVIEW SET 4A

1 **a** $\log_4 64 = \log_4 4^3 = 3$

b $\log_2 256 = \log_2 2^8 = 8$

c $\log_2(0.25) = \log_2(\tfrac{1}{4}) = \log_2 2^{-2} = -2$

d $\log_{25} 5 = \log_{25} 25^{\frac{1}{2}} = \tfrac{1}{2}$

e $\log_8 1 = \log_8 8^0 = 0$

f $\log_{81} 3 = \log_{81} 81^{\frac{1}{4}} = \tfrac{1}{4}$

g $\log_9(0.\overline{1}) = \log_9(\tfrac{1}{9}) = \log_9 9^{-1} = -1$

h $\log_k \sqrt{k} = \log_k k^{\frac{1}{2}} = \tfrac{1}{2}$ provided $k > 0$, $k \neq 1$

2 **a** $\log\sqrt{10}$
$= \log 10^{\frac{1}{2}}$
$= \frac{1}{2}$

b $\log\left(\frac{1}{\sqrt[3]{10}}\right)$
$= \log 10^{-\frac{1}{3}}$
$= -\frac{1}{3}$

c $\log(10^a \times 10^{b+1})$
$= \log 10^{a+b+1}$
$= a+b+1$

3 **a** $4\ln 2 + 2\ln 3$
$= \ln 2^4 + \ln 3^2$
$= \ln(16 \times 9)$
$= \ln 144$

b $\frac{1}{2}\ln 9 - \ln 2$
$= \ln 9^{\frac{1}{2}} - \ln 2$
$= \ln 3 - \ln 2$
$= \ln(\frac{3}{2})$

c $2\ln 5 - 1$
$= \ln 5^2 - \ln e^1$
$= \ln\left(\frac{25}{e}\right)$

d $\frac{1}{4}\ln 81$
$= \ln\left(3^4\right)^{\frac{1}{4}}$
$= \ln 3^1$
$= \ln 3$

4 **a** $\ln\left(e\sqrt{e}\right)$
$= \ln(e^1 e^{\frac{1}{2}})$
$= \ln e^{\frac{3}{2}}$
$= \frac{3}{2}$

b $\ln\left(\frac{1}{e^3}\right)$
$= \ln e^{-3}$
$= -3$

c $\ln(e^{2x}) = 2x$

d $\ln\left(\frac{e}{e^x}\right)$
$= \ln(e^{1-x})$
$= 1-x$

5 **a** $\log 16 + 2\log 3$
$= \log 16 + \log 3^2$
$= \log(16 \times 9)$
$= \log 144$

b $\log_2 16 - 2\log_2 3$
$= \log_2 16 - \log_2 3^2$
$= \log_2(\frac{16}{9})$

c $2 + \log_4 5$
$= \log_4 4^2 + \log_4 5$
$= \log_4(16 \times 5)$
$= \log_4 80$

6 **a** $P = 3 \times b^x$
$\therefore \quad \log P = \log(3 \times b^x)$
$\therefore \quad \log P = \log 3 + \log b^x$
$\therefore \quad \log P = \log 3 + x\log b$

b $m = \dfrac{n^3}{p^2}$
$\therefore \quad \log m = \log\left(\dfrac{n^3}{p^2}\right)$
$\therefore \quad \log m = \log n^3 - \log p^2$
$\therefore \quad \log m = 3\log n - 2\log p$

7 $\log_3 7 \times 2\log_7 x$
$= \cancel{\log_3 7}^{\,1} \times 2 \times \dfrac{\log_3 x}{\cancel{\log_3 7}_{\,1}}$
$= 2\log_3 x$

8 **a** $\log T = 2\log x - \log y$
$\therefore \quad \log T = \log x^2 - \log y$
$\therefore \quad \log T = \log\left(\dfrac{x^2}{y}\right)$
$\therefore \quad T = \dfrac{x^2}{y}$

b $\log_2 K = \log_2 n + \frac{1}{2}\log_2 t$
$\therefore \quad \log_2 K = \log_2 n + \log_2 t^{\frac{1}{2}}$
$\therefore \quad \log_2 K = \log_2(n \times \sqrt{t})$
$\therefore \quad K = n\sqrt{t}$

9 **a** $\ln 32 = \ln 2^5$
$= 5\ln 2$

b $\ln 125 = \ln 5^3$
$= 3\ln 5$

c $\ln 729 = \ln 3^6$
$= 6\ln 3$

10 $\log_2 x$ is defined for all $x > 0$
$\therefore$ the domain is $\{x \mid x > 0\}$
and the range is $y \in \mathbb{R}$.
$\ln(x+5)$ is defined for all $x > -5$
$\therefore$ the domain is $\{x \mid x > -5\}$
and the range is $y \in \mathbb{R}$.

So, the completed table is:

Function	$y = \log_2 x$	$y = \ln(x+5)$
Domain	$x > 0$	$x > -5$
Range	$y \in \mathbb{R}$	$y \in \mathbb{R}$

11 **a** $\log_5 36$
$= \log_5(2^2 \times 3^2)$
$= \log_5 2^2 + \log_5 3^2$
$= 2\log_5 2 + 2\log_5 3$
$= 2A + 2B$

b $\log_5 54$
$= \log_5(2 \times 3^3)$
$= \log_5 2 + \log_5 3^3$
$= \log_5 2 + 3\log_5 3$
$= A + 3B$

c $\log_5(8\sqrt{3})$
$= \log_5(2^3 \times 3^{\frac{1}{2}})$
$= \log_5 2^3 + \log_5 3^{\frac{1}{2}}$
$= 3\log_5 2 + \frac{1}{2}\log_5 3$
$= 3A + \frac{1}{2}B$

d $\log_5(20.25)$
$= \log_5(\frac{81}{4})$
$= \log_5\left(\frac{3^4}{2^2}\right)$
$= \log_5 3^4 - \log_5 2^2$
$= 4\log_5 3 - 2\log_5 2$
$= 4B - 2A$

e $\log_5(0.\overline{8})$
$= \log_5(\frac{8}{9})$
$= \log_5\left(\frac{2^3}{3^2}\right)$
$= \log_5 2^3 - \log_5 3^2$
$= 3\log_5 2 - 2\log_5 3$
$= 3A - 2B$

12 **a**
$$3e^x - 5 = -2e^{-x}$$
$\therefore\ 3e^{2x} - 5e^x = -2$ {multiplying both sides by e^x}
$\therefore\ 3e^{2x} - 5e^x + 2 = 0$
$\therefore\ (3e^x - 2)(e^x - 1) = 0$
$\therefore\ 3e^x - 2 = 0$ or $e^x - 1 = 0$
$\therefore\ e^x = \frac{2}{3}$ or $e^x = 1$
$\therefore\ x = \ln(\frac{2}{3})$ or $x = 0$

b
$$2\ln x - 3\ln\left(\frac{1}{x}\right) = 10$$
$\therefore\ \ln x^2 - \ln\left(\frac{1}{x^3}\right) = 10$
$\therefore\ \ln x^2 + \ln\left(\frac{1}{x^3}\right)^{-1} = 10$
$\therefore\ \ln(x^2 \times x^3) = 10$
$\therefore\ \ln x^5 = 10$
$\therefore\ x^5 = e^{10}$
$\therefore\ x = \sqrt[5]{e^{10}}$
$\therefore\ x = e^2$

REVIEW SET 4B

1 **a** $32 = 10^{\log 32}$
$\approx 10^{1.5051}$

b 0.0013
$= 10^{\log(0.0013)}$
$\approx 10^{-2.8861}$

c 8.963×10^{-5}
$= 10^{\log(8.963)} \times 10^{-5}$
$\approx 10^{0.952} \times 10^{-5}$
$\approx 10^{-4.0475}$

2 **a** $\log_2 x = -3$
$\therefore\ x = 2^{-3}$
$\therefore\ x = \frac{1}{8}$

b $\log_5 x \approx 2.743$
$\therefore\ x \approx 5^{2.743}$
$\therefore\ x \approx 82.7$

c $\log_3 x \approx -3.145$
$\therefore\ x \approx 3^{-3.145}$
$\therefore\ x \approx 0.0316$

3 **a** $\log_2 k \approx 1.699 + x$
$\therefore\ k \approx 2^{1.699+x}$
$\therefore\ k \approx 2^{1.699} \times 2^x$
$\therefore\ k \approx 3.25 \times 2^x$

b $\log_a Q = 3\log_a P + \log_a R$
$= \log_a P^3 + \log_a R$
$= \log_a(P^3 \times R)$
$\therefore\ Q = P^3 R$

c $\log A \approx 5\log B - 2.602$
$\therefore\ \log A - \log B^5 \approx -2.602$
$\therefore\ \log\left(\frac{A}{B^5}\right) \approx -2.602$
$\therefore\ \frac{A}{B^5} \approx 10^{-2.602} \approx 0.0025$
$\therefore\ A \approx \frac{B^5}{400}$

4 **a**
$$\begin{aligned} 5^x &= 7 \\ \therefore\ \log 5^x &= \log 7 \\ \therefore\ x\log 5 &= \log 7 \\ \therefore\ x &= \frac{\log 7}{\log 5} \end{aligned}$$

b
$$\begin{aligned} 20 \times 2^{2x+1} &= 640 \\ \therefore\ 2^{2x+1} &= 32 \\ \therefore\ \log 2^{2x+1} &= \log 32 \\ \therefore\ (2x+1)\log 2 &= \log 32 \\ \therefore\ 2x+1 &= \frac{\log 32}{\log 2} = 5 \\ \therefore\ 2x &= 4 \\ \therefore\ x &= 2 \end{aligned}$$

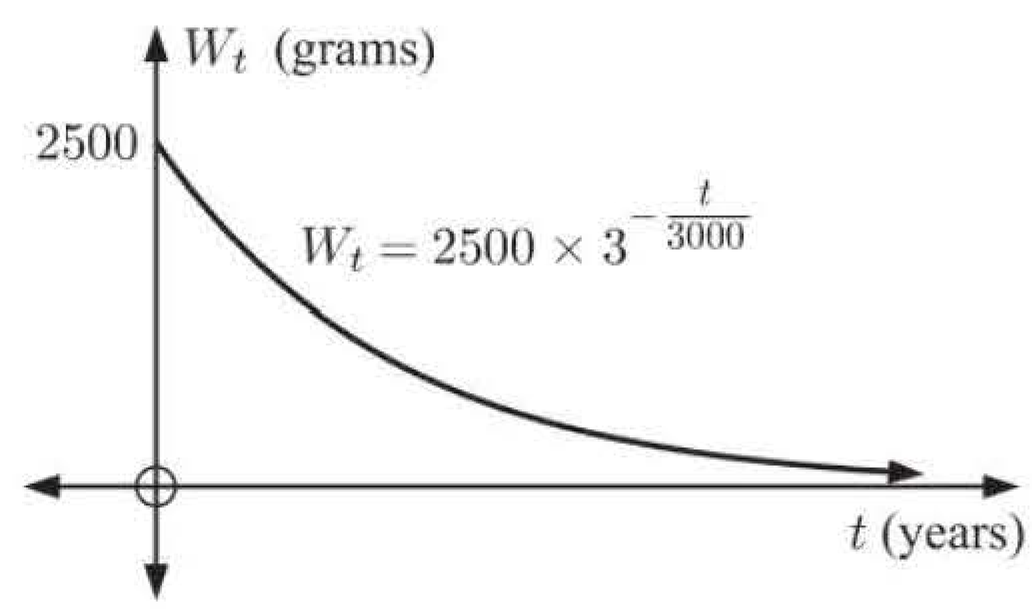
5 $W_t = 2500 \times 3^{-\frac{t}{3000}}$ grams

a
$$\begin{aligned} W_0 &= 2500 \times 3^0 \\ &= 2500 \times 1 \\ &= 2500 \text{ grams} \end{aligned}$$

b We need t when $W_t = 30\%$ of 2500 g
$$\begin{aligned} \therefore\ 2500 \times 3^{-\frac{t}{3000}} &= 0.3 \times 2500 \\ \therefore\ \log 3^{-\frac{t}{3000}} &= \log(0.3) \\ \therefore\ -\frac{t}{3000} \times \log 3 &= \log(0.3) \\ \therefore\ t &= \frac{-\log(0.3) \times 3000}{\log 3} \\ \therefore\ t &\approx 3287.7 \end{aligned}$$
$\therefore$ about 3290 years

c
$$\begin{aligned} &\%\text{ change} \\ &= \left(\frac{W_{1500} - W_0}{W_0}\right) \times 100\% \\ &= \left(\frac{2500 \times 3^{-\frac{1500}{3000}} - 2500}{2500}\right) \times 100\% \\ &= (3^{-\frac{1}{2}} - 1) \times 100\% \\ &\approx -42.3\% \end{aligned}$$
So, a loss of 42.3%.

d



6
$$\begin{aligned} 16^x - 5 \times 8^x &= 0 \\ \therefore\ 2^x \times 8^x - 5 \times 8^x &= 0 \\ \therefore\ 8^x(2^x - 5) &= 0 \\ \therefore\ 2^x &= 5 \quad \text{as} \quad 8^x > 0 \quad \text{for all } x \\ \therefore\ x &= \log_2 5 \quad \text{as required} \end{aligned}$$

7 **a**
$$\begin{aligned} \ln x &= 5 \\ \therefore\ x &= e^5 \end{aligned}$$

b
$$\begin{aligned} 3\ln x + 2 &= 0 \\ \therefore\ 3\ln x &= -2 \\ \therefore\ \ln x &= -\tfrac{2}{3} \\ \therefore\ x &= e^{-\frac{2}{3}} \end{aligned}$$

c
$$\begin{aligned} e^x &= 400 \\ \therefore\ x &= \ln 400 \end{aligned}$$

d
$$\begin{aligned} e^{2x+1} &= 11 \\ \therefore\ 2x+1 &= \ln 11 \\ \therefore\ 2x &= \ln 11 - 1 \\ \therefore\ x &= \frac{\ln 11 - 1}{2} \end{aligned}$$

e
$$\begin{aligned} 25e^{\frac{x}{2}} &= 750 \\ \therefore\ e^{\frac{x}{2}} &= 30 \\ \therefore\ \frac{x}{2} &= \ln 30 \\ \therefore\ x &= 2\ln 30 \end{aligned}$$

8 $P_t = P_0 \times 2^{\frac{t}{3}}, \quad t \geqslant 0$

When $t = 0$, $P_0 = P_0 \times 2^0 = P_0$. So the initial population was P_0.

a If P_t doubles, $P_t = 2P_0$

$$\therefore \; P_0 2^{\frac{t}{3}} = 2P_0$$
$$\therefore \; 2^{\frac{t}{3}} = 2^1$$
$$\therefore \; \frac{t}{3} = 1$$
$$\therefore \; t = 3$$

So, it would take 3 years.

b $$\% \text{ increase} = \left(\frac{P_4 - P_0}{P_0}\right) \times 100\%$$
$$= \left(\frac{P_0 \times 2^{\frac{4}{3}} - P_0}{P_0}\right) \times 100\%$$
$$= (2^{\frac{4}{3}} - 1) \times 100\%$$
$$\approx 151.98\%$$

So, the % increase is 152%.

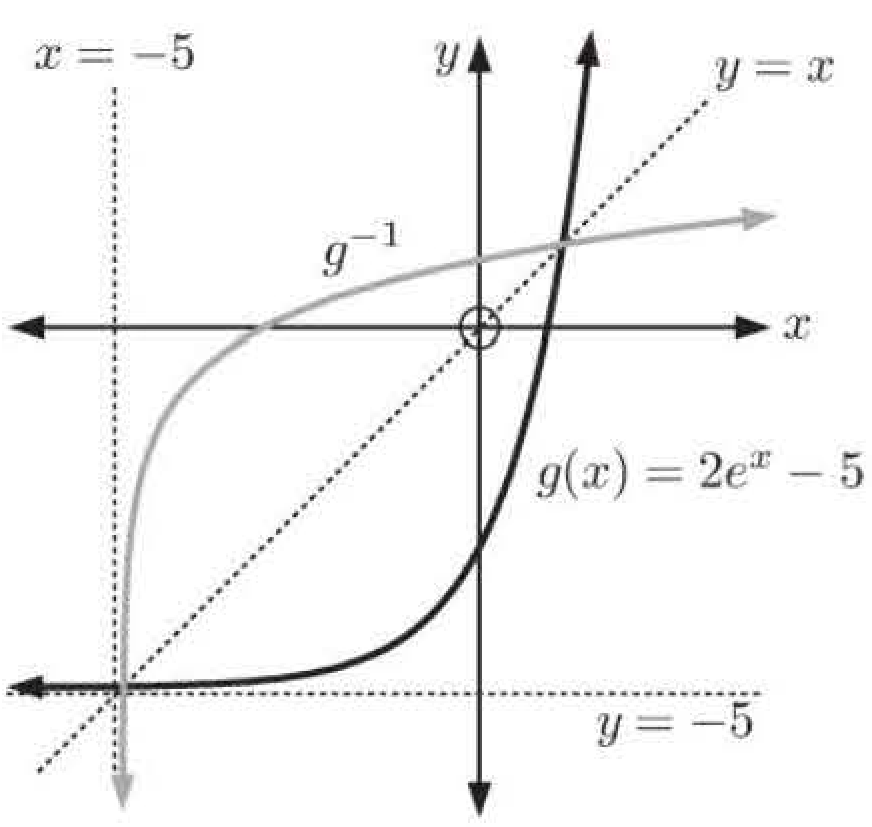
or $$P_4 = P_0 \times 2^{\frac{4}{3}}$$
$$\approx P_0 \times 2.52$$
$$\approx 252\% \text{ of } P_0$$

So, an increase of 152%.

9 **a** $g(x) = 2e^x - 5$ has inverse function $x = 2e^y - 5$

$$\therefore \; 2e^y = x + 5$$
$$\therefore \; e^y = \frac{x+5}{2}$$
$$\therefore \; y = \ln\left(\frac{x+5}{2}\right)$$
$$\therefore \; g^{-1}(x) = \ln\left(\frac{x+5}{2}\right)$$

b



c Domain of g is $\{x \mid x \in \mathbb{R}\}$, range is $\{y \mid y > -5\}$

Domain of g^{-1} is $\{x \mid x > -5\}$, range is $\{y \mid y \in \mathbb{R}\}$

d When $x = 0$, $y = 2e^0 - 5 = -3$

$\therefore$ the y-intercept of g is -3.

When $y = 0$, $2e^x - 5 = 0$

$$\therefore \; e^x = \tfrac{5}{2}$$
$$\therefore \; x = \ln\left(\tfrac{5}{2}\right)$$

$\therefore$ the x-intercept of g is $\ln\left(\frac{5}{2}\right) \approx 0.916$.

$\therefore$ g^{-1} has x-intercept -3 and y-intercept $\ln\left(\frac{5}{2}\right) \approx 0.916$.

As $x \to -\infty$, $y \to -5^+$

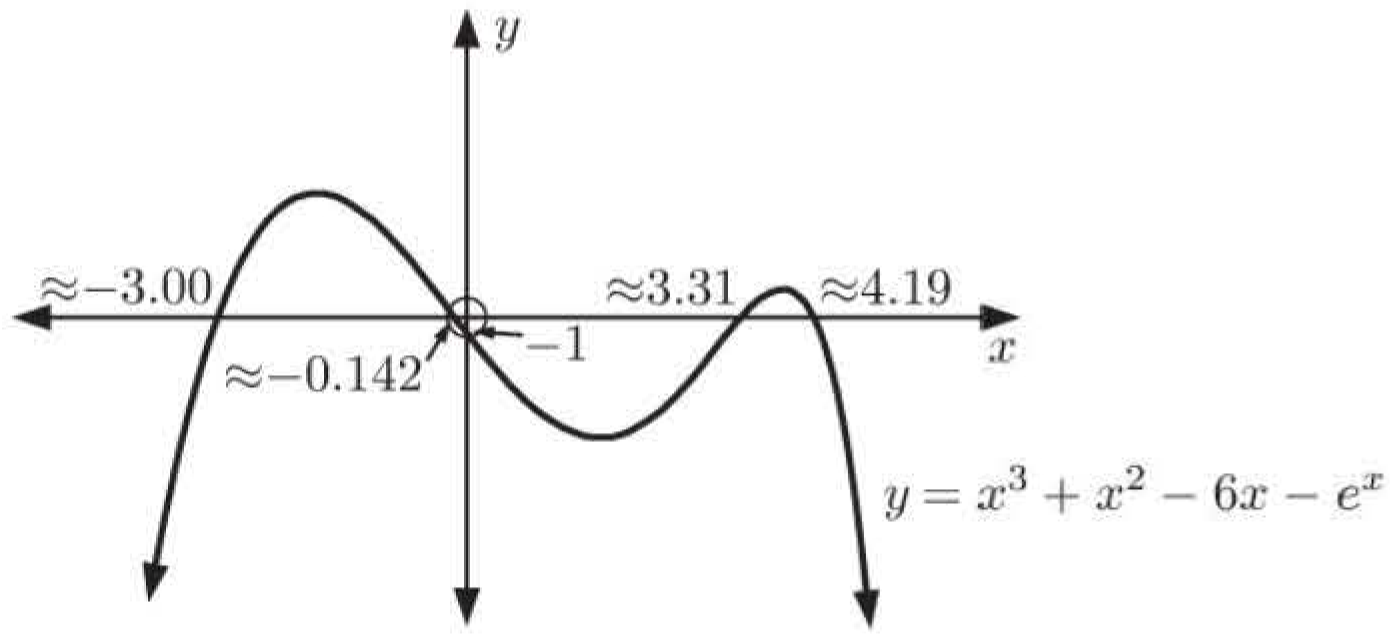
$\therefore$ the H.A. of g is $y = -5$.

$\therefore$ g^{-1} has V.A. $x = -5$.

10 **a** $$g(5) = \ln(5 + 4)$$
$$= \ln 9$$
$$\therefore \; (f \circ g)(5) = f(g(5))$$
$$= e^{\ln 9}$$
$$= 9$$

b $$f(0) = e^0$$
$$= 1$$
$$\therefore \; (g \circ f)(0) = g(f(0))$$
$$= \ln(1 + 4)$$
$$= \ln 5$$

11 **a**



b $e^x < x^3 + x^2 - 6x$ when $x^3 + x^2 - 6x - e^x > 0$

$\therefore$ from the graph, when $-3.00 < x < -0.142$ and $3.31 < x < 4.19$.

REVIEW SET 4C

1 **a** $\log\sqrt{1000}$
$= \log\left(10^3\right)^{\frac{1}{2}}$
$= \log 10^{\frac{3}{2}}$
$= \frac{3}{2}$

b $\log\left(\dfrac{10}{\sqrt[3]{10}}\right)$
$= \log\left(\dfrac{10^1}{10^{\frac{1}{3}}}\right)$
$= \log 10^{\frac{2}{3}} = \frac{2}{3}$

c $\log\left(\dfrac{10^a}{10^{-b}}\right)$
$= \log\left(10^{a-(-b)}\right)$
$= \log 10^{a+b}$
$= a+b$

2 **a** $e^{4\ln x}$
$= \left(e^{\ln x}\right)^4$
$= x^4$

b $\ln(e^5) = 5$
$\{$as $\ln e^a = a\}$

c $\ln\left(\sqrt{e}\right) = \ln e^{\frac{1}{2}}$
$= \frac{1}{2}$

d $10^{\log x + \log 3}$
$= 10^{\log x} \times 10^{\log 3}$
$= x \times 3$
$= 3x$

e $\ln\left(\frac{1}{e^x}\right) = \ln e^{-x}$
$= -x$

f $\dfrac{\log x^2}{\log_3 9}$
$= \dfrac{\log x^2}{2}$
$= \frac{1}{2}\log x^2$
$= \log\left(x^2\right)^{\frac{1}{2}}$
$= \log x$

3 **a** $20 = e^{\ln 20}$
$\approx e^{2.9957}$

b $3000 = e^{\ln 3000}$
$\approx e^{8.0064}$

c $0.075 = e^{\ln(0.075)}$
$\approx e^{-2.5903}$

4 **a** $\log x = 3$
$\therefore\ x = 10^3$
$\therefore\ x = 1000$

b $\log_3(x+2) = 1.732$
$\therefore\ x+2 = 3^{1.732}$
$\therefore\ x+2 \approx 6.7046$
$\therefore\ x \approx 4.7046$
$\therefore\ x \approx 4.70$

c $\log_2\left(\dfrac{x}{10}\right) = -0.671$
$\therefore\ \dfrac{x}{10} = 2^{-0.671}$
$\therefore\ \dfrac{x}{10} \approx 0.628\,07$
$\therefore\ x \approx 6.28$

5 **a** $\ln 60 - \ln 20$
$= \ln(\frac{60}{20})$
$= \ln 3$

b $\ln 4 + \ln 1$
$= \ln 4 + 0$
$= \ln 4$

c $\ln 200 - \ln 8 + \ln 5$
$= \ln(\frac{200}{8}) + \ln 5$
$= \ln\left(\frac{200}{8} \times 5\right)$
$= \ln 125$

6 **a**
$$\begin{aligned} M &= ab^n \\ \therefore\ \log M &= \log(ab^n) \\ \therefore\ \log M &= \log a + \log b^n \\ \therefore\ \log M &= \log a + n\log b \end{aligned}$$

b
$$\begin{aligned} T &= \frac{5}{\sqrt{l}} \\ \therefore\ \log T &= \log\left(\frac{5}{l^{\frac{1}{2}}}\right) \\ \therefore\ \log T &= \log 5 - \log l^{\frac{1}{2}} \\ \therefore\ \log T &= \log 5 - \tfrac{1}{2}\log l \end{aligned}$$

c
$$\begin{aligned} G &= \frac{a^2b}{c} \\ \therefore\ \log G &= \log\left(\frac{a^2b}{c}\right) \\ \therefore\ \log G &= \log(a^2b) - \log c \\ \therefore\ \log G &= \log a^2 + \log b - \log c \\ \therefore\ \log G &= 2\log a + \log b - \log c \end{aligned}$$

7 **a**
$$\begin{aligned} 3^x &= 300 \\ \therefore\ \log 3^x &= \log 300 \\ \therefore\ x\log 3 &= \log 300 \\ \therefore\ x &= \frac{\log 300}{\log 3} \\ \therefore\ x &\approx 5.19 \end{aligned}$$

b
$$\begin{aligned} 30 \times 5^{1-x} &= 0.15 \\ \therefore\ 5^{1-x} &= 0.005 \\ \therefore\ \log 5^{1-x} &= \log(0.005) \\ \therefore\ (1-x)\log 5 &= \log(0.005) \\ \therefore\ 1-x &= \frac{\log(0.005)}{\log 5} \\ \therefore\ 1-x &\approx -3.292 \\ \therefore\ x &\approx 4.29 \end{aligned}$$

c
$$\begin{aligned} 3^{x+2} &= 2^{1-x} \\ \therefore\ \log 3^{x+2} &= \log 2^{1-x} \\ \therefore\ (x+2)\log 3 &= (1-x)\log 2 \\ \therefore\ x\log 3 + 2\log 3 &= \log 2 - x\log 2 \\ \therefore\ x(\log 3 + \log 2) &= \log 2 - 2\log 3 \\ \therefore\ x\log 6 &= \log(\tfrac{2}{9}) \\ \therefore\ x &= \frac{\log(\frac{2}{9})}{\log 6} \\ \therefore\ x &\approx -0.839 \end{aligned}$$

8 **a**
$$\begin{aligned} e^{2x} &= 3e^x \\ \therefore\ e^{2x} - 3e^x &= 0 \\ \therefore\ e^x(e^x - 3) &= 0 \\ \therefore\ e^x - 3 &= 0 \quad \{e^x > 0 \text{ for all } x\} \\ \therefore\ e^x &= 3 \\ \therefore\ x &= \ln 3 \end{aligned}$$

b
$$\begin{aligned} e^{2x} - 7e^x + 12 &= 0 \\ \therefore\ (e^x - 3)(e^x - 4) &= 0 \\ \therefore\ e^x - 3 = 0 \quad &\text{or} \quad e^x - 4 = 0 \\ \therefore\ e^x = 3 \quad &\text{or} \quad e^x = 4 \\ \therefore\ x = \ln 3 \quad &\text{or} \quad \ln 4 \end{aligned}$$

9 **a**
$$\begin{aligned} \ln P &= 1.5\ln Q + \ln T \\ \therefore\ \ln P &= \ln Q^{1.5} + \ln T \\ &= \ln(TQ^{1.5}) \\ \therefore\ P &= TQ^{1.5} \end{aligned}$$

b
$$\begin{aligned} \ln M &= 1.2 - 0.5\ln N \\ \therefore\ \ln M + \ln N^{\frac{1}{2}} &= 1.2 \\ \therefore\ \ln\left(M\sqrt{N}\right) &= 1.2 \\ \therefore\ M\sqrt{N} &= e^{1.2} \\ \therefore\ M &= \frac{e^{1.2}}{\sqrt{N}} \end{aligned}$$

10 **a** $f(x) = e^{x^2} - x^6$

$\therefore\ f(-x) = e^{(-x)^2} - (-x)^6$
$= e^{x^2} - x^6$
$= f(x)$

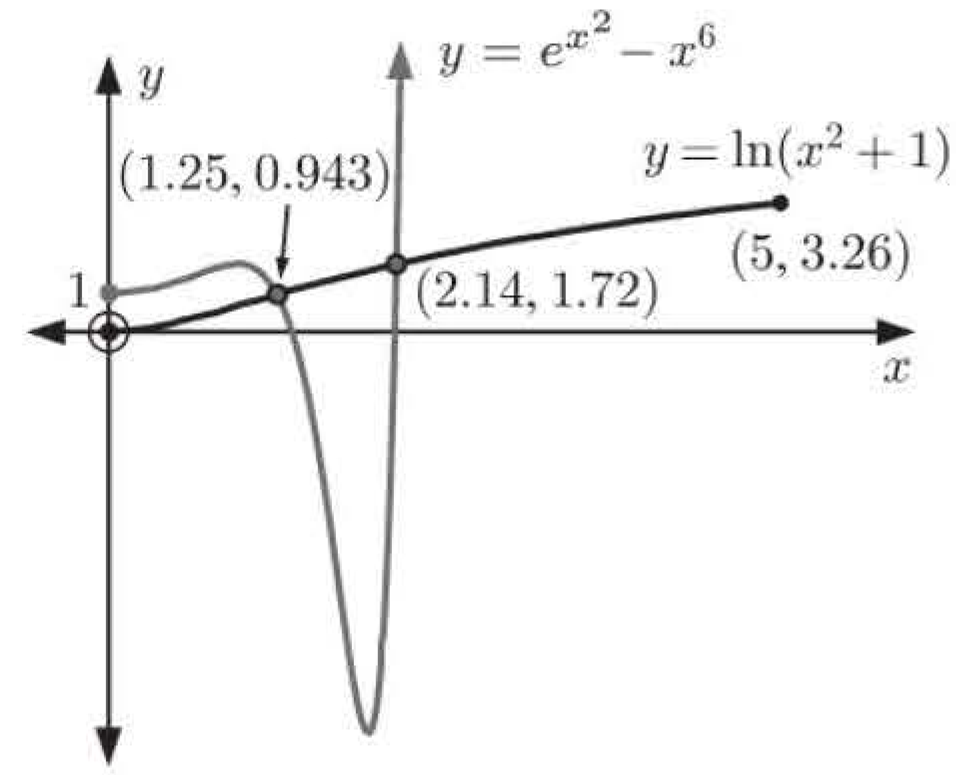
So, $f(x)$ is an even function.

$g(x) = \ln(x^2+1)$

$\therefore\ g(-x) = \ln((-x)^2+1)$
$= \ln(x^2+1)$
$= g(x)$

So, $g(x)$ is an even function.

b



On the domain $0 \leqslant x \leqslant 5$, the graphs intersect at $(1.25, 0.943)$ and $(2.14, 1.72)$.

c $x^6 - e^{x^2} + \ln(x^2+1) > 0$

when $\ln(x^2+1) > e^{x^2} - x^6$

$\therefore$ using **b**, $1.25 < x < 2.14$.

11 $g(x) = \log_3(x+2) - 2$

a We require $x + 2 > 0$, so $x > -2$

$\therefore$ the domain is $\{x \mid x > -2\}$ and the range is $\{y \mid y \in \mathbb{R}\}$.

b If $x \to -2^+$, $y \to -\infty$ $\therefore$ V.A. is $x = -2$.

As $x \to \infty$, $y \to \infty$.

When $x = 0$, $g(0) = \log_3 2 - 2 \approx -1.37$ So, the y-intercept ≈ -1.37

When $y = 0$, $\log_3(x+2) = 2$ $\therefore$ $x + 2 = 3^2$

$\therefore$ $x = 7$ So, the x-intercept is 7.

c g^{-1} is defined by $x = \log_3(y+2) - 2$

$\therefore$ $\log_3(y+2) = x + 2$

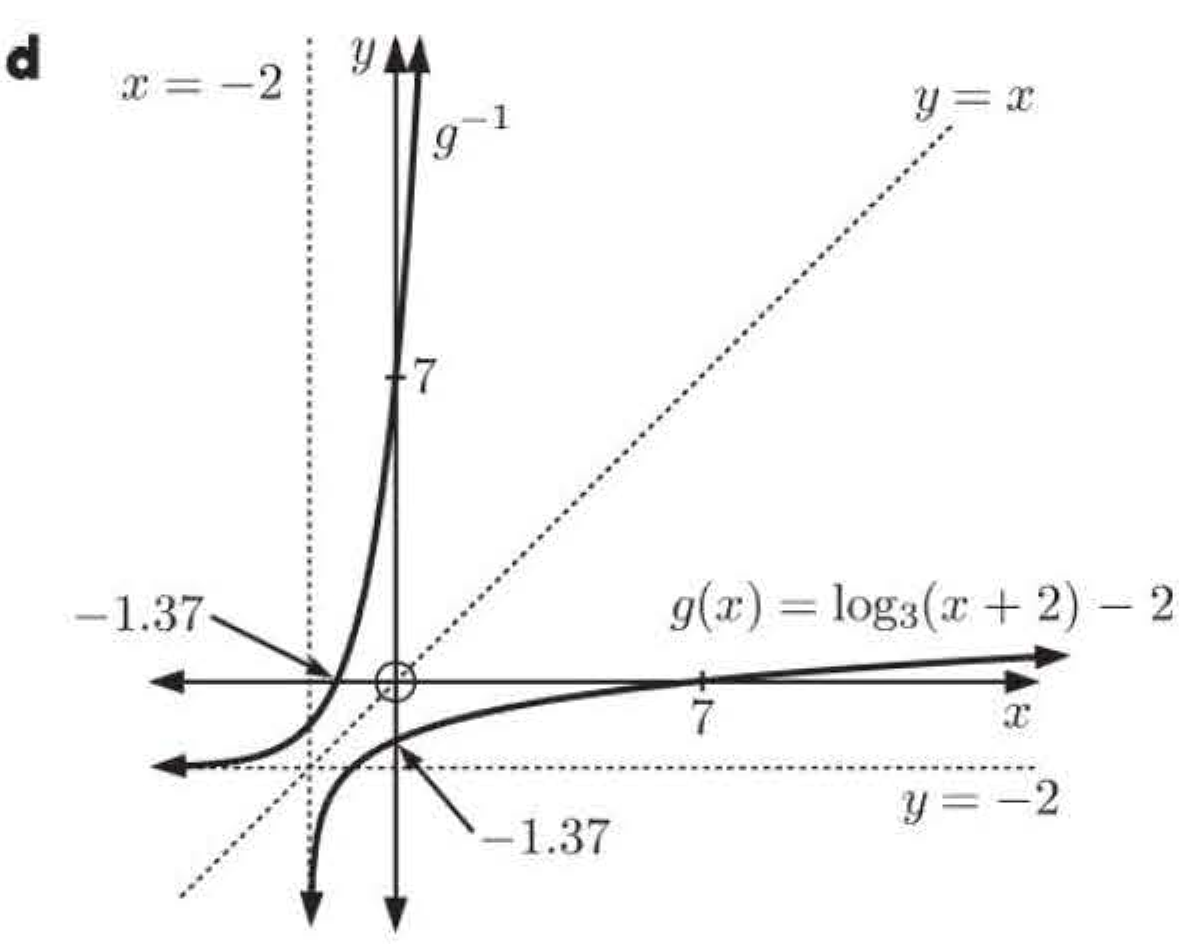
$\therefore$ $y + 2 = 3^{x+2}$

$\therefore$ $y = 3^{x+2} - 2$

$\therefore$ $g^{-1}(x) = 3^{x+2} - 2$

Horizontal asymptote is $y = -2$.
Domain is $x \in \mathbb{R}$.
Range is $\{y \mid y > -2\}$.

d



12 $W_t = 8000 \times e^{-\frac{t}{20}}$ grams

$W_0 = 8000e^0$
$= 8000 \times 1$
$= 8000$ grams

a When $W_t = \frac{1}{2} \times 8000$ grams, $8000e^{-\frac{t}{20}} = 4000$

$\therefore$ $e^{-\frac{t}{20}} = 0.5$

$\therefore$ $-\dfrac{t}{20} = \ln(0.5)$

$\therefore$ $t = -20\ln(0.5) \approx 13.9$ weeks

b When $W_t = 1000$ g,

$8000e^{-\frac{t}{20}} = 1000$

$\therefore$ $e^{-\frac{t}{20}} = \frac{1}{8}$

$\therefore$ $-\dfrac{t}{20} = \ln(\frac{1}{8})$

$\therefore$ $t = -20\ln(\frac{1}{8})$

$\therefore$ $t \approx 41.6$ weeks

c When $W_t = 0.1\%$ of W_0

$= \frac{1}{1000} \times 8000 = 8$ g,

$8000e^{-\frac{t}{20}} = 8$

$\therefore$ $e^{-\frac{t}{20}} = 0.001$

$\therefore$ $-\dfrac{t}{20} = \ln(0.001)$

$\therefore$ $t = -20\ln(0.001) \approx 138$ weeks

Chapter 5

TRANSFORMING FUNCTIONS

EXERCISE 5A

1 $f(x) = x$

a $f(2x) = 2x$

b $f(x)+2$
$= x+2$

c $\frac{1}{2}f(x) = \dfrac{x}{2}$

d $2f(x)+3$
$= 2x+3$

2 $f(x) = x^2$

a $f(3x) = (3x)^2$
$= 9x^2$

b $f\left(\dfrac{x}{2}\right) = \left(\dfrac{x}{2}\right)^2$
$= \dfrac{x^2}{4}$

c $3f(x) = 3x^2$

d $2f(x-1)+5$
$= 2(x-1)^2+5$
$= 2(x^2-2x+1)+5$
$= 2x^2-4x+7$

3 $f(x) = x^3$

a $f(4x)$
$= (4x)^3$
$= 64x^3$

b $\frac{1}{2}f(2x)$
$= \frac{1}{2}(2x)^3$
$= \frac{1}{2}\times 8x^3$
$= 4x^3$

c $f(x+1)$
$= (x+1)^3$
$= x^3+3x^2+3x+1$

d $2f(x+1)-3$
$= 2(x+1)^3-3$
$= 2(x^3+3x^2+3x+1)-3$
$= 2x^3+6x^2+6x-1$

4 $f(x) = 2^x$

a $f(2x) = 2^{2x}$
$= 4^x$

b $f(-x)+1$
$= 2^{-x}+1$

c $f(x-2)+3$
$= 2^{x-2}+3$

d $2f(x)+3$
$= 2\times 2^x+3$
$= 2^{x+1}+3$

5 $f(x) = \dfrac{1}{x}$

a $f(-x)$
$= \dfrac{1}{(-x)}$
$= -\dfrac{1}{x}$

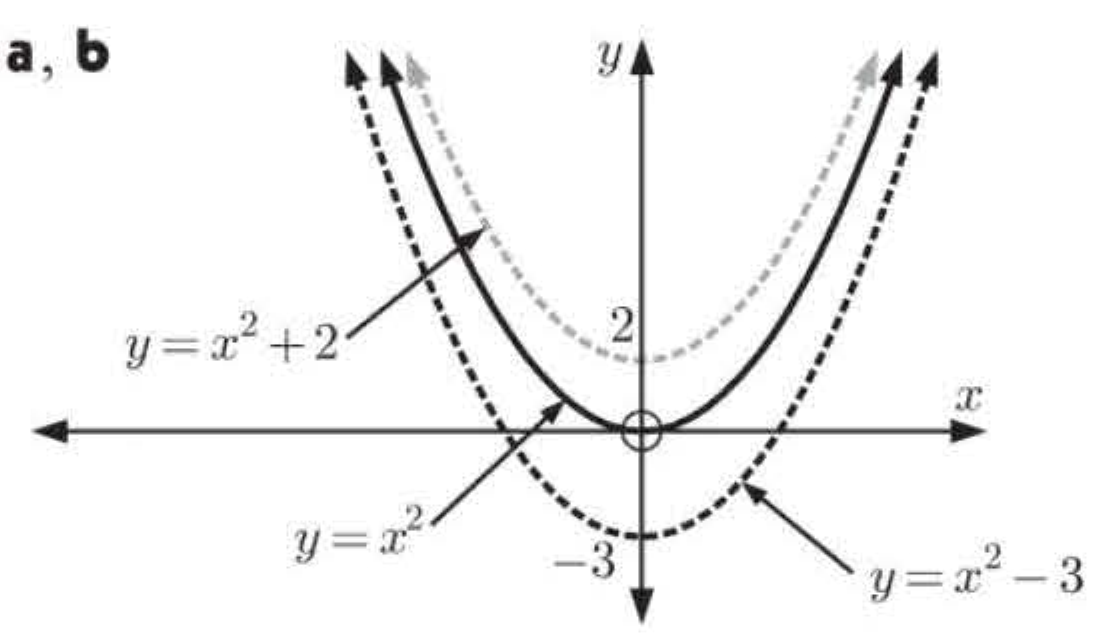
b $f(\frac{1}{2}x)$
$= \dfrac{1}{\frac{1}{2}x}$
$= \dfrac{2}{x}$

c $2f(x)+3$
$= 2\left(\dfrac{1}{x}\right)+3$
$= \dfrac{2}{x}+3$
$= \dfrac{2+3x}{x}$

d $3f(x-1)+2$
$= 3\left(\dfrac{1}{x-1}\right)+2$
$= \dfrac{3+2(x-1)}{x-1}$
$= \dfrac{2x+1}{x-1}$

EXERCISE 5B

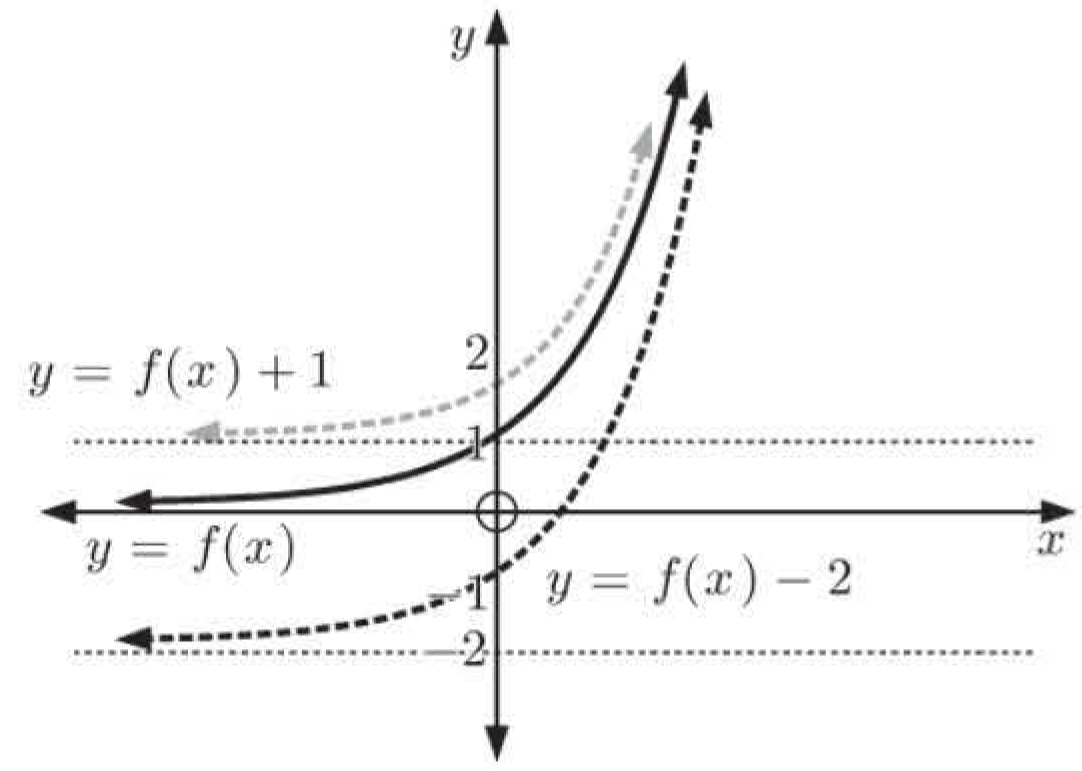
1 **a, b**



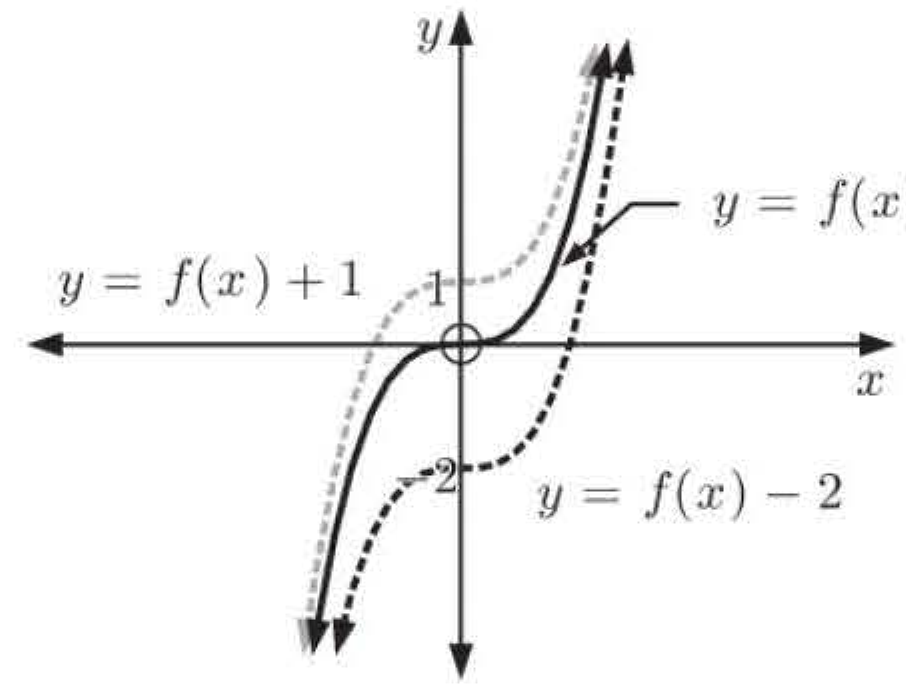
c **i** If $b > 0$, the function is translated vertically upwards through b units.

ii If $b < 0$, the function is translated vertically downwards $|b|$ units.

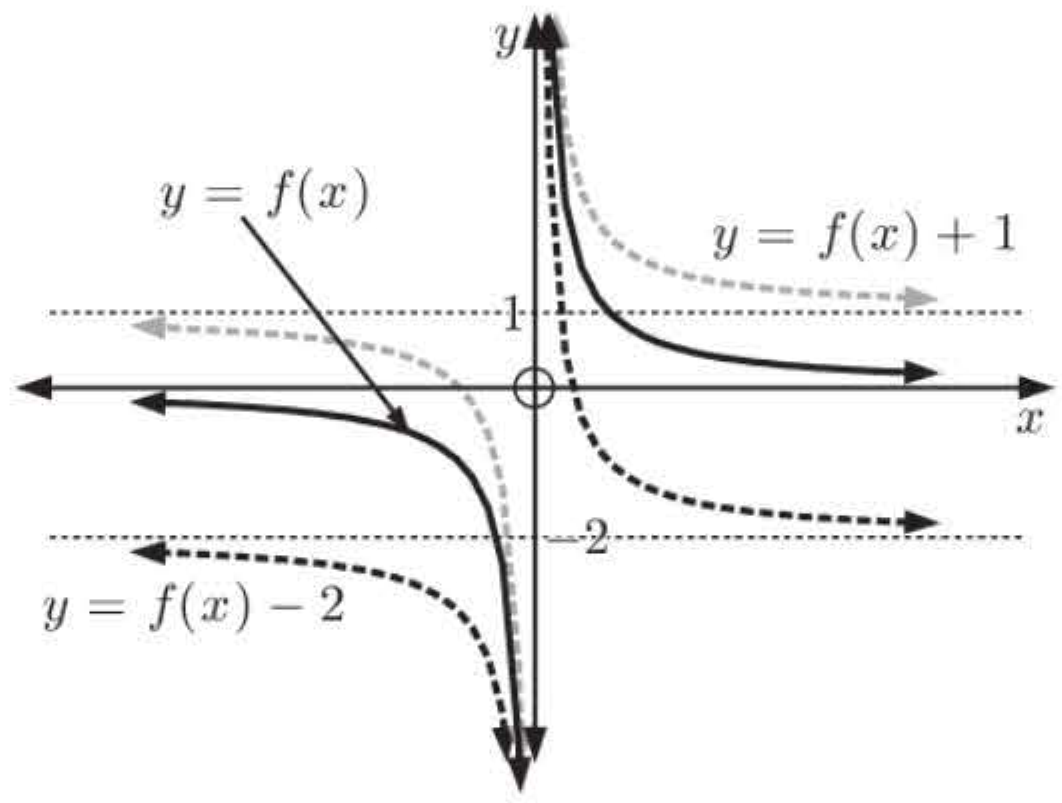
2 **a**



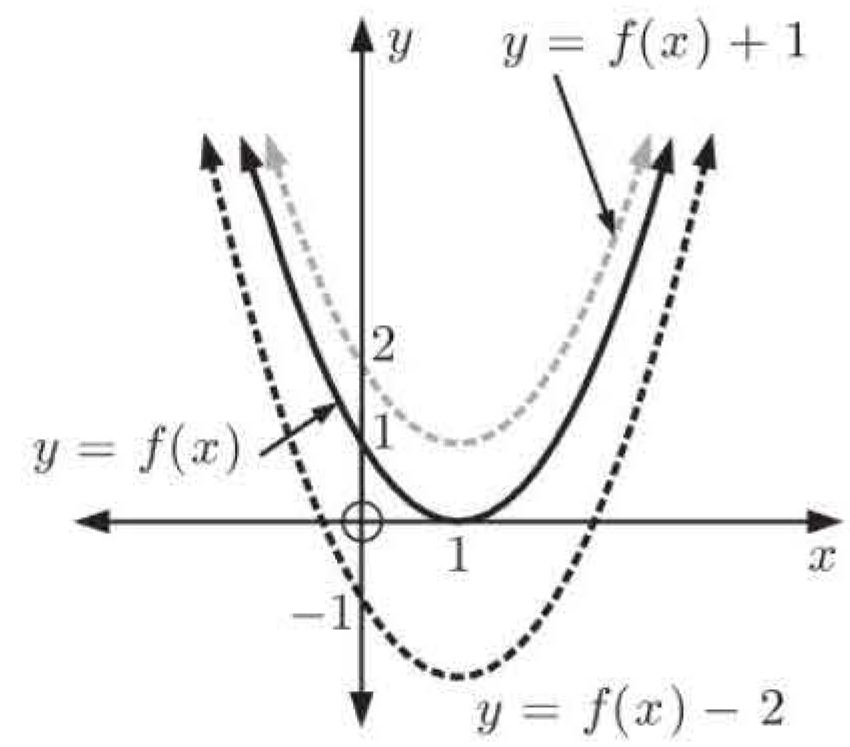
b



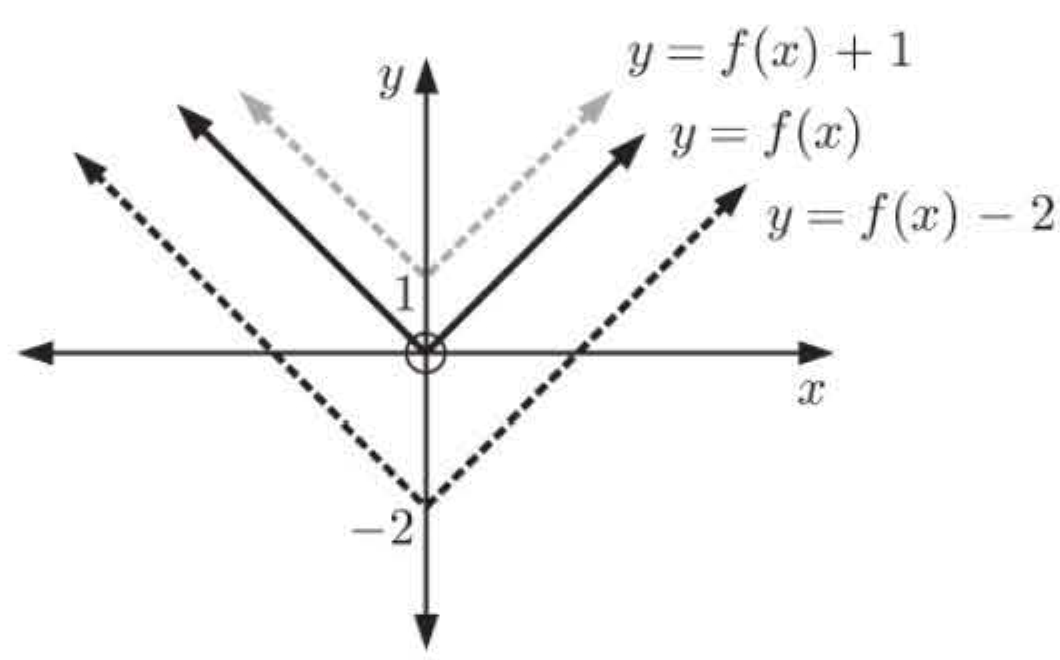
c



d

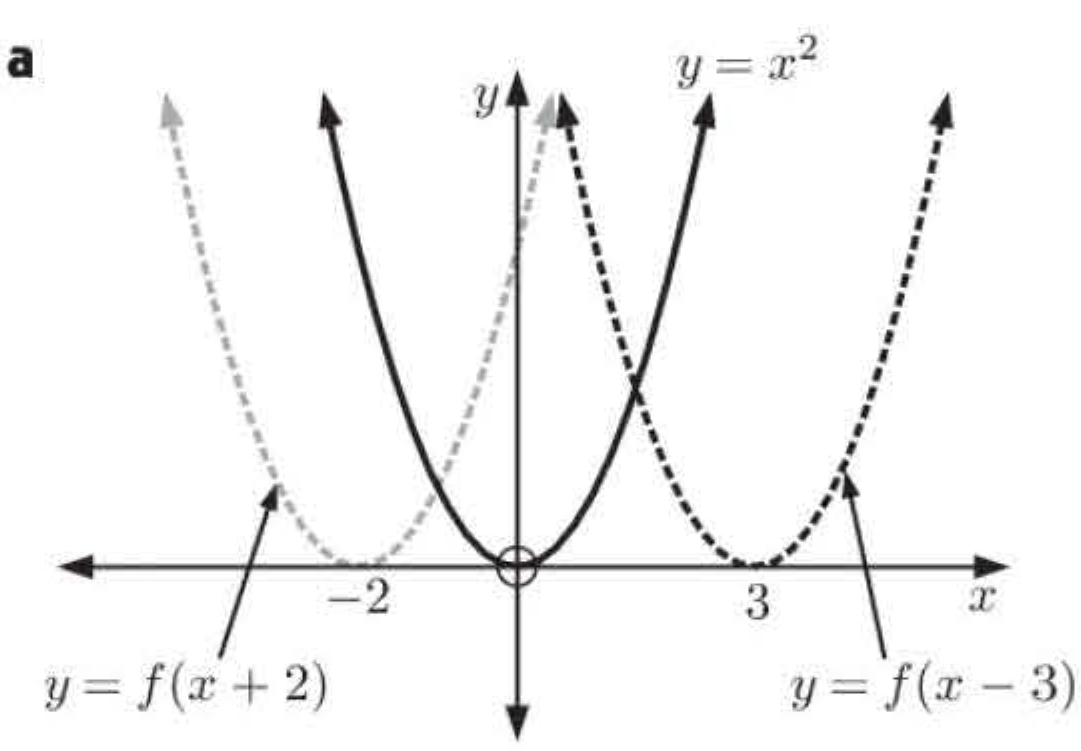


e



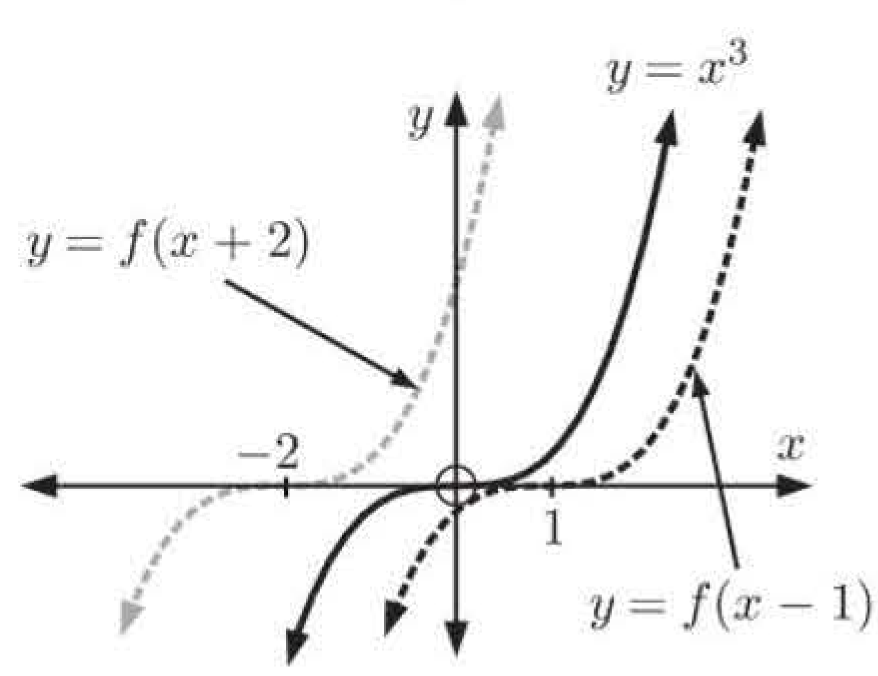
Summary: For $y = f(x) + b$, $y = f(x)$ is translated vertically through b units.
If $b > 0$ movement is vertically upwards b units.
If $b < 0$ movement is vertically downwards $|b|$ units.

3 **a**

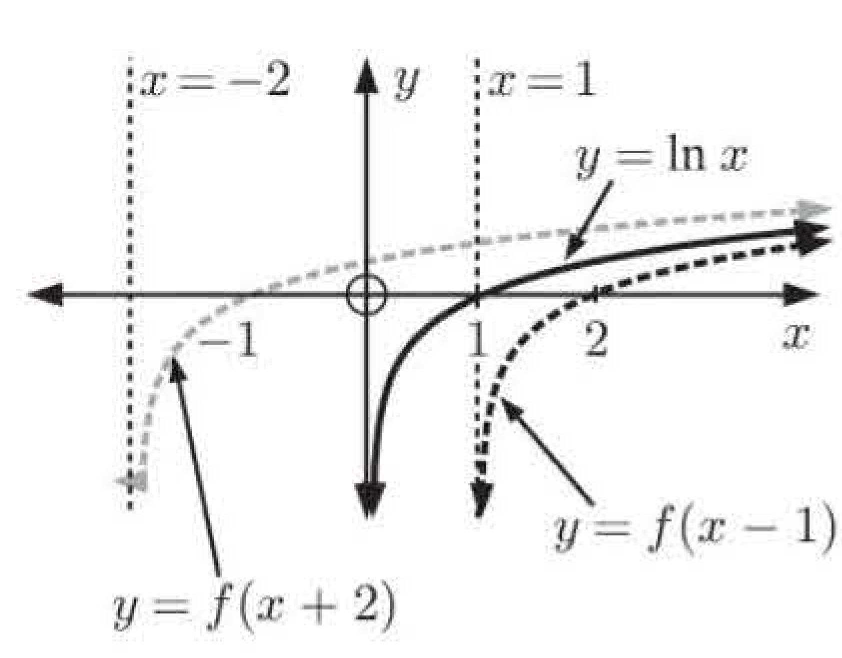


b **i** If $a > 0$, the graph is translated a units right.
ii If $a < 0$, the graph is translated $|a|$ units left.

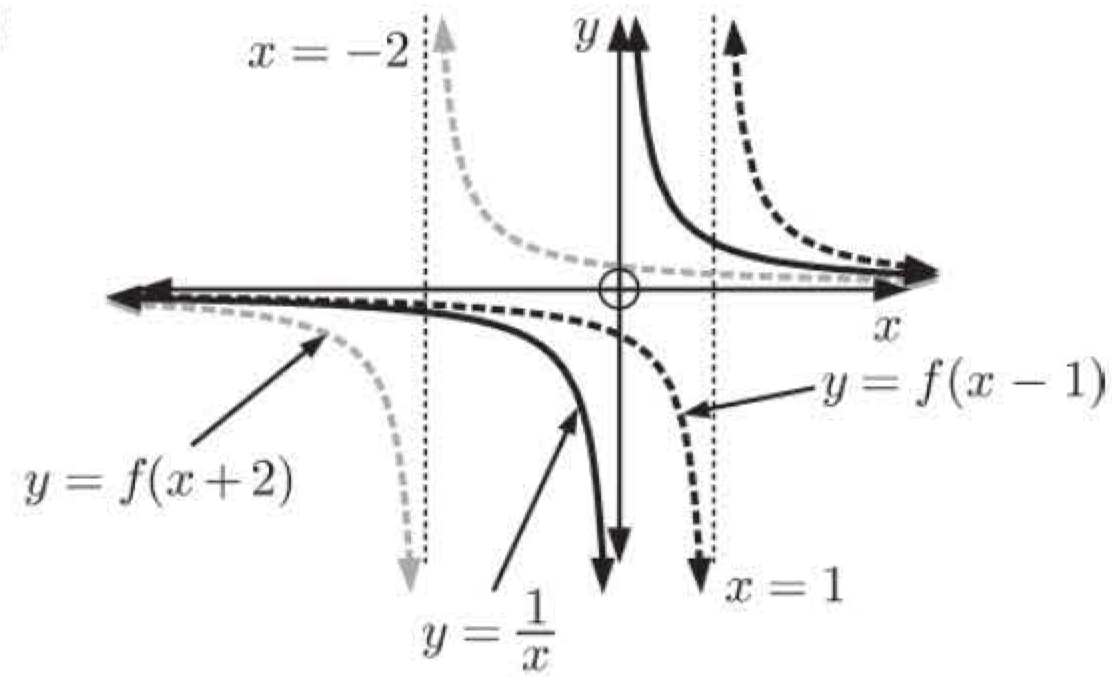
4 **a**



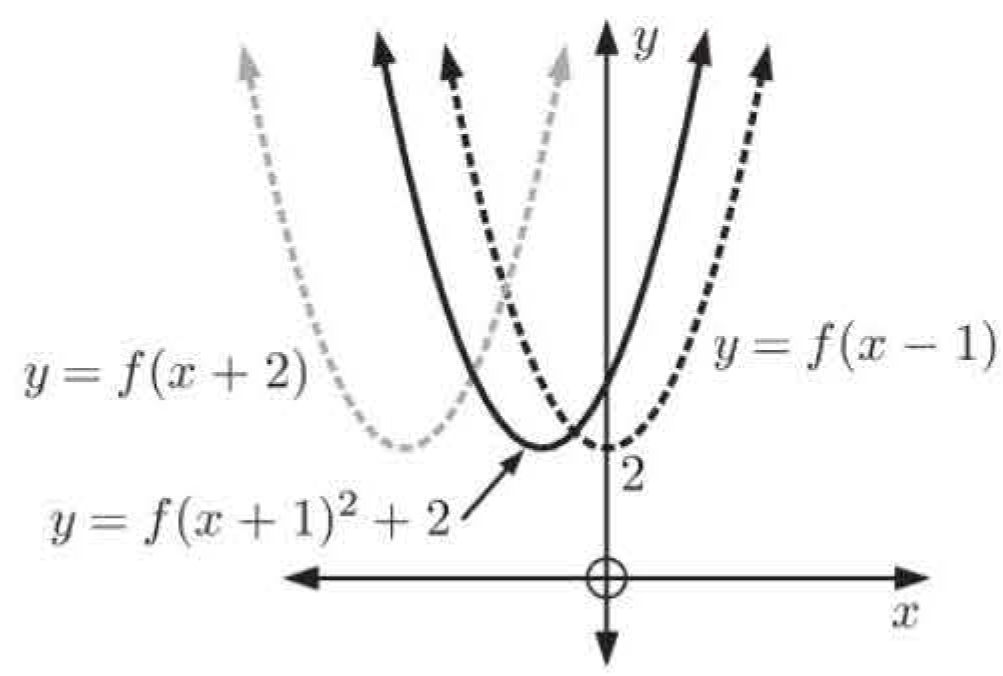
b



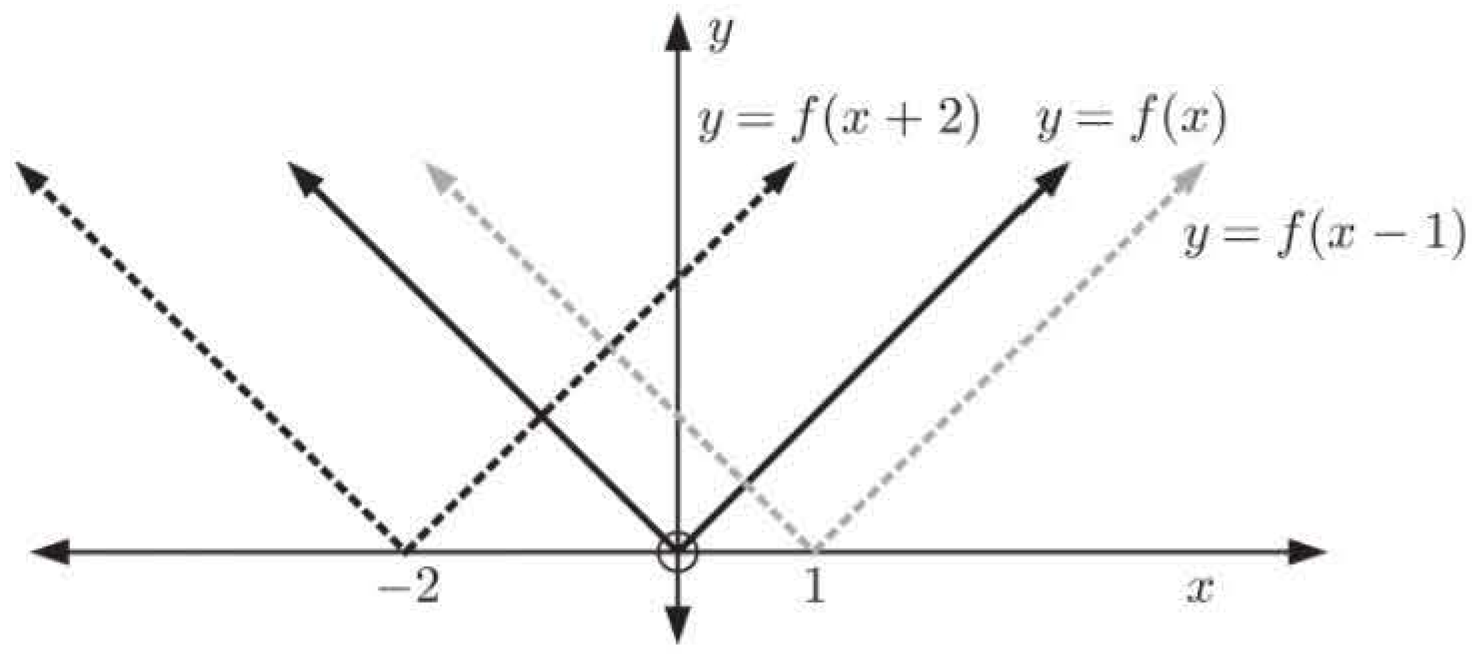
c



d

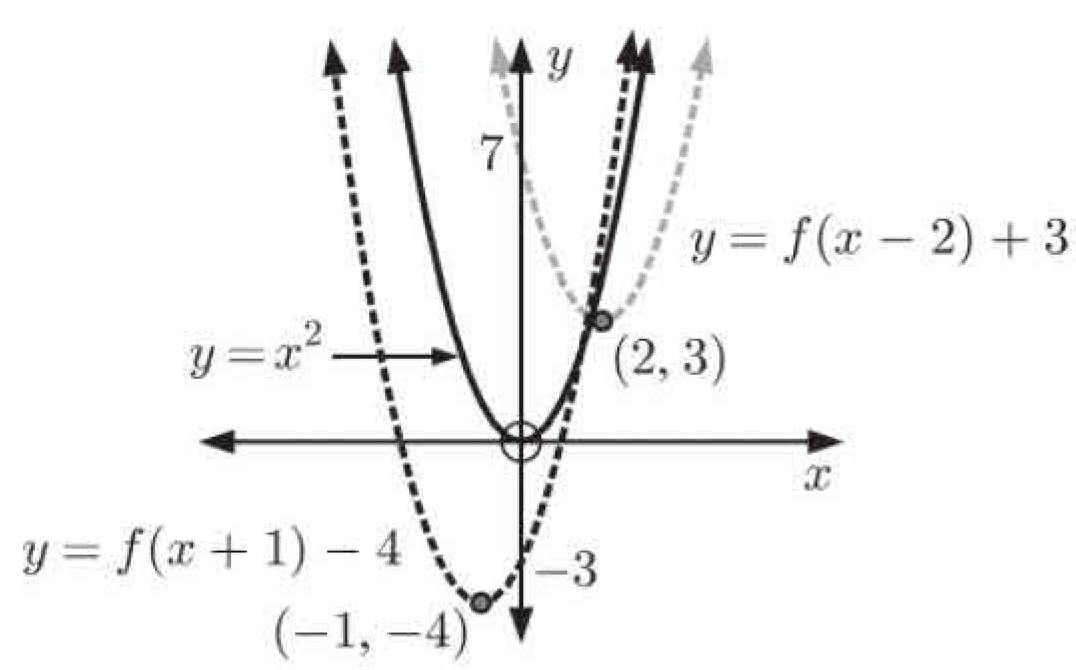


e

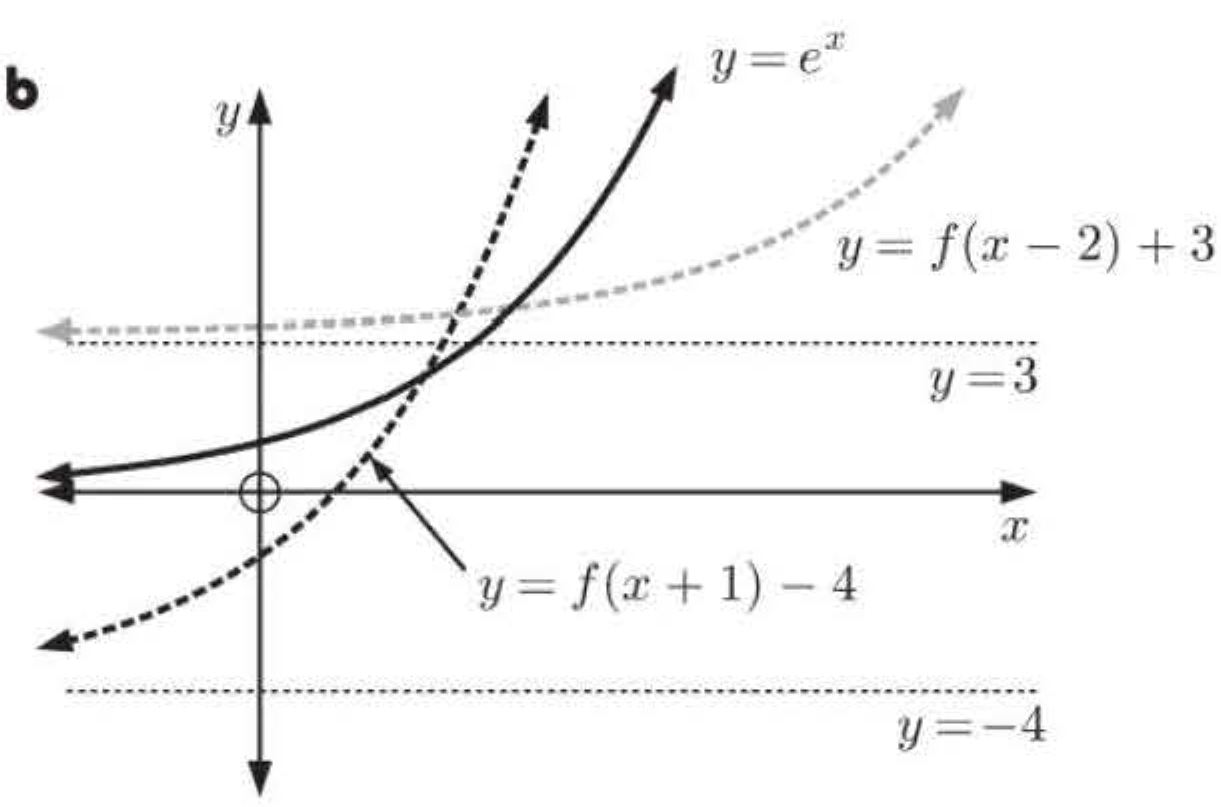


Summary: For $y = f(x - a)$, $y = f(x)$ is translated horizontally through a units. If $a > 0$ movement is to the right. If $a < 0$ movement is to the left.

5 **a**



b



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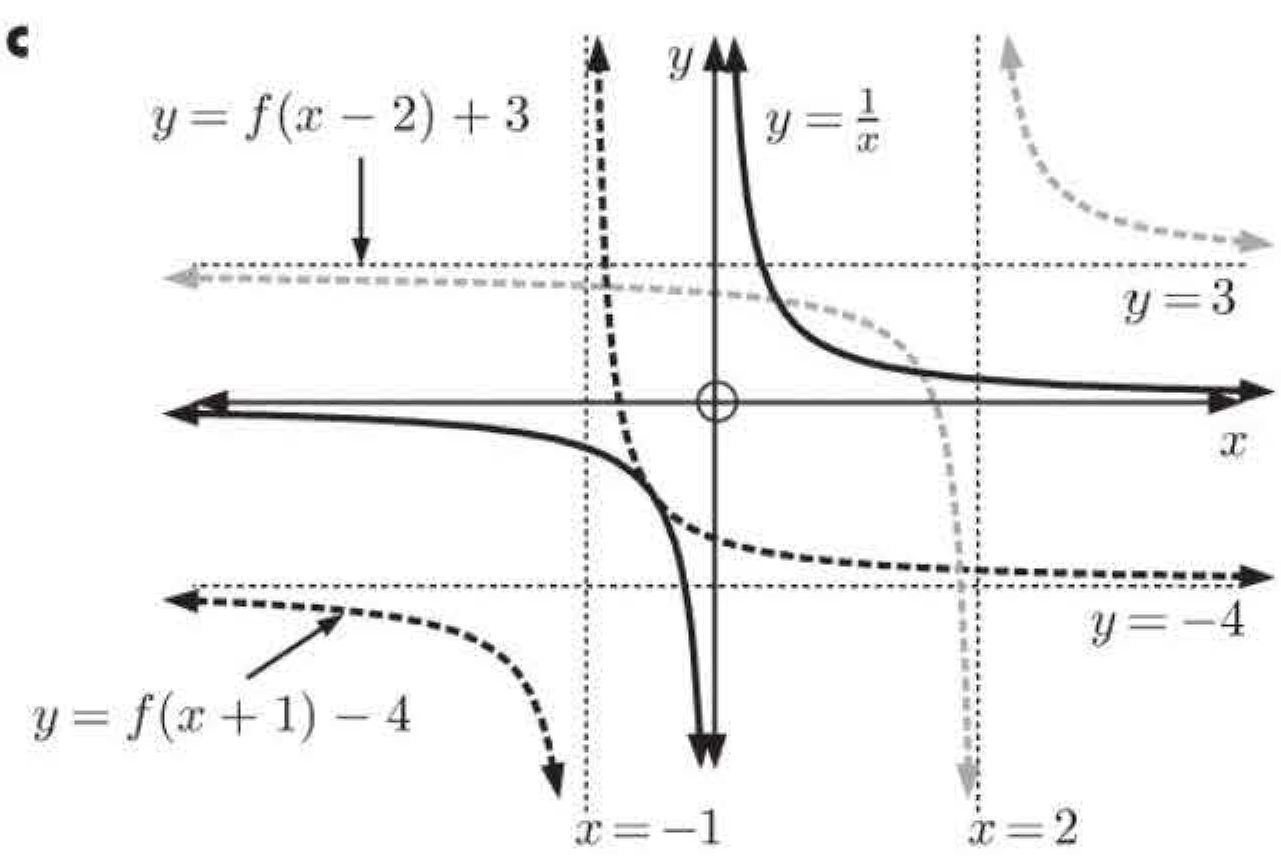
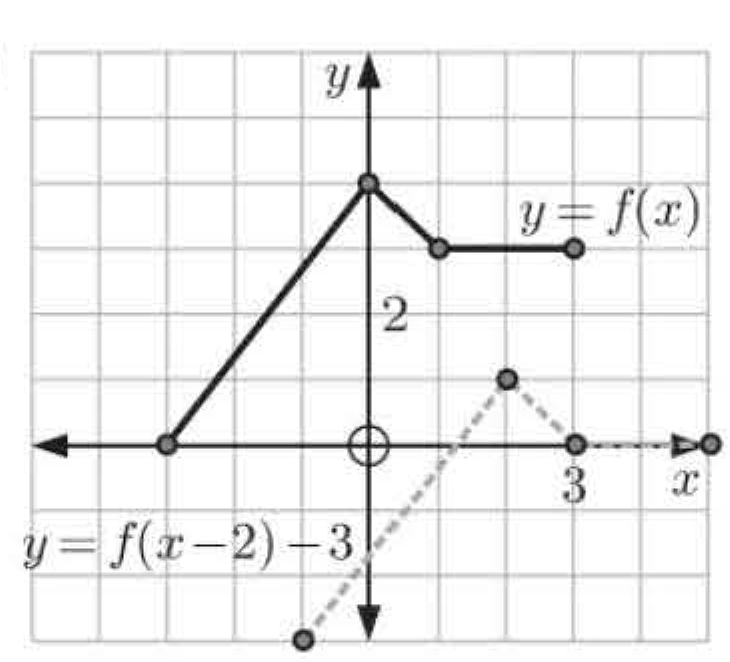
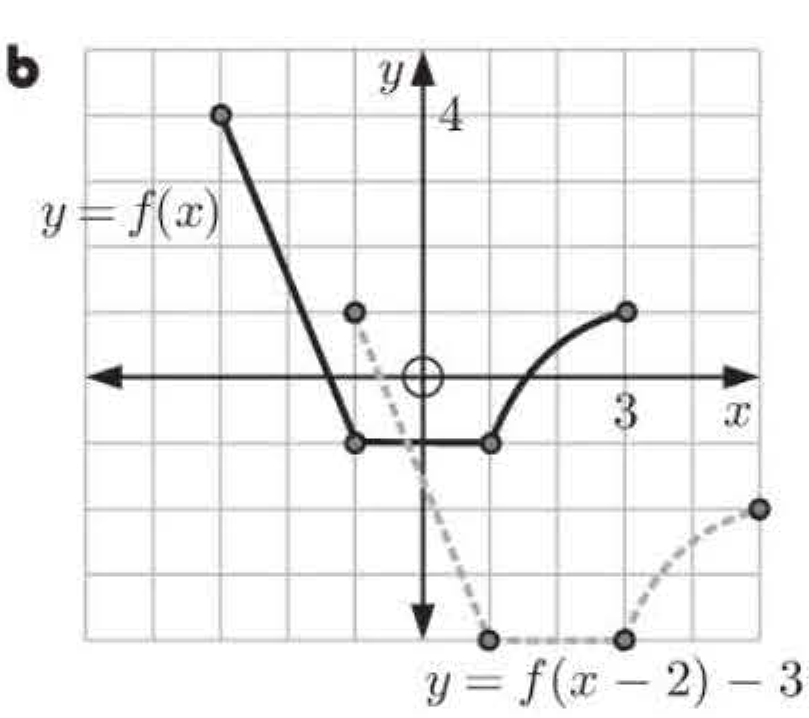


6 A translation of 2 units right and 3 units down or $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$.

a



b



7 To translate $f(x)$ 3 units right, we need to find $f(x-3)$.

$$\begin{aligned}\therefore\ g(x) &= f(x-3)\\ &= (x-3)^2 - 2(x-3) + 2\\ &= x^2 - 6x + 9 - 2x + 6 + 2\end{aligned}$$

$$\therefore\ g(x) = x^2 - 8x + 17$$

8 **a** The transformation from $f(x) = x^2$ to $g(x) = (x-3)^2 + 2$ is a translation of 3 units right and 2 units up.

i $(0, 0)$ is translated to $(3, 2)$. **ii** $(-3, 9)$ is translated to $(0, 11)$.

iii $f(2) = 2^2 = 4$

$\therefore$ $(2, 4)$ is translated to $(5, 6)$.

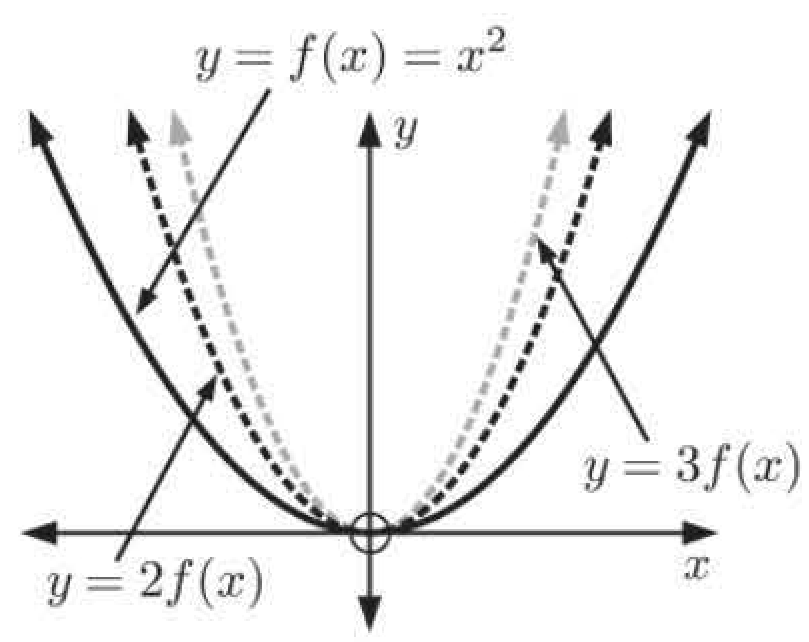
b The transformation from $g(x)$ back to $f(x)$ is a translation of $\begin{pmatrix}-3\\-2\end{pmatrix}$.

i $(1, 6)$ is translated to $(-2, 4)$. **ii** $(-2, 27)$ is translated to $(-5, 25)$.

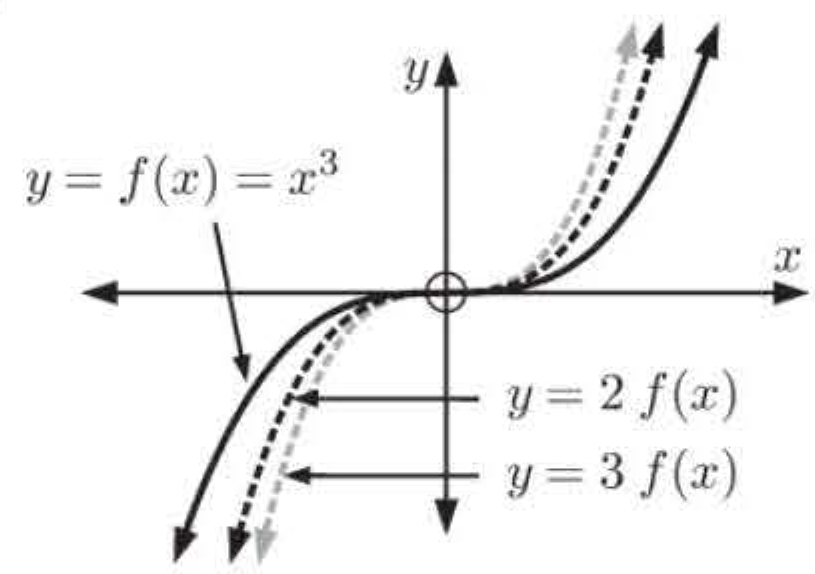
iii $(1\frac{1}{2}, 4\frac{1}{4})$ is translated to $(-1\frac{1}{2}, 2\frac{1}{4})$.

EXERCISE 5C

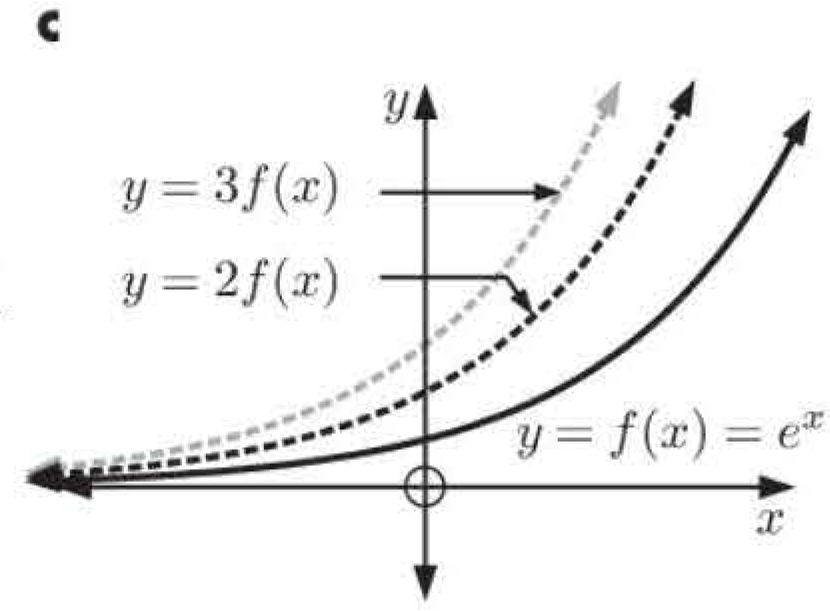
1 **a**



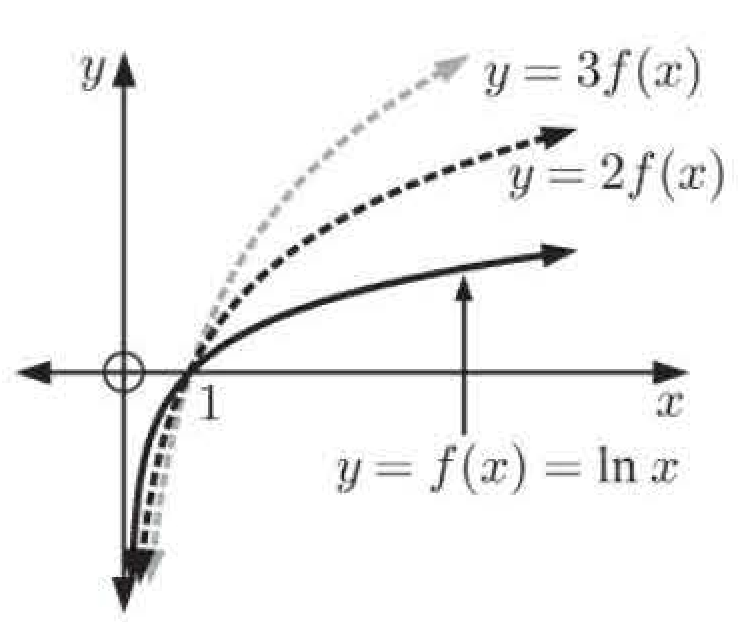
b



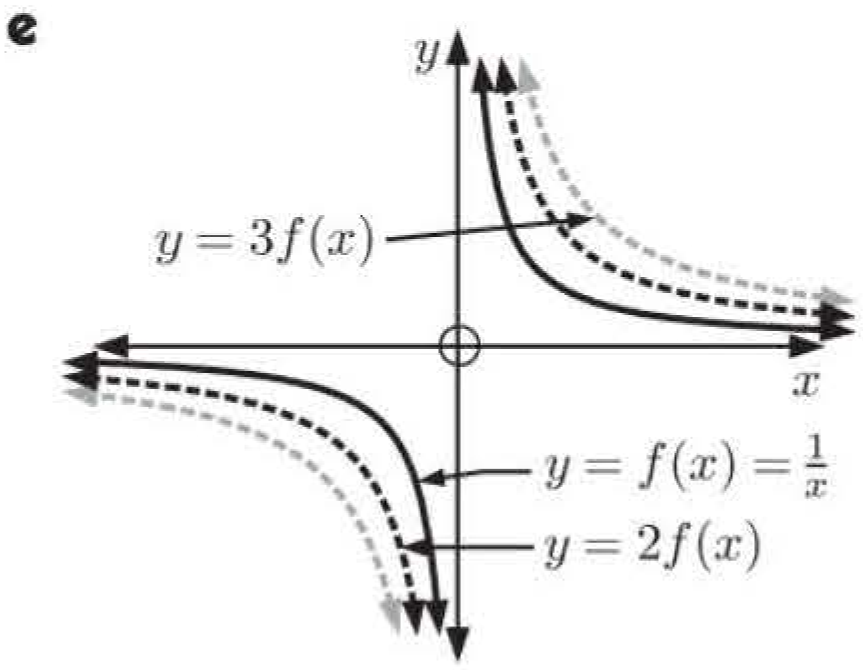
c



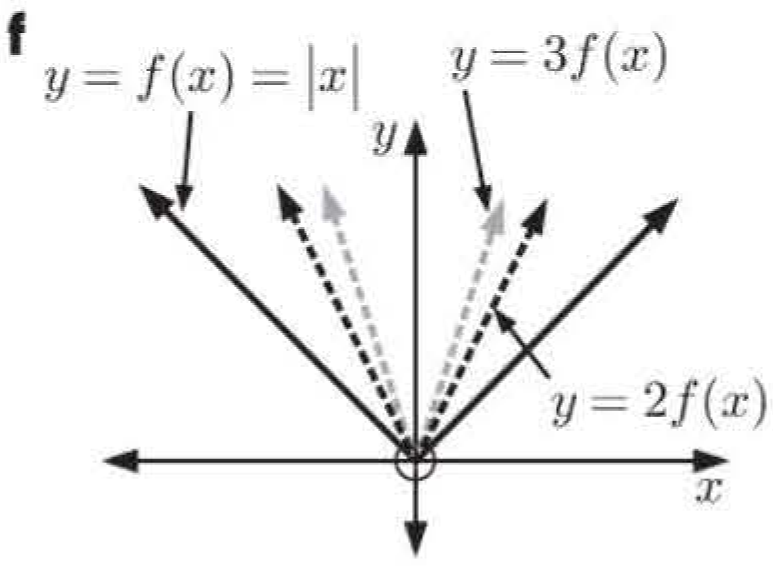
d



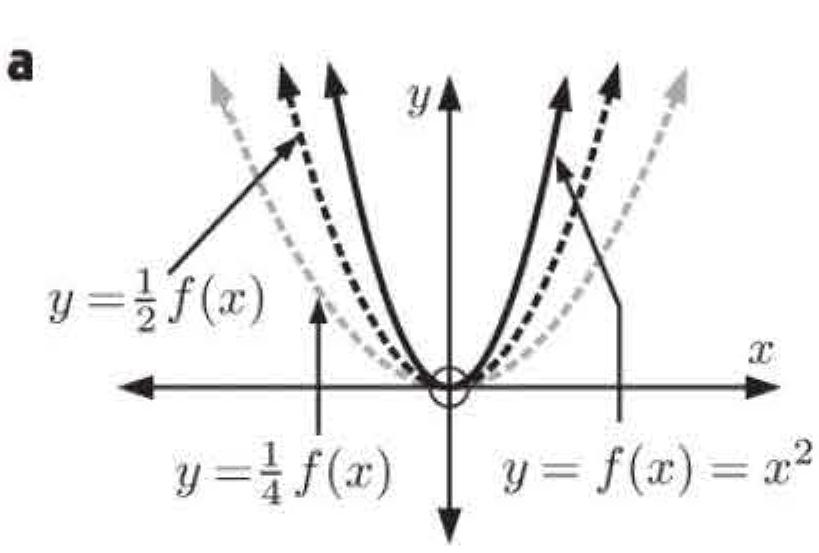
e



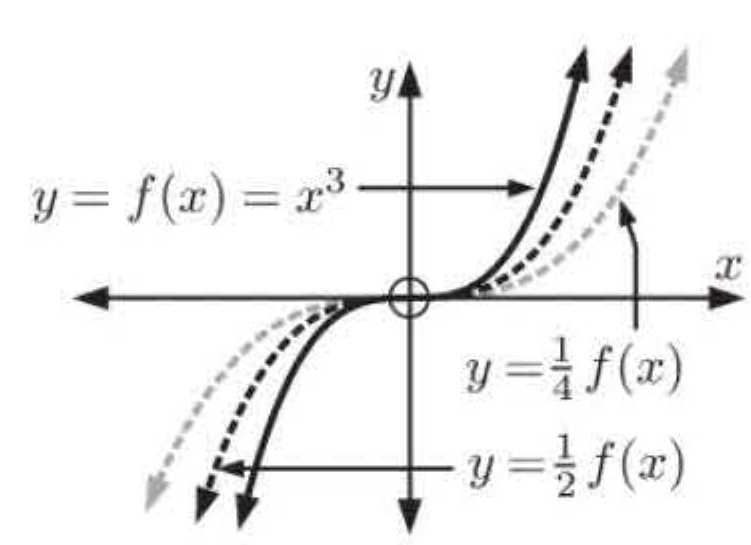
f



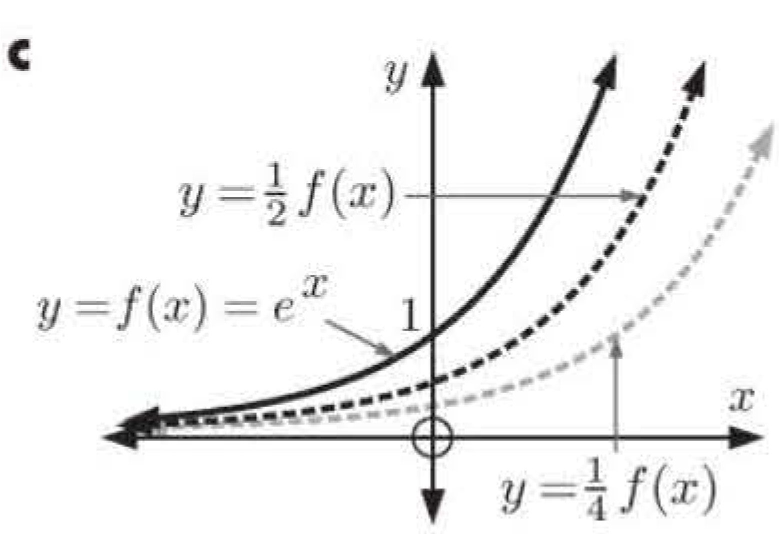
2 **a**



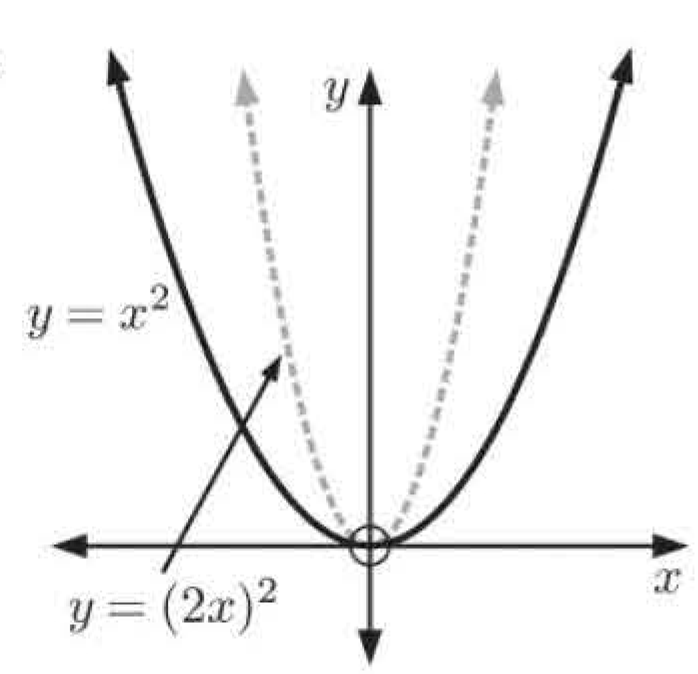
b



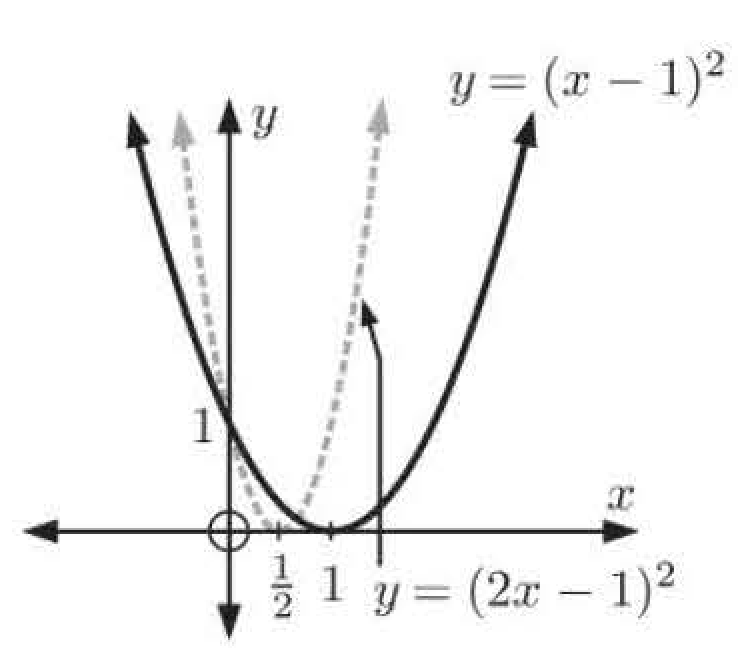
c



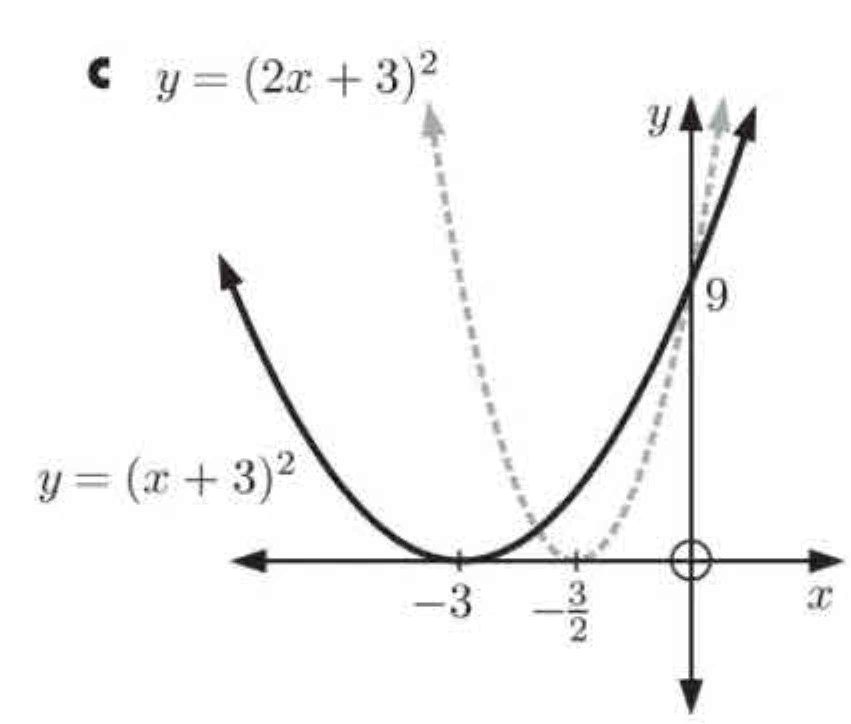
3 **a**



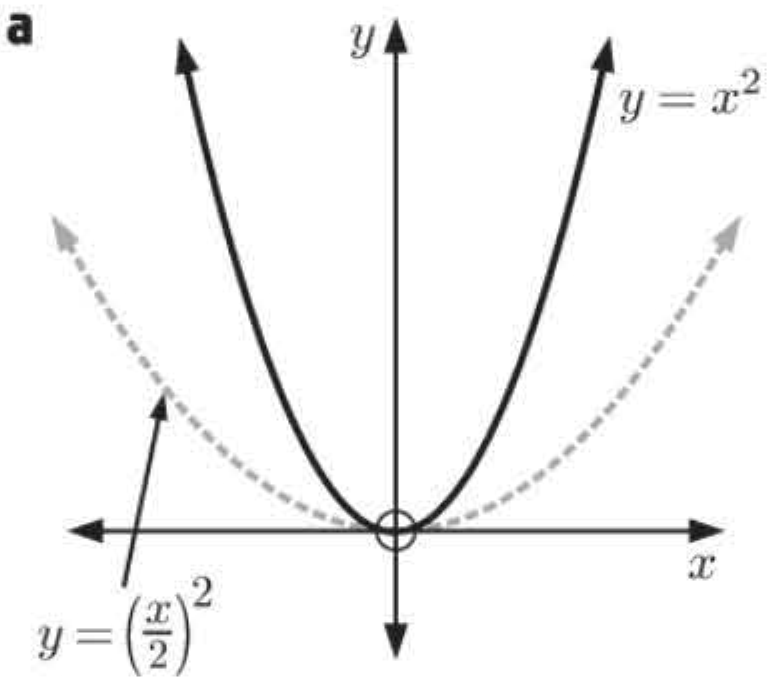
b



c



4 **a**



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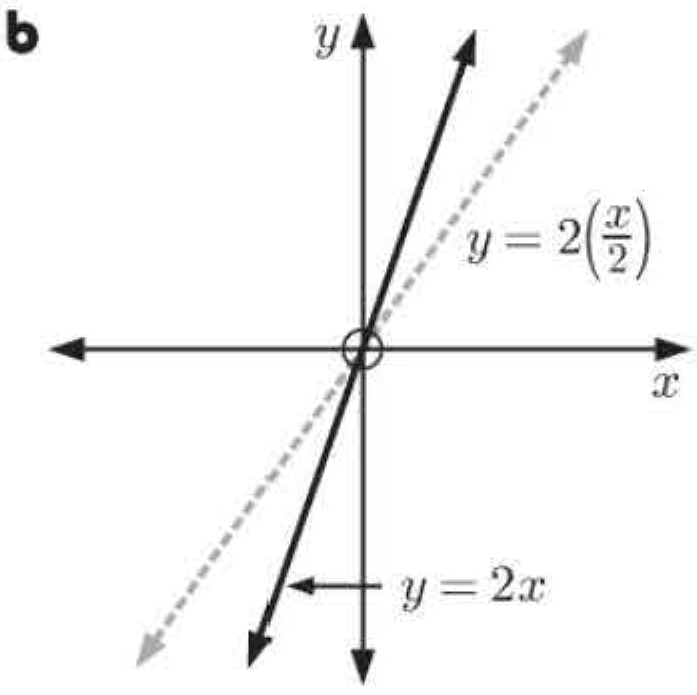
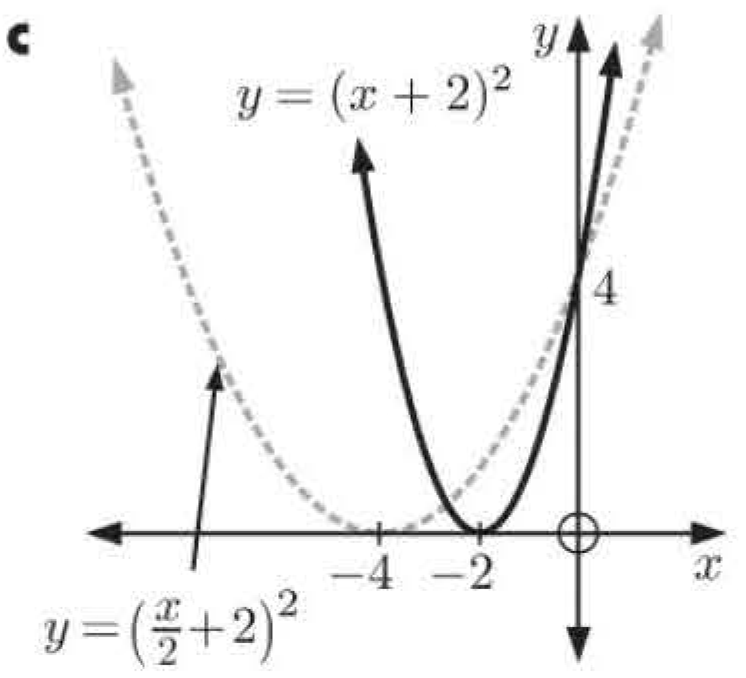
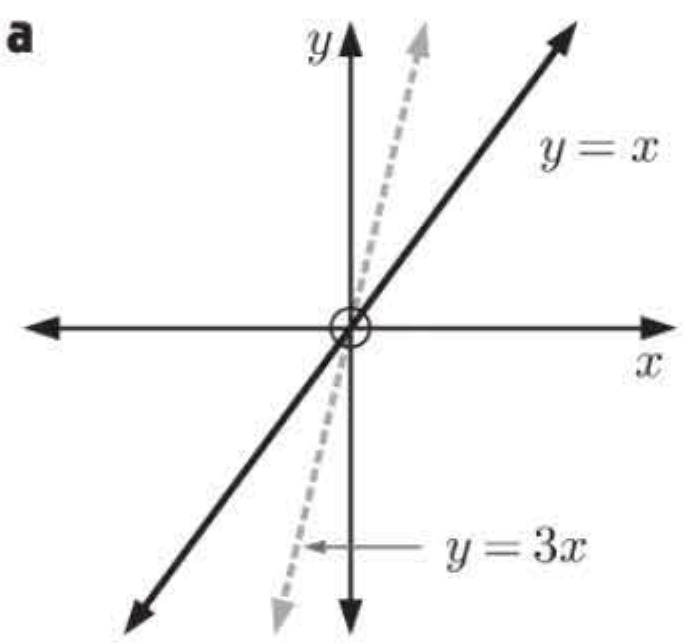


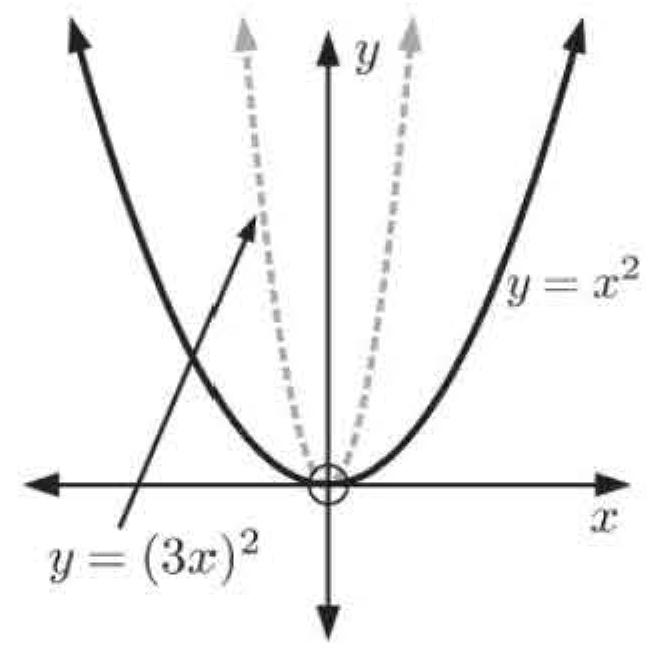
c



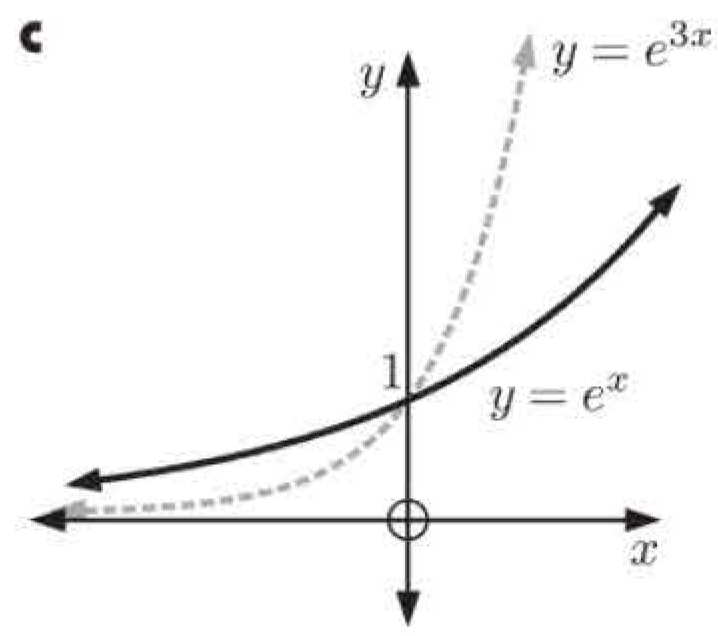
5 **a**



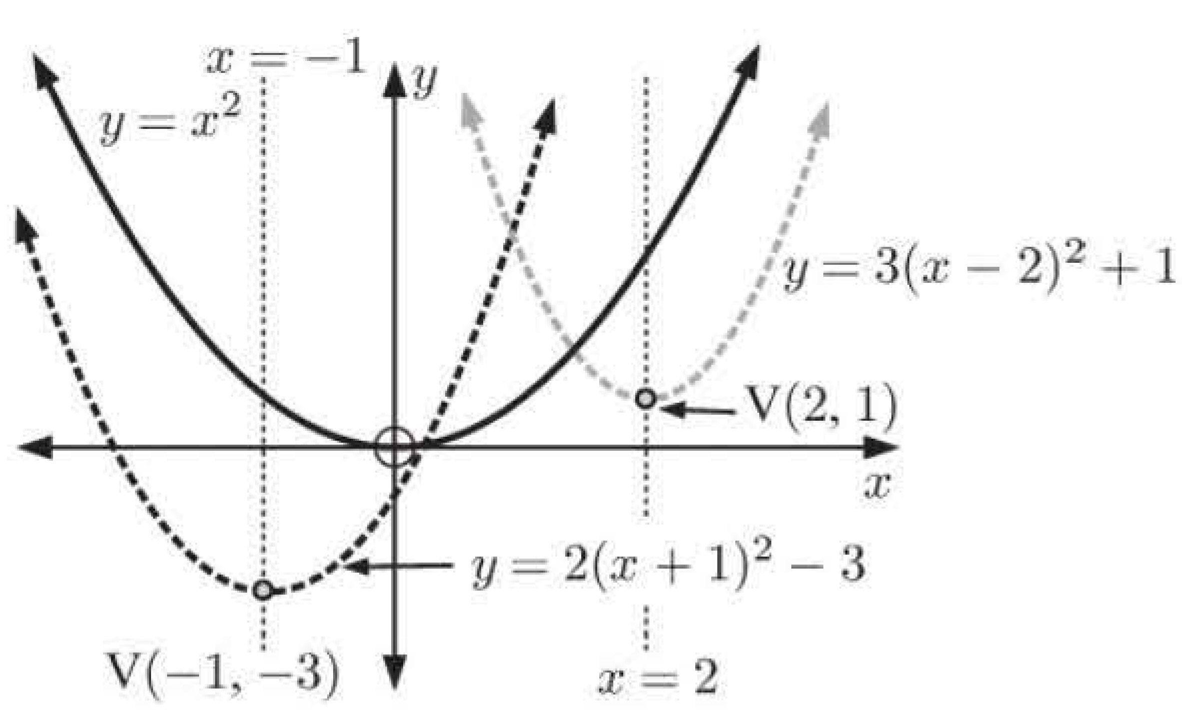
b



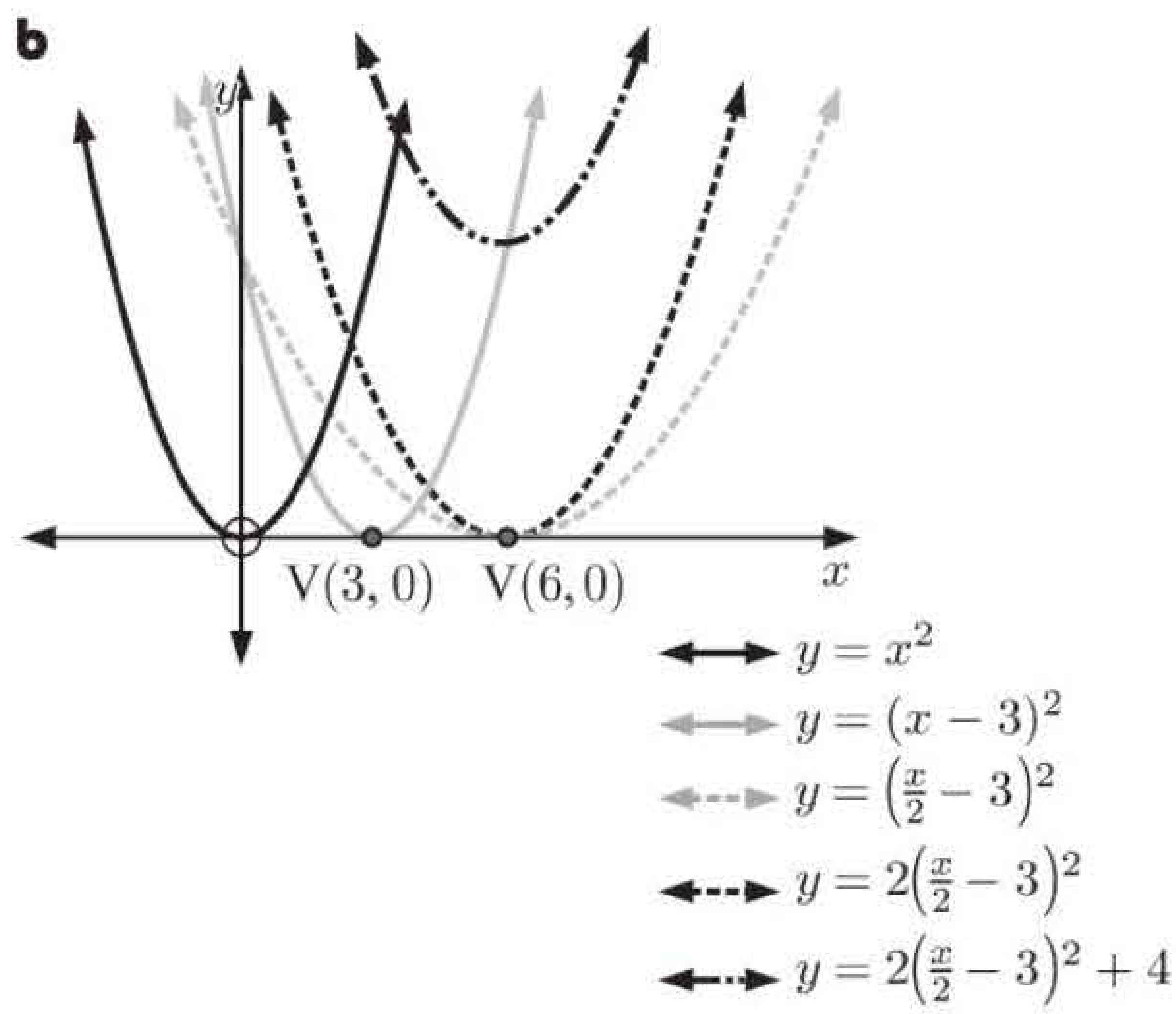
c



6 **a**



b



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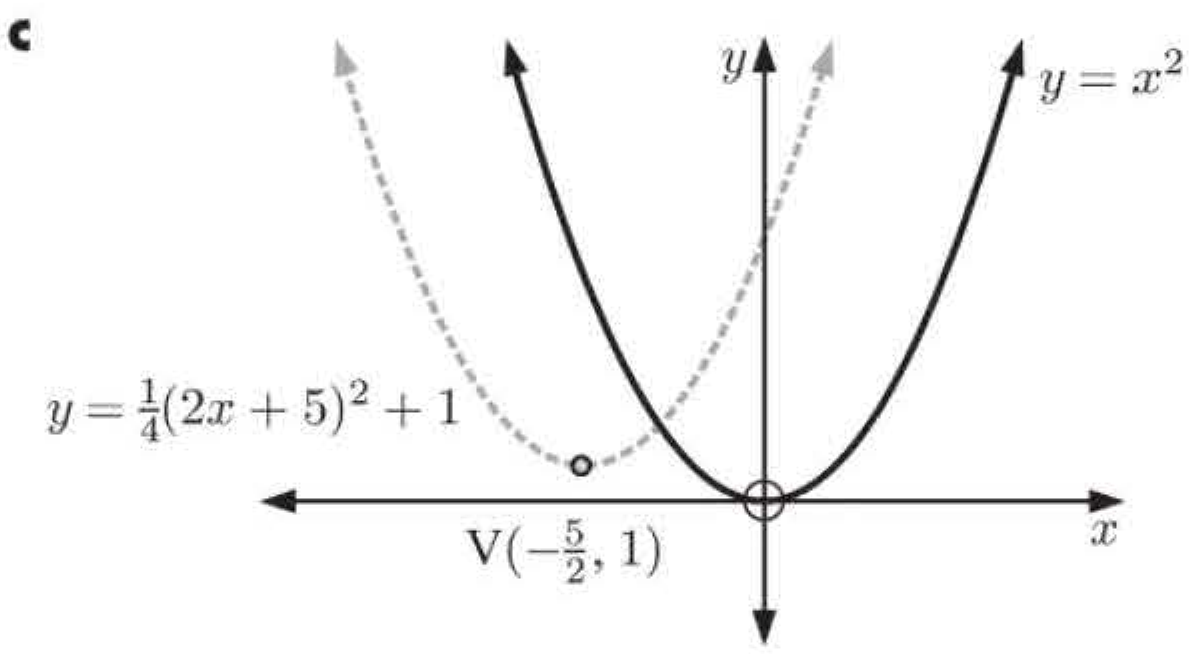


7 **a** The transformation from $y = f(x)$ to $y = 3\,f(2x)$ is a horizontal stretch of factor $\frac{1}{2}$ followed by a vertical stretch of factor 3.

i $(3, -5) \to (\frac{3}{2}, -5) \to (\frac{3}{2}, -15)$ $\quad\therefore$ $(3, -5)$ is transformed to $(\frac{3}{2}, -15)$

ii $(1, 2) \to (\frac{1}{2}, 2) \to (\frac{1}{2}, 6)$ $\quad\therefore$ $(1, 2)$ is transformed to $(\frac{1}{2}, 6)$

iii $(-2, 1) \to (-1, 1) \to (-1, 3)$ $\quad\therefore$ $(-2, 1)$ is transformed to $(-1, 3)$

b The transformation from $y = 3\,f(2x)$ back to $y = f(x)$ is a vertical stretch of factor $\frac{1}{3}$ followed by a horizontal stretch of factor 2.

i $(2, 1) \to (2, \frac{1}{3}) \to (4, \frac{1}{3})$ $\quad\therefore$ $(4, \frac{1}{3})$ is the point on $y = f(x)$

ii $(-3, 2) \to (-3, \frac{2}{3}) \to (-6, \frac{2}{3})$ $\quad\therefore$ $(-6, \frac{2}{3})$ is the point on $y = f(x)$

iii $(-7, 3) \to (-7, 1) \to (-14, 1)$ $\quad\therefore$ $(-14, 1)$ is the point on $y = f(x)$

8 **a** $$f(x) \underset{\substack{\text{horizontal}\\\text{translation}}}{\to} f(x+1) \underset{\substack{\text{horizontal}\\\text{stretch}}}{\to} f(\tfrac{1}{2}x+1) \underset{\substack{\text{vertical}\\\text{stretch}}}{\to} 2\,f(\tfrac{1}{2}x+1) \underset{\substack{\text{vertical}\\\text{translation}}}{\to} 3 + 2\,f(\tfrac{1}{2}x+1)$$

$\therefore$ $f(x)$ is translated horizontally 1 unit left, then horizontally stretched with scale factor 2, then vertically stretched with scale factor 2, then translated 3 units upwards.

b **i** $(1, -3) \to (0, -3) \to (0, -3) \to (0, -6) \to (0, -3)$ $\quad\therefore$ $(1, -3)$ is translated to $(0, -3)$

ii $(2, 1) \to (1, 1) \to (2, 1) \to (2, 2) \to (2, 5)$ $\quad\therefore$ $(2, 1)$ is translated to $(2, 5)$

iii $(-1, -2) \to (-2, -2) \to (-4, -2) \to (-4, -4) \to (-4, -1)$

$\therefore$ $(-1, -2)$ is translated to $(-4, -1)$

c To transform points on $y = 3 + 2f(\frac{1}{2}x + 1)$ back to points on $y = f(x)$, we translate 3 units downwards, then vertically stretch with scale factor $\frac{1}{2}$, then horizontally stretch with scale factor $\frac{1}{2}$, then translate 1 unit right.

i $(-2, -5) \to (-2, -8) \to (-2, -4) \to (-1, -4) \to (0, -4)$

$\therefore$ the point on $f(x)$ is $(0, -4)$.

ii $(1, -1) \to (1, -4) \to (1, -2) \to (\frac{1}{2}, -2) \to (\frac{3}{2}, -2)$ $\quad\therefore$ the point on $f(x)$ is $(\frac{3}{2}, -2)$.

iii $(5, 0) \to (5, -3) \to (5, -\frac{3}{2}) \to (\frac{5}{2}, -\frac{3}{2}) \to (\frac{7}{2}, -\frac{3}{2})$ $\quad\therefore$ the point on $f(x)$ is $(\frac{7}{2}, -\frac{3}{2})$.

EXERCISE 5D

1 **a** If $f(x) = 3x$

then $-f(x) = -(3x)$

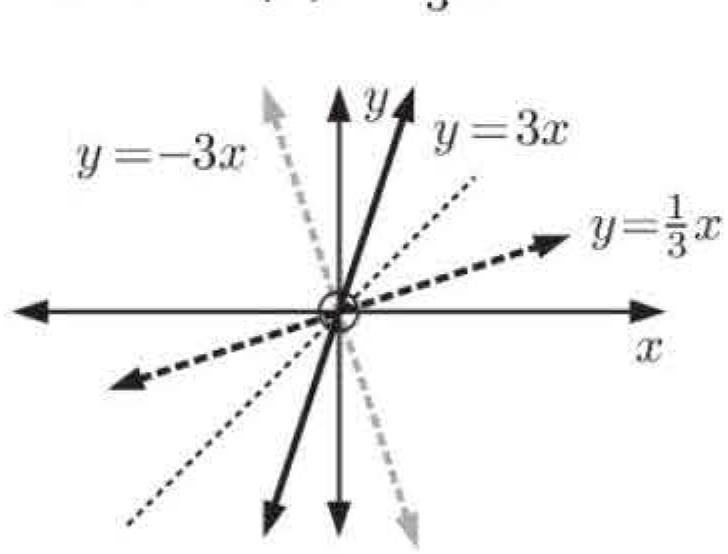
$= -3x$

And $f(x) = 3x$ has inverse function

$x = 3y$

$\therefore$ $y = \frac{1}{3}x$

$\therefore$ $f^{-1}(x) = \frac{1}{3}x$



b If $f(x) = e^x$

then $-f(x) = -(e^x)$

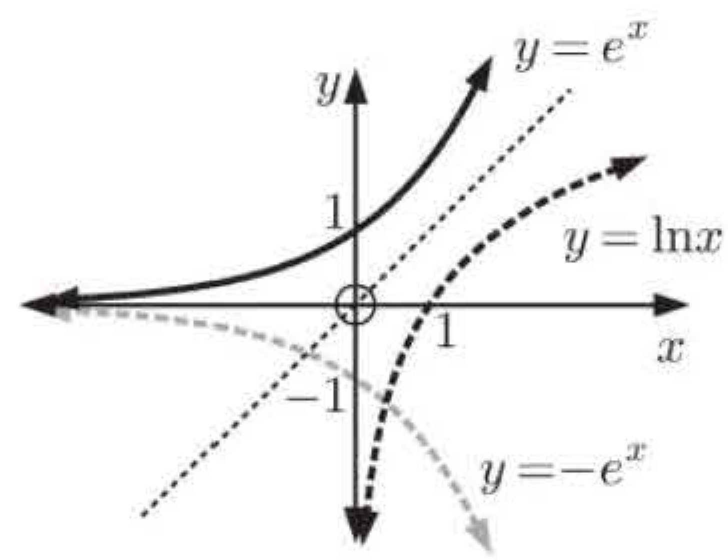
$= -e^x$

And $f(x) = e^x$ has inverse function

$x = e^y$

$\therefore$ $y = \ln x$

$\therefore$ $f^{-1}(x) = \ln x$

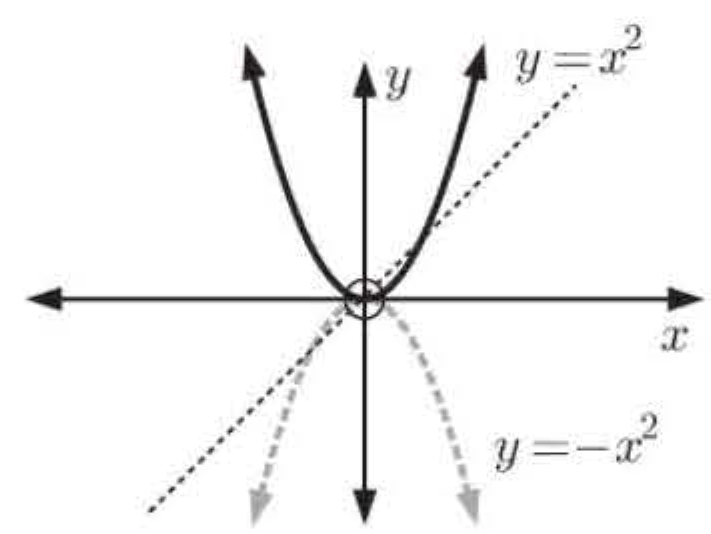


c If $f(x) = x^2$

then $-f(x) = -(x^2)$

$= -x^2$

$f(x) = x^2$ does not have an inverse function as it does not satisfy the horizontal line test.



d If $f(x) = \ln x$

then $-f(x) = -(\ln x)$

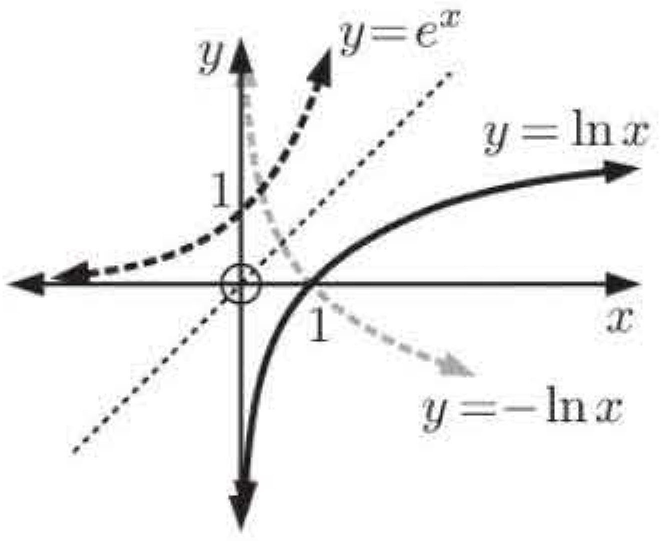
$= -\ln x$

And $f(x) = \ln x$ has inverse function

$x = \ln y$

$\therefore$ $y = e^x$

$\therefore$ $f^{-1}(x) = e^x$



e If $f(x) = x^3 - 2$

then $-f(x) = -(x^3 - 2)$

$= -x^3 + 2$

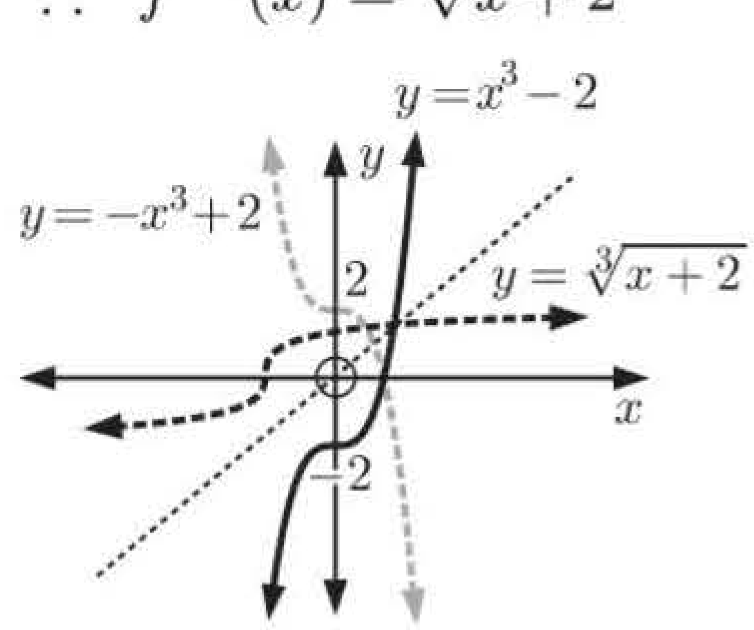
And $f(x) = x^3 - 2$ has inverse function

$x = y^3 - 2$

$\therefore$ $y^3 = x + 2$

$\therefore$ $y = \sqrt[3]{x+2}$

$\therefore$ $f^{-1}(x) = \sqrt[3]{x+2}$

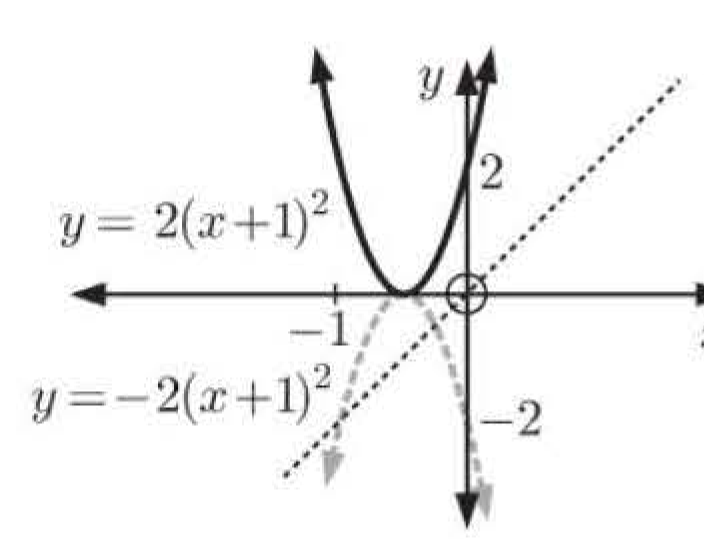


f If $f(x) = 2(x+1)^2$

then $-f(x) = -(2(x+1)^2)$

$= -2(x+1)^2$

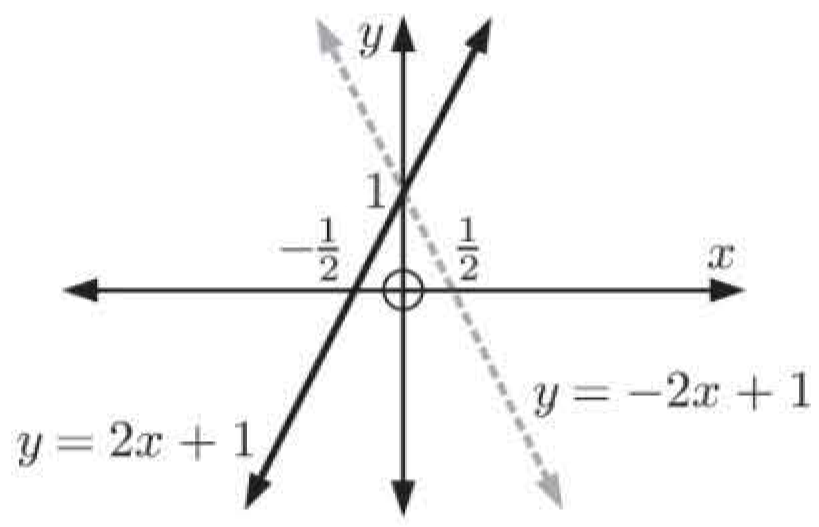
$f(x) = 2(x+1)^2$ does not have an inverse function as it does not satisfy the horizontal line test.



2 **a** $f(x) = 2x + 1$

$\therefore\ f(-x) = 2(-x) + 1$

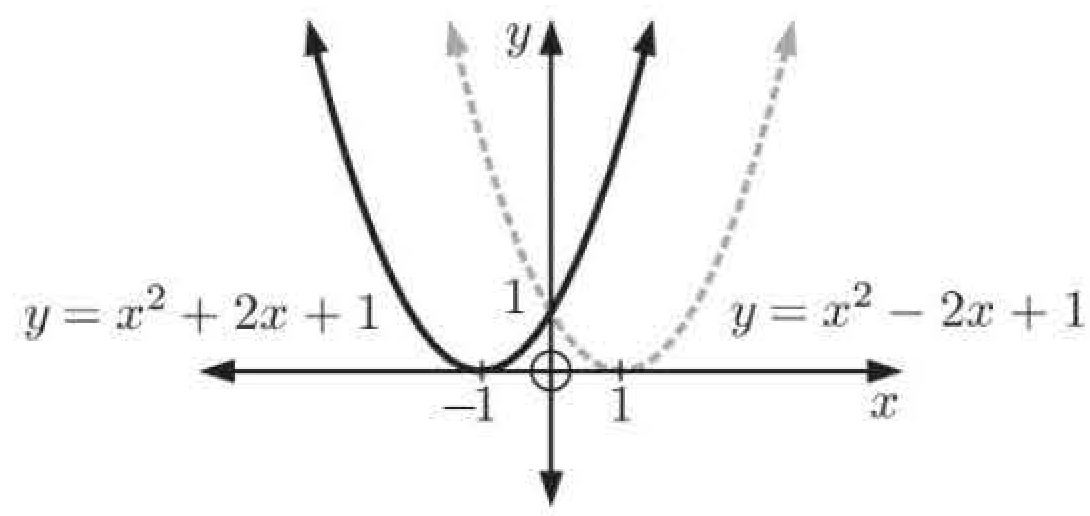
$= -2x + 1$



b $f(x) = x^2 + 2x + 1$

$\therefore\ f(-x) = (-x)^2 + 2(-x) + 1$

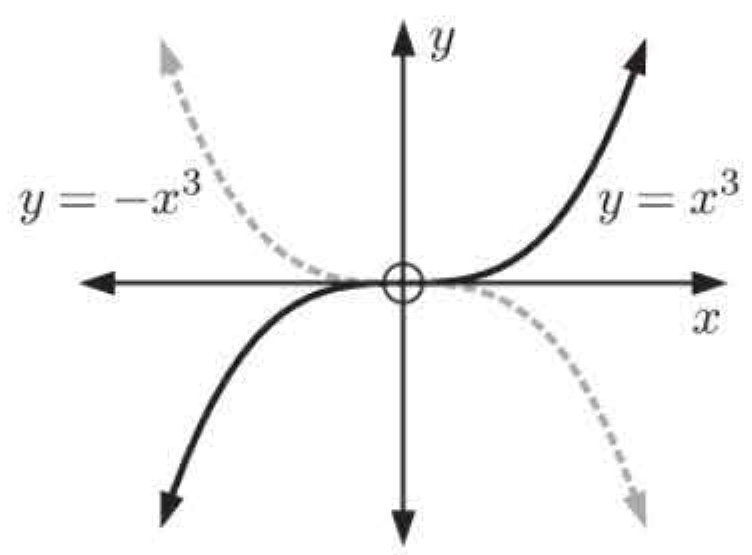
$= x^2 - 2x + 1$



c $f(x) = x^3$

$\therefore\ f(-x) = (-x)^3$

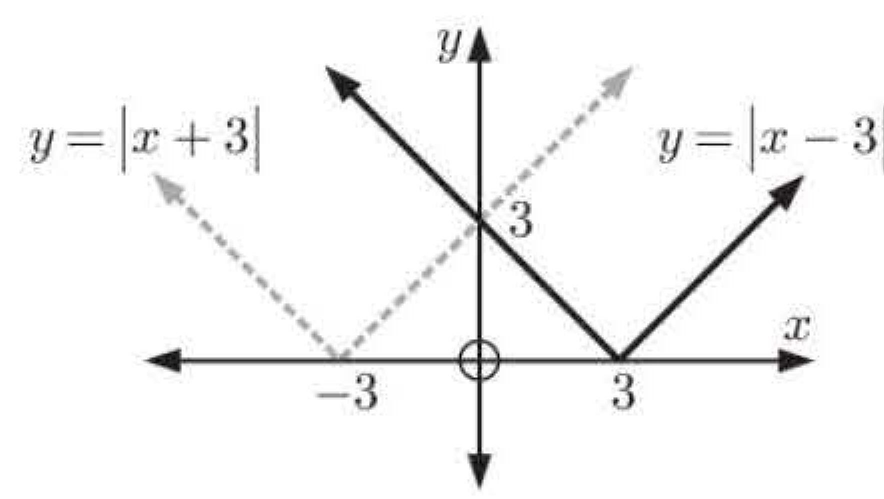
$= -x^3$



d $f(x) = |x - 3|$

$\therefore\ f(-x) = |-x - 3|$

$= |x + 3|$



3 If $g(x)$ is the reflection of $f(x)$ in the x-axis, then $g(x) = -f(x)$

$\therefore\ g(x) = -(x^3 - \ln x)$

$= -x^3 + \ln x$

4 If $g(x)$ is the reflection of $f(x)$ in the y-axis, then $g(x) = f(-x)$

$\therefore\ g(x) = (-x)^4 - 2(-x)^3 - 3(-x)^2 + 5(-x) - 7$

$= x^4 + 2x^3 - 3x^2 - 5x - 7$

5 **a** To transform $y = f(x)$ to $g(x) = -f(x)$, we reflect $y = f(x)$ in the x-axis. To do this we keep the x-coordinates the same and take the negative of the y-coordinates.

i $(3, 0)$ is transformed to $(3, 0)$ **ii** $(2, -1)$ is transformed to $(2, 1)$

iii $(-3, 2)$ is transformed to $(-3, -2)$

b To find the points on $f(x)$ corresponding to $g(x)$, we again take the negative of the y-coordinates.

i The point transformed to $(7, -1)$ is $(7, 1)$.

ii The point transformed to $(-5, 0)$ is $(-5, 0)$.

iii The point transformed to $(-3, -2)$ is $(-3, 2)$.

6 **a** To transform $y = f(x)$ to $h(x) = f(-x)$, we reflect $y = f(x)$ in the y-axis.

To do this we keep the y-coordinates the same and take the negative of the x-coordinates.

i $(2, -1)$ is transformed to $(-2, -1)$. **ii** $(0, 3)$ is transformed to $(0, 3)$.

iii $(-1, 2)$ is transformed to $(1, 2)$. **iv** $(3, 0)$ is transformed to $(-3, 0)$.

b To find the points on $f(x)$ corresponding to $h(x)$, we again take the negative of the x-coordinates.

i The point transformed to $(5, -4)$ is $(-5, -4)$.

ii The point transformed to $(0, 3)$ is $(0, 3)$.

iii The point transformed to $(2, 3)$ is $(-2, 3)$.

iv The point transformed to $(3, 0)$ is $(-3, 0)$.

7 **a** To transform $y = f(x)$ to $m(x) = f^{-1}(x)$, we reflect $y = f(x)$ in the line $y = x$.

To do this we swap the x and y-coordinates.

i $(3, 1)$ is transformed to $(1, 3)$. **ii** $(-2, 4)$ is transformed to $(4, -2)$.

iii $(0, -5)$ is transformed to $(-5, 0)$.

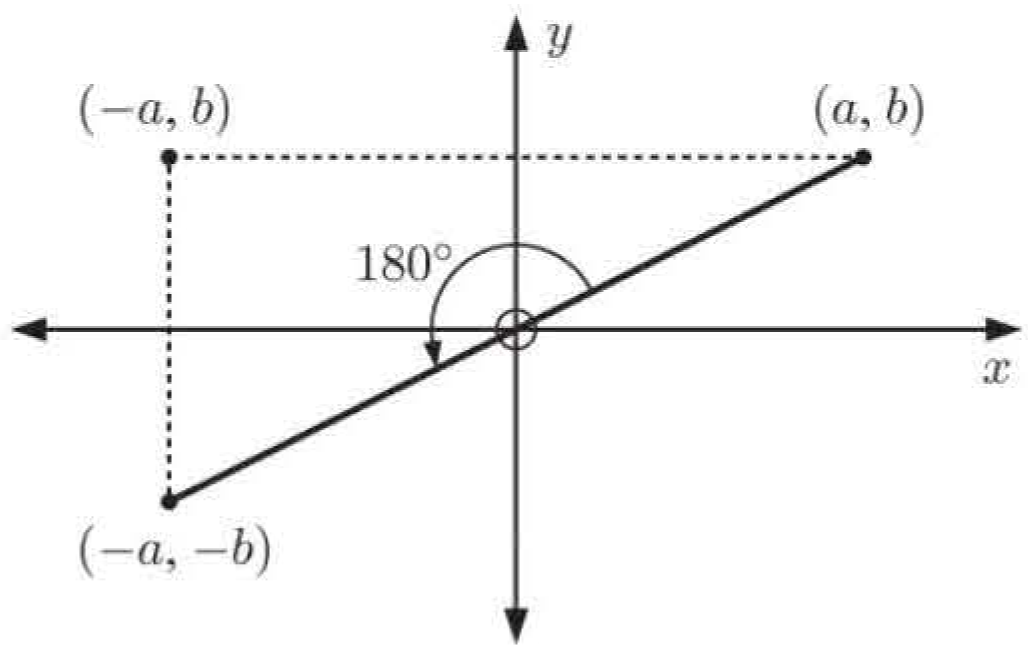
b To find the points on $f(x)$ corresponding to $m(x)$, we again swap the x and y-coordinates.

i The point transformed to $(-1, 1)$ is $(1, -1)$.

ii The point transformed to $(6, 0)$ is $(0, 6)$.

iii The point transformed to $(3, -2)$ is $(-2, 3)$.

8 **a** $f(x)$ is reflected in the y-axis to give $y = f(-x)$, then reflected in the x-axis to give $y = -f(-x)$. This has the effect of rotating the point about the origin through $180°$.

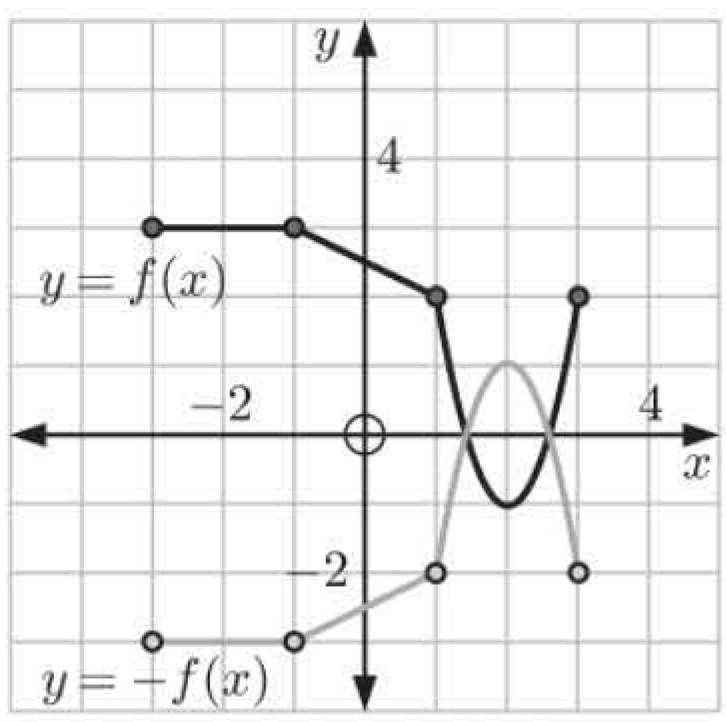


b The point (a, b) is transformed to the point $(-a, -b)$.

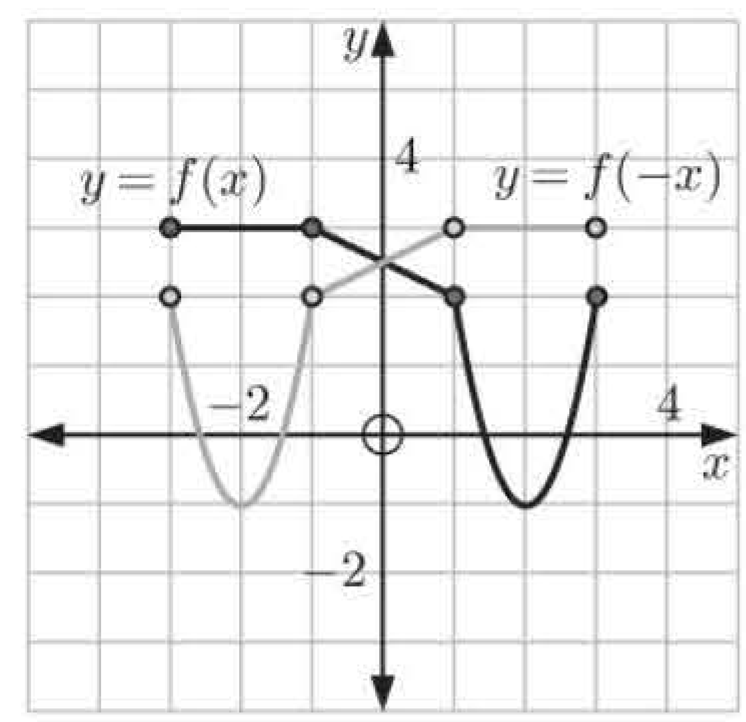
$\therefore$ $(3, -7)$ is transformed to $(-3, 7)$

c The point that transforms to $(-5, -1)$ is $(5, 1)$.

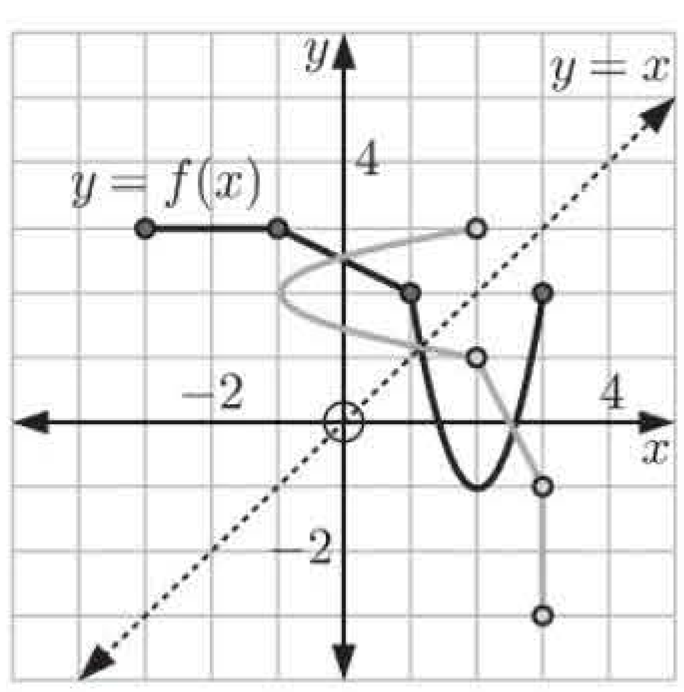
9 **a** **i**



ii



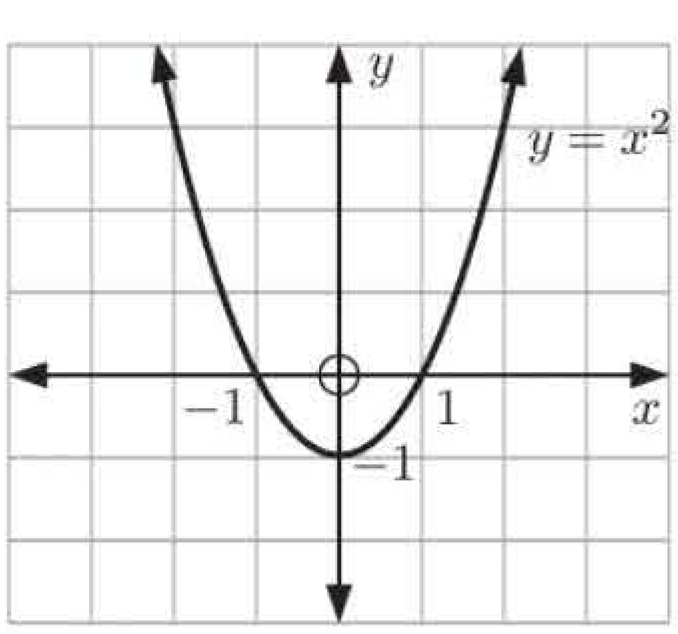
b **i**



ii The reflection of $y = f(x)$ in the line $y = x$ is not $y = f^{-1}(x)$ as $y = f(x)$ is not a function.

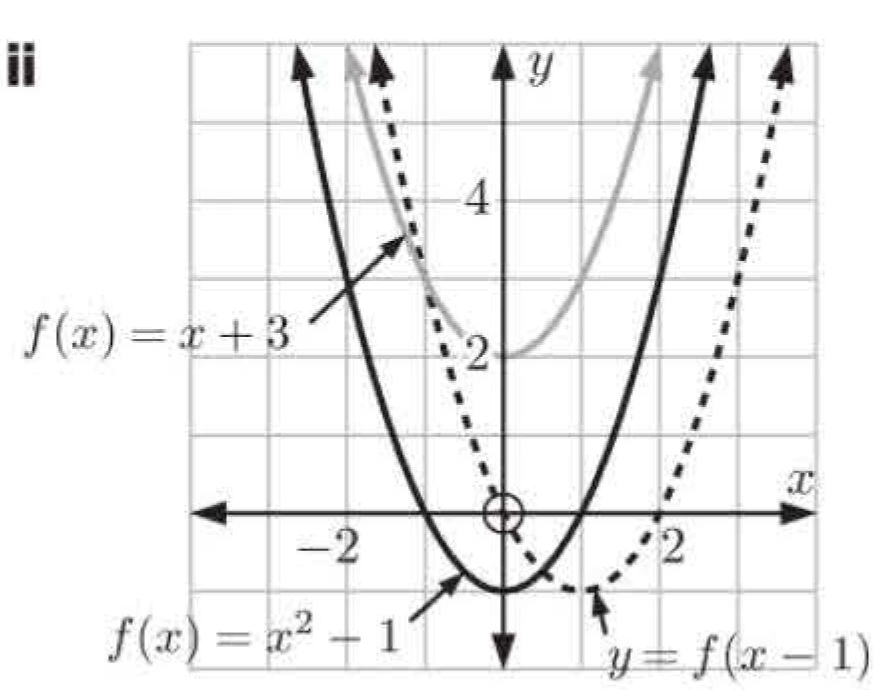
EXERCISE 5E

1 **a**

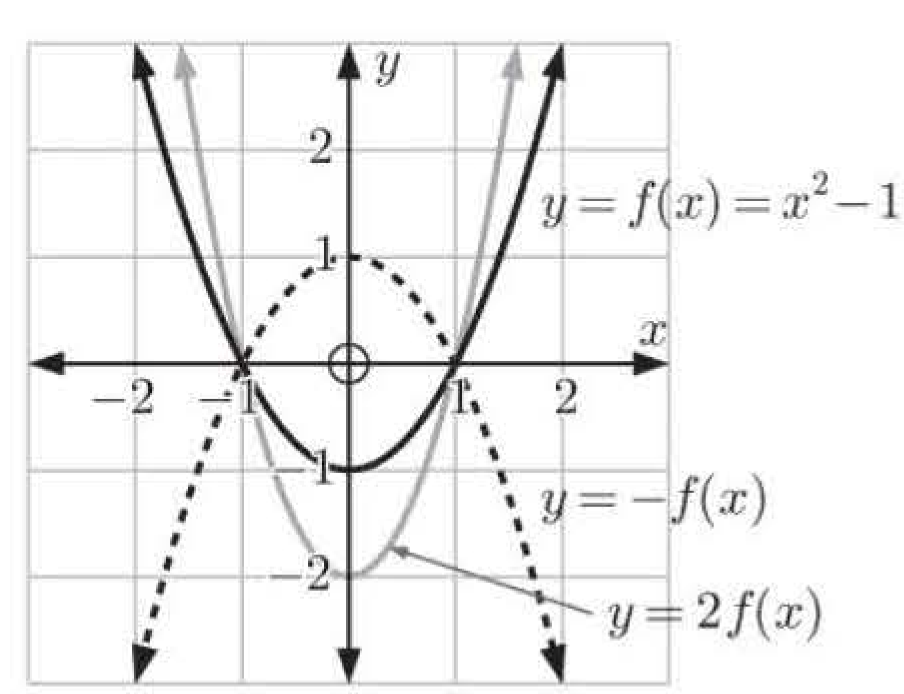


$y = x^2 - 1$ has x-intercepts -1 and 1, and y-intercept -1.

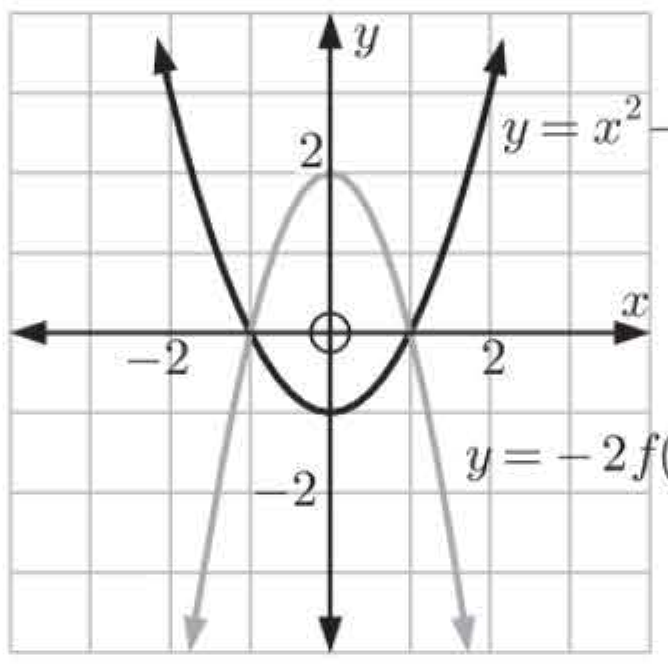
b **i, ii**



iii, iv



c **i** a vertical translation of 3 units upwards **ii** a horizontal translation of 1 unit to the right
iii a vertical stretch with scale factor 2 **iv** a reflection in the x-axis

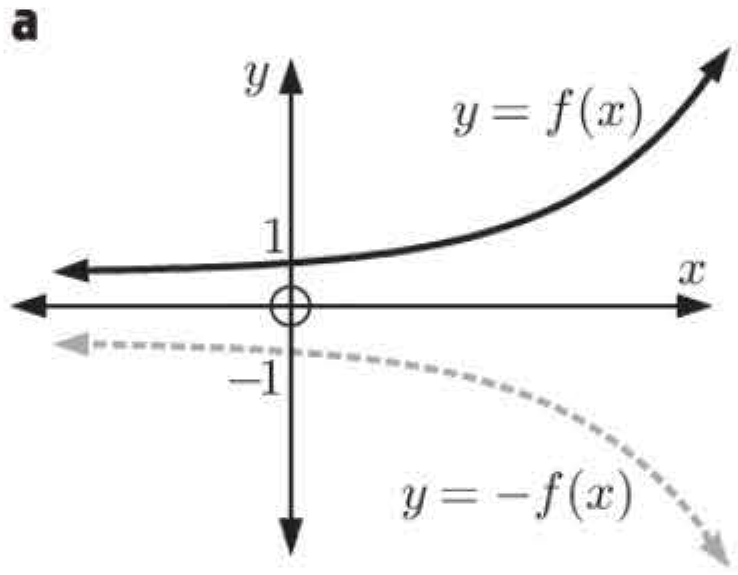
d


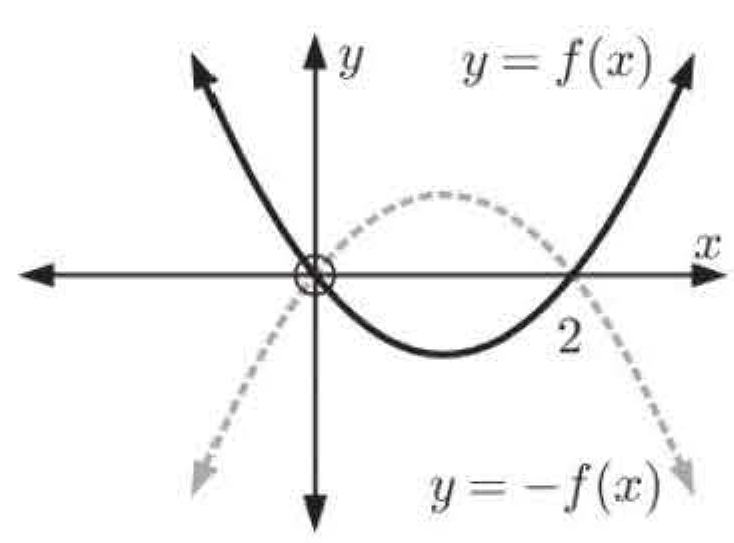
A reflection in the x-axis, followed by a vertical stretch with scale factor 2.

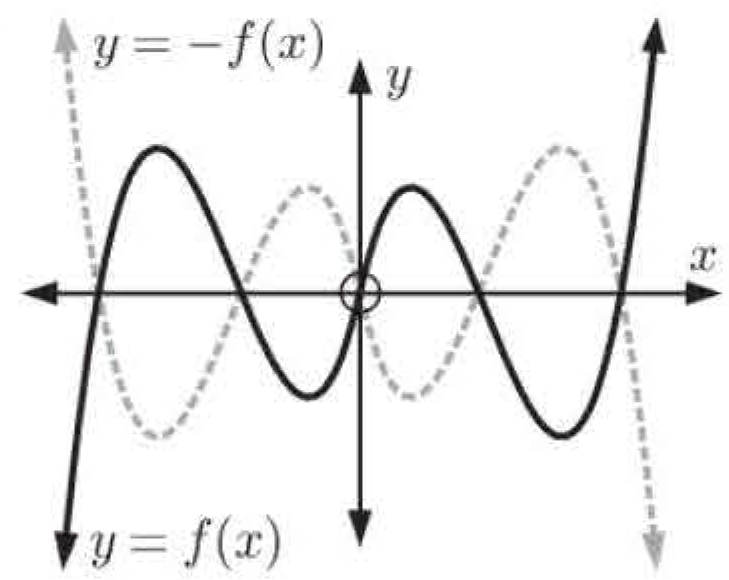
e $(-1, 0)$ and $(1, 0)$

2 **a** **i** A vertical stretch with scale factor 3. **ii** $g(x) = 3f(x)$
b **i** A translation of 2 units downwards. **ii** $g(x) = f(x) - 2$
c **i** A vertical stretch with scale factor $\frac{1}{2}$. **ii** $g(x) = \frac{1}{2}f(x)$
d **i** A reflection in the y-axis. **ii** $g(x) = f(-x)$

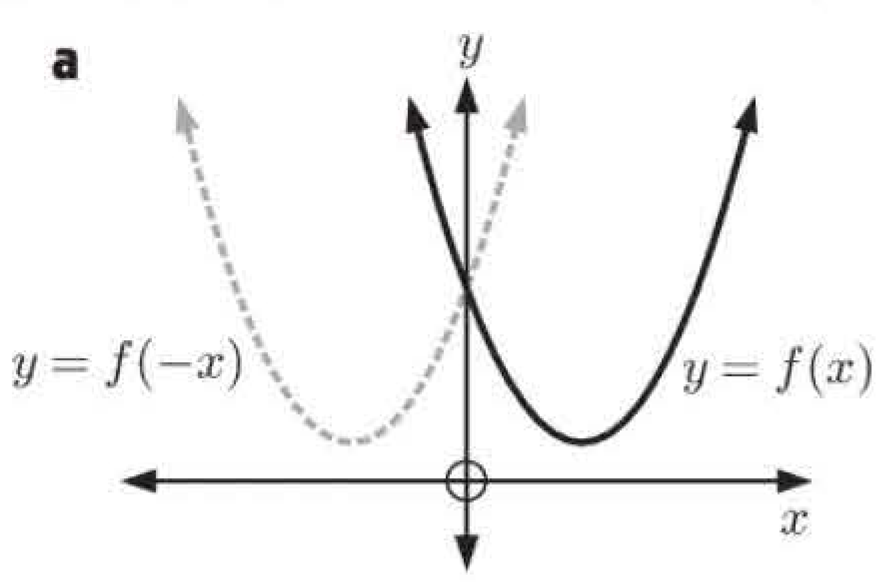
3 $y = -f(x)$ is obtained from $y = f(x)$ by reflecting it in the x-axis.

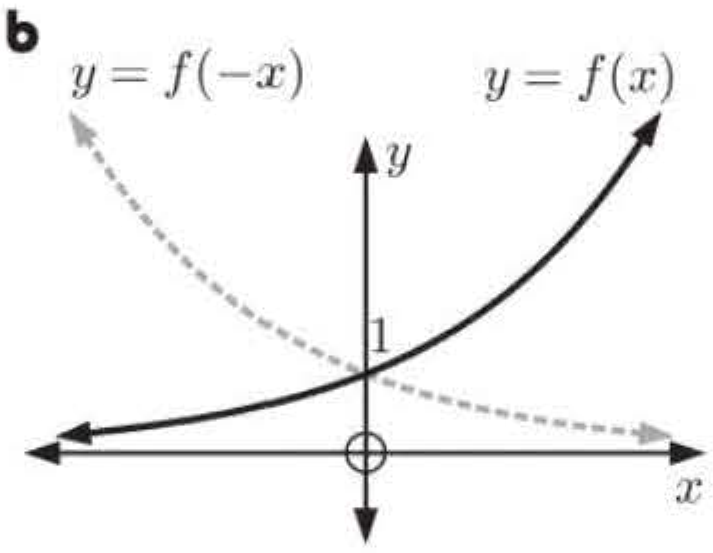
a


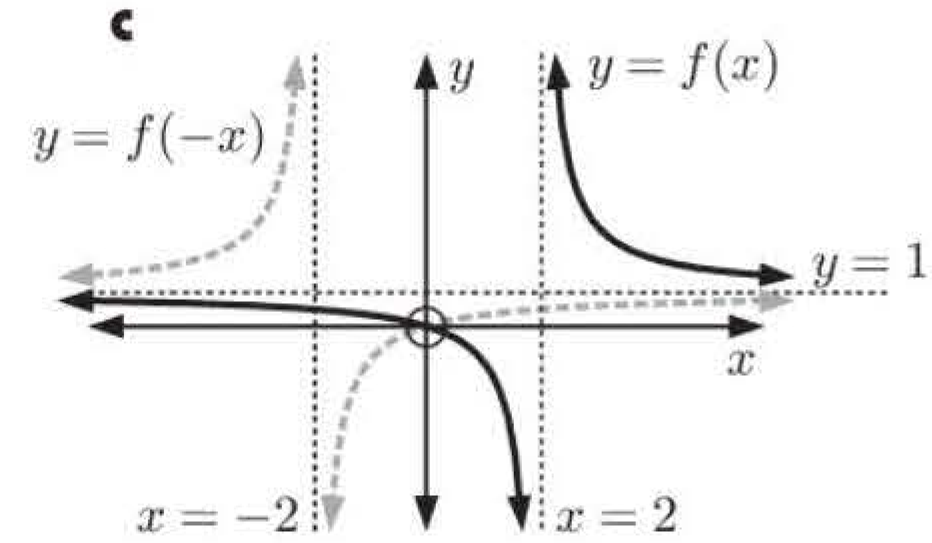
b


c


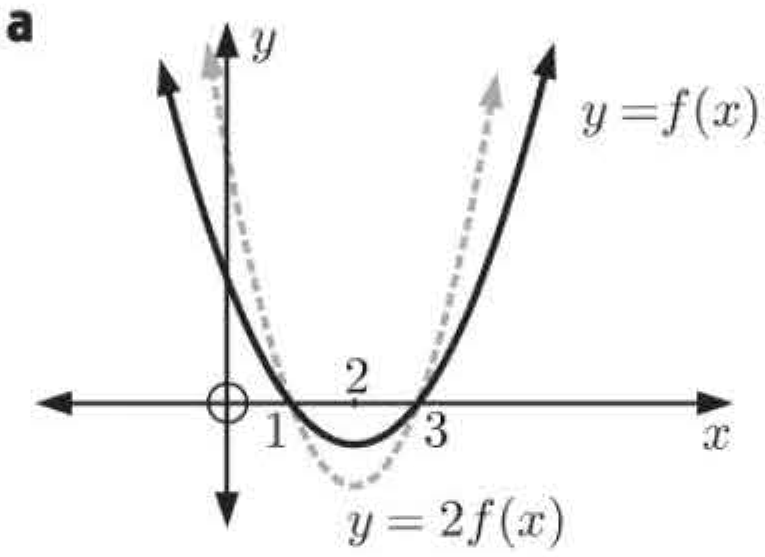
4 $y = f(-x)$ is obtained from $y = f(x)$ by reflecting it in the y-axis.

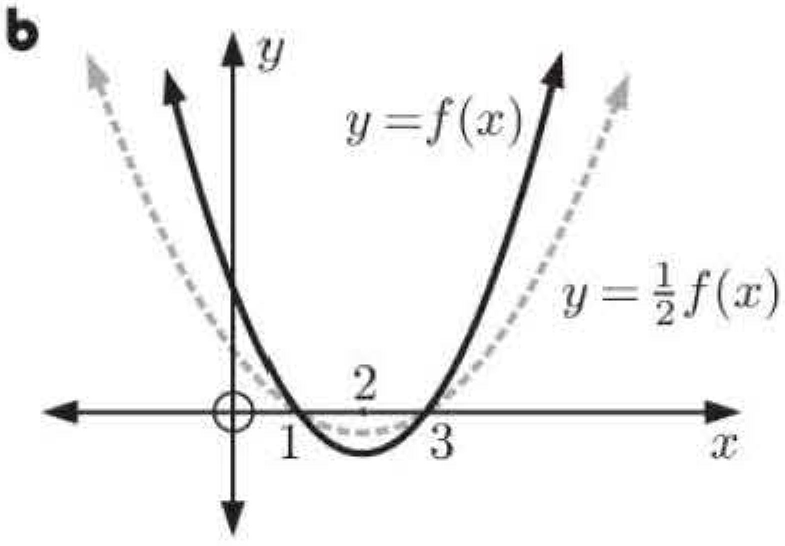
a


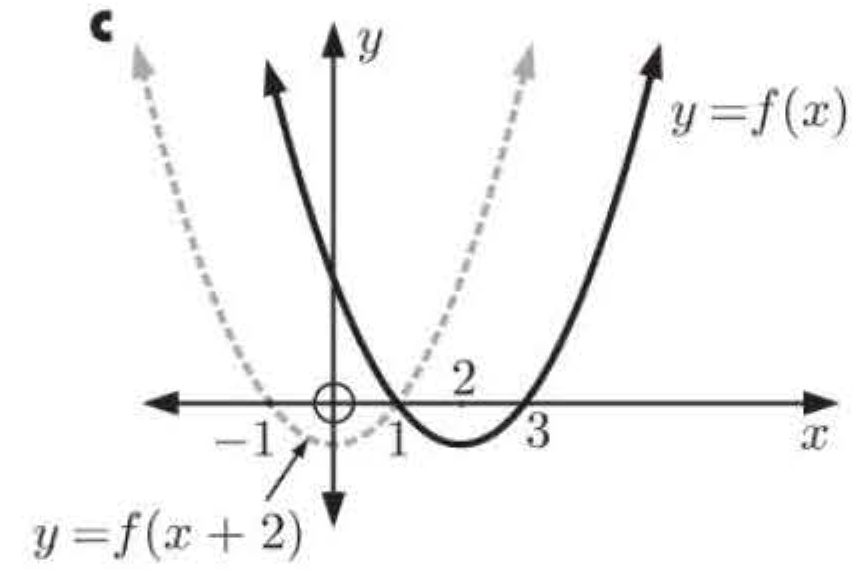
b


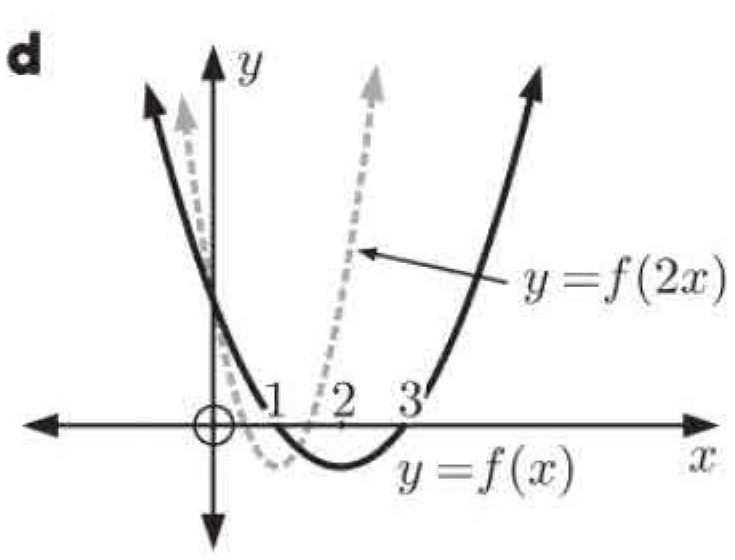
c


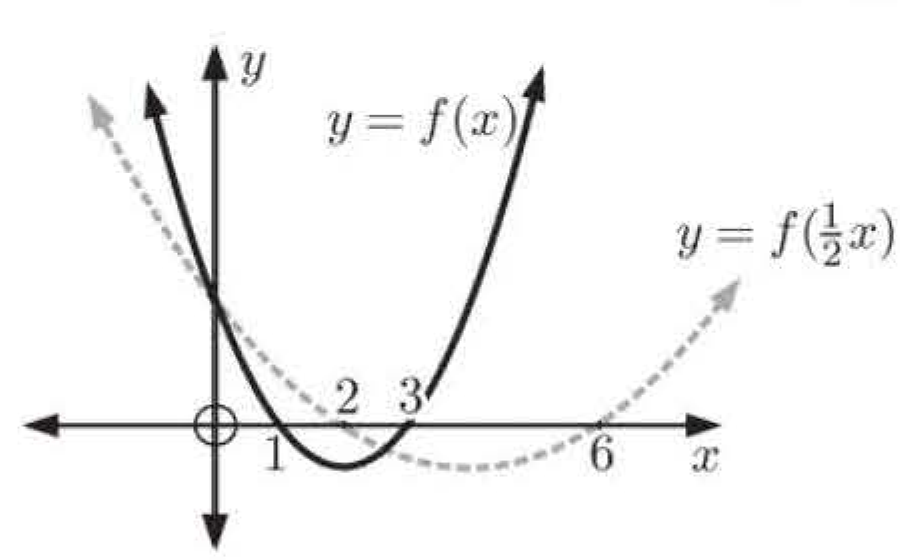
5 $y = 2x^4$ and $y = 6x^4$ are 'thinner' than $y = x^4$ and $y = \frac{1}{2}x^4$ is 'fatter'.
$\therefore$ **a** is **A**, **b** is **B**, **c** is **D**, and **d** is **C**.

6 **a**


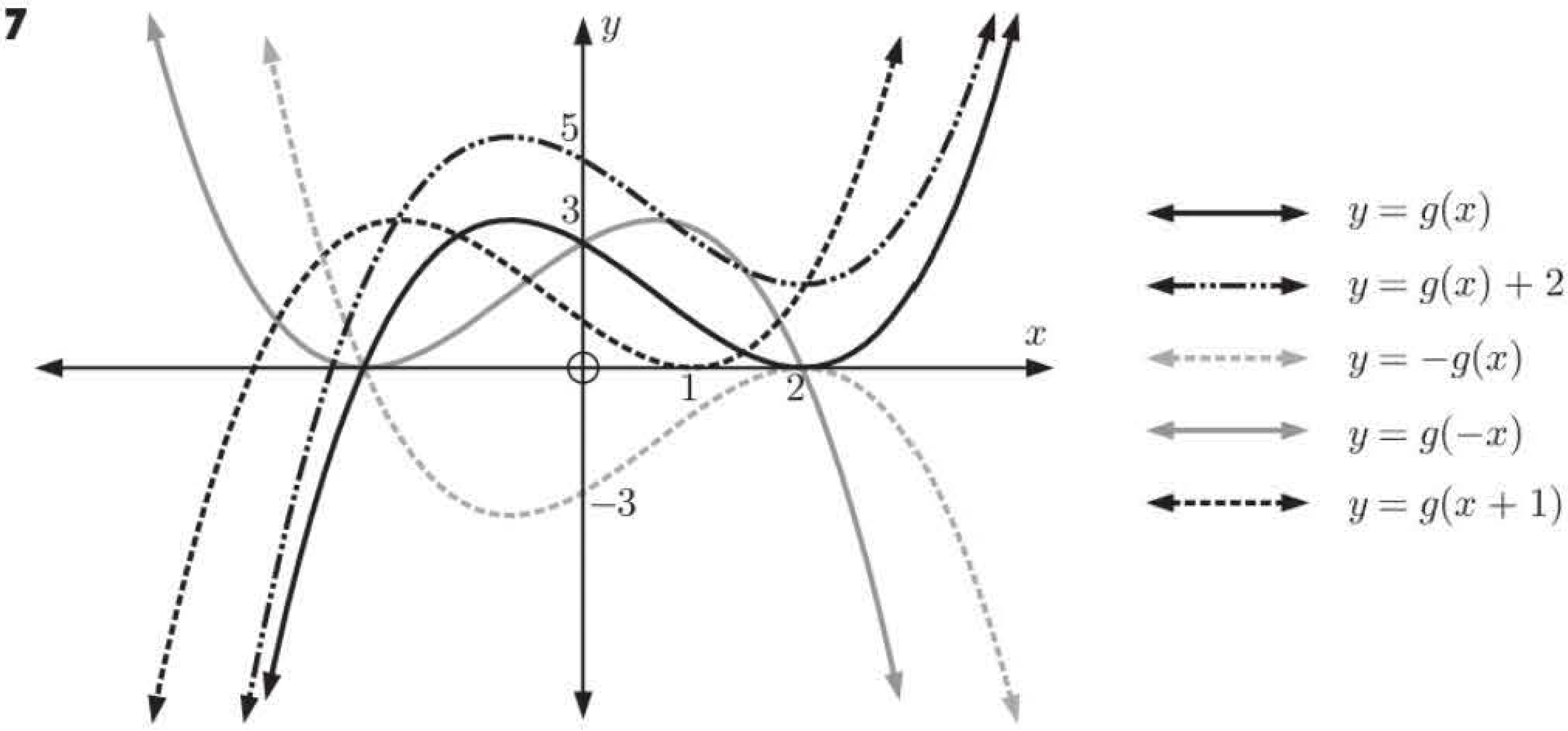
b


c


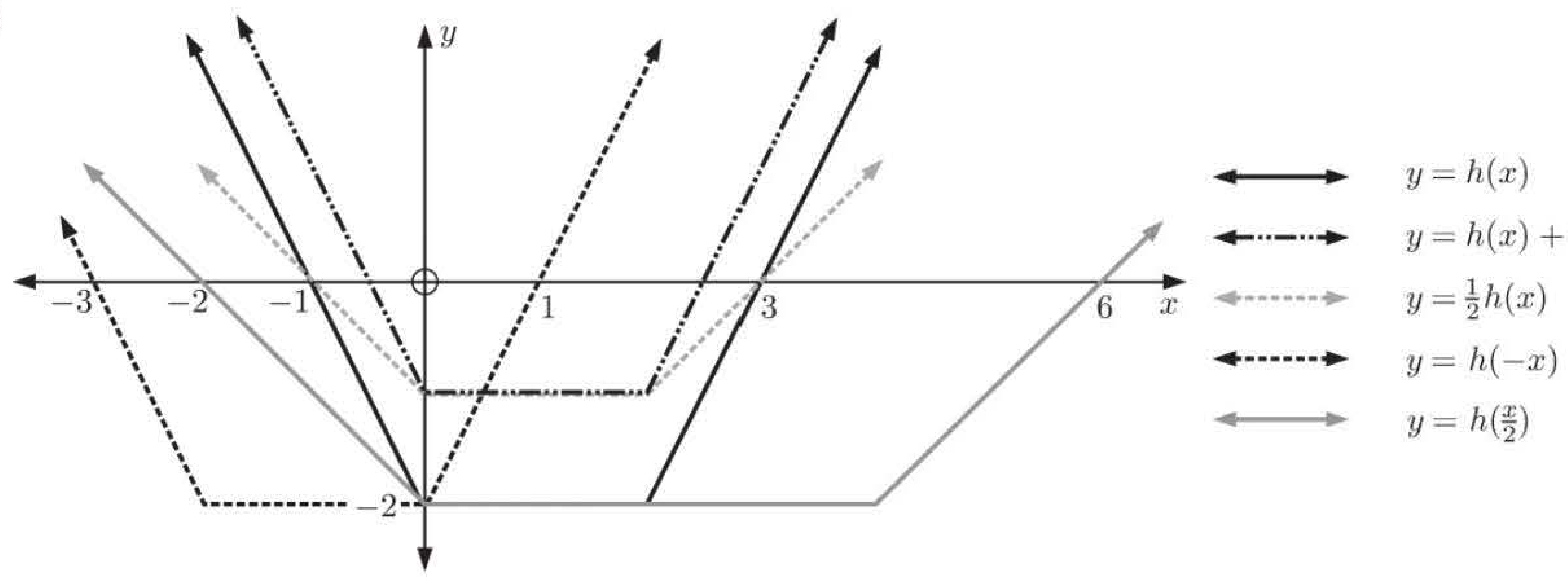
d


e


7



8



EXERCISE 5F

1 **a** Under a vertical stretch with scale factor $\frac{1}{2}$, $y = \dfrac{1}{x}$ becomes $y = \frac{1}{2}\left(\dfrac{1}{x}\right)$. $\therefore\ y = \dfrac{1}{2x}$

b Under a horizontal stretch with scale factor 3, $y = \dfrac{1}{x}$ becomes $y = \dfrac{1}{\left(\frac{x}{3}\right)}$. $\therefore\ y = \dfrac{3}{x}$

c Under a horizontal translation of -3, $y = \dfrac{1}{x}$ becomes $y = \dfrac{1}{x+3}$.

d Under a vertical translation of 4, $y = \dfrac{1}{x}$ becomes $y = \dfrac{1}{x} + 4$. $\therefore\ y = \dfrac{4x+1}{x}$

2 **a** Under a vertical stretch with scale factor 3, $f(x)$ becomes $3f(x)$.

$\therefore\ \dfrac{1}{x}$ becomes $3\left(\dfrac{1}{x}\right) = \dfrac{3}{x}$.

Under a translation of $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $f(x)$ becomes $f(x-1) - 1$.

$\therefore\ \dfrac{3}{x}$ becomes $\dfrac{3}{x-1} - 1$.

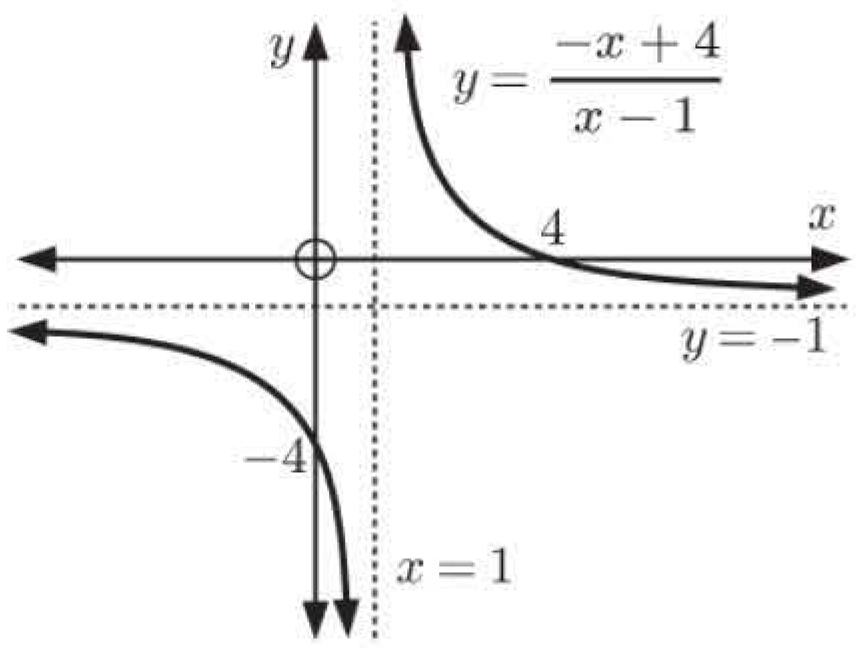
So, $y = \dfrac{1}{x}$ becomes
$$g(x) = \frac{3}{x-1} - 1 = \frac{3 - (x-1)}{x-1} = \frac{-x+4}{x-1}$$

b The asymptotes of $y = \dfrac{1}{x}$ are $x = 0$ and $y = 0$.

These are unchanged by the stretch, and shifted $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ by the translation.

$\therefore$ the vertical asymptote is $x = 1$ and the horizontal asymptote is $y = -1$.

c Domain is $\{x \mid x \neq 1\}$, range is $\{y \mid y \neq -1\}$.

d 

e The graph is not symmetric about $y = x$, so $g(x)$ is not a self-inverse function.

3 **a** **i** $f(x) = \dfrac{2x+4}{x-1}$

$= \dfrac{2(x-1)+6}{x-1}$

$= \dfrac{6}{x-1} + 2$

$y = f(x)$ is a translation of $y = \dfrac{6}{x}$ through $\binom{1}{2}$.

Now $y = \dfrac{6}{x}$ has asymptotes $x = 0$ and $y = 0$.

$\therefore$ $y = f(x)$ has vertical asymptote $x = 1$ and horizontal asymptote $y = 2$.

ii $\dfrac{1}{x}$ becomes $\dfrac{6}{x}$ under a vertical stretch with scale factor 6.

$\dfrac{6}{x}$ becomes $\dfrac{6}{x-1} + 2$ under a translation through $\binom{1}{2}$.

So, $y = \dfrac{1}{x}$ is transformed to $y = f(x)$ under a vertical stretch with scale factor 6, followed by a translation through $\binom{1}{2}$.

b **i** $f(x) = \dfrac{3x-2}{x+1}$

$= \dfrac{3(x+1)-5}{x+1}$

$= -\dfrac{5}{x+1} + 3$

$y = f(x)$ is a translation of $y = -\dfrac{5}{x}$ through $\binom{-1}{3}$.

Now $y = -\dfrac{5}{x}$ has asymptotes $x = 0$ and $y = 0$.

$\therefore$ $y = f(x)$ has vertical asymptote $x = -1$ and horizontal asymptote $y = 3$.

ii $\dfrac{1}{x}$ becomes $\dfrac{5}{x}$ under a vertical stretch with scale factor 5.

$\dfrac{5}{x}$ becomes $-\dfrac{5}{x}$ under a reflection in the x-axis.

$-\dfrac{5}{x}$ becomes $-\dfrac{5}{x+1} + 3$ under a translation through $\binom{-1}{3}$.

So, $y = \dfrac{1}{x}$ is transformed to $y = f(x)$ under a vertical stretch with scale factor 5, followed by a reflection in the x-axis, followed by a translation through $\binom{-1}{3}$.

c **i** $f(x) = \dfrac{2x+1}{2-x}$

$= \dfrac{-2(2-x)+5}{2-x}$

$= \dfrac{5}{2-x} - 2$

$= -\dfrac{5}{x-2} - 2$

$y = f(x)$ is a translation of $y = -\dfrac{5}{x}$ through $\binom{2}{-2}$.

Now $y = -\dfrac{5}{x}$ has asymptotes $x = 0$ and $y = 0$.

$\therefore$ $y = f(x)$ has vertical asymptote $x = 2$ and horizontal asymptote $y = -2$.

ii $\dfrac{1}{x}$ becomes $\dfrac{5}{x}$ under a vertical stretch with scale factor 5.

$\dfrac{5}{x}$ becomes $-\dfrac{5}{x}$ under a reflection in the x-axis.

$-\dfrac{5}{x}$ becomes $-\dfrac{5}{x-2} - 2$ under a translation through $\binom{2}{-2}$.

So, $y = \dfrac{1}{x}$ is transformed to $y = f(x)$ under a vertical stretch with scale factor 5, followed by a reflection in the x-axis, followed by a translation through $\binom{2}{-2}$.

4 **a** **i** $f(x) = \dfrac{2x+3}{x+1}$
$= \dfrac{2(x+1)+1}{x+1}$
$= \dfrac{1}{x+1} + 2$

$y = f(x)$ is a translation of $y = \dfrac{1}{x}$ through $\binom{-1}{2}$.

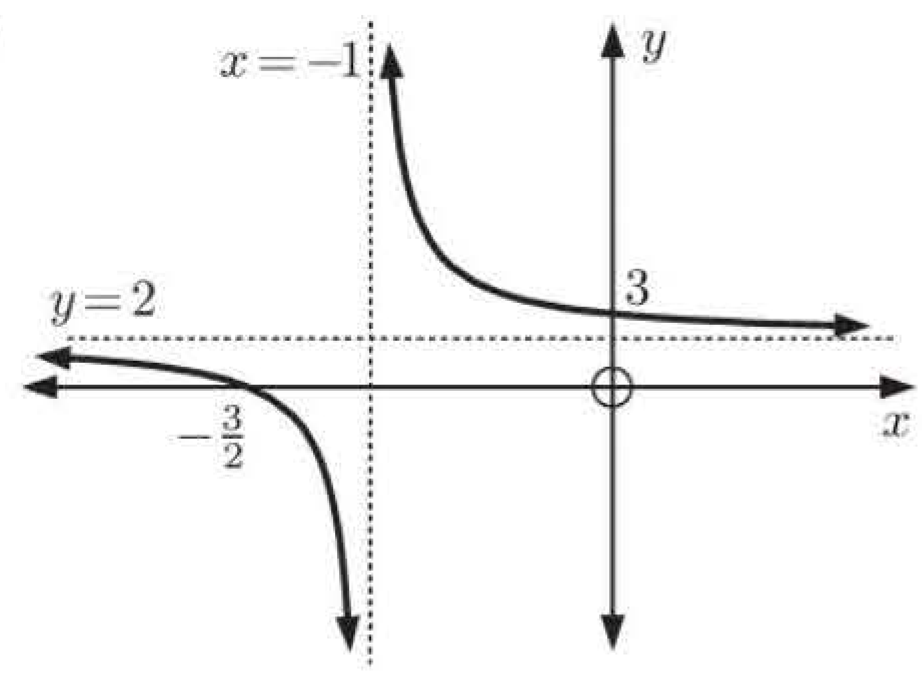
Now $y = \dfrac{1}{x}$ has asymptotes $x = 0$ and $y = 0$.

$\therefore$ $y = f(x)$ has vertical asymptote $x = -1$ and horizontal asymptote $y = 2$.

ii As $x \to -1^-$, $y \to -\infty$.
As $x \to -1^+$, $y \to \infty$.
As $x \to -\infty$, $y \to 2^-$.
As $x \to \infty$, $y \to 2^+$.

iii When $x = 0$, $y = \frac{1}{1} + 2 = 3$.
$\therefore$ the y-intercept is 3.
When $y = 0$, $2x + 3 = 0$
$\therefore$ $x = \frac{-3}{2}$
$\therefore$ the x-intercept is $\frac{-3}{2}$.

iv



v $\dfrac{1}{x}$ becomes $\dfrac{1}{x+1} + 2$ under a translation through $\binom{-1}{2}$.

So, $y = \dfrac{1}{x}$ is transformed to $y = f(x)$ under a translation through $\binom{-1}{2}$.

vi To transform $y = f(x)$ into $y = \dfrac{1}{x}$, we need to reverse the process in **v**.

We need a translation through $\binom{1}{-2}$.

b **i** $f(x) = \dfrac{3}{x-2}$

$y = f(x)$ is a translation of $y = \dfrac{3}{x}$ through $\binom{2}{0}$.

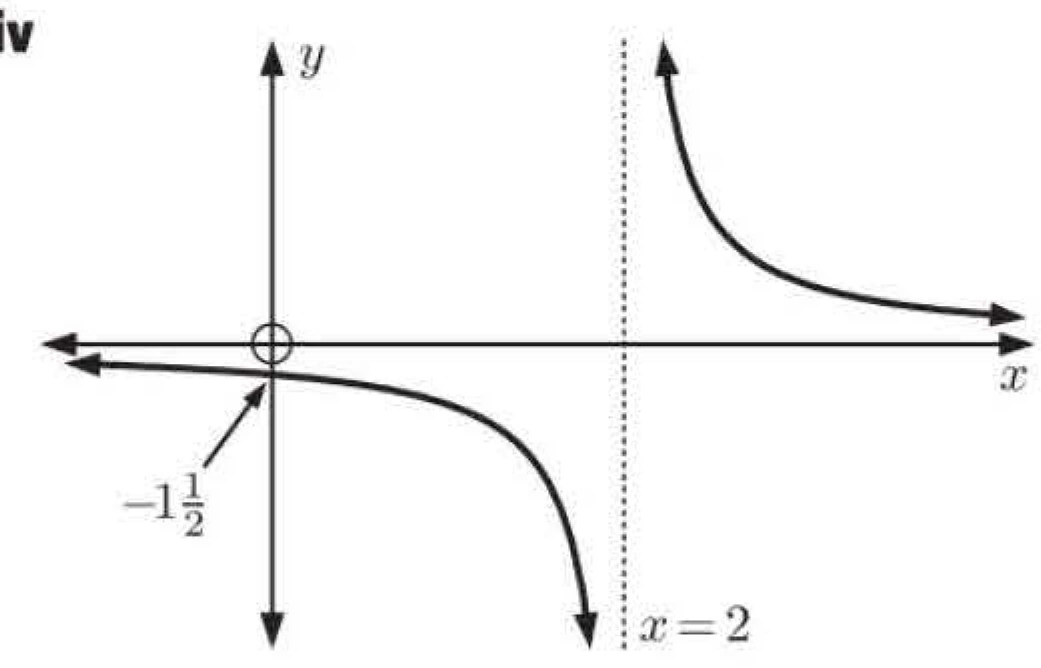
Now $y = \dfrac{3}{x}$ has asymptotes $x = 0$ and $y = 0$.

$\therefore$ $y = f(x)$ has vertical asymptote $x = 2$ and horizontal asymptote $y = 0$.

ii As $x \to 2^-$, $y \to -\infty$.
As $x \to 2^+$, $y \to \infty$.
As $x \to -\infty$, $y \to 0^-$.
As $x \to \infty$, $y \to 0^+$.

iii When $x = 0$, $y = \frac{3}{-2} = -1\frac{1}{2}$.
$\therefore$ the y-intercept is $-1\frac{1}{2}$.
When $y = 0$, $\dfrac{3}{x-2} = 0$
which is not possible
$\therefore$ no x-intercept.

iv



v $\dfrac{1}{x}$ becomes $\dfrac{3}{x}$ under a vertical stretch with scale factor 3.

$\dfrac{3}{x}$ becomes $\dfrac{3}{x-2}$ under a translation through $\binom{2}{0}$.

So, $y = \dfrac{1}{x}$ is transformed to $y = f(x)$ under a vertical stretch with scale factor 3, followed by a translation through $\binom{2}{0}$.

vi To transform $y = f(x)$ into $y = \dfrac{1}{x}$, we need to reverse the process in **v**.

We need a translation through $\binom{-2}{0}$, followed by a vertical stretch with scale factor $\frac{1}{3}$.

c **i** $f(x) = \dfrac{2x-1}{3-x}$

$= \dfrac{-2(3-x)+5}{3-x}$

$= \dfrac{5}{3-x} - 2$

$= -\dfrac{5}{x-3} - 2$

$y = f(x)$ is a translation of $y = -\dfrac{5}{x}$ through $\binom{3}{-2}$.

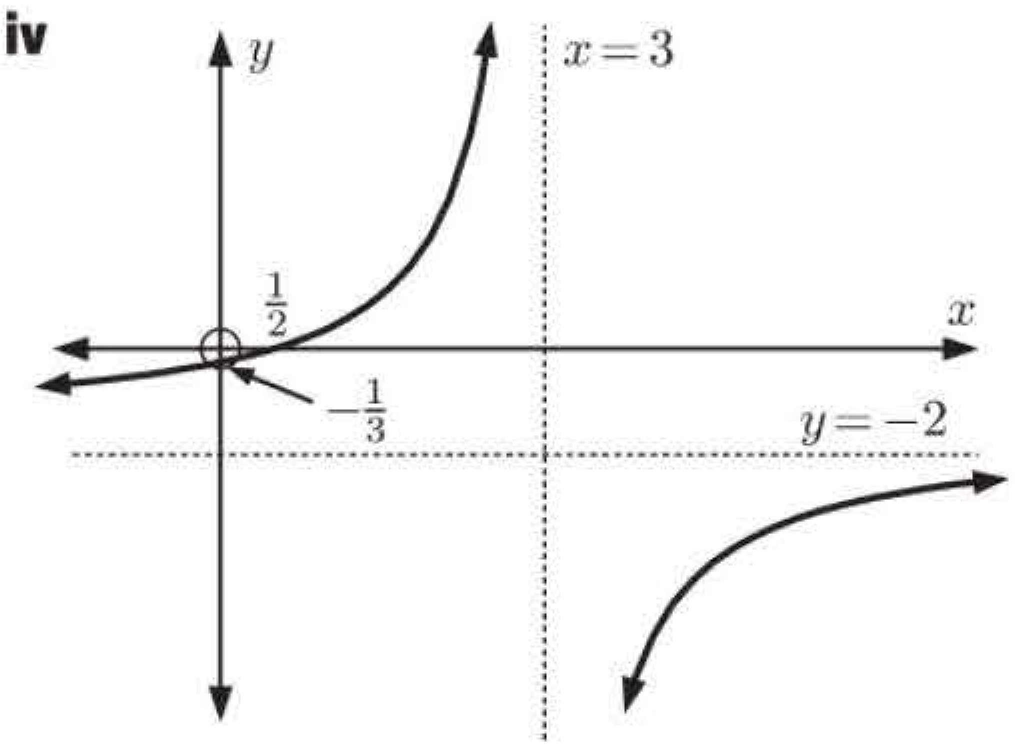
Now $y = -\dfrac{5}{x}$ has asymptotes $x = 0$ and $y = 0$.

$\therefore$ $y = f(x)$ has vertical asymptote $x = 3$ and horizontal asymptote $y = -2$.

ii As $x \to 3^-$, $y \to \infty$.
As $x \to 3^+$, $y \to -\infty$.
As $x \to -\infty$, $y \to -2^+$.
As $x \to \infty$, $y \to -2^-$.

iii When $x = 0$, $y = \frac{-5}{-3} - 2 = -\frac{1}{3}$.
$\therefore$ the y-intercept is $-\frac{1}{3}$.
When $y = 0$, $2x - 1 = 0$
$\therefore$ $x = \frac{1}{2}$
$\therefore$ the x-intercept is $\frac{1}{2}$.

iv



v $\dfrac{1}{x}$ becomes $\dfrac{5}{x}$ under a vertical stretch with scale factor 5.
$\dfrac{5}{x}$ becomes $-\dfrac{5}{x}$ under a reflection in the x-axis.
$-\dfrac{5}{x}$ becomes $-\dfrac{5}{x-3} - 2$ under a translation through $\binom{3}{-2}$.
So, $y = \dfrac{1}{x}$ is transformed to $y = f(x)$ under a vertical stretch with scale factor 5, followed by a reflection in the x-axis, followed by a translation through $\binom{3}{-2}$.

vi To transform $y = f(x)$ into $y = \dfrac{1}{x}$, we need to reverse the process in **v**.
We need a translation through $\binom{-3}{2}$, followed by a reflection in the x-axis, followed by a vertical stretch with scale factor $\frac{1}{5}$.

d **i** $f(x) = \dfrac{5x-1}{2x+1}$

$= \dfrac{\frac{5}{2}x - \frac{1}{2}}{x + \frac{1}{2}}$

$= \dfrac{\frac{5}{2}(x+\frac{1}{2}) - \frac{7}{4}}{x+\frac{1}{2}}$

$= -\dfrac{\frac{7}{4}}{x+\frac{1}{2}} + \frac{5}{2}$

$y = f(x)$ is a translation of $y = -\dfrac{\frac{7}{4}}{x}$ through $\begin{pmatrix} -\frac{1}{2} \\ \frac{5}{2} \end{pmatrix}$.

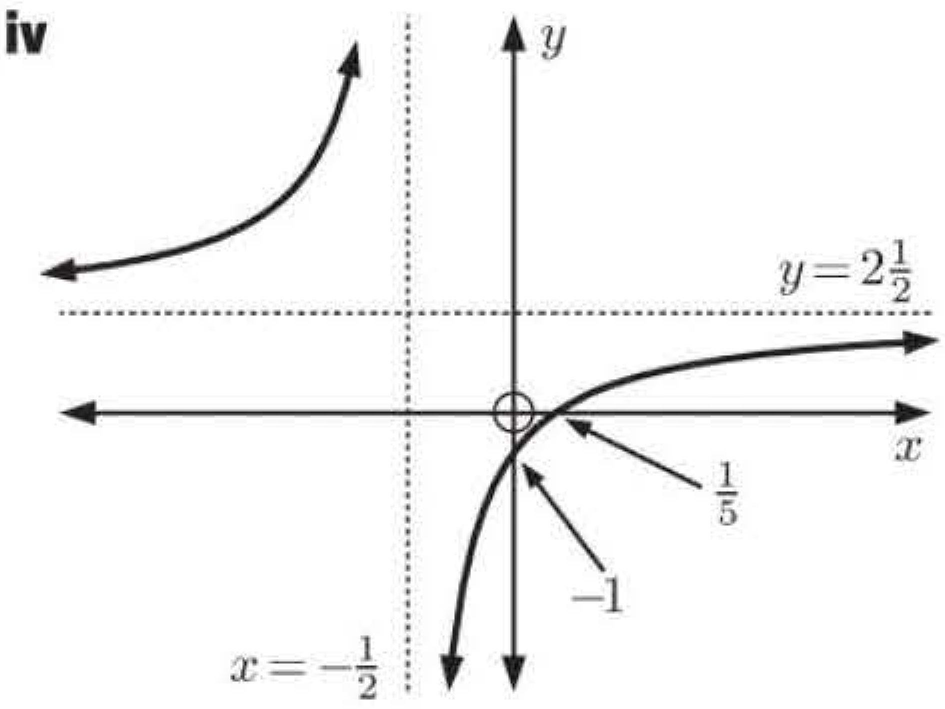
Now $y = -\dfrac{\frac{7}{4}}{x}$ has asymptotes $x = 0$ and $y = 0$.

$\therefore$ $y = f(x)$ has vertical asymptote $x = -\frac{1}{2}$ and horizontal asymptote $y = 2\frac{1}{2}$.

ii As $x \to -\frac{1}{2}^-$, $y \to \infty$.
As $x \to -\frac{1}{2}^+$, $y \to -\infty$.
As $x \to -\infty$, $y \to 2\frac{1}{2}^+$.
As $x \to \infty$, $y \to 2\frac{1}{2}^-$.

iii When $x = 0$, $y = \frac{-1}{1} = -1$.
$\therefore$ the y-intercept is -1.
When $y = 0$, $5x - 1 = 0$
$\therefore$ $x = \frac{1}{5}$
$\therefore$ the x-intercept is $\frac{1}{5}$.

iv



v $\dfrac{1}{x}$ becomes $\dfrac{\frac{7}{4}}{x}$ under a vertical stretch with scale factor $\frac{7}{4}$.

$\dfrac{\frac{7}{4}}{x}$ becomes $-\dfrac{\frac{7}{4}}{x}$ under a reflection in the x-axis.

$-\dfrac{\frac{7}{4}}{x}$ becomes $-\dfrac{\frac{7}{4}}{x+\frac{1}{2}}+\frac{5}{2}$ under a translation through $\begin{pmatrix} -\frac{1}{2} \\ \frac{5}{2} \end{pmatrix}$.

So, $y=\dfrac{1}{x}$ is transformed to $y=f(x)$ under a vertical stretch with scale factor $\frac{7}{4}$, followed by a reflection in the x-axis, followed by a translation through $\begin{pmatrix} -\frac{1}{2} \\ \frac{5}{2} \end{pmatrix}$.

vi To transform $y=f(x)$ into $y=\dfrac{1}{x}$, we need to reverse the process in **v**.

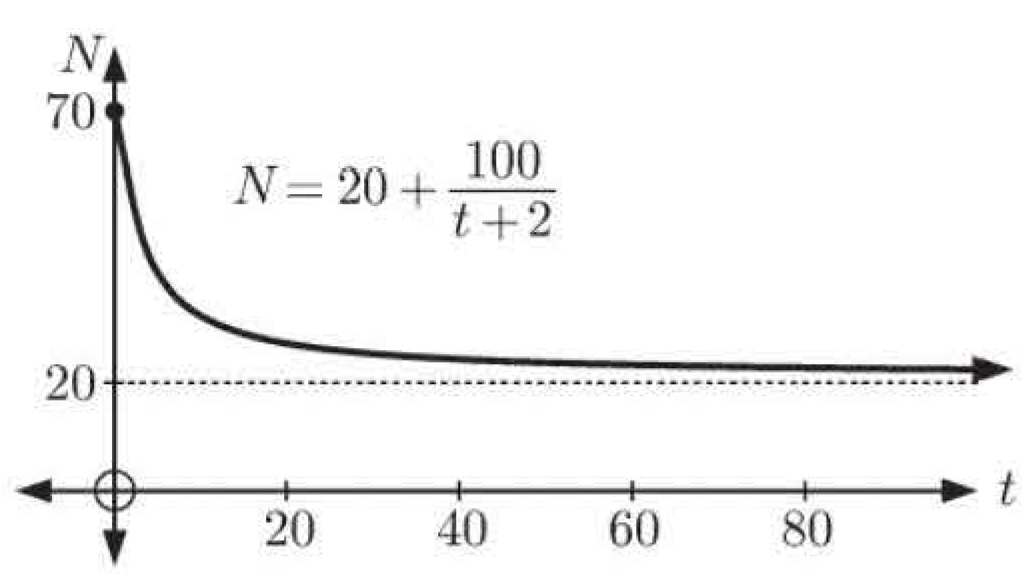
We need a translation through $\begin{pmatrix} \frac{1}{2} \\ -\frac{5}{2} \end{pmatrix}$, followed by a reflection in the x-axis, followed by a vertical stretch with scale factor $\frac{4}{7}$.

5 $N=20+\dfrac{100}{t+2}$ weeds per hectare

a When $t=0$,
$$\begin{aligned} N &= 20+\tfrac{100}{2} \\ &= 20+50 \\ &= 70 \text{ weeds/ha} \end{aligned}$$

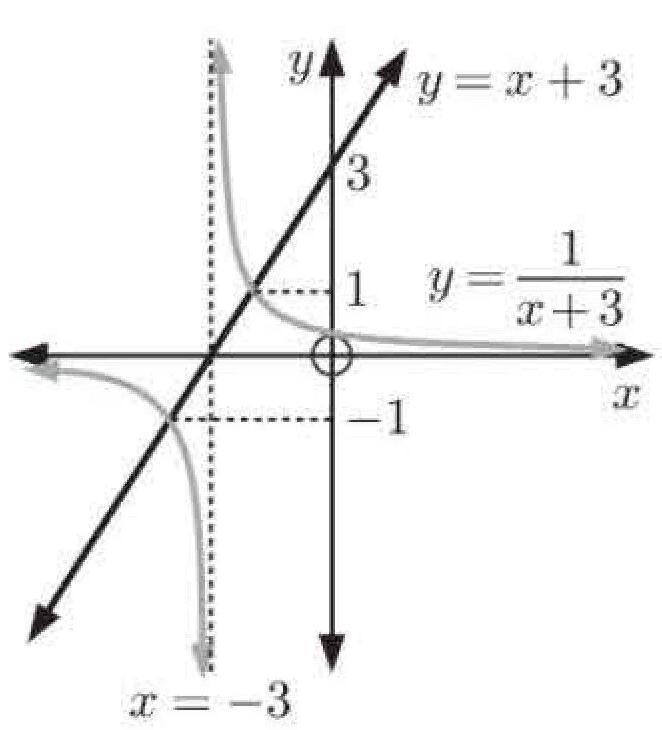
b When $t=8$,
$$\begin{aligned} N &= 20+\tfrac{100}{10} \\ &= 20+10 \\ &= 30 \text{ weeds/ha} \end{aligned}$$

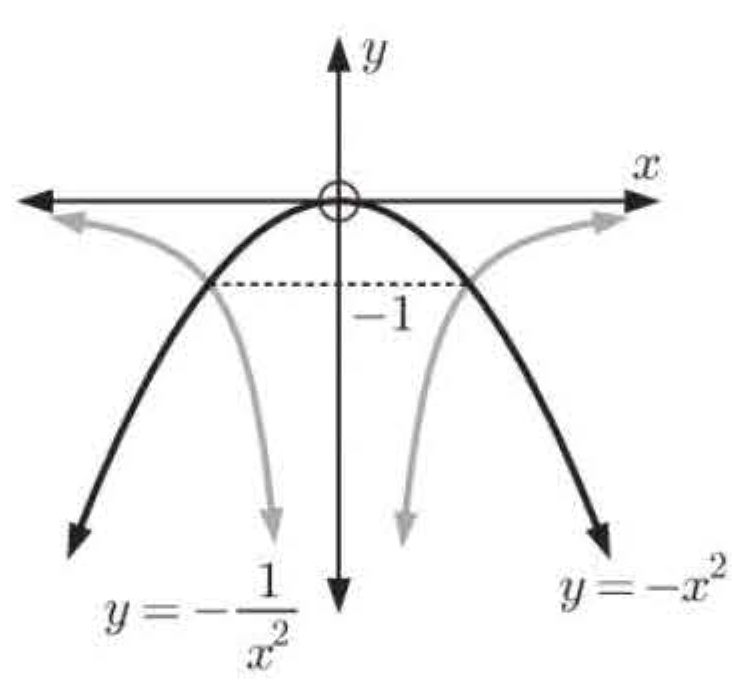
c When $N=40$,
$$\begin{aligned} 20+\frac{100}{t+2} &= 40 \\ \therefore\ \frac{100}{t+2} &= 20 \\ \therefore\ t+2 &= 5 \\ \therefore\ t &= 3 \text{ days} \end{aligned}$$

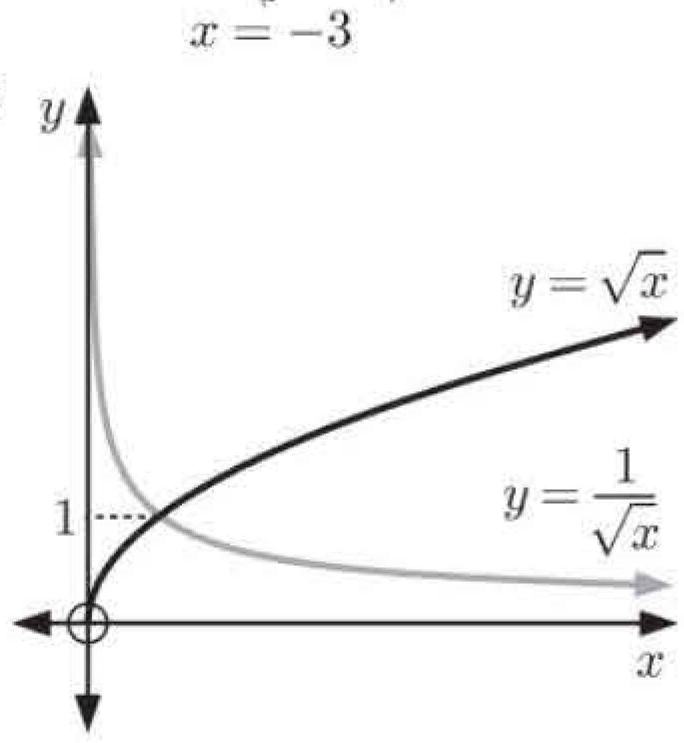
d


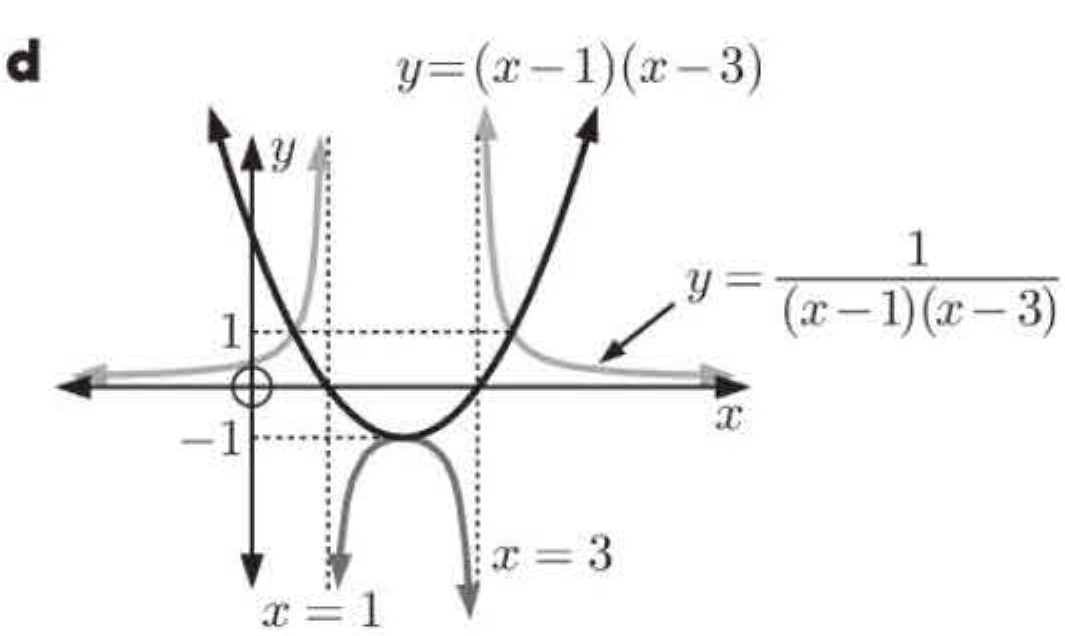
e No, the number of weeds per hectare will approach 20 (from above), so at least 20 weeds will remain..

EXERCISE 5G

1 **a**


b


c


d


2 If $f(x) = \dfrac{1}{f(x)}$ then $y = \dfrac{1}{y}$

$\therefore\ y^2 = 1$

$\therefore\ y = \pm 1$

For **1 a**: When $y = 1$, $x + 3 = 1$ $\therefore\ x = -2$ When $y = -1$, $x + 3 = -1$ $\therefore\ x = -4$

So, the invariant points are $(-2, 1)$ and $(-4, -1)$.

For **1 b**: When $y = 1$, $-x^2 = 1$ which has no real solutions When $y = -1$, $-x^2 = -1$ $\therefore\ x^2 = 1$ $\therefore\ x = \pm 1$

So, the invariant points are $(1, -1)$ and $(-1, -1)$.

For **1 c**: When $y = 1$, $\sqrt{x} = 1$ $\therefore\ x = 1$ When $y = -1$, $\sqrt{x} = -1$ which has no real solutions.

So, the invariant point is $(1, 1)$.

For **1 d**: When $y = 1$, $(x-1)(x-3) = 1$

$\therefore\ x^2 - 4x + 3 = 1$

$\therefore\ x^2 - 4x + 2 = 0$

$\therefore\ x = \dfrac{4 \pm \sqrt{8}}{2} = 2 \pm \sqrt{2}$

$= 3.41$ or 0.586

When $y = -1$, $(x-1)(x-3) = -1$

$\therefore\ x^2 - 4x + 3 = -1$

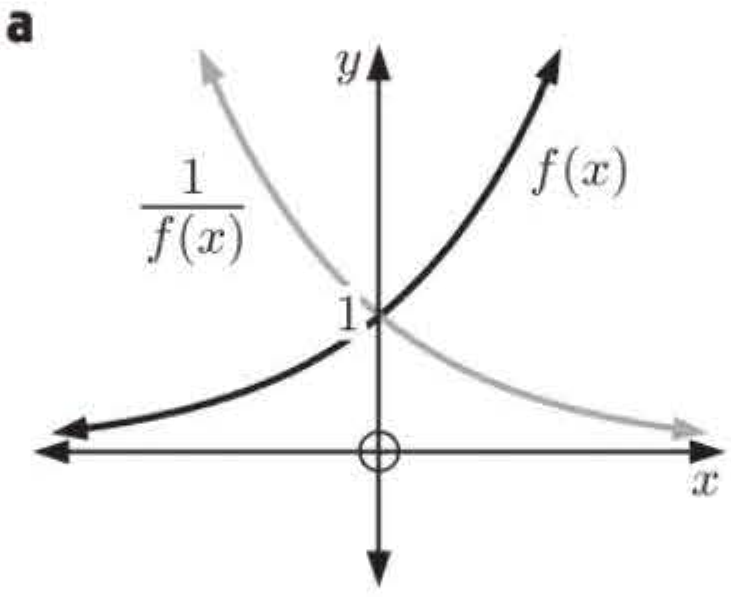
$\therefore\ x^2 - 4x + 4 = 0$

$\therefore\ (x-2)^2 = 0$

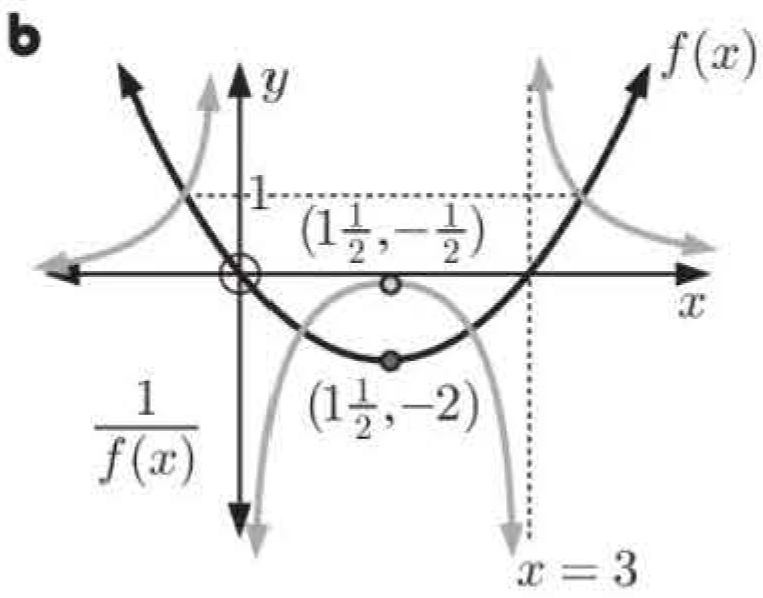
$\therefore\ x = 2$

So, the invariant points are $(3.41, 1)$, $(0.586, 1)$, and $(2, -1)$.

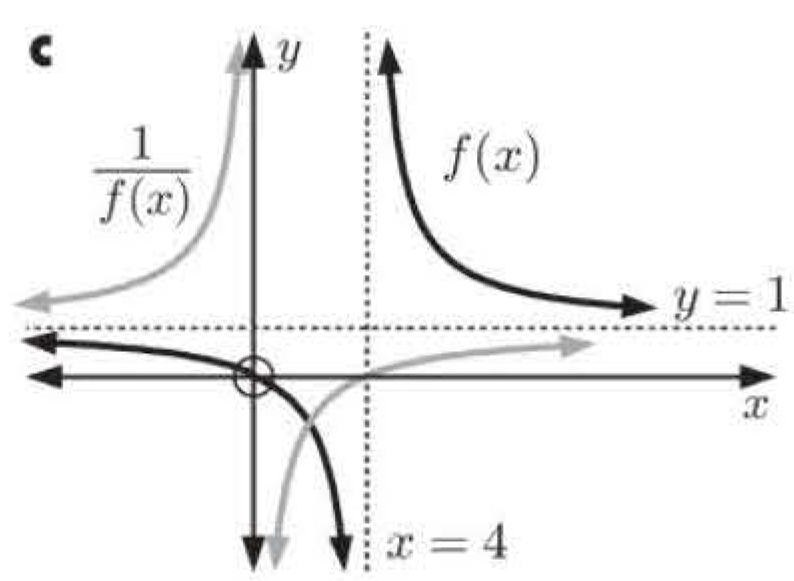
3 **a**



b

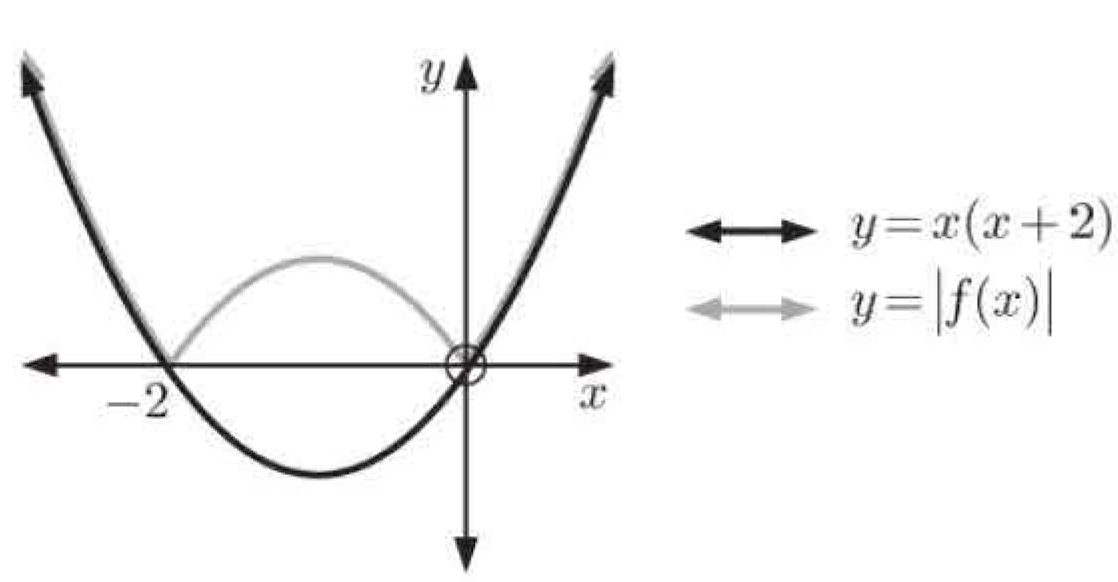


c

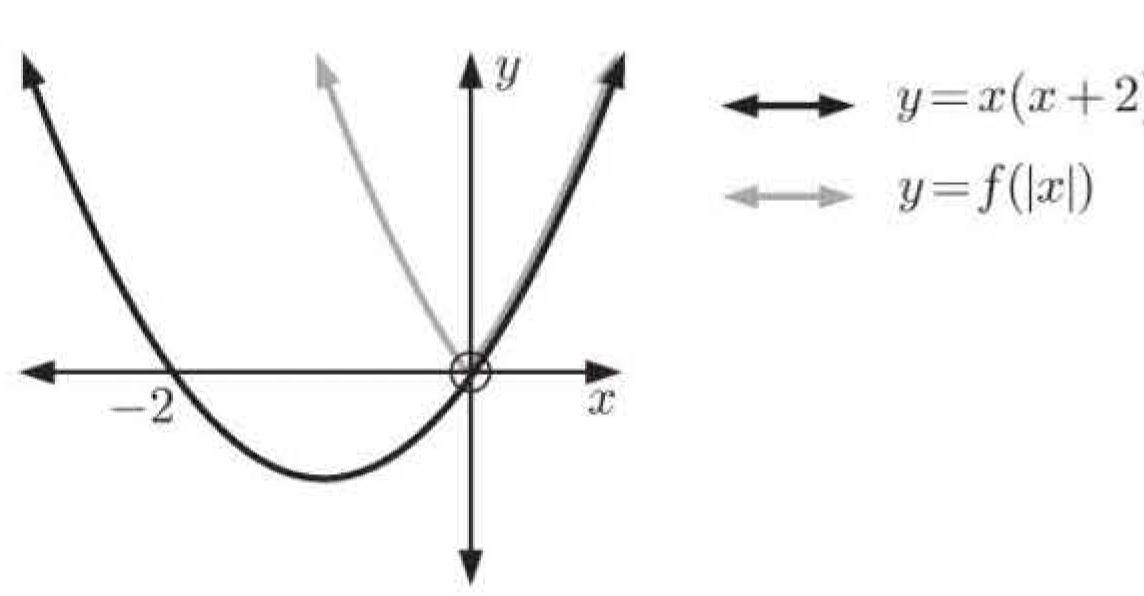


EXERCISE 5H

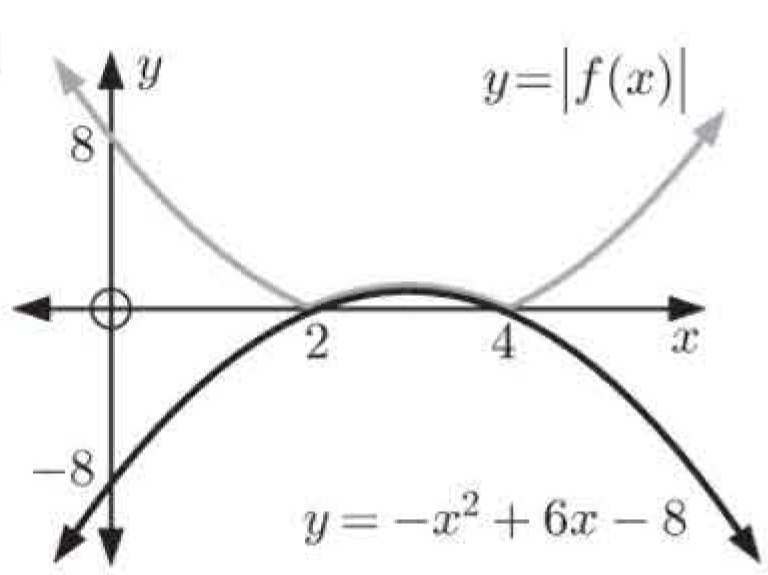
1 **a**



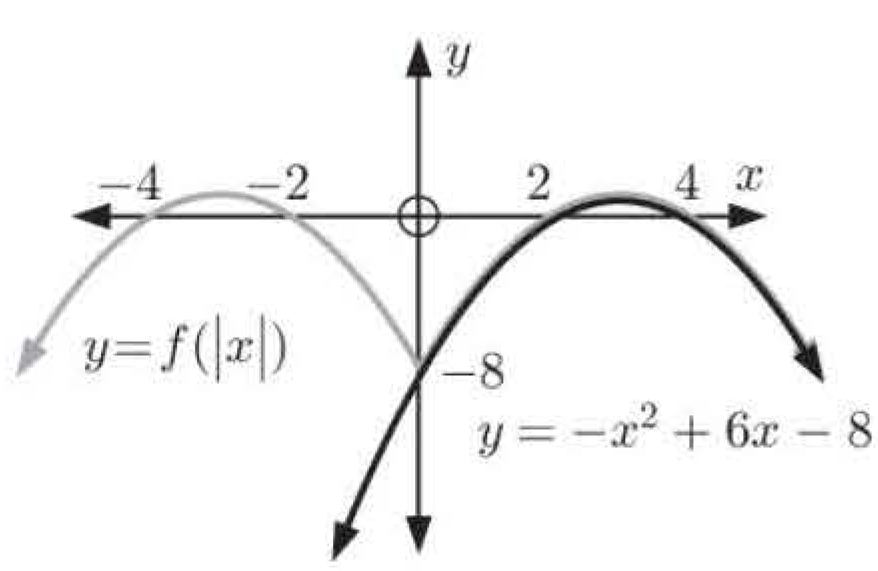
b



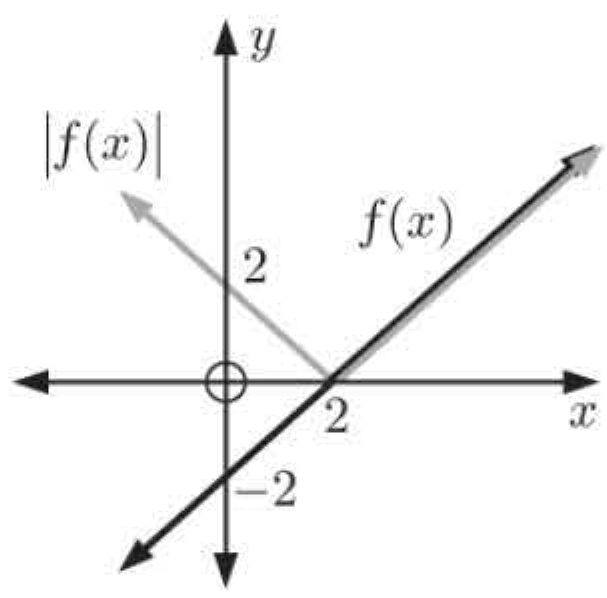
2 **a**



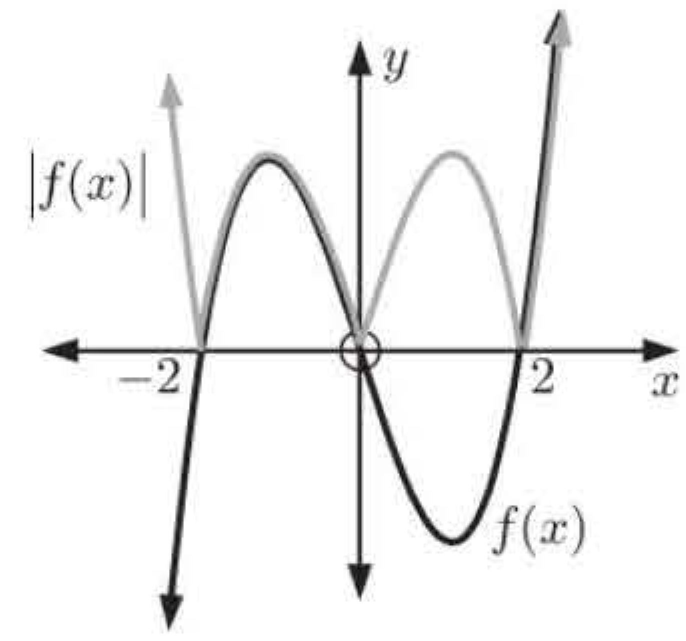
b



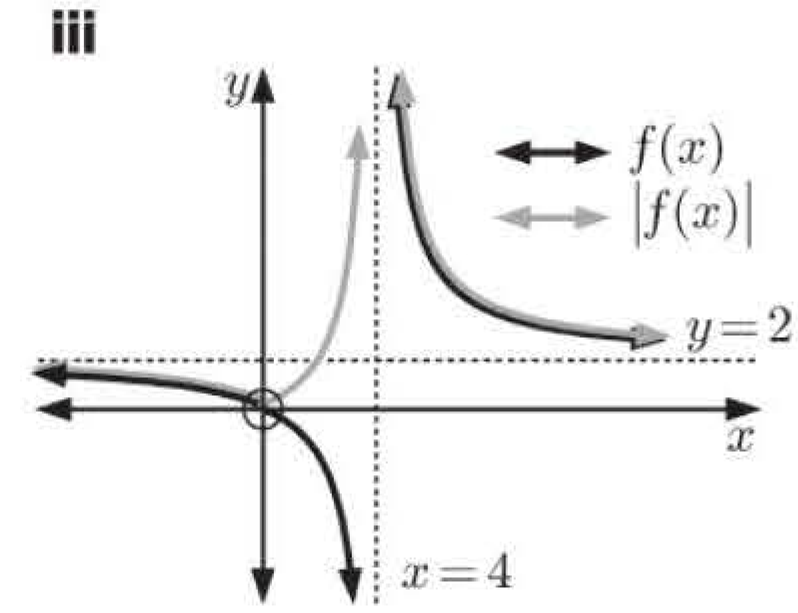
3 **a** **i**



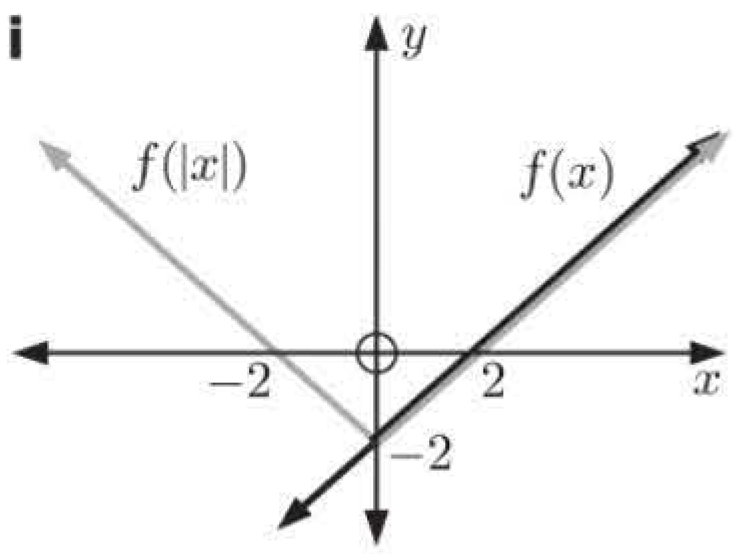
ii



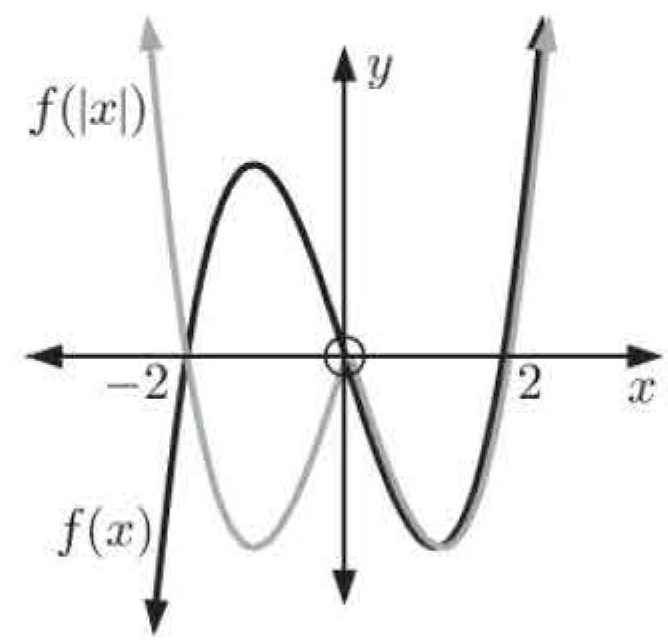
iii



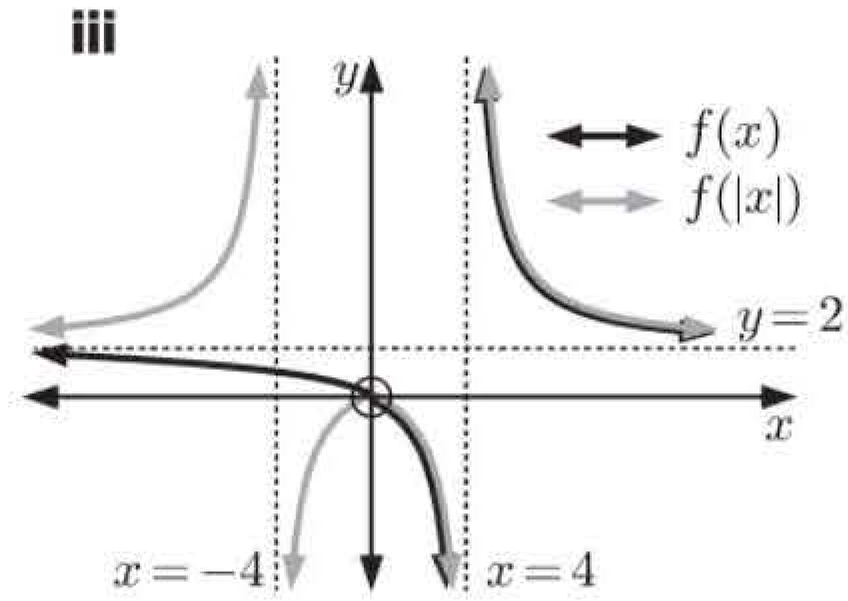
b **i**



ii



iii

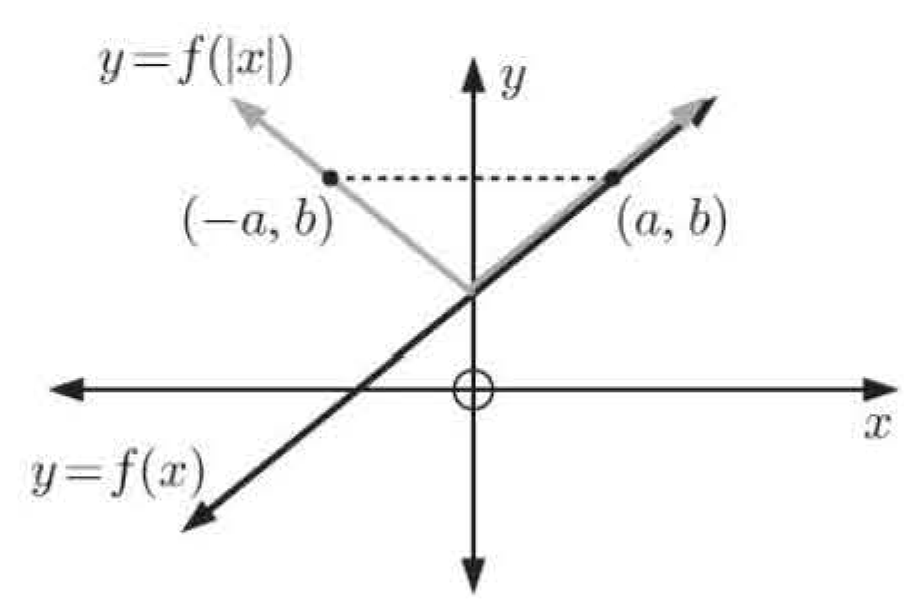


4 To transform $f(x)$ to $|f(x)|$, the point $(a,\ b)$ on $f(x)$ is transformed to $(a,\ |b|)$.

a $(3,\ 0)$ is transformed to $(3,\ 0)$ **b** $(5,\ -2)$ is transformed to $(5,\ 2)$

c $(0,\ 7)$ is transformed to $(0,\ 7)$ **d** $(2,\ 2)$ is transformed to $(2,\ 2)$

5 **a** For any point $(a,\ b)$, $a \geqslant 0$, on $f(x)$, $(a,\ b)$ is also a point on $f(|x|)$, and $(a,\ b)$ is also transformed to $(-a,\ b)$.



i $(0,\ 3)$ is transformed to $(0,\ 3)$

ii $(1,\ 3)$ is transformed to $(1,\ 3)$ and $(-1,\ 3)$

iii $(7,\ -4)$ is transformed to $(7,\ -4)$ and $(-7,\ -4)$

b The point $(a,\ b)$ on $f(|x|)$ has been transformed by the point $(|a|,\ b)$ on $f(x)$.

i $(0,\ 3)$ has been transformed from $(0,\ 3)$

ii $(-1,\ 3)$ has been transformed from $(1,\ 3)$

iii $(10,\ -8)$ has been transformed from $(10,\ -8)$

REVIEW SET 5A

1 $f(x) = x^2 - 2x$

a
$$\begin{aligned} & f(3) \\ &= 3^2 - 2(3) \\ &= 9 - 6 \\ &= 3 \end{aligned}$$

b
$$\begin{aligned} & f(2x) \\ &= (2x)^2 - 2(2x) \\ &= 4x^2 - 4x \end{aligned}$$

c
$$\begin{aligned} & f(-x) \\ &= (-x)^2 - 2(-x) \\ &= x^2 + 2x \end{aligned}$$

d
$$\begin{aligned} & 3\,f(x) - 2 \\ &= 3(x^2 - 2x) - 2 \\ &= 3x^2 - 6x - 2 \end{aligned}$$

2 $f(x) = 5 - x - x^2$

a
$$\begin{aligned} f(-1) &= 5 - (-1) - (-1)^2 \\ &= 5 + 1 - 1 \\ &= 5 \end{aligned}$$

b
$$\begin{aligned} f(x-1) &= 5 - (x-1) - (x-1)^2 \\ &= 5 - x + 1 - [x^2 - 2x + 1] \\ &= 6 - x - x^2 + 2x - 1 \\ &= -x^2 + x + 5 \end{aligned}$$

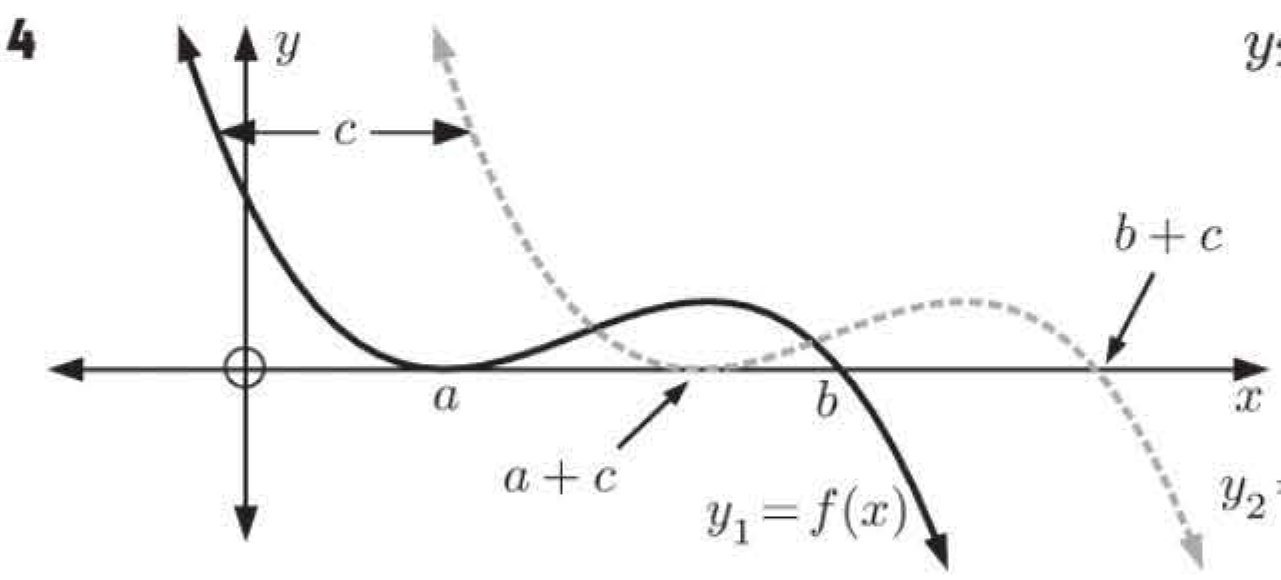
c
$$\begin{aligned} f\left(\tfrac{x}{2}\right) &= 5 - \left(\tfrac{x}{2}\right) - \left(\tfrac{x}{2}\right)^2 \\ &= 5 - \tfrac{1}{2}x - \tfrac{1}{4}x^2 \end{aligned}$$

d
$$\begin{aligned} & 2\,f(x) - f(-x) \\ &= 2(5 - x - x^2) - \left[5 - (-x) - (-x)^2\right] \\ &= 10 - 2x - 2x^2 - [5 + x - x^2] \\ &= 10 - 2x - 2x^2 - 5 - x + x^2 \\ &= -x^2 - 3x + 5 \end{aligned}$$

3 $f(x) = 3x^3 - 2x^2 + x + 2$

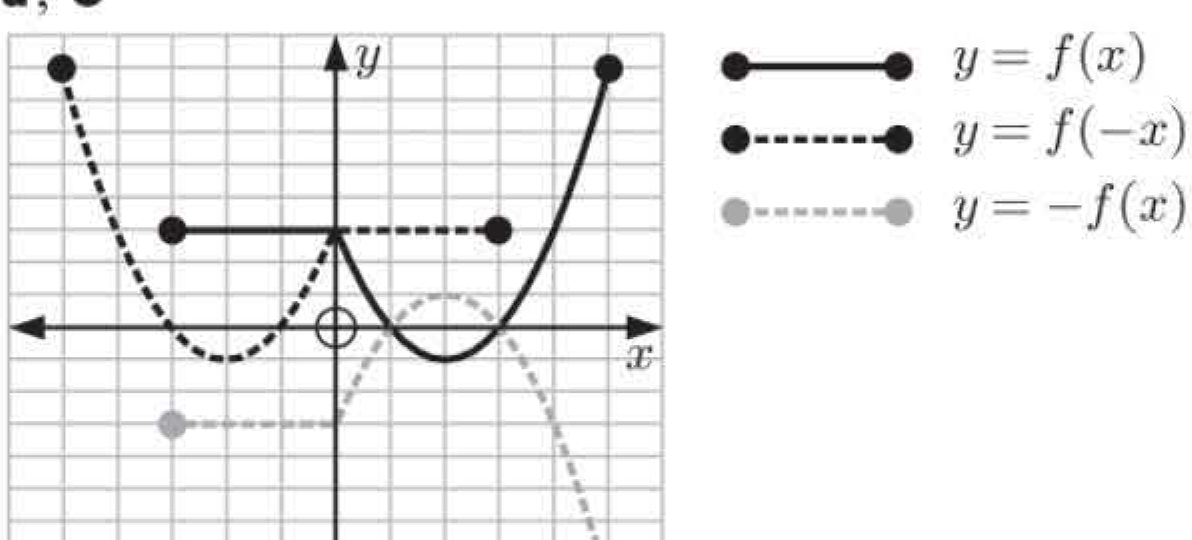
If $g(x)$ is $f(x)$ translated $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$, then
$$\begin{aligned} g(x) &= f(x-1) - 2 \\ &= 3(x-1)^3 - 2(x-1)^2 + (x-1) + 2 - 2 \\ &= 3(x^3 - 3x^2 + 3x - 1) - 2(x^2 - 2x + 1) + x - 1 \\ &= 3x^3 - 9x^2 + 9x - 3 - 2x^2 + 4x - 2 + x - 1 \\ &= 3x^3 - 11x^2 + 14x - 6 \end{aligned}$$

4



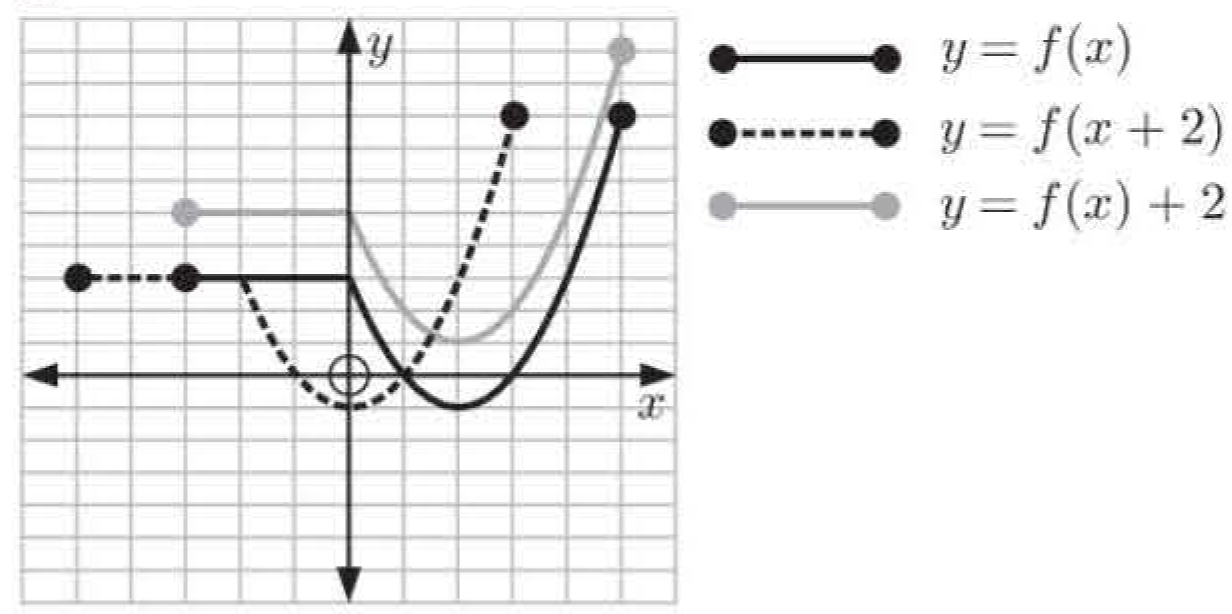
y_2 is obtained by translating y_1 c units to the right.

5 **a**, **b**

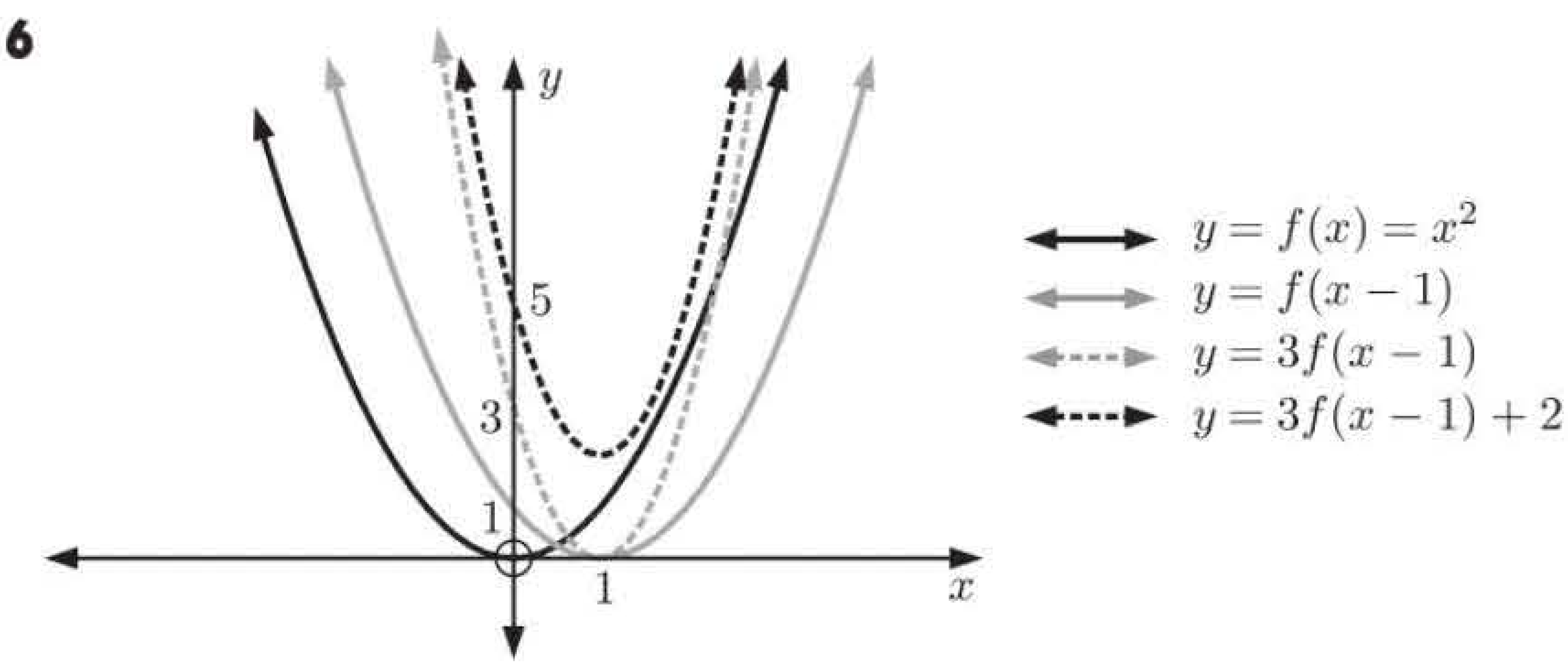


(drawn on two graphs)

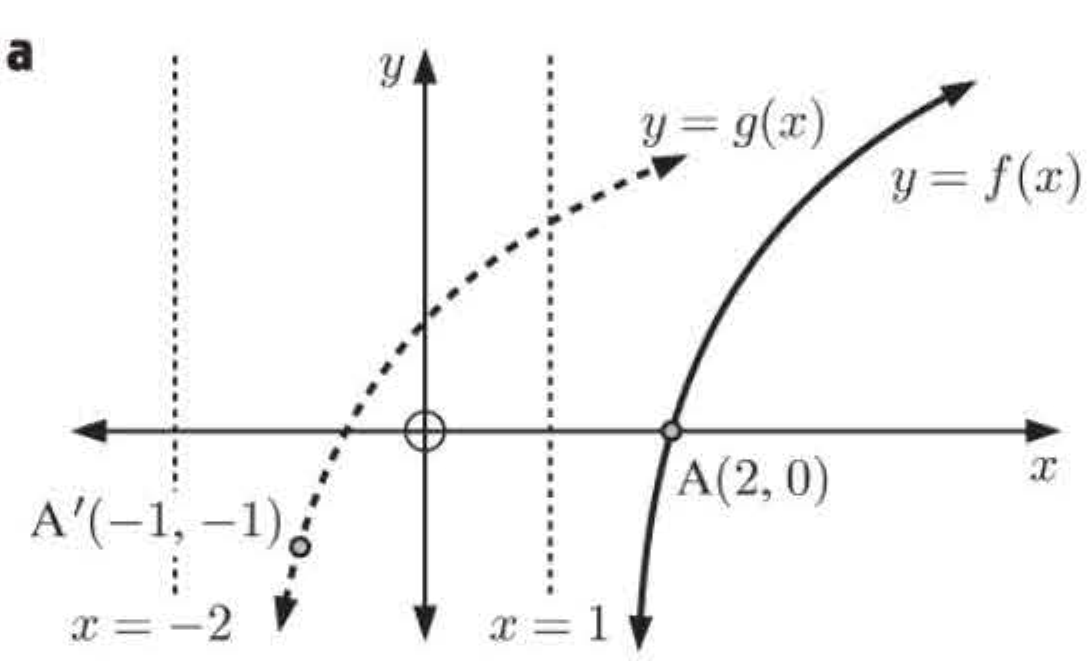
c, **d**



6



7 **a**



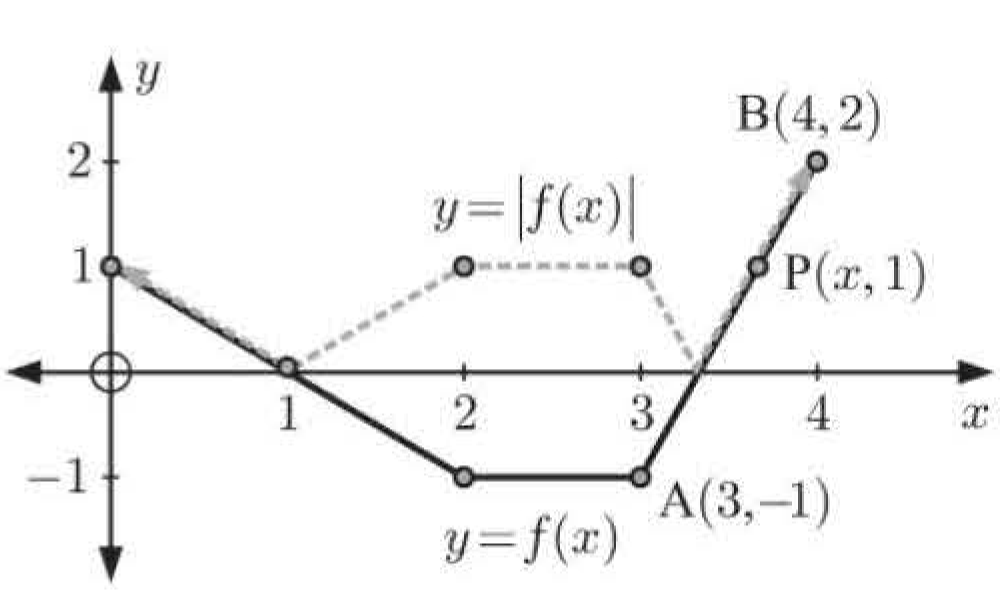
b $f(x+3) - 1$ is a translation of $f(x)$ by $\begin{pmatrix} -3 \\ -1 \end{pmatrix}$.

$\therefore$ vertical asymptote is at $x = 1 - 3 = -2$.

c A(2, 0) translated by $\begin{pmatrix} -3 \\ -1 \end{pmatrix}$ gives

$(2-3,\ 0-1)$ which is A′$(-1, -1)$.

8 **a**



b When $x = 0$,

$$\begin{aligned} \frac{1}{f(x)} &= \frac{1}{f(0)} \\ &= \tfrac{1}{1} \\ &= 1 \end{aligned}$$

$\therefore$ the y-intercept of $\dfrac{1}{f(x)}$ is 1.

c Invariant points for $\dfrac{1}{f(x)}$ occur when $f(x) = \pm 1$.

$f(x) = -1$ for all $x \in [2, 3]$

$f(x) = 1$ when $x = 0$ and at point P.

To find the point P where $f(x) = 1$,

note that the gradient of $[AB] = \dfrac{2-(-1)}{4-3} = 3$,

so $\dfrac{2-1}{4-x} = 3$

$\therefore \quad 1 = 12 - 3x$

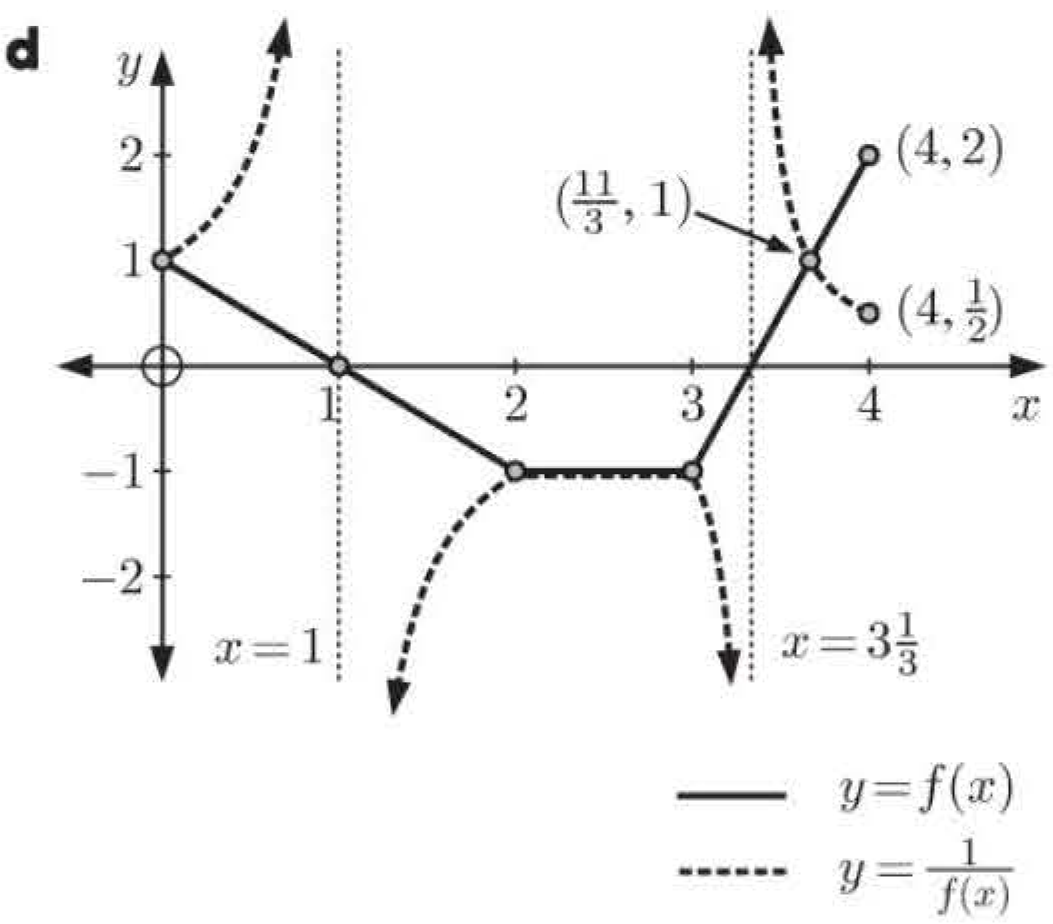
$\therefore \quad 3x = 11$

$\therefore \quad x = \frac{11}{3}$

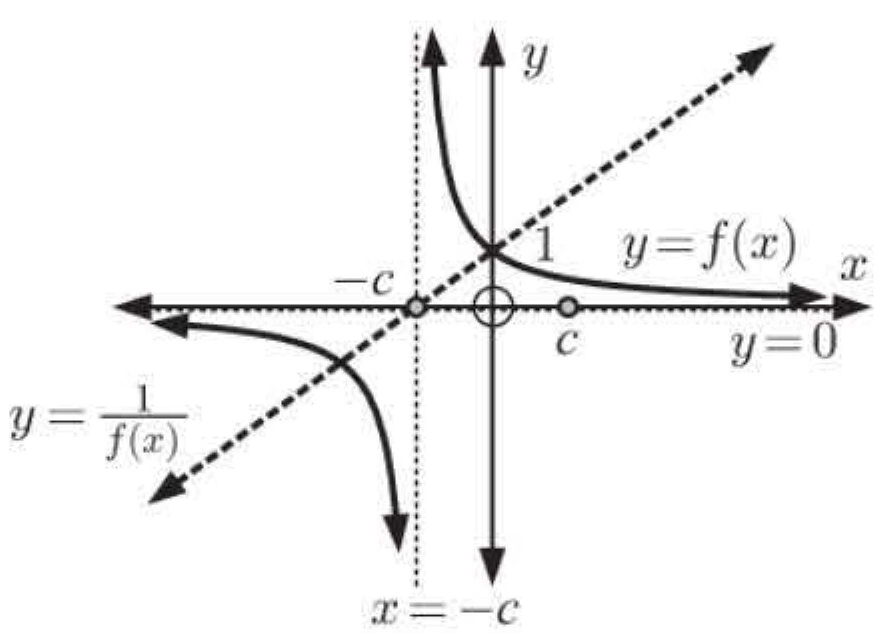
$\therefore \quad f(x)$ is invariant for $\dfrac{1}{f(x)}$ at $(0, 1)$, $(\frac{11}{3}, 1)$,

and all the points on $y = -1$, $x \in [2, 3]$.

d



9



a $f(x) = \dfrac{c}{x+c}$ has a VA $x = -c$ $\{f(x)$ is undefined$\}$

and a HA $y = 0$ $\{$as $|x| \to \infty$, $f(x) \to 0\}$

$f(0) = \dfrac{c}{0+c} = 1$ $\therefore$ the y-intercept is 1

There are no x-intercepts $\{\dfrac{c}{x+c} \neq 0$ for $c > 0\}$

b $\dfrac{1}{f(x)} = \dfrac{x+c}{c}$

$\dfrac{1}{f(0)} = \dfrac{0+c}{c} = 1$ $\therefore$ the y-intercept of $\dfrac{1}{f(x)}$ is 1

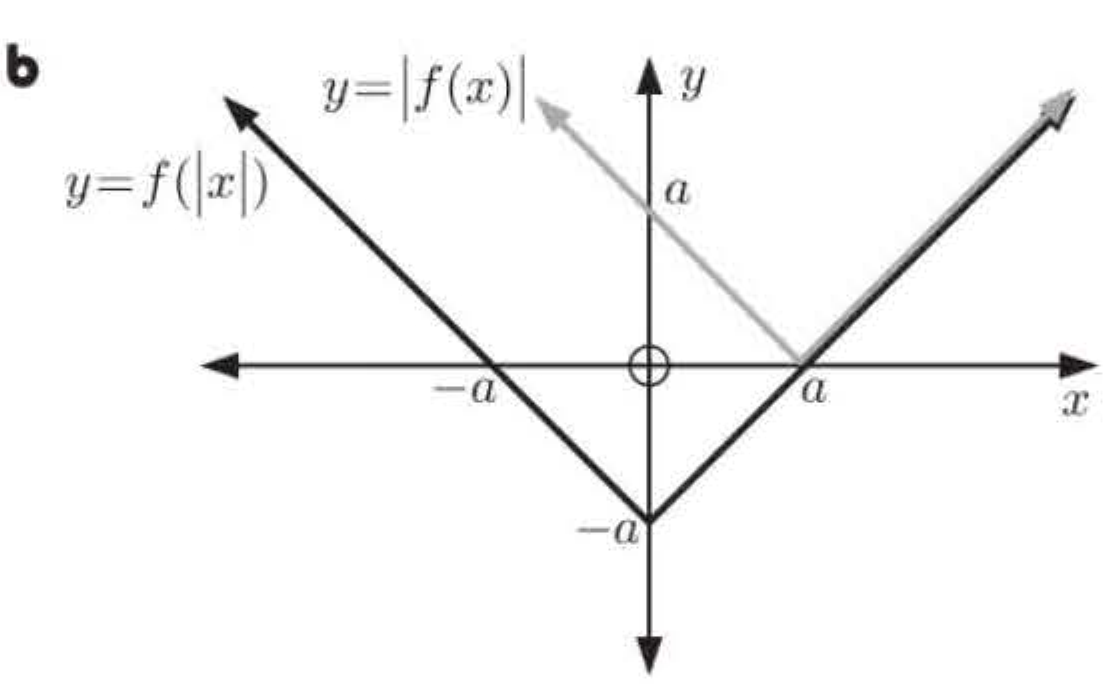
$\dfrac{1}{f(x)} = 0$ when $x + c = 0$ or $x = -c$

$\therefore$ the x-intercept of $\dfrac{1}{f(x)}$ is $-c$.

10 **a** $f(x) = x - a, \quad a > 0$

$\therefore \quad |f(x)| = |x - a|$ and $f(|x|) = |x| - a$

b



c Using the graph in **b**, $|x - a| = |x| - a$ when $x \geqslant a$.

or solving algebraically:

For $x < 0$ and $a > 0$, $|x - a| = a - x$ and $|x| = -x$

If $|x - a| = |x| - a$ then $a - x = -x - a$

$\therefore \quad 2a = 0$

$\therefore \quad a = 0$ which is not true.

For $0 \leqslant x < a$ and $a > 0$, $|x-a| = a-x$ and $|x| = x$

If $|x-a| = |x| - a$ then $a - x = x - a$

$\therefore\ 2x = 2a$

$\therefore\ x = a$ which is not true.

For $x \geqslant a$ and $a > 0$, $|x-a| = x-a$ and $|x| = x$

If $|x-a| = |x| - a$ then $x - a = x - a$ which is true.

So, for $a > 0$, $|x-a| = |x| - a$ is true for all $x \geqslant a$.

REVIEW SET 5B

1 When $y = 0$, $(x+1)^2 - 4 = 0$

$\therefore\ (x+1)^2 = 4$

$\therefore\ x + 1 = \pm 2$

$\therefore\ x = 2 - 1$ or $-2 - 1$

$\therefore\ x = 1$ or -3

When $x = 0$, $y = 1^2 - 4$

$= -3$

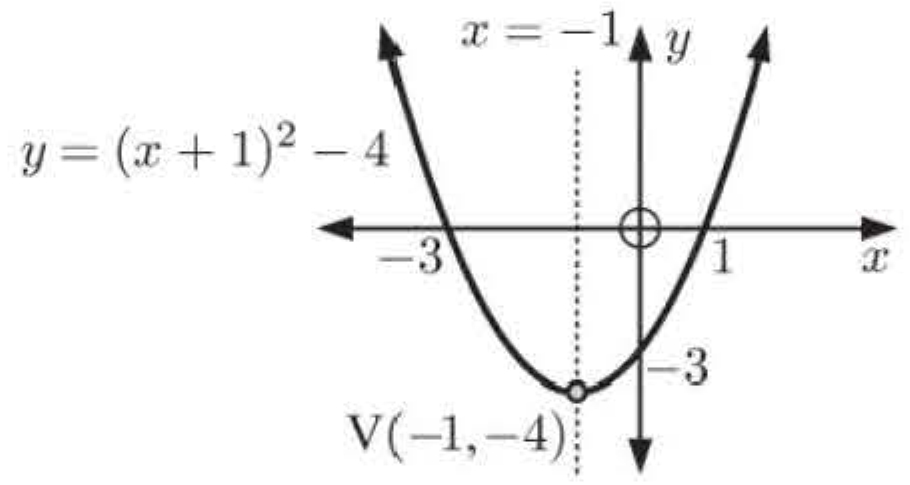
$\therefore$ y-intercept is -3

$\therefore$ x-intercepts are $1, -3$

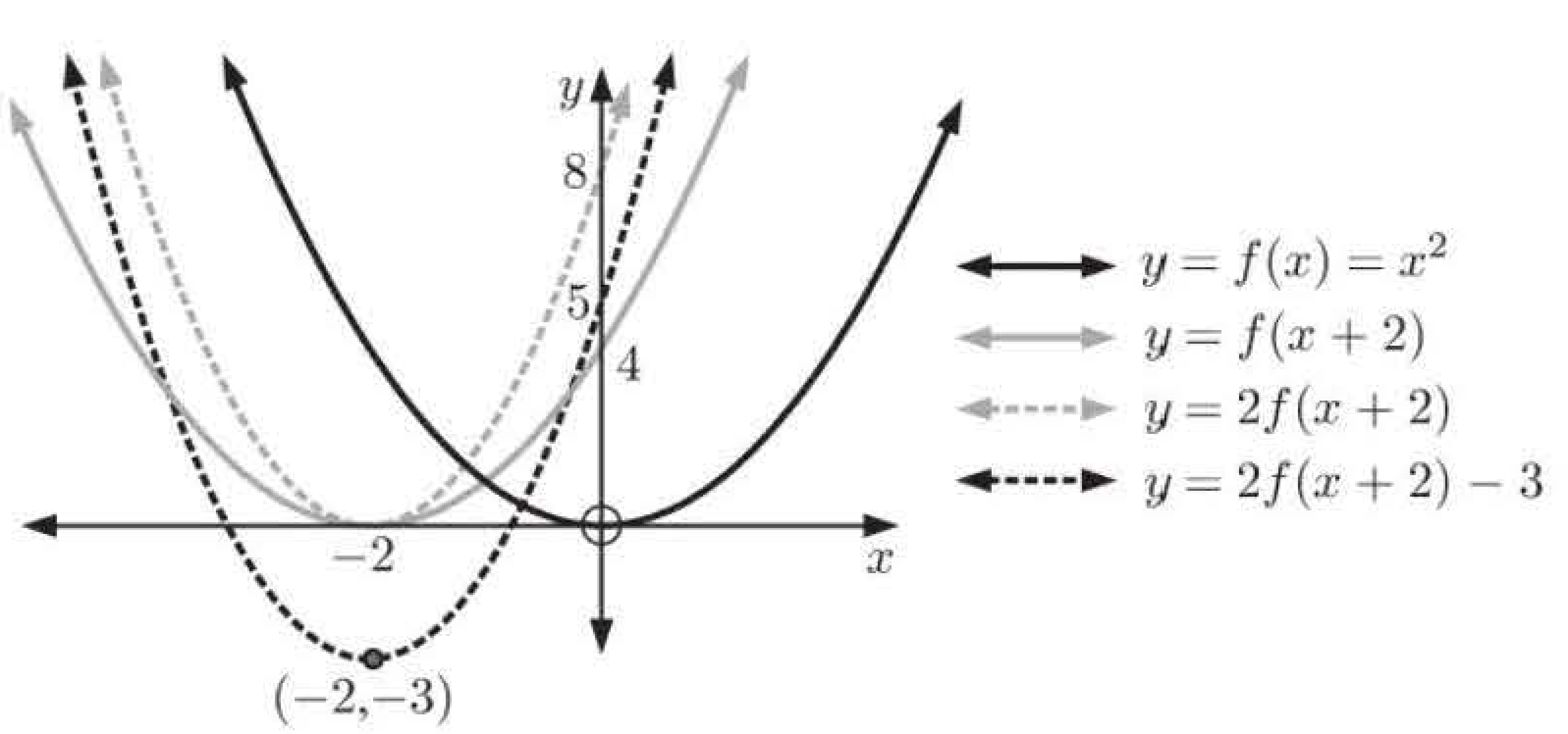
$y = (x+1)^2 - 4$ is obtained from $y = x^2$ under a translation of $\begin{pmatrix} -1 \\ -4 \end{pmatrix}$.

$y = x^2$ has its vertex at $(0, 0)$, so the vertex of $y = (x+1)^2 - 4$ must be $(-1, -4)$.

So, the graph of $y = f(x)$ is:



2



3 **a** The function does not have any axes intercepts.

b As $x \to 0^-$, $y \to -\infty$

As $x \to 0^+$, $y \to \infty$

$\therefore$ the vertical asymptote is $x = 0$.

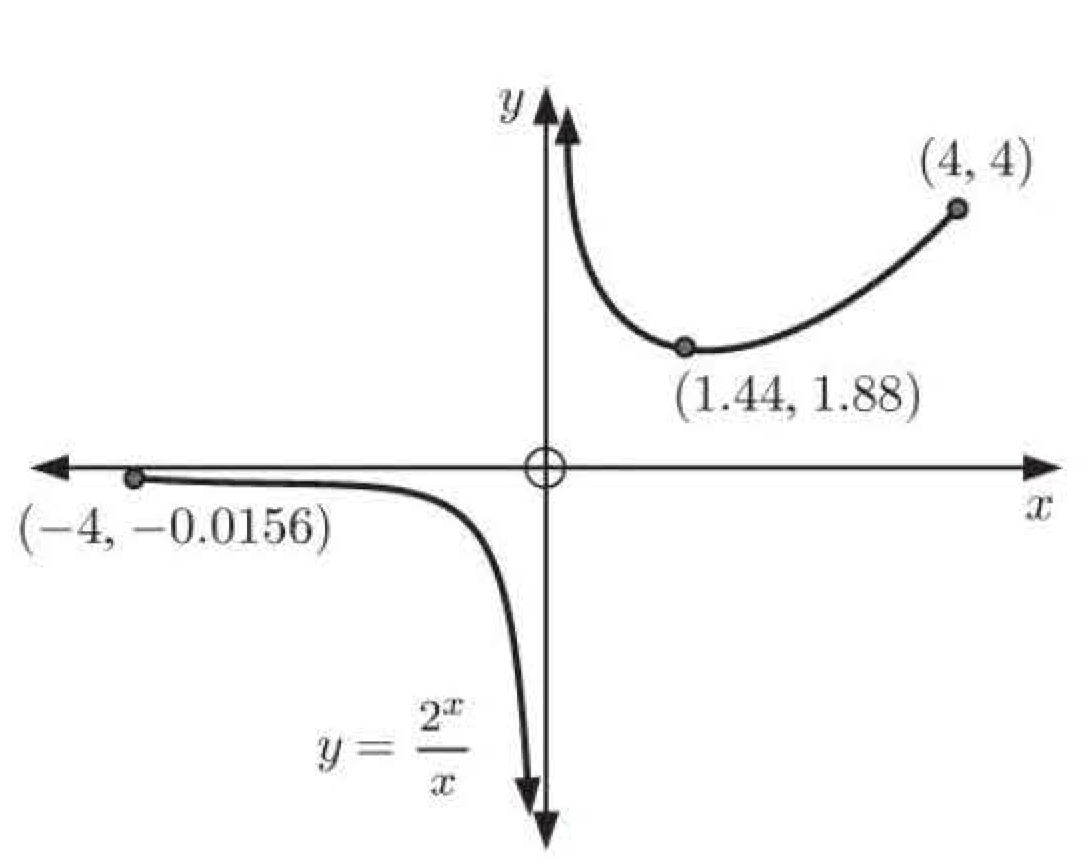
As $x \to \infty$, $y \to \infty$

As $x \to -\infty$, $y \to 0^-$

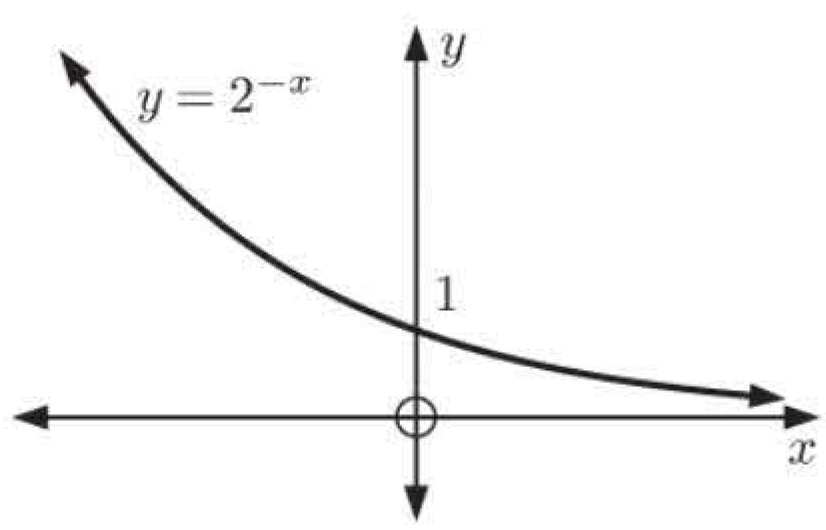
$\therefore$ the horizontal asymptote is $y = 0$.

c There is a local minimum at $(1.44, 1.88)$.

d



4 **a** Graph $y = f(x)$ using a graphics calculator:



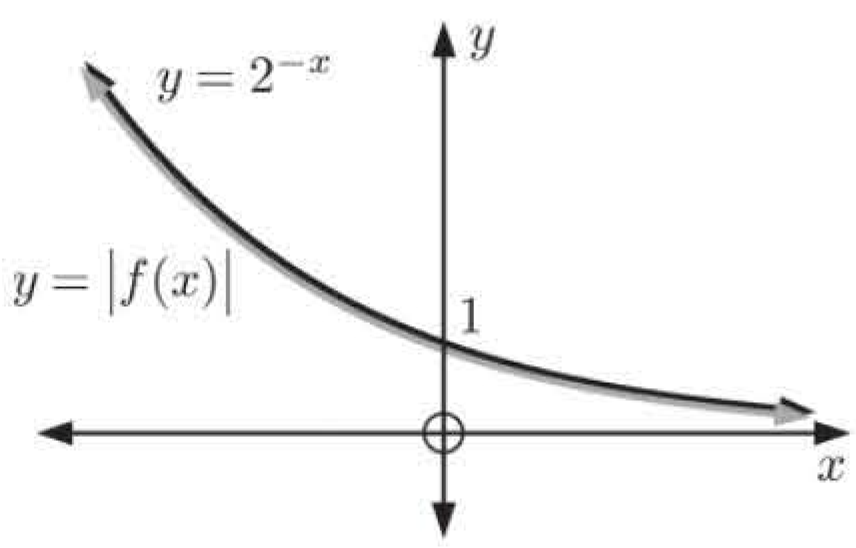
i $x \to \infty$ means x is very large and positive.
We see the graph approaching the x-axis.
$\therefore \ y \to 0 \quad \therefore$ **true**.

ii $x \to -\infty$ means x is very large and negative.
We see the graph heading for ∞. $\therefore$ **false**.

iii When $x = 0$, $y = 2^0 = 1 \neq \frac{1}{2} \quad \therefore$ **false**.

iv The graph is above the x-axis for all x.
$\therefore \ 2^{-x} > 0$ for all $x \quad \therefore$ **true**.

b

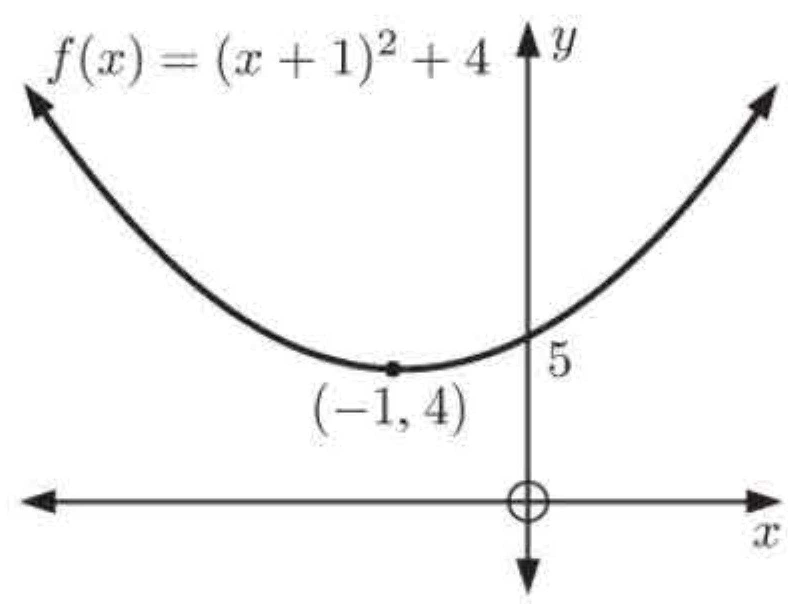


c $y = |f(x)|$ has horizontal asymptote $y = 0$.

5 **a** $f(x) = (x+1)^2 + 4$ is translated by $\binom{2}{4}$ to get $g(x)$.

$$\begin{aligned}\therefore \ g(x) &= f(x-2) + 4 \\ &= \left[((x-2)+1)^2 + 4\right] + 4 \\ &= (x-1)^2 + 8\end{aligned}$$

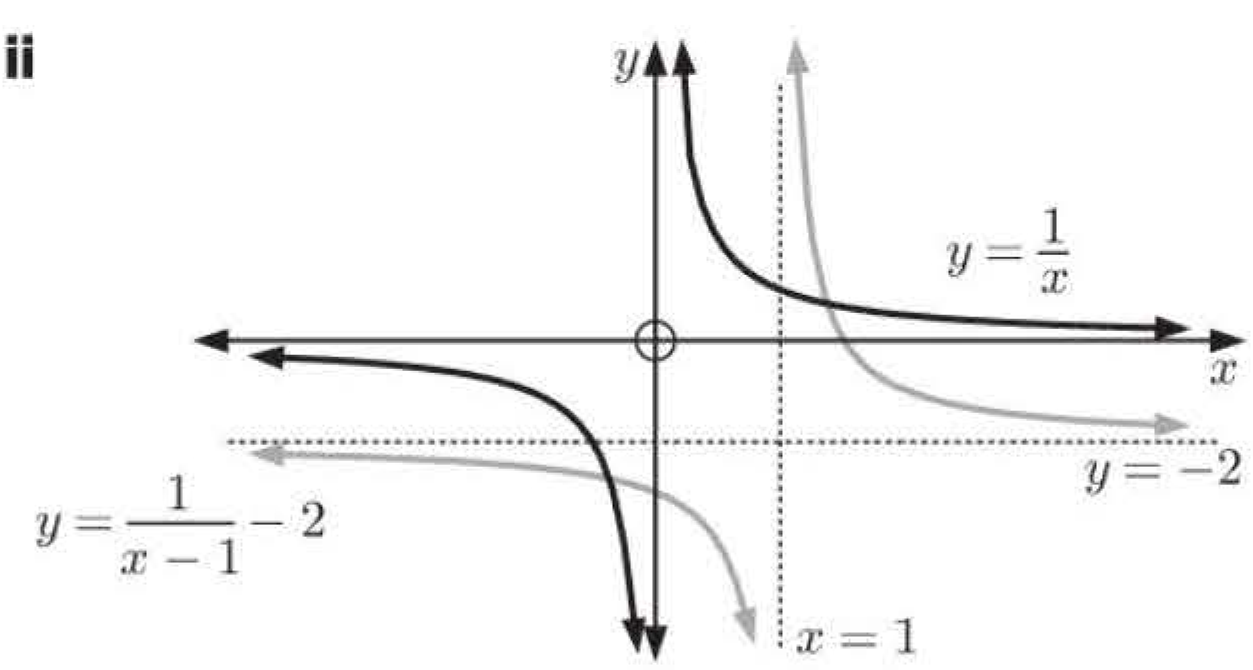
b We graph the function using technology, and from this we can see that the range is $\{y \mid y \geqslant 4\}$.



c $g(x)$ is $f(x)$ translated by $\binom{2}{4}$, so the minimum value of $g(x)$ is $4 + 4 = 8$.
$\therefore$ the range of $g(x)$ is $\{y \mid y \geqslant 8\}$.

6 **a** **i** Under translation $\binom{1}{-2}$, $y = \dfrac{1}{x}$ becomes $y = \dfrac{1}{x-1} - 2$

ii



For $y = \dfrac{1}{x}$, V.A. is $x = 0$, H.A. is $y = 0$.

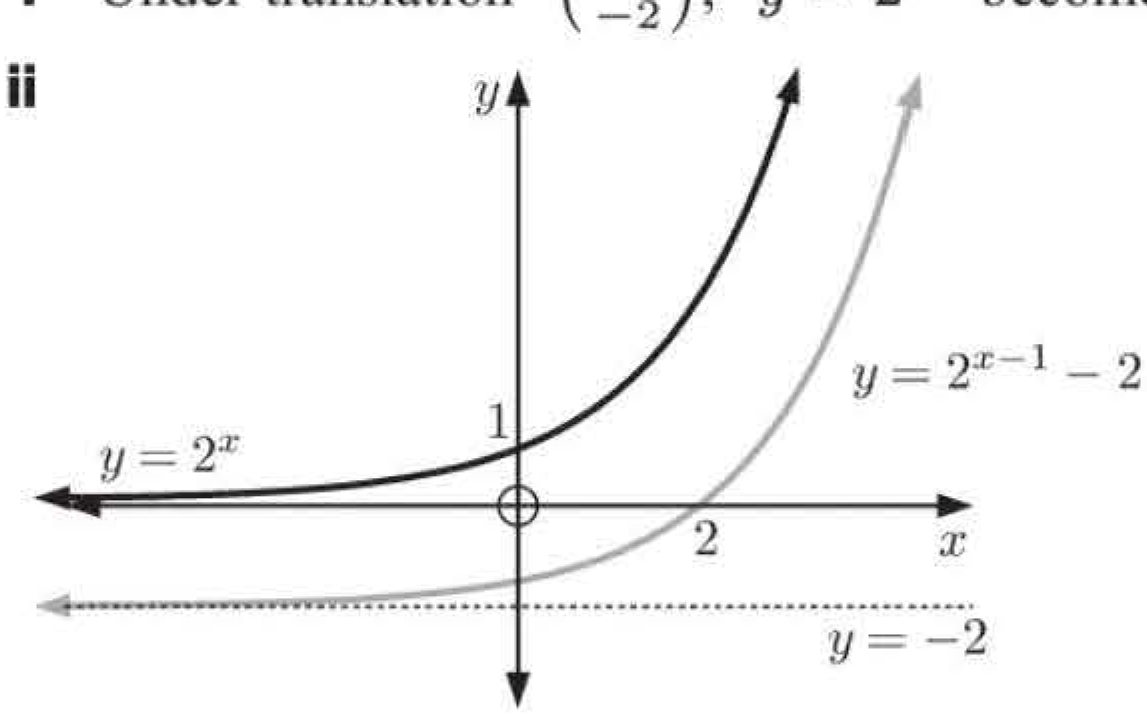
For $y = \dfrac{1}{x-1} - 2$, V.A. is $x = 1$, H.A. is $y = -2$.

iii For $y = \dfrac{1}{x}$, domain is $\{x \mid x \neq 0\}$, range is $\{y \mid y \neq 0\}$.

For $y = \dfrac{1}{x-1} - 2$, domain is $\{x \mid x \neq 1\}$, range is $\{y \mid y \neq -2\}$.

b **i** Under translation $\binom{1}{-2}$, $y = 2^x$ becomes $y = 2^{x-1} - 2$

ii



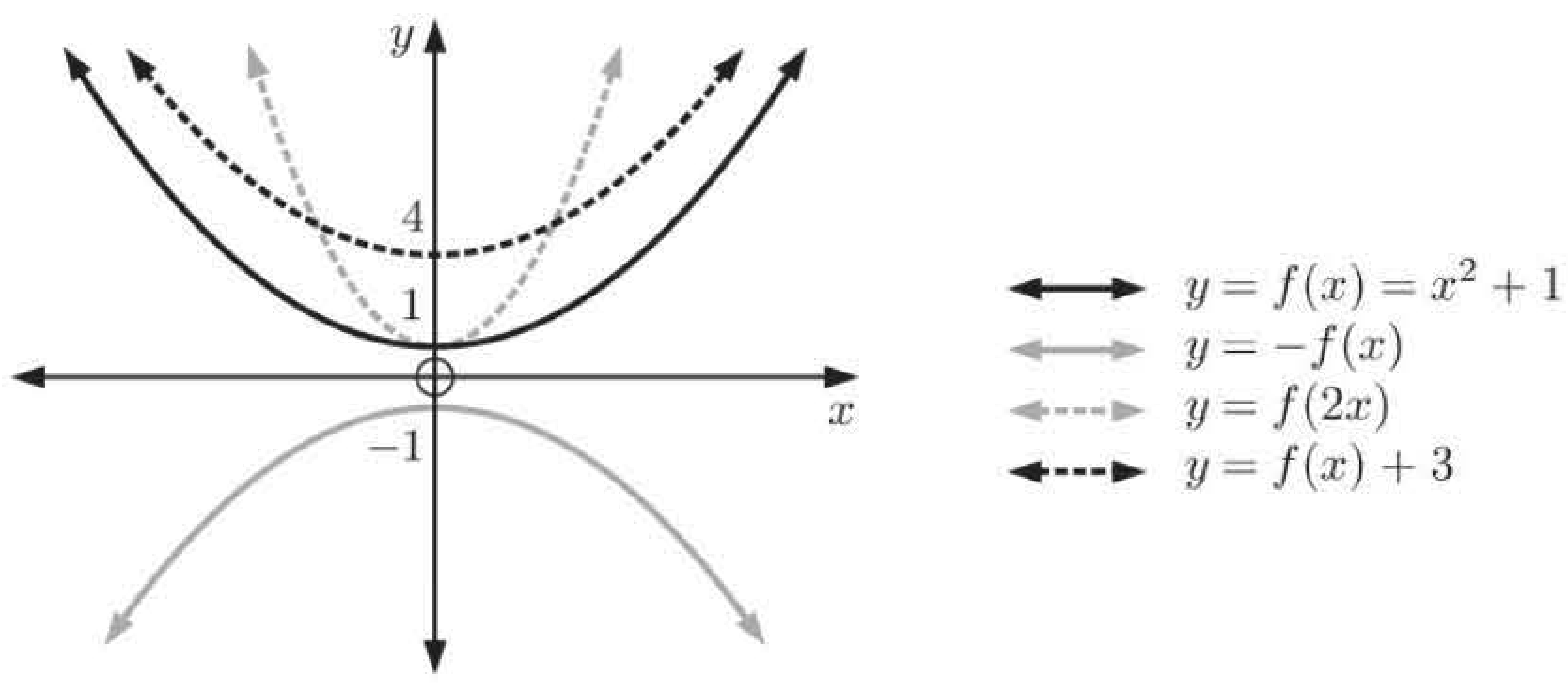
For $y = 2^x$, H.A. is $y = 0$, no V.A.

For $y = 2^{x-1} - 2$, H.A. is $y = -2$, no V.A.

iii For $y = 2^x$, domain is $\{x \mid x \in \mathbb{R}\}$, range is $\{y \mid y > 0\}$.

For $y = 2^{x-1} - 2$, domain is $\{x \mid x \in \mathbb{R}\}$, range is $\{y \mid y > -2\}$.

7



8 **a** $f(x) = x + 2$

stretching $f(x)$ vertically with scale factor 2 becomes $2\,f(x) = 2x + 4$

stretching the function horizontally with scale factor $\frac{1}{2}$ becomes $2\,f(2x) = 2(2x) + 4$
$= 4x + 4$

translating $\frac{1}{2}$ horizontally and -3 vertically, the function becomes $4(x - \frac{1}{2}) + 4 - 3$
$= 4x - 2 + 1$
$\therefore \quad F(x) = 4x - 1$

b $(1, 3) \to (1, 6) \to (\frac{1}{2}, 6) \to (1, 6) \to (1, 3)$

$\therefore$ $(1, 3)$ is an invariant point under the transformation.

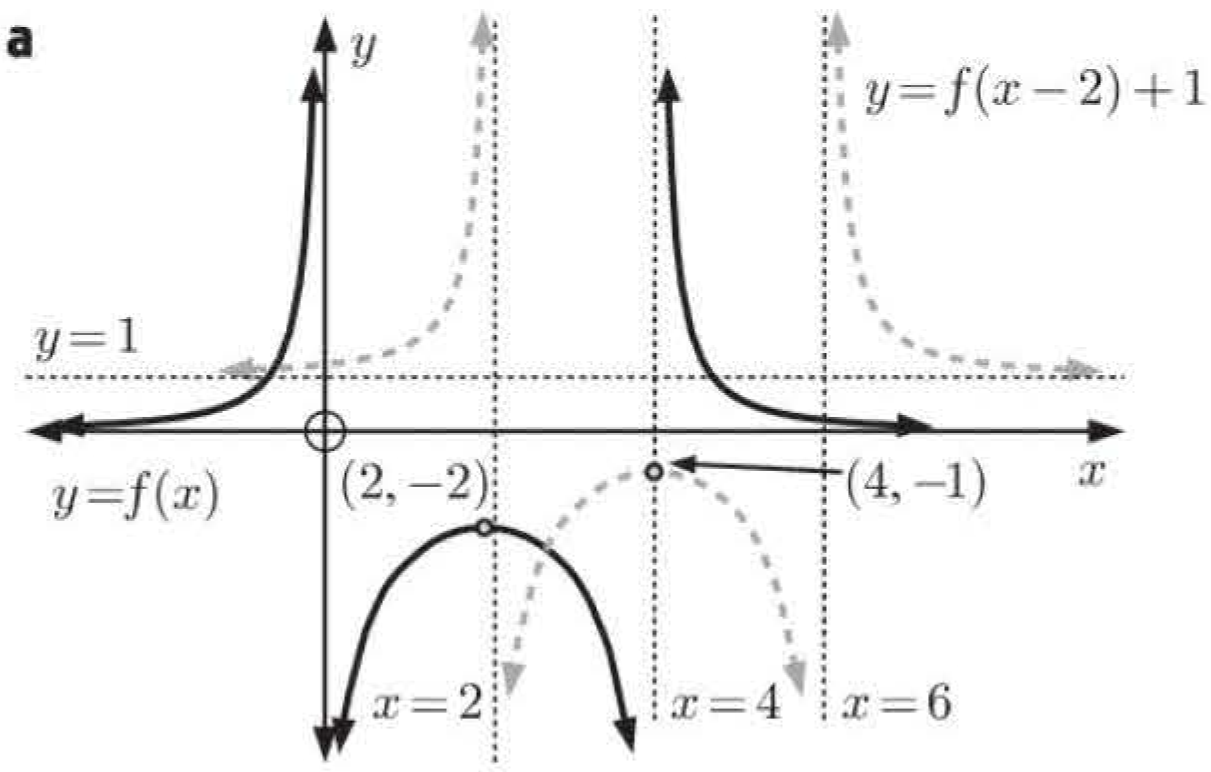
c $(0, 2) \to (0, 4) \to (0, 4) \to (\frac{1}{2}, 4) \to (\frac{1}{2}, 1)$ $\therefore$ $(0, 2)$ transforms to $(\frac{1}{2}, 1)$.

$(-1, 1) \to (-1, 2) \to (-\frac{1}{2}, 2) \to (0, 2) \to (0, -1)$ $\therefore$ $(-1, 1)$ transforms to $(0, -1)$.

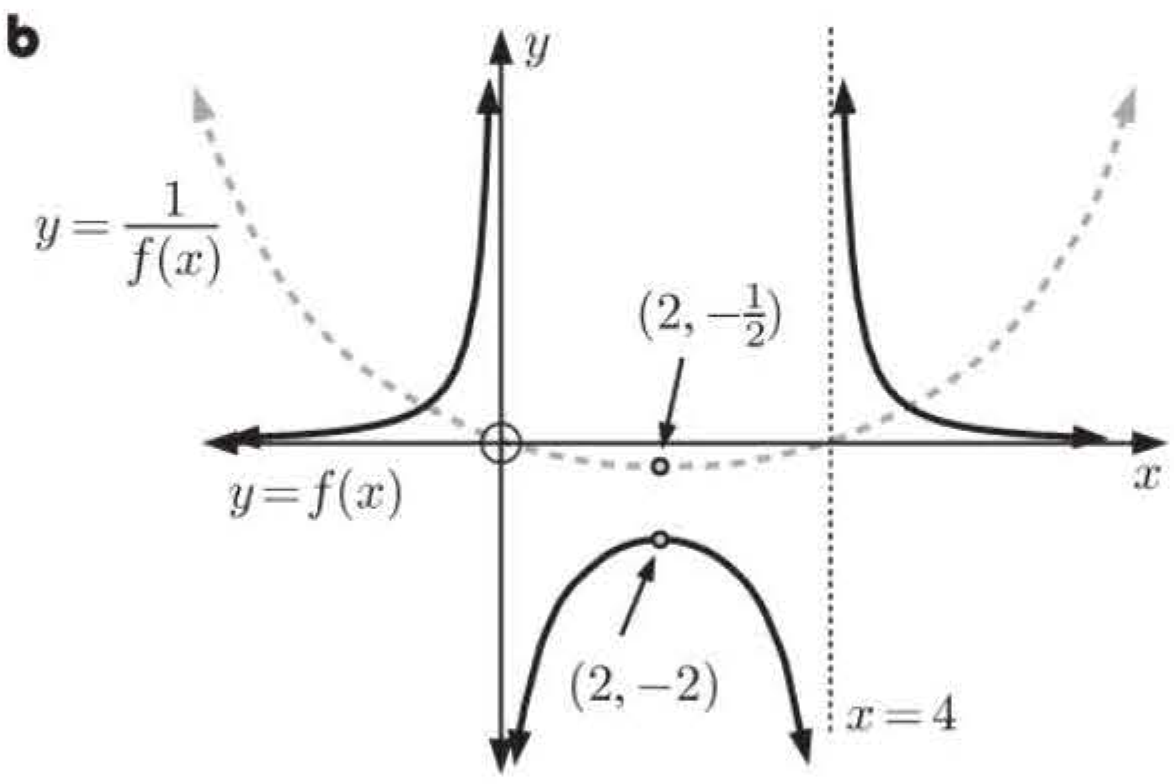
d When $x = \frac{1}{2}$, $F(x) = 4(\frac{1}{2}) - 1 = 1$ $\therefore$ $(\frac{1}{2}, 1)$ lies on $F(x)$.

When $x = 0$, $F(x) = 4(0) - 1 = -1$ $\therefore$ $(0, -1)$ lies on $F(x)$.

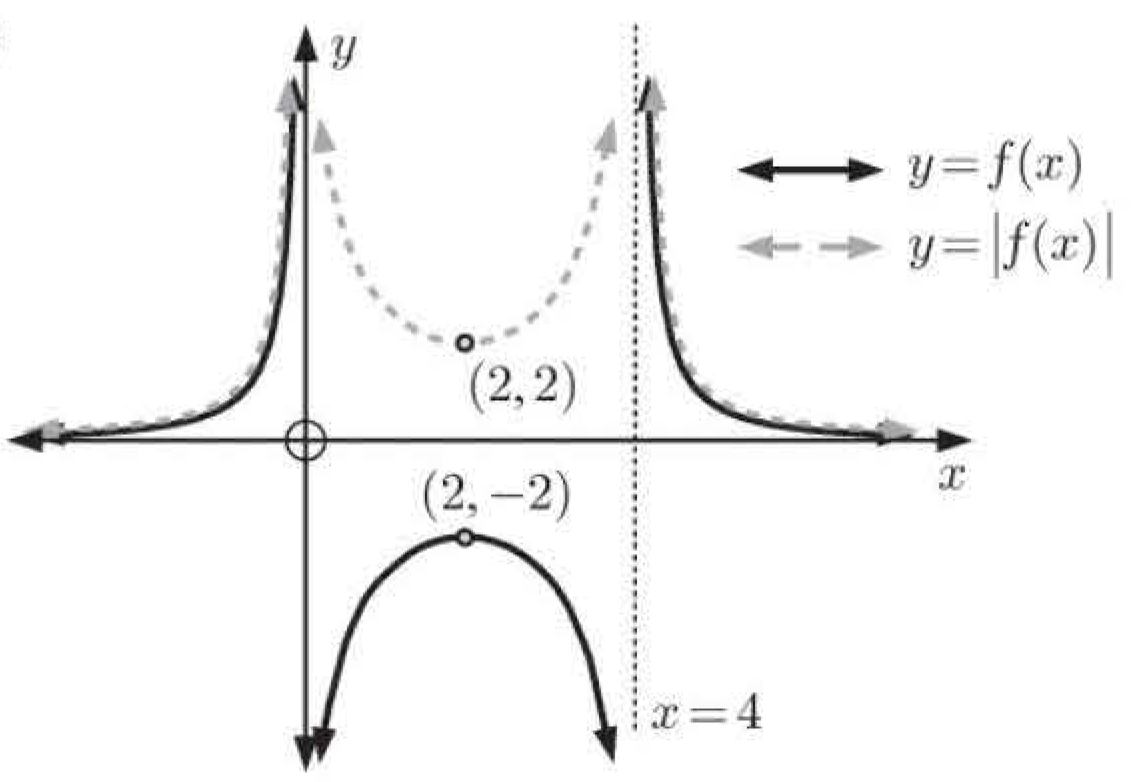
9 **a**



b



c



10 **a** $f(x) = \dfrac{2x-3}{3x+5}$

$= \dfrac{\frac{2}{3}x - 1}{x + \frac{5}{3}}$

$= \dfrac{\frac{2}{3}(x+\frac{5}{3}) - \frac{19}{9}}{x+\frac{5}{3}}$

$= -\dfrac{\frac{19}{9}}{x+\frac{5}{3}} + \frac{2}{3}$

$y = f(x)$ is a translation of $y = -\dfrac{\frac{19}{9}}{x}$ through $\begin{pmatrix} -\frac{5}{3} \\ \frac{2}{3} \end{pmatrix}$.

Now $y = -\dfrac{\frac{19}{9}}{x}$ has asymptotes $x = 0$ and $y = 0$.

$\therefore$ $y = f(x)$ has vertical asymptote $x = -\frac{5}{3}$ and horizontal asymptote $y = \frac{2}{3}$.

b As $x \to -\frac{5}{3}^-$, $y \to \infty$.

As $x \to -\frac{5}{3}^+$, $y \to -\infty$.

As $x \to -\infty$, $y \to \frac{2}{3}^+$.

As $x \to \infty$, $y \to \frac{2}{3}^-$.

c When $x = 0$, $y = -\frac{3}{5}$.

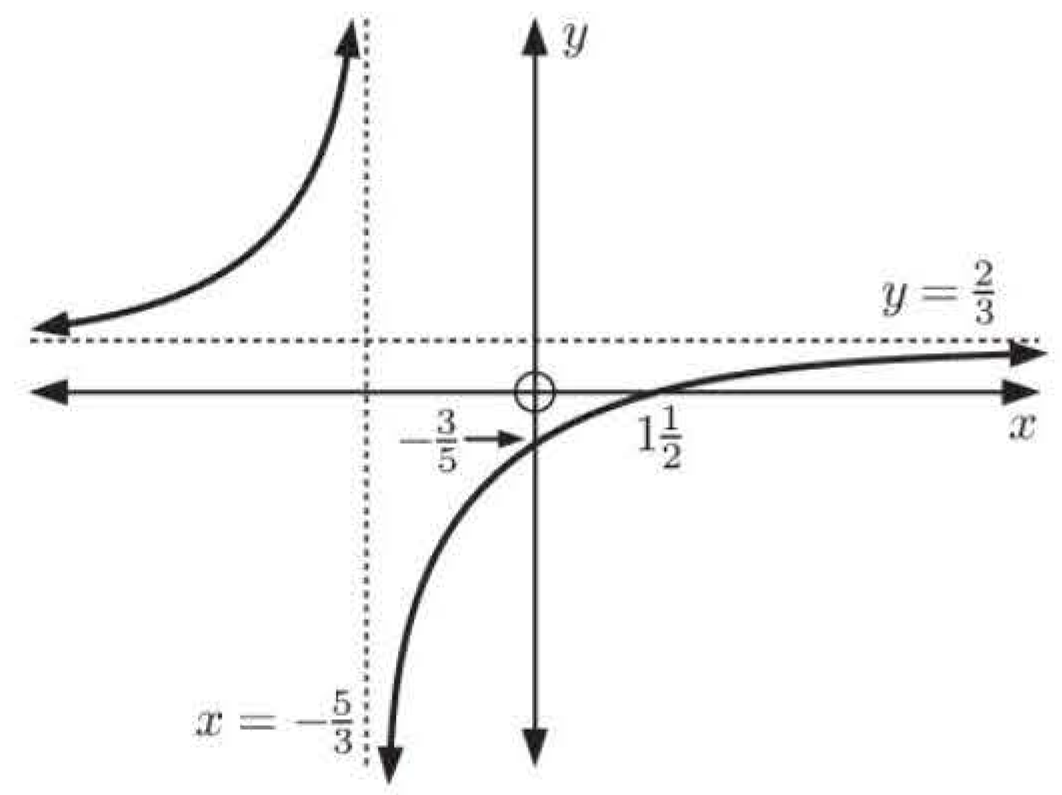
$\therefore$ the y-intercept is $-\frac{3}{5}$.

When $y = 0$, $2x - 3 = 0$

$\therefore$ $x = \frac{3}{2}$

$\therefore$ the x-intercept is $\frac{3}{2}$.

d



e $\dfrac{1}{x}$ becomes $\dfrac{\frac{19}{9}}{x}$ under a vertical stretch with scale factor $\frac{19}{9}$.

$\dfrac{\frac{19}{9}}{x}$ becomes $-\dfrac{\frac{19}{9}}{x}$ under a reflection in the x-axis.

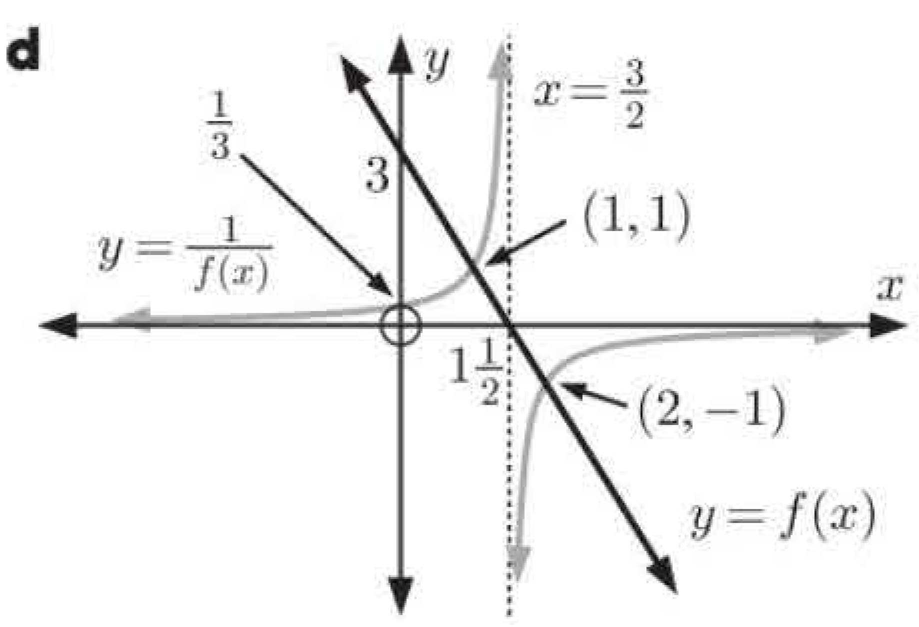
$-\dfrac{\frac{19}{9}}{x}$ becomes $-\dfrac{\frac{19}{9}}{x+\frac{5}{3}} + \frac{2}{3}$ under a translation through $\begin{pmatrix} -\frac{5}{3} \\ \frac{2}{3} \end{pmatrix}$.

So, $y = \dfrac{1}{x}$ is transformed to $y = f(x)$ under a vertical stretch with scale factor $\frac{19}{9}$, followed by a reflection in the x-axis, followed by a translation through $\begin{pmatrix} -\frac{5}{3} \\ \frac{2}{3} \end{pmatrix}$.

f To transform $y = f(x)$ into $y = \dfrac{1}{x}$, we need to reverse the process in **v**.

We need a translation through $\begin{pmatrix} \frac{5}{3} \\ -\frac{2}{3} \end{pmatrix}$, followed by a reflection in the x-axis, followed by a vertical stretch with scale factor $\frac{9}{19}$.

11 **a, d**



$f(0) = -2(0) + 3 = 3$ $\therefore$ the y-intercept of $f(x)$ is 3

$f(x) = 0$ when $-2x + 3 = 0$

$\therefore$ $x = \frac{3}{2}$

$\therefore$ the x-intercept of $f(x)$ is $\frac{3}{2}$.

b The invariant points for $y = \dfrac{1}{f(x)}$ occur when $f(x) = \pm 1$

$f(x) = 1$ when $-2x + 3 = 1$

$\therefore$ $2x = 2$

$\therefore$ $x = 1$

$f(x) = -1$ when $-2x + 3 = -1$

$\therefore$ $2x = 4$

$\therefore$ $x = 2$

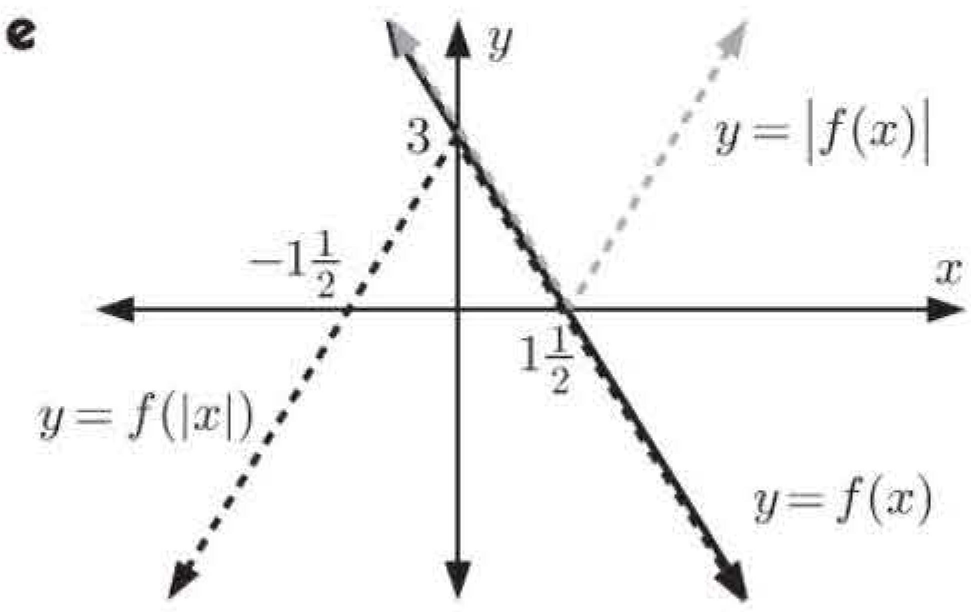
$\therefore$ the invariant points are $(1, 1)$ and $(2, -1)$.

c $\dfrac{1}{f(x)}$ is undefined when $f(x) = 0$

$\therefore$ the vertical asymptote of $y = \dfrac{1}{f(x)}$ is $x = \frac{3}{2}$

The y-intercept of $y = \dfrac{1}{f(x)}$ is $\dfrac{1}{f(0)} = \frac{1}{3}$

e



REVIEW SET 5C

1 $f(x) = \dfrac{4}{x}$

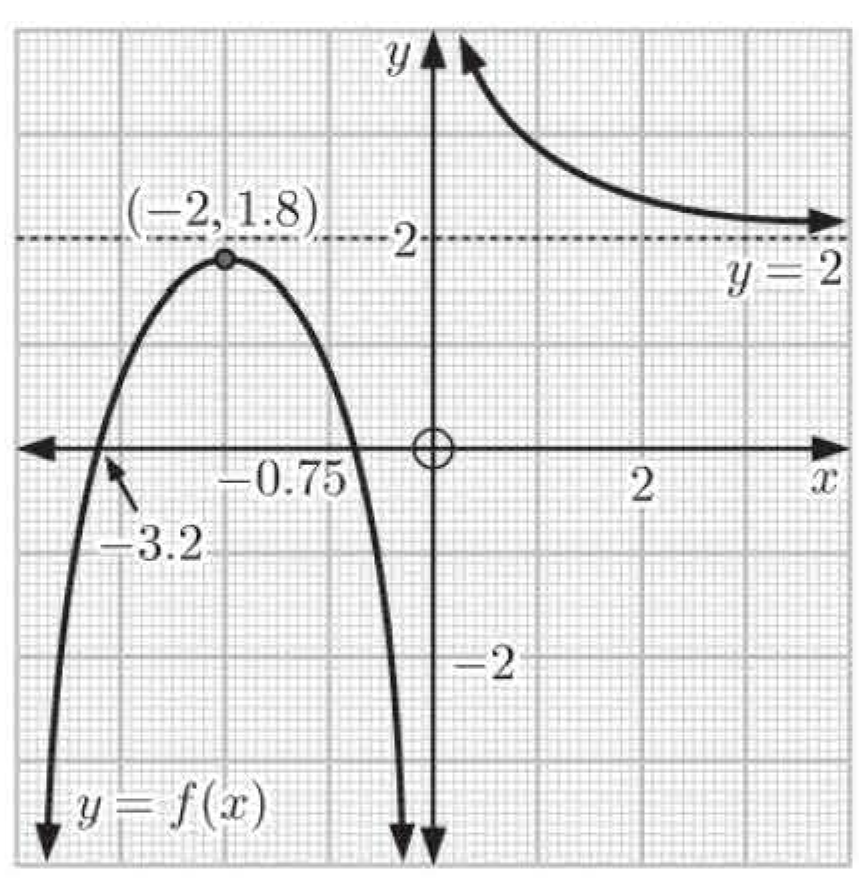
a $f(-4)$
$= \dfrac{4}{-4}$
$= -1$

b $f(2x)$
$= \dfrac{4}{2x}$
$= \dfrac{2}{x}$

c $f\left(\frac{x}{2}\right)$
$= \dfrac{4}{\frac{x}{2}}$
$= 4 \times \dfrac{2}{x}$
$= \dfrac{8}{x}$

d $4\,f(x+2) - 3$
$= 4\left(\dfrac{4}{x+2}\right) - 3$
$= \dfrac{16}{x+2} - 3$
$= \dfrac{16 - 3(x+2)}{x+2} = \dfrac{10 - 3x}{x+2}$

2 **a**



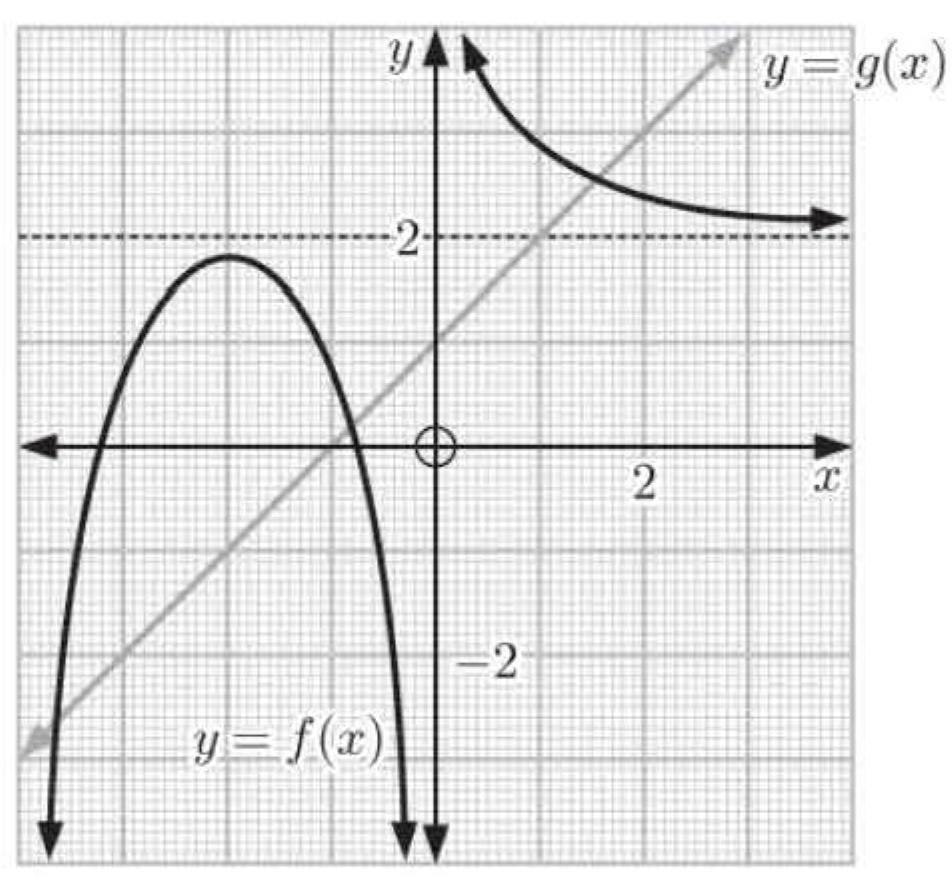
i The coordinates of the turning point are $(-2,\ 1.8)$.

ii The equation of the vertical asymptote is $x = 0$.

iii The equation of the horizontal asymptote is $y = 2$.

iv The x-intercepts are -3.2 and -0.75.

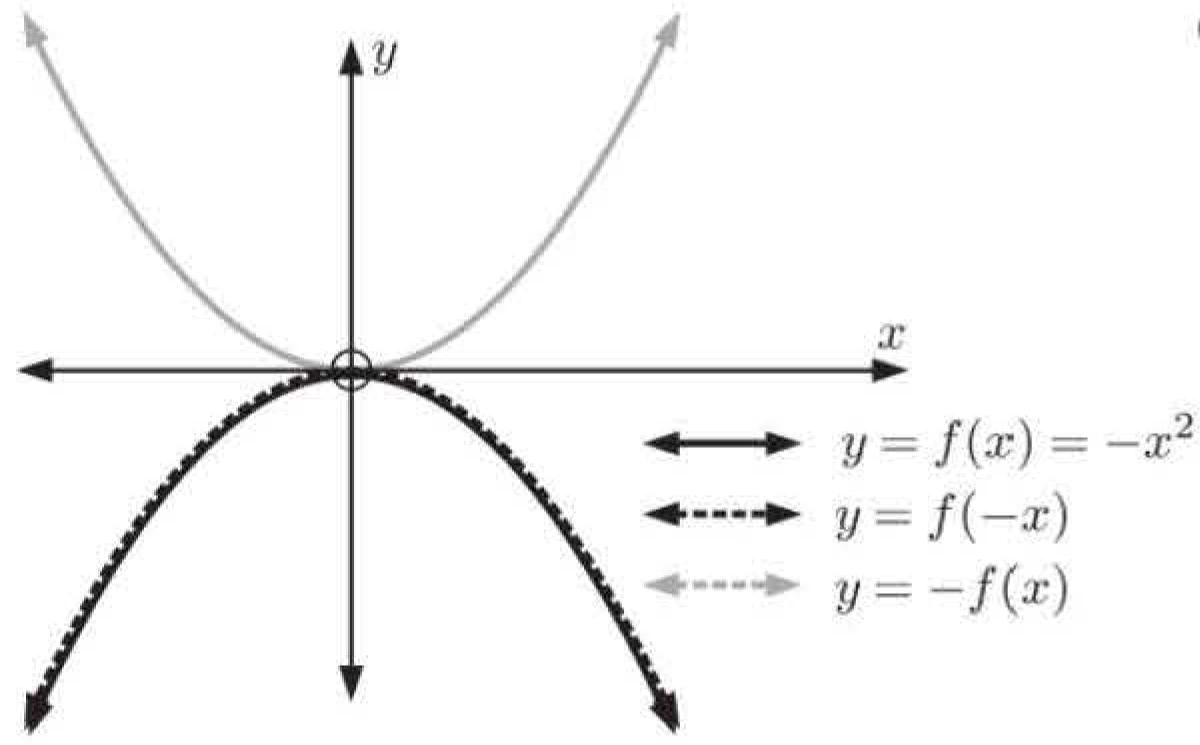
b



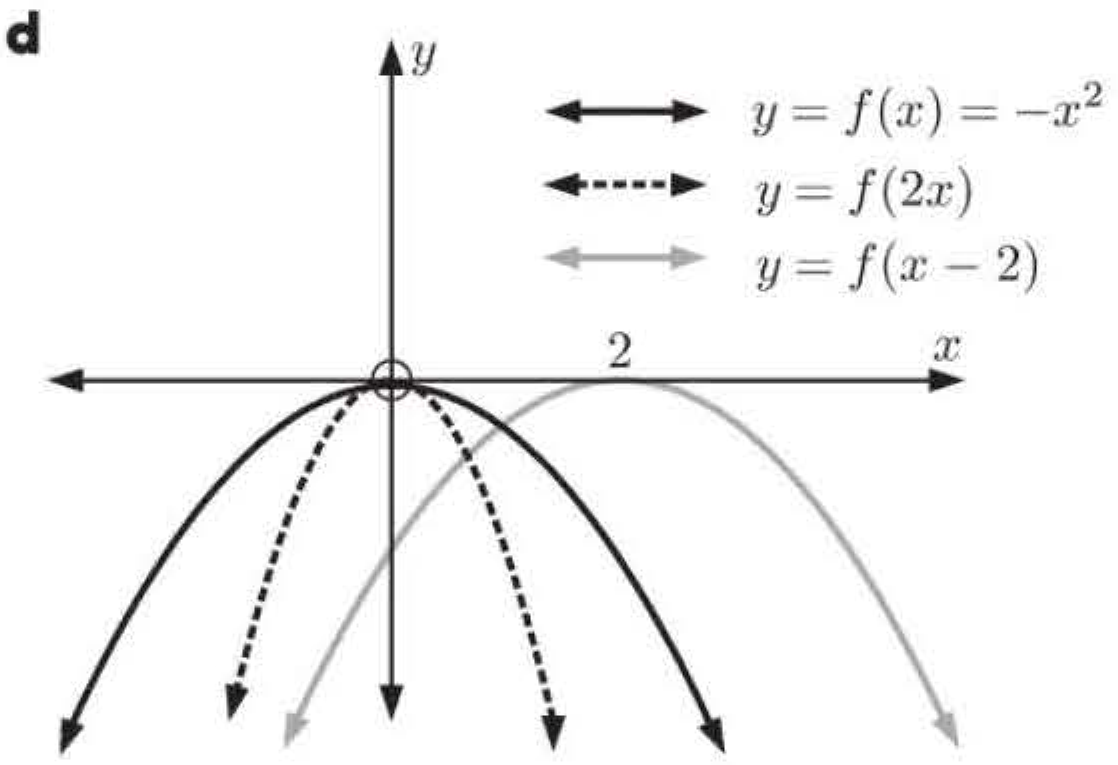
The coordinates of the points of intersection are $(-3.65,\ -2.65)$, $(-0.8,\ 0.2)$, and $(1.55,\ 2.55)$.

3 So that you can see the answers more easily, they have been drawn on two graphs.

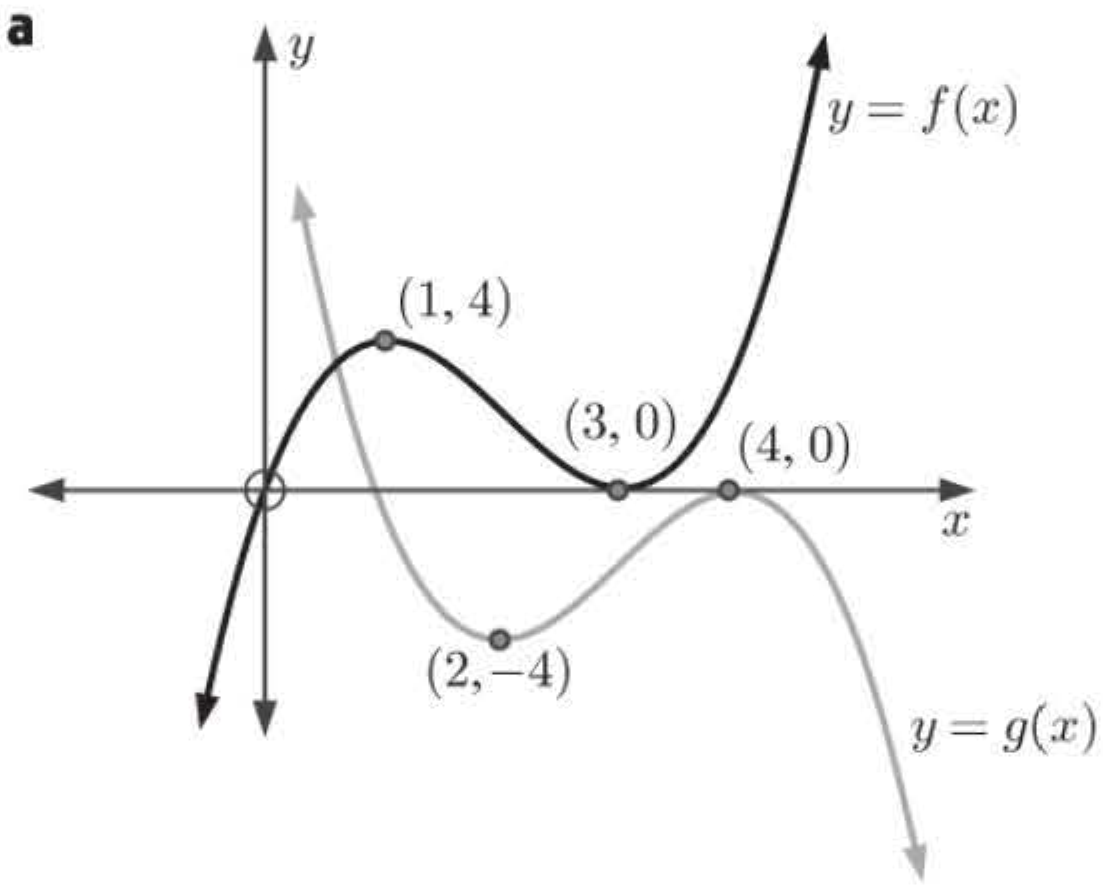
a, b



c, d



4 **a**



b $g(x)$ is obtained from $f(x)$ by a translation of $\binom{1}{0}$ and then a reflection in the x-axis. So, to get the turning point coordinates we add 1 to the x-coordinate and find the negative of the y-coordinate.

$(1, 4) \mapsto (2, -4)$ and $(3, 0) \mapsto (4, 0)$.

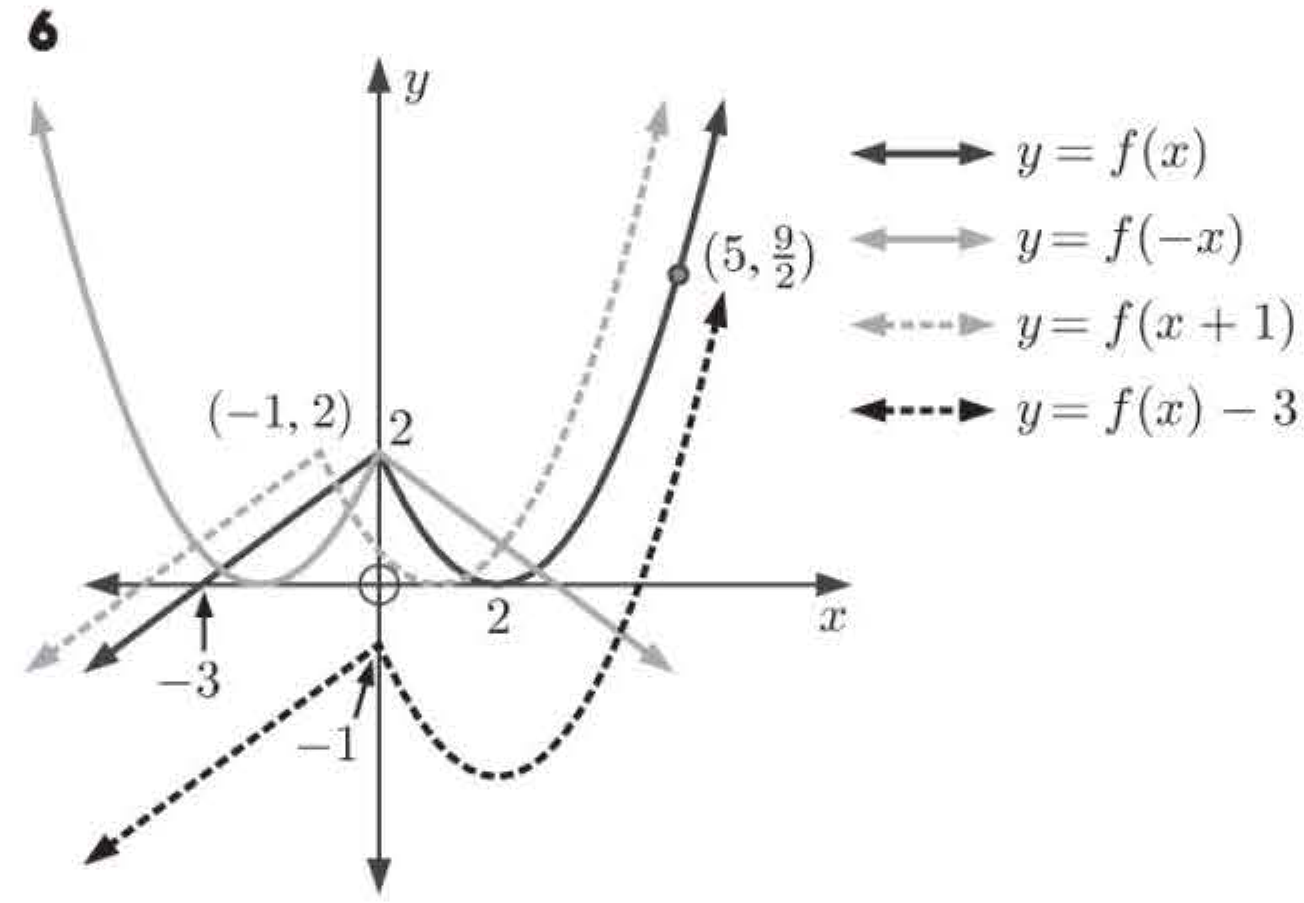
So, the turning points of $g(x)$ are $(2, -4)$ and $(4, 0)$.

5 $f(x) = x^2$ is first reflected in the x-axis to become $-f(x) = -x^2$

The function is then translated by $\binom{-3}{2}$ to become

$$\begin{aligned} -f(x+3)+2 &= -(x+3)^2+2 \\ &= -(x^2+6x+9)+2 \\ \therefore \quad g(x) &= -x^2-6x-7 \end{aligned}$$

6



7 $f(x) = x^3 + 3x^2 - x + 4$

$$\begin{aligned} g(x) &= f(x+1)+3 \\ &= \left[(x+1)^3 + 3(x+1)^2 - (x+1) + 4\right] + 3 \\ &= x^3 + 3x^2 + 3x + 1 + 3(x^2+2x+1) - x - 1 + 4 + 3 \\ &= x^3 + 3x^2 + 3x + 1 + 3x^2 + 6x + 3 - x - 1 + 4 + 3 \\ &= x^3 + 6x^2 + 8x + 10 \end{aligned}$$

8 **a** $f(x) = 3x + 2$

i A translation of 2 units to the left gives

$$\begin{aligned} y &= f(x+2) \\ &= 3(x+2)+2 \\ &= 3x+8 \end{aligned}$$

ii A translation of 6 units upwards gives

$$\begin{aligned} y &= f(x)+6 \\ &= 3x+2+6 \\ &= 3x+8 \end{aligned}$$

b $f(x) = ax + b$ translated k units to the left gives

$$\begin{aligned} y &= f(x+k) \\ &= a(x+k)+b \\ &= ax+ak+b \\ &= (ax+b)+ka \\ &= f(x)+ka \end{aligned}$$

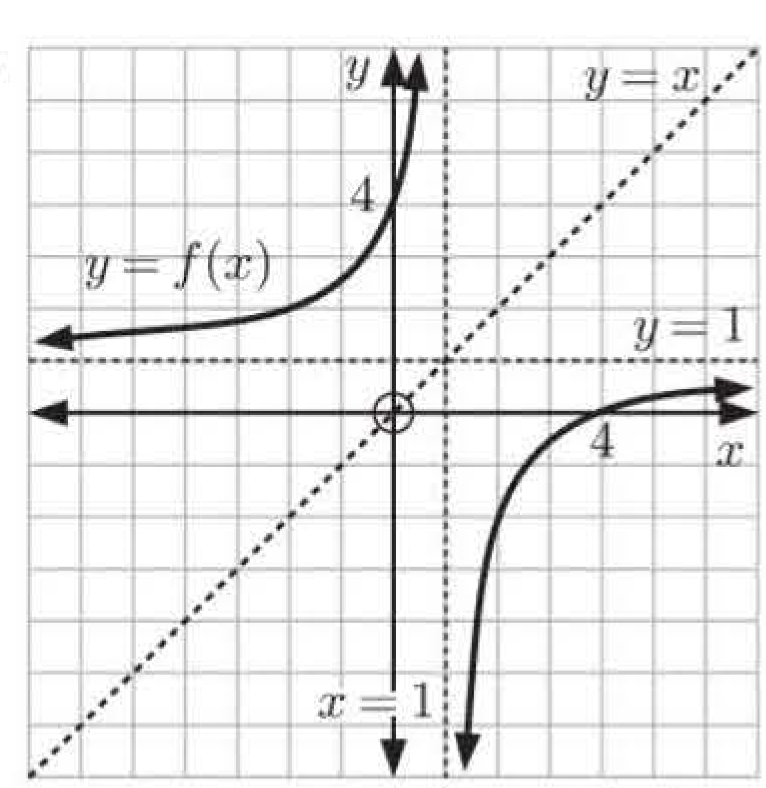
which is $f(x)$ translated ka units upwards.

9 **a** $y = \dfrac{1}{x}$ under a reflection in the y-axis becomes $y = \dfrac{1}{(-x)} = -\dfrac{1}{x}$

$y = -\dfrac{1}{x}$ under a vertical stretch with scale factor 3 becomes $y = 3\left(-\dfrac{1}{x}\right) = -\dfrac{3}{x}$

$y = -\dfrac{3}{x}$ under a translation of $\binom{1}{1}$ becomes $y = \dfrac{-3}{x-1} + 1$

b



domain of $f(x)$ is $\{x \mid x \neq 1\}$

range of $f(x)$ is $\{y \mid y \neq 1\}$

c Yes, since it is a one-to-one function (passes both the vertical and horizontal line tests).

d
$$f(x) = y = \frac{-3}{x-1} + 1$$

$\therefore$ inverse function is $x = \dfrac{-3}{y-1} + 1$

$$\therefore \quad x - 1 = \frac{-3}{y-1}$$

$$\therefore \quad y - 1 = \frac{-3}{x-1}$$

$$\therefore \quad y = \frac{-3}{x-1} + 1$$

$$\therefore \quad f^{-1}(x) = f(x) = \frac{-3}{x-1} + 1$$

$\therefore$ it is a self-inverse function.

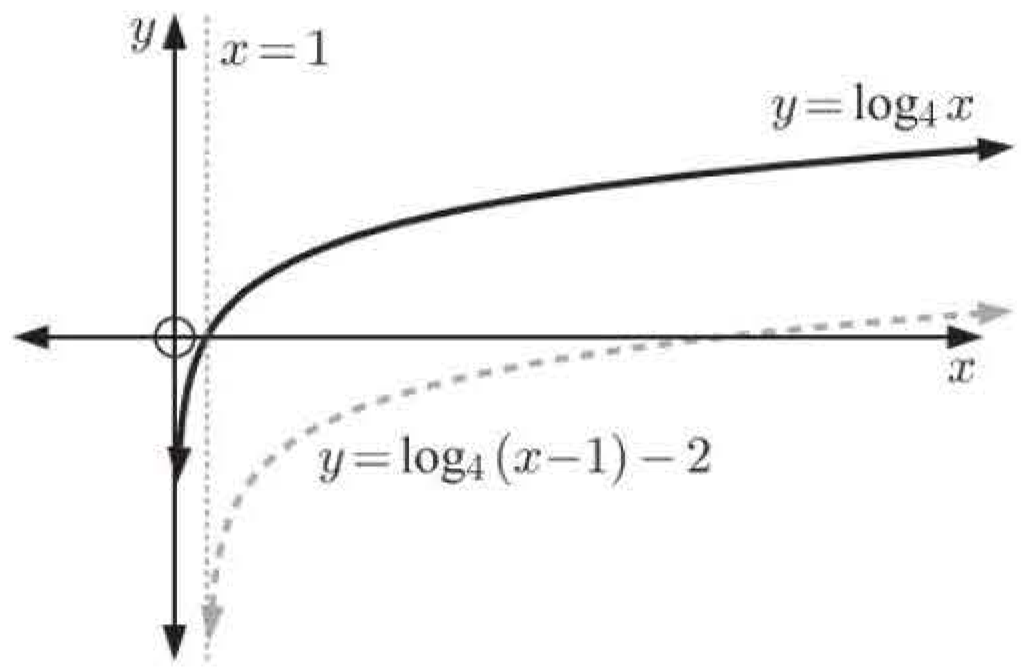
Also, the graph of $f(x)$ is symmetrical about the line $y = x$.

10 **a** Under translation $\binom{1}{-2}$,

$$y = \log_4 x$$

becomes $y = \log_4(x-1) - 2$

b



c For $y = \log_4 x$, VA is $x = 0$, no HA.
For $y = \log_4(x-1) - 2$, VA is $x = 1$, no HA.

d For $y = \log_4 x$,
domain is $\{x \mid x > 0\}$,
range is $\{y \mid y \in \mathbb{R}\}$.

For $y = \log_4(x-1) - 2$,
domain is $\{x \mid x > 1\}$,
range is $\{y \mid y \in \mathbb{R}\}$.

11 **a** Under a vertical stretch with scale factor $\frac{1}{3}$, $f(x)$ becomes $\frac{1}{3}f(x)$.

$\therefore$ $\dfrac{1}{x}$ becomes $\dfrac{1}{3}\left(\dfrac{1}{x}\right) = \dfrac{1}{3x}$

Under a reflection in the y-axis, $f(x)$ becomes $f(-x)$.

$\therefore$ $\dfrac{1}{3x}$ becomes $\dfrac{1}{3(-x)} = \dfrac{-1}{3x}$

Under a translation of 2 units to the right, $f(x)$ becomes $f(x-2)$.

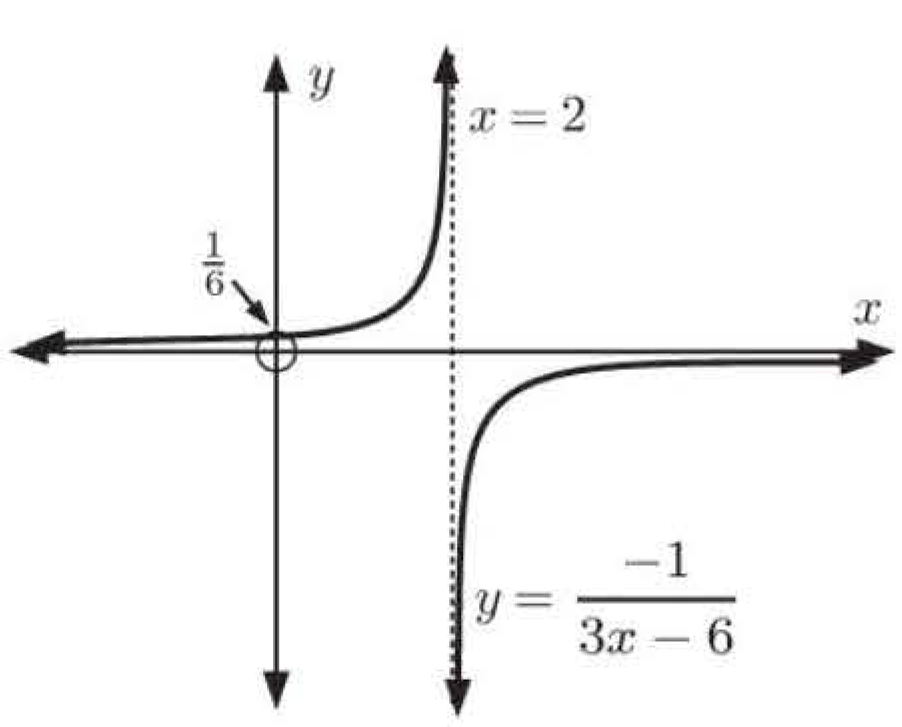
$\therefore$ $\dfrac{-1}{3x}$ becomes $\dfrac{-1}{3(x-2)} = \dfrac{-1}{3x-6}$

So, $y = \dfrac{1}{x}$ becomes $g(x) = \dfrac{-1}{3x-6}$.

b The asymptotes of $y = \dfrac{1}{x}$ are $x = 0$ and $y = 0$.
These are unchanged by the stretch and the reflection, and shifted 2 units to the right by the translation.
$\therefore$ the vertical asymptote is $x = 2$ and the horizontal asymptote is $y = 0$.

c Domain is $\{x \mid x \neq 2\}$,
range is $\{y \mid y \neq 0\}$.

d



Chapter 6
COMPLEX NUMBERS AND POLYNOMIALS

EXERCISE 6A

1 **a** $\sqrt{-9}$
$= \sqrt{9}\times\sqrt{-1}$
$= 3i$

b $\sqrt{-64}$
$= \sqrt{64}\times\sqrt{-1}$
$= 8i$

c $\sqrt{-\frac{1}{4}}$
$= \sqrt{\frac{1}{4}}\times\sqrt{-1}$
$= \frac{1}{2}i$

d $\sqrt{-5}$
$= \sqrt{5}\times\sqrt{-1}$
$= i\sqrt{5}$

e $\sqrt{-8}$
$= \sqrt{8}\times\sqrt{-1}$
$= i\sqrt{8}$

2 **a** x^2-9
$=(x+3)(x-3)$

b x^2+9
$=x^2-9i^2$
$=(x+3i)(x-3i)$

c x^2-7
$=(x+\sqrt{7})(x-\sqrt{7})$

d x^2+7
$=x^2-(i\sqrt{7})^2$
$=(x+i\sqrt{7})(x-i\sqrt{7})$

e $4x^2-1$
$=(2x+1)(2x-1)$

f $4x^2+1$
$=4x^2-i^2$
$=(2x+i)(2x-i)$

g $2x^2-9$
$=(\sqrt{2}x+3)(\sqrt{2}x-3)$

h $2x^2+9$
$=2x^2-9i^2$
$=(\sqrt{2}x+3i)(\sqrt{2}x-3i)$

i x^3-x
$=x(x^2-1)$
$=x(x+1)(x-1)$

j x^3+x
$=x(x^2+1)$
$=x(x^2-i^2)$
$=x(x+i)(x-i)$

k x^4-1
$=(x^2+1)(x^2-1)$
$=(x^2-i^2)(x^2-1)$
$=(x+i)(x-i)(x+1)(x-1)$

l x^4-16
$=(x^2+4)(x^2-4)$
$=(x^2-4i^2)(x^2-4)$
$=(x+2i)(x-2i)(x+2)(x-2)$

3 **a**
$$\begin{aligned} x^2-25&=0\\ \therefore\ (x+5)(x-5)&=0\\ \therefore\ x&=\pm 5 \end{aligned}$$

b
$$\begin{aligned} x^2+25&=0\\ \therefore\ x^2-25i^2&=0\\ \therefore\ (x+5i)(x-5i)&=0\\ \therefore\ x&=\pm 5i \end{aligned}$$

c
$$\begin{aligned} x^2-5&=0\\ \therefore\ (x+\sqrt{5})(x-\sqrt{5})&=0\\ \therefore\ x&=\pm\sqrt{5} \end{aligned}$$

d
$$\begin{aligned} x^2+5&=0\\ \therefore\ x^2-5i^2&=0\\ \therefore\ (x+i\sqrt{5})(x-i\sqrt{5})&=0\\ \therefore\ x&=\pm i\sqrt{5} \end{aligned}$$

e
$$\begin{aligned} 4x^2-9&=0\\ \therefore\ (2x+3)(2x-3)&=0\\ \therefore\ x&=\pm\tfrac{3}{2} \end{aligned}$$

f
$$\begin{aligned} 4x^2+9&=0\\ \therefore\ 4x^2-9i^2&=0\\ \therefore\ (2x+3i)(2x-3i)&=0\\ \therefore\ x&=\pm\tfrac{3}{2}i \end{aligned}$$

g
$$\begin{aligned} x^3-4x&=0\\ \therefore\ x(x^2-4)&=0\\ \therefore\ x(x+2)(x-2)&=0\\ \therefore\ x&=0 \text{ or } \pm 2 \end{aligned}$$

h
$$\begin{aligned} x^3+4x&=0\\ \therefore\ x(x^2+4)&=0\\ \therefore\ x(x^2-4i^2)&=0\\ \therefore\ x(x+2i)(x-2i)&=0\\ \therefore\ x&=0 \text{ or } \pm 2i \end{aligned}$$

i
$$\begin{aligned} x^3 - 3x &= 0 \\ \therefore\ x(x^2-3) &= 0 \\ \therefore\ x(x+\sqrt{3})(x-\sqrt{3}) &= 0 \\ \therefore\ x &= 0 \text{ or } \pm\sqrt{3} \end{aligned}$$

j
$$\begin{aligned} x^3 + 3x &= 0 \\ \therefore\ x(x^2+3) &= 0 \\ \therefore\ x(x^2-3i^2) &= 0 \\ \therefore\ x(x+i\sqrt{3})(x-i\sqrt{3}) &= 0 \\ \therefore\ x &= 0 \text{ or } \pm i\sqrt{3} \end{aligned}$$

k
$$\begin{aligned} x^4 - 1 &= 0 \\ \therefore\ (x^2+1)(x^2-1) &= 0 \\ \therefore\ (x^2-i^2)(x^2-1) &= 0 \\ \therefore\ (x+i)(x-i)(x+1)(x-1) &= 0 \\ \therefore\ x &= \pm i \text{ or } \pm 1 \end{aligned}$$

l
$$\begin{aligned} x^4 &= 81 \\ \therefore\ x^4 - 81 &= 0 \\ \therefore\ (x^2+9)(x^2-9) &= 0 \\ \therefore\ (x^2-9i^2)(x^2-9) &= 0 \\ \therefore\ (x+3i)(x-3i)(x+3)(x-3) &= 0 \\ \therefore\ x &= \pm 3i \text{ or } \pm 3 \end{aligned}$$

4 **a** If $x^2 - 10x + 29 = 0$
$$\begin{aligned} \text{then}\quad x &= \frac{10 \pm \sqrt{100 - 4\times 1\times 29}}{2} \\ &= \frac{10 \pm \sqrt{-16}}{2} \\ &= 5 \pm \sqrt{-4} \\ &= 5 \pm 2i \end{aligned}$$

b If $x^2 + 6x + 25 = 0$
$$\begin{aligned} \text{then}\quad x &= \frac{-6 \pm \sqrt{36 - 4\times 1\times 25}}{2} \\ &= \frac{-6 \pm \sqrt{-64}}{2} \\ &= -3 \pm \sqrt{-16} \\ &= -3 \pm 4i \end{aligned}$$

c If $x^2 + 14x + 50 = 0$,
$$\begin{aligned} \text{then}\quad x &= \frac{-14 \pm \sqrt{14^2 - 4\times 1\times 50}}{2} \\ &= \frac{-14 \pm \sqrt{-4}}{2} \\ &= -7 \pm \sqrt{-1} \\ &= -7 \pm i \end{aligned}$$

d
$$\begin{aligned} \text{If}\quad 2x^2 + 5 &= 6x, \\ \text{then}\quad 2x^2 - 6x + 5 &= 0 \\ x &= \frac{6 \pm \sqrt{36 - 4\times 2\times 5}}{4} \\ &= \frac{6 \pm \sqrt{-4}}{4} \\ &= \frac{3 \pm \sqrt{-1}}{2} \\ &= \tfrac{3}{2} \pm \tfrac{1}{2}i \end{aligned}$$

e If $x^2 - 2\sqrt{3}x + 4 = 0$,
$$\begin{aligned} \text{then}\quad x &= \frac{2\sqrt{3} \pm \sqrt{12 - 4\times 1\times 4}}{2} \\ &= \frac{2\sqrt{3} \pm \sqrt{-4}}{2} \\ &= \sqrt{3} \pm \sqrt{-1} \\ &= \sqrt{3} \pm i \end{aligned}$$

f
$$\begin{aligned} \text{If}\quad 2x + \frac{1}{x} &= 1, \\ \text{then}\quad 2x^2 + 1 &= x \\ \therefore\ 2x^2 - x + 1 &= 0 \\ x &= \frac{1 \pm \sqrt{1 - 4\times 2\times 1}}{4} \\ &= \frac{1 \pm \sqrt{-7}}{4} \\ &= \tfrac{1}{4} \pm i\tfrac{\sqrt{7}}{4} \end{aligned}$$

5 **a**
$$\begin{aligned} x^4 + 2x^2 &= 3 \\ \therefore\ x^4 + 2x^2 - 3 &= 0 \\ \therefore\ (x^2+3)(x^2-1) &= 0 \\ \therefore\ (x^2-3i^2)(x^2-1) &= 0 \\ \therefore\ (x+i\sqrt{3})(x-i\sqrt{3})(x+1)(x-1) &= 0 \\ \therefore\ x &= \pm i\sqrt{3} \text{ or } \pm 1 \end{aligned}$$

b
$$\begin{aligned} x^4 &= x^2 + 6 \\ \therefore\ x^4 - x^2 - 6 &= 0 \\ \therefore\ (x^2-3)(x^2+2) &= 0 \\ \therefore\ (x^2-3)(x^2-2i^2) &= 0 \\ \therefore\ (x+\sqrt{3})(x-\sqrt{3})(x+i\sqrt{2})(x-i\sqrt{2}) &= 0 \\ \therefore\ x &= \pm\sqrt{3} \text{ or } \pm i\sqrt{2} \end{aligned}$$

c
$$\begin{aligned} x^4+5x^2&=36\\ \therefore\ x^4+5x^2-36&=0\\ \therefore\ (x^2+9)(x^2-4)&=0\\ \therefore\ (x^2-9i^2)(x^2-4)&=0\\ \therefore\ (x+3i)(x-3i)(x+2)(x-2)&=0\\ \therefore\ x&=\pm 3i \text{ or } \pm 2 \end{aligned}$$

d
$$\begin{aligned} x^4+9x^2+14&=0\\ \therefore\ (x^2+7)(x^2+2)&=0\\ \therefore\ (x^2-7i^2)(x^2-2i^2)&=0\\ \therefore\ (x+i\sqrt{7})(x-i\sqrt{7})(x+i\sqrt{2})(x-i\sqrt{2})&=0\\ \therefore\ x&=\pm i\sqrt{7} \text{ or } \pm i\sqrt{2} \end{aligned}$$

e
$$\begin{aligned} x^4+1&=2x^2\\ \therefore\ x^4-2x^2+1&=0\\ \therefore\ (x^2-1)^2&=0\\ \therefore\ (x+1)^2(x-1)^2&=0\\ \therefore\ x&=\pm 1 \end{aligned}$$

f
$$\begin{aligned} x^4+2x^2+1&=0\\ \therefore\ (x^2+1)^2&=0\\ \therefore\ (x^2-i^2)^2&=0\\ \therefore\ (x+i)^2(x-i)^2&=0\\ \therefore\ x&=\pm i \end{aligned}$$

EXERCISE 6B.1

1

z	$\mathcal{R}e\,(z)$	$\mathcal{I}m\,(z)$	z	$\mathcal{R}e\,(z)$	$\mathcal{I}m\,(z)$
$3+2i$	3	2	$-3+4i$	-3	4
$5-i$	5	-1	$-7-2i$	-7	-2
3	3	0	$-11i$	0	-11
0	0	0	$i\sqrt{3}$	0	$\sqrt{3}$

2 **a** $z+w$
$=(5-2i)+(2+i)$
$=7-i$

b $2z$
$=2(5-2i)$
$=10-4i$

c $iw=i(2+i)$
$=2i+i^2$
$=-1+2i$

d $z-w$
$=(5-2i)-(2+i)$
$=5-2i-2-i$
$=3-3i$

e $2z-3w$
$=2(5-2i)-3(2+i)$
$=10-4i-6-3i$
$=4-7i$

f zw
$=(5-2i)(2+i)$
$=10-4i+5i-2i^2$
$=12+i$

g $w^2=(2+i)^2$
$=4+4i+i^2$
$=3+4i$

h $z^2=(5-2i)^2$
$=25-20i+4i^2$
$=21-20i$

3 **a** $z+2w$
$=(1+i)+2(-2+3i)$
$=1+i-4+6i$
$=-3+7i$

b z^2
$=(1+i)^2$
$=1+2i+i^2$
$=2i$

c $z^3=z^2\times z$
$=2i(1+i)$
$=2i+2i^2$
$=-2+2i$

d $iz=i(1+i)$
$=i+i^2$
$=-1+i$

e $w^2=(-2+3i)^2$
$=4-12i+9i^2$
$=-5-12i$

f zw
$=(1+i)(-2+3i)$
$=-2+3i-2i+3i^2$
$=-5+i$

g $z^2w=(1+i)^2(-2+3i)$
$=2i(-2+3i)$
$=-4i+6i^2$
$=-6-4i$

h $izw=i(1+i)(-2+3i)$
$=i(-5+i)$
$=-5i+i^2$
$=-1-5i$

4 **a** $i^0 = 1$ $\quad i^4 = 1$ $\quad i^8 = 1$ $\quad i^{-1} = -i$

$i^1 = i$ $\quad i^5 = i$ $\quad i^9 = i$ $\quad i^{-2} = -1$

$i^2 = -1$ $\quad i^6 = -1$ $\quad i^{-3} = i$

$i^3 = -i$ $\quad i^7 = -i$ $\quad i^{-4} = 1$

$i^{-5} = -i$

b $i^{4n+3} = -i$ where n is any integer

5
$$\begin{aligned}(1+i)^4 &= [(1+i)^2]^2\\ &= (1+2i+i^2)^2\\ &= (2i)^2\\ &= -4\end{aligned}$$

$$\begin{aligned}(1+i)^{101} &= (1+i)^{100} \times (1+i)\\ &= \left[(1+i)^4\right]^{25} \times (1+i)\\ &= [-4]^{25}\,(1+i)\\ &= -2^{50}(1+i)\end{aligned}$$

6 **a**
$$\begin{aligned}\frac{z}{w} &= \frac{2-i}{1+3i} \times \frac{1-3i}{1-3i}\\ &= \frac{2-6i-i+3i^2}{1-9i^2}\\ &= \frac{-1-7i}{10}\\ &= -\tfrac{1}{10} - \tfrac{7}{10}i\end{aligned}$$

b
$$\begin{aligned}\frac{i}{z} &= \frac{i}{2-i} \times \frac{2+i}{2+i}\\ &= \frac{2i+i^2}{4-i^2}\\ &= \frac{-1+2i}{5}\\ &= -\tfrac{1}{5} + \tfrac{2}{5}i\end{aligned}$$

c
$$\begin{aligned}\frac{w}{iz} &= \frac{1+3i}{i(2-i)}\\ &= \frac{1+3i}{2i-i^2}\\ &= \frac{1+3i}{1+2i} \times \frac{1-2i}{1-2i}\\ &= \frac{1-2i+3i-6i^2}{1-4i^2}\\ &= \frac{7+i}{5}\\ &= \tfrac{7}{5} + \tfrac{1}{5}i\end{aligned}$$

d
$$\begin{aligned}z^{-2} &= \frac{1}{(2-i)^2}\\ &= \frac{1}{4-4i+i^2}\\ &= \frac{1}{3-4i} \times \frac{3+4i}{3+4i}\\ &= \frac{3+4i}{9-16i^2}\\ &= \frac{3+4i}{25}\\ &= \tfrac{3}{25} + \tfrac{4}{25}i\end{aligned}$$

7 **a**
$$\begin{aligned}&\frac{i}{1-2i}\\ &= \frac{i}{1-2i} \times \frac{1+2i}{1+2i}\\ &= \frac{i+2i^2}{1-4i^2}\\ &= \frac{-2+i}{5}\\ &= -\tfrac{2}{5} + \tfrac{1}{5}i\end{aligned}$$

b
$$\begin{aligned}&\frac{i(2-i)}{3-2i}\\ &= \frac{2i-i^2}{3-2i}\\ &= \frac{1+2i}{3-2i} \times \frac{3+2i}{3+2i}\\ &= \frac{3+2i+6i+4i^2}{9-4i^2}\\ &= \frac{-1+8i}{13}\\ &= -\tfrac{1}{13} + \tfrac{8}{13}i\end{aligned}$$

c
$$\begin{aligned}&\frac{1}{2-i} - \frac{2}{2+i}\\ &= \frac{1}{2-i}\left(\frac{2+i}{2+i}\right) - \frac{2}{2+i}\left(\frac{2-i}{2-i}\right)\\ &= \frac{2+i-2(2-i)}{(2-i)(2+i)}\\ &= \frac{2+i-4+2i}{4-i^2}\\ &= \frac{-2+3i}{5}\\ &= -\tfrac{2}{5} + \tfrac{3}{5}i\end{aligned}$$

8 **a**
$$\begin{aligned}4z-3w &= 4(2+i)-3(-1+2i)\\ &= 8+4i+3-6i\\ &= 11-2i\end{aligned}$$
$\therefore\ \mathcal{Im}\,(4z-3w) = -2$

b
$$\begin{aligned}zw &= (2+i)(-1+2i)\\ &= -2+4i-i+2i^2\\ &= -4+3i\end{aligned}$$
$\therefore\ \mathcal{Re}\,(zw) = -4$

c
$$\begin{aligned} iz^2 &= i(2+i)^2 \\ &= i(4+4i+i^2) \\ &= i(3+4i) \\ &= 3i+4i^2 \\ &= -4+3i \end{aligned}$$
$\therefore \ \mathcal{Im}\,(iz^2) = 3$

d
$$\begin{aligned} \frac{z}{w} &= \frac{2+i}{-1+2i} \times \frac{-1-2i}{-1-2i} \\ &= \frac{-2-4i-i-2i^2}{1-4i^2} \\ &= \frac{0-5i}{5} \\ &= -i \end{aligned}$$
$\therefore \ \mathcal{Re}\left(\frac{z}{w}\right) = 0$

EXERCISE 6B.2

1 **a** $2x+3iy = -x-6i$

Equating real and imaginary parts,
$2x = -x$ and $3y = -6$ $\{x, y$ are real$\}$
$\therefore$ $3x = 0$ and $y = -2$
$\therefore$ $x = 0$ and $y = -2$

b $x^2+ix = 4-2i$

Equating real and imaginary parts,
$x^2 = 4$ and $x = -2$ $\{x$ is real$\}$
$\therefore$ $x = \pm 2$ and $x = -2$
$\therefore$ $x = -2$

c $(x+iy)(2-i) = 8+i$
$$\therefore \ x+iy = \frac{8+i}{2-i} \times \frac{2+i}{2+i}$$
$$\therefore \ x+iy = \frac{16+8i+2i+i^2}{4-i^2}$$
$$\therefore \ x+iy = \frac{15+10i}{5}$$
$$\therefore \ x+iy = 3+2i$$
Equating real and imag. parts, for real x, y
$x = 3$ and $y = 2$

d $(3+2i)(x+iy) = -i$
$$\therefore \ x+iy = \frac{-i}{3+2i} \times \frac{3-2i}{3-2i}$$
$$\therefore \ x+iy = \frac{-3i+2i^2}{9-4i^2}$$
$$\therefore \ x+iy = \frac{-2-3i}{13}$$
Equating real and imag. parts, for real x, y
$x = -\frac{2}{13}$ and $y = -\frac{3}{13}$

2 **a** $2(x+iy) = x-iy$
$\therefore$ $2x+2iy = x-iy$

Equating real and imaginary parts, $2x = x$ and $2y = -y$
$\therefore$ $x = 0$ and $3y = 0$
$\therefore$ $x = 0$ and $y = 0$

b $(x+2i)(y-i) = -4-7i$
$\therefore$ $xy - ix + 2iy - 2i^2 = -4-7i$
$\therefore$ $(xy+2) + i(2y-x) = -4-7i$

Equating real and imaginary parts,
$xy+2 = -4$ and $2y - x = -7$
$\therefore$ $xy = -6$ and $x = 2y+7$
$\therefore$ $(2y+7)y = -6$
$\therefore$ $2y^2+7y = -6$
$\therefore$ $2y^2+7y+6 = 0$
$\therefore$ $(2y+3)(y+2) = 0$
$\therefore$ $y = -\frac{3}{2}$ or $y = -2$

When $y = -2$, $x = 2(-2)+7 = 3$
When $y = -\frac{3}{2}$, $x = 2(-\frac{3}{2})+7 = 4$
$\therefore$ $x = 3$ and $y = -2$
or $x = 4$ and $y = -\frac{3}{2}$

c $(x+i)(3-iy) = 1+13i$
$\therefore$ $3x - ixy + 3i - i^2y = 1+13i$
$\therefore$ $(3x+y) + i(3-xy) = 1+13i$

Equating real and imaginary parts,
$3x+y = 1$ and $3 - xy = 13$
$\therefore$ $y = 1-3x$ and $xy = -10$
$\therefore$ $x(1-3x) = -10$
$\therefore$ $x - 3x^2 = -10$
$\therefore$ $0 = 3x^2 - x - 10$
$\therefore$ $0 = (3x+5)(x-2)$
$\therefore$ $x = -\frac{5}{3}$ or $x = 2$

When $x = -\frac{5}{3}$, $y = 1 - 3(-\frac{5}{3}) = 6$
and when $x = 2$, $y = 1 - 3\times 2 = -5$
$\therefore$ $x = -\frac{5}{3}$ and $y = 6$
or $x = 2$ and $y = -5$

d
$$\begin{aligned} (x+iy)(2+i) &= 2x - i(y+1) \\ \therefore\ 2x + ix + 2iy + yi^2 &= 2x + i(-y-1) \\ \therefore\ (2x-y) + i(x+2y) &= 2x + i(-y-1) \end{aligned}$$

Equating real and imaginary parts, $2x - y = 2x$ and $x + 2y = -y - 1$

$$\begin{aligned} &\therefore\ -y = 2x - 2x \\ &\therefore\ y = 0 \quad \text{and consequently} \\ &x + 0 = 0 - 1 = -1 \\ &\therefore\ x = -1 \quad \text{and} \quad y = 0 \end{aligned}$$

3
$$\begin{aligned} 3z + 17i &= iz + 11 \\ \therefore\ z(3-i) &= 11 - 17i \\ \therefore\ z &= \frac{11-17i}{3-i} \times \frac{(3+i)}{(3+i)} \\ &= \frac{33 + 11i - 51i - 17i^2}{9 - i^2} \\ &= \frac{50 - 40i}{10} \\ &= 5 - 4i \end{aligned}$$

4
$$\begin{aligned} \frac{4}{1+i} &= \frac{4}{(1+i)} \times \frac{(1-i)}{(1-i)} \\ &= \frac{4-4i}{1-i^2} \\ &= \frac{4-4i}{2} = 2 - 2i \\ \therefore\ \sqrt{z} &= (2-2i) + 7 - 2i \\ &= 9 - 4i \\ \therefore\ z &= (9-4i)^2 \\ &= 81 - 72i + 16i^2 \\ &= 65 - 72i \end{aligned}$$

5
$$\begin{aligned} &3(m+ni) = n - 2mi - (1-2i) \\ &\therefore\ 3m + 3ni = n - 2mi - 1 + 2i \\ &\therefore\ 3m + 3ni = (n-1) + i(2-2m) \end{aligned}$$

Equating real and imaginary parts,

$$\begin{aligned} &3m = n - 1 \quad \text{and} \quad 3n = 2 - 2m \\ &\therefore\ n = 3m + 1 \quad \text{and} \quad 3n = 2 - 2m \\ &\therefore\ 3(3m+1) = 2 - 2m \\ &\therefore\ 9m + 3 = 2 - 2m \\ &\therefore\ 11m = -1 \\ &\therefore\ m = -\tfrac{1}{11} \\ &\text{and} \quad n = 3(-\tfrac{1}{11}) + 1 = \tfrac{8}{11} \end{aligned}$$

6
$$\begin{aligned} z &= \frac{3i}{\sqrt{2}-i} + 1 \\ &= \left(\frac{3i}{\sqrt{2}-i}\right)\left(\frac{\sqrt{2}+i}{\sqrt{2}+i}\right) + 1 \\ &= \frac{3i\sqrt{2} + 3i^2}{2 - i^2} + 1 \\ &= \frac{3i\sqrt{2} - 3}{3} + \frac{3}{3} \\ &= \frac{3i\sqrt{2}}{3} \\ &= i\sqrt{2} \end{aligned}$$

7
$$\begin{aligned} (a+bi)^2 &= -16 - 30i \\ a^2 + 2abi + b^2i^2 &= -16 - 30i \\ \therefore\ a^2 - b^2 &= -16 \quad \text{and} \quad 2ab = -30, \\ \therefore\ ab &= -15 \quad \text{and} \quad \therefore\ b = -\frac{15}{a} \end{aligned}$$

So, $a^2 - \left(-\dfrac{15}{a}\right)^2 = -16$

$$\begin{aligned} &\therefore\ a^2 - \frac{225}{a^2} = -16 \\ &\therefore\ a^4 + 16a^2 - 225 = 0 \\ &\therefore\ (a^2+25)(a^2-9) = 0 \\ &\therefore\ a = \pm 3 \ \text{or}\ \pm 5i \end{aligned}$$

But a is real and > 0

$$\therefore\ a = 3 \quad \text{and} \quad b = -\tfrac{15}{3} = -5$$

EXERCISE 6B.3

1 **a** roots α and β are $3 \pm i$ $\quad \therefore \quad \alpha + \beta = 6$ and $\alpha\beta = 9 - i^2 = 10$

$\therefore$ quadratics have form $a(x^2 - 6x + 10) = 0, \quad a \neq 0$

b roots α and β are $1 \pm 3i$ $\quad \therefore \quad \alpha + \beta = 2$ and $\alpha\beta = 1 - 9i^2 = 10$

$\therefore$ quadratics have form $a(x^2 - 2x + 10) = 0, \quad a \neq 0$

c roots α and β are $-2 \pm 5i$ $\quad \therefore \quad \alpha + \beta = -4$ and $\alpha\beta = 4 - 25i^2 = 29$

$\therefore$ quadratics have form $a(x^2 + 4x + 29) = 0, \quad a \neq 0$

d roots α and β are $\sqrt{2} \pm i$ $\quad \therefore \quad \alpha + \beta = 2\sqrt{2}$ and $\alpha\beta = 2 - i^2 = 3$

$\therefore$ quadratics have form $a(x^2 - 2\sqrt{2}x + 3) = 0, \quad a \neq 0$

e roots α and β are $2 \pm \sqrt{3}$ $\quad \therefore \quad \alpha + \beta = 4$ and $\alpha\beta = 4 - 3 = 1$

$\therefore$ quadratics have form $a(x^2 - 4x + 1) = 0, \quad a \neq 0$

f roots α and β are 0 and $-\frac{2}{3}$ $\quad \therefore$ factors are $x, \ 3x + 2$

$\therefore$ quadratics have form $ax(3x + 2) = 0$

$\therefore \quad a(3x^2 + 2x) = 0, \quad a \neq 0$

g roots α and β are $\pm i\sqrt{2}$ $\quad \therefore \quad \alpha + \beta = 0$ and $\alpha\beta = -2i^2 = 2$

$\therefore$ quadratics have form $a(x^2 + 2) = 0, \quad a \neq 0$

h roots α and β are $-6 \pm i$ $\quad \therefore \quad \alpha + \beta = -12$ and $\alpha\beta = 36 - i^2 = 37$

$\therefore$ quadratics have form $a(x^2 + 12x + 37) = 0, \quad a \neq 0$

2 **a** If $3 + i$ is a root then so is $3 - i$ (if a and b are real).

$\therefore \quad \alpha + \beta = 6$ and $\alpha\beta = 9 - i^2 = 10$

$\therefore \quad x^2 - 6x + 10 = 0$ and so $a = -6, \quad b = 10$

b If $1 - \sqrt{2}$ is a root then so is $1 + \sqrt{2}$ if a, b are rational.

$\therefore \quad \alpha + \beta = 2$ and $\alpha\beta = 1 - 2 = -1$

$\therefore \quad x^2 - 2x - 1 = 0$

$\therefore \quad a = -2$ and $b = -1$

c If $a + ai$ is a root then so is $a - ai$ (if a and b are real and $a \neq 0$).

$\therefore \quad \alpha + \beta = 2a$

and $\alpha\beta = (a + ai)(a - ai)$
$= a^2 - (ai)^2$
$= 2a^2$

$\therefore \quad x^2 - 2ax + 2a^2 = x^2 + 4x + b$

$\therefore \quad -2a = 4$ and $b = 2a^2$

$\therefore \quad a = -2$ and $b = 8$

However, if $a = 0, \ a + ai = 0$, which is *not* complex, so the other root could be any real number.

But $\alpha\beta = 0 \quad \therefore \quad b = 0$

$\therefore \quad a = 0, \ b = 0$ is also a solution.

EXERCISE 6B.4

1 *To prove*: $(z_1 - z_2)^* = {z_1}^* - {z_2}^*$

Let $z_1 = a + ib$ and $z_2 = c + id$.

$$\begin{aligned} \therefore \quad (z_1 - z_2)^* &= [(a + ib) - (c + id)]^* \\ &= [(a - c) + i(b - d)]^* \\ &= (a - c) - i(b - d) \\ &= a - c - bi + di \\ &= (a - bi) - (c - di) \\ &= {z_1}^* - {z_2}^* \end{aligned}$$

2
$$\begin{aligned} &(w^* - z)^* - (w - 2z^*) \\ &= w^{**} - z^* - w + 2z^* \\ &= w - z^* - w + 2z^* \\ &= -z^* + 2z^* \\ &= z^* \end{aligned}$$

3 Let $z = a + bi \quad \therefore\ z^* = a - bi$

If $z^* = -z$, then $a - bi = -a - bi$
$$\therefore\ a = -a$$
$$\therefore\ 2a = 0$$
$$\therefore\ a = 0 \text{ and } b \text{ is any real number}$$
$\therefore\ z$ is purely imaginary or $a = 0,\ b = 0 \quad \therefore\ z$ is zero.

4 **a** Let $z_1 = a + bi \quad z_2 = c + di$
$$\begin{aligned}\therefore\ \frac{z_1}{z_2} &= \frac{a+bi}{c+di} \times \frac{c-di}{c-di}\\ &= \frac{(a+bi)(c-di)}{(c+di)(c-di)}\\ &= \frac{ac - adi + bci - bdi^2}{c^2 - i^2d^2}\\ &= \frac{(ac+bd) + i(-ad+bc)}{c^2+d^2}\\ &= \left(\frac{ac+bd}{c^2+d^2}\right) + \left(\frac{bc-ad}{c^2+d^2}\right)i\end{aligned}$$

b
$$\begin{aligned}\frac{{z_1}^*}{{z_2}^*} &= \frac{a-bi}{c-di} \times \frac{c+di}{c+di}\\ &= \frac{(a-bi)(c+di)}{(c-di)(c+di)}\\ &= \frac{ac + adi - bci - bdi^2}{c^2 - i^2d^2}\\ &= \frac{(ac+bd) - i(bc-ad)}{c^2+d^2}\\ &= \left(\frac{ac+bd}{c^2+d^2}\right) - \left(\frac{bc-ad}{c^2+d^2}\right)i\\ &= \left(\frac{z_1}{z_2}\right)^* \quad \text{for all } z_2 \neq 0\end{aligned}$$

5 **a**
$$\begin{aligned}\left(\frac{z_1}{z_2}\right)^* \times {z_2}^* &= \left(\frac{z_1}{z_2} \times z_2\right)^* \quad \{\text{from } \textbf{Example 9}\}\\ &= {z_1}^*\end{aligned}$$
$$\therefore\ \left(\frac{z_1}{z_2}\right)^* = \frac{{z_1}^*}{{z_2}^*} \quad \{\text{dividing both sides by } {z_2}^*\}$$

b $z = a + bi$

i If $z = z^*$,

then $a + bi = a - bi$
$$\therefore\ bi = -bi$$
$$\therefore\ b = -b$$
$$\therefore\ 2b = 0$$
$$\therefore\ b = 0$$
And if $b = 0$, then $z = a + i(0) = a$

$\therefore\ z$ is real.

ii If $z^* = -z$,

then $a - bi = -(a + bi)$
$$\therefore\ a - bi = -a - bi$$
$$\therefore\ a = -a$$
$$\therefore\ 2a = 0$$
$$\therefore\ a = 0$$
And if $a = 0$, then $z = 0 + bi = bi$

$\therefore\ z$ is purely imaginary or zero.

6 **a** Let $z = a + bi$ and $w = c + di$
$$\begin{aligned}\therefore\ zw^* + z^*w &= (a+bi)(c-di) + (a-bi)(c+di)\\ &= ac - adi + bci + bd + ac + adi - bci + bd\\ &= ac + bd + ac + bd\\ &= 2ac + 2bd \quad \text{which is a real number}\end{aligned}$$

b Let $z = a + bi$ and $w = c + di$
$$\begin{aligned}\therefore\ zw^* - z^*w &= (a+bi)(c-di) - (a-bi)(c+di)\\ &= ac - adi + bci + bd - [ac + adi - bci + bd]\\ &= 2bci - 2adi\\ &= (2bc - 2ad)i \quad \text{which is purely imaginary or zero}\end{aligned}$$

7 **a** If $z = a + bi$
$$\begin{aligned}\text{then } z^2 &= (a+bi)(a+bi)\\ &= a^2 + 2abi + b^2i^2\\ &= (a^2 - b^2) + 2abi\end{aligned}$$

b
$$\begin{aligned}(z^*)^2 &= (a-bi)^2\\ &= a^2 - 2abi + b^2i^2\\ &= (a^2 - b^2) - 2abi\\ \text{and } (z^2)^* &= (a^2 - b^2) - 2abi \quad \{\text{from } \textbf{a}\}\\ \therefore\ (z^2)^* &= (z^*)^2 \quad \text{as required}\end{aligned}$$

c
$$\begin{aligned} z^3 &= (z^2)z \\ &= ((a^2-b^2)+2abi)(a+bi) \\ &= a(a^2-b^2)+b(a^2-b^2)i+2a^2bi+2ab^2i^2 \\ &= a^3-ab^2+a^2bi-b^3i+2a^2bi-2ab^2 \\ &= (a^3-3ab^2)+(3a^2b-b^3)i \\ \therefore\ \left(z^3\right)^* &= (a^3-3ab^2)-(3a^2b-b^3)i \\ (z^*)^3 &= (z^*)^2\,z^* \\ &= \left[(a^2-b^2)-2abi\right](a-bi) \\ &= a(a^2-b^2)-b(a^2-b^2)i-2a^2bi+2ab^2i^2 \\ &= a^3-ab^2-a^2bi+b^3i-2a^2bi-2ab^2 \\ &= (a^3-3ab^2)-(3a^2b-b^3)i \quad \text{which is } \left(z^3\right)^* \text{ as required} \end{aligned}$$

8 $w = \dfrac{z-1}{z^*+1}$ where $z = a+bi$

$$\begin{aligned} \therefore\ w &= \frac{(a-1)+bi}{(a+1)-bi} \times \frac{(a+1)+bi}{(a+1)+bi} \\ &= \frac{(a^2-1)+(a-1)bi+(a+1)bi+b^2i^2}{(a+1)^2-b^2i^2} \\ &= \frac{(a^2-b^2-1)+2abi}{(a+1)^2+b^2} \end{aligned}$$

a w is real if $2ab=0$,
that is, if $a=0$ or $b=0$, $a \neq -1$
However, if $b=0$ and $a=-1$,
w is undefined and hence is not real.
$\therefore$ $a=0$ or $(b=0,\ a\neq -1)$.

b w is purely imaginary if
$a^2-b^2-1=0$, and $2ab \neq 0$
that is, if $a^2-b^2=1$
and neither a nor b is zero, and $a \neq -1$.

9 **a**
$$\begin{aligned} (z_1z_2z_3)^* &= [z_1 \times (z_2 \times z_3)]^* \\ &= {z_1}^*\,(z_2 \times z_3)^* && \{\text{as } (zw)^* = z^*w^*\} \\ &= {z_1}^* \times {z_2}^* \times {z_3}^* && \{\text{as } (zw)^* = z^*w^* \text{ again}\} \end{aligned}$$

b
$$\begin{aligned} (z_1z_2z_3z_4)^* &= (z_1z_2z_3)^* \times {z_4}^* && \{\text{as } (zw)^* = z^*w^*\} \\ &= {z_1}^* \times {z_2}^* \times {z_3}^* \times {z_4}^* && \{\text{using } \mathbf{a}\} \end{aligned}$$

c $(z_1 \times z_2 \times z_3 \ldots\ldots z_n)^* = {z_1}^* \times {z_2}^* \times {z_3}^* \ldots\ldots {z_n}^*$

d
$$\begin{aligned} (z^n)^* &= (z \times z \times z \times \ldots\ldots \times z)^* \\ &= z^* \times z^* \times z^* \times \ldots\ldots \times z^* && \{\text{using } \mathbf{c}\} \\ &= (z^*)^n \end{aligned}$$

EXERCISE 6C.1

1 **a**
$$\begin{aligned} &3P(x) \\ &= 3(x^2+2x+3) \\ &= 3x^2+6x+9 \end{aligned}$$

b
$$\begin{aligned} &P(x)+Q(x) \\ &= (x^2+2x+3)+(4x^2+5x+6) \\ &= 5x^2+7x+9 \end{aligned}$$

c
$$\begin{aligned} &P(x)-2Q(x) \\ &= (x^2+2x+3)-2(4x^2+5x+6) \\ &= x^2+2x+3-8x^2-10x-12 \\ &= -7x^2-8x-9 \end{aligned}$$

d
$$\begin{aligned} &P(x)\,Q(x) \\ &= (x^2+2x+3)(4x^2+5x+6) \\ &= 4x^4+8x^3+12x^2+5x^3+10x^2 \\ &\quad +15x+6x^2+12x+18 \\ &= 4x^4+13x^3+28x^2+27x+18 \end{aligned}$$

2 **a**
$$\begin{aligned} &f(x)+g(x) \\ &= (x^2-x+2)+(x^3-3x+5) \\ &= x^3+x^2-4x+7 \end{aligned}$$

b
$$\begin{aligned} &g(x)-f(x) \\ &= (x^3-3x+5)-(x^2-x+2) \\ &= x^3-3x+5-x^2+x-2 \\ &= x^3-x^2-2x+3 \end{aligned}$$

c
$$\begin{aligned}&2f(x)+3g(x)\\&=2(x^2-x+2)+3(x^3-3x+5)\\&=2x^2-2x+4+3x^3-9x+15\\&=3x^3+2x^2-11x+19\end{aligned}$$

d
$$\begin{aligned}&g(x)+x\,f(x)\\&=(x^3-3x+5)+x(x^2-x+2)\\&=x^3-3x+5+x^3-x^2+2x\\&=2x^3-x^2-x+5\end{aligned}$$

e
$$\begin{aligned}&f(x)\,g(x)\\&=(x^2-x+2)(x^3-3x+5)\\&=x^5-x^4+2x^3-3x^3+3x^2\\&\quad-6x+5x^2-5x+10\\&=x^5-x^4-x^3+8x^2-11x+10\end{aligned}$$

f
$$\begin{aligned}&[f(x)]^2\\&=(x^2-x+2)(x^2-x+2)\\&=x^4-x^3+2x^2-x^3+x^2\\&\quad-2x+2x^2-2x+4\\&=x^4-2x^3+5x^2-4x+4\end{aligned}$$

3 **a**
$$\begin{aligned}&(x^2-2x+3)(2x+1)\\&=2x^3-4x^2+6x+x^2-2x+3\\&=2x^3-3x^2+4x+3\end{aligned}$$

b
$$\begin{aligned}&(x-1)^2(x^2+3x-2)\\&=(x^2-2x+1)(x^2+3x-2)\\&=x^4-2x^3+x^2+3x^3-6x^2\\&\quad+3x-2x^2+4x-2\\&=x^4+x^3-7x^2+7x-2\end{aligned}$$

c
$$\begin{aligned}&(x+2)^3\\&=(x+2)(x+2)^2\\&=(x+2)(x^2+4x+4)\\&=x^3+2x^2+4x^2+8x+4x+8\\&=x^3+6x^2+12x+8\end{aligned}$$

d
$$\begin{aligned}&(2x^2-x+3)^2\\&=(2x^2-x+3)(2x^2-x+3)\\&=4x^4-2x^3+6x^2-2x^3+x^2\\&\quad-3x+6x^2-3x+9\\&=4x^4-4x^3+13x^2-6x+9\end{aligned}$$

e
$$\begin{aligned}&(2x-1)^4\\&=(2x-1)^2(2x-1)^2\\&=(4x^2-4x+1)(4x^2-4x+1)\\&=16x^4-16x^3+4x^2-16x^3+16x^2\\&\quad-4x+4x^2-4x+1\\&=16x^4-32x^3+24x^2-8x+1\end{aligned}$$

f
$$\begin{aligned}&(3x-2)^2(2x+1)(x-4)\\&=(9x^2-12x+4)(2x^2-7x-4)\\&=18x^4-24x^3+8x^2-63x^3+84x^2\\&\quad-28x-36x^2+48x-16\\&=18x^4-87x^3+56x^2+20x-16\end{aligned}$$

4 **a**
$$\begin{aligned}&(2x^2-3x+5)(3x-1)\\&=6x^3-11x^2+18x-5\end{aligned}$$

as

	2	−3	5
	×	3	−1
	−2	3	−5
6	−9	15	
6	−11	18	−5

b
$$\begin{aligned}&(4x^2-x+2)(2x+5)\\&=8x^3+18x^2-x+10\end{aligned}$$

as

	4	−1	2
		× 2	5
	20	−5	10
8	−2	4	
8	18	−1	10

c
$$\begin{aligned}&(2x^2+3x+2)(5-x)\\&=-2x^3+7x^2+13x+10\end{aligned}$$

as

	2	3	2
	×	−1	5
	10	15	10
−2	−3	−2	
−2	7	13	10

d
$$\begin{aligned}&(x-2)^2(2x+1)\\&=(x^2-4x+4)(2x+1)\\&=2x^3-7x^2+4x+4\end{aligned}$$

as

	1	−4	4
		× 2	1
	1	−4	4
2	−8	8	
2	−7	4	4

e $(x^2-3x+2)(2x^2+4x-1)$
$= 2x^4-2x^3-9x^2+11x-2$

as
$$\begin{array}{rrrrr}
 & & 1 & -3 & 2 \\
 & \times & 2 & 4 & -1 \\ \hline
 & & -1 & 3 & -2 \\
 & 4 & -12 & 8 & \\
2 & -6 & 4 & & \\ \hline
2 & -2 & -9 & 11 & -2 \\ \hline
\end{array}$$

f $(3x^2-x+2)(5x^2+2x-3)$
$= 15x^4+x^3-x^2+7x-6$

as
$$\begin{array}{rrrrr}
 & & 3 & -1 & 2 \\
 & \times & 5 & 2 & -3 \\ \hline
 & & -9 & 3 & -6 \\
 & 6 & -2 & 4 & \\
15 & -5 & 10 & & \\ \hline
15 & 1 & -1 & 7 & -6 \\ \hline
\end{array}$$

g $(x^2-x+3)^2$
$= x^4-2x^3+7x^2-6x+9$

as
$$\begin{array}{rrrrr}
 & & 1 & -1 & 3 \\
 & \times & 1 & -1 & 3 \\ \hline
 & & 3 & -3 & 9 \\
 & -1 & 1 & -3 & \\
1 & -1 & 3 & & \\ \hline
1 & -2 & 7 & -6 & 9 \\ \hline
\end{array}$$

h $(2x^2+x-4)^2$
$= 4x^4+4x^3-15x^2-8x+16$

as
$$\begin{array}{rrrrr}
 & & 2 & 1 & -4 \\
 & \times & 2 & 1 & -4 \\ \hline
 & & -8 & -4 & 16 \\
 & 2 & 1 & -4 & \\
4 & 2 & -8 & & \\ \hline
4 & 4 & -15 & -8 & 16 \\ \hline
\end{array}$$

i $(2x+5)^3$
$= (2x+5)^2(2x+5)$
$= (4x^2+20x+25)(2x+5)$
$= 8x^3+60x^2+150x+125$

as
$$\begin{array}{rrrr}
 & 4 & 20 & 25 \\
 & \times & 2 & 5 \\ \hline
 & 20 & 100 & 125 \\
8 & 40 & 50 & \\ \hline
8 & 60 & 150 & 125 \\ \hline
\end{array}$$

j $(x^3+x^2-2)^2$
$= x^6+2x^5+x^4-4x^3-4x^2+4$

as
$$\begin{array}{rrrrrrr}
 & & & 1 & 1 & 0 & -2 \\
 & & \times & 1 & 1 & 0 & -2 \\ \hline
 & & & -2 & -2 & 0 & 4 \\
 & & 0 & 0 & 0 & 0 & \\
 & 1 & 1 & 0 & -2 & & \\
1 & 1 & 0 & -2 & & & \\ \hline
1 & 2 & 1 & -4 & -4 & 0 & 4 \\ \hline
\end{array}$$

EXERCISE 6C.2

1 **a**
$$\begin{array}{rl}
 & \qquad x \\ \cline{2-2}
x+2 & \big)\; x^2+2x-3 \\
 & \underline{-(x^2+2x)\quad} \\
 & \qquad\quad -3
\end{array}$$

$\therefore\ Q(x)=x,\quad R=-3$
$\therefore\ x^2+2x-3=x(x+2)-3$

b
$$\begin{array}{rl}
 & \qquad x-4 \\ \cline{2-2}
x-1 & \big)\; x^2-5x+1 \\
 & \underline{-(x^2-\ x)\quad} \\
 & \qquad -4x+1 \\
 & \qquad \underline{-(-4x+4)} \\
 & \qquad\qquad -3
\end{array}$$

$\therefore\ Q(x)=x-4$
$R=-3$

$\therefore\ x^2-5x+1=(x-4)(x-1)-3$

c
$$\begin{array}{rl}
 & \qquad 2x^2+10x+16 \\ \cline{2-2}
x-2 & \big)\; 2x^3+\ 6x^2-\ 4x+\ 3 \\
 & \underline{-(2x^3-\ 4x^2)} \\
 & \qquad 10x^2-\ 4x \\
 & \qquad \underline{-(10x^2-20x)} \\
 & \qquad\qquad 16x+\ 3 \\
 & \qquad\qquad \underline{-(16x-32)} \\
 & \qquad\qquad\qquad 35
\end{array}$$

$\therefore\ Q(x)=2x^2+10x+16$
$R=35$

$\therefore\ 2x^3+6x^2-4x+3=(2x^2+10x+16)(x-2)+35$

2 **a**

$$
\begin{array}{r}
x+1 \\
x-4\,\overline{)\,x^2-3x+6} \\
\underline{-(x^2-4x)} \\
x+6 \\
\underline{-(x-4)} \\
10
\end{array}
$$

$\therefore\ x^2-3x+6=(x+1)(x-4)+10$

b

$$
\begin{array}{r}
x+1 \\
x+3\,\overline{)\,x^2+4x-11} \\
\underline{-(x^2+3x)} \\
x-11 \\
\underline{-(x+\ 3)} \\
-14
\end{array}
$$

$\therefore\ x^2+4x-11=(x+1)(x+3)-14$

c

$$
\begin{array}{r}
2x-3 \\
x-2\,\overline{)\,2x^2-7x+2} \\
\underline{-(2x^2-4x)} \\
-3x+2 \\
\underline{-(-3x+6)} \\
-4
\end{array}
$$

$\therefore\ 2x^2-7x+2=(2x-3)(x-2)-4$

d

$$
\begin{array}{r}
x^2+x-2 \\
2x+1\,\overline{)\,2x^3+3x^2-3x-2} \\
\underline{-(2x^3+\ x^2)} \\
2x^2-3x \\
\underline{-(2x^2+\ x)} \\
-4x-2 \\
\underline{-(-4x-2)} \\
0
\end{array}
$$

$\therefore\ 2x^3+3x^2-3x-2=(x^2+x-2)(2x+1)$

e

$$
\begin{array}{r}
x^2+4x+4 \\
3x-1\,\overline{)\,3x^3+11x^2+8x+7} \\
\underline{-(3x^3-\quad x^2)} \\
12x^2+8x \\
\underline{-(12x^2-4x)} \\
12x+7 \\
\underline{-(12x-4)} \\
11
\end{array}
$$

$\therefore\ 3x^3+11x^2+8x+7$
$=(x^2+4x+4)(3x-1)+11$

f

$$
\begin{array}{r}
x^3-2x^2+\frac{5}{2}x-\frac{1}{4} \\
2x+3\,\overline{)\,2x^4-\ x^3-\ x^2+7x+4} \\
\underline{-(2x^4+3x^3)} \\
-4x^3-\ x^2 \\
\underline{-(-4x^3-6x^2)} \\
5x^2+7x \\
\underline{-(5x^2+\frac{15}{2}x)} \\
-\frac{1}{2}x+4 \\
\underline{-(-\frac{1}{2}x-\frac{3}{4})} \\
\frac{19}{4}
\end{array}
$$

$\therefore\ 2x^4-x^3-x^2+7x+4$
$=(x^3-2x^2+\frac{5}{2}x-\frac{1}{4})(2x+3)+\frac{19}{4}$

3 **a**

$$
\begin{array}{r}
x+2 \\
x-2\,\overline{)\,x^2+0x+5} \\
\underline{-(x^2-2x)} \\
2x+5 \\
\underline{-(2x-4)} \\
9
\end{array}
$$

$$\therefore\ \frac{x^2+5}{x-2}=x+2+\frac{9}{x-2}$$

b

$$
\begin{array}{r}
2x+1 \\
x+1\,\overline{)\,2x^2+3x+0} \\
\underline{-(2x^2+2x)} \\
x+0 \\
\underline{-(x+1)} \\
-1
\end{array}
$$

$$\therefore\ \frac{2x^2+3x}{x+1}=2x+1-\frac{1}{x+1}$$

c

$$
\begin{array}{r}
3x-4 \\
x+2\,\overline{)\,3x^2+2x-5} \\
\underline{-(3x^2+6x)} \\
-4x-5 \\
\underline{-(-4x-8)} \\
3
\end{array}
$$

$$\therefore\ \frac{3x^2+2x-5}{x+2}=3x-4+\frac{3}{x+2}$$

d

$$
\begin{array}{r}
x^2+3x-2 \\
x-1\,\overline{)\,x^3+2x^2-5x+2} \\
\underline{-(x^3-\ x^2)} \\
3x^2-5x \\
\underline{-(3x^2-3x)} \\
-2x+2 \\
\underline{-(-2x+2)} \\
0
\end{array}
$$

$$\therefore\ \frac{x^3+2x^2-5x+2}{x-1}=x^2+3x-2$$

e

$$\begin{array}{r|l} & 2x^2-8x+31 \\ \hline x+4 & 2x^3+0x^2-x+0 \\ & -(2x^3+8x^2) \\ \hline & -8x^2-x \\ & -(-8x^2-32x) \\ \hline & 31x+0 \\ & -(31x+124) \\ \hline & -124 \end{array}$$

$$\therefore \quad \frac{2x^3-x}{x+4}=2x^2-8x+31-\frac{124}{x+4}$$

f

$$\begin{array}{r|l} & x^2+3x+6 \\ \hline x-2 & x^3+x^2+0x-5 \\ & -(x^3-2x^2) \\ \hline & 3x^2+0x \\ & -(3x^2-6x) \\ \hline & 6x-5 \\ & -(6x-12) \\ \hline & 7 \end{array}$$

$$\therefore \quad \frac{x^3+x^2-5}{x-2}=x^2+3x+6+\frac{7}{x-2}$$

EXERCISE 6C.3

1 **a**

$$\begin{array}{r|l} & x+1 \\ \hline x^2+x+1 & x^3+2x^2+x-3 \\ & -(x^3+x^2+x) \\ \hline & x^2+0x-3 \\ & -(x^2+x+1) \\ \hline & -x-4 \end{array}$$

$\therefore \quad Q(x)=x+1, \quad R(x)=-x-4$

b

$$\begin{array}{r|l} & 3 \\ \hline x^2-1 & 3x^2-x+0 \\ & -(3x^2-3) \\ \hline & -x+3 \end{array}$$

$\therefore \quad Q(x)=3, \quad R(x)=-x+3$

c

$$\begin{array}{r|l} & 3x \\ \hline x^2+1 & 3x^3+0x^2+x-1 \\ & -(3x^3+3x) \\ \hline & -2x-1 \end{array}$$

$\therefore \quad Q(x)=3x, \quad R(x)=-2x-1$

d $Q(x)=0, \quad R(x)=x-4$

2 **a**

$$\begin{array}{r|l} & 1 \\ \hline x^2+x+1 & x^2-x+1 \\ & -(x^2+x+1) \\ \hline & -2x \end{array}$$

$$\therefore \quad \frac{x^2-x+1}{x^2+x+1}=1-\frac{2x}{x^2+x+1}$$

$\therefore \quad x^2-x+1=(x^2+x+1)-2x$

b

$$\begin{array}{r|l} & x \\ \hline x^2+2 & x^3+0x^2+0x+0 \\ & -(x^3+2x) \\ \hline & -2x+0 \end{array}$$

$$\therefore \quad \frac{x^3}{x^2+2}=x-\frac{2x}{x^2+2}$$

$\therefore \quad x^3=x(x^2+2)-2x$

c

$$\begin{array}{r|l} & x^2+x+3 \\ \hline x^2-x+1 & x^4+0x^3+3x^2+x-1 \\ & -(x^4-x^3+x^2) \\ \hline & x^3+2x^2+x \\ & -(x^3-x^2+x) \\ \hline & 3x^2+0x-1 \\ & -(3x^2-3x+3) \\ \hline & 3x-4 \end{array}$$

$$\therefore \quad \frac{x^4+3x^2+x-1}{x^2-x+1}=x^2+x+3+\frac{3x-4}{x^2-x+1}$$

$\therefore \quad x^4+3x^2+x-1=(x^2+x+3)(x^2-x+1)+3x-4$

d $\dfrac{2x^3-x+6}{(x-1)^2}=\dfrac{2x^3-x+6}{x^2-2x+1}$

$$\begin{array}{r|l} & \quad\quad\quad\quad\quad 2x+4 \\ \hline x^2-2x+1 & 2x^3+0x^2-\ \ x+6 \\ & -(2x^3-4x^2+2x) \\ \hline & \quad\quad 4x^2-3x+6 \\ & \quad -(4x^2-8x+4) \\ \hline & \quad\quad\quad\quad 5x+2 \end{array}$$

$\therefore\ \dfrac{2x^3-x+6}{(x-1)^2}=2x+4+\dfrac{5x+2}{(x-1)^2}$

$\therefore\ 2x^3-x+6=(2x+4)(x-1)^2+5x+2$

e $\dfrac{x^4}{(x+1)^2}=\dfrac{x^4}{x^2+2x+1}$

$$\begin{array}{r|l} & \quad\quad\quad\quad\quad\quad x^2-2x+3 \\ \hline x^2+2x+1 & x^4+0x^3+0x^2+0x+0 \\ & -(x^4+2x^3+\ \ x^2) \\ \hline & \quad -2x^3-\ \ x^2+0x \\ & -(-2x^3-4x^2-2x) \\ \hline & \quad\quad\quad 3x^2+2x+0 \\ & \quad\quad -(3x^2+6x+3) \\ \hline & \quad\quad\quad\quad\quad -4x-3 \end{array}$$

$\therefore\ \dfrac{x^4}{(x+1)^2}=x^2-2x+3-\dfrac{4x+3}{(x+1)^2}$

$\therefore\ x^4=(x^2-2x+3)(x+1)^2-4x-3$

f $\dfrac{x^4-2x^3+x+5}{(x-1)(x+2)}=\dfrac{x^4-2x^3+x+5}{x^2+x-2}$

$$\begin{array}{r|l} & \quad\quad\quad\quad\quad\quad x^2-3x+5 \\ \hline x^2+x-2 & x^4-2x^3+0x^2+\ \ x+5 \\ & -(x^4+\ \ x^3-2x^2) \\ \hline & \quad -3x^3+2x^2+\ \ x \\ & -(-3x^3-3x^2+6x) \\ \hline & \quad\quad\quad 5x^2-5x+5 \\ & \quad -(5x^2+5x-10) \\ \hline & \quad\quad\quad -10x+15 \end{array}$$

$\therefore\ \dfrac{x^4-2x^3+x+5}{(x-1)(x+2)}=x^2-3x+5+\dfrac{15-10x}{(x-1)(x+2)}$

$\therefore\ x^4-2x^3+x+5=(x^2-3x+5)(x-1)(x+2)-10x+15$

3 $\dfrac{P(x)}{x-2}=\dfrac{(x-2)(x^2+2x+3)+7}{x-2}$

$=x^2+2x+3+\dfrac{7}{x-2}$

$\therefore$ quotient is x^2+2x+3, remainder is 7

4 $\dfrac{f(x)}{x^2+x-2}=\dfrac{(x-1)(x+2)(x^2-3x+5)+15-10x}{(x-1)(x+2)}$

$=x^2-3x+5+\dfrac{15-10x}{(x-1)(x+2)}$

$\therefore$ quotient is x^2-3x+5, remainder is $15-10x$

EXERCISE 6D.1

1 **a** $2x^2-5x-12$ has zeros

$x=\dfrac{5\pm\sqrt{25-4(2)(-12)}}{4}$

$=\dfrac{5\pm\sqrt{121}}{4}$

$=\dfrac{5\pm 11}{4}$

$=4,\ -\frac{6}{4}$

$\therefore$ zeros are $4,\ -\frac{3}{2}$

b $x^2+6x+10$ has zeros

$x=\dfrac{-6\pm\sqrt{36-4(1)(10)}}{2}$

$=\dfrac{-6\pm\sqrt{-4}}{2}$

$=-3\pm i$

$\therefore$ zeros are $-3\pm i$

c $z^2 - 6z + 6$ has zeros

$$z = \frac{6 \pm \sqrt{36 - 4(1)(6)}}{2} = \frac{6 \pm \sqrt{12}}{2} = 3 \pm \sqrt{3}$$

$\therefore$ zeros are $3 \pm \sqrt{3}$

d
$$x^3 - 4x = x(x^2 - 4) = x(x+2)(x-2)$$

$\therefore$ zeros are $0, \pm 2$

e
$$z^3 + 2z = z(z^2 + 2) = z(z^2 - 2i^2) = z(z + i\sqrt{2})(z - i\sqrt{2})$$

$\therefore$ zeros are $0, \pm i\sqrt{2}$

f
$$z^4 + 4z^2 - 5 = (z^2 + 5)(z^2 - 1) = (z^2 - 5i^2)(z^2 - 1) = (z + i\sqrt{5})(z - i\sqrt{5})(z+1)(z-1)$$

$\therefore$ zeros are $\pm i\sqrt{5}, \pm 1$

2 **a**
$$5x^2 = 3x + 2$$
$$\therefore \quad 5x^2 - 3x - 2 = 0$$
$$\therefore \quad (5x+2)(x-1) = 0$$

$\therefore$ roots are $1, -\frac{2}{5}$

b
$$(2x+1)(x^2+3) = 0$$
$$\therefore \quad (2x+1)(x^2 - 3i^2) = 0$$
$$\therefore \quad (2x+1)(x + i\sqrt{3})(x - i\sqrt{3}) = 0$$

$\therefore$ roots are $-\frac{1}{2}, \pm i\sqrt{3}$

c $-2z(z^2 - 2z + 2) = 0$

$$z = 0 \text{ or } \frac{2 \pm \sqrt{4 - 4(1)(2)}}{2}$$
$$= 0 \text{ or } \frac{2 \pm \sqrt{-4}}{2}$$
$$= 0 \text{ or } 1 \pm i$$

$\therefore$ roots are $0, 1 \pm i$

d
$$x^3 = 5x$$
$$\therefore \quad x^3 - 5x = 0$$
$$x(x^2 - 5) = 0$$
$$x(x + \sqrt{5})(x - \sqrt{5}) = 0$$

$\therefore$ roots are $0, \pm\sqrt{5}$

e
$$z^3 + 5z = 0$$
$$z(z^2 + 5) = 0$$
$$z(z^2 - 5i^2) = 0$$
$$z(z + i\sqrt{5})(z - i\sqrt{5}) = 0$$

$\therefore$ roots are $0, \pm i\sqrt{5}$

f
$$z^4 = 3z^2 + 10$$
$$\therefore \quad z^4 - 3z^2 - 10 = 0$$
$$(z^2 - 5)(z^2 + 2) = 0$$
$$(z^2 - 5)(z^2 - 2i^2) = 0$$
$$(z + \sqrt{5})(z - \sqrt{5})(z + i\sqrt{2})(z - i\sqrt{2}) = 0$$

$\therefore$ roots are $\pm\sqrt{5}, \pm i\sqrt{2}$

3 **a**
$$2x^2 - 7x - 15 = (2x+3)(x-5)$$

b $z^2 - 6z + 16$ is zero when

$$z = \frac{6 \pm \sqrt{36 - 4(1)(16)}}{2} = 3 \pm i\sqrt{7}$$

$$\therefore \quad z^2 - 6z + 16 = (z - 3 + i\sqrt{7})(z - 3 - i\sqrt{7})$$

c
$$x^3 + 2x^2 - 4x = x(x^2 + 2x - 4)$$

$x^2 + 2x - 4$ is zero when

$$x = \frac{-2 \pm \sqrt{4 + 16}}{2} = -1 \pm \sqrt{5}$$

$$\therefore \quad x^3 + 2x^2 - 4x = x(x + 1 + \sqrt{5})(x + 1 - \sqrt{5})$$

d
$$6z^3 - z^2 - 2z = z(6z^2 - z - 2) = z(2z+1)(3z-2)$$

e
$$z^4 - 6z^2 + 5 = (z^2 - 1)(z^2 - 5) = (z+1)(z-1)(z + \sqrt{5})(z - \sqrt{5})$$

f
$$z^4 - z^2 - 2 = (z^2 - 2)(z^2 + 1) = (z + \sqrt{2})(z - \sqrt{2})(z + i)(z - i)$$

4 $P(x) = a(x-\alpha)(x-\beta)(x-\gamma)$

$\therefore$ $P(\alpha) = a \times 0 \times (\alpha-\beta)(\alpha-\gamma) = 0$

and $P(\beta) = a(\beta-\alpha) \times 0 \times (\beta-\gamma) = 0$ $\quad\therefore$ α, β, and γ all satisfy $P(x) = 0$

and $P(\gamma) = a(\gamma-\alpha)(\gamma-\beta) \times 0 = 0$ $\quad\therefore$ α, β, and γ are zeros of $P(x)$

5 **a** The zeros ± 2
have sum $= 0$ and product $= -4$
$\therefore$ come from quadratic factor $z^2 - 4$
and zero 3 comes from $(z-3)$
$\therefore$ $P(z) = a(z^2-4)(z-3),\ a \neq 0$

b The zeros $\pm i$
have sum $= 0$ and product $= 1$
$\therefore$ come from quadratic factor $z^2 + 1$
and zero -2 comes from $(z+2)$
$\therefore$ $P(z) = a(z^2+1)(z+2),\ a \neq 0$

c The zeros $-1 \pm i$
have sum $= -2$ and product $= 2$
$\therefore$ come from quadratic factor $z^2 + 2z + 2$
and zero 3 comes from $(z-3)$
$\therefore$ $P(z) = a(z-3)(z^2+2z+2),\ a \neq 0$

d The zeros $-2 \pm \sqrt{2}$
have sum $= -4$ and product $= 2$
$\therefore$ come from quadratic factor $z^2 + 4z + 2$
and zero -1 comes from $(z+1)$
$\therefore$ $P(z) = a(z+1)(z^2+4z+2),\ a \neq 0$

6 **a** For zeros of ± 1, sum $= 0$ and product $= -1$ $\quad\therefore$ come from $z^2 - 1$
For zeros of $\pm\sqrt{2}$, sum $= 0$ and product $= -2$ $\quad\therefore$ come from $z^2 - 2$
$\therefore$ $P(z) = a(z^2-1)(z^2-2),\quad a \neq 0$

b For zeros of $\pm i\sqrt{3}$, sum $= 0$ and product $= 3$ $\quad\therefore$ come from $z^2 + 3$
zeros of $2, -1$ come from $(z-2)(z+1)$
$\therefore$ $P(z) = a(z-2)(z+1)(z^2+3),\ a \neq 0$

c For zeros of $\pm\sqrt{3}$, sum $= 0$ and product $= -3$ $\quad\therefore$ come from $z^2 - 3$
For zeros of $1 \pm i$, sum $= 2$ and product $= 2$ $\quad\therefore$ come from $z^2 - 2z + 2$
$\therefore$ $P(z) = a(z^2-3)(z^2-2z+2),\quad a \neq 0$

d For zeros of $2 \pm \sqrt{5}$, sum $= 4$ and product $= -1$ $\quad\therefore$ come from $z^2 - 4z - 1$
For zeros of $-2 \pm 3i$, sum $= -4$ and product $= 13$ $\quad\therefore$ come from $z^2 + 4z + 13$
$\therefore$ $P(z) = a(z^2-4z-1)(z^2+4z+13),\quad a \neq 0$

EXERCISE 6D.2

1 **a** $2x^2 + 4x + 5 = ax^2 + (2b-6)x + c$
Equating coefficients gives
$a = 2,\ 2b - 6 = 4,$ and $c = 5$

$\therefore$ $2b = 10$
$\therefore$ $b = 5$
$\therefore$ $a = 2,\ b = 5,\ c = 5$

b
$$\begin{aligned} 2x^3 - x^2 + 6 &= (x-1)^2(2x+a) + bx + c \\ &= (x^2-2x+1)(2x+a) + bx + c \\ &= 2x^3 + (a-4)x^2 + (2-2a)x + a + bx + c \\ &= 2x^3 + (a-4)x^2 + (2-2a+b)x + (a+c) \end{aligned}$$

Equating coefficients gives $a - 4 = -1$ $\qquad 2 - 2a + b = 0$ $\qquad a + c = 6$
$\therefore$ $a = 3$ $\qquad \therefore$ $b = 2a - 2$ $\qquad \therefore$ $c = 6 - a$
$\qquad \therefore$ $b = 4$ $\qquad \therefore$ $c = 3$

2 **a**
$$\begin{aligned} z^4 + 4 &= (z^2+az+2)(z^2+bz+2) \\ &= z^4 + (a+b)z^3 + (4+ab)z^2 + (2a+2b)z + 4 \end{aligned}$$

Equating coefficients gives: $a + b = 0$ $\qquad 4 + ab = 0$
$\therefore$ $a = -b$ $\qquad \therefore$ $ab = -4$
By inspection $a = 2$ and $b = -2$
or $a = -2$ and $b = 2$

		1	a	2
	$\times$	1	b	2
		2	$2a$	4
	b	ab	$2b$	
1	a	2		
1	$a+b$	$4+ab$	$2a+2b$	4

b $\quad 2z^4 + 5z^3 + 4z^2 + 7z + 6$

$= (z^2 + az + 2)(2z^2 + bz + 3)$

$= 2z^4 + (2a + b)z^3 + (ab + 7)z^2 + (3a + 2b)z + 6$

$$\begin{array}{ccccc} & & 1 & a & 2 \\ & \times & 2 & b & 3 \\ \hline & & 3 & 3a & 6 \\ & b & ab & 2b & \\ 2 & 2a & 4 & & \\ \hline 2 & 2a+b & ab+7 & 3a+2b & 6 \\ \hline \end{array}$$

Equating coefficients gives: $2a + b = 5$ .... (1)

$3a + 2b = 7$ .... (2)

$ab + 7 = 4$ .... (3)

$\therefore \quad 4a + 2b = 10 \quad \{(1) \times 2\}$

and $\quad 3a + 2b = 7$

and solving these two equations gives $a = 3, \; b = -1$
which checks with (3) as $ab + 7 = -3 + 7 = 4$ ✓

3 Consider

$z^4 + 64$

$= (z^2 + az + 8)(z^2 + bz + 8)$

$= z^4 + (a + b)z^3 + (ab + 16)z^2 + (8a + 8b)z + 64$

$$\begin{array}{ccccc} & & 1 & a & 8 \\ & \times & 1 & b & 8 \\ \hline & & 8 & 8a & 64 \\ & b & ab & 8b & \\ 1 & a & 8 & & \\ \hline 1 & a+b & ab+16 & 8a+8b & 64 \\ \hline \end{array}$$

Equating coefficients gives:

$a + b = 0$ and $ab + 16 = 0$

$\therefore \quad a = -b \qquad \therefore \quad ab = -16$

$\therefore$ by inspection $a = 4$ and $b = -4$

or $a = -4$ and $b = 4$

$\therefore \quad z^4 + 64$ can be factorised into $(z^2 + 4z + 8)(z^2 - 4z + 8)$

Now consider

$z^4 + 64$

$= (z^2 + az + 16)(z^2 + bz + 4)$

$= z^4 + (a + b)z^3 + (ab + 20)z^2 + (4a + 16b)z + 64$

$$\begin{array}{ccccc} & & 1 & a & 16 \\ & \times & 1 & b & 4 \\ \hline & & 4 & 4a & 64 \\ & b & ab & 16b & \\ 1 & a & 16 & & \\ \hline 1 & a+b & ab+20 & 4a+16b & 64 \\ \hline \end{array}$$

Equating coefficients gives:

$a + b = 0$ .... (1) and $ab + 20 = 0$

$4a + 16b = 0$ .... (2) $\qquad \therefore \quad ab = -20$ .... (3)

Solution to (1), (2) is $a = b = 0$

But this does not satisfy (3)

$\therefore$ no values of a and b exist which obey the original assumption

$\therefore$ cannot be factorised in this way.

4 Consider

$x^4 - 4x^2 + 8x - 4$

$= (x^2 + ax + 2)(x^2 + bx - 2)$

$= x^4 + (a + b)x^3 + (ab)x^2 + (2b - 2a)x - 4$

$$\begin{array}{ccccc} & & 1 & a & 2 \\ & \times & 1 & b & -2 \\ \hline & & -2 & -2a & -4 \\ & b & ab & 2b & \\ 1 & a & 2 & & \\ \hline 1 & a+b & ab & 2b-2a & -4 \\ \hline \end{array}$$

Equating coefficients gives:

$a + b = 0$ and $ab = -4$ and $-2a + 2b = 8$

$\therefore \quad 2a + 2b = 0$ .... (1)

$-2a + 2b = 8$ .... (2)

Adding (1) and (2) gives $4b = 8 \quad \therefore \; b = 2$ and hence $a = -2$, which checks with $ab = -4$ ✓

$\therefore \quad P(x) = (x^2 - 2x + 2)(x^2 + 2x - 2)$

Now if $\quad x^4 + 8x = 4x^2 + 4$

then $\quad x^4 - 4x^2 + 8x - 4 = 0$

$\therefore \quad (x^2 - 2x + 2)(x^2 + 2x - 2) = 0$

$\therefore \quad x^2 - 2x + 2 = 0$ or $x^2 + 2x - 2 = 0$

$$\therefore \quad x = \frac{2 \pm \sqrt{4 - 8}}{2} = 1 \pm i \quad \text{or} \quad x = \frac{-2 \pm \sqrt{4 + 8}}{2} = -1 \pm \sqrt{3}$$

$$\therefore \quad x = 1 \pm i, \; -1 \pm \sqrt{3}$$

5 **a** $P(z) = 2z^3 - z^2 + az - 3$
$= (2z-3)(z^2 + bz + 1)$ for some value b
$= 2z^3 + (2b-3)z^2 + (2-3b)z - 3$

	1	b	1
	$\times$	2	-3
	-3	$-3b$	-3
2	$2b$	2	
2	$2b-3$	$2-3b$	-3

Equating coefficients gives:
$2b - 3 = -1$ and $2 - 3b = a$
$2b = 2$ $\therefore$ $a = 2 - 3$
$b = 1$ $a = -1$

$\therefore$ $P(z) = (2z-3)\underbrace{(z^2 + z + 1)}$

this quadratic has zeros $z = \dfrac{-1 \pm \sqrt{1-4}}{2} = \dfrac{-1 \pm i\sqrt{3}}{2}$

$\therefore$ $a = -1$ and zeros are $\frac{3}{2}$, $-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

b $P(z) = 3z^3 - z^2 + (a+1)z + a$
$= (3z+2)(z^2 + bz + c)$
$= 3z^3 + (2+3b)z^2 + (2b+3c)z + 2c$

	1	b	c
	$\times$	3	2
	2	$2b$	$2c$
3	$3b$	$3c$	
3	$2+3b$	$2b+3c$	$2c$

Equating coefficients gives:
$\therefore$ $2 + 3b = -1$, $2b + 3c = a + 1$, and $2c = a$
Now as $2 + 3b = -1$
$\therefore$ $3b = -3$
$\therefore$ $b = -1$

Substituting $b = -1$ and $a = 2c$ into $2b + 3c = a + 1$ gives $2(-1) + 3c = 2c + 1$
$\therefore$ $-2 + 3c = 2c + 1$
$c = 3$
and so $a = 6$

$\therefore$ $P(z) = (3z+2)\underbrace{(z^2 - z + 3)}$

this quadratic has zeros $\dfrac{1 \pm \sqrt{1 - 4(3)(1)}}{2} = \dfrac{1 \pm i\sqrt{11}}{2}$

So, $a = 6$ and the zeros are $-\frac{2}{3}$, $\frac{1}{2} \pm i\frac{\sqrt{11}}{2}$.

6 **a** $P(x) = 2x^4 + ax^3 + bx^2 - 12x - 8$
$= (2x+1)(x-2)(x^2 + cx + 4)$
$= (2x^2 - 3x - 2)(x^2 + cx + 4)$
$= 2x^4 + (2c-3)x^3 + (6-3c)x^2$
$+ (-2c-12)x - 8$

		2	-3	-2
	$\times$	1	c	4
		8	-12	-8
	$2c$	$-3c$	$-2c$	
2	-3	-2		
2	$2c-3$	$6-3c$	$-2c-12$	-8

Equating coefficients: $2c - 3 = a$,
$6 - 3c = b$, and $-2c - 12 = -12$
The last equation has solution $c = 0$, and consequently, $a = -3$ and $b = 6$
$\therefore$ $P(x) = (2x+1)(x-2)(x^2+4) = (2x+1)(x-2)(x+2i)(x-2i)$
$\therefore$ zeros are $-\frac{1}{2}$, 2, and $\pm 2i$ and $a = -3$, $b = 6$.

b $P(x) = 2x^4 + ax^3 + bx^2 + ax + 3$
$= (x+3)(2x-1)(x^2 + cx - 1)$
$= (2x^2 + 5x - 3)(x^2 + cx - 1)$
$= 2x^4 + (2c+5)x^3 + (5c-5)x^2$
$+ (-5-3c)x + 3$

		2	5	-3
	$\times$	1	c	-1
		-2	-5	3
	$2c$	$5c$	$-3c$	
2	5	-3		
2	$2c+5$	$5c-5$	$-5-3c$	3

Equating coefficients: $a = 2c + 5$,
$b = 5c - 5$, $a = -5 - 3c$

$\therefore \quad 2c+5=-5-3c \qquad \{\text{equating } a\text{s}\}$

$\therefore \quad 5c=-10$

$\therefore \quad c=-2$ and so, $a=1 \quad b=-15$

$\therefore \quad P(x)=(x+3)(2x-1)\underbrace{(x^2-2x-1)}_{\text{this quadratic has zeros } \frac{2\pm\sqrt{4+4}}{2}=1\pm\sqrt{2}}$

$\therefore$ zeros are $-3, \frac{1}{2}, 1\pm\sqrt{2}$ and $a=1, \ b=-15$

7 **a** x^3+3x^2-9x+c

$=(x+a)^2(x+b)$

$=(x^2+2ax+a^2)(x+b)$

$=x^3+(b+2a)x^2+(a^2+2ab)x+a^2b$

	1	$2a$	a^2
	$\times$	1	b
	b	$2ab$	a^2b
1	$2a$	a^2	
1	$b+2a$	a^2+2ab	a^2b

Equating coefficients gives

$2a+b=3, \quad a^2+2ab=-9,$ and $c=a^2b$

Substituting $b=3-2a$ into the second equation gives:

$a^2+2a(3-2a)=-9$

$\therefore \quad a^2+6a-4a^2=-9$

$\therefore \quad -3a^2+6a=-9$

$\therefore \quad 3a^2-6a-9=0$

$\therefore \quad a^2-2a-3=0$

$\therefore \quad (a-3)(a+1)=0$

$\therefore \quad \begin{cases} a=3 \\ b=-3 \\ c=-27 \end{cases}$ or $\begin{cases} a=-1 \\ b=5 \\ c=5 \end{cases}$

If $c=-27, \quad P(x)=(x+3)^2(x-3)$

If $c=5, \quad P(x)=(x-1)^2(x+5)$

b $3x^3+4x^2-x+m$

$=(x+a)^2(3x+b)$

$=(x^2+2ax+a^2)(3x+b)$

$=3x^3+(6a+b)x^2+(3a^2+2ab)x+a^2b$

	1	$2a$	a^2
	$\times$	3	b
	b	$2ab$	a^2b
3	$6a$	$3a^2$	
3	$6a+b$	$3a^2+2ab$	a^2b

Equating coefficients gives

$6a+b=4, \ 3a^2+2ab=-1,$ and $a^2b=m$

Substituting $b=4-6a$ into the second equation gives: $3a^2+2a(4-6a)=-1$

$\therefore \quad 3a^2+8a-12a^2=-1$

$\therefore \quad 9a^2-8a-1=0$

$\therefore \quad (9a+1)(a-1)=0$

$\therefore \quad a=-\frac{1}{9}$ or $a=1$

When $a=1, \ b=-2$ and $m=-2$. So, $P(x)=(x+1)^2(3x-2) \quad \therefore$ zeros are $-1, \frac{2}{3}$

When $a=-\frac{1}{9}, \ b=\frac{14}{3}$ and $m=\frac{14}{243}$. So, $P(x)=(x-\frac{1}{9})^2(3x+\frac{14}{3})$

$\therefore$ zeros are $\frac{1}{9}, -\frac{14}{9}$

EXERCISE 6E.1

1 **a** If $P(2)=7$, then $P(x)=Q(x)(x-2)+7$ and $P(x)$ divided by $(x-2)$ leaves a remainder of 7.

b If $P(x)=Q(x)(x+3)-8$, then $P(-3)=-8$ and $P(x)$ divided by $(x+3)$ leaves a remainder of -8.

c If $P(x)$ when divided by $(x-5)$ has a remainder of 11, then $P(5)=11$ and $P(x)=Q(x)(x-5)+11$.

2 **a** $P(x)=x^3+2x^2-7x+5$

$\therefore \quad R=P(1) \quad \{\text{Remainder theorem}\}$

$=1^3+2(1)^2-7+5$

$=1$

b $P(x)=x^4-2x^2+3x-1$

$\therefore \quad R=P(-2) \quad \{\text{Remainder theorem}\}$

$=(-2)^4-2(-2)^2+3(-2)-1$

$=16-8-6-1$

$=1$

3 **a** $P(x) = x^3 - 2x + a$

Now $P(2) = 7$ {Remainder theorem}

$\therefore\ 2^3 - 2(2) + a = 7$

$4 + a = 7$

$\therefore\ a = 3$

b $P(x) = 2x^3 + x^2 + ax - 5$

Now $P(-1) = -8$

$\therefore\ 2(-1)^3 + (-1)^2 + a(-1) - 5 = -8$

$-2 + 1 - a - 5 = -8$

$\therefore\ -a - 6 = -8$

$-a = -2$

$\therefore\ a = 2$

4 $P(x) = x^3 + 2x^2 + ax + b$

Now $P(1) = 4$ and $P(-2) = 16$ {Remainder theorem}

If $P(1) = 4$ then $1 + 2 + a + b = 4$ and so $a + b = 1$ .... (1)

If $P(-2) = 16$ then $(-2)^3 + 2(-2)^2 + a(-2) + b = 16$

$\therefore\ -8 + 8 - 2a + b = 16$

$\therefore\ -2a + b = 16$ .... (2)

Solving (1) and (2) $-a - b = -1$

$-2a + b = 16$

$\therefore\ -3a = 15$ {adding}

$\therefore\ a = -5$ and so $b = 6$

$\therefore\ a = -5$ and $b = 6$

5 $P(x) = 2x^n + ax^2 - 6$

By the Remainder theorem, $P(1) = -7$ $\therefore\ 2(1)^n + a(1)^2 - 6 = -7$

$\therefore\ 2 + a - 6 = -7$

$\therefore\ a = -3$

So, $P(x) = 2x^n - 3x^2 - 6$

and since $P(-3) = 129$, $\therefore\ 2(-3)^n - 3(-3)^2 - 6 = 129$

$2(-3)^n - 27 - 6 = 129$

$2(-3)^n = 162$

$(-3)^n = 81$

$\therefore\ n = 4$

$\therefore\ a = -3$ and $n = 4$

6 $P(z) = Q(z)(z^2 - 3z + 2) + (4z - 7) = Q(z)(z - 2)(z - 1) + (4z - 7)$

a Remainder is $P(1)$ {Remainder theorem}

$\therefore\ R = Q(1) \times (1 - 2) \times 0 + (4 - 7)$

$= -3$

b Remainder is $P(2)$ {Remainder theorem}

$\therefore\ R = Q(2) \times 0 \times (2 - 1) + [4(2) - 7]$

$= 0 + 1$

$= 1$

7 Suppose $P(z)$ is divided by $(z - 3)(z + 1)$

$\therefore\ P(z) = Q(z) \times (z - 3)(z + 1) + (Az + B)$

↑ the remainder must be of this form

Now $P(-1) = -8$ $\therefore\ Q(-1) \times 0 + (-A + B) = -8$

$\therefore\ -A + B = -8$ .... (1)

and $P(3) = 4$ $\therefore\ Q(3) \times 0 + (3A + B) = 4$

$\therefore\ 3A + B = 4$ .... (2)

Solving (1) and (2), $-A + B = -8$

$-3A - B = -4$

$-4A = -12$

$\therefore\ A = 3$ and so $B = -5$

$\therefore\ R(z) = 3z - 5$

8 Suppose $P(x)$ is divided by $(x-a)(x-b)$ and has remainder $Ex+F$

hence $P(x) = Q(x) \times (x-a)(x-b) + Ex + F$

Now $P(a) = Ea + F$ .... (1) and $P(b) = Eb + F$ .... (2)

Subtracting (1) from (2), $P(b) - P(a) = Eb - Ea = E(b-a)$

$$\therefore \quad E = \frac{P(b)-P(a)}{b-a}$$

$$\therefore \text{ from (1), } \quad F = P(a) - Ea = P(a) - \left(\frac{P(b)-P(a)}{b-a}\right)a$$

$$\text{Now} \quad R(x) = Ex + F$$

$$\therefore \quad R(x) = \left(\frac{P(b)-P(a)}{b-a}\right)x + P(a) - \left(\frac{P(b)-P(a)}{b-a}\right)a$$

$$\therefore \quad R(x) = \left(\frac{P(b)-P(a)}{b-a}\right)(x-a) + P(a)$$

EXERCISE 6E.2

1 **a** $P(x) = 2x^3 + x^2 + kx - 4$

$$\begin{array}{r|rrr|r} -2 & 2 & 1 & k & -4 \\ & 0 & -4 & 6 & -2k-12 \\ \hline & 2 & -3 & k+6 & -2k-16 \end{array}$$

if $(x+2)$ is a factor then $P(-2) = 0$

$\therefore \quad -2k - 16 = 0$

$\therefore \quad k = -8$

$\therefore \quad P(x) = 2x^3 + x^2 - 8x - 4$

$= (x+2)(2x^2 - 3x - 2)$ $\{$as when $k=-8$, $k+6=-2\}$

$\therefore \quad P(x) = (x+2)(2x+1)(x-2)$ and $k = -8$

b $P(x) = x^4 - 3x^3 - kx^2 + 6x$

$$\begin{array}{r|rrrr|r} 3 & 1 & -3 & -k & 6 & 0 \\ & 0 & 3 & 0 & -3k & 18-9k \\ \hline & 1 & 0 & -k & 6-3k & 18-9k \end{array}$$

if $(x-3)$ is a factor then $P(3) = 0$

$\therefore \quad 18 - 9k = 0$

$\therefore \quad 9k = 18$

$\therefore \quad k = 2$

$\therefore \quad P(x) = x^4 - 3x^3 - 2x^2 + 6x$

$\therefore \quad P(x) = (x-3)(x^3 - 2x)$ $\{$as when $k = 2$, $-k = -2$, and $6 - 3k = 0\}$

$= x(x-3)(x^2-2)$

$= x(x-3)(x+\sqrt{2})(x-\sqrt{2})$ and $k = 2$

2 $P(x) = 2x^3 + ax^2 + bx + 5$

if $(x-1)$ is a factor, $P(1) = 0$

$\therefore \quad 2(1)^3 + a(1)^2 + b(1) + 5 = 0$

$2 + a + b + 5 = 0$

$\therefore \quad a + b = -7$ .... (1)

if $(x+5)$ is a factor, $P(-5) = 0$

$2(-5)^3 + a(-5)^2 + b(-5) + 5 = 0$

$-250 + 25a - 5b + 5 = 0$

$25a - 5b = 245$

$\therefore \quad 5a - b = 49$ .... (2)

Adding (1) and (2) gives: $6a = 42$

$\therefore \quad a = 7$ and $b = -14$

3 **a** $P(z) = z^3 - z^2 + (k-5)z + (k^2 - 7)$

$$\begin{array}{r|rrr|r} 3 & 1 & -1 & k-5 & k^2-7 \\ & 0 & 3 & 6 & 3k+3 \\ \hline & 1 & 2 & k+1 & k^2+3k-4 \end{array}$$

if 3 is a zero, $R = P(3) = 0$

$\therefore \quad k^2 + 3k - 4 = 0$

$(k+4)(k-1) = 0$

$\therefore \quad k = -4$ or $k = 1$

if $k=1$, $P(z)=(z-3)(z^2+2z+2)$

the quadratic has zeros:

$$\frac{-2\pm\sqrt{4-8}}{2}=-1\pm i$$

$\therefore$ zeros are $3, -1\pm i$

if $k=-4$, $P(z)=(z-3)(z^2+2z-3)$
$=(z-3)(z+3)(z-1)$

$\therefore$ zeros are $3, -3$, and 1

b $P(z)=z^3+mz^2+(3m-2)z-10m-4$

if $(z-2)$ is a factor, $P(2)=0$

since $*=0$, R_1 is always 0

$\therefore$ $(z-2)$ is always a factor

now for $(z-2)^2$ to be a factor

$7m+10=0$ $\{R_2$ is also $0\}$

$\therefore$ $m=-\frac{10}{7}$

$$\begin{array}{c|cccc}
2 & 1 & m & 3m-2 & -10m-4\\
 & 0 & 2 & 2m+4 & 10m+4\\
\hline
2 & 1 & m+2 & 5m+2 & 0\ldots\ (*)\\
 & 0 & 2 & 2m+8 & \\
\hline
 & 1 & m+4 & 7m+10 &
\end{array}$$

4 **a** **i** $P(x)=x^3-a^3$

$P(a)=a^3-a^3$
$=0$

$\therefore$ $(x-a)$ is a linear factor of $P(x)$ for all a

ii
$$\begin{array}{c|cccc}
a & 1 & 0 & 0 & -a^3\\
 & 0 & a & a^2 & a^3\\
\hline
 & 1 & a & a^2 & 0
\end{array}$$

$\therefore$ $P(x)=(x-a)(x^2+ax+a^2)$

b **i** $P(x)=x^3+a^3$

$P(-a)=-a^3+a^3$
$=0$

$\therefore$ $(x+a)$ is a factor of $P(x)$ for all a

ii
$$\begin{array}{c|cccc}
-a & 1 & 0 & 0 & a^3\\
 & 0 & -a & a^2 & -a^3\\
\hline
 & 1 & -a & a^2 & 0
\end{array}$$

$\therefore$ $P(x)=(x+a)(x^2-ax+a^2)$

5 **a** Consider $P(x)=x^n+1$

if $(x+1)$ is a factor then

$P(-1)=0$

$\therefore$ $(-1)^n+1=0$

$\therefore$ $(-1)^n=-1$

which is only true if n is odd

$\therefore$ $(x+1)$ is a factor of $x^n+1 \Leftrightarrow n$ is odd.

if n is odd, $(-1)^n=-1$

$\therefore$ $(-1)^n+1=0$

then $P(-1)=0$ if $P(x)=x^n+1$

$\therefore$ $x=-1$ is a zero of $P(x)$

$\therefore$ $(x+1)$ is a factor of $P(x)$

b $P(x)=x^3-3ax-9$ and if $(x-1-a)$ is a factor then $P(1+a)=0$

$$\begin{array}{c|cccc}
1+a & 1 & 0 & -3a & -9\\
 & 0 & 1+a & 1+2a+a^2 & a^3+1\\
\hline
 & 1 & 1+a & a^2-a+1 & a^3-8
\end{array}$$

$\therefore$ $a^3-8=0$

$\therefore$ $a=2$ {the only real solution}

EXERCISE 6E.3

1 Since it is a real polynomial, the zeros must be $-\frac{1}{2}$, $1-3i$, and $1+3i$.

For $1\pm 3i$, $\alpha+\beta=2$, and $\alpha\beta=1-9i^2=10$

$\therefore$ factors are $(2x+1)$ and $(x^2-2x+10)$

$\therefore$ $P(x)=a(2x+1)(x^2-2x+10)$, $a\neq 0$

2 $p(1) = p(2+i) = 0$

Hence zeros of $p(x)$ must be $1, 2 \pm i$ {as $p(x)$ is real}

For $2 \pm i$, $\alpha + \beta = 4$ and $\alpha\beta = 4 - i^2 = 5$

$\therefore$ factors must be $(x-1)$ and $(x^2 - 4x + 5)$

$\therefore$ $p(x) = k(x-1)(x^2 - 4x + 5)$

Since $p(0) = -20$ then $-20 = k(-1)(5)$

$\therefore$ $k = 4$

$\therefore$ $p(x) = 4(x-1)(x^2 - 4x + 5)$

$\therefore$ $p(x) = 4x^3 - 20x^2 + 36x - 20$

$$\begin{array}{cccc} & 1 & -4 & 5 \\ & \times & 4 & -4 \\ \hline & -4 & 16 & -20 \\ 4 & -16 & 20 & \\ \hline 4 & -20 & 36 & -20 \\ \hline \end{array}$$

3 **a** $2 - 3i$ is a zero of $z^3 + pz + q$ and as the cubic has real coefficients, $2 + 3i$ is also a zero.

For $2 \pm 3i$, $\alpha + \beta = 4$ and $\alpha\beta = 4 - 9i^2 = 13$

$\therefore$ $z^2 - 4z + 13$ is a factor

$\therefore$ $z^3 + pz + q = (z^2 - 4z + 13)(z + a)$ for some a

$= z^3 + (a-4)z^2 + (13 - 4a)z + 13a$

$$\begin{array}{cccc} & 1 & -4 & 13 \\ & \times & 1 & a \\ \hline & a & -4a & 13a \\ 1 & -4 & 13 & \\ \hline 1 & a-4 & 13-4a & 13a \\ \hline \end{array}$$

Equating coefficients:

$a - 4 = 0$, $\quad 13 - 4a = p$, $\quad$ and $\quad 13a = q$

$\therefore$ $a = 4$, $\quad p = -3$, $\quad q = 52$

$\therefore$ the other zeros are -4 and $2 + 3i$

b *Check*: Since $P(2-3i) = 0$, $(2-3i)^3 + p(2-3i) + q = 0$

Expanding, $(-46 - 9i) + p(2 - 3i) + q = 0$

$\therefore$ $(-46 + 2p + q) + (-9 - 3p)i = 0$

Equating real and imaginary parts, $-46 + 2p + q = 0$ .... (1)

and $-9 - 3p = 0$ .... (2)

From (2), $p = -3$, so in (1), $-46 - 6 + q = 0$ $\therefore$ $p = -3$, $q = 52$ ✓

4 $3 + i$ is a root of $z^4 - 2z^3 + az^2 + bz + 10 = 0$ where the coefficients are real.

$\therefore$ $3 - i$ is also a root

For $3 \pm i$, $\alpha + \beta = 6$ and $\alpha\beta = 9 - i^2 = 10$

$\therefore$ $z^2 - 6z + 10$ is a factor

$\therefore$ $z^4 - 2z^3 + az^2 + bz + 10$

$= (z^2 - 6z + 10)(z^2 + sz + 1)$ for some s

$$\begin{array}{ccccc} & & 1 & -6 & 10 \\ & \times & 1 & s & 1 \\ \hline & & 1 & -6 & 10 \\ & s & -6s & 10s & \\ 1 & -6 & 10 & & \\ \hline 1 & s-6 & 11-6s & 10s-6 & 10 \\ \hline \end{array}$$

Equating coefficients:

$s - 6 = -2$, $\quad 11 - 6s = a$ $\quad$ and $\quad 10s - 6 = b$

$\therefore$ $s = 4$ $\quad a = 11 - 6(4) = -13$ $\quad b = 10(4) - 6 = 34$

$\therefore$ the other factor is $z^2 + 4z + 1$ which has zeros $\dfrac{-4 \pm \sqrt{16 - 4}}{2} = -2 \pm \sqrt{3}$

$\therefore$ $a = -13$, $b = 34$ and the other roots are $3 - i$, $-2 \pm \sqrt{3}$.

5 Let the purely imaginary zero be bi. Since $P(z)$ is real, another zero is $-bi$ (b is real).

$\therefore$ $z^2 + b^2$ is a factor of $P(z)$

$\therefore$ $z^3 + az^2 + 3z + 9 = (z^2 + b^2)(z + c)$

$= z^3 + cz^2 + b^2 z + b^2 c$

Equating coefficients, $b^2 = 3$, $b^2 c = 9$, and $a = c$

$\therefore$ $c = 3$, $a = 3$, and $b = \pm\sqrt{3}$

$\therefore$ $P(z) = (z+3)(z^2 + 3)$

$\therefore$ $P(z) = (z+3)(z + i\sqrt{3})(z - i\sqrt{3})$, $a = 3$

6 Let ai be the purely imaginary zero of $3x^3+kx^2+15x+10$

$\therefore$ as $P(x)$ is real, $-ai$ is also a zero

For $\pm ai$, $\alpha+\beta=0$ and $\alpha\beta=-a^2i^2=a^2$

$\therefore$ x^2+a^2 is a factor

$\therefore$ $3x^3+kx^2+15x+10=(x^2+a^2)(3x+b)$

$=3x^3+bx^2+3a^2x+a^2b$

	1	0	a^2
	$\times$	3	b
	b	0	a^2b
3	0	$3a^2$	
3	b	$3a^2$	a^2b

Equating coefficients, $k=b$

and $3a^2=15$ $\therefore$ $a^2=5$

and $a^2b=10$ $\therefore$ $b=2$ $\therefore$ $k=2$

$\therefore$ $P(x)=(x^2+5)(3x+2)$

$\therefore$ $P(x)=(3x+2)(x-i\sqrt{5})(x+i\sqrt{5})$, $k=2$

EXERCISE 6E.4

1 **a** $2x^2-3x+4=0$

$\therefore$ the sum of the roots $=-\dfrac{(-3)}{2}=\frac{3}{2}$

The polynomial equation has degree 2.

$\therefore$ the product of the roots $=\dfrac{(-1)^2 4}{2}=2$

b $3x^3-4x^2+8x-5=0$

$\therefore$ the sum of the roots $=-\dfrac{(-4)}{3}=\frac{4}{3}$

The polynomial equation has degree 3.

$\therefore$ the product of the roots $=\dfrac{(-1)^3(-5)}{3}=\frac{5}{3}$

c $x^4-x^3+2x^2+3x-4=0$

$\therefore$ the sum of the roots $=-\dfrac{(-1)}{1}=1$

The polynomial equation has degree 4.

$\therefore$ the product of the roots $=\dfrac{(-1)^4(-4)}{1}=-4$

d $2x^5-3x^4+x^2-8=0$

$\therefore$ the sum of the roots $=-\dfrac{(-3)}{2}=\frac{3}{2}$

The polynomial equation has degree 5.

$\therefore$ the product of the roots $=\dfrac{(-1)^5(-8)}{2}=4$

e $x^7-x^5+2x-9=0$

$\therefore$ $x^7+(0)x^6-x^5+2x-9=0$

$\therefore$ the sum of the roots $=-\frac{0}{1}=0$

The polynomial equation has degree 7.

$\therefore$ the product of the roots $=\dfrac{(-1)^7(-9)}{1}=9$

f $x^6-1=0$

$\therefore$ $x^6+(0)x^5-1=0$

$\therefore$ the sum of the roots $=-\dfrac{0}{1}=0$

The polynomial equation has degree 6.

$\therefore$ the product of the roots $=\dfrac{(-1)^6(-1)}{1}=-1$

2 **a** The cubic polynomial $P(x)$ has zeros $3\pm i\sqrt{2}$ and $\frac{2}{3}$.

$\therefore$ the sum of the zeros $=(3+i\sqrt{2})+(3-i\sqrt{2})+\frac{2}{3}$

$=6+\frac{2}{3}$

$=\frac{20}{3}$

The product of the zeros $=\frac{2}{3}\times(3+i\sqrt{2})(3-i\sqrt{2})$

$=\frac{2}{3}(9+2)$

$=\frac{22}{3}$

b Let $P(x)=6x^3+ax^2+bx+c$ {leading coefficient is 6}.

$\therefore$ the sum of the zeros is $-\dfrac{a}{6}=\dfrac{20}{3}$

$\therefore$ $-a=\frac{120}{3}$

$\therefore$ $a=-40$

So, the coefficient of x^2 is -40.

c The product of the zeros is $\dfrac{(-1)^3c}{6}=\dfrac{22}{3}$

$\therefore$ $-c=\frac{132}{3}$

$\therefore$ $c=-44$

So, the constant term is -44.

3 **a** A polynomial has zeros $-2,\ 3 \pm i,$ and $\sqrt{k} \pm 1$.

The constant term is the y-intercept. $\therefore$ the constant term is 18.

$\therefore$ the product of the zeros is $-2(3+i)(3-i)(\sqrt{k}+1)(\sqrt{k}-1) = \dfrac{(-1)^5 18}{-1} = 18$

$$\begin{aligned} \therefore\ -2(9+1)(k-1) &= 18 \\ \therefore\ -20(k-1) &= 18 \\ \therefore\ k-1 &= \tfrac{-9}{10} \\ \therefore\ k &= \tfrac{1}{10} \end{aligned}$$

b Let a be the coefficient of x^4.

$\therefore$ the sum of the zeros is

$$-2 + (3+i) + (3-i) + \left(\sqrt{\tfrac{1}{10}} + 1\right) + \left(\sqrt{\tfrac{1}{10}} - 1\right) = -\frac{a}{-1} = a$$

$$\begin{aligned} \therefore\ a &= -2 + 6 + 2\sqrt{\tfrac{1}{10}} \\ &= 4 + \frac{2}{\sqrt{10}} \\ &= \frac{4\sqrt{10}+2}{\sqrt{10}}\left(\frac{\sqrt{10}}{\sqrt{10}}\right) \\ &= \frac{40+2\sqrt{10}}{10} \\ &= \frac{20+\sqrt{10}}{5} \end{aligned}$$

So, the coefficient of x^4 is $\dfrac{20+\sqrt{10}}{5}$.

4 The polynomial has zeros $\frac{1}{2},\ 1 \pm \sqrt{2},$ and $m \pm ni$.

$\therefore$ the sum of the zeros is $\frac{1}{2} + (1+\sqrt{2}) + (1-\sqrt{2}) + (m+ni) + (m-ni) = -\frac{3}{2}$

$$\begin{aligned} \therefore\ \tfrac{1}{2} + 2 + 2m &= -\tfrac{3}{2} \\ \therefore\ \tfrac{5}{2} + 2m &= -\tfrac{3}{2} \\ \therefore\ 2m &= -4 \\ \therefore\ m &= -2 \end{aligned}$$

The constant term is the y-intercept. $\therefore$ the constant term is 5.

$\therefore$ the product of the zeros is $\frac{1}{2} \times (1+\sqrt{2})(1-\sqrt{2})(-2+ni)(-2-ni) = \dfrac{(-1)^5 5}{2}$

$$\begin{aligned} \therefore\ \tfrac{1}{2}(1-2)(4+n^2) &= -\tfrac{5}{2} \\ \therefore\ -\tfrac{1}{2}(4+n^2) &= -\tfrac{5}{2} \\ \therefore\ 4+n^2 &= 5 \\ \therefore\ n^2 &= 1 \\ \therefore\ n &= 1 \qquad \{n > 0\} \end{aligned}$$

So, $m = -2$ and $n = 1$.

5 The quartic polynomial has zeros $a \pm i$ and $3 \pm a$.

$\therefore$ the product of the zeros is $(a+i)(a-i)(3+a)(3-a) = \dfrac{(-1)^4 25}{1}$

$$\begin{aligned}
\therefore\ (a^2+1)(9-a^2) &= 25\\
\therefore\ -a^4+8a^2+9 &= 25\\
\therefore\ a^4-8a^2+16 &= 0\\
\therefore\ (a^2)^2-8(a^2)+16 &= 0\\
\therefore\ (a^2-4)^2 &= 0\\
\therefore\ a^2 &= 4\\
\therefore\ a &= \pm 2
\end{aligned}$$

6 **a** $x^3 - px^2 + qx - r = 0$ has non-zero roots p, q, and r.

$\therefore$ the sum of the roots is

$$\begin{aligned}
p+q+r &= -\frac{-p}{1}\\
\therefore\ p+q+r &= p\\
\therefore\ q+r &= 0\\
\therefore\ q &= -r
\end{aligned}$$

The product of the roots is

$$\begin{aligned}
pqr &= \frac{(-1)^3(-r)}{1}\\
\therefore\ pqr &= r\\
\therefore\ pq &= 1\\
\therefore\ p &= \frac{1}{q}\\
\therefore\ p &= -\frac{1}{r} \quad \{\text{as } q=-r\}
\end{aligned}$$

So, $q = -r$ and $p = -\dfrac{1}{r}$.

b By the Factor Theorem, if p, q, and r are zeros of the polynomial, then $(x-p)$, $(x-q)$, and $(x-r)$ are factors.

$$\begin{aligned}
\therefore\ x^3-px^2+qx-r &= (x-p)(x-q)(x-r)\\
&= (x^2-(p+q)x+pq)(x-r)\\
&= x^3-(p+q+r)x^2+(pq+pr+qr)x-pqr
\end{aligned}$$

Equating coefficients of x:

$$\begin{aligned}
pq+pr+qr &= q\\
\therefore\ \left(-\frac{1}{r}\right)(-r)+\left(-\frac{1}{r}\right)r+(-r)r &= -r \quad \{\text{substituting } -\tfrac{1}{r} \text{ for } p, \text{ and } -r \text{ for } q\}\\
\therefore\ 1-1-r^2 &= -r\\
\therefore\ r^2-r &= 0\\
\therefore\ r(r-1) &= 0\\
\therefore\ r=0 \text{ or } r &= 1\\
\therefore\ r &= 1 \quad \{r \text{ is non-zero}\}
\end{aligned}$$

$\therefore\ p = -\dfrac{1}{r} = -1$ and $q = -r = -1$

So, $p=-1$, $q=-1$, and $r=1$.

EXERCISE 6F.1

1 **a** A single factor such as $(x-\alpha)$ indicates that the graph *cuts* the x-axis at α.

b A squared factor such as $(x-\alpha)^2$ indicates that the graph *touches* the x-axis at α.

c A cubed factor such as $(x-\alpha)^3$ indicates that the graph *cuts* the x-axis at α, and at α the graph changes shape.

2 **a** The x-intercepts are: -1, 2, and 3

As the curve passes through (0, 12),

$\therefore\ P(x) = a(x+1)(x-2)(x-3),\ a \neq 0$

$12 = a(1)(-2)(-3) \quad \therefore\ a = 2$

$\therefore\ P(x) = 2(x+1)(x-2)(x-3)$

b The x-intercepts are: -3, $-\frac{1}{2}$, and $\frac{1}{2}$ $\therefore\ P(x) = a(x+3)(2x+1)(2x-1),\quad a \neq 0$

As the curve passes through $(0, 6)$, $6 = a(3)(1)(-1) \quad \therefore\ a = -2$

$\therefore\ P(x) = -2(x+3)(2x+1)(2x-1)$

c The x-intercepts are: -4, -4, and 3 $\therefore\ P(x) = a(x+4)^2(x-3),\quad a \neq 0$

As the curve passes through $(0, -12)$, $-12 = a(4)^2(-3) \quad \therefore\ a = \frac{1}{4}$

$\therefore\ P(x) = \frac{1}{4}(x+4)^2(x-3)$

d The x-intercepts are: -5, -2, and 5 $\therefore\ P(x) = a(x+5)(x+2)(x-5),\quad a \neq 0$

As the curve passes through $(0, -5)$, $-5 = a(5)(2)(-5) \quad \therefore\ a = \frac{1}{10}$

$\therefore\ P(x) = \frac{1}{10}(x+5)(x+2)(x-5)$

e The x-intercepts are: -4, 3, and 3 $\therefore\ P(x) = a(x+4)(x-3)^2,\quad a \neq 0$

As the curve passes through $(0, 9)$, $9 = a(4)(-3)^2 \quad \therefore\ a = \frac{1}{4}$

$\therefore\ P(x) = \frac{1}{4}(x+4)(x-3)^2$

f The x-intercepts are: -3, -2, and $-\frac{1}{2}$ $\therefore\ P(x) = a(x+3)(x+2)(2x+1),\quad a \neq 0$

As the curve passes through $(0, -12)$, $-12 = a(3)(2)(1) \quad \therefore\ a = -2$

$\therefore\ P(x) = -2(x+3)(x+2)(2x+1)$

3 **a** $P(x) = a(x-3)(x-1)(x+2)$

Since $P(x)$ passes through $(2, -4)$,

$-4 = a(-1)(1)(4)$

$\therefore\ -4 = -4a$

$\therefore\ a = 1$

$\therefore\ P(x) = (x-3)(x-1)(x+2)$

b $P(x) = ax(x+2)(2x-1)$

Since $P(x)$ passes through $(-3, -21)$,

$-21 = -3a(-1)(-7)$

$\therefore\ -21 = -21a$

$\therefore\ a = 1$

$\therefore\ P(x) = x(x+2)(2x-1)$

c $P(x) = a(x-1)^2(x+2)$

Since $P(x)$ passes through $(4, 54)$,

$54 = a(9)(6)$

$\therefore\ a = 1$

$\therefore\ P(x) = (x-1)^2(x+2)$

d $P(x) = a(3x+2)^2(x-4)$

Since $P(x)$ passes through $(-1, -5)$,

$-5 = a(1)(-5)$

$\therefore\ a = 1$

$\therefore\ P(x) = (3x+2)^2(x-4)$

4 **a** $y = 2(x-1)(x+2)(x+4)$

has x-intercepts $1, -2, -4$

has y-intercept $2(-1)(2)(4) = -16$

$\therefore$ matches graph **F**

b $y = -(x+1)(x-2)(x-4)$

has x-intercepts $-1, 2, 4$

has y-intercept $-(1)(-2)(-4) = -8$

$\therefore$ matches graph **C**

c $y = (x-1)(x-2)(x+4)$

has x-intercepts $1, 2, -4$

has y-intercept $(-1)(-2)(4) = 8$

$\therefore$ matches graph **A**

d $y = -2(x-1)(x+2)(x+4)$

has x-intercepts $1, -2, -4$

has y-intercept $-2(-1)(2)(4) = 16$

$\therefore$ matches graph **E**

e $y = -(x-1)(x+2)(x+4)$

has x-intercepts $1, -2, -4$

has y-intercept $-(-1)(2)(4) = 8$

$\therefore$ matches graph **D**

f $y = 2(x-1)(x-2)(x+4)$

has x-intercepts $1, 2, -4$

has y-intercept $2(-1)(-2)(4) = 16$

$\therefore$ matches graph **B**

5 **a** $\frac{1}{2}$ and -3 are zeros, and so $(2x-1)$ and $(x+3)$ are factors

$\therefore\ P(x) = (2x-1)(x+3)(ax+b)$

But $P(0) = 30$

$\therefore\ b(-1)(3) = 30$ and so $b = -10$

$\therefore\ P(x) = (2x-1)(x+3)(ax-10)$

Now $P(1) = (1)(4)(a-10) = -20$

$\therefore\ a - 10 = -5$ and so $a = 5$

$\therefore\ P(x) = (2x-1)(x+3)(5x-10)$

$\therefore\ P(x) = 5(x-2)(2x-1)(x+3)$

b 1 is a zero and so $(x-1)$ is a factor, touches at -2 indicates that $(x+2)^2$ is a factor

$\therefore\ P(x) = k(x-1)(x+2)^2$

But $P(0) = 8$ $\therefore\ 8 = k(-1)(2)^2$ and so $k = -2$

$\therefore\ P(x) = -2(x-1)(x+2)^2$

c cuts the x-axis at $(2, 0)$ and so $(x-2)$ is a factor

$\therefore\ P(x) = (x-2)(ax^2+bx+c)$

But $P(0) = -4$ $\therefore\ -2c = -4$ and so $c = 2$

Also $P(1) = -1$ $\therefore\ (-1)(a+b+2) = -1$

$\therefore\ a+b+2 = 1$ $\therefore\ a+b = -1$ .... (1)

Also $P(-1) = -21$ $\therefore\ (-3)(a-b+2) = -21$

$\therefore\ a-b+2 = 7$ $\therefore\ a-b = 5$ .... (2)

Adding (1) and (2) gives $2a = 4$ $\therefore\ a = 2$ and so $b = -3$

$\therefore\ P(x) = (x-2)(2x^2-3x+2)$

EXERCISE 6F.2

1 **a** $P(x) = a(x+1)^2(x-1)^2$

where $a \neq 0$, and passes through $(0, 2)$

$2 = a(1)(1)$

$\therefore\ a = 2$

$\therefore\ P(x) = 2(x+1)^2(x-1)^2$

b $P(x) = a(x+3)(x+1)^2(3x-2)$

where $a \neq 0$, and passes through $(0, -6)$

$-6 = a(3)(1)(-2)$

$\therefore\ a = 1$

$\therefore\ P(x) = (x+3)(x+1)^2(3x-2)$

c $P(x) = a(x+2)(x+1)(x-2)^2$

where $a \neq 0$, and passes through $(0, -16)$

$-16 = a(2)(1)(4)$

$\therefore\ a = -2$

$\therefore\ P(x) = -2(x+2)(x+1)(x-2)^2$

d $P(x) = a(x+3)(x+1)(2x-3)(x-3)$

where $a \neq 0$, and passes through $(0, -9)$

$-9 = a(3)(1)(-3)(-3)$

$\therefore\ a = -\frac{1}{3}$

$\therefore\ P(x) = -\frac{1}{3}(x+3)(x+1)(2x-3)(x-3)$

e $P(x) = a(x+1)(x-4)^3$

where $a \neq 0$, and passes through $(0, -16)$

$-16 = a(1)(-4)^3$

$\therefore\ a = \frac{1}{4}$

$\therefore\ P(x) = \frac{1}{4}(x+1)(x-4)^3$

f $P(x) = ax^2(x+2)(x-3)$

where $a \neq 0$, and passes through $(-3, 54)$

$54 = a(9)(-1)(-6)$

$\therefore\ 54 = 54a$

$\therefore\ a = 1$

$\therefore\ P(x) = x^2(x+2)(x-3)$

2 **a** $y = (x-1)^2(x+1)(x+3)$

has x-intercepts $-1, -3$, touches at 1

has y-intercept $(-1)^2(1)(3) = 3$ (> 0)

$\therefore$ matches graph **C**

b $y = -2(x-1)^2(x+1)(x+3)$

has x-intercepts $-1, -3$, touches at 1

has y-intercept $-2(-1)^2(1)(3) = -6$ (< 0)

$\therefore$ matches graph **F**

c $y = (x-1)(x+1)^2(x+3)$

has x-intercepts $1, -3$, touches at -1

has y-intercept $(-1)(1)^2(3) = -3$ (< 0)

$\therefore$ matches graph **A**

d $y = (x-1)(x+1)^2(x-3)$

has x-intercepts $1, 3$, touches at -1

has y-intercept $(-1)(1)^2(-3) = 3$ (> 0)

$\therefore$ matches graph **E**

e $y = -\frac{1}{3}(x-1)(x+1)(x+3)^2$

has x-intercepts $1, -1$, touches at -3

has y-intercept $-\frac{1}{3}(-1)(1)(3)^2 = 3$ (> 0)

$\therefore$ matches graph **B**

f $y = -(x-1)(x+1)(x-3)^2$

has x-intercepts $1, -1$, touches at 3

has y-intercept $-(-1)(1)(3)^2 = 9$ (> 0)

$\therefore$ matches graph **D**

3 **a** $P(x) = a(x+4)(2x-1)(x-2)^2$
where $a \neq 0$, and passes through $(1, 5)$
$5 = a \times 5 \times 1 \times 1$
$\therefore \quad a = 1$
$\therefore \quad P(x) = (x+4)(2x-1)(x-2)^2$

b $P(x) = a(3x-2)^2(x+3)^2$
where $a \neq 0$, and passes through $(-4, 49)$
$49 = a(-14)^2(1)$
$\therefore \quad a = \frac{1}{4}$
$\therefore \quad P(x) = \frac{1}{4}(3x-2)^2(x+3)^2$

c $P(x) = a(2x+1)(2x-1)(x+2)(x-2)$
where $a \neq 0$, and passes through $(1, -18)$
$-18 = a(3)(1)(3)(-1)$
$\therefore \quad a = 2$
$\therefore \quad P(x) = 2(2x+1)(2x-1)(x+2)(x-2)$

d $P(x) = (x-1)^2(ax^2+bx+c)$
where $a \neq 0$, and cuts y-axis at $(0, -1)$
$-1 = 1 \times (0+0+c)$
$\therefore \quad c = -1$
$\therefore \quad P(x) = (x-1)^2(ax^2+bx-1)$
But $P(-1) = -4$
$\therefore \quad -4 = 4(a-b-1)$
$\therefore \quad a - b = 0$ .... (1)
Also $P(2) = 15$
$\therefore \quad 15 = 1(4a+2b-1)$
$\therefore \quad 16 = 4a+2b$
$\therefore \quad 2a + b = 8$ .... (2)
Adding (1) and (2) we get:
$\therefore \quad a = \frac{8}{3}$ and so $b = \frac{8}{3}$ also
$\therefore \quad P(x) = (x-1)^2(\frac{8}{3}x^2 + \frac{8}{3}x - 1)$

EXERCISE 6F.3

1 **a** $P(x) = x^3 - 3x^2 - 3x + 1$
From technology, -1 is a zero.
Check: $P(-1) = -1 - 3 + 3 + 1 = 0$ ✓
$\therefore \quad x + 1$ is a factor
$\therefore \quad x^3 - 3x^2 - 3x + 1 = (x+1)(x^2 - 4x + 1)$
and the quadratic has zeros of $\dfrac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}$
$\therefore$ zeros are $-1, \ 2 \pm \sqrt{3}$

$$\begin{array}{r|rrr|r} -1 & 1 & -3 & -3 & 1 \\ & 0 & -1 & 4 & -1 \\ \hline & 1 & -4 & 1 & 0 \end{array}$$

b $P(x) = x^3 - 3x^2 + 4x - 2$
From technology, 1 is a zero
Check: $P(1) = 1 - 3 + 4 - 2 = 0$ ✓
$\therefore \quad x - 1$ is a factor
From the division process $x^2 - 2x + 2$ is a quadratic factor
and it has zeros of $\dfrac{2 \pm \sqrt{4 - 4 \times 2}}{2} = \dfrac{2 \pm \sqrt{-4}}{2} = 1 \pm i$
$\therefore$ zeros are $1, \ 1 \pm i$

$$\begin{array}{r|rrr|r} 1 & 1 & -3 & 4 & -2 \\ & 0 & 1 & -2 & 2 \\ \hline & 1 & -2 & 2 & 0 \end{array}$$

c $P(x) = 2x^3 - 3x^2 - 4x - 35$
From technology, $\frac{7}{2}$ is a zero.
Check: $P(\frac{7}{2}) = \frac{343}{4} - \frac{147}{4} - 14 - 35$
$= \dfrac{343 - 147 - 56 - 140}{4}$
$= 0$ ✓

$$\begin{array}{r|rrr|r} \frac{7}{2} & 2 & -3 & -4 & -35 \\ & 0 & 7 & 14 & 35 \\ \hline & 2 & 4 & 10 & 0 \end{array}$$

From the division process $2x^2 + 4x + 10$ is a quadratic factor
$\therefore \quad P(x) = (x - \frac{7}{2})(2x^2 + 4x + 10)$
$= (2x - 7)(x^2 + 2x + 5)$
where the quadratic has zeros $\dfrac{-2 \pm \sqrt{4 - 20}}{2} = -1 \pm 2i$
$\therefore$ zeros are $\frac{7}{2}, \ -1 \pm 2i$

d $P(x) = 2x^3 - x^2 + 20x - 10$

From technology, $\frac{1}{2}$ is a zero.

Check: $P(\frac{1}{2}) = \frac{1}{4} - \frac{1}{4} + 10 - 10 = 0$ ✓

$\frac{1}{2}$	2	−1	20	−10
	0	1	0	10
	2	0	20	0

$\therefore\ P(x) = (x - \frac{1}{2})(2x^2 + 20)$
$\quad = (2x - 1)(x^2 + 10)$

$\therefore$ zeros are $\frac{1}{2},\ \pm i\sqrt{10}$

e $P(x) = 4x^4 - 4x^3 - 25x^2 + x + 6$

From technology, -2 and 3 are zeros

Check: $P(-2) = 64 + 32 - 100 - 2 + 6 = 0$ ✓
$P(3) = 324 - 108 - 225 + 3 + 6 = 0$ ✓

$\therefore\ P(x) = (x + 2)(x - 3)(4x^2 - 1)$

$\therefore$ zeros are $-2,\ 3,\ \pm\frac{1}{2}$

−2	4	−4	−25	1	6
	0	−8	24	2	−6
3	4	−12	−1	3	0
	0	12	0	−3	
	4	0	−1	0	

f $P(x) = x^4 - 6x^3 + 22x^2 - 48x + 40$

From technology, 2 seems to be a double zero.
{Graph touches the x-axis at 2}

Check: $P(2) = 16 - 48 + 88 - 96 + 40 = 0$ ✓

$\therefore\ P(x) = (x - 2)^2(x^2 - 2x + 10)$

where the quadratic has zeros of $\dfrac{2 \pm \sqrt{4 - 40}}{2}$
$= 1 \pm 3i$

$\therefore$ zeros are 2 (repeated), $1 \pm 3i$

2	1	−6	22	−48	40
	0	2	−8	28	−40
2	1	−4	14	−20	0
	0	2	−4	20	
	1	−2	10	0	

2 **a** $P(x) = x^3 + 2x^2 + 3x + 6$

From technology, -2 is a zero.

Check: $P(-2) = -8 + 8 - 6 + 6 = 0$ ✓

$\therefore\ P(x) = (x + 2)(x^2 + 3)$
$\quad = (x + 2)(x + i\sqrt{3})(x - i\sqrt{3})$

$\therefore$ roots of $P(x) = 0$ are $x = -2$ and $x = \pm i\sqrt{3}$

−2	1	2	3	6
	0	−2	0	−6
	1	0	3	0

b $P(x) = 2x^3 + 3x^2 - 3x - 2$

From technology, 1 is a zero.

Check: $P(1) = 2 + 3 - 3 - 2 = 0$ ✓

$\therefore\ P(x) = (x - 1)(2x^2 + 5x + 2)$
$\quad = (x - 1)(2x + 1)(x + 2)$

$\therefore$ roots of $P(x) = 0$ are $1,\ -\frac{1}{2},\ -2$

1	2	3	−3	−2
	0	2	5	2
	2	5	2	0

c $P(x) = x^3 - 6x^2 + 12x - 8$

From technology, 2 is a zero.

Check: $P(2) = 8 - 24 + 24 - 8 = 0$ ✓

$\therefore\ P(x) = (x - 2)(x^2 - 4x + 4)$
$\quad = (x - 2)(x - 2)(x - 2)$

$\therefore$ only root of $P(x) = 0$ is $x = 2$ (a treble root)

2	1	−6	12	−8
	0	2	−8	8
	1	−4	4	0

d $P(x) = 2x^3 - 5x^2 - 9x + 18$

From technology, 3 is a zero.

Check: $P(3) = 54 - 45 - 27 + 18 = 0$ ✓

$\therefore\ P(x) = (x - 3)(2x^2 + x - 6)$
$\quad = (x - 3)(2x - 3)(x + 2)$

$\therefore$ roots of $P(x) = 0$ are $3,\ \frac{3}{2}$, and -2

3	2	−5	−9	18
	0	6	3	−18
	2	1	−6	0

e $P(x) = x^4 - x^3 - 9x^2 + 11x + 6$

From technology, 2 and -3 are zeros.

Check: $P(2) = 16 - 8 - 36 + 22 + 6 = 0$ ✓

$P(-3) = 81 + 27 - 81 - 33 + 6 = 0$ ✓

$\therefore\ P(x) = (x-2)(x+3)(x^2 - 2x - 1)$

2	1	-1	-9	11	6
	0	2	2	-14	-6
-3	1	1	-7	-3	0
	0	-3	6	3	
	1	-2	-1	0	

where the quadratic has zeros of $\dfrac{2 \pm \sqrt{4+4}}{2} = 1 \pm \sqrt{2}$

$\therefore$ roots of $P(x) = 0$ are $2, -3, 1 \pm \sqrt{2}$

f $P(x) = 2x^4 - 13x^3 + 27x^2 - 13x - 15$

From technology, $-\frac{1}{2}$ and 3 are zeros.

Check: $P(-\frac{1}{2}) = \frac{1}{8} + \frac{13}{8} + \frac{27}{4} + \frac{13}{2} - 15 = 0$ ✓

$P(3) = 162 - 351 + 243 - 39 - 15 = 0$ ✓

$\therefore\ P(x) = (x + \frac{1}{2})(x-3)(2x^2 - 8x + 10)$

$= (2x+1)(x-3)(x^2 - 4x + 5)$

$-\frac{1}{2}$	2	-13	27	-13	-15
	0	-1	7	-17	15
3	2	-14	34	-30	0
	0	6	-24	30	
	2	-8	10	0	

where the quadratic has zeros of $\dfrac{4 \pm \sqrt{16-20}}{2} = 2 \pm i$

$\therefore$ roots of $P(x) = 0$ are $-\frac{1}{2}, 3, 2 \pm i$

3 **a** Consider $P(x) = x^3 - 3x^2 + 4x - 2$

From technology, 1 is a zero.

Check: $P(1) = 1 - 3 + 4 - 2 = 0$ ✓

Now $P(x) = (x-1)(x^2 - 2x + 2)$

1	1	-3	4	-2
	0	1	-2	2
	1	-2	2	0

where the quadratic has zeros of $\dfrac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$

$\therefore\ P(x) = (x-1)(x-1+i)(x-1-i)$

b Consider $P(x) = x^3 + 3x^2 + 4x + 12$

From technology, -3 is a zero.

Check: $P(-3) = -27 + 27 - 12 + 12 = 0$ ✓

Now $P(x) = (x+3)(x^2+4)$

$\therefore\ P(x) = (x+3)(x-2i)(x+2i)$

-3	1	3	4	12
	0	-3	0	-12
	1	0	4	0

c Consider $P(x) = 2x^3 - 9x^2 + 6x - 1$

From technology, $\frac{1}{2}$ is a zero.

Check: $P(\frac{1}{2}) = \frac{1}{4} - \frac{9}{4} + 3 - 1 = 0$ ✓

Now $P(x) = (x - \frac{1}{2})(2x^2 - 8x + 2)$

$= (2x-1)(x^2 - 4x + 1)$

$\frac{1}{2}$	2	-9	6	-1
	0	1	-4	1
	2	-8	2	0

where the quadratic has zeros of $\dfrac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}$

$\therefore\ P(x) = (2x-1)(x - 2 + \sqrt{3})(x - 2 - \sqrt{3})$

d $P(x) = x^3 - 4x^2 + 9x - 10$

From technology, 2 is a zero.

Check: $P(2) = 8 - 16 + 18 - 10 = 0$ ✓

Now $P(x) = (x-2)(x^2 - 2x + 5)$

2	1	-4	9	-10
	0	2	-4	10
	1	-2	5	0

where the quadratic has zeros of $\dfrac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2i$

$\therefore\ P(x) = (x-2)(x-1+2i)(x-1-2i)$

e $P(x) = 4x^3 - 8x^2 + x + 3$

From technology, 1 is a zero.

Check: $P(1) = 4 - 8 + 1 + 3 = 0$ ✓

$$\begin{array}{c|cccc} 1 & 4 & -8 & 1 & 3 \\ & 0 & 4 & -4 & -3 \\ \hline & 4 & -4 & -3 & \multicolumn{1}{|c}{0} \end{array}$$

Now $P(x) = (x-1)(4x^2 - 4x - 3)$

$= (x-1)(2x-3)(2x+1)$

$\therefore$ $P(x) = (x-1)(2x+1)(2x-3)$

f $P(x) = 3x^4 + 4x^3 + 5x^2 + 12x - 12$

From technology, -2 and $\frac{2}{3}$ are zeros.

Check: $P(-2) = 48 - 32 + 20 - 24 - 12 = 0$ ✓

$P(\frac{2}{3}) = \frac{16}{27} + \frac{32}{27} + \frac{20}{9} + 8 - 12 = 0$ ✓

$$\begin{array}{c|ccccc} -2 & 3 & 4 & 5 & 12 & -12 \\ & 0 & -6 & 4 & -18 & 12 \\ \hline \frac{2}{3} & 3 & -2 & 9 & -6 & \multicolumn{1}{|c}{0} \\ & 0 & 2 & 0 & 6 & \\ \hline & 3 & 0 & 9 & \multicolumn{1}{|c}{0} & \end{array}$$

Now $P(x) = (x+2)(x - \frac{2}{3})(3x^2 + 9)$

$= (x+2)(3x-2)(x^2+3)$

$\therefore$ $P(x) = (x+2)(3x-2)(x + i\sqrt{3})(x - i\sqrt{3})$

g $P(x) = 2x^4 - 3x^3 + 5x^2 + 6x - 4$

From technology, -1 and $\frac{1}{2}$ are zeros.

Check: $P(-1) = 2 + 3 + 5 - 6 - 4 = 0$ ✓

$P(\frac{1}{2}) = \frac{1}{8} - \frac{3}{8} + \frac{5}{4} + 3 - 4 = 0$ ✓

$$\begin{array}{c|ccccc} -1 & 2 & -3 & 5 & 6 & -4 \\ & 0 & -2 & 5 & -10 & 4 \\ \hline \frac{1}{2} & 2 & -5 & 10 & -4 & \multicolumn{1}{|c}{0} \\ & 0 & 1 & -2 & 4 & \\ \hline & 2 & -4 & 8 & \multicolumn{1}{|c}{0} & \end{array}$$

Now $P(x) = (x+1)(x - \frac{1}{2})(2x^2 - 4x + 8)$

$= (x+1)(2x-1)(x^2 - 2x + 4)$

where the quadratic has zeros of $\dfrac{2 \pm \sqrt{4-16}}{2} = 1 \pm i\sqrt{3}$

$\therefore$ $P(x) = (x+1)(2x-1)(x - 1 + i\sqrt{3})(x - 1 - i\sqrt{3})$

h $P(x) = 2x^3 + 5x^2 + 8x + 20$

From technology, $-\frac{5}{2}$ is a zero.

Check: $P(-\frac{5}{2}) = -\frac{125}{4} + \frac{125}{4} - 20 + 20 = 0$ ✓

$$\begin{array}{c|cccc} -\frac{5}{2} & 2 & 5 & 8 & 20 \\ & 0 & -5 & 0 & -20 \\ \hline & 2 & 0 & 8 & \multicolumn{1}{|c}{0} \end{array}$$

Now $P(x) = (x + \frac{5}{2})(2x^2 + 8)$

$= (2x+5)(x^2+4)$

$\therefore$ $P(x) = (2x+5)(x-2i)(x+2i)$

4 **a** Using technology, $x^3 + 2x^2 - 6x - 6$ has zeros of -0.860, 2.13, and -3.27

b Using technology, $x^3 + x^2 - 7x - 8$ has zeros of -2.52, -1.18, and 2.70

5 **a** $f(t) = kt(t-a)^2$

From the graph a is the t-value at the point where the graph touches the t-axis.

$\therefore$ $a = 700$ milliseconds

This represents the time when the barrier has returned to its original position.

b when $t = 100$ ms $f(t) = 85$ mm

$\therefore$ $85 = k \times 100(100 - 700)^2$

$85 = 100 \times k \times 360\,000$

$k = \dfrac{85}{36\,000\,000}$

$\therefore$ $f(t) = \dfrac{85t}{36\,000\,000}(t - 700)^2$

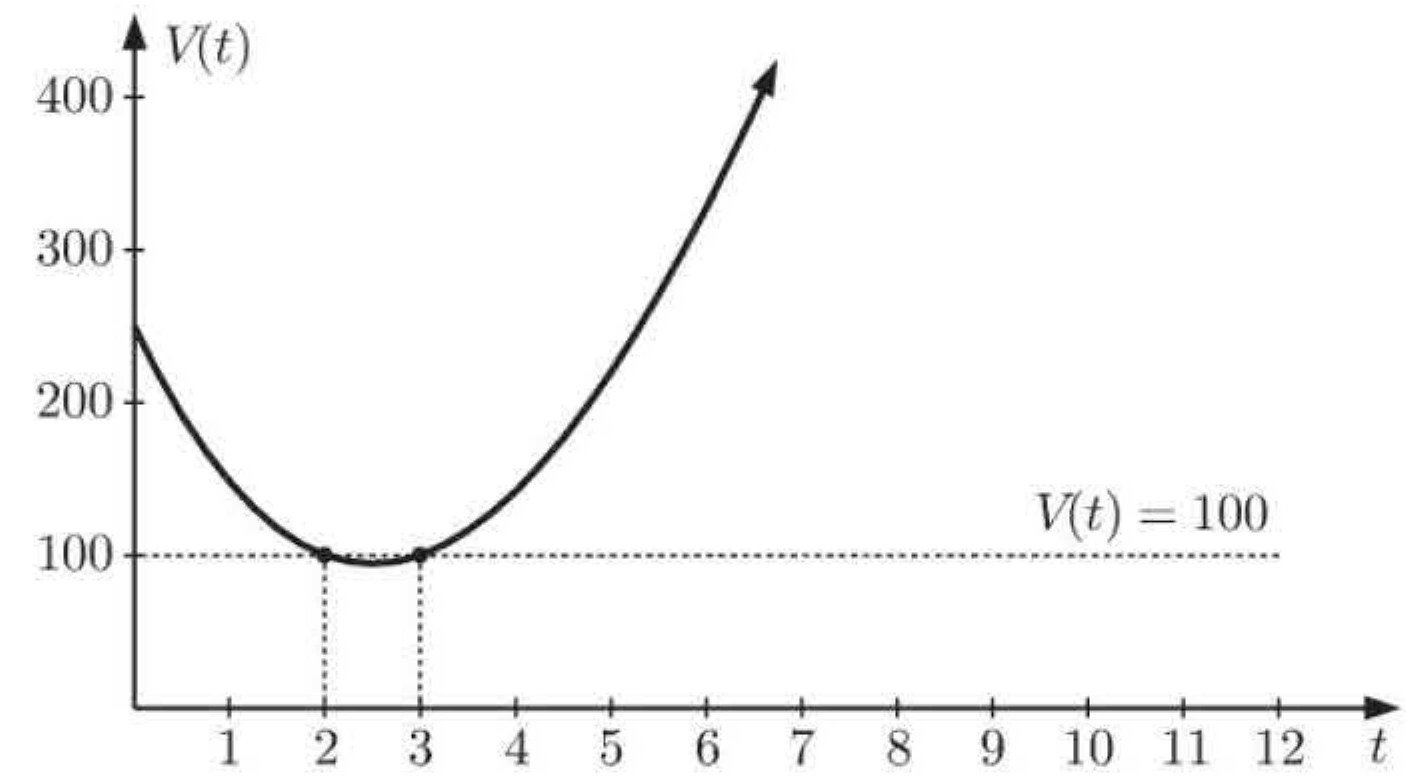
6 $V(t) = -t^3 + 30t^2 - 131t + 250$

We graph $V(t)$ against t and add the graph of $V(t) = 100$.

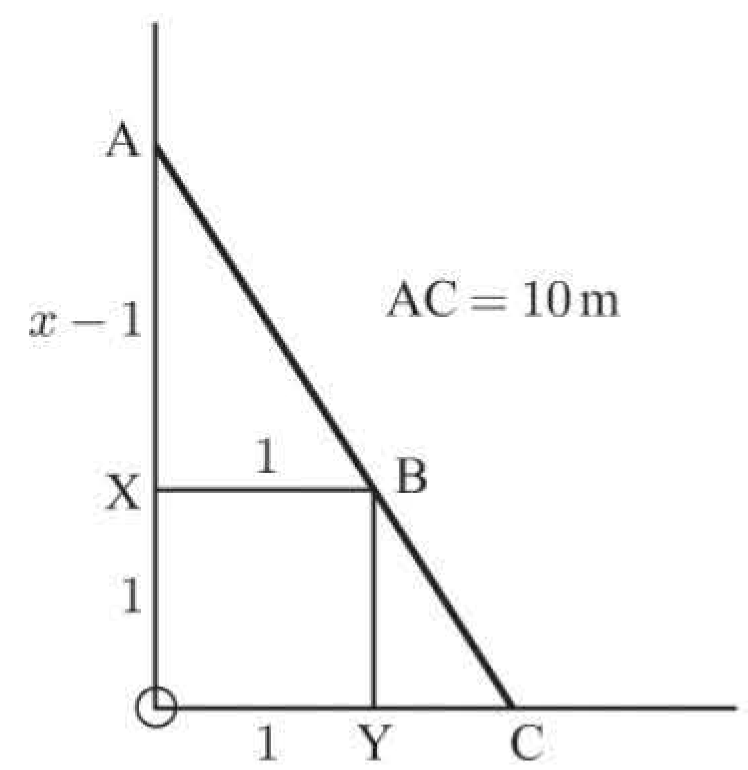
From the graph, the level drops below 100 ML when $t = 2$ and rises above 100 ML again when $t = 3$.

Now if $t = 0$ is Jan 1st, $0 \leqslant t < 1$ is January.

$\therefore$ as irrigation is prohibited for $2 < t < 3$, it is banned during March.



7



Let the height of the wall where the ladder touches be x m.

Using similar triangles AXB, AOC:

$$\frac{x-1}{1} = \frac{x}{\text{OC}}$$

$$\therefore \quad \text{OC} = \frac{x}{x-1}$$

but $x^2 + \text{OC}^2 = 10^2$

$$x^2 + \left(\frac{x}{x-1}\right)^2 = 100$$

Using technology to find the intersection of $y = x^2 + \left(\frac{x}{x-1}\right)^2$ and $y = 100$

$x \approx 1.112$ or 9.938

So, distance ≈ 9.938 m or 1.112 m

REVIEW SET 6A

1 **a** $a + bi = 4 = 4 + 0i, \quad \therefore \quad a = 4, \quad b = 0$

b $(1 - 2i)(a + bi) = -5 - 10i$

$$\therefore \quad a + bi = \frac{-5-10i}{1-2i} \times \frac{1+2i}{1+2i}$$
$$= \frac{-5 - 10i - 10i - 20i^2}{1 - 4i^2}$$
$$= \frac{15 - 20i}{5}$$
$$= 3 - 4i$$

$\therefore \quad a = 3 \quad b = -4$

c $(a + 2i)(1 + bi) = 17 - 19i$

$\therefore \quad a + 2i + abi + 2i^2 b = 17 - 19i$

$\therefore \quad (a - 2b) + i(ab + 2) = 17 - 19i$

Equating real and imaginary parts,

$a - 2b = 17$ and $ab + 2 = -19$

$\therefore \quad a = 2b + 17$ $\qquad ab = -21$

$\therefore \quad b(2b + 17) = -21$

$\therefore \quad 2b^2 + 17b + 21 = 0$

$\therefore \quad (2b + 3)(b + 7) = 0$

$\therefore \quad b = -\frac{3}{2}$ or $b = -7$

When $b = -7$, $a = 3$ and when $b = -\frac{3}{2}$, $a = 14$

2 $z = 3 + i, \quad w = -2 - i$

a $2z - 3w$
$= 2(3 + i) - 3(-2 - i)$
$= 6 + 2i + 6 + 3i$
$= 12 + 5i$

b $\dfrac{z^*}{w}$
$$= \frac{3-i}{-2-i} \times \frac{-2+i}{-2+i}$$
$$= \frac{-6 + 2i + 3i - i^2}{4 - i^2}$$
$$= \frac{-5 + 5i}{5}$$
$$= -1 + i$$

c z^3
$= (3 + i)^3$
$= 3^3 + 3(3^2)(i) + 3(3)(i^2) + i^3$
$= 27 + 27i + 9i^2 - i$
$= 27 - 9 + 26i$
$= 18 + 26i$

3
$$\begin{aligned} z &= \frac{3}{i+\sqrt{3}} + \sqrt{3} \\ &= \frac{3}{i+\sqrt{3}}\,\frac{(i-\sqrt{3})}{(i-\sqrt{3})} + \sqrt{3} \\ &= \frac{3i-3\sqrt{3}}{i^2-3} + \sqrt{3} \\ &= \frac{3i-3\sqrt{3}}{-4} - \frac{4\sqrt{3}}{-4} \\ &= \frac{3i-7\sqrt{3}}{-4} \end{aligned}$$
$\therefore\ \mathcal{Re}(z) = \frac{7\sqrt{3}}{4}$, $\mathcal{Im}(z) = -\frac{3}{4}$

4
$$\begin{aligned} 2z-1 &= iz-i \\ \therefore\ 2(a+bi)-1 &= i(a+bi)-i \\ \therefore\ 2a+2bi-1 &= ai+bi^2-i \\ \therefore\ (2a-1)+2bi &= -b+i(a-1) \end{aligned}$$
Equating real and imaginary parts,

$2a-1=-b$ and $2b=a-1$

$\therefore\ b=1-2a$ and $2b=a-1$

$\therefore\ 2(1-2a)=a-1$

$\therefore\ 2-4a=a-1$

$\therefore\ 3=5a$

$\therefore\ a=\frac{3}{5}$

and $b=1-2(\frac{3}{5})=-\frac{1}{5}$

$\therefore\ z=\frac{3}{5}-\frac{1}{5}i$

5 Let $z=a+bi,\quad w=c+di$
$$\begin{aligned} \therefore\ zw^*-z^*w &= (a+bi)(c-di)-(a-bi)(c+di) \\ &= ac-adi+bci-bdi^2-ac-adi+bci+bdi^2 \\ &= 2bci-2adi \\ &= 2i(bc-ad) \end{aligned}$$
which is purely imaginary if $bc-ad\neq 0$ and zero if $bc-ad=0$.

6
$$\begin{aligned} w &= \frac{z+1}{z^*+1} \\ &= \frac{a+1+bi}{a+1-bi}\times\frac{(a+1+bi)}{(a+1+bi)} \\ &= \frac{(a+1)^2+2(a+1)bi+b^2i^2}{(a+1)^2-b^2i^2} \\ &= \frac{(a+1)^2+2(a+1)bi-b^2}{(a+1)^2+b^2} \\ &= \frac{(a+1)^2-b^2}{(a+1)^2+b^2}+i\left(\frac{2(a+1)b}{(a+1)^2+b^2}\right) \end{aligned}$$
w is purely imaginary when

$(a+1)^2-b^2=0$ and $2(a+1)b\neq 0$

$\therefore\ b^2=(a+1)^2$ and $a\neq -1,\ b\neq 0$

$\therefore\ b=\pm(a+1),\ a\neq -1,\ b\neq 0$

7 **a**
$$\begin{aligned} &(3x^3+2x-5)(4x-3) \\ &= 12x^4-9x^3+8x^2-6x-20x+15 \\ &= 12x^4-9x^3+8x^2-26x+15 \end{aligned}$$

b $(2x^2-x+3)^2 = 4x^4-4x^3+13x^2-6x+9$

$$\begin{array}{rrrrr} & & 2 & -1 & 3 \\ & \times & 2 & -1 & 3 \\ \hline & & 6 & -3 & 9 \\ & -2 & 1 & -3 & \\ 4 & -2 & 6 & & \\ \hline 4 & -4 & 13 & -6 & 9 \\ \hline \end{array}$$

8 **a**
$$\begin{array}{r|l} & x^2-2x+4 \\ \hline x+2 & x^3+0x^2+0x+0 \\ & -(x^3+2x^2) \\ & \quad -2x^2+0x \\ & \quad -(-2x^2-4x) \\ & \qquad 4x+0 \\ & \qquad -(4x+8) \\ & \qquad\quad -8 \end{array}$$
$$\therefore\ \frac{x^3}{x+2} = x^2-2x+4-\frac{8}{x+2}$$

b $(x+2)(x+3)=x^2+5x+6$
$$\begin{array}{r|l} & x-5 \\ \hline x^2+5x+6 & x^3+0x^2+0x+0 \\ & -(x^3+5x^2+6x) \\ & \quad -5x^2-6x+0 \\ & \quad -(-5x^2-25x-30) \\ & \qquad 19x+30 \end{array}$$
$$\therefore\ \frac{x^3}{(x+2)(x+3)} = x-5+\frac{19x+30}{(x+2)(x+3)}$$

9 **a** $3x^4 - 4x^3 + 3x^2 + 8$

$\therefore$ the sum of the zeros is $-\dfrac{-4}{3} = \frac{4}{3}$

The polynomial has degree 4.

$\therefore$ the product of the zeros is $\dfrac{(-1)^4 8}{3} = \frac{8}{3}$

b $2x^6 + 2x^4 - x^3 + 7x - 10$

$= 2x^6 + (0)x^5 + 2x^4 - x^3 + 7x - 10$

$\therefore$ the sum of the zeros is $-\dfrac{0}{2} = 0$

The polynomial has degree 6.

$\therefore$ the product of the zeros is $\dfrac{(-1)^6(-10)}{2} = -5$

10 The Remainder theorem:

"When a polynomial $P(x)$ is divided by $x-k$ until a constant remainder R is obtained then $R = P(k)$."

Proof: From the division process, $P(x) = (x-k)Q(x) + R$

Now, letting $x = k$, $P(k) = (k-k) \times Q(k) + R$

$\therefore\ P(k) = 0 \times Q(k) + R$

$\therefore\ P(k) = R$

$\therefore\ R = P(k)$

11 Let $P(z) = z^2 + az + (3 + a)$

if $-2 + bi$ is a zero then $P(-2+bi) = 0$

$$\therefore\ (-2+bi)^2 + a(-2+bi) + 3 + a = 0$$
$$4 - 4bi + b^2i^2 - 2a + abi + 3 + a = 0$$
$$(4 - b^2 - 2a + 3 + a) + i(-4b + ab) = 0$$

$\therefore\ 4 - b^2 - 2a + 3 + a = 0$ and $-4b + ab = 0$

$a = 7 - b^2$ $\quad\therefore\ b(a-4) = 0$

$\therefore\ b = 0$ or $a = 4$

If $b = 0$ then $a = 7 - 0 = 7$.

If $a = 4$ then $b^2 = 3$ and so $b = \pm\sqrt{3}$.

12 As $(x-3)(x+2) = x^2 - x - 6$,

$P(x) = Q(x)(x^2 - x - 6) + (Ax + B)$, where $Q(x)$ is the quotient and $Ax + B$ is the remainder.

Now $P(x)$ has remainder 2 when divided by $x - 3$ and so $P(3) = 2$ {Remainder theorem}

$\therefore\ Q(3)(9 - 3 - 6) + (3A + B) = 2$

$\therefore\ 3A + B = 2$ .... (1)

Also $P(x)$ has remainder -13 when divided by $x + 2$ and so $P(-2) = -13$

$\therefore\ Q(-2)(4 + 2 - 6) + (-2A + B) = -13$

$\therefore\ -2A + B = -13$ .... (2)

Solving (1) and (2): $5A = 15$

$\therefore\ A = 3$ and $B = -7$

So, the remainder is $3x - 7$.

13 $2 - i\sqrt{3}$ and $\sqrt{2} + 1$

Since the quartic has real rational coefficients, $2 - i\sqrt{3}$ and $2 + i\sqrt{3}$ are zeros

$\sqrt{2} + 1$ and $-\sqrt{2} + 1$ are zeros

$\therefore$ the four zeros are: $2 \pm i\sqrt{3}$ and $\pm\sqrt{2} + 1$

For $2 \pm i\sqrt{3}$, $\alpha + \beta = 4$

$\alpha\beta = 4 - 3i^2 = 7$

$\therefore\ P(z) = (z^2 - 4z + 7)(z^2 - 2z - 1)$

$\therefore\ P(z) = z^4 - 6z^3 + 14z^2 - 10z - 7$

For $\pm\sqrt{2} + 1$, $\alpha + \beta = 2$

$\alpha\beta = -2 + 1 = -1$

$$\begin{array}{rrrrr}
 & & 1 & -4 & 7 \\
 & \times & 1 & -2 & -1 \\
\hline
 & & -1 & 4 & -7 \\
 & -2 & 8 & -14 & \\
1 & -4 & 7 & & \\
\hline
1 & -6 & 14 & -10 & -7 \\
\hline
\end{array}$$

14 $f(x) = x^3 - 3x^2 - 9x + b$ .... (1)

$= (x-k)^2(x+a)$

$= (x^2 - 2kx + k^2)(x+a)$

$= x^3 + (a-2k)x^2 + (k^2 - 2ak)x + ak^2$ .... (2)

	1	$-2k$	k^2
	$\times$	1	a
	a	$-2ak$	ak^2
1	$-2k$	k^2	
1	$a-2k$	k^2-2ak	ak^2

Equating coefficients of (1) and (2) gives

$\therefore \quad a - 2k = -3, \quad k^2 - 2ak = -9, \quad$ and $\quad ak^2 = b$

Since $a = 2k-3$, then as $k^2 - 2ak = -9$

$$k^2 - 2k(2k-3) = -9$$
$$k^2 - 4k^2 + 6k = -9$$
$$\therefore \quad 3k^2 - 6k - 9 = 0$$
$$k^2 - 2k - 3 = 0$$
$$(k-3)(k+1) = 0$$
$$\therefore \quad k = -1 \text{ or } k = 3$$

If $k = -1$, $a = -5$ and $b = ak^2 = -5$ and so $f(x) = (x+1)^2(x-5)$

and the roots of $f(x) = 0$ are $-1, 5$

If $k = 3$, $a = 3$ and $b = ak^2 = 3 \times 9 = 27$ and so $f(x) = (x-3)^2(x+3)$

and the roots of $f(x) = 0$ are $3, -3$

15 Since the coefficients are rational, $3 + i\sqrt{2}$ and $1 + \sqrt{2}$ also have to be zeros.

For zeros of $3 \pm i\sqrt{2}$, $\alpha + \beta = 6$ and $\alpha\beta = 9 - 2i^2 = 11$

For zeros of $1 \pm \sqrt{2}$, $\alpha + \beta = 2$ and $\alpha\beta = 1 - 2 = -1$

$\therefore \quad P(x) = a(x^2 - 6x + 11)(x^2 - 2x - 1), \quad a \neq 0$

$\therefore \quad P(x) = a(x^4 - 8x^3 + 22x^2 - 16x - 11), \quad a \neq 0$

		1	-6	11
	$\times$	1	-2	-1
		-1	6	-11
	-2	12	-22	
1	-6	11		
1	-8	22	-16	-11

16 $P(x) = x^n + 3x^2 + kx + 6$

$P(-1) = 12$ {Remainder theorem} $\quad \therefore \quad 12 = (-1)^n + 3 - k + 6$

$\therefore \quad k = (-1)^n - 3$ .... (1)

$P(1) = 8$ {Remainder theorem} $\quad \therefore \quad 8 = 1^n + 3 + k + 6$

$\therefore \quad 8 = 10 + k$

$\therefore \quad k = -2$

$\therefore$ (1) becomes $-2 = (-1)^n - 3$

$\therefore \quad (-1)^n = 1 \qquad \therefore \quad n$ is even

If $34 < n < 38$, then $n = 36$.

17 Let $x^3 - x + 1 = (x-\alpha)(x-\beta)(x-\gamma)$

$= (x^2 - [\alpha+\beta]x + \alpha\beta)(x-\gamma)$

$= x^3 - (\alpha+\beta+\gamma)x^2 + (\alpha\beta + \beta\gamma + \alpha\gamma) - \alpha\beta\gamma$

$\alpha + \beta + \gamma = 0$

$\alpha\beta + \beta\gamma + \alpha\gamma = -1$

$\therefore \quad \alpha\beta\gamma = -1$

Now $\gamma = \dfrac{-1}{\alpha\beta}$ .... (1) and $\alpha\beta + \gamma(\alpha+\beta) = -1$ .... (2)

$$\therefore \quad \alpha\beta + \gamma(-\gamma) = -1$$
$$\therefore \quad \alpha\beta - \gamma^2 = -1$$
$$\therefore \quad \alpha\beta - \frac{1}{(\alpha\beta)^2} = -1 \quad \{\text{using (1)}\}$$
$$\therefore \quad (\alpha\beta)^3 - 1 = -(\alpha\beta)^2$$
$$\therefore \quad (\alpha\beta)^3 + (\alpha\beta)^2 - 1 = 0$$

$\therefore \quad \alpha\beta$ is a root of $x^3 + x^2 - 1 = 0$

REVIEW SET 6B

1
$$\begin{aligned}\frac{5}{2-i} &= \frac{5}{2-i}\,\frac{(2+i)}{(2+i)}\\ &= \frac{10+5i}{4-i^2}\\ &= \frac{10+5i}{5} = 2+i\\ \therefore\ \sqrt[3]{z} &= (2+i)-3-2i\\ &= -1-i\end{aligned}$$

$$\begin{aligned}\therefore\ z &= (-1-i)^3\\ &= -(i+1)^3\\ &= -(i^3+3i^2+3i+1)\\ &= -(-i-3+3i+1)\\ &= -(2i-2)\\ &= 2-2i\\ \therefore\ x &= 2,\ y=-2\end{aligned}$$

2 Let $z = a+bi = \sqrt{5-12i}$, a, b real

$$\begin{aligned}\therefore\ (a+bi)^2 &= 5-12i\\ \therefore\ a^2+2abi+b^2i^2 &= 5-12i\\ \therefore\ a^2-b^2+2abi &= 5-12i\end{aligned}$$

Equating real and imaginary parts,

$a^2-b^2=5$ and $2ab=-12$

$\therefore\ a^2-b^2=5$ and $b=-\dfrac{6}{a}$

$$\begin{aligned}\therefore\ a^2-\left(-\frac{6}{a}\right)^2 &= 5\\ \therefore\ a^2-\frac{36}{a^2} &= 5\\ \therefore\ a^4-36 &= 5a^2\\ \therefore\ a^4-5a^2-36 &= 0\\ \therefore\ (a^2-9)(a^2+4) &= 0\\ \therefore\ a &= \pm 3\ \ (a \text{ real})\end{aligned}$$

using $b=-\dfrac{6}{a}$, if $a=3,\ b=-2$

and if $a=-3,\ b=2$

$\therefore\ z=3-2i$ or $z=-3+2i$

3 If $z=a+bi$

then $z+z^* = (a+bi)+(a-bi)$
$= 2a$ which is real.

Also, $zz^* = (a+bi)(a-bi)$
$= a^2+abi-abi-b^2i^2$
$= a^2+b^2$ which is real.

4 $z=4+i \qquad w=3-2i$

$$\begin{aligned}2w^*-iz &= 2(3+2i)-i(4+i)\\ &= 6+4i-4i-i^2\\ &= 7\end{aligned}$$

5 If $\dfrac{2-3i}{2a+bi}=3+2i$ then

$$\begin{aligned}2a+bi &= \frac{2-3i}{3+2i}\times\frac{3-2i}{3-2i}\\ \therefore\ 2a+bi &= \frac{6-4i-9i+6i^2}{9-4i^2} = \frac{-13i}{13}\\ \therefore\ 2a+bi &= 0-i\\ \therefore\ a=0 &\text{ and } b=-1\end{aligned}$$

6 If $a+ai$ is a root of $x^2-6x+b=0$ then

$$\begin{aligned}(a+ai)^2-6(a+ai)+b &= 0\\ \therefore\ a^2+2a^2i-a^2-6a-6ai+b &= 0\\ \therefore\ (b-6a)+(2a^2-6a)i &= 0\\ \therefore\ b=6a \text{ and } 2a^2-6a &= 0\\ \therefore\ b=6a \text{ and } 2a(a-3) &= 0\end{aligned}$$

$\therefore\ a=0$ or 3, and when $a=0,\ b=0$
and when $a=3,\ b=18$

7 Let $P(x)=x^{47}-3x^{26}+5x^3+11$

$$\begin{aligned}\therefore\ R &= P(-1) \qquad \{\text{Remainder theorem}\}\\ &= (-1)^{47}-3(-1)^{26}+5(-1)^3+11\\ &= -1-3-5+11\end{aligned}$$

$\therefore$ remainder $=2$

8 $P(z) = 2z^3 + z^2 + 10z + 5$

Using technology, $-\frac{1}{2}$ is a zero.

Check: $P(-\frac{1}{2}) = -\frac{1}{4} + \frac{1}{4} - 5 + 5 = 0$ ✓

$$\begin{array}{r|rrr|r} -\frac{1}{2} & 2 & 1 & 10 & 5 \\ & 0 & -1 & 0 & -5 \\ \hline & 2 & 0 & 10 & 0 \end{array}$$

$\therefore\ P(z) = (z + \frac{1}{2})(2z^2 + 10)$

$P(z) = (2z + 1)(z^2 + 5)$

$P(z) = (2z + 1)(z - i\sqrt{5})(z + i\sqrt{5})$

9 Touches the x-axis at $(-2, 0)$ and cuts it at $(1, 0)$.

$\therefore\ P(x) = (x + 2)^2(x - 1)(ax + b)$

But $P(0) = 12 \quad \therefore\ 4(-1)b = 12 \quad \therefore\ b = -3$

$\therefore\ P(x) = (x + 2)^2(x - 1)(ax - 3)$

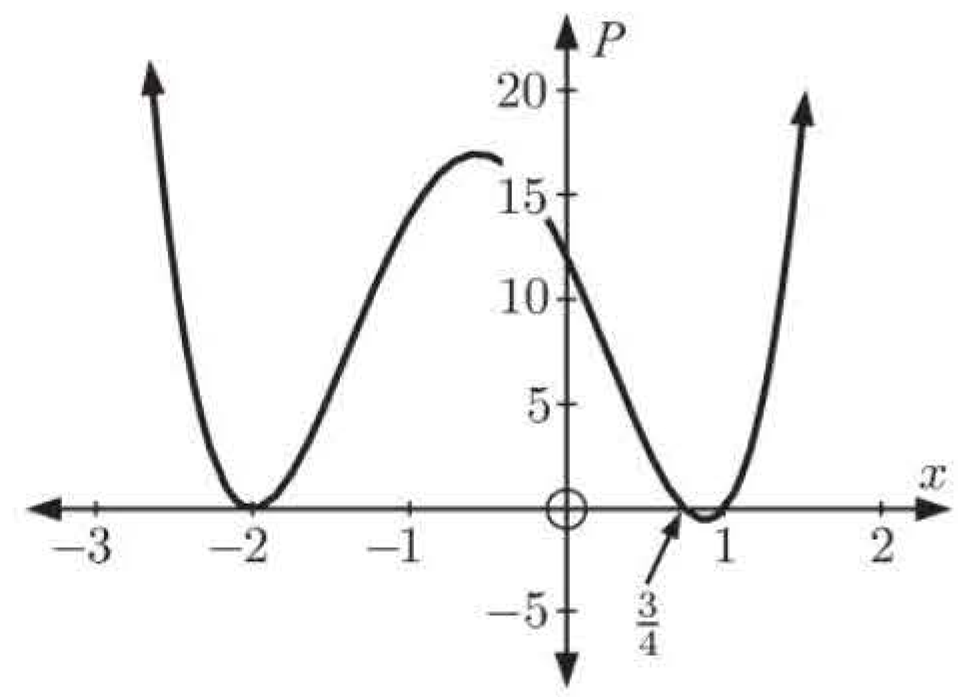
Also $P(2) = 80 \quad \therefore\ 80 = 16(1)(2a - 3)$

$\therefore\ 2a - 3 = 5$

$\therefore\ 2a = 8$

$\therefore\ a = 4$

$\therefore\ P(x) = (x + 2)^2(x - 1)(4x - 3)$



10 Let $P(z) = 2z^4 - 5z^3 + 13z^2 - 4z - 6$

From technology, 1 and $-\frac{1}{2}$ are zeros.

Check: $P(1) = 2 - 5 + 13 - 4 - 6 = 0$ ✓

$P(-\frac{1}{2}) = \frac{1}{8} + \frac{5}{8} + \frac{13}{4} + 2 - 6 = 0$ ✓

$$\begin{array}{r|rrrr|r} 1 & 2 & -5 & 13 & -4 & -6 \\ & 0 & 2 & -3 & 10 & 6 \\ \hline -\frac{1}{2} & 2 & -3 & 10 & 6 & 0 \\ & 0 & -1 & 2 & -6 & \\ \hline & 2 & -4 & 12 & 0 & \end{array}$$

$\therefore\ P(z) = (z - 1)(z + \frac{1}{2})(2z^2 - 4z + 12)$

$= (z - 1)(2z + 1)(z^2 - 2z + 6)$

where the quadratic has zeros of $\dfrac{2 \pm \sqrt{4 - 24}}{2} = 1 \pm i\sqrt{5}$. $\therefore$ zeros are $1, -\frac{1}{2}, 1 \pm i\sqrt{5}$.

11 Let $P(z) = z^4 + 2z^3 - 2z^2 + 8$

From technology, -2 seems to be a double zero.

Check: $P(-2) = 16 - 16 - 8 + 8 = 0$ ✓

$$\begin{array}{r|rrrr|r} -2 & 1 & 2 & -2 & 0 & 8 \\ & 0 & -2 & 0 & 4 & -8 \\ \hline -2 & 1 & 0 & -2 & 4 & 0 \\ & 0 & -2 & 4 & -4 & \\ \hline & 1 & -2 & 2 & 0 & \end{array}$$

$\therefore\ P(z) = (z + 2)^2(z^2 - 2z + 2)$

where the quadratic has zeros of $\dfrac{2 \pm \sqrt{4 - 8}}{2} = 1 \pm i$

$\therefore\ P(z) = (z + 2)^2(z - 1 + i)(z - 1 - i)$

12 Zeros are $2 + i$ and $-1 + 3i$.

Since we have a real polynomial, the other zeros are $2 - i$ and $-1 - 3i$.

For zeros of $2 \pm i$, $\alpha + \beta = 4$ and $\alpha\beta = 4 - i^2 = 5$

For zeros of $-1 \pm 3i$, $\alpha + \beta = -2$ and $\alpha\beta = 1 - 9i^2 = 10$

$\therefore\ P(z) = a(z^2 - 4z + 5)(z^2 + 2z + 10), \quad a \neq 0$

13 Since $3 - 2i$ is a zero, so is $3 + 2i$. These have $\alpha + \beta = 6$ and $\alpha\beta = 9 - 4i^2 = 13$.

$\therefore\ z^2 - 6z + 13$ is a factor

$\therefore\ P(z) = (z^2 - 6z + 13)(z^2 + Az + B)$

$= z^4 + kz^3 + 32z + 3k - 1$

$$\begin{array}{ccccc} & & 1 & -6 & 13 \\ & \times & 1 & A & B \\ \hline & & B & -6B & 13B \\ & A & -6A & 13A & \\ 1 & -6 & 13 & & \\ \hline 1 & A-6 & B-6A+13 & 13A-6B & 13B \\ \hline \end{array}$$

Equating coefficients gives

$$\begin{cases} A - 6 = k \\ B - 6A + 13 = 0 \\ 13A - 6B = 32 \\ 3k - 1 = 13B \end{cases}$$

$\therefore \quad -6A + B = -13 \quad .... (1)$

and $\quad 13A - 6B = 32 \quad .... (2)$

(1) $\times$ 6 gives $\quad -36A + 6B = -78 \quad .... (3)$

Adding (2) and (3) gives: $\quad -23A = -46 \quad \therefore \quad A = 2$

$\therefore \quad k = A - 6 = 2 - 6 = -4$ and $B = 6A - 13 = 12 - 13 = -1$

$$\therefore \quad P(z) = (z^2 - 6z + 13)\underbrace{(z^2 + 2z - 1)}_{\text{this quadratic has zeros } \frac{-2 \pm \sqrt{4+4}}{2} = -1 \pm \sqrt{2}}$$

$\therefore \quad k = -4$, zeros are $3 \pm 2i$, $\; -1 \pm \sqrt{2}$

14 Consider $P(z) = z^4 + 2z^3 + 6z^2 + 8z + 8$.

If one zero is purely imaginary, for example, ai, then $-ai$ is also a zero (a is real).

$\pm ai$ have $\alpha + \beta = 0$ and $\alpha\beta = -a^2 i^2 = a^2$

$\therefore \quad z^2 + a^2$ is a factor, $\therefore \quad z^2 + A$ is a factor

$\therefore \quad P(z) = (z^2 + A)(z^2 + Bz + C)$

$\qquad = z^4 + Bz^3 + (A + C)z^2 + ABz + AC$

		1	B	C
	$\times$	1	0	A
		A	AB	AC
1	B	C		
1	B	$A+C$	AB	AC

Equating coefficients gives:

$B = 2, \quad A + C = 6, \quad AB = 8$ and $AC = 8$

$\therefore \quad B = 2 \qquad \therefore \quad A = 4 \qquad \therefore \quad C = 2$

$$\therefore \quad P(z) = (z^2 + 4)\underbrace{(z^2 + 2z + 2)}_{\text{this quadratic has zeros } \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i}$$

$\therefore \quad$ zeros are $\pm 2i$, $\; -1 \pm i$

15 $\dfrac{P(x)}{x^2 - 3x + 2} = Q(x) + \dfrac{2x + 3}{x^2 - 3x + 2}$

$\therefore \quad P(x) = Q(x)(x^2 - 3x + 2) + 2x + 3$

and $\quad P(2) = Q(2)(4 - 6 + 2) + 4 + 3$

$\qquad = Q(2) \times (0) + 7$

$\qquad = 7$, so the remainder is 7.

16 They meet where $\quad x^2 + (2x + k)^2 + 8x - 4(2x + k) + 2 = 0$

$\therefore \quad x^2 + 4x^2 + 4kx + k^2 + 8x - 8x - 4k + 2 = 0$

$\therefore \quad 5x^2 + 4kx + (k^2 - 4k + 2) = 0$

Now $\quad \Delta = (4k)^2 - 4(5)(k^2 - 4k + 2)$

$\qquad = 16k^2 - 20k^2 + 80k - 40$

$\qquad = -4k^2 + 80k - 40$

and $\Delta < 0$ when $-4k^2 + 80k - 40 < 0$

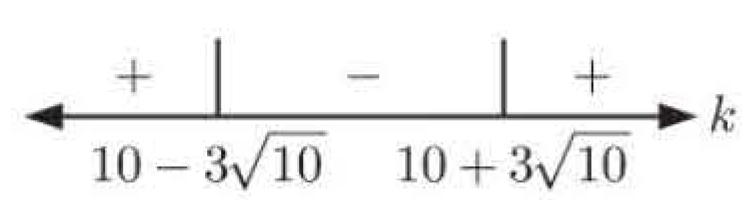
$\therefore \quad k^2 - 20k + 10 > 0$

$\therefore \quad k^2 - 20k + 10^2 + 10 - 10^2 > 0$

$\therefore \quad (k - 10)^2 - 90 > 0$

$\therefore \quad (k - 10 + 3\sqrt{10})(k - 10 - 3\sqrt{10}) > 0$

$\therefore \quad k \in \,]-\infty, 10 - 3\sqrt{10}[$ or $k \in \,]10 + 3\sqrt{10}, \infty[$



17 **a** $P(x) = 2x^4 - 8x^3 + ax^2 + bx - 110$ has zeros $m \pm 2i$ and $1 \pm n\sqrt{3}$.

$\therefore \quad$ the sum of the zeros is $\quad (m + 2i) + (m - 2i) + (1 + n\sqrt{3}) + (1 - n\sqrt{3}) = -\frac{-8}{2}$

$\therefore \quad 2m + 2 = 4$

$\therefore \quad 2m = 2$

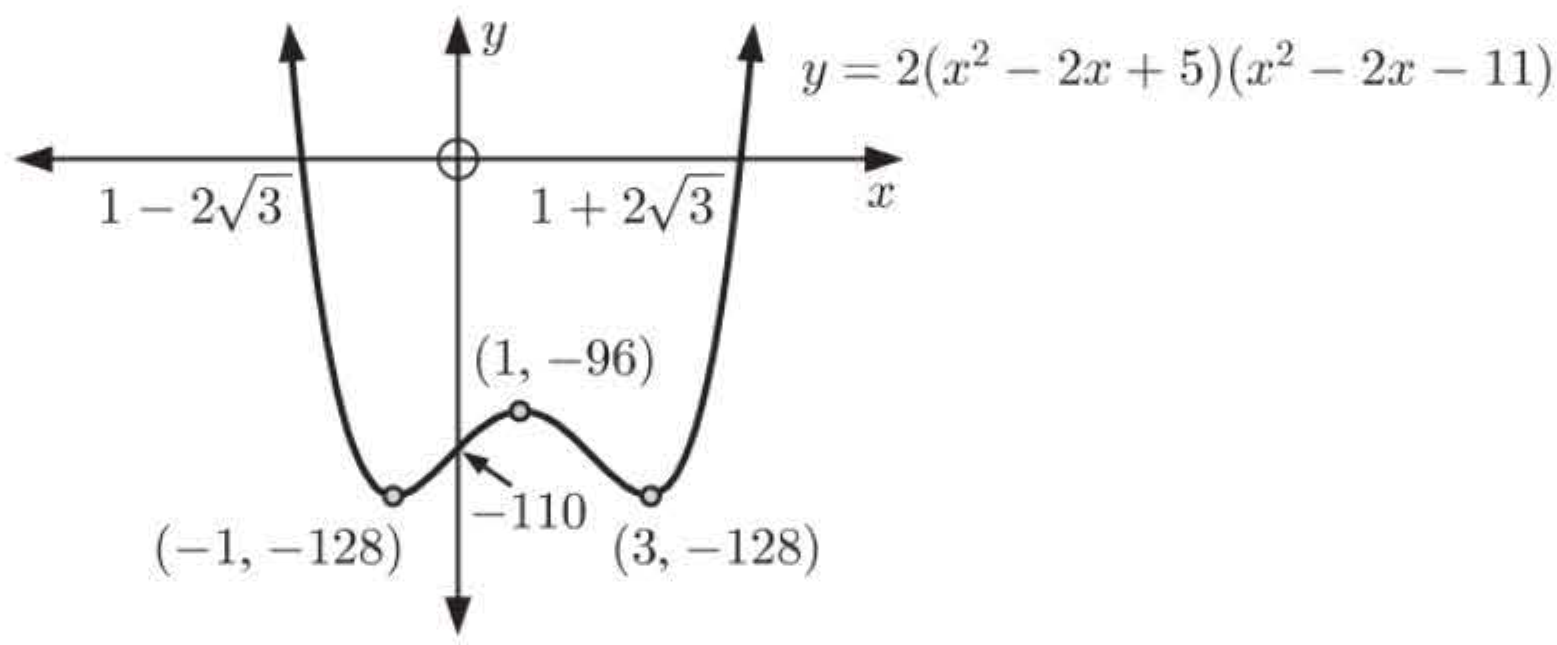
$\therefore \quad m = 1$

$\therefore$ the product of the zeros is $(1+2i)(1-2i)(1+n\sqrt{3})(1-n\sqrt{3}) = \dfrac{(-1)^4(-110)}{2}$

$$\begin{aligned}
\therefore\ (1+4)(1-3n^2) &= -55\\
\therefore\ 5(1-3n^2) &= -55\\
\therefore\ 1-3n^2 &= -11\\
\therefore\ 3n^2 &= 12\\
\therefore\ n^2 &= 4\\
\therefore\ n &= \pm 2
\end{aligned}$$

So, $m = 1$ and $n = \pm 2$.

b $P(x) = 2(x-1+2i)(x-1-2i)(x-1+2\sqrt{3})(x-1-2\sqrt{3})$
$= 2(x^2-2x+5)(x^2-2x-11)$



REVIEW SET 6C

1
$$\begin{aligned}
(3x+2yi)(1-i) &= (3y+1)i - x\\
\therefore\ 3x - 3xi + 2yi - 2yi^2 &= 3yi + i - x\\
\therefore\ (3x+2y) + i(2y-3x) &= -x + i(3y+1)
\end{aligned}$$

Equating real and imaginary parts, $3x + 2y = -x$ and $2y - 3x = 3y + 1$

$\therefore\ 4x = -2y$ and $-1 - 3x = y$

$\therefore\ 4x = -2(-1-3x)$

$\therefore\ 4x = 2 + 6x$

$\therefore\ -2x = 2$

$\therefore\ x = -1$ and $y = -1 - 3(-1) = 2$

2
$$\begin{aligned}
z^2 + iz + 10 &= 6z\\
\therefore\ z^2 + (i-6)z + 10 &= 0\\
\therefore\ z &= \frac{6 - i \pm \sqrt{(i-6)^2 - 4(10)}}{2}\\
&= \frac{6 - i \pm \sqrt{i^2 - 12i + 36 - 40}}{2}\\
&= \frac{6 - i \pm \sqrt{-5 - 12i}}{2}
\end{aligned}$$

We must now find $\sqrt{-5-12i}$

Let $z = a + bi = \sqrt{-5-12i}$, a, b real

$\therefore\ (a+bi)^2 = -5 - 12i$

$\therefore\ a^2 + 2abi + b^2i^2 = -5 - 12i$

$\therefore\ a^2 - b^2 + 2abi = -5 - 12i$

Equating real and imaginary parts,

$a^2 - b^2 = -5$ and $2ab = -12$

$\therefore\ a^2 - b^2 = -5$ and $b = -\dfrac{6}{a}$

$\therefore\ a^2 - \left(-\dfrac{6}{a}\right)^2 = -5$

$\therefore\ a^2 - \dfrac{36}{a^2} = -5$

$\therefore\ a^4 + 5a^2 - 36 = 0$

$\therefore\ (a^2+9)(a^2-4) = 0$

$\therefore\ a = 2$ (a real, $a > 0$)

and $b = \frac{-6}{2} = -3$

$\therefore\ \sqrt{-5-12i} = 2 - 3i$

so $z = \dfrac{6 - i \pm (2-3i)}{2}$

$\therefore\ z = \dfrac{8-4i}{2}$ or $\dfrac{4+2i}{2}$

$\therefore\ z = 4 - 2i$ or $2 + i$

3 Let $z = a + bi$ and $w = c + di$

$\therefore \quad zw^* + z^*w$
$= (a + bi)(c - di) + (a - bi)(c + di)$
$= ac - adi + bci - bdi^2 + ac + adi - bci - bdi^2$
$= ac + bd + ac + bd$
$= 2ac + 2bd$ which is a real number

4 **a** If $x + iy = 0$ then $x = 0$ and $y = 0$ {equating real and imaginary parts}

b
$$(3 - 2i)(x + i) = 17 + yi$$
$$3x + 3i - 2xi - 2i^2 = 17 + yi$$
$$(3x + 2) + i(3 - 2x) = 17 + yi$$
$\therefore \quad 3x + 2 = 17$ and $y = 3 - 2x$
$\therefore \quad 3x = 15$
$\therefore \quad x = 5$
and so $\therefore \quad y = 3 - 10$
$\therefore \quad y = -7$

c
$$(x + iy)^2 = x - iy$$
$$x^2 + i^2y^2 + 2xyi = x - iy$$
$$x^2 - y^2 = x \quad \text{and} \quad 2xy = -y$$
Now if $2xy + y = 0$
then $y(2x + 1) = 0$
$\therefore \quad y = 0$ or $x = -\frac{1}{2}$

If $y = 0$, then $x^2 = x$ and so $x = 0$ or 1
If $x = -\frac{1}{2}$, then $\frac{1}{4} - y^2 = -\frac{1}{2}$,
$\therefore \quad y^2 = \frac{3}{4}$
and so $y = \pm\frac{\sqrt{3}}{2}$

Possible solutions are:

x	0	1	$-\frac{1}{2}$	$-\frac{1}{2}$
y	0	0	$\frac{\sqrt{3}}{2}$	$\frac{-\sqrt{3}}{2}$

5 Let $z = a + bi$ and $w = c + di$ where $b \neq 0$ and $d \neq 0$ .... (1)

Now $z + w = (a + c) + (b + d)i$ and $zw = (a + bi)(c + di)$
$= (ac - bd) + i(bc + ad)$

As $z + w$ is real, $b + d = 0 \quad \therefore \quad b = -d$ .... (2)
As zw is real, $bc + ad = 0$ .... (3)

Substituting (2) into (3) $\quad -dc + ad = 0$
$\therefore \quad d(a - c) = 0$
Since $d \neq 0$ {from (1)}
$\therefore \quad a = c$ and $b = -d$
$\therefore \quad z^* = a - bi = c + di = w$

6 **a** $2x^3 + 3x^2 - 4x + 6 = 0$

$\therefore$ the sum of the roots is $-\frac{3}{2}$
The polynomial equation has degree 3.
$\therefore$ the product of the roots is $\dfrac{(-1)^3 6}{2} = -3$

b $4x^4 = x^2 + 2x - 6$

$\therefore \quad 4x^4 + (0)x^3 - x^2 - 2x + 6 = 0$
$\therefore$ the sum of the roots is $-\frac{0}{4} = 0$
The polynomial equation has degree 4.
$\therefore$ the product of the roots is $\dfrac{(-1)^4 6}{4} = \frac{3}{2}$

7
$$\sqrt{z} = \frac{2}{3 - 2i} + 2 + 5i$$
$$= \frac{2}{3 - 2i} \times \frac{3 + 2i}{3 + 2i} + 2 + 5i$$
$$= \frac{6 + 4i}{9 - 4i^2} + 2 + 5i$$
$$= \frac{6 + 4i}{13} + 2 + 5i$$
$$= \tfrac{32}{13} + \tfrac{69}{13}i$$

$$\therefore \quad z = \left(\tfrac{32}{13} + \tfrac{69}{13}i\right)^2$$
$$= \frac{32^2 - 69^2}{169} + \frac{2 \times 32 \times 69}{169}i$$
$$= -\tfrac{3737}{169} + \tfrac{4416}{169}i$$

8 Let $P(x) = 2x^{17} + 5x^{10} - 7x^3 + 6$

Now $R = P(2)$ {Remainder theorem}

$= 2^{18} + 5 \times 2^{10} - 7 \times 2^3 + 6$

$= 267\,214$

9 Let $P(z) = 2z^3 + az^2 + 62z + [a - 5]$

If $5 - i$ is a zero, then so is $5 + i$

{as $P(z)$ is a real polynomial}

and for these zeros $\alpha + \beta = 10$

$\alpha\beta = 25 - i^2 = 26$

	1	-10	26
	$\times$	2	b
	b	$-10b$	$26b$
2	-20	52	
2	$b - 20$	$52 - 10b$	$26b$

$\therefore$ $P(z) = (z^2 - 10z + 26)(2z + b)$

Equating coefficients gives $a = b - 20$, $52 - 10b = 62$, and $a - 5 = 26b$.

From $52 - 10b = 62$,

$-10b = 10$

$\therefore$ $b = -1$, and since $a = b - 20$ we find $a = -21$.

Check: $a - 5 = -26$ and $26b = -26$ ✓

$\therefore$ $P(z) = 2z^3 - 21z^2 + 62z - 26$

$= (z^2 - 10z + 26)(2z - 1)$

$\therefore$ other two zeros are $\frac{1}{2}$, $5 + i$, and $a = -21$.

10 **a** Two zeros are $i\sqrt{2}$ and $\frac{1}{2}$, so another zero must be $-i\sqrt{2}$

Now $\pm i\sqrt{2}$ have $\alpha + \beta = 0$ and $\alpha\beta = -2i^2 = 2$

$\therefore$ $x^2 + 2$ is a factor

$\therefore$ $P(x) = a(2x - 1)(x^2 + 2)$, $a \neq 0$

b Two zeros are $1 - i$ and $-3 - i$

$\therefore$ the other zeros must be $1 + i$ and $-3 + i$

$1 \pm i$ have $\alpha + \beta = 2$ and $\alpha\beta = 1 - i^2 = 2$

and $-3 \pm i$ have $\alpha + \beta = -6$ and $\alpha\beta = 9 - i^2 = 10$

$\therefore$ $P(x) = a(x^2 - 2x + 2)(x^2 + 6x + 10)$, $a \neq 0$

11 **a** $P(x) = 2x^3 + 7x^2 + kx - k$ .... (1)

$= (x + a)^2(2x + b)$ .... (2)

$= (x^2 + 2ax + a^2)(2x + b)$

$= 2x^3 + (b + 4a)x^2 + (2ab + 2a^2)x + a^2b$ .... (3)

	1	$2a$	a^2
	$\times$	2	b
	b	$2ab$	a^2b
2	$4a$	$2a^2$	
2	$b + 4a$	$2ab + 2a^2$	a^2b

Equating coefficients gives in (1) and (3):

$b + 4a = 7$, $2ab + 2a^2 = k$ and $a^2b = -k$

$\therefore$ $2ab + 2a^2 = -a^2b$ {equating ks}

$\therefore$ $2a(7 - 4a) + 2a^2 = -a^2(7 - 4a)$

$\therefore$ $14a - 8a^2 + 2a^2 + 7a^2 - 4a^3 = 0$

$\therefore$ $4a^3 - a^2 - 14a = 0$

$\therefore$ $a(4a^2 - a - 14) = 0$

$\therefore$ $a(4a + 7)(a - 2) = 0$

$\therefore$ $a = 0, -\frac{7}{4}$, or 2

If $a = 0$, $b = 7$ and $k = 0$.

If $a = 2$, $b = -1$ and $k = 4$.

If $a = -\frac{7}{4}$, $b = 14$ and $k = -\frac{343}{8}$.

b Largest value of k is $k = 4$ and $P(x) = (x + 2)^2(2x - 1)$. {substituting into (2)}

12 $$2z^4 - 3z^3 + 2z^2 = 6z + 4$$

$\therefore \quad 2z^4 - 3z^3 + 2z^2 - 6z - 4 = 0$

From technology, 2 and $-\frac{1}{2}$ are solutions.

Check: $P(2) = 32 - 24 + 8 - 12 - 4 = 0 \quad \checkmark$

$P(-\frac{1}{2}) = \frac{1}{8} + \frac{3}{8} + \frac{1}{2} + 3 - 4 = 0 \quad \checkmark$

$$\begin{array}{r|rrrr|r} 2 & 2 & -3 & 2 & -6 & -4 \\ & 0 & 4 & 2 & 8 & 4 \\ \hline -\frac{1}{2} & 2 & 1 & 4 & 2 & 0 \\ & 0 & -1 & 0 & -2 & \\ \hline & 2 & 0 & 4 & 0 & \end{array}$$

$\therefore \quad P(z) = (z-2)(z+\frac{1}{2})(2z^2+4)$

$= (z-2)(2z+1)(z^2+2) \qquad \therefore$ roots are $2, -\frac{1}{2}, \pm i\sqrt{2}$

13 $P(z) = z^3 + az^2 + kz + ka$

$P(-a) = -a^3 + a^3 - ka + ka = 0$

$\therefore \quad z + a$ is a factor

$$\begin{array}{r|rrr|r} -a & 1 & a & k & ka \\ & 0 & -a & 0 & -ka \\ \hline & 1 & 0 & k & 0 \end{array}$$

$\therefore \quad P(z) = (z+a)(z^2+k)$

or $P(z) = z^3 + az^2 + kz + ka$

$= z^2(z+a) + k(z+a)$

$= (z+a)(z^2+k)$

a $P(z) = 0$ has one real root if $k > 0, \ a \in \mathbb{R}$

b and 3 real roots if $k \leqslant 0, \ a \in \mathbb{R}$

14 Let $P(x) = 6x^3 + ax^2 - 4ax + b$

$3x + 2$ and $x - 2$ are factors

$\therefore \quad 6x^3 + ax^2 - 4ax + b = (3x+2)(x-2)(2x+c)$

$= (3x^2 - 4x - 4)(2x + c)$

$= 6x^3 - 8x^2 - 8x + 3cx^2 - 4cx - 4c$

$= 6x^3 + (3c - 8)x^2 + (-8 - 4c)x - 4c$

Equating coefficients gives: $a = 3c - 8, \quad -4a = -8 - 4c,$ and $b = -4c$

Equating as gives: $3c - 8 = 2 + c$

$\therefore \quad 2c = 10$

$\therefore \quad c = 5$

Consequently, $a = 3(5) - 8 = 7$ and $b = -20$.

15 $y = x - k$ meets $(x-2)^2 + (y+3)^2 = 4$ where

$(x-2)^2 + (x - k + 3)^2 = 4$

$\therefore \quad x^2 - 4x + 4 + (x-k)^2 + 6(x-k) + 9 - 4 = 0$

$\therefore \quad x^2 - 4x + 4 + x^2 - 2kx + k^2 + 6x - 6k + 5 = 0$

$\therefore \quad 2x^2 + (-4 - 2k + 6)x + (4 + k^2 - 6k + 5) = 0$

$\therefore \quad 2x^2 + (2 - 2k)x + (k^2 - 6k + 9) = 0$

which is a quadratic in x and in the tangent case $\Delta = 0$

$\therefore \quad (2 - 2k)^2 - 4(2)(k^2 - 6k + 9) = 0$

$\therefore \quad 4 - 8k + 4k^2 - 8k^2 + 48k - 72 = 0$

$\therefore \quad -4k^2 + 40k - 68 = 0$

$\therefore \quad k^2 - 10k + 17 = 0$

$\therefore \quad k = 5 \pm 2\sqrt{2}$

16

$$\begin{array}{r l}
 & \phantom{x^4+{}} x^2 + 3x - 9 \\
x^2+2 \,\big) & x^4 + 3x^3 - 7x^2 + 11x - 1 \\
 & -(x^4 + 2x^2) \\
\hline
 & \phantom{x^4+{}} 3x^3 - 9x^2 + 11x \\
 & \phantom{x^4+{}} -(3x^3 + 6x) \\
\hline
 & -9x^2 + 5x - 1 \\
 & -(-9x^2 - 18) \\
\hline
 & \phantom{x^4+3x^3-9x^2+{}} 5x + 17
\end{array}$$

$\therefore$ $Q(x) = x^2 + 3x - 9$ $R(x) = 5x + 17$ and the new function would be divisible by $x^2 + 2$ if $x^4 + 3x^3 - 7x^2 + (2 + a)x + b = P(x) - R(x)$

and as $P(x) - R(x) = x^4 + 3x^3 - 7x^2 + 11x - 1 - (5x + 17)$

$= x^4 + 3x^3 - 7x^2 + 6x - 18$

then $2 + a = 6$ and $b = -18$ {equating coefficients}

$\therefore$ $a = 4$ and $b = -18$

17 **a** The polynomial $P(x)$ of degree 5 has zeros $m \pm 2i$, $1 \pm mi$, and 2.
The constant term is the y-intercept. $\therefore$ the constant term is -56.

$\therefore$ the product of the zeros is $2(m + 2i)(m - 2i)(1 + mi)(1 - mi) = \dfrac{(-1)^5(-56)}{1}$

$$\therefore\ 2(m^2 + 4)(1 + m^2) = 56$$
$$\therefore\ m^4 + 5m^2 + 4 = 28$$
$$\therefore\ (m^2)^2 + 5(m^2) - 24 = 0$$
$$\therefore\ (m^2 + 8)(m^2 - 3) = 0$$
$$\therefore\ m^2 = -8 \text{ or } m^2 = 3$$
$$\therefore\ m^2 = 3 \qquad \{m^2 \geqslant 0\}$$
$$\therefore\ m = \pm\sqrt{3}$$

b Let the coefficient of x^4 be a.
If $m = \sqrt{3}$, then the sum of the zeros is

$$2 + (\sqrt{3} + 2i) + (\sqrt{3} - 2i) + (1 + i\sqrt{3}) + (1 - i\sqrt{3}) = -\frac{a}{1}$$
$$\therefore\ 2 + 2\sqrt{3} + 2 = -a$$
$$\therefore\ a = -4 - 2\sqrt{3}$$

If $m = -\sqrt{3}$, then the sum of the zeros is

$$2 + (-\sqrt{3} + 2i) + (-\sqrt{3} - 2i) + (1 - i\sqrt{3}) + (1 + i\sqrt{3}) = -a$$
$$\therefore\ 2 - 2\sqrt{3} + 2 = -a$$
$$\therefore\ a = -4 + 2\sqrt{3}$$

So, the coefficient of x^4 is $-4 - 2\sqrt{3}$ if $m = \sqrt{3}$,
and $-4 + 2\sqrt{3}$ if $m = -\sqrt{3}$.

Chapter 7

SEQUENCES AND SERIES

EXERCISE 7A

1 **a** 4, 13, 22, 31 **b** 45, 39, 33, 27 **c** 2, 6, 18, 54 **d** 96, 48, 24, 12

2 **a** The sequence starts at 8 and each term is 8 more than the previous term. The next two terms are 40 and 48.

b The sequence starts at 2 and each term is 3 more than the previous term. The next two terms are 14 and 17.

c The sequence starts at 36 and each term is 5 less than the previous term. The next two terms are 16 and 11.

d The sequence starts at 96 and each term is 7 less than the previous term. The next two terms are 68 and 61.

e The sequence starts at 1 and each term is 4 times the previous term. The next two terms are 256 and 1024.

f The sequence starts at 2 and each term is 3 times the previous term. The next two terms are 162 and 486.

g The sequence starts at 480 and each term is half the previous term. The next two terms are 30 and 15.

h The sequence starts at 243 and each term is one third of the previous term. The next two terms are 3 and 1.

i The sequence starts at 50 000 and each term is one fifth of the previous term. The next two terms are 80 and 16.

3 **a** Each term is the square of the number of the term. The next three terms are 25, 36, and 49.

b Each term is the cube of the number of the term. The next three terms are 125, 216, and 343.

c Each term is $n \times (n+1)$ where n is the number of the term. The next three terms are 30, 42, and 56.

4 **a** 79, 75 (subtracting 4 each time) **b** 1280, 5120 (multiplying by 4 each time)

c 625, 1296 $(1^4, 2^4, 3^4, 4^4, ...)$ **d** 13, 17 (prime numbers)

e 16, 22 (the difference between terms increases by 1) **f** 6, 12 $(-1, +2, -3, +4, ...)$

EXERCISE 7B.1

1 **a** $u_1 = 3(1) - 2$
$= 3 - 2$
$= 1$

b $u_5 = 3(5) - 2$
$= 15 - 2$
$= 13$

c $u_{27} = 3(27) - 2$
$= 81 - 2$
$= 79$

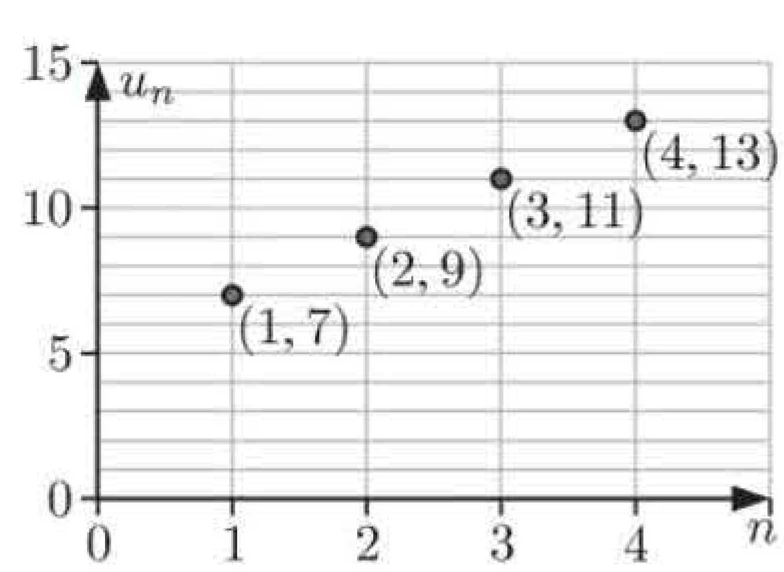
2 **a** $u_1 = 2(1) + 5$
$= 2 + 5$
$= 7$

$u_2 = 2(2) + 5$
$= 4 + 5$
$= 9$

$u_3 = 2(3) + 5$
$= 6 + 5$
$= 11$

$u_4 = 2(4) + 5$
$= 8 + 5$
$= 13$

b



3 **a** The sequence $\{2n\}$ begins 2, 4, 6, 8, 10 (letting $n = 1, 2, 3, 4, 5, ...$).

b The sequence $\{2n + 2\}$ begins 4, 6, 8, 10, 12 (letting $n = 1, 2, 3, 4, 5, ...$).

c The sequence $\{2n - 1)$ begins 1, 3, 5, 7, 9 (letting $n = 1, 2, 3, 4, 5, ...$).

d The sequence $\{2n - 3\}$ begins -1, 1, 3, 5, 7 (letting $n = 1, 2, 3, 4, 5, ...$).

e The sequence $\{2n + 3\}$ begins 5, 7, 9, 11, 13 (letting $n = 1, 2, 3, 4, 5, ...$).

f The sequence $\{2n + 11\}$ begins 13, 15, 17, 19, 21 (letting $n = 1, 2, 3, 4, 5, ...$).

g The sequence $\{3n+1\}$ begins $4, 7, 10, 13, 16$ (letting $n = 1, 2, 3, 4, 5, ...$).

h The sequence $\{4n-3\}$ begins $1, 5, 9, 13, 17$ (letting $n = 1, 2, 3, 4, 5, ...$).

4 **a** The sequence $\{2^n\}$ begins $2, 4, 8, 16, 32$ (letting $n = 1, 2, 3, 4, 5, ...$).

b The sequence $\{3 \times 2^n\}$ begins $6, 12, 24, 48, 96$ (letting $n = 1, 2, 3, 4, 5, ...$).

c The sequence $\{6 \times (\frac{1}{2})^n\}$ begins $3, \frac{3}{2}, \frac{3}{4}, \frac{3}{8}, \frac{3}{16}$ (letting $n = 1, 2, 3, 4, 5, ...$).

d The sequence $\{(-2)^n\}$ begins $-2, 4, -8, 16, -32$ (letting $n = 1, 2, 3, 4, 5, ...$).

5 $\{15-(-2)^n\}$ generates the sequence with first five terms:
$t_1 = 15-(-2)^1 = 17$, $t_2 = 15-(-2)^2 = 11$, $t_3 = 15-(-2)^3 = 23$,
$t_4 = 15-(-2)^4 = -1$, $t_5 = 15-(-2)^5 = 47$

EXERCISE 7B.2

1 $u_1 = 3,\ u_n = u_{n-1} - 4,\ n > 1$

a $u_2 = u_1 - 4$
$= 3 - 4$
$= -1$

b $u_3 = u_2 - 4$
$= -1 - 4$
$= -5$

c $u_4 = u_3 - 4$
$= -5 - 4$
$= -9$

d $u_5 = u_4 - 4$
$= -9 - 4$
$= -13$

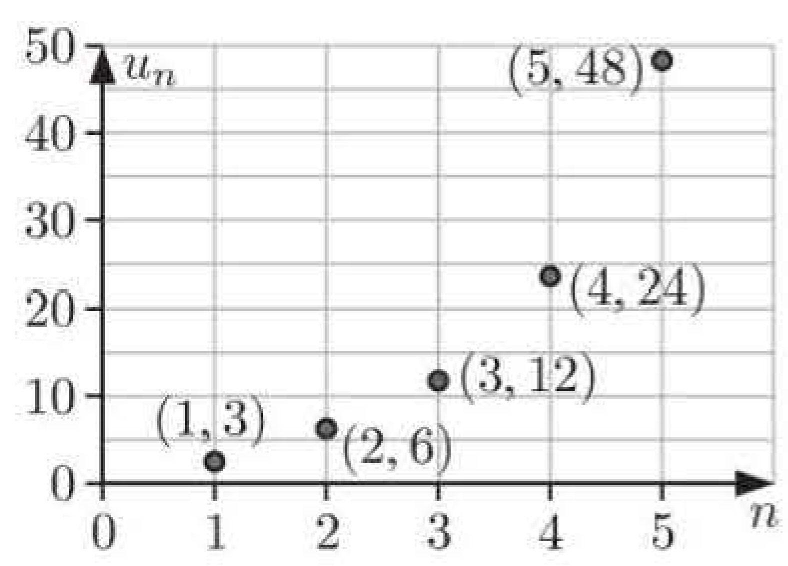
2 **a** The first term of the sequence is 3. So, $u_1 = 3$.
Each subsequent term is double the previous one. So, $u_n = 2u_{n-1}$.
$\therefore$ the sequence is defined by $u_1 = 3,\ u_n = 2u_{n-1},\ n > 1$.

b $u_2 = 2u_1$
$= 2 \times 3$
$= 6$

$u_3 = 2u_2$
$= 2 \times 6$
$= 12$

$u_4 = 2u_3$
$= 2 \times 12$
$= 24$

$u_5 = 2u_4$
$= 2 \times 24$
$= 48$



3 $u_1 = 1,\ u_n = \dfrac{1}{1+u_{n-1}},\ n > 1$.

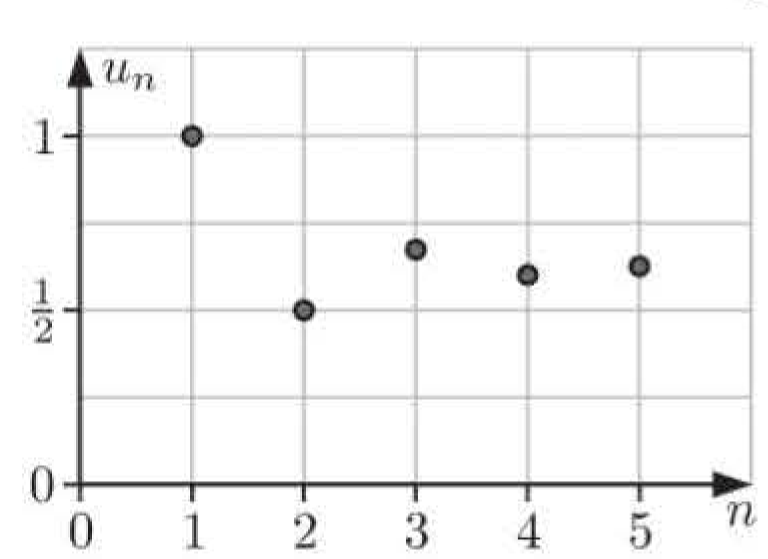
a $u_2 = \dfrac{1}{1+u_1} = \dfrac{1}{1+1} = \frac{1}{2}$

$u_3 = \dfrac{1}{1+u_2} = \dfrac{1}{1+\frac{1}{2}} = \dfrac{1}{\frac{3}{2}} = \frac{2}{3}$

$u_4 = \dfrac{1}{1+u_3} = \dfrac{1}{1+\frac{2}{3}} = \dfrac{1}{\frac{5}{3}} = \frac{3}{5}$

$u_5 = \dfrac{1}{1+u_4} = \dfrac{1}{1+\frac{3}{5}} = \dfrac{1}{\frac{8}{5}} = \frac{5}{8}$

b



c **i** As $n \to \infty$, $u_{n-1} \to u$ and $u_n \to u$

$\therefore$ as $n \to \infty$, $u_n = \dfrac{1}{1+u_{n-1}} \to \dfrac{1}{1+u}$

$\therefore$ as $n \to \infty$, $u_n \to \dfrac{1}{1+u}$ and also $u_n \to u$

$$\therefore \quad u = \frac{1}{1+u}$$
$$\therefore \quad u(1+u) = 1$$
$$\therefore \quad u + u^2 = 1$$
$$\therefore \quad u^2 + u - 1 = 0$$

ii $u^2 + u - 1 = 0$

$$\therefore \quad u = \frac{-1 \pm \sqrt{1-4(1)(-1)}}{2(1)}$$
$$= \frac{-1 \pm \sqrt{5}}{2}$$

But $u_n > 0$ for all n, so $u = \dfrac{-1+\sqrt{5}}{2}$

Now, $u_n = \dfrac{1}{1+u_{n-1}}$

$$= \frac{1}{1+\frac{1}{1+u_{n-2}}}$$
$$= \frac{1}{1+\frac{1}{1+\frac{1}{1+u_{n-3}}}}$$
$$= \ldots.$$

$$\therefore \quad u = \frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\ldots}}}} = \frac{-1+\sqrt{5}}{2}$$

4 $u_1 = 1, \ u_n = \sqrt{\dfrac{1}{1+u_{n-1}}}, \quad n > 1$

a $u_2 = \sqrt{\dfrac{1}{1+u_1}} = \sqrt{\dfrac{1}{1+1}} = \sqrt{\frac{1}{2}} \approx 0.707\,11$

$u_3 = \sqrt{\dfrac{1}{1+u_2}} = \sqrt{\dfrac{1}{1+\frac{1}{\sqrt{2}}}} \approx 0.765\,37$

$u_4 = \sqrt{\dfrac{1}{1+u_3}} \approx 0.752\,63$

$u_5 = \sqrt{\dfrac{1}{1+u_4}} \approx 0.755\,36$

b As $n \to \infty$, $u_{n-1} \to u$ and $u_n \to u$

$\therefore$ as $n \to \infty$, $u_n = \sqrt{\dfrac{1}{1+u_{n-1}}} \to \sqrt{\dfrac{1}{1+u}}$

So, as $n \to \infty$, $u_n \to \sqrt{\dfrac{1}{1+u}}$ and $u_n \to u$

$$\therefore \quad u = \sqrt{\frac{1}{1+u}}$$
$$\therefore \quad u^2 = \frac{1}{1+u}$$
$$\therefore \quad u^2(1+u) = 1$$
$$\therefore \quad u^2 + u^3 = 1$$
$$\therefore \quad u^3 + u^2 - 1 = 0$$

c $u^3 + u^2 - 1 = 0$

$\therefore\ u \approx 0.754\,88$ {using technology}

and $u = \sqrt{\dfrac{1}{1+\sqrt{\frac{1}{1+\sqrt{\frac{1}{1+\sqrt{....}}}}}}} \approx 0.754\,88$

5 Suppose a sequence is defined by $u_1 = 1,\ u_n = \dfrac{1}{1+u_{n-1}^2},\ n > 0.$

Assume that the sequence terms tend to the constant value u, so that as $n \to \infty,\ u_{n-1} \to u$ and $u_n \to u$.

$\therefore\ u_n = \dfrac{1}{1+u_{n-1}^2} \to \dfrac{1}{1+u^2}$ {since as $n \to \infty,\ u_{n-1} \to u$}

So, as $n \to \infty,\ u_n \to \dfrac{1}{1+u^2}$ and $u_n \to u$

$$\therefore\ u = \frac{1}{1+u^2}$$

$$\therefore\ u(1+u^2) = 1$$

$$\therefore\ u + u^3 = 1$$

$$\therefore\ u^3 + u - 1 = 0$$

$\therefore\ u \approx 0.682\,33$ {using technology}

and $u = \dfrac{1}{1+\left(\frac{1}{1+\left(\frac{1}{1+\left(\frac{1}{1+(....)}\right)^2}\right)^2}\right)^2} \approx 0.682\,33$

EXERCISE 7C

1 **a** The sequence begins with 19 and the common difference is $25 - 19 = 6$.

$$\begin{aligned} u_n &= u_1 + (n-1)d \\ \therefore\ u_n &= 19 + 6(n-1) \\ \text{So,}\quad u_{10} &= 19 + 6(10-1) \\ &= 19 + 6 \times 9 \\ &= 73 \end{aligned}$$

b The sequence begins with 101 and the common difference is $97 - 101 = -4$.

$$\begin{aligned} u_n &= u_1 + (n-1)d \\ \therefore\ u_n &= 101 + (-4)(n-1) \\ \text{So,}\quad u_{10} &= 101 + (-4)(10-1) \\ &= 101 - 4 \times 9 \\ &= 65 \end{aligned}$$

c The sequence begins with 8 and the common difference is $9\frac{1}{2} - 8 = \frac{3}{2}$.

$$\begin{aligned} u_n &= u_1 + (n-1)d \\ \therefore\ u_n &= 8 + \tfrac{3}{2}(n-1) \\ \text{So,}\quad u_{10} &= 8 + \tfrac{3}{2}(10-1) \\ &= 8 + \tfrac{3}{2} \times 9 \\ &= 21\tfrac{1}{2} \end{aligned}$$

2 **a** The first term of the arithmetic sequence is 31 and the common difference is $36 - 31 = 5$.

$$\begin{aligned} u_n &= u_1 + (n-1)d \\ \therefore\ u_n &= 31 + 5(n-1) \\ \text{So, } u_{15} &= 31 + 5(15-1) \\ &= 31 + 5 \times 14 \\ &= 101 \end{aligned}$$

b The first term of the arithmetic sequence is 5 and the common difference is $-3 - 5 = -8$.

$$\begin{aligned} u_n &= u_1 + (n-1)d \\ \therefore\ u_n &= 5 + (-8)(n-1) \\ \text{So, } u_{15} &= 5 + (-8)(15-1) \\ &= 5 - 8 \times 14 \\ &= -107 \end{aligned}$$

c The first term of the arithmetic sequence is a and the common difference is $a + d - a = d$.

$$\begin{aligned} u_n &= u_1 + (n-1)d \\ \therefore\ u_n &= a + (n-1)d \\ \text{So, } u_{15} &= a + d(15-1) \\ &= a + 14d \end{aligned}$$

3 **a** $17 - 6 = 11$
$28 - 17 = 11$
$39 - 28 = 11$
$50 - 39 = 11$

Assuming that the pattern continues, consecutive terms differ by 11.
$\therefore$ the sequence is arithmetic with $u_1 = 6$, $d = 11$.

b
$$\begin{aligned} u_n &= u_1 + (n-1)d \\ &= 6 + (n-1)11 \\ &= 11n - 5 \end{aligned}$$

c
$$\begin{aligned} u_{50} &= 11(50) - 5 \\ &= 545 \end{aligned}$$

d Let $u_n = 325 = 11n - 5$
$\therefore\ 330 = 11n$
$\therefore\ n = 30$
So, 325 is the 30th member.

e Let $u_n = 761 = 11n - 5$
$\therefore\ 766 = 11n$
$\therefore\ n = 69\frac{7}{11}$, but n must be an integer, so 761 is not a member of the sequence.

4 **a** $83 - 87 = -4$
$79 - 83 = -4$
$75 - 79 = -4$

Assuming that the pattern continues, consecutive terms differ by -4.
$\therefore$ the sequence is arithmetic with $u_1 = 87$, $d = -4$.

b
$$\begin{aligned} u_n &= u_1 + (n-1)d \\ &= 87 + (n-1)(-4) \\ &= 87 - 4n + 4 \\ &= 91 - 4n \end{aligned}$$

c
$$\begin{aligned} u_{40} &= 91 - 4(40) \\ &= 91 - 160 \\ &= -69 \end{aligned}$$

d Let $u_n = -297 = 91 - 4n$
$\therefore\ 4n = 388$
$\therefore\ n = 97$
So, -297 is the 97th term of the sequence.

5 **a** $u_n = 3n - 2,\quad u_{n+1} = 3(n+1) - 2 = 3n + 1$

$$\begin{aligned} u_{n+1} - u_n &= (3n+1) - (3n-2) \\ &= 3, \quad \text{a constant} \end{aligned}$$

Consecutive terms differ by 3.
$\therefore$ the sequence is arithmetic.

b $u_1 = 3(1) - 2 = 1,\quad d = 3$

c $u_{57} = 3(57) - 2 = 169$

d Let $u_n = 450 = 3n - 2$, so $3n = 452$ and hence $n = 150\frac{2}{3}$.
We try the two values on either side of $n = 150\frac{2}{3}$, which are $n = 150$ and $n = 151$:
$u_{150} = 3(150) - 2 = 448$ and $u_{151} = 3(151) - 2 = 451$
So, $u_{150} = 448$ is the largest term which is smaller than 450.

6 **a** $u_n = \dfrac{71 - 7n}{2} = 35\frac{1}{2} - \frac{7}{2}n$

$$u_{n+1} = \frac{71 - 7(n+1)}{2} = \frac{71 - 7n - 7}{2} = \frac{64 - 7n}{2} = 32 - \tfrac{7}{2}n$$

$$u_{n+1} - u_n = (32 - \tfrac{7}{2}n) - (35\tfrac{1}{2} - \tfrac{7}{2}n) = -\tfrac{7}{2}, \quad \text{a constant}$$

So, consecutive terms differ by $-\frac{7}{2}$. $\therefore$ the sequence is arithmetic.

b $u_1 = \dfrac{71 - 7(1)}{2} = 32,\quad d = -\frac{7}{2}$

c $u_{75} = \dfrac{71 - 7(75)}{2} = -227$

d Let $u_n = -200 = \dfrac{71-7n}{2}$ so $-400 = 71 - 7n$ $\therefore\ 7n = 471$
$\therefore\ n = 67\frac{2}{7}$

We try the two values on either side of $n = 67\frac{2}{7}$, which are $n = 67$ and $n = 68$:

$u_{67} = \dfrac{71-7(67)}{2} = -199$ and $u_{68} = \dfrac{71-7(68)}{2} = -202\frac{1}{2}$

So, the terms of the sequence are less than -200 for $n \geqslant 68$.

7 **a** The terms are consecutive, so we equate common differences:

$k - 32 = 3 - k$
$\therefore\ 2k = 35$
$\therefore\ k = 17\frac{1}{2}$

b The terms are consecutive, so we equate common differences:

$7 - k = 10 - 7$
$\therefore\ 7 - k = 3$
$\therefore\ k = 4$

c The terms are consecutive, so we equate common differences:

$(2k+1) - (k+1) = 13 - (2k+1)$
$\therefore\ k = 12 - 2k$
$\therefore\ 3k = 12$
$\therefore\ k = 4$

d The terms are consecutive, so we equate common differences:

$(2k+3) - (k-1) = (7-k) - (2k+3)$
$\therefore\ k + 4 = 4 - 3k$
$\therefore\ 4k = 0$
$\therefore\ k = 0$

e The terms are consecutive, so we equate common differences:

$k^2 - k = (k^2+6) - k^2$
$\therefore\ k^2 - k - 6 = 0$
$\therefore\ (k+2)(k-3) = 0$
$\therefore\ k = -2$ or 3

f The terms are consecutive, so we equate common differences:

$k - 5 = k^2 - 8 - k$
$\therefore\ k^2 - 2k - 3 = 0$
$\therefore\ (k-3)(k+1) = 0$
$\therefore\ k = -1$ or 3

8 **a** $u_7 = 41$ $\therefore\ u_1 + 6d = 41$ .... (1)
$u_{13} = 77$ $\therefore\ u_1 + 12d = 77$ .... (2)

Solving simultaneously,

$-u_1 - 6d = -41$
$u_1 + 12d = 77$

$\therefore\ 6d = 36$ {adding the equations}
$\therefore\ d = 6$

So in (1), $u_1 + 6(6) = 41$
$\therefore\ u_1 + 36 = 41$ $\therefore\ u_1 = 5$

Now $u_n = u_1 + (n-1)d$
$\therefore\ u_n = 5 + (n-1)6$
$\therefore\ u_n = 6n - 1$

b $u_5 = -2$ $\therefore\ u_1 + 4d = -2$ .... (1)
$u_{12} = -12\frac{1}{2}$ $\therefore\ u_1 + 11d = -12\frac{1}{2}$ .... (2)

Solving simultaneously,

$-u_1 - 4d = 2$
$u_1 + 11d = -12\frac{1}{2}$

$\therefore\ 7d = -10\frac{1}{2}$ {adding the equations}
$\therefore\ d = -\frac{3}{2}$

So in (1), $u_1 + 4(-\frac{3}{2}) = -2$ $\therefore\ u_1 = 4$

Now $u_n = u_1 + (n-1)d$
$\therefore\ u_n = 4 + (n-1)(-\frac{3}{2})$
$\therefore\ u_n = -\frac{3}{2}n + \frac{11}{2}$

c $u_7 = 1$ $\therefore\ u_1 + 6d = 1$ .... (1)
$u_{15} = -39$ $\therefore\ u_1 + 14d = -39$ .... (2)

Solving simultaneously,

$-u_1 - 6d = -1$
$u_1 + 14d = -39$

$\therefore\ 8d = -40$ {adding the equations}
$\therefore\ d = -5$

So in (1), $u_1 + 6(-5) = 1$
$\therefore\ u_1 - 30 = 1$ $\therefore\ u_1 = 31$

Now $u_n = u_1 + (n-1)d$
$\therefore\ u_n = 31 + (n-1)(-5)$
$\therefore\ u_n = 31 - 5n + 5$
$\therefore\ u_n = -5n + 36$

d $u_{11} = -16$ $\therefore\ u_1 + 10d = -16$ .... (1)
$u_8 = -11\frac{1}{2}$ $\therefore\ u_1 + 7d = -11\frac{1}{2}$ .... (2)

Solving simultaneously,

$-u_1 - 10d = 16$
$u_1 + 7d = -11\frac{1}{2}$

$\therefore\ -3d = 4\frac{1}{2}$ {adding the equations}
$\therefore\ d = -\frac{3}{2}$

So in (1), $u_1 + 10(-\frac{3}{2}) = -16$
$\therefore\ u_1 - 15 = -16$ $\therefore\ u_1 = -1$

Now $u_n = u_1 + (n-1)d$
$\therefore\ u_n = -1 + (n-1)(-\frac{3}{2})$
$\therefore\ u_n = -\frac{3}{2}n + \frac{1}{2}$

9 **a** Let the numbers be
$5,\ 5+d,\ 5+2d,\ 5+3d,\ 10.$
Then $5+4d=10$
$\therefore\ 4d=5$
$\therefore\ d=\frac{5}{4}=1\frac{1}{4}$
So, the numbers are
$5, 6\frac{1}{4}, 7\frac{1}{2}, 8\frac{3}{4}, 10.$

b Let the numbers be $-1,\ -1+d,\ -1+2d,$
$-1+3d,\ -1+4d,\ -1+5d,\ -1+6d,\ 32.$
Then $-1+7d=32$
$\therefore\ 7d=33$
$\therefore\ d=\frac{33}{7}=4\frac{5}{7}$
So, the numbers are
$-1,\ 3\frac{5}{7},\ 8\frac{3}{7},\ 13\frac{1}{7},\ 17\frac{6}{7},\ 22\frac{4}{7},\ 27\frac{2}{7},\ 32.$

10 **a** $u_1=36,\ \ 35\frac{1}{3}-36=-\frac{2}{3},$
$34\frac{2}{3}-35\frac{1}{3}=-\frac{2}{3}$
So, $d=-\frac{2}{3}.$

b $u_n=u_1+(n-1)d$
$\therefore\ -30=36+(n-1)(-\frac{2}{3})$ {letting $u_n=-30$}
$\therefore\ -66=-\frac{2}{3}n+\frac{2}{3}$
$\therefore\ \frac{2}{3}n=66\frac{2}{3}$
$\therefore\ n=100$ So, -30 is the 100th term of the sequence.

11 $u_1=23,\ \ 36-23=13$
$49-36=13$
$62-49=13$
so $d=13$

$u_n=u_1+(n-1)d$
$\therefore\ u_n=23+(n-1)13$
$=23+13n-13$
$\therefore\ u_n=13n+10$

Let $u_n=100\,000$
$=13n+10$
$\therefore\ 99\,990=13n$
$\therefore\ n=7691\frac{7}{13}$

We try the two values on either side of $n=7691\frac{7}{13}$, which are $n=7691$ and $n=7692$:
$u_{7691}=13(7691)+10=99\,993$ and $u_{7692}=13(7692)+10=100\,006$
So, the first term to exceed 100 000 is $u_{7692}=100\,006$.

12 **a** *Month 1*: 5 cars *Month 2*: $5+13=18$ cars *Month 3*: $18+13=31$ cars
Month 4: $31+13=44$ cars *Month 5*: $44+13=57$ cars *Month 6*: $57+13=70$ cars

b Every month after the first, the factory assembles 13 cars, so the difference between successive months is always 13. Thus we have an arithmetic sequence with $u_1=5$ and $d=13$.

c $u_n=u_1+(n-1)d$
$=5+(n-1)\times 13$
$=13n-8$
$\therefore\ u_{12}=13\times 12-8$ {12 months = 1 year}
$=148$
So, 148 cars are made in the first year.

d $u_n=250=13n-8$
$\therefore\ 258=13n$
$\therefore\ n=\dfrac{258}{13}\approx 19.85$
So, the 250th car is made in the 20th month.

13 **a** $41-34=48-41=55-48=7$
Assuming the pattern continues, there is a common difference of 7, and so the number of Valéria's friends forms an arithmetic sequence with $u_1=34$ and $d=7$.

b $u_n=u_1+(n-1)d$
$=34+(n-1)\times 7$
$=27+7n$
$\therefore\ u_{12}=27+7\times 12=111$
So, after 12 weeks Valéria will have 111 online friends.

c $u_n=150=27+7n$
$\therefore\ 123=7n$
$\therefore\ n=\frac{123}{7}\approx 17.57$
So, Valéria will have 150 online friends after the 18th week.

14 **a** *July 1st*: $100-2.7\times 1=97.3$ tonnes of hay
July 2nd: $100-2.7\times 2=94.6$ tonnes of hay
July 3rd: $100-2.7\times 3=91.9$ tonnes of hay

b $94.6-97.3=91.9-94.6=-2.7$
So $d=-2.7$, which means the cows eat 2.7 tonnes of hay per day.

c $u_{25}=100-2.7\times 25=32.5$ tonnes
So, at the end of July 25th there are 32.5 tonnes of hay remaining in the barn.

d July has 31 days, so the end of day 31 is the start of August.
$u_{31} = 100 - 2.7 \times 31 = 16.3$ tonnes.
Hence there are 16.3 tonnes of hay in the barn at the beginning of August.

EXERCISE 7D.1

1 **a** $\frac{6}{2} = 3 \quad \therefore \quad r = 3, \quad u_1 = 2 \qquad \therefore \quad b = 6 \times 3 = 18$ and $c = 18 \times 3 = 54$

b $\frac{5}{10} = \frac{1}{2} \quad \therefore \quad r = \frac{1}{2}, \quad u_1 = 10 \qquad \therefore \quad b = 5 \times \frac{1}{2} = 2\frac{1}{2}$ and $c = 2\frac{1}{2} \times \frac{1}{2} = 1\frac{1}{4}$

c $\frac{-6}{12} = -\frac{1}{2} \quad \therefore \quad r = -\frac{1}{2}, \quad u_1 = 12 \qquad \therefore \quad b = -6 \times -\frac{1}{2} = 3$ and $c = 3 \times -\frac{1}{2} = -1\frac{1}{2}$

2 **a** $\frac{6}{3} = 2 \quad \therefore \quad r = 2, \quad u_1 = 3 \quad \therefore \quad u_6 = 3 \times 2^{6-1} = 96$

b $\frac{10}{2} = 5 \quad \therefore \quad r = 5, \quad u_1 = 2 \quad \therefore \quad u_6 = 2 \times 5^{6-1} = 6250$

c $\frac{256}{512} = \frac{1}{2} \quad \therefore \quad r = \frac{1}{2}, \quad u_1 = 512 \quad \therefore \quad u_6 = 512 \times (\frac{1}{2})^{6-1} = 16$

3 **a** $\frac{3}{1} = 3 \quad \therefore \quad r = 3, \quad u_1 = 1 \quad \therefore \quad u_9 = 1 \times 3^{9-1} = 6561$

b $\frac{18}{12} = \frac{3}{2} \quad \therefore \quad r = \frac{3}{2}, \quad u_1 = 12 \quad \therefore \quad u_9 = 12 \times (\frac{3}{2})^{9-1} = 307\frac{35}{64}$ or $\frac{19\,683}{64}$

c $\dfrac{-\frac{1}{8}}{\frac{1}{16}} = -2 \quad \therefore \quad r = -2, \quad u_1 = \frac{1}{16} \quad \therefore \quad u_9 = \frac{1}{16} \times (-2)^{9-1} = 16$

d $\dfrac{ar}{a} = r \quad \therefore \quad r = r, \quad u_1 = a \quad \therefore \quad u_9 = a \times r^{9-1} = ar^8$

4 **a** $\frac{10}{5} = \frac{20}{10} = \frac{40}{20} = 2$

Assuming the pattern continues, consecutive terms have a common ratio of 2.
$\therefore$ the sequence is geometric with $u_1 = 5$ and $r = 2$.

b
$$u_n = u_1 r^{n-1}$$
$$\therefore \quad u_n = 5 \times 2^{n-1}$$
$$\text{so} \quad u_{15} = 5 \times 2^{14} = 81\,920$$

5 **a**
$$\frac{-6}{12} = -\frac{1}{2}$$
$$\frac{3}{-6} = -\frac{1}{2}$$
$$\frac{(-\frac{3}{2})}{3} = -\frac{1}{2}$$

Assuming the pattern continues, consecutive terms have a common ratio of $-\frac{1}{2}$.
$\therefore$ the sequence is geometric with $u_1 = 12$ and $r = -\frac{1}{2}$.

b
$$u_n = u_1 r^{n-1}$$
$$\therefore \quad u_n = 12 \times \left(-\tfrac{1}{2}\right)^{n-1}$$

$$\begin{aligned} \text{so} \quad u_{13} &= 12 \times (-\tfrac{1}{2})^{13-1} \\ &= 12 \times (-\tfrac{1}{2})^{12} \\ &= 12 \times \tfrac{1}{4096} \\ &= 3 \times \tfrac{1}{1024} = \tfrac{3}{1024} \end{aligned}$$

6
$$\frac{-6}{8} = -\frac{3}{4}$$
$$\frac{4.5}{-6} = -\frac{(\frac{9}{2})}{6} = -\frac{3}{4}$$
$$\frac{-3.375}{4.5} = \frac{(-\frac{27}{8})}{(\frac{9}{2})} = -\frac{3}{4}$$

Assuming the pattern continues, consecutive terms have a common ratio of $-\frac{3}{4}$.
$\therefore$ the sequence is geometric with $u_1 = 8$ and $r = -\frac{3}{4}$.

$u_n = u_1 r^{n-1} = 8 \times (-\frac{3}{4})^{n-1}$ So, $u_{10} = 8 \times (-\frac{3}{4})^9 \approx -0.600\,677\,490\,2$

7 $\dfrac{4\sqrt{2}}{8} = \dfrac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$

$\dfrac{4}{4\sqrt{2}} = \frac{1}{\sqrt{2}}$

$\dfrac{2\sqrt{2}}{4} = \dfrac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$

Assuming the pattern continues, consecutive terms have a common ratio of $\frac{1}{\sqrt{2}}$.

$\therefore$ the sequence is geometric with $u_1 = 8$ and $r = \frac{1}{\sqrt{2}}$.

$$u_n = u_1 r^{n-1} = 8\left(\tfrac{1}{\sqrt{2}}\right)^{n-1} = 2^3 \times \left(2^{-\frac{1}{2}}\right)^{n-1} = 2^3 \times 2^{-\frac{1}{2}n+\frac{1}{2}}$$

So, $u_n = 2^{\frac{7}{2}-\frac{1}{2}n}$

8 **a** Since the terms are geometric,

$\dfrac{k}{7} = \dfrac{28}{k}$

$\therefore\ k^2 = 196$

$\therefore\ k = \pm 14$

b Since the terms are geometric,

$\dfrac{3k}{k} = \dfrac{20-k}{3k} = 3$

$\therefore\ 20 - k = 9k$

$\therefore\ 20 = 10k$

$\therefore\ k = 2$

c Since the terms are geometric,

$\dfrac{k+8}{k} = \dfrac{9k}{k+8}$

$\therefore\ (k+8)^2 = 9k^2$

$\therefore\ k^2 + 16k + 64 = 9k^2$

$\therefore\ 8k^2 - 16k - 64 = 0$

$\therefore\ 8(k^2 - 2k - 8) = 0$

$\therefore\ 8(k+2)(k-4) = 0$ and so $k = -2$ or 4

9 **a** $u_4 = 24 \quad \therefore\ u_1 \times r^3 = 24 \quad .... (1)$

$u_7 = 192 \quad \therefore\ u_1 \times r^6 = 192 \quad .... (2)$

So, $\dfrac{u_1 r^6}{u_1 r^3} = \dfrac{192}{24} \quad \{(2) \div (1)\}$

$\therefore\ r^3 = 8 \quad \therefore\ r = 2$

So in (1), $u_1 \times 2^3 = 24$

$\therefore\ u_1 \times 8 = 24$

$\therefore\ u_1 = 3$

$\therefore\ u_n = 3 \times 2^{n-1}$

b $u_3 = 8 \quad \therefore\ u_1 \times r^2 = 8 \quad .... (1)$

$u_6 = -1 \quad \therefore\ u_1 \times r^5 = -1 \quad .... (2)$

So, $\dfrac{u_1 r^5}{u_1 r^2} = -\dfrac{1}{8} \quad \{(2) \div (1)\}$

$\therefore\ r^3 = -\frac{1}{8} \quad \therefore\ r = -\frac{1}{2}$

So in (1), $u_1 \times (-\frac{1}{2})^2 = 8$

$\therefore\ u_1 \times \frac{1}{4} = 8$

$\therefore\ u_1 = 32$

$\therefore\ u_n = 32 \times (-\frac{1}{2})^{n-1}$

c $u_7 = 24 \quad \therefore\ u_1 \times r^6 = 24 \quad .... (1)$

$u_{15} = 384 \quad \therefore\ u_1 \times r^{14} = 384 \quad .... (2)$

So, $\dfrac{u_1 r^{14}}{u_1 r^6} = \dfrac{384}{24} \quad \{(2) \div (1)\}$

$\therefore\ r^8 = 16 \quad \therefore\ r = \pm\sqrt{2}$

So in (1), $u_1 \times (\pm\sqrt{2})^6 = 24$

$\therefore\ u_1 \times 8 = 24$

$\therefore\ u_1 = 3$

Now $u_n = u_1 r^{n-1}$

$\therefore\ u_n = 3 \times (\sqrt{2})^{n-1}$

or $u_n = 3 \times (-\sqrt{2})^{n-1}$

d $u_3 = 5 \quad \therefore\ u_1 \times r^2 = 5 \quad .... (1)$

$u_7 = \frac{5}{4} \quad \therefore\ u_1 \times r^6 = \frac{5}{4} \quad .... (2)$

So, $\dfrac{u_1 r^6}{u_1 r^2} = \dfrac{(\frac{5}{4})}{5} \quad \{(2) \div (1)\}$

$\therefore\ r^4 = \frac{1}{4} \quad \therefore\ r = \pm\frac{1}{\sqrt{2}}$

So in (1), $u_1 \times (\pm\frac{1}{\sqrt{2}})^2 = 5$

$\therefore\ u_1 \times \frac{1}{2} = 5$

$\therefore\ u_1 = 10$

Now $u_n = u_1 r^{n-1}$

$\therefore\ u_n = 10 \times (\frac{1}{\sqrt{2}})^{n-1}$

$= 10 \times (\sqrt{2})^{1-n}$

or $u_n = 10 \times (-\frac{1}{\sqrt{2}})^{n-1}$

$= 10 \times (-\sqrt{2})^{1-n}$

10 **a** 2, 6, 18, 54, ... has $u_1 = 2$ and $r = 3$

$u_n = u_1 r^{n-1}$ $\therefore$ $u_n = 2 \times 3^{n-1}$

Let $u_n = 10\,000 = 2 \times 3^{n-1}$, so $5000 = 3^{n-1}$

$\therefore$ $n \approx 8.7527$ {using technology}

We try the two values on either side of $n = 8.7527$, which are $n = 8$ and $n = 9$:

$u_8 = 2 \times 3^7 = 4374$ and $u_9 = 2 \times 3^8 = 13\,122$

So, the first term to exceed 10 000 is $u_9 = 13\,122$.

b $4, 4\sqrt{3}, 12, 12\sqrt{3}, ...$ has $u_1 = 4$ and $r = \sqrt{3}$

$u_n = u_1 r^{n-1}$ $\therefore$ $u_n = 4 \times \left(\sqrt{3}\right)^{n-1}$

Let $u_n = 4800 = 4 \times \left(\sqrt{3}\right)^{n-1}$, so $1200 = \left(\sqrt{3}\right)^{n-1}$

$\therefore$ $n \approx 13.91$ {using technology}

We try the two values on either side of $n \approx 13.91$, which are $n = 13$ and $n = 14$:

$u_{13} = 4 \times (\sqrt{3})^{12} = 2916$ and $u_{14} = 4 \times (\sqrt{3})^{13} = 2916\sqrt{3} \approx 5050.7$

So, the first term to exceed 4800 is $u_{14} = 2916\sqrt{3} \approx 5050.7$.

c 12, 6, 3, 1.5, ... has $u_1 = 12$ and $r = \frac{1}{2}$

$u_n = u_1 r^{n-1}$ $\therefore$ $u_n = 12 \times (\frac{1}{2})^{n-1}$ Let $0.0001 = u_n = 12 \times \left(\frac{1}{2}\right)^{n-1}$

$\therefore$ $0.000\,008\,\overline{3} = \left(\frac{1}{2}\right)^{n-1}$

$\therefore$ $n \approx 17.87$ {using technology}

We try the two values on either side of $n \approx 17.87$, which are $n = 17$ and $n = 18$:

$u_{17} = 12 \times \left(\frac{1}{2}\right)^{16} \approx 0.000\,183\,1$ and $u_{18} = 12 \times \left(\frac{1}{2}\right)^{17} \approx 0.000\,091\,55$

So, the first term of the sequence which is less than 0.0001 is $u_{18} \approx 0.000\,091\,55$.

EXERCISE 7D.2

1 There is a fixed percentage increase each week, so the population forms a geometric sequence.

$u_{n+1} = u_1 \times r^n$, where $u_1 = 500$, $r = 1.12$

a **i** $u_{11} = 500 \times (1.12)^{10}$
≈ 1552.92
There will be approximately 1550 ants.

ii $u_{21} = 500 \times (1.12)^{20}$
≈ 4823.15
There will be approximately 4820 ants.

b For the population to reach 2000, $u_{n+1} = 500 \times (1.12)^n = 2000$

$\therefore$ $(1.12)^n = 4$

$\therefore$ $n \approx 12.23$ {using technology}

It will take approximately 12.2 weeks.

2 $u_{n+1} = u_1 \times r^n$, where $u_1 = 555$, $r = 0.955$

a $u_{16} = 555 \times (0.955)^{15}$
≈ 278.19 The population is approximately 278 animals in the year 2010.

b For the population to have declined to 50,

$u_{n+1} = 555 \times (0.955)^n = 50$

$\therefore$ $(0.955)^n = 0.0\overline{900}$

$\therefore$ $n \approx 52.3$ {using technology}

So, in the 53rd year the population is 50. This is the year 2047.

3 $u_{n+1} = u_1 \times r^n$, where $u_1 = 32$, $r = 1.18$

a **i** $u_6 = 32 \times (1.18)^5$
≈ 73.21
There will be approximately 73 deer.

ii $u_{11} = 32 \times (1.18)^{10}$
≈ 167.48
There will be approximately 167 deer.

b For the population to reach 5000, $u_{n+1} = 32 \times (1.18)^n = 5000$

$\therefore \; n \approx 30.52$ {using technology}

So, it will take approximately 30.5 years.

4 $u_{n+1} = u_1 \times r^n$, where $u_1 = 178$, $r = 1.32$

a **i** $u_{11} = 178 \times (1.32)^{10}$
≈ 2858.6

There will be approximately 2860 marsupials.

ii $u_{26} = 178 \times (1.32)^{25}$
$\approx 183\,979.0$

There will be approximately 184 000 marsupials.

b For the population to reach 10 000, $u_{n+1} = 178 \times (1.32)^n = 10\,000$

$\therefore \; n \approx 14.5$ {using technology}

So, it will take approximately 14.5 years.

EXERCISE 7D.3

1 **a** $u_{n+1} = u_1 \times r^n$
where $u_1 = 3000$, $r = 1.1$, $n = 3$

$\therefore \; u_4 = 3000 \times (1.1)^3$
$= 3993$

The investment will amount to \$3993.

b Interest = amount after 3 years − initial amount
= \$3993 − \$3000
= \$993

2 $u_{n+1} = u_1 \times r^n$ where $u_1 = 20\,000$, $r = 1.12$, $n = 4$

$\therefore \; u_5 = 20\,000 \times (1.12)^4$
$\approx 31\,470.39$

Interest = €31 470.39 − €20 000
= €11 470.39

3 **a** $u_{n+1} = u_1 \times r^n$
where $u_1 = 30\,000$, $r = 1.1$, $n = 4$

$\therefore \; u_5 = 30\,000 \times (1.1)^4$
$= 43\,923$

The investment amounts to ¥43 923.

b Interest = amount after 4 years − initial amount
= ¥43 923 − ¥30 000
= ¥13 923

4 $u_{n+1} = u_1 \times r^n$ where $u_1 = 80\,000$, $r = 1.09$, $n = 3$

$\therefore \; u_4 = 80\,000 \times (1.09)^3$
$= 103\,602.32$

Interest = amount after 3 years − initial amount
= \$103 602.32 − \$80 000
= \$23 602.32

5 $u_{n+1} = u_1 \times r^n$ where $u_1 = 100\,000$, $r = 1 + \dfrac{0.08}{2} = 1.04$, $n = 10$

$\therefore \; u_{11} = 100\,000 \times (1.04)^{10}$
$\approx 148\,024.43$ It amounts to ¥148 024.43.

6 $u_{n+1} = u_1 \times r^n$ where $u_1 = 45\,000$, $r = 1 + \dfrac{0.075}{4} = 1.018\,75$, $n = 7$ {21 months = 7 'quarters'}

$\therefore \; u_{10} = 45\,000 \times (1.018\,75)^7$
$\approx 51\,249.06$ It amounts to £51 249.06.

7 The initial investment u_1 is unknown.
The interest rate is compounded annually, so the multiplier $r = 1 + 0.075 = 1.075$.
There are 4 compounding periods, so $n = 4$

$\therefore \; u_{n+1} = u_5$

Now $u_5 = u_1 \times r^4$ {using $u_{n+1} = u_1 \times r^n$}

$\therefore\ 20\,000 = u_1 \times (1.075)^4$

$\therefore\ u_1 = \dfrac{20\,000}{(1.075)^4}$

$\therefore\ u_1 \approx 14\,976.01$ So, \$14 976.01 should be invested now.

8 The initial investment is unknown.

The interest rate is compounded annually, so the multiplier $r = 1 + 0.055 = 1.055$.

There are $\frac{60}{12} = 5$ compounding periods, so $n = 5$

$\therefore\ u_{n+1} = u_6$

Now $u_6 = u_1 \times r^5$ {using $u_{n+1} = u_1 \times r^n$}

$\therefore\ 15\,000 = u_1 \times (1.055)^5$

$\therefore\ u_1 = \dfrac{15\,000}{(1.055)^5}$

$\therefore\ u_1 \approx 11\,477.02$ The initial investment required is £11 477.02.

9 The initial investment is unknown.

The interest rate is compounded quarterly, so the multiplier $r = 1 + \dfrac{0.08}{4} = 1.02$.

There are $3 \times 4 = 12$ compounding periods, so $n = 12$

$\therefore\ u_{n+1} = u_{13}$

Now $u_{13} = u_1 \times r^{12}$ {using $u_{n+1} = u_1 \times r^n$}

$\therefore\ 25\,000 = u_1 \times (1.02)^{12}$

$\therefore\ u_1 = \dfrac{25\,000}{(1.02)^{12}}$

$\therefore\ u_1 \approx 19\,712.33$ I should invest €19 712.33 now.

10 The initial investment is unknown.

The interest rate is compounded monthly, so the multiplier $r = 1 + \dfrac{0.09}{12} = 1.0075$.

There are $8 \times 12 = 96$ compounding periods, so $n = 96$

$\therefore\ u_{n+1} = u_{97}$

Now $u_{97} = u_1 \times r^{96}$ {using $u_{n+1} = u_1 \times r^n$}

$\therefore\ 40\,000 = u_1 \times (1.0075)^{96}$

$\therefore\ u_1 = \dfrac{40\,000}{(1.0075)^{96}}$

$\therefore\ u_1 \approx 19\,522.47$ The initial investment should be ¥19 522.47.

EXERCISE 7E

1 **a** **i** 3, 11, 19, 27, ... is arithmetic with $u_1 = 3,\ d = 8,$ so $u_n = 3 + (n-1)8 = 8n - 5$

$S_n = \sum_{k=1}^{n} (8k - 5)$

ii $S_5 = 3 + 11 + 19 + 27 + 35 = 95$

b **i** 42, 37, 32, 27, ... is arithmetic with $u_1 = 42,\ d = -5,$

so $u_n = 42 + (n-1)(-5) = 47 - 5n$

$\therefore\ S_n = \sum_{k=1}^{n} (47 - 5k)$

ii $S_5 = 42 + 37 + 32 + 27 + 22 = 160$

c **i** 12, 6, 3, $1\frac{1}{2}$, ... is geometric with $u_1 = 12$, $r = \frac{1}{2}$, so $u_n = 12 \times (\frac{1}{2})^{n-1}$

$S_n = \sum_{k=1}^{n} 12(\frac{1}{2})^{k-1}$

ii $S_5 = 12 + 6 + 3 + 1\frac{1}{2} + \frac{3}{4} = 23\frac{1}{4}$

d **i** 2, 3, $4\frac{1}{2}$, $6\frac{3}{4}$, ... is geometric with $u_1 = 2$, $r = \frac{3}{2}$, so $u_n = 2 \times (\frac{3}{2})^{n-1}$

$S_n = \sum_{k=1}^{n} 2(\frac{3}{2})^{k-1}$

ii $S_5 = 2 + 3 + 4\frac{1}{2} + 6\frac{3}{4} + 10\frac{1}{8} = 26\frac{3}{8}$

e **i** 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, ... is geometric with $u_1 = 1$, $r = \frac{1}{2}$, so $u_n = 1 \times (\frac{1}{2})^{n-1} = \dfrac{1}{2^{n-1}}$

$S_n = \sum_{k=1}^{n} \dfrac{1}{2^{k-1}}$

ii $S_5 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = 1\frac{15}{16}$

f **i** 1, 8, 27, 64, ...

$S_n = \sum_{k=1}^{n} k^3$ {since $1 = 1^3$, $8 = 2^3$, $27 = 3^3$, $64 = 4^3$}

ii $S_5 = 1 + 8 + 27 + 64 + 125 = 225$

2 **a** $\sum_{k=1}^{3} 4k = 4 + 8 + 12 = 24$

b $\sum_{k=1}^{6} (k+1) = 2 + 3 + 4 + 5 + 6 + 7 = 27$

c $\sum_{k=1}^{4} (3k-5) = -2 + 1 + 4 + 7 = 10$

d $\sum_{k=1}^{5} (11-2k) = 9 + 7 + 5 + 3 + 1 = 25$

e $\sum_{k=1}^{7} k(k+1) = 2 + 6 + 12 + 20 + 30 + 42 + 56 = 168$

f $\sum_{k=1}^{5} 10 \times 2^{k-1} = 10 + 20 + 40 + 80 + 160 = 310$

3 $u_n = 3n - 1$

$$\begin{aligned}\therefore \quad u_1 + u_2 + u_3 + ... + u_{20} &= \sum_{n=1}^{20} (3n-1) \\ &= 2 + 5 + 8 + 11 + 14 + 17 + 20 + 23 + 26 + 29 + 32 + 35 \\ &\quad + 38 + 41 + 44 + 47 + 50 + 53 + 56 + 59 \\ &= 610\end{aligned}$$

4 **a** $\sum_{k=1}^{n} c = \underbrace{c + c + c + ... + c}_{n \text{ times}} = cn$

b $$\begin{aligned}\sum_{k=1}^{n} ca_k &= ca_1 + ca_2 + ... + ca_n \\ &= c(a_1 + a_2 + ... + a_n) \\ &= c\sum_{k=1}^{n} a_k\end{aligned}$$

c $$\begin{aligned}\sum_{k=1}^{n} (a_k + b_k) &= (a_1 + b_1) + (a_2 + b_2) + ... + (a_n + b_n) \\ &= (a_1 + a_2 + ... + a_n) + (b_1 + b_2 + ... + b_n) \\ &= \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k\end{aligned}$$

5 **a** $\sum_{k=1}^{5} k(k+1)(k+2) = 6 + 24 + 60 + 120 + 210 = 420$

b $\displaystyle\sum_{k=6}^{12} \left(100(1.2)^{k-3}\right) = 172.8 + 207.36 + 248.832 + 298.5984 + 358.318\,08$
$+\,429.981\,696 + 515.978\,0352$
≈ 2232

6 **a** $\displaystyle\sum_{k=1}^{n} k = 1 + 2 + 3 + ... + (n-2) + (n-1) + n$

b
$$\begin{array}{l} \ \ 1 \quad + \quad 2 \quad + \quad 3 \quad + ... + (n-2) + (n-1) + \quad n \\ \ \ n \quad + (n-1) + (n-2) + ... + \quad 3 \quad + \quad 2 \quad + \quad 1 \\ \hline (n+1) + (n+1) + (n+1) + ... + (n+1) + (n+1) + (n+1) = n(n+1) \end{array}$$

c $S_n = \displaystyle\sum_{k=1}^{n} k = 1 + 2 + 3 + ... + (n-2) + (n-1) + n$ {from **a**}

and $2[1 + 2 + 3 + ... + (n-2) + (n-1) + n] = n(n+1)$ {from **b**}

$\therefore\ 2S_n = n(n+1)$

$\therefore\ S_n = \dfrac{n(n+1)}{2}$

d $\displaystyle\sum_{k=1}^{n} (ak+b) = \sum_{k=1}^{n} ak + \sum_{k=1}^{n} b$
$= a\displaystyle\sum_{k=1}^{n} k + nb$
$= \dfrac{an(n+1)}{2} + nb$ {using **c**}

But $\displaystyle\sum_{k=1}^{n} (ak+b) = 8n^2 + 11n$

$\therefore\ \dfrac{an(n+1)}{2} + nb = 8n^2 + 11n$

$\therefore\ \dfrac{an^2 + an}{2} + nb = 8n^2 + 11n$

$\therefore\ \dfrac{a}{2}n^2 + \dfrac{a}{2}n + nb = 8n^2 + 11n$

$\therefore\ \dfrac{a}{2}n^2 + \left(\dfrac{a}{2} + b\right) n = 8n^2 + 11n$

Comparing coefficients, we get $\dfrac{a}{2} = 8$ and $\dfrac{a}{2} + b = 11$

$\therefore\ a = 16$ $\quad\therefore\ 8 + b = 11$

$\therefore\ b = 3$

7 **a** $\displaystyle\sum_{k=1}^{n} (3k^2 + 4k - 3) = \sum_{k=1}^{n} 3k^2 + \sum_{k=1}^{n} 4k - \sum_{k=1}^{n} 3$ {using **4 c**}

$= 3\displaystyle\sum_{k=1}^{n} k^2 + 4\sum_{k=1}^{n} k - 3n$ {using **4 a**, **b**}

b $\displaystyle\sum_{k=1}^{n} (k+1)(k+2) = \sum_{k=1}^{n} (k^2 + 3k + 2)$

$= \displaystyle\sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k + 2n$

$= \dfrac{n(n+1)(2n+1)}{6} + 3\dfrac{n(n+1)}{2} + 2n$

$= \dfrac{n(n+1)(2n+1) + 9n(n+1) + 12n}{6}$

$= \dfrac{2n^3 + n^2 + 2n^2 + n + 9n^2 + 9n + 12n}{6}$

$= \dfrac{2n^3 + 12n^2 + 22n}{6}$

$= \dfrac{n(n^2 + 6n + 11)}{3}$

When $n = 10$,

$$\begin{aligned}\text{LHS} &= \sum_{k=1}^{10}(k+1)(k+2)\\ &= 2\times3+3\times4+4\times5+5\times6\\ &\quad +6\times7+7\times8+8\times9\\ &\quad +9\times10+10\times11+11\times12\\ &= 6+12+20+30+42+56\\ &\quad +72+90+110+132\\ &= 570\end{aligned}$$

$$\begin{aligned}\text{RHS} &= \frac{10(10^2+6(10)+11)}{3}\\ &= \frac{10\times171}{3}\\ &= 570 \quad \checkmark\end{aligned}$$

EXERCISE 7F

1 **a** The series is arithmetic with
$u_1 = 3, \quad d = 4, \quad n = 20$

$$S_n = \frac{n}{2}(2u_1 + (n-1)d)$$

$$\begin{aligned}\text{So,}\quad S_{20} &= \tfrac{20}{2}(2\times3+19\times4)\\ &= 10(6+76)\\ &= 820\end{aligned}$$

b The series is arithmetic with
$u_1 = \frac{1}{2}, \quad d = \frac{5}{2}, \quad n = 50$

$$S_n = \frac{n}{2}(2u_1 + (n-1)d)$$

$$\begin{aligned}\text{So,}\quad S_{50} &= \tfrac{50}{2}\left(2\times\tfrac{1}{2}+49\times\tfrac{5}{2}\right)\\ &= 25(1+122\tfrac{1}{2})\\ &= 3087\tfrac{1}{2}\end{aligned}$$

c The series is arithmetic with
$u_1 = 100, \quad d = -7, \quad n = 40$

$$S_n = \frac{n}{2}(2u_1 + (n-1)d)$$

$$\begin{aligned}\text{So,}\quad S_{40} &= \tfrac{40}{2}(2\times100+39\times(-7))\\ &= 20(200-273)\\ &= -1460\end{aligned}$$

d The series is arithmetic with
$u_1 = 50, \quad d = -\frac{3}{2}, \quad n = 80$

$$S_n = \frac{n}{2}(2u_1 + (n-1)d)$$

$$\begin{aligned}\text{So,}\quad S_{80} &= \tfrac{80}{2}\left(2\times50+79\times(-\tfrac{3}{2})\right)\\ &= 40(100-\tfrac{237}{2})\\ &= -740\end{aligned}$$

2 **a** The series is arithmetic with
$u_1 = 5, \quad d = 3, \quad u_n = 101$

$$\begin{aligned}\text{Since}\quad u_n &= 101,\\ \text{then}\quad u_1+(n-1)d &= 101\\ \therefore\quad 5+3(n-1) &= 101\\ \therefore\quad 5+3n-3 &= 101\\ \therefore\quad 3n &= 99\\ \therefore\quad n &= 33\end{aligned}$$

$$\begin{aligned}\text{So,}\quad S_n &= \frac{n}{2}(u_1+u_n)\\ &= \tfrac{33}{2}(5+101)\\ &= 1749\end{aligned}$$

b The series is arithmetic with
$u_1 = 50, \quad d = -\frac{1}{2}, \quad u_n = -20$

$$\begin{aligned}\text{Since}\quad u_n &= -20,\\ \text{then}\quad u_1+(n-1)d &= -20\\ \therefore\quad 50+(-\tfrac{1}{2})(n-1) &= -20\\ \therefore\quad -\tfrac{1}{2}n+\tfrac{1}{2} &= -70\\ \therefore\quad -\tfrac{1}{2}n &= -\tfrac{141}{2}\\ \therefore\quad n &= 141\end{aligned}$$

$$\begin{aligned}\text{So,}\quad S_n &= \frac{n}{2}(u_1+u_n)\\ &= \tfrac{141}{2}(50+(-20))\\ &= 2115\end{aligned}$$

c The series is arithmetic with
$u_1 = 8, \quad d = \frac{5}{2}, \quad u_n = 83$

$$\begin{aligned}\text{Since}\quad u_n &= 83,\\ \text{then}\quad u_1+(n-1)d &= 83\\ \therefore\quad 8+\tfrac{5}{2}(n-1) &= 83\\ \therefore\quad \tfrac{5}{2}n-\tfrac{5}{2} &= 75\\ \therefore\quad \tfrac{5}{2}n &= \tfrac{155}{2}\\ \therefore\quad n &= 31\end{aligned}$$

$$\begin{aligned}\text{So,}\quad S_n &= \frac{n}{2}(u_1+u_n)\\ &= \tfrac{31}{2}(8+83)\\ &= 1410\tfrac{1}{2}\end{aligned}$$

3 **a** $\sum_{k=1}^{10}(2k+5) = 7+9+11+ .... +25$

This series is arithmetic with $u_1 = 7$, $d = 2$, and $n = 10$.

$\therefore$ sum $= \frac{n}{2}[2u_1 + (n-1)d] = \frac{10}{2}[14 + 9 \times 2] = 160$

b $\sum_{k=1}^{15}(k-50) = (-49) + (-48) + (-47) + ... + (-35)$

This series is arithmetic with $u_1 = -49$, $d = 1$, and $n = 15$.

$\therefore$ sum $= \frac{n}{2}[2u_1 + (n-1)d] = \frac{15}{2}[-98 + 14 \times 1] = -630$

c $\sum_{k=1}^{20}\left(\frac{k+3}{2}\right) = 2 + \frac{5}{2} + 3 + ... + \frac{23}{2}$

This series is arithmetic with $u_1 = 2$, $r = \frac{1}{2}$, and $n = 20$.

$\therefore$ sum $= \frac{n}{2}[2u_1 + (n-1)d] = \frac{20}{2}[4 + 19 \times \frac{1}{2}] = 135$

4 $u_1 = 5$, $n = 7$, $u_n = 53$

$$\begin{aligned} S_n &= \frac{n}{2}(u_1 + u_n) \\ &= \tfrac{7}{2}(5 + 53) \\ &= 203 \end{aligned}$$

5 $u_1 = 6$, $n = 11$, $u_n = -27$

$$\begin{aligned} S_n &= \frac{n}{2}(u_1 + u_n) \\ &= \tfrac{11}{2}(6 + (-27)) \\ &= -115\tfrac{1}{2} \end{aligned}$$

6 The total number of bricks can be expressed as an arithmetic series: $1 + 2 + 3 + 4 + ... + n$

We know that the total number of bricks is 171, so $S_n = 171$.

Also, $u_1 = 1$, $d = 1$ and we need to find n, the number of members (layers) of the series.

$$S_n = \frac{n}{2}(2u_1 + (n-1)d) = 171$$

$$\therefore \quad \frac{n}{2}(2 \times 1 + (n-1) \times 1) = 171$$

$$\therefore \quad n(2 + n - 1) = 342$$

$$\therefore \quad n(n+1) = 342$$

$$\therefore \quad n^2 + n - 342 = 0$$

$$\therefore \quad (n-18)(n+19) = 0$$

$$\therefore \quad n = -19 \text{ or } 18$$

But $n > 0$, so $n = 18$. So, there are 18 layers placed.

7 The total number of seats in n rows can be expressed as an arithmetic series:

$22 + 23 + 24 + ... + u_n$

Row 1 has 22 seats, so $u_1 = 22$. Row 2 has 23 seats, so $d = 1$.

$$\begin{aligned} S_n &= \frac{n}{2}(2u_1 + (n-1)d) \\ &= \frac{n}{2}(2 \times 22 + 1(n-1)) \\ &= \frac{n}{2}(44 + n - 1) \end{aligned}$$

$\therefore \quad S_n = \frac{n}{2}(n + 43)$ which is the total number of seats in n rows.

a Number of seats in row 44 of one section $=$ total no. of seats in every row $-$ no. of seats in the first 43 rows

$$\begin{aligned} &= S_{44} - S_{43} \\ &= \tfrac{44}{2}(44 + 43) - \tfrac{43}{2}(43 + 43) \\ &= 1914 - 1849 \\ &= 65 \end{aligned}$$

b Number of seats in a section $= S_{44} = 1914$ (from **a**)

c Number of seats in 25 sections $= S_{44} \times 25 = 1914 \times 25 = 47\,850$

8 **a** The first 50 multiples of 11 can be expressed as an arithmetic series:

$11 + 22 + 33 + ... + u_{50}$ where $u_1 = 11, \quad d = 11, \quad n = 50$

$S_n = \frac{n}{2}(2u_1 + (n-1)d) \qquad \therefore \quad S_{50} = \frac{50}{2}(2 \times 11 + 11(50-1))$

$$= 25(22 + 539)$$
$$= 14\,025$$

b The multiples of 7 between 0 and 1000 can be expressed as an arithmetic series:

$7 + 14 + 21 + 28 + ... + u_n$ where $u_1 = 7, \quad d = 7$

To find u_n, we need to find the largest multiple of 7 less than 1000.

Now $\frac{1000}{7} \approx 142.9$, so $u_n = 142 \times 7 = 994$

But $u_n = u_1 + (n-1)d$

$\therefore \quad 994 = 7 + 7(n-1)$

$\therefore \quad 987 = 7n - 7$

$\therefore \quad 7n = 994$

$\therefore \quad n = 142$

So, $S_{142} = \frac{142}{2}(7 + 994) = 71\,071$

c The integers between 1 and 100 which are not divisible by 3 can be expressed as:

1, 2, 4, 5, 7, 8, ..., 100 where $u_1 = 1, \quad u_n = 100$.

Alternatively, these integers can be expressed as two separate arithmetic series:

$S_A = 1 + 4 + 7 + ... + 97 + 100$ where $u_1 = 1, \quad d = 3, \quad u_n = 100$

and $S_B = 2 + 5 + 8 + ... + 95 + 98$ where $u_1 = 2, \quad d = 3, \quad u_n = 98$

Now for S_A, $u_n = u_1 + (n-1)d$

$\therefore \quad 100 = 1 + 3(n-1)$

$\therefore \quad 99 = 3n - 3$

$\therefore \quad 3n = 102$

$\therefore \quad n = 34$

and for S_B, $u_n = u_1 + (n-1)d$

$\therefore \quad 98 = 2 + 3(n-1)$

$\therefore \quad 96 = 3n - 3$

$\therefore \quad 3n = 99$

$\therefore \quad n = 33$

So, $S_A = \frac{34}{2}(1 + 100) = 1717$ and $S_B = \frac{33}{2}(2 + 98) = 1650$

The total sum $= S_A + S_B = 1717 + 1650 = 3367$

9 The series of odd numbers can be expressed as an arithmetic series:

$1 + 3 + 5 + 7 + ...$ where $u_1 = 1, \quad d = 2$

a Now $u_n = u_1 + (n-1)d = 1 + 2(n-1)$

$\therefore \quad u_n = 2n - 1$

b We need to show that S_n is n^2.

The sum of the first n odd numbers can be expressed as an arithmetic series:

$1 + 3 + 5 + 7 + ... + (2n-1)$ $\quad \{$using $u_n = 2n - 1$ from **a**$\}$

So, $S_n = \frac{n}{2}(u_1 + u_n)$

$$= \frac{n}{2}(1 + (2n-1))$$

$$= \frac{n}{2}(2n)$$

Hence $S_n = n^2$ as required.

c $S_1 = 1 \qquad\qquad\qquad = 1 = 1^2 = n^2$ for $n = 1$ ✓

$S_2 = 1 + 3 \qquad\qquad = 4 = 2^2 = n^2$ for $n = 2$ ✓

$S_3 = 1 + 3 + 5 \qquad = 9 = 3^2 = n^2$ for $n = 3$ ✓

$S_4 = 1 + 3 + 5 + 7 = 16 = 4^2 = n^2$ for $n = 4$ ✓

10 $u_6 = 21$, $S_{17} = 0$. We need to find u_1 and u_2.

$S_n = \frac{n}{2}(2u_1 + (n-1)d)$

$\therefore\ S_{17} = \frac{17}{2}(2u_1 + 16d) = 0$

$\therefore\ u_1 + 8d = 0$

$\therefore\ u_1 = -8d$ .... (1)

Also, $u_n = u_1 + (n-1)d$

$\therefore\ u_6 = u_1 + 5d$

$\therefore\ 21 = -8d + 5d$ {using (1)}

$\therefore\ 21 = -3d$

$\therefore\ d = -7$

So, $u_1 = -8(-7) = 56$ and $u_2 = 56 - 7 = 49$

The first two terms are 56 and 49.

11 Let the three consecutive terms be $x - d$, x, and $x + d$.

Now, sum of terms $= 12$

$\therefore\ (x-d) + x + (x+d) = 12$

$\therefore\ 3x = 12$

$\therefore\ x = 4$

So, the terms are $4-d$, 4, $4+d$

Also, product of terms $= -80$

$\therefore\ (4-d)4(4+d) = -80$

$\therefore\ 4(4^2 - d^2) = -80$

$\therefore\ 16 - d^2 = -20$

$\therefore\ d^2 = 36$ $\therefore\ d = \pm 6$

So, the three terms could be $4-6$, 4, $4+6$, which are $-2, 4, 10$

or $4-(-6)$, 4, $4+(-6)$, which are $10, 4, -2$.

12 Let the five consecutive terms be $n-2d$, $n-d$, n, $n+d$, $n+2d$.

Now, sum of terms $= 40$ $\therefore\ (n-2d) + (n-d) + n + (n+d) + (n+2d) = 40$

$\therefore\ 5n = 40$

$\therefore\ n = 8$

So the terms are $8-2d$, $8-d$, 8, $8+d$, $8+2d$

Also, the product of the first, middle and last terms $= (8-2d) \times 8 \times (8+2d) = 224$

$\therefore\ 8(8^2 - 4d^2) = 224$

$\therefore\ 64 - 4d^2 = 28$

$\therefore\ 4d^2 = 36$

$\therefore\ d^2 = 9$

$\therefore\ d = \pm 3$

So, the five terms could be $8-2(3)$, $8-3$, 8, $8+3$, $8+2(3)$, which are $2, 5, 8, 11, 14$

or $8-2(-3)$, $8-(-3)$, 8, $8+(-3)$, $8+2(-3)$, which are $14, 11, 8, 5, 2$.

13 $11, 14, 17, 20, \ldots$ is arithmetic with $u_1 = 11$, $d = 3$

$$\therefore\ S_n = \frac{n}{2}(2u_1 + (n-1)d) = \frac{n}{2}(22 + 3(n-1)) = \frac{n}{2}(3n + 19)$$

Suppose $S_n = 2000$

$\therefore\ \frac{n}{2}(3n+19) = 2000$

$\therefore\ 3n^2 + 19n = 4000$

Using technology to list the terms of $3n^2 + 19n$:

$n = 33$ gives $3n^2 + 19n = 3894$ $\therefore\ S_{33} = 1947$

$n = 34$ gives $3n^2 + 19n = 4114$ $\therefore\ S_{34} = 2057$

$\therefore$ Henk would sell the 2000th TV set in week 34.

14 **a** $S_n = \dfrac{n(3n+11)}{2}$

$\therefore\ S_1 = u_1 = \dfrac{1(14)}{2} = 7$

and $S_2 = u_1 + u_2 = \dfrac{2(17)}{2} = 17$

$\therefore\ u_1 = 7$ and $u_2 = 10$

b $u_1 = 7$ and $d = 3$

$\therefore\ u_{20} = u_1 + 19d = 7 + 19 \times 3 = 64$

$\therefore$ the twentieth term is 64.

EXERCISE 7G.1

1 **a** The series is geometric with $u_1 = 12,\quad r = \frac{1}{2},\quad n = 10$

Now $S_n = \dfrac{u_1(1-r^n)}{1-r}$

$\therefore\quad S_{10} = \dfrac{12\left(1-(\frac{1}{2})^{10}\right)}{1-\frac{1}{2}}$

$\approx 23.9766 \approx 24.0$

b The series is geometric with $u_1 = \sqrt{7},\quad r = \sqrt{7},\quad n = 12$

Now $S_n = \dfrac{u_1(r^n-1)}{r-1}$

$\therefore\quad S_{12} = \dfrac{\sqrt{7}\left((\sqrt{7})^{12}-1\right)}{\sqrt{7}-1}$

$\approx 189\,134$

c The series is geometric with $u_1 = 6,\quad r = -\frac{1}{2},\quad n = 15$

Now $S_n = \dfrac{u_1(1-r^n)}{1-r}$

$\therefore\quad S_{15} = \dfrac{6\left(1-(-\frac{1}{2})^{15}\right)}{1-(-\frac{1}{2})} \approx 4.000$

d The series is geometric with $u_1 = 1,\quad r = -\frac{1}{\sqrt{2}},\quad n = 20$

Now $S_n = \dfrac{u_1(1-r^n)}{1-r}$

$\therefore\quad S_{20} = \dfrac{1\left(1-(-\frac{1}{\sqrt{2}})^{20}\right)}{1-(-\frac{1}{\sqrt{2}})} \approx 0.5852$

2 **a** The series is geometric with $u_1 = \sqrt{3},\quad r = \sqrt{3}$

$$S_n = \frac{u_1(r^n-1)}{r-1}$$
$$= \frac{\sqrt{3}\left((\sqrt{3})^n-1\right)}{\sqrt{3}-1} \times \left(\frac{\sqrt{3}+1}{\sqrt{3}+1}\right)$$
$$= \frac{\left(3+\sqrt{3}\right)\left((\sqrt{3})^n-1\right)}{3-1}$$
$$= \frac{3+\sqrt{3}}{2}\left((\sqrt{3})^n-1\right)$$

b The series is geometric with $u_1 = 12,\quad r = \frac{1}{2}$

$$S_n = \frac{u_1(1-r^n)}{1-r}$$
$$= \frac{12\left(1-(\frac{1}{2})^n\right)}{1-\frac{1}{2}}$$
$$= 24\left(1-(\tfrac{1}{2})^n\right)$$

c The series is geometric with $u_1 = 0.9,\quad r = 0.1$

$$S_n = \frac{u_1(1-r^n)}{1-r}$$
$$= \frac{0.9\left(1-(0.1)^n\right)}{1-0.1}$$
$$= 1-(0.1)^n$$

d The series is geometric with $u_1 = 20,\quad r = -\frac{1}{2}$

$$S_n = \frac{u_1(1-r^n)}{1-r} = \frac{20\left(1-(-\frac{1}{2})^n\right)}{1-(-\frac{1}{2})}$$
$$= \frac{20\left(1-(-\frac{1}{2})^n\right)}{(\frac{3}{2})}$$
$$= \tfrac{40}{3}\left(1-(-\tfrac{1}{2})^n\right)$$

3 **a** $S_1 = u_1 \quad \therefore\quad u_1 = 3$

b $u_2 = S_2 - S_1$
$= 4 - 3 = 1$

So, $r = \frac{1}{3}$

c $u_1 = 3,\quad r = \frac{1}{3}$

so $u_n = 3 \times (\frac{1}{3})^{n-1}$

$\therefore\quad u_5 = 3 \times (\frac{1}{3})^4 = \frac{1}{27}$

4 **a** $\displaystyle\sum_{k=1}^{10} 3 \times 2^{k-1} = 3 + 6 + 12 + ... + 384 + 768 + 1536$

This series is geometric with $u_1 = 3,\quad r = 2,\quad$ and $\quad n = 10.$

$\therefore\quad \text{sum} = \dfrac{u_1(r^n-1)}{r-1} = \dfrac{3(2^{10}-1)}{1} = 3069$

b $\sum_{k=1}^{12} \left(\frac{1}{2}\right)^{k-2} = 2 + 1 + \frac{1}{2} + ... + \frac{1}{256} + \frac{1}{512} + \frac{1}{1024}$

This series is geometric with $u_1 = 2$, $r = \frac{1}{2}$, and $n = 12$.

$\therefore \quad \text{sum} = \dfrac{u_1(1-r^n)}{1-r} = \dfrac{2\left(1-(\frac{1}{2})^{12}\right)}{\frac{1}{2}} = 4\left(1-\left(\frac{1}{2}\right)^{12}\right) = \dfrac{2^{12}-1}{2^{10}}$

$\therefore \quad \text{sum} = \dfrac{4095}{1024} \approx 4.00$

c $\sum_{k=1}^{25} 6 \times (-2)^k = -12 + 24 + (-48) + ... + 100\,663\,296 + (-201\,326\,592)$

This series is geometric with $u_1 = -12$, $r = -2$, and $n = 25$.

$\therefore \quad \text{sum} = \dfrac{u_1(1-r^n)}{1-r} = \dfrac{-12\left(1-(-2)^{25}\right)}{1+2} = -4\left(1-(-2)^{25}\right)$

$\therefore \quad \text{sum} = -134\,217\,732$

5 **a**
$$\begin{aligned} A_2 &= A_1 \times 1.06 + 2000 \\ &= (A_0 \times 1.06 + 2000) \times 1.06 + 2000 \\ &= (2000 \times 1.06 + 2000) \times 1.06 + 2000 \end{aligned}$$
$\therefore \quad A_2 = 2000 + 2000 \times 1.06 + 2000 \times (1.06)^2$ as required

b
$$\begin{aligned} A_3 &= A_2 \times 1.06 + 2000 \\ &= \left[2000 + 2000 \times 1.06 + 2000 \times (1.06)^2\right] \times 1.06 + 2000 \quad \{\text{from } \mathbf{a}\} \end{aligned}$$
$\therefore \quad A_3 = 2000\left[1 + 1.06 + (1.06)^2 + (1.06)^3\right]$ as required

c
$$\begin{aligned} A_9 = 2000[1 &+ 1.06 + (1.06)^2 + (1.06)^3 + (1.06)^4 + (1.06)^5 + (1.06)^6 + (1.06)^7 \\ &+ (1.06)^8 + (1.06)^9] \end{aligned}$$
$\therefore \quad A_9 \approx 26\,361.59$

$\therefore$ the total bank balance after 10 years is $\$26\,361.59$

6 **a** $S_1 = \frac{1}{2}$, $S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$, $S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$, $S_4 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}$,
$S_5 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} = \frac{31}{32}$

b $S_n = \dfrac{2^n - 1}{2^n}$

c
$$\begin{aligned} S_n &= \frac{u_1(1-r^n)}{1-r}, \quad \text{where } u_1 = \tfrac{1}{2},\ r = \tfrac{1}{2} \\ &= \frac{\frac{1}{2}\left(1-(\frac{1}{2})^n\right)}{1-\frac{1}{2}} \end{aligned}$$
$\therefore \quad S_n = 1 - (\frac{1}{2})^n = 1 - \dfrac{1}{2^n} = \dfrac{2^n - 1}{2^n}$

d As $n \to \infty$, $\left(\frac{1}{2}\right)^n \to 0$,
and so $S_n \to 1$ (from below)

e The diagram represents one whole unit divided into smaller and smaller fractions.
As $n \to \infty$, the area which the fraction represents becomes smaller and smaller, and the total area approaches the area of a 1×1 unit square.

7 $u_2 = u_1 r = 6$ and so $u_1 = \dfrac{6}{r}$

$S_3 = u_1 + u_1 r + u_1 r^2 = -14$

$\therefore \quad \dfrac{6}{r} + 6 + 6r = -14$

$\therefore \quad \dfrac{6}{r} + 20 + 6r = 0$

$\therefore \quad 6 + 20r + 6r^2 = 0$

$\therefore \quad 3r^2 + 10r + 3 = 0$

$\therefore \quad (3r+1)(r+3) = 0$

$\therefore \quad r = -\frac{1}{3}$ or -3

When $r = -\frac{1}{3}$, $u_1 = -18$

When $r = -3$, $u_1 = -2$

Since $u_4 = u_1 r^3$,
$u_4 = -18 \times \left(-\frac{1}{3}\right)^3$ or $-2 \times (-3)^3$

$\therefore \quad u_4 = \frac{2}{3}$ or 54

8 $u_1 = 1$ Now $u_2 = u_1 + d$ {for arithmetic} and $u_2 = u_1 r$ {for geometric}

$\therefore\ u_1 + d = u_1 r$

$\therefore\ 1 + d = r$ .... (1)

Also, $u_{14} = u_1 + 13d$ {for arithmetic} and $u_3 = u_1 r^2$ {for geometric}

$\therefore\ u_1 + 13d = 3u_1 r^2$

$\therefore\ 1 + 13d = 3r^2$ .... (2)

From (1), $d = r - 1$ $\therefore$ in (2) $1 + 13(r-1) = 3r^2$

$\therefore\ 1 + 13r - 13 = 3r^2$

$\therefore\ 3r^2 - 13r + 12 = 0$

$\therefore\ (3r-4)(r-3) = 0$

$\therefore\ r = \frac{4}{3}$ or 3

When $r = \frac{4}{3}$, $d = \frac{1}{3}$: $u_{20} = u_1 + 19d = 1 + 19(\frac{1}{3}) = \frac{22}{3} = 7\frac{1}{3}$ {arithmetic}

$u_{20} = u_1 r^{19} = 1 \times \left(\frac{4}{3}\right)^{19} = \left(\frac{4}{3}\right)^{19}$ {geometric}

When $r = 3$, $d = 2$: $u_{20} = u_1 + 19d = 1 + 19(2) = 39$ {arithmetic}

$u_{20} = u_1 r^{19} = 1 \times 3^{19} = 3^{19}$ {geometric}

So, the 20th terms are $7\frac{1}{3}$ for arithmetic and $\left(\frac{4}{3}\right)^{19}$ for geometric or

39 for arithmetic and 3^{19} for geometric.

9 **a** The LHS is arithmetic with $u_1 = 5$, $d = 2$, 'n' $= n$.

Now $\frac{n}{2}(2u_1 + (n-1)d) = 1517$

$\therefore\ \frac{n}{2}(10 + 2(n-1)) = 1517$

$\therefore\ \frac{n}{2}(2n+8) = 1517$

$\therefore\ n(n+4) - 1517 = 0$

$\therefore\ n^2 + 4n - 1517 = 0$ where $n > 1$

and $n = 37$ {using technology}

b The LHS is geometric with $u_1 = 2$, $r = 3$, 'n' $= n$.

Now $u_1\left(\frac{r^n - 1}{r - 1}\right) = 177\,146$

$\therefore\ 2\left(\frac{3^n - 1}{3 - 1}\right) = 177\,146$

$\therefore\ 3^n - 1 = 177\,146$

$\therefore\ 3^n = 177\,147$

$\therefore\ n = 11$ {using technology}

10 $u_1, u_2, ..., u_n$ is a geometric sequence with common ratio r.

$\therefore\ u_n = u_1 r^{n-1}$

Now

$$(u_1+u_2)^2 + (u_2+u_3)^2 + (u_3+u_4)^2 + ... + (u_{n-1}+u_n)^2$$

$$= (u_1^2 + 2u_1u_2 + u_2^2) + (u_2^2 + 2u_2u_3 + u_3^2) + (u_3^2 + 2u_3u_4 + u_4^2) + ... + (u_{n-1}^2 + 2u_{n-1}u_n + u_n^2)$$

$$= (2u_1u_2 + 2u_2u_3 + 2u_3u_4 + ... + 2u_{n-1}u_n) + (u_1^2 + 2u_2^2 + 2u_3^2 + 2u_4^2 + ... + 2u_{n-1}^2 + u_n^2)$$

$$= \underbrace{2(u_1u_2 + u_2u_3 + u_3u_4 + ... + u_{n-1}u_n)}_{(1)} + \underbrace{2(u_1^2 + u_2^2 + u_3^2 + ... + u_{n-1}^2 + u_n^2)}_{(2)} - (u_1^2 + u_n^2) \quad .... (*)$$

Consider (1):

$$2(u_1u_2 + u_2u_3 + u_3u_4 + ... + u_{n-1}u_n)$$

$$= 2\left[u_1(u_1r) + (u_1r)(u_1r^2) + (u_1r^2(u_1r^3) + ... + (u_1r^{n-2})(u_1r^{n-1})\right]$$

{using the sequence definition $u_n = u_1r^{n-1}$ to substitute values for $u_2, u_3, u_4, ..., u_{n-1}, u_n$}

$$= 2u_1(u_1r + u_1r^3 + u_1r^5 + ... + u_1r^{2n-3})$$ {taking out common factor u_1}

Consider (2):

$$2(u_1^2 + u_2^2 + u_3^2 + ... + u_{n-1}^2 + u_n^2)$$

$$= 2\left[u_1^2 + (u_1r)^2 + (u_1r^2)^2 + ... + (u_1r^{n-2})^2 + (u_1r^{n-1})^2\right]$$

{using the sequence definition $u_n = u_1r^{n-1}$ to substitute values for $u_2, u_3, ..., u_{n-1}, u_n$}

$$= 2(u_1^2 + u_1^2r^2 + u_1^2r^4 + ... + u_1^2r^{2n-4} + u_1^2r^{2n-2})$$

$$= 2u_1(u_1 + u_1r^2 + u_1r^4 + ... + u_1r^{2n-4} + u_1r^{2n-2})$$ {taking out common factor u_1}

Now $(*)$:

$$2(u_1u_2+u_2u_3+u_3u_4+....+u_{n-1}u_n)+2(u_1^2+u_2^2+u_3^2+....+u_{n-1}^2+u_n^2)-(u_1^2+u_n^2)$$
$$=2u_1(u_1r+u_1r^3+u_1r^5+....+u_1r^{2n-3})+2u_1(u_1+u_1r^2+u_1r^4+....+u_1r^{2n-4}+u_1r^{2n-2})$$
$$-(u_1^2+u_n^2)$$
$$=2u_1\underbrace{(u_1+u_1r+u_1r^2+u_1r^3+u_1r^4+u_1r^5+....+u_1r^{2n-4}+u_1r^{2n-3}+u_1r^{2n-2})}_{(3)}-(u_1^2+u_n^2)$$

{Notice that (3) is the sum of the first $(2n-1)$ terms of a geometric sequence with first term u_1 and common ratio r.}

$$=2u_1\times\frac{u_1(r^{2n-1}-1)}{r-1}-(u_1^2+u_n^2)$$
$$=\frac{2u_1^2(r^{2n-1}-1)}{r-1}-(u_1^2+u_n^2)$$

11 **a** $A_3=A_2\times1.03-R$
$=(\$8000\times(1.03)^2-1.03R-R)\times1.03-R$
$=\$8000\times(1.03)^3-(1.03)^2R-(1.03)R-R$

b $A_8=\$8000\times(1.03)^8-(1.03)^7R-(1.03)^6R-(1.05)^5R-....-1.03R-R$

c $A_8=0$

$\therefore\ R\left(1+1.03+(1.03)^2+(1.03)^3+....+(1.03)^7\right)=\$8000\times(1.03)^8$

$$\therefore\ R\left(1\left[\frac{(1.03)^8-1}{1.03-1}\right]\right)=\$8000\times(1.03)^8$$

$$\therefore\ R=\frac{\$8000\times(1.03)^8\times0.03}{(1.03)^8-1}\approx\$1139.65$$

d Notice in **c** that $\$8000=P$ and $(1.03)^8=\left(1+\frac{3}{100}\right)^8=\left(1+\frac{r}{100}\right)^m$

$$0.03=\tfrac{3}{100}=\frac{r}{100}$$

and $(1.03)^8-1=\left(1+\frac{r}{100}\right)^m-1$

So, in the general case $R=\dfrac{P\times\left(1+\frac{r}{100}\right)^m\times\frac{r}{100}}{\left(1+\frac{r}{100}\right)^m-1}$

EXERCISE 7G.2

1 **a** **i** $u_1=\frac{3}{10}$

ii $r=\dfrac{\left(\frac{3}{100}\right)}{\left(\frac{3}{10}\right)}=\frac{1}{10}$
$=0.1$

b We need to show that $0.\overline{3}=\frac{1}{3}$.

Now $0.\overline{3}=\frac{3}{10}+\frac{3}{100}+\frac{3}{1000}+....$

So, let $S_n=\frac{3}{10}+\frac{3}{100}+\frac{3}{1000}+....$

Since $n\to\infty$, then $S=\dfrac{u_1}{1-r}=\dfrac{\frac{3}{10}}{1-\left(\frac{1}{10}\right)}=\frac{1}{3}$

So, $0.\overline{3}=\frac{1}{3}$ as required.

2 **a** $0.\overline{4} = 0.444\,444\,....$
$= \frac{4}{10} + \frac{4}{100} + \frac{4}{1000} + ...$
which is a geometric series with
$u_1 = 0.4, \quad r = 0.1$
$\therefore \quad S = \dfrac{u_1}{1-r} = \dfrac{0.4}{1-0.1} = \dfrac{0.4}{0.9}$
$= \frac{4}{9}$
So, $\quad 0.\overline{4} = \frac{4}{9}$

b $0.\overline{16} = 0.161\,616\,....$
$= \dfrac{16}{10^2} + \dfrac{16}{10^4} + \dfrac{16}{10^6} + ...$
which is a geometric series with
$u_1 = 0.16, \ r = 0.01$
$\therefore \quad S = \dfrac{u_1}{1-r} = \dfrac{0.16}{0.99} = \dfrac{16}{99}$
So, $\quad 0.\overline{16} = \frac{16}{99}$

c $0.\overline{312} = 0.312\,312\,312\,....$
$= \dfrac{312}{10^3} + \dfrac{312}{10^6} + \dfrac{312}{10^9} + ...$
which is a geometric series with $\quad u_1 = 0.312, \quad r = 0.001$
$\therefore \quad S = \dfrac{u_1}{1-r} = \dfrac{0.312}{0.999} = \dfrac{312}{999} = \dfrac{104}{333}$ So, $\quad 0.\overline{312} = \frac{104}{333}$

3 Checking **Exercise 7G.1 6b**: $\quad S = \dfrac{u_1}{1-r} = \dfrac{\frac{1}{2}}{1-\frac{1}{2}} = 1 \quad \checkmark$

4 **a** $18 + 12 + 8 + ...$ is an infinite geometric series with $u_1 = 18, \ r = \frac{2}{3}$.
$\therefore \quad S = \dfrac{u_1}{1-r} = \dfrac{18}{\frac{1}{3}} = 54$

b $18.9 - 6.3 + 2.1 - ....$ is an infinite geometric series with $u_1 = 18.9, \ r = -\frac{1}{3}$.
$\therefore \quad S = \dfrac{u_1}{1-r} = \dfrac{18.9}{\frac{4}{3}} = 14.175$

5 **a** $\displaystyle\sum_{k=1}^{\infty} \frac{3}{4^k} = \frac{3}{4} + \frac{3}{16} + \frac{3}{64} + ...$
is an infinite geometric series with
$u_1 = \frac{3}{4}, \ r = \frac{1}{4}$.
$\therefore \quad S = \dfrac{u_1}{1-r} = \dfrac{\frac{3}{4}}{\frac{3}{4}} = 1$

b $\displaystyle\sum_{k=0}^{\infty} 6(-\tfrac{2}{5})^k = 6 - 6 \times (\tfrac{2}{5}) + 6 \times (\tfrac{2}{5})^2 - ....$
is an infinite geometric series with
$u_1 = 6, \ r = -\frac{2}{5}$.
$\therefore \quad S = \dfrac{u_1}{1-r} = \dfrac{6}{\frac{7}{5}} = \frac{30}{7} = 4\frac{2}{7}$

6 Let the terms of the geometric series be $\quad u_1, \ u_1 r, \ u_1 r^2, \ ....$
Then $\quad u_1 + u_1 r + u_1 r^2 = 19 \quad$ and $\quad \dfrac{u_1}{1-r} = 27$
$\therefore \quad u_1(1 + r + r^2) = 19 \qquad \therefore \quad u_1 = 27(1-r) \quad .... (2)$
$\therefore \quad u_1 = \dfrac{19}{1 + r + r^2} \quad .... (1)$

Equating (1) and (2), $\quad \dfrac{19}{1+r+r^2} = 27(1-r)$
$\therefore \quad \frac{19}{27} = (1-r)(1+r+r^2)$
$\therefore \quad \frac{19}{27} = 1 + r + r^2 - r - r^2 - r^3$
$\therefore \quad \frac{19}{27} = 1 - r^3$
$\therefore \quad r^3 = \frac{8}{27}$
$\therefore \quad r = \frac{2}{3}$
Substituting $r = \frac{2}{3}$ into (2) gives $\quad u_1 = 27(1 - \frac{2}{3}) = 9$
$\therefore \quad$ the first term is 9 and the common ratio is $\frac{2}{3}$.

7 Let the terms of the geometric series be $u_1,\ u_1r,\ u_1r^2,\ ....$

Then $u_1r = \frac{8}{5}$ and $\dfrac{u_1}{1-r} = 10$

$\therefore\ u_1 = \dfrac{8}{5r}$ .... (1) $\qquad \therefore\ u_1 = 10 - 10r$ .... (2)

Equating (1) and (2), $\dfrac{8}{5r} = 10 - 10r$

$\therefore\ 8 = 50r - 50r^2$

$\therefore\ 50r^2 - 50r + 8 = 0$

$\therefore\ 2(25r^2 - 25r + 4) = 0$

$\therefore\ 2(5r-1)(5r-4) = 0$

$\therefore\ r = \frac{1}{5}$ or $\frac{4}{5}$

Using (2), if $r = \frac{1}{5}$, $u_1 = 10 - 10(\frac{1}{5}) = 8$

if $r = \frac{4}{5}$, $u_1 = 10 - 10(\frac{4}{5}) = 2$

$\therefore$ either $u_1 = 8,\ r = \frac{1}{5}$ or $u_1 = 2,\ r = \frac{4}{5}$

8 **a** Total time of motion $= 1 + (90\% \times 1) + (90\% \times 1) + (90\% \times 90\% \times 1)$

$+ (90\% \times 90\% \times 1) + (90\% \times 90\% \times 90\% \times 1) + ...$

$= 1 + 0.9 + 0.9 + (0.9)^2 + (0.9)^2 + (0.9)^3 + ...$

$= 1 + 2(0.9) + 2(0.9)^2 + 2(0.9)^3 + ...$ as required

b The total time of motion can be written as $[2 + 2(0.9) + 2(0.9)^2 + 2(0.9)^3 + ...] - 1$

So, $S_n = \dfrac{u_1(1-r^n)}{1-r} - 1$, where $u_1 = 2,\ r = 0.9$

$\therefore\ S_n = \dfrac{2(1-0.9^n)}{1-0.9} - 1$

$\therefore\ S_n = \dfrac{2(1-0.9^n)}{0.1} - 1$

$\therefore\ S_n = 20(1-0.9^n) - 1$

$\therefore\ S_n = 20 - 20 \times 0.9^n - 1$

$\therefore\ S_n = 19 - 20 \times 0.9^n$

c For the ball to come to rest, n must approach infinity.

As $n \to \infty$, $0.9^n \to 0$ and so $20 \times 0.9^n \to 0$ also.

$\therefore\ S_n \to 19^-$.

So, it takes 19 seconds for the ball to come to rest.

9 **a** $18 - 9 + 4.5 - ....$ is an infinite geometric series with $u_1 = 18,\ r = -\frac{1}{2}$.

Since $|r| < 1$, the series converges.

$\therefore\ S = \dfrac{u_1}{1-r}$

$= \dfrac{18}{\frac{3}{2}}$

$= 12$

$\therefore$ the series is convergent, and its sum is 12.

b $1.2 + 1.8 + 2.7 + ...$ is an infinite geometric series with $u_1 = 1.2,\ r = 1.5$.

Since $|r| > 1$, the series is divergent.

$S_n > 100$ when $\dfrac{u_1(r^n - 1)}{r-1} > 100$

$\therefore\ \dfrac{1.2(1.5^n - 1)}{\frac{1}{2}} > 100$

$\therefore\ 2.4(1.5^n - 1) > 100$

$\therefore\ 1.5^n - 1 > \frac{125}{3}$

$\therefore\ 1.5^n > \frac{128}{3}$

$\therefore\ n > 9.26$

{using technology}

$\therefore\ n = 10$ is the smallest value of n such that $S_n > 100$.

10 Total distance travelled

$= h + 2\left(\frac{3}{4}\right)h + 2\left(\frac{3}{4}\right)\left(\frac{3}{4}\right)h + ...$

$= h + 2\left(\frac{3}{4}\right)h\left[1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 + ...\right]$

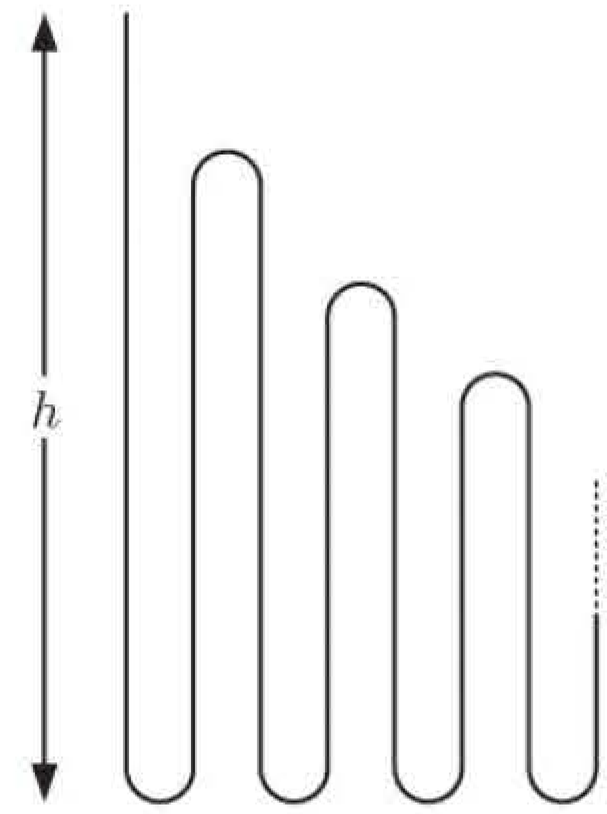
$= h + \frac{3}{2}h\left(\frac{1}{1-\frac{3}{4}}\right) \quad \left\{\text{as } |r| = \left|\frac{3}{4}\right| < 1 \text{ and } S = \frac{u_1}{1-r}\right\}$

$= h + \frac{3}{2}h(4)$

$= 7h$

But $7h = 490$, so $h = 70$.

The ball was dropped from a height of 70 cm.



11

$$\sum_{k=1}^{\infty}\left(\frac{3x}{2}\right)^{k-1} = \left(\frac{3x}{2}\right)^0 + \left(\frac{3x}{2}\right)^1 + \left(\frac{3x}{2}\right)^2 + \left(\frac{3x}{2}\right)^3 + ...$$

$$= 1 + \frac{3x}{2} + \left(\frac{3x}{2}\right)^2 + \left(\frac{3x}{2}\right)^3 + ...$$

$$= \frac{u_1}{1-r} \quad \{\text{as it converges to 4 and is geometric}\}$$

$$= \frac{1}{1-\frac{3x}{2}} = \frac{2}{2-3x}$$

$\therefore \frac{2}{2-3x} = 4$ and so $2 - 3x = \frac{1}{2}$

$\therefore 3x = 1\frac{1}{2}$

$\therefore x = \frac{1}{2}$

REVIEW SET 7A

1 **a** arithmetic, $d = -8$ **b** geometric, $r = 1$ or arithmetic, $d = 0$

c geometric, $r = -\frac{1}{2}$ **d** neither

e 4, 8, 12, 16, ... arithmetic, $d = 4$

2 Since the terms are consecutive,

$(k-2) - 3k = k + 7 - (k-2)$ {equating common differences}

$\therefore k - 2 - 3k = k + 7 - k + 2$

$\therefore -2 - 2k = 9$

$\therefore 2k = -11$

$\therefore k = -\frac{11}{2}$

3 28, 23, 18, 13, ...

$23 - 28 = -5$

$18 - 23 = -5$

$13 - 18 = -5$

Assuming that the pattern continues, consecutive terms differ by -5.

$\therefore$ the sequence is arithmetic with $u_1 = 28$, $d = -5$.

$u_n = u_1 + (n-1)d$

$= 28 + (n-1)(-5)$

$= 28 - 5n + 5$

$= 33 - 5n$

$S_n = \frac{n}{2}(2u_1 + (n-1)d)$

$= \frac{n}{2}(2 \times 28 + (n-1)(-5))$

$= \frac{n}{2}(56 - 5n + 5)$

$= \frac{n}{2}(61 - 5n)$

4 The terms are geometric, so $\dfrac{k}{4} = \dfrac{k^2-1}{k}$

$$\therefore\ k^2 = 4(k^2-1)$$
$$\therefore\ 3k^2 = 4$$
$$\therefore\ k^2 = \tfrac{4}{3}$$
$$\therefore\ k = \pm\tfrac{2}{\sqrt{3}} = \pm\tfrac{2\sqrt{3}}{3}$$

5 $u_6 = \frac{16}{3}$ $\quad\therefore\ u_1 \times r^5 = \frac{16}{3}$ .... (1)

$u_{10} = \frac{256}{3}$ $\quad\therefore\ u_1 \times r^9 = \frac{256}{3}$ .... (2)

So, $\dfrac{u_1 r^9}{u_1 r^5} = \dfrac{\left(\frac{256}{3}\right)}{\left(\frac{16}{3}\right)}$ $\quad\{(2) \div (1)\}$

$$\therefore\ r^4 = 16$$
$$\therefore\ r = \pm 2$$

Substituting $r = 2$ into (1) gives

$$u_1 \times 2^5 = \tfrac{16}{3}$$
$$\therefore\ u_1 \times 32 = \tfrac{16}{3}$$
$$\therefore\ u_1 = \tfrac{1}{6}$$

Substituting $r = -2$ into (1) gives

$$u_1 \times (-2)^5 = \tfrac{16}{3}$$
$$\therefore\ u_1 \times (-32) = \tfrac{16}{3}$$
$$\therefore\ u_1 = -\tfrac{1}{6}$$

Now $u_n = u_1 r^{n-1}$ $\quad\therefore\ u_n = \frac{1}{6} \times 2^{n-1}$ or $-\frac{1}{6} \times (-2)^{n-1}$

6 Let the numbers be $23,\ 23+d,\ 23+2d,\ 23+3d,\ 23+4d,\ 23+5d,\ 23+6d,\ 9$

Then $23 + 7d = 9$

$$\therefore\ 7d = -14$$
$$\therefore\ d = -2$$

So, the numbers are 23, 21, 19, 17, 15, 13, 11, 9.

7 **a** The sequence 86, 83, 80, 77, ... is arithmetic with $u_1 = 86,\ d = -3$.

$$u_n = u_1 + (n-1)d$$
$$\therefore\ u_n = 86 + (n-1)(-3) = 86 - 3n + 3$$
$$\therefore\ u_n = 89 - 3n$$

b $\frac{3}{4}, 1, \frac{7}{6}, \frac{9}{7}$, ... can also be written as $\frac{3}{4}, \frac{5}{5}, \frac{7}{6}, \frac{9}{7}$, ...

So, the numerator starts at 3 and increases by 2 each time, whilst the denominator starts at 4 and increases by 1 each time.

The nth term is $\dfrac{2n+1}{n+3}$, and so $u_n = \dfrac{2n+1}{n+3}$

c The sequence 100, 90, 81, 72.9, ... is geometric with $u_1 = 100,\ r = \frac{90}{100} = 0.9$

$$u_n = u_1 r^{n-1}$$
$$\therefore\ u_n = 100(0.9)^{n-1}$$

8 **a** $\displaystyle\sum_{k=1}^{7} k^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2$

$$= 1 + 4 + 9 + 16 + 25 + 36 + 49$$
$$= 140$$

b $\displaystyle\sum_{k=1}^{4} \frac{k+3}{k+2} = \tfrac{4}{3} + \tfrac{5}{4} + \tfrac{6}{5} + \tfrac{7}{6}$

$$= \tfrac{99}{20}$$

9 **a** $18 - 12 + 8 - ....$

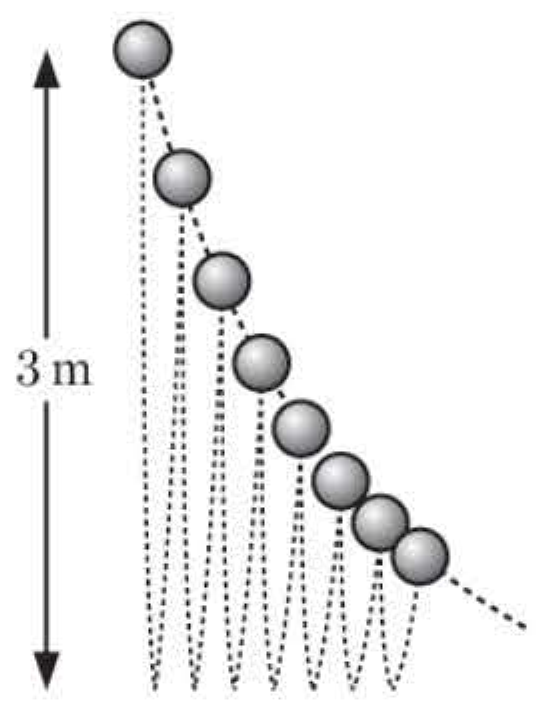
The series is geometric with $u_1 = 18,\ r = -\frac{2}{3}$

$$\therefore\ S = \frac{u_1}{1-r}$$
$$= \frac{18}{\frac{5}{3}}$$
$$= \tfrac{54}{5}$$
$$= 10\tfrac{4}{5}$$

b $8+4\sqrt{2}+4+....$

The series is geometric with $u_1=8,\ r=\frac{1}{\sqrt{2}}$

$$\begin{aligned}\therefore\ S&=\frac{u_1}{1-r}\\&=\frac{8}{(1-\frac{1}{\sqrt{2}})}\times\frac{(1+\frac{1}{\sqrt{2}})}{(1+\frac{1}{\sqrt{2}})}\\&=\frac{8+\frac{8}{\sqrt{2}}}{1-\frac{1}{2}}\\&=\frac{8+4\sqrt{2}}{\frac{1}{2}}\\&=16+8\sqrt{2}\end{aligned}$$

10 

Total distance travelled

$$\begin{aligned}&=3+3\times0.8\times2+3\times(0.8)^2\times2+3\times(0.8)^3\times2+....\\&=3+3\times0.8\times2\left[1+0.8+(0.8)^2+(0.8)^3+....\right]\\&=3+4.8\times\frac{1}{1-0.8}\quad\left\{\text{as } r=0.8,\quad |r|<1 \text{ so converges to } \frac{u_1}{1-r}\right\}\\&=3+\tfrac{4.8}{0.2}\\&=3+24=27\text{ metres}\end{aligned}$$

11 **a** $S_n=\dfrac{3n^2+5n}{2}$

$$\begin{aligned}\therefore\ u_n&=S_n-S_{n-1}\\&=\frac{3n^2+5n}{2}-\frac{3(n-1)^2+5(n-1)}{2}\\&=\frac{3n^2+5n-3(n^2-2n+1)-5(n-1)}{2}\\&=\frac{3n^2+5n-3n^2+6n-3-5n+5}{2}\\&=\frac{6n+2}{2}\\\therefore\ u_n&=3n+1\end{aligned}$$

b Using part **a**,

$$\begin{aligned}u_n-u_{n-1}&=[3n+1]-[3(n-1)+1]\\&=3n+1-3n+3-1\\&=3\end{aligned}$$

The difference between consecutive terms is constant for all n, so the sequence is arithmetic.

12 If a, b, and c are consecutive terms of a geometric sequence with constant ratio r, then $b=ar$ and $c=ar^2$ .... (1)

If a, b, and c are consecutive terms of an arithmetic sequence then

$$\begin{aligned}b-a&=c-b\\\therefore\ ar-a&=ar^2-ar\quad\{\text{using (1)}\}\\\therefore\ ar^2-2ar+a&=0\\\therefore\ a(r^2-2r+1)&=0\\\therefore\ a(r-1)^2&=0\\\therefore\ a=0 &\text{ or } r=1\end{aligned}$$

If $a=0$ then $b=0$ and $c=0$ {using (1)}

If $r=1$ then $b=a(1)=a$ and $c=a(1)^2=a$

In either case, $a=b=c$.

13 If x, y, and z are consecutive terms of a geometric sequence, then

$$\frac{y}{x} = \frac{z}{y} \quad \{\text{equating constant ratios}\}$$

$$\therefore \quad y^2 = xz \quad .... \ (1)$$

Now $x + y + z = \frac{7}{3}$ .... (2)

$$\therefore \quad (x+y+z)^2 = \tfrac{49}{9}$$

$$\therefore \quad x^2 + y^2 + z^2 + 2xy + 2xz + 2yz = \tfrac{49}{9} \qquad \{\text{expanding LHS}\}$$

$$\therefore \quad \tfrac{91}{9} + 2(xy + xz + yz) = \tfrac{49}{9} \qquad \{x^2 + y^2 + z^2 = \tfrac{91}{9}\}$$

$$\therefore \quad 2(xy + xz + yz) = -\tfrac{42}{9}$$

$$\therefore \quad xy + xz + yz = -\tfrac{7}{3}$$

$$\therefore \quad xy + y^2 + yz = -\tfrac{7}{3} \qquad \{\text{using (1)}\}$$

$$\therefore \quad y(x+y+z) = -(x+y+z) \qquad \{x + y + z = \tfrac{7}{3}\}$$

$$\therefore \quad y = -1$$

Substituting $y = -1$ into (1) and (2) gives $xz = 1$ and $x + z = \frac{10}{3}$ .... (4)

$$\therefore \quad z = \frac{1}{x} \quad .... \ (3)$$

Substituting (3) into (4) gives $x + \dfrac{1}{x} = \frac{10}{3}$

$$\therefore \quad 3x^2 - 10x + 3 = 0$$

$$\therefore \quad (3x-1)(x-3) = 0$$

$$\therefore \quad x = \tfrac{1}{3} \text{ or } 3$$

Using (3), if $x = \frac{1}{3}$, $z = 3$ and if $x = 3$, $z = \frac{1}{3}$

$$\therefore \quad x = \tfrac{1}{3},\ y = -1,\ z = 3 \quad \text{or} \quad x = 3,\ y = -1,\ z = \tfrac{1}{3}$$

14 The first two terms of a geometric series are $2x$ and $x - 2$, so $u_1 = 2x$ and $r = \dfrac{x-2}{2x}$

Now $$S = \frac{u_1}{1-r} = \frac{2x}{1 - (\frac{x-2}{2x})} = \frac{4x^2}{2x - (x-2)} = \frac{4x^2}{x+2}$$

The sum of the series is $\frac{18}{7}$, so $\dfrac{4x^2}{x+2} = \frac{18}{7}$

$$\therefore \quad 28x^2 = 18x + 36$$

$$\therefore \quad 28x^2 - 18x - 36 = 0$$

$$\therefore \quad 2(14x^2 - 9x - 18) = 0$$

$$\therefore \quad 2(7x+6)(2x-3) = 0$$

$$\therefore \quad x = -\tfrac{6}{7} \text{ or } \tfrac{3}{2}$$

When $x = -\frac{6}{7}$, $$r = \frac{-\frac{6}{7} - 2}{2(-\frac{6}{7})} = \frac{-\frac{20}{7}}{-\frac{12}{7}} = \tfrac{5}{3}$$

When $x = \frac{3}{2}$, $$r = \frac{\frac{3}{2} - 2}{2(\frac{3}{2})} = \frac{-\frac{1}{2}}{3} = -\tfrac{1}{6}$$

$|r| < 1$ only when $x = \frac{3}{2}$, so $x = \frac{3}{2}$ is the only solution.

15 a, b, and c are arithmetic, so $a - b = b - c = d$ where d is a constant.

a $$(c+a) - (b+c) = c + a - b - c = a - b = d$$

$$(a+b) - (c+a) = a + b - c - a = b - c = d$$

$\therefore$ the differences between the terms are equal.

$\therefore$ $b + c$, $c + a$, and $a + b$ are also consecutive terms of an arithmetic sequence.

b $$\frac{1}{\sqrt{b}+\sqrt{c}} = \frac{1}{(\sqrt{b}+\sqrt{c})}\frac{(\sqrt{b}-\sqrt{c})}{(\sqrt{b}-\sqrt{c})} = \frac{\sqrt{b}-\sqrt{c}}{b-c} = \frac{\sqrt{b}-\sqrt{c}}{d} \quad .... (1)$$

$$\frac{1}{\sqrt{c}+\sqrt{a}} = \frac{1}{(\sqrt{c}+\sqrt{a})}\frac{(\sqrt{c}-\sqrt{a})}{(\sqrt{c}-\sqrt{a})} = \frac{\sqrt{c}-\sqrt{a}}{c-a} = \frac{\sqrt{c}-\sqrt{a}}{-2d} \quad .... (2)$$

$$\frac{1}{\sqrt{a}+\sqrt{b}} = \frac{1}{(\sqrt{a}+\sqrt{b})}\frac{(\sqrt{a}-\sqrt{b})}{(\sqrt{a}-\sqrt{b})} = \frac{\sqrt{a}-\sqrt{b}}{a-b} = \frac{\sqrt{a}-\sqrt{b}}{d} \quad .... (3)$$

Using (2) and (1):

$$\frac{1}{\sqrt{c}+\sqrt{a}} - \frac{1}{\sqrt{b}+\sqrt{c}} = \frac{\sqrt{c}-\sqrt{a}}{-2d} - \frac{\sqrt{b}-\sqrt{c}}{d} = \frac{\sqrt{a}-\sqrt{c}-2\sqrt{b}+2\sqrt{c}}{2d}$$
$$= \frac{\sqrt{a}-2\sqrt{b}+\sqrt{c}}{2d}$$

Using (3) and (2):

$$\frac{1}{\sqrt{a}+\sqrt{b}} - \frac{1}{\sqrt{c}+\sqrt{a}} = \frac{\sqrt{a}-\sqrt{b}}{d} - \frac{\sqrt{c}-\sqrt{a}}{-2d} = \frac{2\sqrt{a}-2\sqrt{b}+\sqrt{c}-\sqrt{a}}{2d}$$
$$= \frac{\sqrt{a}-2\sqrt{b}+\sqrt{c}}{2d}$$

$\therefore$ the differences between the terms are equal.

$\therefore$ $\dfrac{1}{\sqrt{b}+\sqrt{c}}$, $\dfrac{1}{\sqrt{c}+\sqrt{a}}$ and $\dfrac{1}{\sqrt{a}+\sqrt{b}}$ are also consecutive terms of an arithmetic sequence.

REVIEW SET 7B

1 **a** $u_n = 3^{n-2}$ $\quad\therefore\ u_1 = 3^{-1} = \frac{1}{3},\ \ u_2 = 3^0 = 1,\ \ u_3 = 3^1 = 3,\ \ u_4 = 3^2 = 9$

b $u_n = \dfrac{3n+2}{n+3}$ $\quad\therefore\ u_1 = \frac{5}{4},\ \ u_2 = \frac{8}{5},\ \ u_3 = \frac{11}{6},\ \ u_4 = \frac{14}{7} = 2$

c $u_n = 2^n - (-3)^n$

$\therefore\ u_1 = 2-(-3) = 5,\ \ u_2 = 4-9 = -5,\ \ u_3 = 8-(-27) = 35,\ \ u_4 = 16-81 = -65$

2 $u_n = 6\left(\frac{1}{2}\right)^{n-1}$

a $\dfrac{u_{n+1}}{u_n} = \dfrac{6\left(\frac{1}{2}\right)^{n+1-1}}{6\left(\frac{1}{2}\right)^{n-1}} = \frac{1}{2}$ for all n

$\therefore$ $\{u_n\}$ is a geometric sequence.

b $u_1 = 6,$
$r = \frac{1}{2}$

c $u_{16} = 6\left(\frac{1}{2}\right)^{15}$
$\approx 0.000\,183$

3 **a** Given $24, 23\frac{1}{4}, 22\frac{1}{2}, ..., -36$ we have $u_1 = 24,\ \ u_n = -36,$ and we need to find n.

The sequence is arithmetic with $d = -\frac{3}{4}$.

Now $u_n = u_1 + (n-1)d$

$\therefore\ -36 = 24 + (n-1)(-\frac{3}{4})$

$\therefore\ -60 = -\frac{3}{4}n + \frac{3}{4}$

$\therefore\ \frac{3}{4}n = \frac{243}{4}$

$\therefore\ n = 81$ $\quad$ So, -36 is the 81st term in the sequence.

b $u_{35} = 24 + (35-1)(-\frac{3}{4})$
$= 24 - \frac{102}{4}$
$= -\frac{3}{2}$
$= -1\frac{1}{2}$

c $S_n = \dfrac{n}{2}(2u_1 + (n-1)d)$

$\therefore\ S_{40} = \frac{40}{2}(2\times 24 + (40-1)(-\frac{3}{4}))$
$= 20(48 - \frac{117}{4})$
$= 375$

4 **a** $3+9+15+21+....$
The series is arithmetic with
$u_1=3, \quad d=6, \quad n=23$

Now $S_n=\frac{n}{2}(2u_1+(n-1)d)$

$\therefore \quad S_{23}=\frac{23}{2}(2\times 3+6(23-1))$

$\therefore \quad S_{23}=\frac{23}{2}(6+132)$
$=1587$

b $24+12+6+3+....$
The series is geometric with
$u_1=24, \quad r=\frac{1}{2}, \quad n=12$

$$S_n=\frac{u_1(1-r^n)}{1-r}$$

$$\therefore \quad S_{12}=\frac{24\left(1-(\frac{1}{2})^{12}\right)}{1-\frac{1}{2}}$$
$$=48\left(1-(\tfrac{1}{2})^{12}\right)$$
$$=47\tfrac{253}{256}\approx 48.0$$

5 5, 10, 20, 40, ... The sequence is geometric with $u_1=5, \quad r=2$
$u_n=u_1r^{n-1}=5\times 2^{n-1}$
Let $u_n=10\,000=5\times 2^{n-1}$
$\therefore \quad 2000=2^{n-1}$
$\therefore \quad n\approx 11.97$ {using technology}
We try the two values on either side of $n\approx 11.97$, which are $n=11$ and $n=12$:
$u_{11}=5\times 2^{10}$ and $u_{12}=5\times 2^{11}$
$=5120$ $\quad$ $=10\,240$
So, the first term to exceed 10 000 is $u_{12}=10\,240$.

6 **a** $u_6=u_1\times r^5$ is the amount after 5 years, where $u_1=6000$, $r=1.07$
$=6000\times(1.07)^5$
≈ 8415.31 So, the value of the investment will be €8415.31.

b If interest is compounded quarterly, then $r=1+\frac{0.07}{4}=1.0175$
and $n=5\times 4=20$

$\therefore \quad u_{21}=u_1\times r^{20}$
$=6000\times(1.0175)^{20}$
≈ 8488.67 So, the value of the investment will be €8488.67.

c If interest is compounded monthly, then $r=1+\frac{0.07}{12}=1.005\,8\overline{3}$
and $n=5\times 12=60$

$\therefore \quad u_{61}=u_1\times r^{60}$
$=6000\times(1.005\,8\overline{3})^{60}$
≈ 8505.75 So, the value of the investment will be €8505.75

7 **a** $u_n=5n-8$
$\therefore \quad u_{10}=5\times 10-8=42$

b $u_{n+1}-u_n=(5(n+1)-8)-(5n-8)$
$=5n+5-8-5n+8$
$=5$

c The difference between consecutive terms u_n and u_{n+1} is constant for all n, so the sequence is arithmetic.

d $S_n=\frac{n}{2}(2u_1+(n-1)d)$ where $d=5$ and $u_1=5\times 1-8=-3$
Now, $u_{15}+u_{16}+u_{17}+....+u_{30}=S_{30}-S_{14}$
$=\frac{30}{2}(2(-3)+(30-1)\times 5)-\frac{14}{2}(2(-3)+(14-1)\times 5)$
$=1672$

8 $u_6 = 24$ $\therefore\ u_1 \times r^5 = 24$ .... (1)
$u_{11} = 768$ $\therefore\ u_1 \times r^{10} = 768$ .... (2)

So $\dfrac{u_1 r^{10}}{u_1 r^5} = \dfrac{768}{24}$ $\{(2) \div (1)\}$

$\therefore\ r^5 = 32$

$\therefore\ r = 2$

Substituting $r = 2$ into (1) gives $u_1 \times 2^5 = 24$

$\therefore\ u_1 = \frac{24}{32} = \frac{3}{4}$

$u_n = u_1 r^{n-1} = \left(\frac{3}{4}\right) 2^{n-1}$

a $u_{17} = \left(\frac{3}{4}\right) 2^{17-1}$
$= 49\,152$

b $S_n = \dfrac{u_1(r^n - 1)}{r - 1} = \dfrac{\frac{3}{4}(2^n - 1)}{2 - 1}$
$= \frac{3}{4}(2^n - 1)$
$\therefore\ S_{15} = \frac{3}{4}(2^{15} - 1) = 24\,575.25$

9 $24, 8, \frac{8}{3}, \frac{8}{9}, ...$ is geometric with $u_1 = 24,\ r = \frac{1}{3}$

$u_n = u_1 r^{n-1} = 24\left(\frac{1}{3}\right)^{n-1}$

Given $u_n = 0.001$, we need to find n, so $u_n = 24\left(\frac{1}{3}\right)^{n-1} = 0.001$

$\therefore\ \left(\frac{1}{3}\right)^{n-1} = \dfrac{0.001}{24}$

$\therefore\ n \approx 10.18$ {using technology}

We try the two values on either side of $n \approx 10.18$, which are $n = 10$ and $n = 11$:

$u_{10} = 24\left(\frac{1}{3}\right)^9 = \dfrac{8}{6561} \approx 0.001\,22$ and $u_{11} = 24\left(\frac{1}{3}\right)^{10} = \dfrac{8}{19\,683} \approx 0.000\,406$

$\therefore\ u_{11} \approx 0.000\,406$ is the first term of the sequence which is less than 0.001.

10 **a** $128, 64, 32, 16, ..., \frac{1}{512}$
is geometric with:
$u_1 = 128,\ r = \frac{1}{2},\ u_n = \frac{1}{512}$

$u_n = u_1 r^{n-1}$
$= 128\left(\frac{1}{2}\right)^{n-1}$
$= 2^7 \times 2^{1-n}$

$\therefore\ \frac{1}{512} = 2^7 \times 2^{1-n}$
$\therefore\ 2^{-9} = 2^{8-n}$
$\therefore\ -9 = 8 - n$
$\therefore\ n = 17$

So, there are 17 terms in the sequence.

b $S_n = \dfrac{u_1(1 - r^n)}{1 - r}$

$\therefore\ S_{17} = \dfrac{128\left(1 - \left(\frac{1}{2}\right)^{17}\right)}{1 - \frac{1}{2}}$
$= 255\frac{511}{512}$
≈ 255.998
≈ 256

11 **a** $1.21 - 1.1 + 1 - ....$ is an infinite geometric series with $u_1 = 1.21,\ r = -\frac{10}{11}$.

$\therefore\ S = \dfrac{u_1}{1 - r} = \dfrac{1.21}{\frac{21}{11}} = \frac{1331}{2100}$

$\therefore\ S \approx 0.634$

b $\frac{14}{3} + \frac{4}{3} + \frac{8}{21} + ...$ is an infinite geometric series with $u_1 = \frac{14}{3},\ r = \frac{2}{7}$.

$\therefore\ S = \dfrac{u_1}{1 - r} = \dfrac{\frac{14}{3}}{\frac{5}{7}}$

$\therefore\ S = \frac{98}{15} = 6\frac{8}{15}$

12 $u_{n+1} = u_1 \times r^n$ where $u_{n+1} = 20\,000$, $r = 1 + \dfrac{0.09}{12} = 1.0075$, $n = 4 \times 12 = 48$

$\therefore\ 20\,000 = u_1 \times (1.0075)^{48}$

$\therefore\ u_1 = \dfrac{20\,000}{(1.0075)^{48}}$

$\therefore\ u_1 \approx 13\,972.28$ So, $13 972.28 should be invested.

13 **a** $u_{n+1} = u_1 \times r^n$ where $u_1 = 3000$, $r = 1.05$, $n = 3$

$\therefore\ u_{n+1} = 3000 \times (1.05)^3$

$= 3472.875$ There were approximately 3470 iguanas.

b $u_{n+1} = u_1 \times r^n$ where $u_1 = 3000$, $u_{n+1} = 5000$, $r = 1.05$

$\therefore\ 10\,000 = 3000 \times (1.05)^n$

$\therefore\ n \approx 24.68$ {using technology}

After 24.68 years the population will exceed 10 000. This is during the year 2029.

14 $u_1 = x + 3$, $u_2 = u_1 r = x - 2$

$\therefore\ r = \dfrac{u_2}{u_1} = \dfrac{x-2}{x+3}$

The series will converge if $|r| < 1$

$\therefore\ \left|\dfrac{x-2}{x+3}\right| < 1$

$\therefore\ |x-2| < |x+3|$ $\left\{\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|}\right\}$

$\therefore\ (x-2)^2 < (x+3)^2$

$\therefore\ x^2 - 4x + 4 < x^2 + 6x + 9$

$\therefore\ -10x < 5$

$\therefore\ x > -\frac{1}{2}$

REVIEW SET 7C

1 $u_n = 68 - 5n$

a $u_{n+1} - u_n = [68 - 5(n+1)] - [68 - 5n]$

$= 68 - 5n - 5 - 68 + 5n$

$= -5$ for all n

$\therefore$ the sequence is arithmetic with common difference $d = -5$.

b $u_1 = 68 - 5(1) = 63$, $d = -5$

c $u_{37} = 68 - 5(37) = -117$

d Let $u_n = -200$, and we need to find n.

$u_n = 68 - 5n = -200$

$\therefore\ 5n = 268$

$\therefore\ n = 53\frac{3}{5}$

We try the two values on either side of $n = 53\frac{3}{5}$, which are $n = 53$ and $n = 54$:

$u_{53} = 68 - 5(53) = -197$ and $u_{54} = 68 - 5(54) = -202$

So, the first term of the sequence less than -200 is $u_{54} = -202$.

2 **a** 3, 12, 48, 192, ...

$\frac{12}{3} = 4$ $\frac{48}{12} = 4$ $\frac{192}{48} = 4$

Assuming the pattern continues, consecutive terms have a common ratio of 4.

$\therefore$ the sequence is geometric with $u_1 = 3$ and $r = 4$.

b $u_n = u_1 r^{n-1}$

$\therefore \quad u_n = 3 \times 4^{n-1}$

$\therefore \quad u_9 = 3 \times 4^8 = 196\,608$

3 $u_7 = 31 \quad \therefore \quad u_1 + 6d = 31 \quad .... (1)$

$u_{15} = -17 \quad \therefore \quad u_1 + 14d = -17 \quad .... (2)$

So, $(u_1 + 14d) - (u_1 + 6d) = -17 - 31 \qquad \{(2) - (1)\}$

$\therefore \quad 8d = -48$

$\therefore \quad d = -6$

So in (1), $u_1 + 6(-6) = 31$

$\therefore \quad u_1 - 36 = 31$

$\therefore \quad u_1 = 67$

Now $u_n = u_1 + (n-1)d$

$\therefore \quad u_n = 67 + (n-1)(-6)$

$\therefore \quad u_n = 67 - 6n + 6$

$\therefore \quad u_n = 73 - 6n$

So, $u_{34} = 73 - 6(34) = -131$

4 **a** $4 + 11 + 18 + 25 + ...$

The series is arithmetic with $u_1 = 4,\ d = 7,\ u_k = u_1 + (k-1)d$

$$= 4 + 7(k-1)$$
$$= 7k - 3$$

So, the series is $\sum_{k=1}^{n} (7k - 3)$.

b $\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + ...$

The series is geometric with $u_1 = \frac{1}{4},\ r = \frac{1}{2}$,

$$u_k = u_1 r^{k-1} = \tfrac{1}{4} \times \left(\tfrac{1}{2}\right)^{k-1} = \left(\tfrac{1}{2}\right)^2 \left(\tfrac{1}{2}\right)^{k-1} = \left(\tfrac{1}{2}\right)^{k+1}$$

So, the series is $\sum_{k=1}^{n} (\frac{1}{2})^{k+1}$.

5 **a** $\sum_{k=1}^{8} \left(\frac{31 - 3k}{2}\right) = 14 + 12\frac{1}{2} + 11 + 9\frac{1}{2} + 8 + 6\frac{1}{2} + 5 + 3\frac{1}{2}$

This series is arithmetic with $u_1 = 14,\quad n = 8,\quad$ and $\quad u_n = 3\frac{1}{2}$.

$\therefore$ the sum is $\frac{8}{2}(14 + 3\frac{1}{2}) = 70$

b $\sum_{k=1}^{15} 50(0.8)^{k-1} \approx 50 + 40 + 32 + ... + 3.436 + 2.749 + 2.199$

This series is geometric with $u_1 = 50,\quad r = 0.8,\quad$ and $\quad n = 15$.

$\therefore$ the sum is $\dfrac{50\left[1 - (0.8)^{15}\right]}{1 - 0.8} \approx 241$

c $\sum_{k=7}^{\infty} 5(\frac{2}{5})^{k-1} = 5(\frac{2}{5})^6 + 5(\frac{2}{5})^7 + 5(\frac{2}{5})^8 + ...$

The series is an infinite geometric series with $u_1 = 5(\frac{2}{5})^6,\quad r = \frac{2}{5}$

$$\therefore \quad S = \frac{u_1}{1 - r} = \frac{5(\frac{2}{5})^6}{\frac{3}{5}}$$

$$= \frac{2^6}{3 \times 5^4}$$

$$= \tfrac{64}{1875}$$

6 $11+16+21+26+....$ is arithmetic with $u_1=11,\quad d=5$

$$\begin{aligned}\therefore\quad S_n &= \frac{n}{2}(2u_1+(n-1)d)\\ &= \frac{n}{2}(2\times 11+5(n-1))\\ &= \frac{n}{2}(22+5n-5)\\ &= \frac{n}{2}(5n+17)\end{aligned}$$

Given $S_n=450$, we need to find n,

so $S_n=\frac{n}{2}(5n+17)=450$

$\therefore\quad \frac{5}{2}n^2+\frac{17}{2}n-450=0$

$\therefore\quad 5n^2+17n-900=0$

$\therefore\quad n\approx -15.2,\ 11.8$ {using technology}

But $n>0$, so $n\approx 11.8$

We try the two values on either side of $n\approx 11.8$, which are $n=11$ and $n=12$:

$S_{11}=\frac{11}{2}(5(11)+17)=396$ and $S_{12}=\frac{12}{2}(5(12)+17)=462$

$\therefore$ 12 terms of the series are required to exceed a sum of 450.

7 **a** $u_{n+1}=u_1\times r^n$ where $u_1=12\,500$, $r=1+\dfrac{0.0825}{2}=1.041\,25$, $n=5\times 2=10$

So, $u_{n+1}=12\,500\times(1.041\,25)^{10}$
$\approx 18\,726.65$ The value of the investment is £18 726.65 .

b $u_{n+1}=u_1\times r^n$ where $u_1=12\,500$, $r=1+\dfrac{0.0825}{12}=1.006\,875$, $n=5\times 12=60$

So, $u_{n+1}=12\,500\times(1.006\,875)^{60}$
$\approx 18\,855.74$ The value of the investment is £18 855.74 .

8 **a** Let the terms of the geometric series be $u_1,\ u_1r,\ u_1r^2,\ ....$

Then $u_1+u_1r=90$ and $u_1r^2=24$

$\therefore\quad u_1(1+r)=90$

$\therefore\quad u_1=\dfrac{90}{1+r}$ .... (1)

$\therefore\quad u_1=\dfrac{24}{r^2}$ .... (2)

Equating (1) and (2) gives $\dfrac{90}{1+r}=\dfrac{24}{r^2}$

$$\begin{aligned}\therefore\quad 90r^2&=24r+24\\ \therefore\quad 90r^2-24r-24&=0\\ \therefore\quad 6(15r^2-4r-4)&=0\\ \therefore\quad 6(5r+2)(3r-2)&=0\\ \therefore\quad r&=-\tfrac{2}{5}\text{ or }\tfrac{2}{3}\end{aligned}$$

Using (2), if $r=-\frac{2}{5}$ then $u_1=\dfrac{24}{(-\frac{2}{5})^2}=\dfrac{24}{\frac{4}{25}}=150$

if $r=\frac{2}{3}$ then $u_1=\dfrac{24}{(\frac{2}{3})^2}=\dfrac{24}{\frac{4}{9}}=54$

$\therefore$ either $u_1=150,\ r=-\frac{2}{5}$ or $u_1=54,\ r=\frac{2}{3}$

b Since $|r|<1$ in each case, both series converge.

When $u_1=150,\ r=-\frac{2}{5}$

$$\therefore\quad S=\frac{u_1}{1-r}=\frac{150}{\frac{7}{5}}=\frac{750}{7}=107\tfrac{1}{7}$$

When $u_1=54,\ r=\frac{2}{3}$

$$\therefore\quad S=\frac{u_1}{1-r}=\frac{54}{\frac{1}{3}}=162$$

9 Since Seve walks an additional 500 m $=$ 0.5 km each week, we have an arithmetic sequence with $u_1 = 10$ and constant difference $d = 0.5$.

$$u_n = u_1 + (n-1)d$$
$$\therefore\ u_n = 10 + (n-1)0.5$$

a $u_{52} = 10 + (52-1)0.5$ {52 weeks in a year}
$= 35.5$

$\therefore$ Seve walks 35.5 km in the last week.

b In total, Seve walks $10 + 10.5 + 11 + ... + 35.5$, which is an arithmetic series.

$$S_n = \frac{n}{2}(u_1 + u_n)$$
$$\therefore\ S_{52} = \tfrac{52}{2}(10 + 35.5)$$
$$= 1183$$

$\therefore$ Seve walks 1183 km in total.

10 **a** $\sum\limits_{k=1}^{\infty} 50\,(2x-1)^{k-1}$ is a geometric series with $r = 2x - 1$ and converges if $-1 < r < 1$

$\therefore\ -1 < 2x - 1 < 1$
$\therefore\ 0 < 2x < 2$
$\therefore\ 0 < x < 1$

b When $x = 0.3$, $2x - 1 = 0.6 - 1 = -0.4$

and $\sum\limits_{k=1}^{\infty} 50\,(2x-1)^{k-1} = 50(-0.4)^0 + 50(-0.4)^1 + 50(-0.4)^2 + ...$

which is geometric with $u_1 = 50$, $r = -0.4$

Now as $0 < 0.3 < 1$, the series converges and $S = \dfrac{u_1}{1-r} = \dfrac{50}{1+0.4} = \dfrac{50}{\frac{7}{5}} = 35\frac{5}{7}$

11 Since $a, b, c, d,$ and e are consecutive terms of an arithmetic sequence,
$b - a = c - b = d - c = e - d$

Taking the first and last equalities, $b - a = e - d$
$\therefore\ b + d = a + e$

Taking the middle equalities, $c - b = d - c$
$\therefore\ 2c = b + d$

$\therefore\ a + e = b + d = 2c$

12 Let the geometric sequence be $1, \underbrace{r, r^2, r^3, ..., r^{n-1}, r^n}_{n \text{ terms}}, 2$

$\therefore\ r^{n+1} = 2$ and so $r = 2^{\frac{1}{n+1}}$

The required sum is $r + r^2 + r^3 + ... + r^{n-1} + r^n$,
which is geometric with $u_1 = r$, 'r' $= r$, and 'n' $= n$.

$$\therefore\ S_n = \frac{u_1(r^n - 1)}{r - 1}$$
$$= \frac{r(r^n - 1)}{r - 1}$$
$$= \frac{r^{n+1} - r}{r - 1}$$
$$= \frac{2 - 2^{\frac{1}{n+1}}}{2^{\frac{1}{n+1}} - 1}$$

13 Let the first three terms of the arithmetic sequence be $u_1,\ u_1+d,\ u_1+2d,$ and the first three terms of the geometric sequence be $u_1,\ u_1r,\ u_1r^2.$

The second terms are equal

$\therefore\quad u_1 + d = u_1 r$

$\therefore\quad d = u_1(r-1)$

$\therefore\quad u_1 = \dfrac{d}{r-1}$ .... (1)

Now
$$\frac{u_1r^2}{u_1+2d} = \frac{(\frac{d}{r-1})r^2}{(\frac{d}{r-1})+2d} \quad \{\text{using (1)}\}$$
$$= \frac{dr^2}{d+2d(r-1)}$$
$$= \frac{dr^2}{d(2r-1)}$$
$$= \frac{r^2}{2r-1}$$

$\therefore$ the third term of the geometric sequence is $\dfrac{r^2}{2r-1}$ times the third term of the arithmetic sequence.

14 The value of $\underbrace{111....11}_{2n \text{ lots of } 1}$ is $1+10+10^2+....+10^{2n-1}$, which is a geometric series with $u_1 = 1,\ r = 10,\ `n\text{'} = 2n.$

$\therefore$ the value of this sum is
$$S_1 = \frac{u_1(r^n-1)}{r-1} = \frac{1(10^{2n}-1)}{10-1}$$
$$= \frac{10^{2n}-1}{9}$$

The value of $\underbrace{222....22}_{n \text{ lots of } 2}$ is $2+2\times 10+2\times 10^2+....+2\times 10^{n-1}$,

which is a geometric series with $u_1 = 2,\ r = 10,\ `n\text{'} = n.$

$\therefore$ the value of this sum is
$$S_2 = \frac{u_1(r^n-1)}{r-1} = \frac{2(10^n-1)}{10-1}$$
$$= \frac{2\times 10^n - 2}{9}$$

$\therefore$ the value of
$$\underbrace{111....11}_{2n \text{ lots of } 1} - \underbrace{222....22}_{n \text{ lots of } 2} = S_1 - S_2 = \frac{10^{2n}-1}{9} - \frac{2\times 10^n - 2}{9}$$
$$= \frac{10^{2n}-1-2\times 10^n+2}{9}$$
$$= \frac{10^{2n}-2\times 10^n+1}{9}$$
$$= \frac{(10^n-1)^2}{3^2}$$
$$= \left(\frac{10^n-1}{3}\right)^2$$

$10^n - 1 = \underbrace{99....9}_{n \text{ lots of } 9}$ is divisible by 3, so $\dfrac{10^n-1}{3}$ is an integer

$\therefore\quad \underbrace{111....11}_{2n \text{ lots of } 1} - \underbrace{222....22}_{n \text{ lots of } 2}$ is a perfect square.

Chapter 8

COUNTING AND THE BINOMIAL EXPANSION

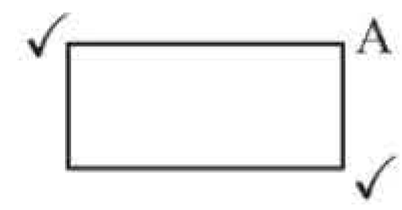
EXERCISE 8A

1 There are 3 paths from P to Q
3 paths from Q to R
2 paths from R to S

$\therefore$ number of routes possible
$= 3 \times 3 \times 2$ {product principle}
$= 18$

2 **a** There are 4 choices for A. But once A is located, there is 1 choice for B, 1 for C, and 1 for D.
$\therefore$ there are $4 \times 1 \times 1 \times 1 = 4$ ways.

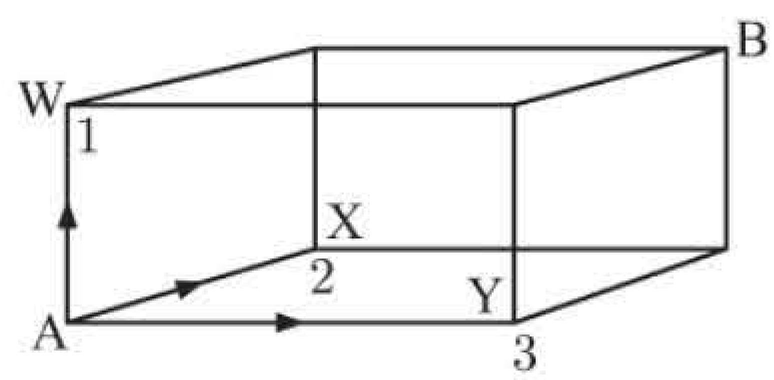
b There are 4 choices for A. But once A is located there are 2 choices for B.



Once B is located there is 1 choice for C and 1 for D.
$\therefore$ there are $4 \times 2 \times 1 \times 1 = 8$ ways.

c There are 4 choices for A. Once A is located there are 3 choices for B. Once B is located there are 2 choices for C and then 1 for D.
$\therefore$ there are $4 \times 3 \times 2 \times 1 = 24$ ways.

3



From A there are 3 possible first leg paths, to W, X, or Y. Then there are 2 second leg paths to B.
$\therefore$ total number $= 3 \times 2 = 6$ paths.

4 Any of the 7 teams could be in 'top' position. Then there are 6 left which could be in the 'second' position.
So, there are $7 \times 6 = 42$ possible ways.

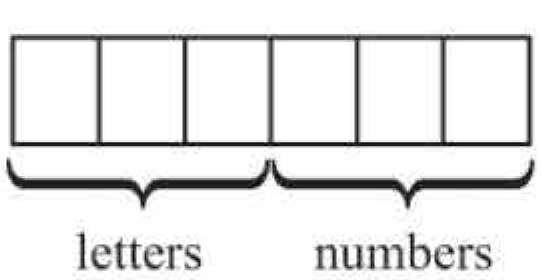
5 Any of the 8 teams could be 'top'.
Any of the remaining 7 could be second.
Any of the remaining 6 could be third.
Any of the remaining 5 could be fourth.
$\therefore$ there are $8 \times 7 \times 6 \times 5$
$= 1680$ ways.

6 There are 5 digits to choose from.

a Number of ways $= 5 \times 5 \times 5 = 125$

b Number of ways $= 5 \times 4 \times 3 = 60$

7



Repetitions are allowed.
$\therefore$ total number of ways
$= 26 \times 26 \times 26 \times 10 \times 10 \times 10$
$= 17\,576\,000$

8 **a** The 1st letter could go into either of the 2 boxes, and the second could go into either of the 2 boxes,
$\therefore$ there are $2 \times 2 = 4$ ways.

These are:

Box X	Box Y
A, B	-
A	B
B	A
-	A, B

b There are $3 \times 3 = 9$ ways.

c There are $3 \times 3 \times 3 \times 3 = 81$ ways.

EXERCISE 8B

1 **a** There are $2 \times 2 + 3 \times 3$
$= 13$ different paths

b There are $4 \times 2 + 3 \times 2 \times 2$
$= 20$ different paths

c There are $2 + 4 \times 2 + 3 \times 3$
$= 19$ different paths

d There are $2 \times 2 + 2 \times 2 + 2 \times 3 \times 4$
$= 32$ different paths

2 There are $3 \times 2 + 3 \times 1 + 2 \times 2$
$= 13$ different train journeys

EXERCISE 8C.1

1 $0! = 1$

$1! = 1$

$2! = 2 \times 1 = 2$

$3! = 3 \times 2 \times 1 = 6$

$4! = 4 \times 3 \times 2 \times 1 = 24$

$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

$6! = 6 \times 5! = 6 \times 120 = 720$

$7! = 7 \times 6! = 7 \times 720 = 5040$

$8! = 8 \times 7! = 8 \times 5040 = 40\,320$

$9! = 9 \times 8! = 9 \times 40\,320 = 362\,880$

$10! = 10 \times 9! = 10 \times 362\,880 = 3\,628\,800$

2 **a** $\dfrac{6!}{5!} = \dfrac{6 \times \cancel{5!}}{\cancel{5!}_1} = 6$

b $\dfrac{6!}{4!} = \dfrac{6 \times 5 \times \cancel{4!}}{\cancel{4!}_1} = 30$

c $\dfrac{6!}{7!} = \dfrac{\cancel{6!}^1}{7 \times \cancel{6!}} = \frac{1}{7}$

d $\dfrac{4!}{6!} = \dfrac{\cancel{4!}^1}{6 \times 5 \times \cancel{4!}} = \frac{1}{30}$

e $\dfrac{100!}{99!} = \dfrac{100 \times \cancel{99!}}{\cancel{99!}_1} = 100$

f $\dfrac{7!}{5! \times 2!} = \dfrac{7 \times 6 \times \cancel{5!}}{\cancel{5!} \times 2}$
$= 21$

3 **a** $\dfrac{n!}{(n-1)!}$
$= \dfrac{n \times \cancel{(n-1)!}}{\cancel{(n-1)!}_1}$
$= n, \quad n \geqslant 1$

b $\dfrac{(n+2)!}{n!}$
$= \dfrac{(n+2)(n+1)\cancel{n!}}{\cancel{n!}_1}$
$= (n+2)(n+1), \quad n \geqslant 0$

c $\dfrac{(n+1)!}{(n-1)!}$
$= \dfrac{(n+1)(n)\cancel{(n-1)!}}{\cancel{(n-1)!}_1}$
$= n(n+1), \quad n \geqslant 1$

4 **a** $7 \times 6 \times 5$
$= \dfrac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{4 \times 3 \times 2 \times 1}$
$= \dfrac{7!}{4!}$

b 10×9
$= \dfrac{10 \times 9 \times 8!}{8!}$
$= \dfrac{10!}{8!}$

c $11 \times 10 \times 9 \times 8 \times 7$
$= \dfrac{11 \times 10 \times 9 \times 8 \times 7 \times 6!}{6!}$
$= \dfrac{11!}{6!}$

d $\dfrac{13 \times 12 \times 11}{3 \times 2 \times 1}$
$= \dfrac{13 \times 12 \times 11 \times 10!}{10! \times 3 \times 2 \times 1}$
$= \dfrac{13!}{10! \times 3!}$

e $\dfrac{1}{6 \times 5 \times 4}$
$= \dfrac{3!}{6 \times 5 \times 4 \times 3!}$
$= \dfrac{3!}{6!}$

f $\dfrac{4 \times 3 \times 2 \times 1}{20 \times 19 \times 18 \times 17}$
$= \dfrac{4! \times 16!}{20 \times 19 \times 18 \times 17 \times 16!}$
$= \dfrac{4! \times 16!}{20!}$

5 **a** $5! + 4!$
$= 5 \times 4! + 4!$
$= 4!(5+1)$
$= 6 \times 4!$

b $11! - 10!$
$= 11 \times 10! - 10!$
$= 10!(11-1)$
$= 10 \times 10!$

c $6! + 8!$
$= 6! + 8 \times 7 \times 6!$
$= 6!(1 + 8 \times 7)$
$= 57 \times 6!$

d $12! - 10!$
$= 12 \times 11 \times 10! - 10!$
$= 10!(12 \times 11 - 1)$
$= 131 \times 10!$

e $9! + 8! + 7!$
$= 9 \times 8 \times 7! + 8 \times 7! + 7!$
$= 7!(72 + 8 + 1)$
$= 81 \times 7!$

f $7! - 6! + 8!$
$= 7 \times 6! - 6! + 8 \times 7 \times 6!$
$= 6!(7 - 1 + 56)$
$= 62 \times 6!$

g $12! - 2 \times 11!$
$= 12 \times 11! - 2 \times 11!$
$= 11!(12 - 2)$
$= 10 \times 11!$

h $3 \times 9! + 5 \times 8!$
$= 3 \times 9 \times 8! + 5 \times 8!$
$= 8!(27 + 5)$
$= 32 \times 8!$

6 **a** $\dfrac{12!-11!}{11}$
$=\dfrac{12\times 11!-11!}{11}$
$=\dfrac{11!(12-1)}{11}$
$=\dfrac{11!\times 11}{11}$
$=11!$

b $\dfrac{10!+9!}{11}$
$=\dfrac{10\times 9!+9!}{11}$
$=\dfrac{9!(10+1)}{11}$
$=\dfrac{9!\times 11}{11}$
$=9!$

c $\dfrac{10!-8!}{89}$
$=\dfrac{10\times 9\times 8!-8!}{89}$
$=\dfrac{8!(90-1)}{89}$
$=\dfrac{8!\times 89}{89}$
$=8!$

d $\dfrac{10!-9!}{9!}$
$=\dfrac{10\times 9!-9!}{9!}$
$=\dfrac{9!(10-1)}{9!}$
$=9$

e $\dfrac{6!+5!-4!}{4!}$
$=\dfrac{6\times 5\times 4!+5\times 4!-4!}{4!}$
$=\dfrac{4!(30+5-1)}{4!}$
$=34$

f $\dfrac{n!+(n-1)!}{(n-1)!}$
$=\dfrac{n\times(n-1)!+(n-1)!}{(n-1)!}$
$=\dfrac{(n-1)!(n+1)}{(n-1)!}$
$=n+1$

g $\dfrac{n!-(n-1)!}{n-1}$
$=\dfrac{n\times(n-1)!-(n-1)!}{n-1}$
$=\dfrac{(n-1)!(n-1)}{n-1}$
$=(n-1)!$

h $\dfrac{(n+2)!+(n+1)!}{n+3}$
$=\dfrac{(n+2)(n+1)!+(n+1)!}{n+3}$
$=\dfrac{(n+1)!(n+2+1)}{n+3}$
$=(n+1)!$

EXERCISE 8C.2

1 **a** $\binom{3}{1}=\dfrac{3!}{1!(3-1)!}$
$=\dfrac{3!}{1!\times 2!}$
$=\dfrac{3\times\cancel{2\times 1}}{1\times\cancel{2\times 1}}$
$=3$

b $\binom{4}{2}=\dfrac{4!}{2!(4-2)!}$
$=\dfrac{4!}{2!\times 2!}$
$=\dfrac{4\times 3\times\cancel{2\times 1}}{2\times 1\times\cancel{2\times 1}}$
$=\dfrac{12}{2}$
$=6$

c $\binom{7}{3}=\dfrac{7!}{3!(7-3)!}$
$=\dfrac{7!}{3!\times 4!}$
$=\dfrac{7\times 6\times 5\times\cancel{4\times 3\times 2\times 1}}{3\times 2\times 1\times\cancel{4\times 3\times 2\times 1}}$
$=\dfrac{210}{6}$
$=35$

d $\binom{10}{4}=\dfrac{10!}{4!(10-4)!}$
$=\dfrac{10!}{4!\times 6!}$
$=\dfrac{10\times 9\times 8\times 7\times\cancel{6\times 5\times 4\times 3\times 2\times 1}}{4\times 3\times 2\times 1\times\cancel{6\times 5\times 4\times 3\times 2\times 1}}$
$=\dfrac{5040}{24}$
$=210$

2 **a** **i** $\binom{8}{2} = \dfrac{8!}{2!(8-2)!}$
$= \dfrac{8!}{2! \times 6!}$
$= \dfrac{8 \times 7 \times \cancel{6 \times 5 \times 4 \times 3 \times 2 \times 1}}{2 \times 1 \times \cancel{6 \times 5 \times 4 \times 3 \times 2 \times 1}}$
$= \dfrac{56}{2}$
$= 28$

ii $\binom{8}{6} = \dfrac{8!}{6!(8-6)!}$
$= \dfrac{8!}{6! \times 2!}$
$= \dfrac{8 \times 7 \times \cancel{6 \times 5 \times 4 \times 3 \times 2 \times 1}}{\cancel{6 \times 5 \times 4 \times 3 \times 2 \times 1} \times 2 \times 1}$
$= \dfrac{56}{2}$
$= 28$

b $\binom{n}{r} = \dfrac{n!}{r!(n-r)!}$ for all $n \in \mathbb{Z}^+$, $r = 0, 1, 2, ..., n.$

$\binom{n}{n-r} = \dfrac{n!}{(n-r)!(n-(n-r))!}$ for all $n \in \mathbb{Z}^+$, $r = 0, 1, 2, ..., n.$
$= \dfrac{n!}{(n-r)!(n-n+r)!}$
$= \dfrac{n!}{(n-r)!r!}$
$= \dfrac{n!}{r!(n-r)!}$

$\therefore \binom{n}{r} = \binom{n}{n-r}$ for all $n \in \mathbb{Z}^+$, $r = 0, 1, 2, ..., n.$

3 $\binom{9}{k} = 4\binom{7}{k-1}$

$\therefore \dfrac{9!}{k!(9-k)!} = 4\left(\dfrac{7!}{(k-1)!(7-[k-1])!}\right)$

$\therefore \dfrac{9!}{k!(9-k)!} = \dfrac{4 \times 7!}{(k-1)!(8-k)!}$

$\therefore \dfrac{9!}{4 \times 7!} = \dfrac{k!(9-k)!}{(k-1)!(8-k)!}$

$\therefore \dfrac{9 \times 8 \times \cancel{7!}}{4 \times \cancel{7!}} = \dfrac{k\cancel{(k-1)!} \times (9-k)\cancel{(8-k)!}}{\cancel{(k-1)!(8-k)!}_{1}}$

$\therefore \dfrac{9 \times 8}{4} = k(9-k)$

$\therefore 18 = 9k - k^2$

$\therefore k^2 - 9k + 18 = 0$

$\therefore (k-3)(k-6) = 0$

$\therefore k = 3$ or 6

EXERCISE 8D

1 **a** W, X, Y, Z

b WX, WY, WZ, XW, XY, XZ, YW, YX, YZ, ZW, ZX, ZY

c WXY, WXZ, WYX, WYZ, WZX, WZY, XWY, XWZ, XYW, XYZ, XZW, XZY, YWX, YWZ, YXW, YXZ, YZW, YZX, ZWX, ZWY, ZXW, ZXY, ZYW, ZYX

2 **a** AB, AC, AD, AE, BA, BC, BD, BE, CA, CB, CD, CE, DA, DB, DC, DE, EA, EB, EC, ED

b ABC, ABD, ABE, ACB, ACD, ACE, ADB, ADC, ADE, AEB, AEC, AED, BAC, BAD, BAE, BCA, BCD, BCE, BDA, BDC, BDE, BEA, BEC, BED, CAB, CAD, CAE, CBA, CBD, CBE, CDA, CDB, CDE, CEA, CEB, CED, DAB, DAC, DAE, DBA, DBC, DBE, DCA, DCB, DCE, DEA, DEB, DEC, EAB, EAC, EAD, EBA, EBC, EBD, ECA, ECB, ECD, EDA, EDB, EDC

(2 at a time: 20   3 at a time: 60)

3 **a** There are $5! = 120$ different orderings.

b There are $8 \times 7 \times 6 = 336$ different orderings.

c There are $10 \times 9 \times 8 \times 7 = 5040$ different signals.

4 **a** $\boxed{4}\boxed{3}$ $\therefore$ there are $4 \times 3 = 12$ different signals

b $\boxed{4}\boxed{3}\boxed{2}$ $\therefore$ there are $4 \times 3 \times 2 = 24$ different signals

c $12 + 24 = 36$ different signals {using **a** and **b**}

5 There are 6 different letters $\therefore$ $6! = 720$ permutations.

a $\boxed{4}\boxed{3}\boxed{2}\boxed{1}\boxed{1}\boxed{1}$ $\therefore$ there are $4 \times 3 \times 2 \times 1 \times 1 \times 1 = 24$ permutations

E D

b $\boxed{1}\boxed{4}\boxed{3}\boxed{2}\boxed{1}\boxed{1}$ $\therefore$ there are $1 \times 4 \times 3 \times 2 \times 1 \times 1 = 24$ permutations

F A

c $\boxed{2}\boxed{4}\boxed{3}\boxed{2}\boxed{1}\boxed{1}$ $\therefore$ there are $2 \times 4 \times 3 \times 2 \times 1 \times 1 = 48$ permutations

A or E the other one

6 **a** $\boxed{7}\boxed{7}\boxed{7}$ So, there are $7^3 = 343$ different numbers.

b $\boxed{7}\boxed{6}\boxed{5}$ So, there are $7 \times 6 \times 5 = 210$ different numbers.

c $\boxed{6}\boxed{5}\boxed{4}$ So, there are $6 \times 5 \times 4 = 120$ different numbers.

6 remain fill 1st with any of 4 odds

7 **a** $\boxed{9}\boxed{9}\boxed{8}$

cannot use 0

So, there are
$9 \times 9 \times 8 = 648$ numbers.

b $\boxed{8}\boxed{8}\boxed{1}$

cannot use 0 or 5 a 5

So, there are
$8 \times 8 \times 1 = 64$ numbers.

c $\boxed{9}\boxed{8}\boxed{1}$

0

So, there are
$9 \times 8 \times 1 = 72$
different numbers.

d $64 + 72 = 136$ different numbers.
{using **b** and **c**}

8 **a** As there are no restrictions, the number of ways is $5! = 120$.

b X and Y are together in $2!$ ways {XY or YX}
They, together with the other three books can be permutated in $4!$ ways.
$\therefore$ total number $= 2! \times 4! = 48$ ways.

c $120 - 48 = 72$ ways {using **a** and **b**}

9 **a** As there are no restrictions, the number of ways is $10! = 3\,628\,800$.

b A, B and C are together in $3!$ ways {ABC, ACB, BAC, BCA, CAB, CBA}
They, together with the other 7 can be permutated in $8!$ ways.
$\therefore$ total number is $3! \times 8! = 241\,920$ ways.

10 **a** $\boxed{4}\boxed{4}\boxed{3}$ So, there are $4 \times 4 \times 3 = 48$ different numbers.

not 0

b $\boxed{2}\boxed{4}\boxed{3}$ So, there are $2 \times 4 \times 3 = 24$ different numbers.

1 or 3

c The last digit must be a 0 or 8.

If it is 0: $\boxed{3}\boxed{3}\boxed{1}$ $\therefore\ 3 \times 3 \times 1 = 9$ different numbers

3, 5, or 8   0

If it is 8: $\boxed{2}\boxed{3}\boxed{1}$ $\therefore\ 2 \times 3 \times 1 = 6$ different numbers

3 or 5   8

$\therefore$ in total there are $9 + 6 = 15$ different numbers.

11 **a** $\boxed{6}\boxed{5}\boxed{4}\boxed{3}$ So, there are $6 \times 5 \times 4 \times 3 = 360$ different arrangements.

b If no vowels are used, there are 4 letters to choose from.

$\therefore$ $\boxed{4}\boxed{3}\boxed{2}\boxed{1}$ So, there are $4! = 24$ different arrangements.

$\therefore$ if at least one vowel must be used, there are $360 - 24$ {from **a**}
$= 336$ different arrangements

c We first count the number of ways two vowels are adjacent.

A and O can be put together in 2! ways {AO or OA}

These vowels can be placed in any one of 3 positions {1st and 2nd, 2nd and 3rd, or 3rd and 4th}

The remaining 2 places can be filled from the other 4 letters in 4×3 different ways.

$\therefore$ two vowels are adjacent in $2! \times 3 \times 4 \times 3 = 72$ ways

$\therefore$ no two vowels are adjacent in $360 - 72 = 288$ ways

12 **a** $\boxed{9}\boxed{8}\boxed{7}\boxed{6}\boxed{5}$ So, there are $9 \times 8 \times 7 \times 6 \times 5 = 15\,120$ different ways.

b $\boxed{4}\boxed{3}\boxed{2}\boxed{6}\boxed{5}$ So, there are $4 \times 3 \times 2 \times 6 \times 5 = 720$ different ways.

2, 4, 6, or 8

13 **a** $\boxed{10}\boxed{9}\boxed{8}\boxed{7}\boxed{6}\boxed{5}\boxed{4}\boxed{3}\boxed{2}\boxed{1}$ $\therefore\ 10! = 3\,628\,800$ different ways.

b **i** $\boxed{10}\boxed{9}\boxed{8}\boxed{7}\boxed{6}\boxed{5}$ $\therefore\ 10 \times 9 \times 8 \times 7 \times 6 \times 5 = 151\,200$ different ways

Alice can sit in any of the 10 seats

Her friend can sit in any of the remaining 9 seats

ii Alice can sit in any of the 8 middle seats.

She can choose the two friends to sit next to her in 5×4 different ways.

The remaining 3 friends can occupy the other 7 seats in $7 \times 6 \times 5$ different ways.

$\therefore$ there are $8 \times 5 \times 4 \times 7 \times 6 \times 5 = 33\,600$ different ways.

EXERCISE 8E

1 ABCD, ABCE, ABCF, ABDE, ABDF, ABEF, ACDE, ACDF, ACEF, ADEF, BCDE, BCDF, BCEF, BDEF, CDEF, and $\binom{6}{4} = 15$ ✓

2 There are $\binom{17}{11} = 12\,376$ different teams.

3 **a** There are $\binom{9}{5} = 126$ different possible selections.

b If question 1 is compulsory there are $\binom{1}{1}\binom{8}{4} = 1 \times 70 = 70$ possible selections.

4 **a** If no restrictions, there are $\binom{13}{3} = 286$ different committees.

b $\binom{1}{1}\binom{12}{2} = 66$ of them consist of the president and two others.

5 **a** If no restrictions, there are $\binom{12}{5} = 792$ different teams.

b **i** Those containing the captain and vice-captain number $\binom{2}{2}\binom{10}{3} = 1 \times 120 = 120$.

ii Those containing exactly one of the captain and vice captain number $\binom{2}{1}\binom{10}{4} = 2 \times 210 = 420$.

6 Number of different teams $= \binom{3}{3}\binom{1}{0}\binom{11}{6} = 1 \times 1 \times 462 = 462$.

7 **a** If 1 person must be selected, number of ways $= \binom{1}{1}\binom{9}{3} = 84$

b If 2 are always excluded, the number of ways $= \binom{2}{0}\binom{8}{4} = 70$

c If 1 is always 'in' and 2 are always 'out', the number of ways is $\binom{1}{1}\binom{2}{0}\binom{7}{3} = 35$

8 **a** If there are no restrictions the number of ways $= \binom{16}{5} = 4368$

b The three men can be chosen in $\binom{10}{3}$ ways and the 2 women in $\binom{6}{2}$ ways.

$\therefore$ total number of ways $= \binom{10}{3} \times \binom{6}{2} = 120 \times 15 = 1800$ ways.

c If it contains all men, the number of ways $= \binom{10}{5} \times \binom{6}{0} = 252$

d If it contains at least 3 men it would contain
3 men and 2 women or 4 men and 1 woman or 5 men and 0 women
and this can be done in $\binom{10}{3}\binom{6}{2} + \binom{10}{4}\binom{6}{1} + \binom{10}{5}\binom{6}{0}$ ways $= 3312$ ways.

e If it contains at least one of each sex, the total number of ways
$= \binom{10}{1}\binom{6}{4} + \binom{10}{2}\binom{6}{3} + \binom{10}{3}\binom{6}{2} + \binom{10}{4}\binom{6}{1} = 4110$
or $\binom{16}{5} - \binom{10}{0}\binom{6}{5} - \binom{10}{5}\binom{6}{0} = 4110$

9 **a** The 2 doctors can be chosen in $\binom{6}{2}$ ways
The 1 dentist can be chosen in $\binom{3}{1}$ ways
The 2 others can be chosen in $\binom{7}{2}$ ways
$\therefore$ the total number of ways $= \binom{6}{2} \times \binom{3}{1} \times \binom{7}{2} = 945$

b If it contains 2 doctors, 3 must be chosen from the other 10,
$\therefore$ there are $\binom{6}{2}\binom{10}{3} = 1800$ ways.

c If it contains at least one person from each of the two professions this can be done in
$\binom{9}{1}\binom{7}{4} + \binom{9}{2}\binom{7}{3} + \binom{9}{3}\binom{7}{2} + \binom{9}{4}\binom{7}{1} = 4347$ *or* $\binom{16}{5} - \binom{9}{0}\binom{7}{5} = 4347$

10 There are 20 points (for vertices) to choose from and any 2 form a line.
This can be done in $\binom{20}{2}$ ways. But this count includes the 20 lines joining the vertices.
$\therefore$ the number of diagonals $= \binom{20}{2} - 20 = 190 - 20 = 170$

11 **a** **i** $\binom{12}{2} = 66$ lines can be determined.

ii Of the lines in **a i** $\binom{1}{1}\binom{11}{1} = 11$ pass through B.

b **i** $\binom{12}{3} = 220$ triangles can be determined.

ii Of the triangles in **b i** $\binom{1}{1}\binom{11}{2} = 55$ have one vertex B.

12 The digits must be from 1 to 9. So, there are 9 of them, and we want any 4.

This can be done in $\binom{9}{4} = 126$ ways.

Once they have been selected they can be put in one ascending order

$\therefore$ total number $= 126 \times 1 = 126$.

13 **a** The different committees of 4, consisting of selections from 5 men and 6 women *in all possible ways* are

(4 men, 0 women) or (3 men, 1 woman) or (2 men, 2 women) or (1 man, 3 women) or (0 men, 4 women)

$\therefore \binom{5}{4}\binom{6}{0} + \binom{5}{3}\binom{6}{1} + \binom{5}{2}\binom{6}{2} + \binom{5}{1}\binom{6}{3} + \binom{5}{0}\binom{6}{4} = \binom{11}{4} \longleftarrow$ total number unrestricted

b The generalisation is:

$$\binom{m}{0}\binom{n}{r} + \binom{m}{1}\binom{n}{r-1} + \binom{m}{2}\binom{n}{r-2} + \dots + \binom{m}{r-2}\binom{n}{2} + \binom{m}{r-1}\binom{n}{1} + \binom{m}{r}\binom{n}{0} = \binom{m+n}{r}$$

14 **a** Consider a simpler case of 4 people (A, B, C, and D) going into two equal groups.

AB	CD	(1)	(1) and (6) are the same division.
AC	BD	(2)	(2) and (5) are the same division.
AD	BC	(3)	(3) and (4) are the same division.
BC	AD	(4)	
BD	AC	(5)	So, the number of ways is $\frac{1}{2}$ of $\binom{4}{2}$.
CD	AB	(6)	

So, for 2 equal groups of 6, the number of ways $\frac{1}{2}$ of $\binom{12}{6}\binom{6}{6}$

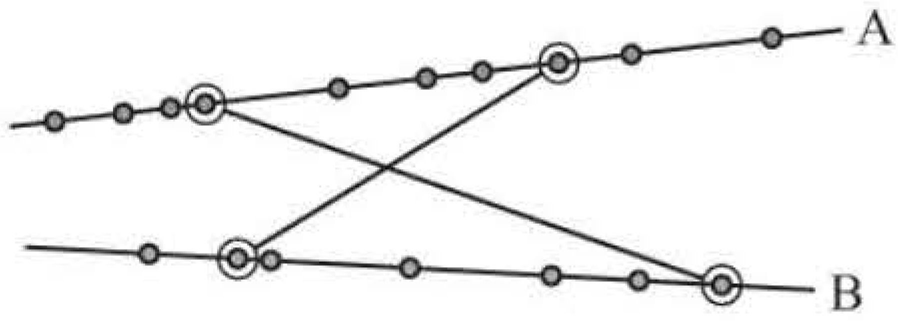
$= \frac{1}{2} \times 924$

$= 462$

b For 3 equal groups of 4, the number of ways $= \dfrac{1}{3!} \times \binom{12}{4} \times \binom{8}{4} \times \binom{4}{4}$

$= 5775$

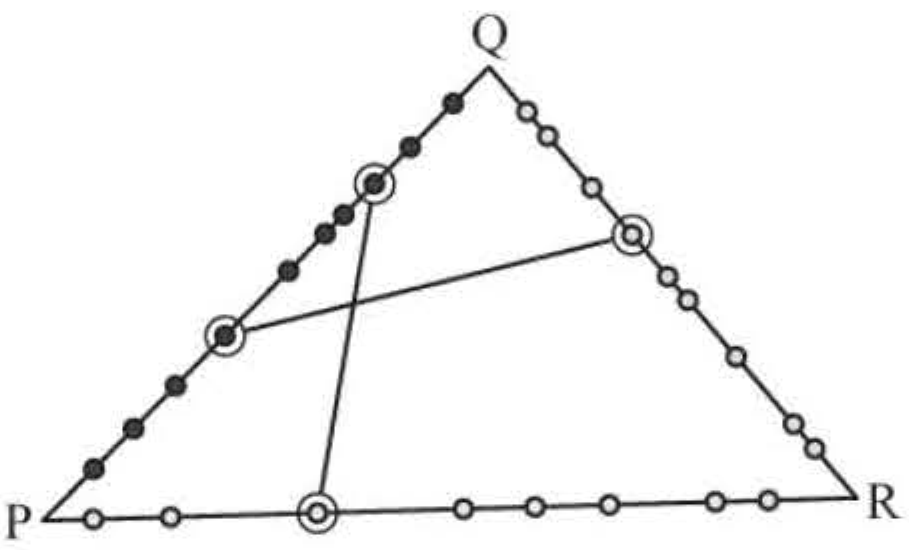
15 There is one point of intersection for every combination of 4 points (2 from A, 2 from B) as shown.



There are $\binom{10}{2} \times \binom{7}{2}$ ways to choose these points.

$\therefore$ the maximum number of points of intersection (when none of the intersection points coincide) is $\binom{10}{2} \times \binom{7}{2} = 945$

16 There is one point of intersection for every combination of 4 points (no more than 2 from any one line) as shown.



$\therefore$ the maximum number of points of intersection (when none of the intersection points coincide) is

$\binom{10}{2}\binom{9}{2}\binom{8}{0} + \binom{10}{2}\binom{9}{0}\binom{8}{2} + \binom{10}{0}\binom{9}{2}\binom{8}{2}$

$+ \binom{10}{2}\binom{9}{1}\binom{8}{1} + \binom{10}{1}\binom{9}{2}\binom{8}{1} + \binom{10}{1}\binom{9}{1}\binom{8}{2}$

$= 12\,528$

17 **a** Each handshake occurs between 2 people. So we need the number of ways 2 people can be selected from 10.

$\therefore$ the number of handshakes between committee members is $\binom{10}{2} = 45$.

A 10-sided polygon can be used to solve this problem by counting the total number of lines (including edges) between each pair of vertices. Each line represents a handshake between 2 of the 10 committee members.

b The number of handshakes between 273 delegates is $\binom{273}{2} = 37\,128$.

c The number of different orders in which the committee members can line up on stage is $10! = 3\,628\,800$.

EXERCISE 8F

1 **a** $(p+q)^3$
$= p^3 + 3p^2q + 3pq^2 + q^3$

b $(x+1)^3$
$= x^3 + 3x^2(1)^1 + 3x(1)^2 + (1)^3$
$= x^3 + 3x^2 + 3x + 1$

c $(x-3)^3$
$= x^3 + 3x^2(-3) + 3x(-3)^2 + (-3)^3$
$= x^3 - 9x^2 + 27x - 27$

d $(2+x)^3$
$= 2^3 + 3(2)^2x + 3(2)x^2 + x^3$
$= 8 + 12x + 6x^2 + x^3$

e $(3x-1)^3$
$= (3x)^3 + 3(3x)^2(-1) + 3(3x)(-1)^2 + (-1)^3$
$= 27x^3 - 27x^2 + 9x - 1$

f $(2x+5)^3$
$= (2x)^3 + 3(2x)^2(5) + 3(2x)(5)^2 + (5)^3$
$= 8x^3 + 60x^2 + 150x + 125$

g $(2a-b)^3 = (2a)^3 + 3(2a)^2(-b) + 3(2a)(-b)^2 + (-b)^3$
$= 8a^3 - 12a^2b + 6ab^2 - b^3$

h $(3x - \frac{1}{3})^3 = (3x)^3 + 3(3x)^2(-\frac{1}{3}) + 3(3x)(-\frac{1}{3})^2 + (-\frac{1}{3})^3$
$= 27x^3 - 9x^2 + x - \frac{1}{27}$

i $\left(2x + \dfrac{1}{x}\right)^3 = (2x)^3 + 3(2x)^2\left(\dfrac{1}{x}\right) + 3(2x)\left(\dfrac{1}{x}\right)^2 + \left(\dfrac{1}{x}\right)^3$
$= 8x^3 + 12x + \dfrac{6}{x} + \dfrac{1}{x^3}$

2 **a** $(1+x)^4 = 1^4 + 4(1)^3x + 6(1)^2x^2 + 4(1)x^3 + x^4$
$= 1 + 4x + 6x^2 + 4x^3 + x^4$

b $(p-q)^4 = p^4 + 4p^3(-q) + 6p^2(-q)^2 + 4p(-q)^3 + (-q)^4$
$= p^4 - 4p^3q + 6p^2q^2 - 4pq^3 + q^4$

c $(x-2)^4 = x^4 + 4x^3(-2)^1 + 6x^2(-2)^2 + 4x(-2)^3 + (-2)^4$
$= x^4 - 8x^3 + 24x^2 - 32x + 16$

d $(3-x)^4 = 3^4 + 4(3)^3(-x) + 6(3)^2(-x)^2 + 4(3)(-x)^3 + (-x)^4$
$= 81 - 108x + 54x^2 - 12x^3 + x^4$

e $(1+2x)^4 = 1^4 + 4(1)^3(2x) + 6(1)^2(2x)^2 + 4(1)(2x)^3 + (2x)^4$
$= 1 + 8x + 24x^2 + 32x^3 + 16x^4$

f $(2x-3)^4 = (2x)^4 + 4(2x)^3(-3)^1 + 6(2x)^2(-3)^2 + 4(2x)(-3)^3 + (-3)^4$
$= 16x^4 - 12 \times 8x^3 + 54 \times 4x^2 - 108 \times 2x + 81$
$= 16x^4 - 96x^3 + 216x^2 - 216x + 81$

g $(2x+b)^4 = (2x)^4 + 4(2x)^3b + 6(2x)^2b^2 + 4(2x)b^3 + b^4$
$= 16x^4 + 32x^3b + 24x^2b^2 + 8xb^3 + b^4$

h $\left(x + \dfrac{1}{x}\right)^4 = x^4 + 4x^3\left(\dfrac{1}{x}\right) + 6x^2\left(\dfrac{1}{x}\right)^2 + 4x\left(\dfrac{1}{x}\right)^3 + \left(\dfrac{1}{x}\right)^4$
$= x^4 + 4x^2 + 6 + \dfrac{4}{x^2} + \dfrac{1}{x^4}$

i $\left(2x - \dfrac{1}{x}\right)^4 = (2x)^4 + 4(2x)^3\left(-\dfrac{1}{x}\right) + 6(2x)^2\left(-\dfrac{1}{x}\right)^2 + 4(2x)\left(-\dfrac{1}{x}\right)^3 + \left(-\dfrac{1}{x}\right)^4$
$= 16x^4 - 32x^2 + 24 - \dfrac{8}{x^2} + \dfrac{1}{x^4}$

3 **a** $(x+2)^5 = x^5 + 5x^4(2) + 10x^3(2)^2 + 10x^2(2)^3 + 5x(2)^4 + 2^5$
$= x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 32$

b $(x-2y)^5 = x^5 + 5x^4(-2y) + 10x^3(-2y)^2 + 10x^2(-2y)^3 + 5x(-2y)^4 + (-2y)^5$
$= x^5 - 10x^4y + 40x^3y^2 - 80x^2y^3 + 80xy^4 - 32y^5$

c $(1+2x)^5 = 1^5 + 5(1)^4(2x) + 10(1)^3(2x)^2 + 10(1)^2(2x)^3 + 5(1)(2x)^4 + (2x)^5$
$= 1 + 10x + 40x^2 + 80x^3 + 80x^4 + 32x^5$

d $\left(x-\frac{1}{x}\right)^5 = x^5 + 5x^4\left(-\frac{1}{x}\right) + 10x^3\left(-\frac{1}{x}\right)^2 + 10x^2\left(-\frac{1}{x}\right)^3 + 5x\left(-\frac{1}{x}\right)^4 + \left(-\frac{1}{x}\right)^5$
$= x^5 - 5x^3 + 10x - \frac{10}{x} + \frac{5}{x^3} - \frac{1}{x^5}$

4 **a** 1 5 10 10 5 1 ⟵ the 5th row
1 6 15 20 15 6 1 ⟵ the 6th row

b **i** $(x+2)^6 = x^6 + 6x^5(2) + 15x^4(2)^2 + 20x^3(2)^3 + 15x^2(2)^4 + 6x(2)^5 + (2)^6$
$= x^6 + 12x^5 + 60x^4 + 160x^3 + 240x^2 + 192x + 64$

ii $(2x-1)^6 = (2x)^6 + 6(2x)^5(-1) + 15(2x)^4(-1)^2 + 20(2x)^3(-1)^3 + 15(2x)^2(-1)^4$
$+ 6(2x)(-1)^5 + (-1)^6$
$= 64x^6 - 6 \times 32x^5 + 15 \times 16x^4 - 20 \times 8x^3 + 15 \times 4x^2 - 6 \times 2x + 1$
$= 64x^6 - 192x^5 + 240x^4 - 160x^3 + 60x^2 - 12x + 1$

iii $\left(x+\frac{1}{x}\right)^6 = x^6 + 6x^5\left(\frac{1}{x}\right) + 15x^4\left(\frac{1}{x}\right)^2 + 20x^3\left(\frac{1}{x}\right)^3 + 15x^2\left(\frac{1}{x}\right)^4 + 6x\left(\frac{1}{x}\right)^5 + \left(\frac{1}{x}\right)^6$
$= x^6 + 6x^4 + 15x^2 + 20 + \frac{15}{x^2} + \frac{6}{x^4} + \frac{1}{x^6}$

5 **a** $\left(1+\sqrt{2}\right)^3 = (1)^3 + 3(1)^2\left(\sqrt{2}\right) + 3(1)\left(\sqrt{2}\right)^2 + \left(\sqrt{2}\right)^3$
$= 1 + 3\sqrt{2} + 3 \times 2 + 2 \times \sqrt{2}$
$= 1 + 3\sqrt{2} + 6 + 2\sqrt{2}$
$= 7 + 5\sqrt{2}$

b $\left(\sqrt{5}+2\right)^4 = \left(\sqrt{5}\right)^4 + 4\left(\sqrt{5}\right)^3(2) + 6\left(\sqrt{5}\right)^2(2)^2 + 4\left(\sqrt{5}\right)(2)^3 + 2^4$
$= 25 + 8 \times 5\sqrt{5} + 24 \times 5 + 32\sqrt{5} + 16$
$= 25 + 40\sqrt{5} + 120 + 32\sqrt{5} + 16$
$= 161 + 72\sqrt{5}$

c $\left(2-\sqrt{2}\right)^5$
$= (2)^5 + 5(2)^4\left(-\sqrt{2}\right) + 10(2)^3\left(-\sqrt{2}\right)^2 + 10(2)^2\left(-\sqrt{2}\right)^3 + 5(2)^1\left(-\sqrt{2}\right)^4 + \left(-\sqrt{2}\right)^5$
$= 32 - 80\sqrt{2} + 160 - 80\sqrt{2} + 40 - 4\sqrt{2}$
$= 232 - 164\sqrt{2}$

6 **a** $(2+x)^6 = (2)^6 + 6(2)^5x + 15(2)^4x^2 + 20(2)^3x^3 + 15(2)^2x^4 + 6(2)x^5 + x^6$
$= 64 + 192x + 240x^2 + 160x^3 + 60x^4 + 12x^5 + x^6$

b $(2.01)^6$ is obtained by letting $x = 0.01$
$\therefore \quad (2.01)^6 = 64 + 192 \times (0.01) + 240 \times (0.01)^2 + 160 \times (0.01)^3$
$+ 60 \times (0.01)^4 + 12 \times (0.01)^5 + (0.01)^6$
$= 65.944\,160\,601\,201$

$$\begin{array}{rl} & 64 \\ & 1.92 \\ & 0.024 \\ & 0.000\,16 \\ & 0.000\,000\,6 \\ & 0.000\,000\,001\,2 \\ + & 0.000\,000\,000\,001 \\ \hline & 65.944\,160\,601\,201 \end{array}$$

7 **a** $(a+b)^3 = 8 + 12e^x + ...$

and $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$\therefore\ a^3 = 8$

$\therefore\ a = 2$

and $3a^2b = 12e^x$

$\therefore\ 3(2)^2b = 12e^x$

$\therefore\ 12b = e^x$

$\therefore\ b = e^x$

So, $a = 2,\ b = e^x$.

b $(a+b)^3 = (2+e^x)^3$

$= 2^3 + 3(2)^2e^x + 3(2)(e^x)^2 + (e^x)^3$

$= 8 + 12e^x + 6e^{2x} + e^{3x}$

So, the remaining two terms are $6e^{2x}$ and e^{3x}.

8 $(2x+3)(x+1)^4$

$= (2x+3)(x^4 + 4x^3 + 6x^2 + 4x + 1)$

$= 2x^5 + 8x^4 + 12x^3 + 8x^2 + 2x + 3x^4 + 12x^3 + 18x^2 + 12x + 3$

$= 2x^5 + 11x^4 + 24x^3 + 26x^2 + 14x + 3$

9 **a** $(3a+b)^5 = (3a)^5 + 5(3a)^4b + 10(3a)^3b^2 + ...$

$\therefore$ the coefficient of a^3b^2 is $10 \times 3^3 = 270$

b $(2a+3b)^6 = (2a)^6 + 6(2a)^5(3b) + 15(2a)^4(3b)^2 + 20(2a)^3(3b)^3 + ...$

$\therefore$ the coefficient of a^3b^3 is $20 \times 2^3 \times 3^3 = 4320$

EXERCISE 8G

1 **a** $(1+2x)^{11} = 1^{11} + \binom{11}{1}(2x)^1 + \binom{11}{2}(2x)^2 + ... + \binom{11}{10}(2x)^{10} + (2x)^{11}$

b $\left(3x + \dfrac{2}{x}\right)^{15}$

$= (3x)^{15} + \binom{15}{1}(3x)^{14}\left(\dfrac{2}{x}\right)^1 + \binom{15}{2}(3x)^{13}\left(\dfrac{2}{x}\right)^2 + ... + \binom{15}{14}(3x)^1\left(\dfrac{2}{x}\right)^{14} + \left(\dfrac{2}{x}\right)^{15}$

c $\left(2x - \dfrac{3}{x}\right)^{20}$

$= (2x)^{20} + \binom{20}{1}(2x)^{19}\left(-\dfrac{3}{x}\right)^1 + \binom{20}{2}(2x)^{18}\left(-\dfrac{3}{x}\right)^2 + ... + \binom{20}{19}(2x)^1\left(-\dfrac{3}{x}\right)^{19} + \left(-\dfrac{3}{x}\right)^{20}$

2 **a** For $(2x+5)^{15}$, $a = (2x)$, $b = 5$, and $n = 15$

Now $T_{r+1} = \binom{n}{r}a^{n-r}b^r$ and letting $r = 5$ gives $T_6 = \binom{15}{5}(2x)^{10}5^5$.

b For $\left(x^2+y\right)^9$, $a = (x^2)$, $b = y$, and $n = 9$

Now $T_{r+1} = \binom{n}{r}a^{n-r}b^r$ and letting $r = 3$ gives $T_4 = \binom{9}{3}(x^2)^6(y)^3$.

c For $\left(x - \dfrac{2}{x}\right)^{17}$, $a = x$, $b = \left(-\dfrac{2}{x}\right)$, and $n = 17$

Now $T_{r+1} = \binom{n}{r}a^{n-r}b^r$ and letting $r = 9$ gives $T_{10} = \binom{17}{9}x^8\left(-\dfrac{2}{x}\right)^9$.

d For $\left(2x^2 - \dfrac{1}{x}\right)^{21}$, $a = (2x^2)$, $b = \left(-\dfrac{1}{x}\right)$, and $n = 21$

Now $T_{r+1} = \binom{n}{r}a^{n-r}b^r$ and letting $r = 8$ gives $T_9 = \binom{21}{8}(2x^2)^{13}\left(-\dfrac{1}{x}\right)^8$.

3 **a** For $(x+b)^7$, $a = x$, $b = b$, and $n = 7$

$\therefore$ the general term $T_{r+1} = \binom{7}{r}x^{7-r}b^r$

b If $x^{7-r} = x^4$ then $7 - r = 4$

$\therefore \ r = 3$

Now $T_4 = \binom{7}{3} x^4 b^3$

$\therefore$ the coefficient of x^4 is $\binom{7}{3} b^3 = 35b^3$

But the coefficient of x^4 is -280

So, $35b^3 = -280$

$\therefore \ b^3 = -8$

$\therefore \ b = \sqrt[3]{-8}$

$\therefore \ b = -2$

4 **a** For $\left(x + \dfrac{2}{x^2}\right)^{15}$, $a = x$, $b = \dfrac{2}{x^2}$, and $n = 15$

Now
$$\begin{aligned} T_{r+1} &= \binom{n}{r} a^{n-r} b^r \\ &= \binom{15}{r} x^{15-r} \left(\frac{2}{x^2}\right)^r \\ &= \binom{15}{r} x^{15-r} \frac{2^r}{x^{2r}} \\ &= \binom{15}{r} 2^r x^{15-3r} \end{aligned}$$

The constant term does not contain x.

$\therefore \ 15 - 3r = 0$

$\therefore \ r = 5$

so $T_6 = \binom{15}{5} 2^5 x^0$

$\therefore$ the constant term is $\binom{15}{5} 2^5$.

b For $\left(x - \dfrac{3}{x^2}\right)^9$, $a = x$, $b = \left(-\dfrac{3}{x^2}\right)$, and $n = 9$

Now
$$\begin{aligned} T_{r+1} &= \binom{n}{r} a^{n-r} b^r \\ &= \binom{9}{r} x^{9-r} \left(-\frac{3}{x^2}\right)^r \\ &= \binom{9}{r} x^{9-r} \frac{(-3)^r}{x^{2r}} \\ &= \binom{9}{r} (-3)^r x^{9-3r} \end{aligned}$$

The constant term does not contain x.

$\therefore \ 9 - 3r = 0$

$\therefore \ r = 3$

so $T_4 = \binom{9}{3} (-3)^3 x^0$

$\therefore$ the constant term is $\binom{9}{3} (-3)^3$.

5 **a**

Row 1	1 1	⟵	**b** **i** sum $= 1 + 1 = 2 = 2^1$
Row 2	1 2 1	⟵	**ii** sum $= 1 + 2 + 1 = 4 = 2^2$
Row 3	1 3 3 1	⟵	**iii** sum $= 1 + 3 + 3 + 1 = 8 = 2^3$
Row 4	1 4 6 4 1	⟵	**iv** sum $= 1 + 4 + 6 + 4 + 1 = 16 = 2^4$
Row 5	1 5 10 10 5 1	⟵	**v** sum $= 1 + 5 + 10 + 10 + 5 + 1 = 32 = 2^5$

c The sum of the numbers in row n of Pascal's triangle is 2^n.

d
$$\begin{aligned} &(1 + x)^n \\ &= \binom{n}{0} 1^n + \binom{n}{1} 1^{n-1} x + \binom{n}{2} 1^{n-2} x^2 + \binom{n}{3} 1^{n-3} x^3 + \dots + \binom{n}{n-1} 1^1 x^{n-1} + \binom{n}{n} x^n \\ &= \binom{n}{0} + \binom{n}{1} x + \binom{n}{2} x^2 + \binom{n}{3} x^3 + \dots + \binom{n}{n-1} x^{n-1} + \binom{n}{n} x^n \qquad \{\text{as all powers of 1 are 1}\} \end{aligned}$$

Now letting $x = 1$ gives LHS $= (1 + 1)^n = 2^n$

and RHS $= \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \dots + \binom{n}{n-1} + \binom{n}{n}$

$\therefore \ \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n-1} + \binom{n}{n} = 2^n$

6 **a** In $\left(3 + 2x^2\right)^{10}$, $a = 3$, $b = (2x^2)$, and $n = 10$

Now
$$\begin{aligned} T_{r+1} &= \binom{n}{r} a^{n-r} b^r \\ &= \binom{10}{r} 3^{10-r} (2x^2)^r \\ &= \binom{10}{r} 3^{10-r} 2^r x^{2r} \end{aligned}$$

We now let $2r = 10$

$\therefore \ r = 5$

So, $T_6 = \underbrace{\binom{10}{5} 3^5 2^5}\, x^{10}$

$\therefore$ the coefficient is $\binom{10}{5} 3^5 2^5$.

b In $\left(2x^2 - \frac{3}{x}\right)^6$, $a = (2x^2)$, $b = \left(-\frac{3}{x}\right)$, and $n = 6$

Now
$$\begin{aligned} T_{r+1} &= \binom{n}{r} a^{n-r} b^r \\ &= \binom{6}{r} \left(2x^2\right)^{6-r} \left(-\frac{3}{x}\right)^r \\ &= \binom{6}{r} 2^{6-r} x^{12-2r} \frac{(-3)^r}{x^r} \\ &= \binom{6}{r} 2^{6-r} (-3)^r x^{12-3r} \end{aligned}$$

We now let $12 - 3r = 3$
$\therefore\ 3r = 9$
$\therefore\ r = 3$

So, $T_4 = \underbrace{\binom{6}{3} 2^3 (-3)^3}\, x^3$

$\therefore$ the coefficient is $\binom{6}{3} 2^3 (-3)^3$.

c In $\left(2x^2 - 3y\right)^6$, $a = (2x^2)$, $b = (-3y)$, and $n = 6$

Now
$$\begin{aligned} T_{r+1} &= \binom{n}{r} a^{n-r} b^r \\ &= \binom{6}{r} \left(2x^2\right)^{6-r} (-3y)^r \\ &= \binom{6}{r} 2^{6-r} x^{12-2r} (-3)^r y^r \\ &= \binom{6}{r} 2^{6-r} (-3)^r x^{12-2r} y^r \end{aligned}$$

We find r such that
$12 - 2r = 6$ and $r = 3$
$\therefore\ r = 3$ is the solution

So, $T_4 = \underbrace{\binom{6}{3} 2^3 (-3)^3}\, x^6 y^3$

$\therefore$ the coefficient is $\binom{6}{3} 2^3 (-3)^3$.

d In $\left(2x^2 - \frac{1}{x}\right)^{12}$, $a = (2x^2)$, $b = \left(-\frac{1}{x}\right)$, and $n = 12$

Now
$$\begin{aligned} T_{r+1} &= \binom{n}{r} a^{n-r} b^r \\ &= \binom{12}{r} \left(2x^2\right)^{12-r} \left(-\frac{1}{x}\right)^r \\ &= \binom{12}{r} 2^{12-r} x^{24-2r} \frac{(-1)^r}{x^r} \\ &= \binom{12}{r} 2^{12-r} (-1)^r x^{24-3r} \end{aligned}$$

We now let $24 - 3r = 12$
$\therefore\ 3r = 12$
$\therefore\ r = 4$

So, $T_5 = \underbrace{\binom{12}{4} 2^8 (-1)^4}\, x^{12}$

$\therefore$ the coefficient is $\binom{12}{4} 2^8 (-1)^4$.

7 **a** $(x+2)(x^2+1)^8$

$$= (x+2)\left[(x^2)^8 + \binom{8}{1}(x^2)^7 + \binom{8}{2}(x^2)^6 + \dots + \binom{8}{6}(x^2)^2 + \binom{8}{7}(x^2)^1 + \binom{8}{8}\right]$$

only terms which when multiplied give an x^5

$\therefore$ coefficient of x^5 is $1 \times \binom{8}{6} = \binom{8}{6} = 28$.

b $(2-x)(3x+1)^9$

$$= (2-x)\left[(3x)^9 + \binom{9}{1}(3x)^8 + \binom{9}{2}(3x)^7 + \binom{9}{3}(3x)^6 + \binom{9}{4}(3x)^5 + \dots\right]$$

$\therefore$ term containing x^6 is $2 \times \binom{9}{3} 3^6 x^6 + (-x) \times \binom{9}{4} 3^5 x^5 = 2\binom{9}{3} 3^6 x^6 - \binom{9}{4} 3^5 x^6 = 91\,854 x^6$

8 In $(x^2y - 2y^2)^6$, $a = (x^2y)$, $b = (-2y^2)$, and $n = 6$.

Now
$$\begin{aligned} T_{r+1} &= \binom{n}{r} a^{n-r} b^r \\ &= \binom{6}{r} (x^2y)^{6-r} (-2y^2)^r \\ &= \binom{6}{r} x^{12-2r} y^{6-r} (-2)^r y^{2r} \\ &= \binom{6}{r} (-2)^r x^{12-2r} y^{6+r} \end{aligned}$$

Since x and y are raised to the same power,
$12 - 2r = 6 + r$
$\therefore\ 3r = 6$
$\therefore\ r = 2$

$T_3 = \binom{6}{2} (-2)^2 x^8 y^8 = 60 x^8 y^8$

9 **a** $(1+x)^n$ has $T_3 = \binom{n}{2}1^{n-2}x^2 = \binom{n}{2}x^2$ and $n \geqslant 2$

But this term is $36x^2$ $\quad \therefore \ \binom{n}{2} = 36$

$$\therefore \ \frac{n(n-1)}{2} = 36$$

$$\therefore \ n(n-1) = 72$$

$$\therefore \ n^2 - n - 72 = 0$$

$$\therefore \ (n-9)(n+8) = 0$$

$$\therefore \ n = 9 \text{ or } -8$$

But $n \geqslant 2$, so $n = 9$

and $T_4 = \binom{n}{3}1^{n-3}x^3$

$$\therefore \ T_4 = \binom{9}{3}x^3 = 84x^3$$

b $(1+kx)^n = 1^n + \binom{n}{1}1^{n-1}(kx)^1 + \binom{n}{2}1^{n-2}(kx)^2 + ...$

$$= 1 + \binom{n}{1}kx + \binom{n}{2}k^2x^2 + ...$$

$\therefore \ \binom{n}{1}k = -12$ and $\binom{n}{2}k^2 = 60$

$\therefore \ nk = -12$ and $\dfrac{n(n-1)}{2}k^2 = 60$

$$\therefore \ n(n-1)k^2 = 120$$

But $k = -\dfrac{12}{n}$ $\quad \therefore \ n(n-1)\dfrac{144}{n^2} = 120$

$$\therefore \ 144(n-1) = 120n \quad \{n \geqslant 2\}$$

$$\therefore \ 144n - 120n = 144$$

$$\therefore \ 24n = 144$$

$$\therefore \ n = 6 \quad \text{and so } k = -2$$

10 $T_{r+1} = \binom{n}{r}a^{n-r}b^r$ where $n = 10$, $a = (x^2)$, $b = \left(\dfrac{1}{ax}\right)$

$$= \binom{10}{r}(x^2)^{10-r}\left(\frac{1}{ax}\right)^r$$

$$= \binom{10}{r}x^{20-2r} \times \frac{1}{a^r x^r}$$

$$= \binom{10}{r}x^{20-3r} \times \frac{1}{a^r}$$

We let $20 - 3r = 11$

$$\therefore \ 3r = 9$$

$$\therefore \ r = 3$$

and $T_4 = \binom{10}{3}x^{11} \times \dfrac{1}{a^3}$

$$= \frac{\binom{10}{3}}{a^3}x^{11}$$

So, $\dfrac{\binom{10}{3}}{a^3} = 15$

$$\therefore \ \frac{120}{a^3} = 15$$

$$\therefore \ a^3 = 8$$

$$\therefore \ a = 2$$

11 $(1+2x-x^2)^5$

$= \left([1+2x] - x^2\right)^5$

$= (1+2x)^5 + 5(1+2x)^4(-x^2) + 10(1+2x)^3(-x^2)^2 + ...$

$\quad$ {all further terms contain higher powers of x than x^4}

$= 1^5 + 5(1^4)(2x) + 10(1^3)(2x)^2 + 10(1^2)(2x)^3 + 5(1)(2x)^4 + ...$

$\quad - 5x^2(1^4 + 4(1^3)(2x) + 6(1^2)(2x)^2 + ...) + 10x^4(1^3 + ...) + ...$

$= 1 + 10x + 40x^2 + 80x^3 + 80x^4 - 5x^2 - 40x^3 - 120x^4 + 10x^4 + ...$

$= 1 + 10x + 35x^2 + 40x^3 - 30x^4 + ...$

12 $\binom{n}{1} = \dfrac{n!}{1!(n-1)!} = \dfrac{n\cancel{(n-1)!}}{1\cancel{(n-1)!}} = \dfrac{n}{1} = n$

and $\binom{n}{2} = \dfrac{n!}{2!(n-2)!} = \dfrac{n(n-1)\cancel{(n-2)!}}{2\cancel{(n-2)!}} = \dfrac{n(n-1)}{2}$ for $n \in \mathbb{Z}^+,\ n \geqslant 2.$

13 **a** From **5 d**, $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \dots + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n$

Now, letting $x = -1$ gives $\text{LHS} = (1+(-1))^n = 0$

and $\text{RHS} = \binom{n}{0} + \binom{n}{1}(-1) + \binom{n}{2}(-1)^2 + \binom{n}{3}(-1)^3 + \dots$
$+ \binom{n}{n-1}(-1)^{n-1} + \binom{n}{n}(-1)^n$
$= \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \dots + (-1)^n\binom{n}{n}$

$\therefore\ \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \dots + (-1)^n\binom{n}{n} = 0$

b As $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \dots + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n,$

$(1+x)^{2n+1} = \binom{2n+1}{0} + \binom{2n+1}{1}x + \binom{2n+1}{2}x^2 + \dots + \binom{2n+1}{2n}x^{2n} + \binom{2n+1}{2n+1}x^{2n+1}$

Now letting $x = 1$ gives $\text{LHS} = 2^{2n+1} = 2^{2n} \times 2^1 = 4^n \times 2$

and $\text{RHS} = \binom{2n+1}{0} + \binom{2n+1}{1} + \binom{2n+1}{2} + \dots + \binom{2n+1}{2n} + \binom{2n+1}{2n+1}$
$= 2\left[\binom{2n+1}{0} + \binom{2n+1}{1} + \binom{2n+1}{2} + \dots + \binom{2n+1}{n}\right]$
$\left\{\binom{2n+1}{2n+1} = \binom{2n+1}{0},\ \binom{2n+1}{2n} = \binom{2n+1}{1},\ \dots,\ \binom{2n+1}{n+1} = \binom{2n+1}{n}\right\}$

$\therefore\ 2\left[\binom{2n+1}{0} + \binom{2n+1}{1} + \binom{2n+1}{2} + \dots + \binom{2n+1}{n}\right] = 4^n \times 2$

$\therefore\ \binom{2n+1}{0} + \binom{2n+1}{1} + \binom{2n+1}{2} + \dots + \binom{2n+1}{n} = 4^n$

14 $$\sum_{r=0}^{n} 2^r\binom{n}{r} = 2^0\binom{n}{0} + 2^1\binom{n}{1} + 2^2\binom{n}{2} + \dots + 2^{n-1}\binom{n}{n-1} + 2^n\binom{n}{n}$$

Using **5 d**, $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \dots + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n$

$\therefore$ letting $x = 2$, $(1+2)^n = \binom{n}{0} + \binom{n}{1}2 + \binom{n}{2}2^2 + \dots + \binom{n}{n-1}2^{n-1} + \binom{n}{n}2^n$

$\therefore\ 3^n = 2^0\binom{n}{0} + 2^1\binom{n}{1} + 2^2\binom{n}{2} + \dots + 2^{n-1}\binom{n}{n-1} + 2^n\binom{n}{n}$

$\therefore\ \displaystyle\sum_{r=0}^{n} 2^r\binom{n}{r} = 3^n$

15 For any polynomial $f(x)$, the sum of its coefficients is $f(1)$.

Let $f(x) = x^3 + 2x^2 + 3x - 7$

$\therefore$ the sum of the coefficients of $f(x) = f(1)$
$= 1^3 + 2(1)^2 + 3(1) - 7$
$= 1 + 2 + 3 - 7 = -1$

Now consider the function $g(x) = (x^3 + 2x^2 + 3x - 7)^{100}$
$= [f(x)]^{100}$

The sum of the coefficients of $g(x) = g(1)$
$= [f(1)]^{100}$
$= (-1)^{100} = 1$

$\therefore$ the sum of the coefficients of $(x^3 + 2x^2 + 3x - 7)^{100}$ is 1.

16 From **5 d**, $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + ... + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n$

$\therefore\ (1+x)^{2n} = \binom{2n}{0} + \binom{2n}{1}x + ... + \binom{2n}{n-1}x^{n-1} + \binom{2n}{n}x^n + ... + \binom{2n}{2n-1}x^{2n-1} + \binom{2n}{2n}x^{2n}$

Now $(1+x)^n(1+x)^n = (1+x)^{2n}$

$\therefore\ \left[\binom{n}{0} + \binom{n}{1}x + ... + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n\right]\left[\binom{n}{0} + \binom{n}{1}x + ... + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n\right]$

$= \binom{2n}{0} + \binom{2n}{1}x + ... + \binom{2n}{n-1}x^{n-1} + \binom{2n}{n}x^n + ... + \binom{2n}{2n-1}x^{2n-1} + \binom{2n}{2n}x^{2n}$

Equating coefficients of x^n,

$\binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + ... + \binom{n}{n-1}\binom{n}{1} + \binom{n}{n}\binom{n}{0} = \binom{2n}{n}$

But $\binom{n}{n} = \binom{n}{0}$, $\binom{n}{n-1} = \binom{n}{1}$, and so on.

$\therefore\ \binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + ... + \binom{n}{n-1}^2 + \binom{n}{n}^2 = \binom{2n}{n}$

17 **a** $(3+x)^n = 3^n + \binom{n}{1}3^{n-1}x + \binom{n}{2}3^{n-2}x^2 + \binom{n}{3}3^{n-3}x^3 + ... + \binom{n}{n-1}3x^{n-1} + x^n$

b Now letting $x=1$ in **a**,

$\text{LHS} = (3+1)^n = 4^n$

and $\text{RHS} = 3^n + \binom{n}{1}3^{n-1} + \binom{n}{2}3^{n-2} + \binom{n}{3}3^{n-3} + ... + 3n + 1$

$\therefore\ 3^n + \binom{n}{1}3^{n-1} + \binom{n}{2}3^{n-2} + \binom{n}{3}3^{n-3} + ... + 3n + 1 = 4^n.$

18 **a**
$$\begin{aligned} n\binom{n-1}{r-1} &= n\frac{(n-1)!}{(r-1)!(n-1-[r-1])!} \\ &= \frac{n\times(n-1)!}{(r-1)!(n-r)!} \\ &= r\times\frac{n!}{r!(n-r)!} \\ &= r\binom{n}{r} \end{aligned}$$

b
$$\begin{aligned} &\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + ... + n\binom{n}{n} \\ &= n\binom{n-1}{0} + n\binom{n-1}{1} + n\binom{n-1}{2} + ... + n\binom{n-1}{n-1} \quad \{\text{using } \mathbf{a}\} \\ &= n\left[\binom{n-1}{0} + \binom{n-1}{1} + \binom{n-1}{2} + ... + \binom{n-1}{n-1}\right] \\ &= n\,2^{n-1} \quad \{\text{second part of } \mathbf{5\,d}\} \end{aligned}$$

c **i**
$$\begin{aligned} \sum_{r=0}^{n} P_r &= P_0 + P_1 + P_2 + ... + P_n \\ &= \binom{n}{0}p^0(1-p)^n + \binom{n}{1}p^1(1-p)^{n-1} + \binom{n}{2}p^2(1-p)^{n-2} + ... + \binom{n}{n}p^n(1-p)^0 \\ &= (p+[1-p])^n \quad \{\text{binomial expansion}\} \\ &= 1^n = 1 \end{aligned}$$

ii
$$\begin{aligned} \sum_{r=1}^{n} r\,P_r &= 1P_1 + 2P_2 + 3P_3 + ... + nP_n \\ &= 1\binom{n}{1}p^1(1-p)^{n-1} + 2\binom{n}{2}p^2(1-p)^{n-2} + 3\binom{n}{3}p^3(1-p)^{n-3} + ... \\ &\qquad + n\binom{n}{n}p^n(1-p)^0 \\ &= n\binom{n-1}{0}p^1(1-p)^{n-1} + n\binom{n-1}{1}p^2(1-p)^{n-2} + n\binom{n-1}{2}p^3(1-p)^{n-3} + ... \\ &\qquad + n\binom{n-1}{n-1}p^n \quad \{\text{using } \mathbf{a}\} \\ &= np[\binom{n-1}{0}p^0(1-p)^{n-1} + \binom{n-1}{1}p^1(1-p)^{n-2} + \binom{n-1}{2}p^2(1-p)^{n-3} + ... \\ &\qquad + \binom{n-1}{n-1}p^{n-1}] \\ &= np\left[(p+(1-p))^{n-1}\right] \\ &= np\times 1^{n-1} \\ &= np \end{aligned}$$

REVIEW SET 8A

1 **a** $\dfrac{n!}{(n-2)!} = \dfrac{n(n-1)(n-2)!}{(n-2)!}$

$= n(n-1), \quad n \geqslant 2$

b $\dfrac{n! + (n+1)!}{n!} = \dfrac{n! + (n+1)n!}{n!}$

$= \dfrac{n!(1+n+1)}{n!}$

$= n+2$

2 There are $\binom{8}{2} = 28$ handshakes made.

3 **a**

4	3	2	1	1
				T

$\therefore\ 4 \times 3 \times 2 \times 1 \times 1 = 24$ arrangements end in T

b

1	3	2	1	1
P				T

$\therefore\ 1 \times 3 \times 2 \times 1 \times 1 = 6$ arrangements begin with P and end with T

4 **a**

9	10	10
cannot use 0		

So, $9 \times 10 \times 10 = 900$ three digit numbers can be formed.

b To be divisible by 5 the last digit must be 0 or 5.

either

9	10	1
		0

or

9	10	1
		5

$\therefore\ 9 \times 10 \times 1 + 9 \times 10 \times 1 = 180$ of them are divisible by 5.

5 **a** $(a+b)^4 = e^{4x} - 4e^{2x} + \ldots.$

and $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

$\therefore\ a^4 = e^{4x}$ and $4a^3b = -4e^{2x}$

$\therefore\ a^4 = (e^x)^4$ $\qquad \therefore\ 4e^{3x}b = -4e^{2x}$

$\therefore\ a = e^x$ $\qquad \therefore\ b = -e^{-x}$

b $(a+b)^4 = e^{4x} - 4e^{2x} + 6(e^x)^2(-e^{-x})^2 + 4e^x(-e^{-x})^3 + (-e^{-x})^4$

$= e^{4x} - 4e^{2x} + 6e^{2x}e^{-2x} - 4e^xe^{-3x} + e^{-4x}$

$= e^{4x} - 4e^{2x} + 6 - 4e^{-2x} + e^{-4x}$

6 $\left(\sqrt{3}+2\right)^5 = \left(\sqrt{3}\right)^5 + 5\left(\sqrt{3}\right)^4(2) + 10\left(\sqrt{3}\right)^3(2)^2 + 10\left(\sqrt{3}\right)^2(2)^3 + 5\left(\sqrt{3}\right)^1(2)^4 + 2^5$

$= 9\sqrt{3} + 90 + 120\sqrt{3} + 240 + 80\sqrt{3} + 32$

$= 362 + 209\sqrt{3}$

7 For $\left(3x^2 + \dfrac{1}{x}\right)^8$, $a = (3x^2)$, $b = \left(\dfrac{1}{x}\right)$, $n = 8$

$T_{r+1} = \binom{n}{r}a^{n-r}b^r$

$= \binom{8}{r}\left(3x^2\right)^{8-r}\left(\dfrac{1}{x}\right)^r$

$= \binom{8}{r}3^{8-r}x^{16-2r-r}$

$= \binom{8}{r}3^{8-r}x^{16-3r}$

Now a constant term does not contain x

$\therefore\ 16 - 3r = 0$

$\therefore\ 3r = 16$

$\therefore\ r = 5\frac{1}{3}$

which is impossible as r is in $\mathbb{Z}$

$\therefore$ no constant term exists.

8 $(1+cx)(1+x)^4 = (1+cx)(1^4 + \binom{4}{1}1^3x + \binom{4}{2}1^2x^2 + \binom{4}{3}1x^3 + x^4)$

$\therefore$ coefficient of x^3 is $1 \times \binom{4}{3} \times 1 + c \times \binom{4}{2} \times 1^2 = 4 + 6c$

But the coefficient of x^3 is 22, so $4 + 6c = 22$

$$\therefore \quad 6c = 18$$
$$\therefore \quad c = 3$$

9 **a** $(2+x)^n = 2^n + \binom{n}{1}2^{n-1}x^1 + \binom{n}{2}2^{n-2}x^2 + \binom{n}{3}2^{n-3}x^3 + ... + \binom{n}{n-1}2^1x^{n-1} + x^n$

b $2^n + \binom{n}{1}2^{n-1} + \binom{n}{2}2^{n-2} + \binom{n}{3}2^{n-3} + ... + 2n + 1$

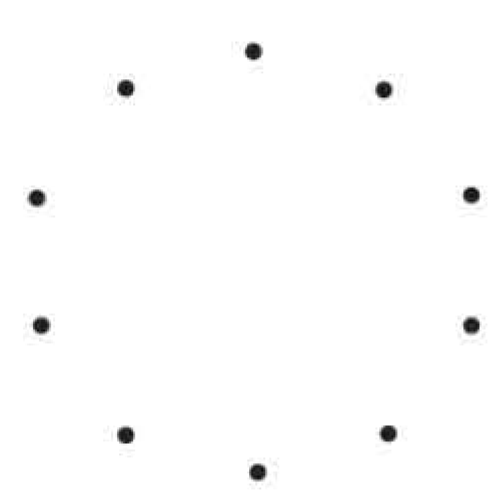
$= 2^n + \binom{n}{1}2^{n-1}x^1 + \binom{n}{2}2^{n-2}x^2 + \binom{n}{3}2^{n-3}x^3 + ... + \binom{n}{n-1}2^1x^{n-1} + x^n$, where $x = 1$

$= (2+1)^n$

$= 3^n$

REVIEW SET 8B

1



a To form a line we need to select any two points from the 10.

$\therefore$ total is $\binom{10}{2} = 45$ lines.

b To form a triangle we need to select any three points from the 10.

$\therefore$ total is $\binom{10}{3} = 120$ triangles.

2 $(4+x)^3 = 4^3 + 3(4)^2x^1 + 3(4)^1x^2 + x^3$
$= 64 + 48x + 12x^2 + x^3$

Letting $x = 0.02$ gives $(4.02)^3 = 64 + 48(0.02) + 12(0.02)^2 + (0.02)^3$
$= 64 + 0.96 + 0.0048 + 0.000\,008$
$= 64.964\,808$

3
$$T_{r+1} = \binom{6}{r}(3x)^{6-r}\left(\frac{-2}{x^2}\right)^r$$
$$= \binom{6}{r}3^{6-r}x^{6-r}(-2)^r x^{-2r}$$
$$= \binom{6}{r}3^{6-r}x^{6-3r}(-2)^r$$

If we let $6 - 3r = 0$ then $r = 2$

$\therefore \quad T_3 = \underbrace{\binom{6}{2}3^4(-2)^2}\,x^0$

$\therefore$ constant term $= \binom{6}{2}3^4(-2)^2 = 4860$

4 In the expansion of $(x+5)^6$, $a = x$, $b = 5$, $n = 6$

$T_{r+1} = \binom{n}{r}a^{n-r}b^r$
$= \binom{6}{r}x^{6-r}5^r$

For the coefficient of x^3 we let $6 - r = 3$
$\therefore \quad r = 3$

and $T_4 = \underbrace{\binom{6}{3}5^3}\,x^3$

$\therefore$ the coefficient is $\binom{6}{3}5^3 = 2500$.

5 **a** With no restrictions there are $\binom{10}{5} = 252$ different teams.

b Those consisting of at least one of each sex

$= \binom{10}{5} - \binom{6}{5}\binom{4}{0}$ {there are no teams consisting of 5 women}

$= 246$

6 **a**

9	9	8	7

cannot use 0

So, $9 \times 9 \times 8 \times 7 = 4536$ numbers are possible.

b Either they end in 0 or in 5

9	8	7	1

0

or

8	8	7	1

cannot use 0 5

So, $9 \times 8 \times 7 \times 1 + 8 \times 8 \times 7 \times 1$
$= 952$ numbers are possible.

7 The sixth row of Pascal's triangle is 1 6 15 20 15 6 1

$\therefore \quad (a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$

a
$$\begin{aligned}(x-3)^6 &= x^6 + 6x^5(-3) + 15x^4(-3)^2 + 20x^3(-3)^3 + 15x^2(-3)^4 + 6x(-3)^5 + (-3)^6 \\ &= x^6 - 18x^5 + 135x^4 - 540x^3 + 1215x^2 - 1458x + 729\end{aligned}$$

b
$$\begin{aligned}&\left(1+\frac{1}{x}\right)^6 \\ &= (1)^6 + 6(1)^5\left(\frac{1}{x}\right) + 15(1)^4\left(\frac{1}{x}\right)^2 + 20(1)^3\left(\frac{1}{x}\right)^3 + 15(1)^2\left(\frac{1}{x}\right)^4 + 6(1)\left(\frac{1}{x}\right)^5 + \left(\frac{1}{x}\right)^6 \\ &= 1 + \frac{6}{x} + \frac{15}{x^2} + \frac{20}{x^3} + \frac{15}{x^4} + \frac{6}{x^5} + \frac{1}{x^6}\end{aligned}$$

8 In $\left(2x - \dfrac{3}{x^2}\right)^{12}$, $a = (2x)$, $b = \left(-\dfrac{3}{x^2}\right)$, $n = 12$

$$\begin{aligned}T_{r+1} &= \binom{n}{r}a^{n-r}b^r \\ &= \binom{12}{r}(2x)^{12-r}\left(-\frac{3}{x^2}\right)^r \\ &= \binom{12}{r}2^{12-r}x^{12-r}\frac{(-3)^r}{x^{2r}} \\ &= \binom{12}{r}2^{12-r}(-3)^r x^{12-3r}\end{aligned}$$

For the coefficient of x^{-6} we let $12 - 3r = -6$

$\therefore \quad 3r = 18$

$\therefore \quad r = 6$

So, $T_7 = \underbrace{\binom{12}{6}2^6(-3)^6}\, x^{-6}$

$\therefore$ the coefficient is $\binom{12}{6}2^6(-3)^6 = 43\,110\,144.$

9
$$\begin{aligned}&(2x+3)(x-2)^6 \\ &= (2x+3)\left[x^6 + \binom{6}{1}x^5(-2) + \binom{6}{2}x^4(-2)^2 + ...\right]\end{aligned}$$

$\therefore$ coefficient of x^5 is $2 \times \binom{6}{2} \times (-2)^2 + 3 \times \binom{6}{1} \times (-2) = 8\binom{6}{2} - 6\binom{6}{1} = 84$

10
$$\begin{aligned}T_{r+1} &= \binom{9}{r}(2x)^{9-r}\left(\frac{1}{ax^2}\right)^r \\ &= \binom{9}{r}2^{9-r}x^{9-r} \times \frac{1}{a^r x^{2r}} \\ &= \binom{9}{r}2^{9-r}a^{-r}x^{9-3r}\end{aligned}$$

Letting $r = 2$, $T_3 = \underbrace{\binom{9}{2}2^7a^{-2}}\, x^3$

$\therefore \quad \dfrac{\binom{9}{2}2^7}{a^2} = 288$

$\therefore \quad a^2 = \dfrac{\binom{9}{2}2^7}{288} = 16$

$\therefore \quad a = \pm 4$

REVIEW SET 8C

1 **a**

26	26	10	10	10	10

$\therefore$ there are $26^2 \times 10^4 = 6\,760\,000$ if there are no restrictions

b

5	26	10	10	10	10

a vowel

$\therefore$ there are $5 \times 26 \times 10^4 = 1\,300\,000$ possibilities if the first letter is a vowel

c

26	25	10	9	8	7

$\therefore$ there are $26 \times 25 \times 10 \times 9 \times 8 \times 7 = 3\,276\,000$, if there are no repetitions

2 **a** Total number $= \binom{8+7}{5} = \binom{15}{5} = 3003$ committees

b Total with 2 men and 3 women $= \binom{8}{2}\binom{7}{3} = 980$ committees

c Total with at least one man = total unrestricted − total with all women

$= 3003 - \binom{8}{0}\binom{7}{5} = 2982$ committees

3 **a** $(x-2y)^3 = x^3 + 3x^2(-2y) + 3x(-2y)^2 + (-2y)^3$
$= x^3 - 6x^2y + 12xy^2 - 8y^3$

b $(3x+2)^4 = (3x)^4 + 4(3x)^3(2) + 6(3x)^2(2)^2 + 4(3x)(2)^3 + (2)^4$
$= 81x^4 + 216x^3 + 216x^2 + 96x + 16$

4 In the expansion of $(2x+5)^6$, $a = (2x)$, $b = 5$, $n = 6$

$$\begin{aligned} T_{r+1} &= \binom{n}{r}a^{n-r}b^r \\ &= \binom{6}{r}(2x)^{6-r}5^r \\ &= \binom{6}{r}2^{6-r}x^{6-r}5^r \end{aligned}$$

For the coefficient of x^3 we let $6 - r = 3$
$\therefore\ r = 3$

and $T_4 = \underbrace{\binom{6}{3}2^3 5^3}\, x^3$

$\therefore$ the coefficient is $\binom{6}{3}2^3 5^3 = 20\,000$.

5 In the expansion of $\left(2x^2 - \frac{1}{x}\right)^6$, $a = 2x^2$, $b = -\frac{1}{x}$, $n = 6$

$$\begin{aligned} T_{r+1} &= \binom{n}{r}a^{n-r}b^r \\ &= \binom{6}{r}(2x^2)^{6-r}\left(-\frac{1}{x}\right)^r \\ &= \binom{6}{r}2^{6-r}x^{12-2r}(-1)^r x^{-r} \\ &= \binom{6}{r}2^{6-r}(-1)^r x^{12-3r} \end{aligned}$$

For the constant term we let $12 - 2r = 0$
$\therefore\ r = 4$

and $T_5 = \underbrace{\binom{6}{4}2^2(-1)^4}\, x^0$

$\therefore$ the constant term is $\binom{6}{4}2^2(-1)^4 = 60$.

6 $(1+kx)^n = 1^n + \binom{n}{1}1^{n-1}(kx)^1 + \binom{n}{2}1^{n-2}(kx)^2 + \ldots.$
$= 1 + nkx + \binom{n}{2}k^2x^2 + \ldots.$

$\therefore\ nk = -4$ and $\binom{n}{2}k^2 = \frac{15}{2}$

$\therefore\ k = -\frac{4}{n}$ and $\frac{n(n-1)}{2}k^2 = \frac{15}{2}$

$\therefore\ n(n-1)k^2 = 15$

Using $k = -\frac{4}{n}$, $n(n-1)\left(\frac{16}{n^2}\right) = 15$

$\therefore\ 16(n-1) = 15n \quad \{n \geqslant 2\}$

$\therefore\ 16n - 16 = 15n$

$\therefore\ n = 16$ and so $k = \frac{-4}{16} = -\frac{1}{4}$

7 The three sisters can sit together in 3! ways. They as a group, plus the other 5 people, make 6 items which can be permutated in 6! ways.

$\therefore$ total number of orderings $= 3! \times 6! = 4320$

8 **a** If no restrictions there are $\binom{18}{8} = 43\,758$ different teams possible.

b If 4 of each sex are needed there are $\binom{11}{4} \times \binom{7}{4} = 11\,550$ different teams.

c If at least 2 women are needed, total number $= \binom{18}{8} - \binom{11}{8}\binom{7}{0} - \binom{11}{7}\binom{7}{1}$

$= 41\,283$ different teams

9 $(m-2n)^{10} = m^{10} + \binom{10}{1}m^9(-2n) + \binom{10}{2}m^8(-2n)^2 + ... + (-2n)^{10}$

$= m^{10} - 20m^9n + 45m^8(4n^2) + ... + 1024n^{10}$

$= m^{10} - 20m^9n + 180m^8n^2 + ... + 1024n^{10}$

$\therefore\ k = 180$

10 $\left(x^3+\dfrac{q}{x^3}\right)^8$ has $T_{r+1} = \binom{8}{r}(x^3)^{8-r}\left(\dfrac{q}{x^3}\right)^r$

$= \binom{8}{r}x^{24-3r}\dfrac{q^r}{x^{3r}}$

$= \binom{8}{r}x^{24-6r}q^r$

which has constant term $\binom{8}{4}q^4$ $\{24-6r=0$ when $r=4\}$

$\left(x^3+\dfrac{q}{x^3}\right)^4$ has $T_{r+1} = \binom{4}{r}(x^3)^{4-r}\left(\dfrac{q}{x^3}\right)^r$

$= \binom{4}{r}x^{12-3r}q^r x^{-3r}$

$= \binom{4}{r}x^{12-6r}q^r$

which has constant term $\binom{4}{2}q^2$ $\{12-6r=0$ when $r=2\}$

$\therefore\ \binom{8}{4}q^4 = \binom{4}{2}q^2$

$\therefore\ 70q^4 - 6q^2 = 0$

$\therefore\ q^2(70q^2-6) = 0$

$\therefore\ 70q^2 - 6 = 0$ $\{q=0$ gives a trivial solution$\}$

$\therefore\ q^2 = \frac{6}{70} = \frac{3}{35}$

$\therefore\ q = \pm\sqrt{\frac{3}{35}}$

Chapter 9

MATHEMATICAL INDUCTION

EXERCISE 9A

1 The nth term of the sequence $3, 7, 11, 15, 19, ...$ is $4n-1$ for $n \in \mathbb{Z}^+$.

2 **a**
$3^1 = 3 \qquad 1+2(1) = 3$
$3^2 = 9 \qquad 1+2(2) = 5$
$3^3 = 27 \qquad 1+2(3) = 7$
$3^4 = 81 \qquad 1+2(4) = 9$

Our proposition is:
$3^n > 1+2n$ for $n = 2, 3, 4, 5, ...$
or for all $n \in \mathbb{Z}^+$, $n \geqslant 2$

b
$11^1 - 1 = 10$
$11^2 - 1 = 121 - 1 = 120$
$11^3 - 1 = 1331 - 1 = 1330$
$11^4 - 1 = 14\,641 - 1 = 14\,640$

Our proposition is:
$11^n - 1$ is divisible by 10 for all $n \in \mathbb{Z}^+$

c
$7^1 + 2 = 7 + 2 = 9 = 3 \times 3$
$7^2 + 2 = 49 + 2 = 51 = 3 \times 17$
$7^3 + 2 = 343 + 2 = 345 = 3 \times 115$
$7^4 + 2 = 2401 + 2 = 2403 = 3 \times 801$

Our proposition is:
$7^n + 2$ is divisible by 3 for all $n \in \mathbb{Z}^+$

d
$(1 - \frac{1}{2}) = \frac{1}{2}$
$(1 - \frac{1}{2})(1 - \frac{1}{3}) = \frac{1}{2} \times \frac{2}{3} = \frac{1}{3}$
$(1 - \frac{1}{2})(1 - \frac{1}{3})(1 - \frac{1}{4}) = \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} = \frac{1}{4}$
$(1 - \frac{1}{2})(1 - \frac{1}{3})(1 - \frac{1}{4})(1 - \frac{1}{5}) = \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} \times \frac{4}{5} = \frac{1}{5}$

Our proposition is: $(1 - \frac{1}{2})(1 - \frac{1}{3})(1 - \frac{1}{4}) .... \left(1 - \dfrac{1}{n+1}\right) = \dfrac{1}{n+1}$ for all $n \in \mathbb{Z}^+$

3 **a**
$2 = 2 = 1 \times 2$
$2 + 4 = 6 = 2 \times 3$
$2 + 4 + 6 = 12 = 3 \times 4$
$2 + 4 + 6 + 8 = 20 = 4 \times 5$
$2 + 4 + 6 + 8 + 10 = 30 = 5 \times 6$

Our proposition is:
$2 + 4 + 6 + 8 + 10 + ... + \underbrace{2n}_{n\text{th term}} = n(n+1)$ for all $n \in \mathbb{Z}^+$

$\therefore \displaystyle\sum_{i=1}^{n} 2i = n(n+1)$ for all $n \in \mathbb{Z}^+$

b
$1! = 1$
$1! + 2 \times 2! = 1 + 2(2) = 5$
$1! + 2 \times 2! + 3 \times 3! = 1 + 4 + 18 = 23$
$1! + 2 \times 2! + 3 \times 3! + 4 \times 4! = 1 + 4 + 18 + 96 = 119$

where each number result is 1 less than a factorial number

$1 = 2! - 1$
$5 = 3! - 1$
$23 = 4! - 1$
$119 = 5! - 1$

Our proposition is:
$1! + 2 \times 2! + 3 \times 3! + 4 \times 4! + ... + n \times n! = (n+1)! - 1$
for all $n \in \mathbb{Z}^+$

$\therefore \displaystyle\sum_{i=1}^{n} i \times i! = (n+1)! - 1$ for all $n \in \mathbb{Z}^+$

c
$$\frac{1}{2!} = \frac{1}{2} = \frac{2! - 1}{2!}$$
$$\frac{1}{2!} + \frac{2}{3!} = \frac{1}{2} + \frac{2}{6} = \frac{5}{6} = \frac{3! - 1}{3!}$$
$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} = \frac{1}{2} + \frac{2}{6} + \frac{3}{24} = \frac{23}{24} = \frac{4! - 1}{4!}$$
$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \frac{4}{5!} = \frac{1}{2} + \frac{2}{6} + \frac{3}{24} + \frac{4}{120} = \frac{119}{120} = \frac{5! - 1}{5!}$$

Our proposition is: $\dfrac{1}{2!}+\dfrac{2}{3!}+\dfrac{3}{4!}+\dfrac{4}{5!}+\ldots.+\dfrac{n}{(n+1)!}=\dfrac{(n+1)!-1}{(n+1)!}$ for all $n\in\mathbb{Z}^+$

$$\therefore\ \sum_{i=1}^{n}\frac{i}{(i+1)!}=\frac{(n+1)!-1}{(n+1)!}\quad\text{for all } n\in\mathbb{Z}^+$$

d $\dfrac{1}{2\times5}=\dfrac{1}{10}$

$\dfrac{1}{2\times5}+\dfrac{1}{5\times8}=\dfrac{1}{10}+\dfrac{1}{40}=\dfrac{5}{40}=\dfrac{1}{8}=\dfrac{2}{16}$

$\dfrac{1}{2\times5}+\dfrac{1}{5\times8}+\dfrac{1}{8\times11}=\dfrac{1}{10}+\dfrac{1}{40}+\dfrac{1}{88}=\dfrac{3}{22}$

$\dfrac{1}{2\times5}+\dfrac{1}{5\times8}+\dfrac{1}{8\times11}+\dfrac{1}{11\times14}=\dfrac{1}{7}=\dfrac{4}{28}$

10, 16, 22, 28 is arithmetic with $u_1=10,\ d=6$

$$\begin{aligned}u_n&=u_1+(n-1)d\\&=10+6(n-1)\\&=6n+4\end{aligned}$$

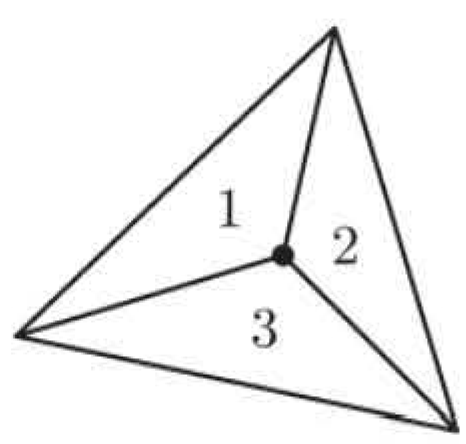
Our proposition is:

$\dfrac{1}{2\times5}+\dfrac{1}{5\times8}+\dfrac{1}{8\times11}+\dfrac{1}{11\times14}+\ldots.+\dfrac{1}{(3n-1)(3n+2)}=\dfrac{n}{6n+4}$ for all $n\in\mathbb{Z}^+$

{2, 5, 8, 11 are arithmetic with $u_1=2,\ d=3\ \therefore\ u_n=2+(n-1)3=3n-1$}

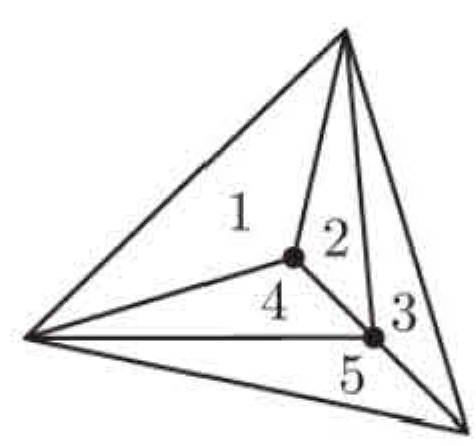
$$\therefore\ \sum_{i=1}^{n}\frac{1}{(3i-1)(3i+2)}=\frac{n}{6n+4}\quad\text{for all } n\in\mathbb{Z}^+$$

4 For $n=1$



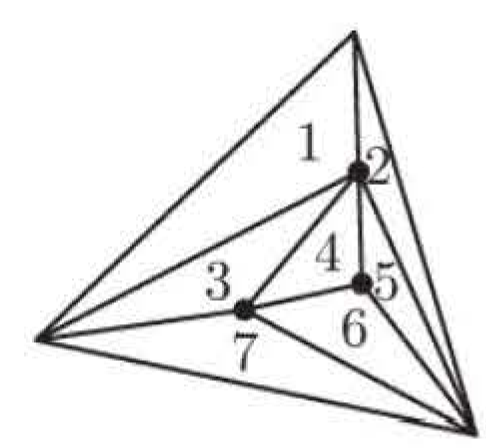
$T_1=3=2\times1+1$

For $n=2$



$T_2=5=2\times2+1$

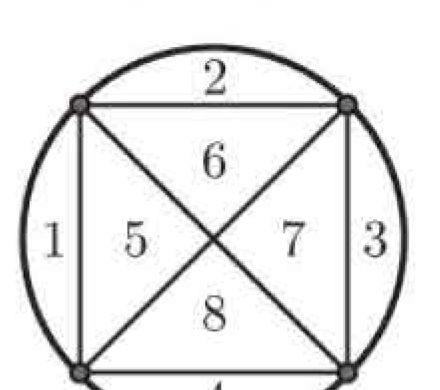
For $n=3$



$T_3=7=2\times3+1$

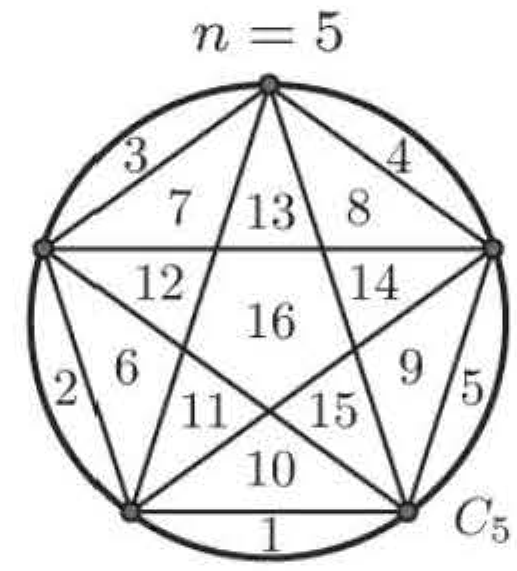
Our proposition is: The maximum number of triangles for n points within the original triangle is given by $T_n=2n+1$ for all $n\in\mathbb{Z}^+$.

5 **a** $n=4$



$C_4=8$

$n=5$



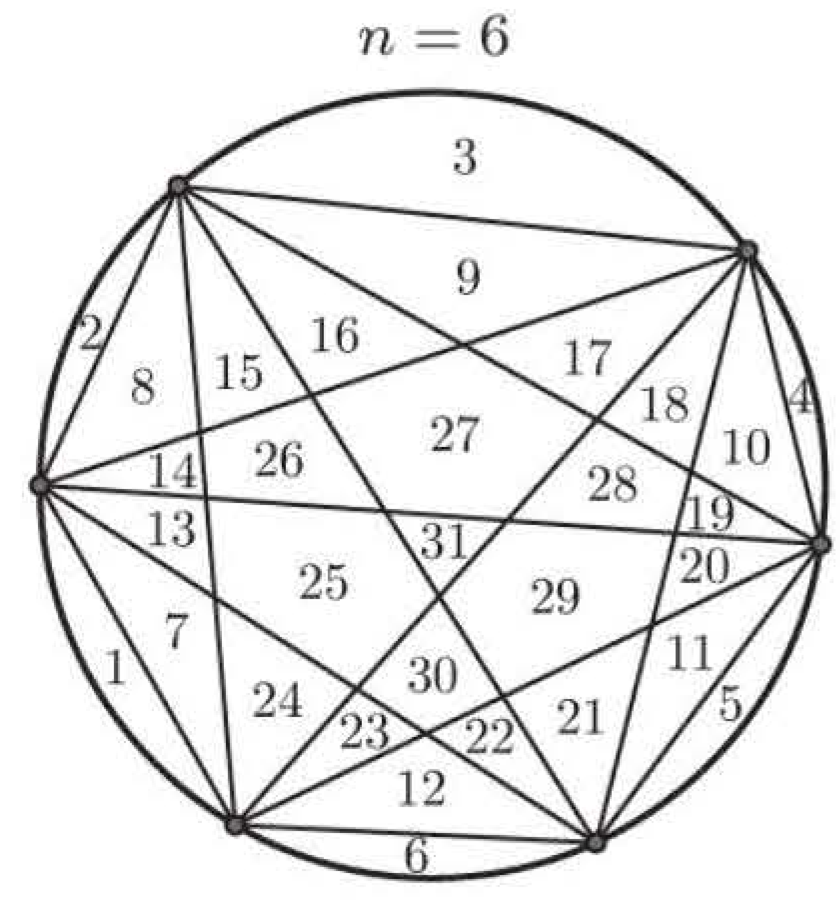
$C_5=16$

b When $n=1,\ C_1=1=2^0=2^{1-1}$
$n=2,\ C_2=2=2^1=2^{2-1}$
$n=3,\ C_3=4=2^2=2^{3-1}$
$n=4,\ C_4=8=2^3=2^{4-1}$
$n=5,\ C_5=16=2^4=2^{5-1}$

So, from the cases $n=1, 2, 3, 4, 5$, our conjecture is:

The number of regions for n points placed around a circle is given by $C_n=2^{n-1}$ for all $n\in\mathbb{Z}^+$.

c $n=6$



By the conjecture we expect $2^{6-1}=2^5=32$ regions, but there are only 31.

So, we no longer believe the conjecture.

EXERCISE 9B.1

1 **a** If $n=0$, $3^n+1=3^0+1=2$ which is divisible by 2.

$$\begin{aligned}3^n+1 &= (1+2)^n+1\\ &= 1^n+\binom{n}{1}2+\binom{n}{2}2^2+\binom{n}{3}2^3+....+\binom{n}{n-1}2^{n-1}+\binom{n}{n}2^n+1\\ &= 2+\binom{n}{1}2+\binom{n}{2}2^2+\binom{n}{3}2^3+....+\binom{n}{n-1}2^{n-1}+\binom{n}{n}2^n\\ &= 2\left(1+\binom{n}{1}+\binom{n}{2}2+\binom{n}{3}2^2+....+\binom{n}{n-1}2^{n-2}+\binom{n}{n}2^{n-1}\right)\end{aligned}$$

where the contents of the brackets is an integer.

$\therefore$ 3^n+1 is divisible by 2.

b P_n is: 3^n+1 is divisible by 2 for all integers $n \geqslant 0$.

Proof: (By the principle of mathematical induction)

(1) If $n=0$, $3^0+1=2=1\times 2$ $\therefore$ P_0 is true.

(2) If P_k is true, then $3^k+1=2A$ where A is an integer, and $A \geqslant 1$.

Now $$\begin{aligned}3^{k+1}+1 &= 3^1 3^k+1\\ &= 3(2A-1)+1 \quad \{\text{using } P_k\}\\ &= 6A-3+1\\ &= 6A-2\\ &= 2(3A-1) \quad \text{where } 3A-1 \text{ is an integer as } A\in\mathbb{Z}\end{aligned}$$

Thus $3^{k+1}+1$ is divisible by 2 if 3^k+1 is divisible by 2.

Since P_0 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all integers $n \geqslant 0$ {Principle of mathematical induction}

2 **a** If $n=0$, $6^n-1=6^0-1=0$ which is divisible by 5.

$$\begin{aligned}6^n-1 &= (5+1)^n-1\\ &= 5^n+\binom{n}{1}5^{n-1}+\binom{n}{2}5^{n-2}+\binom{n}{3}5^{n-3}+....+\binom{n}{n-1}5+1^n-1\\ &= 5^n+\binom{n}{1}5^{n-1}+\binom{n}{2}5^{n-2}+\binom{n}{3}5^{n-3}+....+\binom{n}{n-1}5\\ &= 5\left(5^{n-1}+\binom{n}{1}5^{n-2}+\binom{n}{2}5^{n-3}+\binom{n}{3}5^{n-4}+....+\binom{n}{n-1}\right)\end{aligned}$$

where the contents of the brackets is an integer.

$\therefore$ 6^n-1 is divisible by 5.

b P_n is: 6^n-1 is divisible by 5 for all integers $n \geqslant 0$

Proof: (By the principle of mathematical induction)

(1) If $n=0$, $6^0-1=0$ which is divisible by 5 $\therefore$ P_0 is true

(2) If P_k is true, then $6^k-1=5A$ where $A\in\mathbb{N}$

Now $6^{k+1}-1$
$$\begin{aligned}&= 6^1 6^k-1\\ &= 6(5A+1)-1 \quad \{\text{using } P_k\}\\ &= 30A+6-1\\ &= 30A+5\\ &= 5(6A+1) \quad \text{where } 6A+1 \text{ is an integer as } A\in\mathbb{N}\end{aligned}$$

Thus, $6^{k+1}-1$ is divisible by 5 if 6^k-1 is divisible by 5.

Since P_0 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all integers $n \geqslant 0$ {Principle of mathematical induction}

3 **a** P_n is: $n^3 + 2n$ is divisible by 3 for all positive integers n

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, $1^3 + 2(1) = 3$ which is divisible by 3

(2) If P_k is true, then $k^3 + 2k = 3A$ where $A \in \mathbb{Z}$

Now $(k+1)^3 + 2(k+1)$

$= k^3 + 3k^2 + 3k + 1 + 2k + 2$

$= (k^3 + 2k) + 3k^2 + 3k + 3$

$= 3A + 3k^2 + 3k + 3$ {using P_k}

$= 3(A + k^2 + k + 1)$ where $A + k^2 + k + 1$ is an integer as A and k are integers

Thus $(k+1)^3 + 2(k+1)$ is divisible by 3 if $k^3 + 2k$ is divisible by 3.

Since P_1 is true, and P_{k+1} is true whenever P_k is true, then P_n is true for all positive integers n {Principle of mathematical induction}

b P_n is: $n(n^2 + 5)$ is divisible by 6 for all integers $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, $1(1^2 + 5) = 1 \times 6 = 6$ which is divisible by 6 $\therefore$ P_1 is true

(2) If P_k is true, then $k(k^2 + 5) = 6A$ where A is an integer

Now $(k+1)[(k+1)^2 + 5] = (k+1)(k^2 + 2k + 1 + 5)$

$= (k+1)(k^2 + 2k + 6)$

$= k^3 + 2k^2 + 6k + k^2 + 2k + 6$

$= k^3 + 5k + [3k^2 + 3k + 6]$

$= k(k^2 + 5) + 3k(k+1) + 6$

$= 6A + 6 + 3k(k+1)$

We notice that $k(k+1)$ is the product of consecutive integers, one of which must be even $\therefore$ $k(k+1) = 2B$ where $B \in \mathbb{Z}$

$\therefore$ $(k+1)[(k+1)^2 + 5] = 6A + 6 + 3(2B)$

$= 6(A + 1 + B)$ where $A + 1 + B \in \mathbb{Z}$

Thus $(k+1)[(k+1)^2 + 5]$ is divisible by 6 if $k(k^2 + 5)$ is divisible by 6.

Since P_1 is true, and P_{k+1} is true whenever P_k is true, then P_n is true for all integers $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

c P_n is: $7^n - 4^n - 3^n$ is divisible by 12 for all $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, $7^1 - 4^1 - 3^1 = 0$ which is divisible by 12 $\therefore$ P_1 is true

(2) If P_k is true, then $7^k - 4^k - 3^k = 12A$ where $A \in \mathbb{Z}$

Now $7^{k+1} - 4^{k+1} - 3^{k+1}$

$= 7(7^k) - 4(4^k) - 3(3^k)$

$= 7[12A + 4^k + 3^k] - 4(4^k) - 3(3^k)$ {using P_k}

$= 84A + 7(4^k) + 7(3^k) - 4(4^k) - 3(3^k)$

$= 84A + 3(4^k) + 4(3^k)$

$= 84A + 3 \times 4 \times 4^{k-1} + 4 \times 3 \times 3^{k-1}$

$= 12(7A + 4^{k-1} + 3^{k-1})$ where $k \geqslant 2$, $k \in \mathbb{Z}^+$

$= 12 \times$ an integer {as 4^{k-1} and 3^{k-1} are integers}

Thus $7^{k+1} - 4^{k+1} - 3^{k+1}$ is divisible by 12 if $7^k - 4^k - 3^k$ is divisible by 12.

Since P_1 is true, and P_{k+1} is true whenever P_k is true, then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

EXERCISE 9B.2

1 **a** P_n is: $1+2+3+4+....+n=\dfrac{n(n+1)}{2}$ for all $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n=1$, LHS $=1$ and RHS $=\dfrac{1(2)}{2}=1$, $\therefore$ P_1 is true

(2) If P_k is true then $1+2+3+4+....+k=\dfrac{k(k+1)}{2}$

Thus $1+2+3+4+....+k+(k+1)$

$$=\frac{k(k+1)}{2}+k+1 \qquad \{\text{using } P_k\}$$

$$=\frac{k(k+1)}{2}+\frac{2(k+1)}{2} \qquad \{\text{to equalise denominators}\}$$

$$=\frac{(k+1)(k+2)}{2} \qquad \{\text{common factor of } \frac{(k+1)}{2}\}$$

$$=\frac{(k+1)([k+1]+1)}{2}$$

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

b P_n is: $1\times 2+2\times 3+3\times 4+....+n(n+1)=\dfrac{n(n+1)(n+2)}{3}$ for all $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n=1$, LHS $=1\times 2=2$ and RHS $=\dfrac{1(2)(3)}{3}=2$, $\therefore$ P_1 is true

(2) If P_k is true then

$$1\times 2+2\times 3+3\times 4+....+k(k+1)=\frac{k(k+1)(k+2)}{3}$$

$\therefore$ $1\times 2+2\times 3+3\times 4+....+k(k+1)+(k+1)(k+2)$

$$=\frac{k(k+1)(k+2)}{3}+(k+1)(k+2) \qquad \{\text{using } P_k\}$$

$$=\frac{k(k+1)(k+2)}{3}+\frac{3(k+1)(k+2)}{3} \qquad \{\text{to equalise denominators}\}$$

$$=\frac{(k+1)(k+2)(k+3)}{3} \qquad \{\text{common factor of } (k+1)(k+2)\}$$

$$=\frac{[k+1]([k+1]+1)([k+1]+2)}{3}$$

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

c P_n is: $3\times 5+6\times 6+9\times 7+12\times 8+....+3n(n+4)=\dfrac{n(n+1)(2n+13)}{2}$
for all $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n=1$, LHS $=3\times 5=15$, RHS $=\dfrac{1\times 2\times(2+13)}{2}=15$, $\therefore$ P_1 is true

(2) If P_k is true, then

$$3\times 5+6\times 6+9\times 7+....+3k(k+4)=\frac{k(k+1)(2k+13)}{2}$$

Now $3 \times 5 + 6 \times 6 + 9 \times 7 + ... + 3k(k+4) + 3(k+1)(k+5)$

$$= \frac{k(k+1)(2k+13)}{2} + 3(k+1)(k+5) \quad \{\text{using } P_k\}$$

$$= \frac{k(k+1)(2k+13)}{2} + \frac{6(k+1)(k+5)}{2} \quad \{\text{to equalise denominators}\}$$

$$= \frac{(k+1)[k(2k+13)+6(k+5)]}{2} \quad \{\text{common factor}\}$$

$$= \frac{(k+1)[2k^2+19k+30]}{2}$$

$$= \frac{(k+1)(k+2)(2k+15)}{2}$$

$$= \frac{(k+1)([k+1]+1)(2[k+1]+13)}{2}$$

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

d P_n is: $1^3 + 2^3 + 3^3 + 4^3 + ... + n^3 = \dfrac{n^2(n+1)^2}{4}$ for all $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, LHS $= 1^3 = 1$, RHS $= \dfrac{1^2(2)^2}{4} = 1$ $\therefore$ P_1 is true

(2) If P_k is true, then $1^3 + 2^3 + 3^3 + ... + k^3 = \dfrac{k^2(k+1)^2}{4}$

Now $1^3 + 2^3 + 3^3 + ... + k^3 + (k+1)^3$

$$= \frac{k^2(k+1)^2}{4} + (k+1)^3 \quad \{\text{using } P_k\}$$

$$= \frac{k^2(k+1)^2}{4} + \frac{4(k+1)^3}{4} \quad \{\text{equalising denominators}\}$$

$$= \frac{(k+1)^2[k^2+4(k+1)]}{4} \quad \{\text{common factor}\}$$

$$= \frac{(k+1)^2(k^2+4k+4)}{4}$$

$$= \frac{(k+1)^2(k+2)^2}{4}$$

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

2 **a** The sum of the first n odd numbers $1 + 3 + 5 + 7 + ... + 2n - 1$ is the sum of the first n terms of an arithmetic series.

$u_1 = 1$, $d = 2$. $\therefore$ $u_n = u_1 + (n-1)d = 1 + 2(n-1) = 2n - 1$.

Thus $S_n = \dfrac{n}{2}(2u_1 + (n-1)d)$

$$= \frac{n}{2}(2 \times 1 + 2(n-1))$$

$$= \frac{n}{2}(2 + 2n - 2)$$

$$= \frac{n}{2}(2n)$$

$$= n^2$$

So, the sum of the first n odd numbers is n^2.

b P_n is: $1+3+5+7+ .... +(2n-1) = n^2$ for all $n \in \mathbb{Z}^+$.

Proof: (By the principle of mathematical induction)

(1) If $n=1$, LHS $=1$, RHS $=1^2=1$ $\therefore$ P_1 is true.

(2) If P_k is true, then $1+3+5+7+ .... +(2k-1) = k^2$

Now $1+3+5+7+ .... +(2k-1)+[2(k+1)-1]$

$= k^2+[2(k+1)-1]$ {using P_k}

$= k^2+2k+2-1$

$= k^2+2k+1$

$= (k+1)^2$

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

3 P_n is: $1\times 2^0 + 2\times 2 + 3\times 2^2 + 4\times 2^3 + ... + n\times 2^{n-1} = (n-1)\times 2^n + 1$ for all $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n=1$, LHS $=1$, RHS $=(0)2^0+1=1$, $\therefore$ P_1 is true

(2) If P_k is true, then $1\times 2^0 + 2\times 2 + 3\times 2^2 + 4\times 2^3 + ... + k\times 2^{k-1} = (k-1)2^k+1$

Now $1\times 2^0 + 2\times 2 + 3\times 2^2 + 4\times 2^3 + ... + k\,2^{k-1} + (k+1)2^k$

$= (k-1)2^k + 1 + (k+1)2^k$ {using P_k}

$= 2^k(k-1+k+1)+1$

$= 2^k(2k)+1$

$= k\,2^{k+1}+1$

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

4 **a** P_n is: $\dfrac{1}{1\times 2}+\dfrac{1}{2\times 3}+\dfrac{1}{3\times 4}+ .... +\dfrac{1}{n(n+1)} = \dfrac{n}{n+1}$ for all $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n=1$, LHS $=\dfrac{1}{1\times 2}=\dfrac{1}{2}$, RHS $=\dfrac{1}{1+1}=\dfrac{1}{2}$ $\therefore$ P_1 is true

(2) If P_k is true, then $\dfrac{1}{1\times 2}+\dfrac{1}{2\times 3}+\dfrac{1}{3\times 4}+ .... +\dfrac{1}{k(k+1)} = \dfrac{k}{k+1}$

Now $\dfrac{1}{1\times 2}+\dfrac{1}{2\times 3}+\dfrac{1}{3\times 4}+ .... +\dfrac{1}{k(k+1)}+\dfrac{1}{(k+1)(k+2)}$

$= \dfrac{k}{k+1}+\dfrac{1}{(k+1)(k+2)}$ {using P_k}

$= \dfrac{k}{k+1}\left(\dfrac{k+2}{k+2}\right)+\dfrac{1}{(k+1)(k+2)}$ {equalising denominators}

$= \dfrac{k(k+2)+1}{(k+1)(k+2)}$

$= \dfrac{k^2+2k+1}{(k+1)(k+2)}$

$= \dfrac{(k+1)^2}{(k+1)(k+2)}$

$= \dfrac{k+1}{k+2} = \dfrac{k+1}{(k+1)+1}$

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

b $\dfrac{1}{10\times 11}+\dfrac{1}{11\times 12}+\dfrac{1}{12\times 13}+ .... +\dfrac{1}{20\times 21}=S_{20}-S_9=\frac{20}{21}-\frac{9}{10}=\frac{11}{210}$

c P_n is: $\dfrac{1}{1\times 2\times 3}+\dfrac{1}{2\times 3\times 4}+\dfrac{1}{3\times 4\times 5}+ .... +\dfrac{1}{n(n+1)(n+2)}=\dfrac{n(n+3)}{4(n+1)(n+2)}$
for all $n\in\mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n=1$, LHS $=\dfrac{1}{1\times 2\times 3}=\dfrac{1}{6}$, RHS $=\dfrac{1(4)}{4(2)(3)}=\dfrac{1}{6}$ $\therefore$ P_1 is true.

(2) If P_k is true, then

$$\frac{1}{1\times 2\times 3}+\frac{1}{2\times 3\times 4}+\frac{1}{3\times 4\times 5}+ .... +\frac{1}{k(k+1)(k+2)}=\frac{k(k+3)}{4(k+1)(k+2)}$$

Now

$$\frac{1}{1\times 2\times 3}+\frac{1}{2\times 3\times 4}+\frac{1}{3\times 4\times 5}+ .... +\frac{1}{k(k+1)(k+2)}+\frac{1}{(k+1)(k+2)(k+3)}$$

$$=\frac{k(k+3)}{4(k+1)(k+2)}+\frac{1}{(k+1)(k+2)(k+3)} \qquad \{\text{using } P_k\}$$

$$=\frac{k(k+3)}{4(k+1)(k+2)}\left(\frac{k+3}{k+3}\right)+\frac{4}{4(k+1)(k+2)(k+3)} \qquad \{\text{equalising denominators}\}$$

$$=\frac{k(k+3)^2+4}{4(k+1)(k+2)(k+3)}$$

$$=\frac{k(k^2+6k+9)+4}{4(k+1)(k+2)(k+3)}$$

$$=\frac{k^3+6k^2+9k+4}{4(k+1)(k+2)(k+3)} \qquad \begin{array}{r|rrr|r} -1 & 1 & 6 & 9 & 4 \\ & 0 & -1 & -5 & -4 \\ \hline & 1 & 5 & 4 & 0 \end{array}$$

$$=\frac{\cancel{(k+1)}(k^2+5k+4)}{4\cancel{(k+1)}(k+2)(k+3)}$$

$$=\frac{(k+1)(k+4)}{4(k+2)(k+3)}$$

$$=\frac{(k+1)([k+1]+3)}{4([k+1]+1)([k+1]+2)}$$

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n\in\mathbb{Z}^+$ {Principle of mathematical induction}

5 **a** P_n is: $1\times 1!+2\times 2!+3\times 3!+ .... +n\times n!=(n+1)!-1$ for all $n\in\mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n=1$, LHS $=1\times 1!$ $\qquad$ RHS $=2!-1$
$=1\times 1$ $\qquad$ $=2-1$
$=1$ $\qquad$ $=1$ $\therefore$ P_1 is true

(2) If P_k is true, then $1\times 1!+2\times 2!+3\times 3!+ .... +k\times k!=(k+1)!-1$

Now $1\times 1!+2\times 2!+3\times 3!+ .... +k\times k!+(k+1)(k+1)!$

$=(k+1)!-1+(k+1)(k+1)!$ {using P_k}

$=(k+1)!(1+k+1)-1$

$=(k+1)!(k+2)-1$

$=(k+2)!-1$

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n\in\mathbb{Z}^+$ {Principle of mathematical induction}

b P_n is: $\dfrac{1}{2!}+\dfrac{2}{3!}+\dfrac{3}{4!}+\,....\,+\dfrac{n}{(n+1)!}=\dfrac{(n+1)!-1}{(n+1)!}$ for all $n\in\mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n=1$, LHS $=\dfrac{1}{2!}=\dfrac{1}{2}$, RHS $=\dfrac{2!-1}{2!}=\dfrac{2-1}{2}=\dfrac{1}{2}$ $\therefore$ P_1 is true

(2) If P_k is true, then $\dfrac{1}{2!}+\dfrac{2}{3!}+\dfrac{3}{4!}+\,....\,+\dfrac{k}{(k+1)!}=\dfrac{(k+1)!-1}{(k+1)!}$

Now $\dfrac{1}{2!}+\dfrac{2}{3!}+\dfrac{3}{4!}+\,....\,+\dfrac{k}{(k+1)!}+\dfrac{k+1}{(k+2)!}$

$$=\frac{(k+1)!-1}{(k+1)!}+\frac{k+1}{(k+2)!}\qquad\{\text{using } P_k\}$$

$$=\left(\frac{k+2}{k+2}\right)\left[\frac{(k+1)!-1}{(k+1)!}\right]+\frac{k+1}{(k+2)!}\qquad\{\text{equalising denominators}\}$$

$$=\frac{(k+2)!-(k+2)+k+1}{(k+2)!}$$

$$=\frac{(k+2)!-k-2+k+1}{(k+2)!}$$

$$=\frac{(k+2)!-1}{(k+2)!}$$

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n\in\mathbb{Z}^+$ {Principle of mathematical induction}

c $\dfrac{1}{2!}+\dfrac{2}{3!}+\dfrac{3}{4!}+\,....\,+\dfrac{9}{10!}=\dfrac{10!-1}{10!}=\dfrac{3\,628\,799}{3\,628\,800}$

6 P_n is: $1\times n+2\times(n-1)+3\times(n-2)+\,....\,+(n-1)\times 2+n\times 1=\dfrac{n(n+1)(n+2)}{6}$
for all $n\in\mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n=1$, LHS $=1\times 1=1$, RHS $=\dfrac{1(2)(3)}{6}=1$ $\therefore$ P_1 is true

(2) If P_k is true, then $1\times k+2(k-1)+3(k-2)+\,....\,+(k-1)2+k\times 1=\dfrac{k(k+1)(k+2)}{6}$

Now $1(k+1)+2(k)+3(k-1)+\,....\,+k\,2+(k+1)1$

$$=1(k)+2(k-1)+3(k-2)+\,....\,+k\,1+1+2+3+\,....\,+k+k+1$$

{using the hint}

$$=\frac{k(k+1)(k+2)}{6}+\frac{(k+1)(k+2)}{2}\qquad\{\text{using } P_k \text{ and the sum of an arithmetic series}\}$$

$$=\frac{k(k+1)(k+2)}{6}+\frac{3(k+1)(k+2)}{6}\qquad\{\text{equalising denominators}\}$$

$$=\frac{(k+1)(k+2)(k+3)}{6}$$

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n\in\mathbb{Z}^+$ {Principle of mathematical induction}

7 **a** P_n is: if $u_1=5$ and $u_{n+1}=u_n+8n+5$ for all $n\in\mathbb{Z}^+$, then $u_n=4n^2+n$

Proof: (By the principle of mathematical induction)

(1) If $n=1$, $u_1=4(1)^2+1=5$ which is true and so P_1 is true.

(2) If P_k is true, then $u_k = 4k^2 + k$ and
$$\begin{aligned} u_{k+1} &= u_k + 8k + 5 \\ &= 4k^2 + k + 8k + 5 \quad \{\text{using } P_k\} \\ &= 4(k^2 + 2k + 1) + k + 1 \\ &= 4(k+1)^2 + (k+1) \end{aligned}$$

$\therefore$ P_{k+1} is also true.

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

b P_n is: if $u_1 = 1$ and $u_{n+1} = 2 + 3u_n$ for all $n \in \mathbb{Z}^+$, then $u_n = 2(3^{n-1}) - 1$

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, $u_1 = 2(3^{1-1}) - 1 = 1$ which is true and so P_1 is true.

(2) If P_k is true, then $u_k = 2(3^{k-1}) - 1$ and
$$\begin{aligned} u_{k+1} &= 2 + 3u_k \\ &= 2 + 3(2[3^{k-1}] - 1) \quad \{\text{using } P_k\} \\ &= 2 + 2 \times 3^k - 3 \\ &= 2(3^k) - 1 \end{aligned}$$

$\therefore$ P_{k+1} is also true.

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

c P_n is: if $u_1 = 2$ and $u_{n+1} = \dfrac{u_n}{2(n+1)}$ for all $n \in \mathbb{Z}^+$, then $u_n = \dfrac{2^{2-n}}{n!}$

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, $u_1 = \dfrac{2^{2-1}}{1!} = 2^1 = 2$ which is true and so P_1 is true.

(2) If P_k is true, then $u_k = \dfrac{2^{2-k}}{k!}$ and
$$\begin{aligned} u_{k+1} = \frac{u_k}{2(k+1)} &= \frac{2^{2-k}}{k!\,2(k+1)} \quad \{\text{using } P_k\} \\ &= \frac{2^{2-k-1}}{(k+1)k!} \\ &= \frac{2^{2-(k+1)}}{(k+1)!} \end{aligned}$$

$\therefore$ P_{k+1} is also true.

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

d P_n is: if $u_1 = 1$ and $u_{n+1} = u_n + (-1)^n(n+1)^2$ for all $n \in \mathbb{Z}^+$,
then $u_n = \dfrac{(-1)^{n-1}n(n+1)}{2}$

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, $u_1 = \dfrac{(-1)^0 \times 1 \times 2}{2} = 1$ which is true and so P_1 is true.

(2) If P_k is true, then $u_k = \dfrac{(-1)^{k-1}k(k+1)}{2}$

and
$$\begin{aligned} u_{k+1} &= u_k + (-1)^k(k+1)^2 \\ &= \frac{(-1)^{k-1}k(k+1)}{2} + (-1)^k(k+1)^2 \quad \{\text{using } P_k\} \\ &= \frac{(-1)^{k-1}k(k+1) + 2(-1)^k(k+1)^2}{2} \\ &= \frac{2(-1)^k(k+1)^2 - (-1)^k k(k+1)}{2} \end{aligned}$$

$$\therefore \quad u_{k+1} = \frac{(-1)^k(k+1)[2(k+1)-k]}{2}$$
$$= \frac{(-1)^k(k+1)(k+2)}{2} \qquad \therefore \quad P_{k+1} \text{ is also true.}$$

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

8 **a** $u_1 = 1 = 1^2$
$u_2 = u_1 + 2(1) + 1 = 1 + 2 + 1 = 4 = 2^2$
$u_3 = u_2 + 2(2) + 1 = 4 + 4 + 1 = 9 = 3^2$
$u_4 = u_3 + 2(3) + 1 = 9 + 6 + 1 = 16 = 4^2$
We conjecture that $u_n = n^2$ for all $n \in \mathbb{Z}^+$

b P_n is: if $u_1 = 1$ and $u_{n+1} = u_n + (2n+1)$ for all $n \in \mathbb{Z}^+$, then $u_n = n^2$

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, then $u_1 = 1 = 1^2$, so P_1 is true.

(2) If P_k is true, then $u_k = k^2$ and
$$u_{k+1} = u_k + (2k+1)$$
$$= k^2 + 2k + 1 \quad \{\text{using } P_k\}$$
$$= (k+1)^2 \quad \therefore \quad P_{k+1} \text{ is also true.}$$

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

9 **a** $u_1 = \frac{1}{3}$
$$u_2 = u_1 + \frac{1}{(2(1)+1)(2(1)+3)} = \frac{1}{3} + \frac{1}{3 \times 5} = \frac{5+1}{15} = \frac{6}{15} = \frac{2}{5}$$
$$u_3 = u_2 + \frac{1}{(2(2)+1)(2(2)+3)} = \frac{2}{5} + \frac{1}{5 \times 7} = \frac{14+1}{35} = \frac{15}{35} = \frac{3}{7}$$
$$u_4 = u_3 + \frac{1}{(2(3)+1)(2(3)+3)} = \frac{3}{7} + \frac{1}{7 \times 9} = \frac{27+1}{63} = \frac{28}{63} = \frac{4}{9}$$

We conjecture that $u_n = \dfrac{n}{2n+1}$ for all $n \in \mathbb{Z}^+$

b P_n is: if $u_1 = \frac{1}{3}$ and $u_{n+1} = u_n + \dfrac{1}{(2n+1)(2n+3)}$ for all $n \in \mathbb{Z}^+$, then $u_n = \dfrac{n}{2n+1}$

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, then $u_1 = \dfrac{1}{3} = \dfrac{1}{2(1)+1}$, so P_1 is true.

(2) If P_k is true, then $u_k = \dfrac{k}{2k+1}$

and
$$u_{k+1} = u_k + \frac{1}{(2k+1)(2k+3)}$$
$$= \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)} \quad \{\text{using } P_k\}$$
$$= \frac{k(2k+3)+1}{(2k+1)(2k+3)}$$
$$= \frac{2k^2+3k+1}{(2k+1)(2k+3)}$$
$$= \frac{\cancel{(2k+1)}(k+1)}{\cancel{(2k+1)}(2k+3)}$$
$$= \frac{k+1}{2(k+1)+1} \qquad \therefore \quad P_{k+1} \text{ is also true.}$$

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

10 **a** $\left(2+\sqrt{3}\right)^1 = 2+\sqrt{3}$ $\qquad \therefore\ A_1 = 2,\quad B_1 = 1$

$\left(2+\sqrt{3}\right)^2 = 4+4\sqrt{3}+3$

$= 7+4\sqrt{3}$ $\qquad \therefore\ A_2 = 7,\quad B_2 = 4$

$\left(2+\sqrt{3}\right)^3 = (7+4\sqrt{3})(2+\sqrt{3})$

$= 14+7\sqrt{3}+8\sqrt{3}+4(3)$

$= 26+15\sqrt{3}$ $\qquad \therefore\ A_3 = 26,\quad B_3 = 15$

$\left(2+\sqrt{3}\right)^4 = \left(7+4\sqrt{3}\right)^2$

$= 49+56\sqrt{3}+16(3)$

$= 97+56\sqrt{3}$ $\qquad \therefore\ A_4 = 97,\quad B_4 = 56$

b
$$\begin{aligned}\left(2+\sqrt{3}\right)^n &= A_n + B_n\sqrt{3}\\ \therefore\ \left(2+\sqrt{3}\right)^{n+1} &= \left(2+\sqrt{3}\right)^n\left(2+\sqrt{3}\right)\\ &= \left(A_n + B_n\sqrt{3}\right)\left(2+\sqrt{3}\right)\\ &= 2A_n + A_n\sqrt{3} + 2B_n\sqrt{3} + B_n(3)\\ &= 2A_n + 3B_n + (A_n+2B_n)\sqrt{3}\end{aligned}$$

$\therefore\ A_{n+1} = 2A_n + 3B_n, \quad B_{n+1} = A_n + 2B_n$

c $A_1^{\,2} - 3B_1^{\,2} = 2^2 - 3(1)^2 = 4-3 = 1$

$A_2^{\,2} - 3B_2^{\,2} = 7^2 - 3(4)^2 = 49 - 3\times 16 = 1$

$A_3^{\,2} - 3B_3^{\,2} = 26^2 - 3(15)^2 = 676 - 3\times 225 = 1$

$A_4^{\,2} - 3B_4^{\,2} = 97^2 - 3(56)^2 = 9409 - 3\times 3136 = 1$

$\therefore$ we conjecture $(A_n)^2 - 3(B_n)^2 = 1$ for all $n \in \mathbb{Z}^+$

d P_n is: if $\left(2+\sqrt{3}\right)^n = A_n + B_n\sqrt{3}$ for all $n\in\mathbb{Z}^+$, then $A_n^{\,2} - 3B_n^{\,2} = 1$

Proof: (By the principle of mathematical induction)

(1) If $n=1$, $A_1 = 2$, $B_1 = 1$, and $A_1^{\,2} - 3B_1^{\,2} = 2^2 - 3(1)^2 = 1$, so P_1 is true.

(2) If P_k is true, then $A_k^{\,2} - 3B_k^{\,2} = 1$, and

$$\begin{aligned}A_{k+1}^{\,2} - 3B_{k+1}^{\,2} &= (2A_k+3B_k)^2 - 3(A_k+2B_k)^2 \quad \{\text{using } \mathbf{b}\}\\ &= 4A_k^{\,2} + 12A_kB_k + 9B_k^{\,2} - 3(A_k^{\,2} + 4A_kB_k + 4B_k^{\,2})\\ &= 4A_k^{\,2} + \cancel{12A_kB_k} + 9B_k^{\,2} - 3A_k^{\,2} - \cancel{12A_kB_k} - 12B_k^{\,2}\\ &= A_k^{\,2} - 3B_k^{\,2}\\ &= 1 \qquad \{\text{using } P_k\}\end{aligned}$$

$\therefore\ P_{k+1}$ is true.

Since P_1 is true, and P_{k+1} is true whenever P_k is true, then P_n is true for all $n\in\mathbb{Z}^+$ {Principle of mathematical induction}

11 P_n is: $\dfrac{2^n - (-1)^n}{3}$ is an odd number for all $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n=1$, $\dfrac{2^1 - (-1)^1}{3} = \dfrac{3}{3} = 1$ which is odd $\therefore\ P_1$ is true

(2) If P_k is true, then $\dfrac{2^k - (-1)^k}{3} = 2A+1$ where $A\in\mathbb{Z}$

Now
$$\begin{aligned}\frac{2^{k+1} - (-1)^{k+1}}{3} &= \frac{2(2^k) - (-1)^{k+1}}{3}\\ &= \frac{2[6A+3+(-1)^k] - (-1)^{k+1}}{3} \quad \{\text{using } P_k\}\end{aligned}$$

$$\therefore \quad \frac{2^{k+1}-(-1)^{k+1}}{3} = \frac{12A+6+2(-1)^k-(-1)(-1)^k}{3}$$
$$= \frac{12A+6+2(-1)^k+(-1)^k}{3}$$
$$= \frac{12A+6+3(-1)^k}{3}$$
$$= 4A+2+(-1)^k$$

Now $4A+2$ is always even and $(-1)^k$ is either $+1$ or -1

$\therefore$ $4A+2+(-1)^k$ is odd

$\therefore$ $\dfrac{2^{k+1}-(-1)^{k+1}}{3}$ is odd

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

12 **a** P_n is: if $u_1 = 11$, $u_2 = 37$, and $u_{n+2} = 5u_{n+1} - 6u_n$ for all $n \in \mathbb{Z}^+$,
then $u_n = 5(3^n) - 2^{n+1}$

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, $5(3^1) - 2^{1+1} = 15 - 2^2 = 11 = u_1$, so P_1 is true.
If $n = 2$, $5(3^2) - 2^{2+1} = 5 \times 9 - 2^3 = 37 = u_2$, so P_2 is true.

(2) If P_k and P_{k+1} are true, then $u_k = 5(3^k) - 2^{k+1}$ and $u_{k+1} = 5(3^{k+1}) - 2^{k+2}$
and $u_{k+2} = 5u_{k+1} - 6u_k$
$$= 5[5(3^{k+1}) - 2^{k+2}] - 6[5(3^k) - 2^{k+1}] \quad \{\text{using } P_k \text{ and } P_{k+1}\}$$
$$= 25(3^{k+1}) - 5(2^{k+2}) - 30(3^k) + 6(2^{k+1})$$
$$= 25(3^{k+1}) - 5(2^{k+2}) - 10(3^{k+1}) + 3(2^{k+2})$$
$$= 15(3^{k+1}) - 2(2^{k+2})$$
$$= 5(3^{k+2}) - 2^{k+3} \qquad \therefore \quad P_{k+2} \text{ is true.}$$

Since P_1 and P_2 are true, and P_{k+2} is true whenever P_k and P_{k+1} are true,
then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

b
$$u_{n+2} = au_{n+1} + bu_n$$
$$\therefore \quad \left(3+\sqrt{5}\right)^{n+2} + \left(3-\sqrt{5}\right)^{n+2} = a\left[\left(3+\sqrt{5}\right)^{n+1} + \left(3-\sqrt{5}\right)^{n+1}\right] + b\left[\left(3+\sqrt{5}\right)^{n} + \left(3-\sqrt{5}\right)^{n}\right]$$
$$\therefore \quad \left(3+\sqrt{5}\right)^2\left(3+\sqrt{5}\right)^n + \left(3-\sqrt{5}\right)^2\left(3-\sqrt{5}\right)^n = a\left[\left(3+\sqrt{5}\right)\left(3+\sqrt{5}\right)^n + \left(3-\sqrt{5}\right)\left(3-\sqrt{5}\right)^n\right] + b\left[\left(3+\sqrt{5}\right)^n + \left(3-\sqrt{5}\right)^n\right]$$
$$\therefore \quad \left(14+6\sqrt{5}\right)\left(3+\sqrt{5}\right)^n + \left(14-6\sqrt{5}\right)\left(3-\sqrt{5}\right)^n = \left(3a+a\sqrt{5}\right)\left(3+\sqrt{5}\right)^n + \left(3a-a\sqrt{5}\right)\left(3-\sqrt{5}\right)^n + b\left(3+\sqrt{5}\right)^n + b\left(3-\sqrt{5}\right)^n$$
$$\therefore \quad \left(14+6\sqrt{5}\right)\left(3+\sqrt{5}\right)^n + \left(14-6\sqrt{5}\right)\left(3-\sqrt{5}\right)^n = \left(3a+b+a\sqrt{5}\right)\left(3+\sqrt{5}\right)^n + \left(3a+b-a\sqrt{5}\right)\left(3-\sqrt{5}\right)^n$$

Equating coefficients of $\left(3+\sqrt{5}\right)^n$, $14+6\sqrt{5} = 3a+b+a\sqrt{5}$

Equating rational and irrational parts,

$3a+b = 14$ and $a = 6$

$\therefore$ $a = 6$ and $b = -4$ {this checks with the coefficients of $\left(3-\sqrt{5}\right)^n$}

$\therefore$ $u_{n+2} = 6u_{n+1} - 4u_n$

P_n is: if $u_n = \left(3+\sqrt{5}\right)^n + \left(3-\sqrt{5}\right)^n$ where $n \in \mathbb{Z}^+$, then u_n is a multiple of 2^n

Proof: (By the principle of mathematical induction)

(1) If $n=1$, $u_1 = (3+\sqrt{5}) + (3-\sqrt{5})$
$= 6$ which is a multiple of $2^1 = 2$, so P_1 is true.

If $n=2$, $u_2 = (3+\sqrt{5})^2 + (3-\sqrt{5})^2$
$= 9 + \cancel{6\sqrt{5}} + 5 + 9 - \cancel{6\sqrt{5}} + 5$
$= 28$ which is a multiple of $2^2 = 4$, so P_2 is true.

(2) If P_k and P_{k+1} are true, then $u_k = A \times 2^k$, where $A \in \mathbb{Z}$
and $u_{k+1} = B \times 2^{k+1}$, where $B \in \mathbb{Z}$

Now
$$\begin{aligned} u_{k+2} &= 6u_{k+1} - 4u_k \quad \{\text{using the above result}\} \\ &= 6(B \times 2^{k+1}) - 4(A \times 2^k) \\ &= 3B \times 2^{k+2} - A \times 2^{k+2} \\ &= (3B - A) \times 2^{k+2} \end{aligned}$$

which is a multiple of 2^{k+2} since $3B - A \in \mathbb{Z}$ {as $A, B \in \mathbb{Z}$}

$\therefore$ P_{k+2} is true.

Since P_1 and P_2 are true, and P_{k+2} is true whenever P_k and P_{k+1} are true, then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

EXERCISE 9B.3

1 **a** P_n is: $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right) \dots \left(1-\frac{1}{n+1}\right) = \frac{1}{n+1}$ for all $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

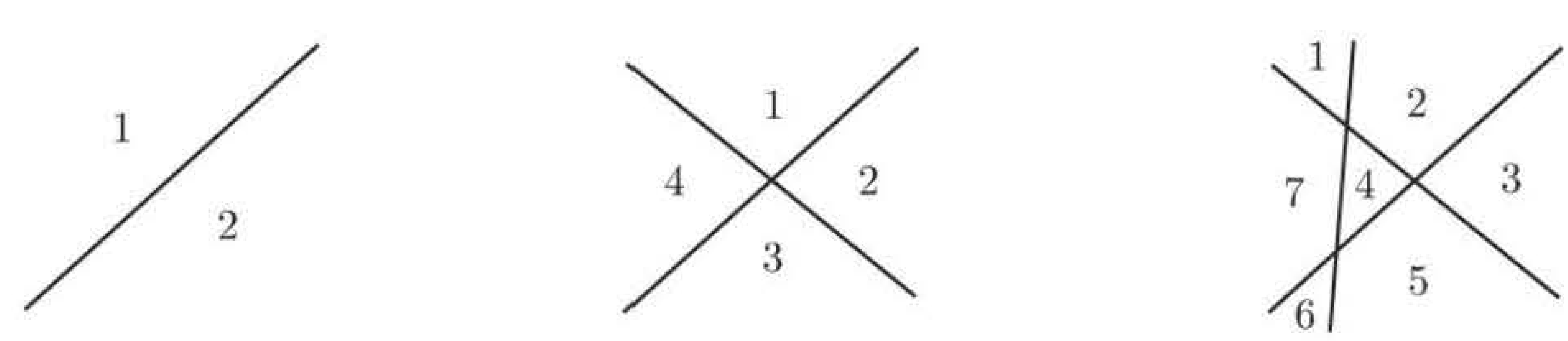
(1) If $n=1$, LHS $= \left(1-\frac{1}{2}\right) = \frac{1}{2}$, RHS $= \frac{1}{1+1} = \frac{1}{2}$ $\therefore$ P_1 is true

(2) If P_k is true, then $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right) \dots \left(1-\frac{1}{k+1}\right) = \frac{1}{k+1}$

$$\begin{aligned} \therefore \quad & \left(1-\tfrac{1}{2}\right)\left(1-\tfrac{1}{3}\right)\left(1-\tfrac{1}{4}\right) \dots \left(1-\frac{1}{k+1}\right)\left(1-\frac{1}{k+2}\right) \\ &= \frac{1}{k+1}\left(1-\frac{1}{k+2}\right) \quad \{\text{using } P_k\} \\ &= \frac{1}{k+1}\left(\frac{k+2-1}{k+2}\right) \\ &= \frac{1}{k+1}\left(\frac{k+1}{k+2}\right) \\ &= \frac{1}{k+2} \end{aligned}$$

Since P_1 is true, and P_{k+1} is true whenever P_k is true, then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

b



1 line, $R_1 = 2$ 2 lines, $R_2 = 4$ 3 lines, $R_3 = 7$, and so on.

P_n is: For n lines as described, $R_n = \frac{n(n+1)}{2} + 1$ for all $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, $R_1 = \dfrac{1(2)}{2} + 1 = 1 + 1 = 2$ $\therefore$ P_1 is true

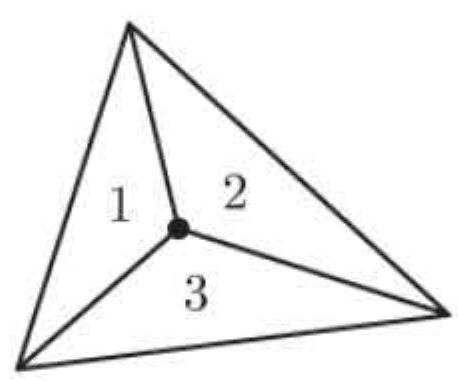
(2) If P_k is true, then $R_k = \dfrac{k(k+1)}{2} + 1$

The addition of another line creates another $k+1$ regions

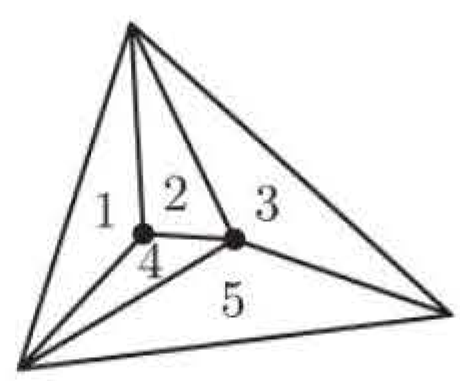
$$\begin{aligned}\therefore\quad R_{k+1} &= \frac{k(k+1)}{2} + 1 + k + 1 \qquad \{\text{using } P_k\}\\ &= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} + 1\\ &= \frac{k^2+k+2k+2}{2} + 1\\ &= \frac{k^2+3k+2}{2} + 1\\ &= \frac{(k+1)(k+2)}{2} + 1\end{aligned}$$

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

c



$n = 1$, $T_1 = 3$



$n = 2$, $T_2 = 5$

P_n is: For n points inside the triangle (as described) there are $T_n = 2n + 1$ triangular partitions for all $n \in \mathbb{Z}^+$.

Proof: (By the principle of mathematical induction)

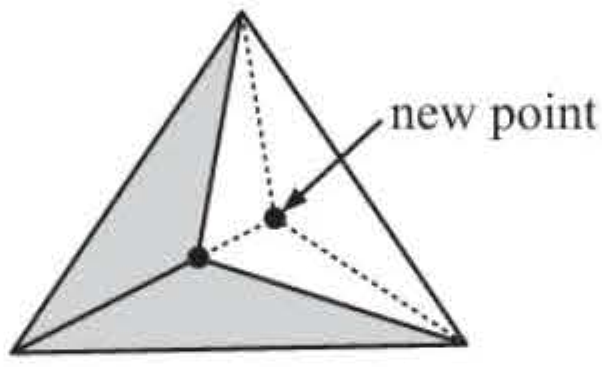
(1) If $n = 1$, $T_1 = 2(1) + 1 = 3$ $\therefore$ P_1 is true

(2) If P_k is true, then $T_k = 2k + 1$

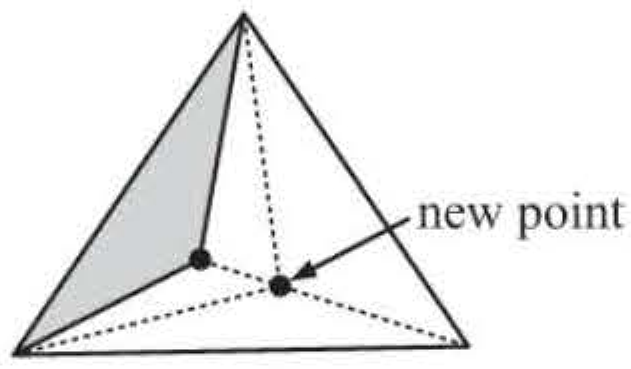
Adding an extra point within the triangle gives the $(k+1)$th case.

This point could be either

- in an existing triangle or
- on an existing line between 2 triangles



So, 1 triangle becomes 3, a net increase of 2.



So, 2 triangles become 4, a net increase of 2.

In each case 2 triangles are added

$$\begin{aligned}\therefore\quad T_{k+1} &= T_k + 2\\ &= 2k + 1 + 2 \quad \{\text{using } P_k\}\\ &= 2(k+1) + 1\end{aligned}$$

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

d P_n is: $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)....\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}$ for all $n \geqslant 2,\ n \in \mathbb{Z}$

Proof: (By the principle of mathematical induction)

(1) If $n=2$, LHS $=1-\frac{1}{2^2}$ RHS $=\frac{2+1}{2(2)}$

$=1-\frac{1}{4}=\frac{3}{4}$ $=\frac{3}{4}$ $\therefore$ P_2 is true

(2) If P_k is true, then $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)....\left(1-\frac{1}{k^2}\right)=\frac{k+1}{2k}$

Now $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)....\left(1-\frac{1}{k^2}\right)\left(1-\frac{1}{(k+1)^2}\right)$

$$\begin{aligned}
&=\frac{k+1}{2k}\left(1-\frac{1}{(k+1)^2}\right) \quad \{\text{using } P_k\}\\
&=\frac{k+1}{2k}\left(\frac{(k+1)^2-1}{(k+1)^2}\right)\\
&=\frac{k+1}{2k}\left(\frac{k^2+2k+1-1}{(k+1)^2}\right)\\
&=\frac{k^2+2k}{2k(k+1)}\\
&=\frac{k(k+2)}{2k(k+1)}\\
&=\frac{k+2}{2(k+1)}
\end{aligned}$$

Since P_2 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \geqslant 2,\ n \in \mathbb{Z}$ {Principle of mathematical induction}

2 **a** P_n is: $3^n \geqslant 1+2n$ for $n \in \mathbb{N}$

Proof: (By the principle of mathematical induction)

(1) If $n=0$, we have $3^0 \geqslant 1+2(0)$

$\therefore$ $1 \geqslant 1$ which is true $\therefore$ P_0 is true

(2) If P_k is true, then $3^k \geqslant 1+2k$

$$\begin{aligned}
\text{Now } 3^{k+1}=3^k \times 3 &\geqslant (1+2k)\times 3 \quad \{\text{using } P_k\}\\
&\geqslant 3+6k\\
&\geqslant 3+2k \quad \{k \geqslant 0\}\\
&\geqslant 1+2(k+1)
\end{aligned}$$

$\therefore$ P_{k+1} is true.

Since P_0 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{N}$ {Principle of mathematical induction}

b P_n is: $n! \geqslant 2^n$ for $n \in \mathbb{Z},\ n \geqslant 4$

Proof: (By the principle of mathematical induction)

(1) If $n=4$, we have $4! \geqslant 2^4$

$\therefore$ $24 \geqslant 16$ which is true $\therefore$ P_4 is true

(2) If P_k is true then $k! \geqslant 2^k$

$$\begin{aligned}
\text{Now } (k+1)!=(k+1)\times k! &\geqslant (k+1)\times 2^k \quad \{\text{using } P_k\}\\
&\geqslant 2 \times 2^k \quad \{k \geqslant 4\}\\
&\geqslant 2^{k+1}
\end{aligned}$$

$\therefore$ P_{k+1} is true.

Since P_4 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z},\ n \geqslant 4$ {Principle of mathematical induction}

c P_n is: $8^n \geqslant n^3$ for $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, we have $8^1 \geqslant 1^3$

$\therefore \; 8 \geqslant 1$ which is true, so P_1 is true

(2) If P_k is true then $8^k \geqslant k^3$

$$\begin{aligned} \text{Now } 8^{k+1} &= 8 \times 8^k \geqslant 8 \times k^3 && \{\text{using } P_k\} \\ &\geqslant (2k)^3 \\ &\geqslant (k+1)^3 && \{k \geqslant 1, \text{ so } 2k \geqslant k+1\} \end{aligned}$$

$\therefore \; P_{k+1}$ is true.

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

d P_n is: $(1-h)^n \leqslant \dfrac{1}{1+nh}$ for $0 \leqslant h \leqslant 1$ for $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, we have $(1-h) \leqslant \dfrac{1}{1+h}$

$$\begin{aligned} &\therefore \; (1-h)(1+h) \leqslant 1 \qquad \{1+h \geqslant 0\} \\ &\therefore \; 1-h^2 \leqslant 1 \\ &\therefore \; h^2 \geqslant 0 \end{aligned}$$

which is true for $0 \leqslant h \leqslant 1$, so P_1 is true

(2) If P_k is true then $(1-h)^k \leqslant \dfrac{1}{1+kh}$ for $0 \leqslant h \leqslant 1$

Now $(1-h)^{k+1} = (1-h)(1-h)^k$

$$\therefore \; (1-h)^{k+1} \leqslant (1-h)\left(\frac{1}{1+kh}\right) \qquad \{\text{using } P_k\}$$

$$\therefore \; (1-h)^{k+1} \leqslant \left(\frac{1-h}{1+kh}\right) \times \left(\frac{1+kh+h}{1+kh+h}\right)$$

$$\therefore \; (1-h)^{k+1} \leqslant \frac{1+kh+h-h-kh^2-h^2}{(1+kh)(1+kh+h)}$$

$$\therefore \; (1-h)^{k+1} \leqslant \frac{(1+kh)-(kh^2+h^2)}{(1+kh)(1+(k+1)h)}$$

$$\therefore \; (1-h)^{k+1} \leqslant \frac{1-\frac{kh^2+h^2}{1+kh}}{1+(k+1)h}$$

$$\therefore \; (1-h)^{k+1} \leqslant \frac{1}{1+(k+1)h} \qquad \left\{k, h \geqslant 0, \text{ so } \frac{kh^2+h^2}{1+kh} \geqslant 0\right\}$$

$\therefore \; P_{k+1}$ is true.

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

3 **a** P_n is: $(z_1 + z_2 + ... + z_n)^* = {z_1}^* + {z_2}^* + ... + {z_n}^*$ for all $n \in \mathbb{Z}^+$ and complex $z_1, z_2, ..., z_n$

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, $(z_1)^* = {z_1}^*$ $\therefore$ P_1 is true

(2) If P_k is true, then $(z_1 + z_2 + ... + z_k)^* = {z_1}^* + {z_2}^* + ... + {z_k}^*$

$$\begin{aligned} \text{Now } \; & (z_1 + z_2 + ... + z_k + z_{k+1})^* \\ &= ((z_1 + z_2 + ... + z_k) + z_{k+1})^* \\ &= (z_1 + z_2 + ... + z_k)^* + z_{k+1}^* && \{\text{using } (z_1+z_2)^* = {z_1}^* + {z_2}^*\} \\ &= {z_1}^* + {z_2}^* + ... + z_k^* + z_{k+1}^* && \{\text{using } P_k\} \end{aligned}$$

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

b P_n is: $(z_1z_2....z_n)^* = {z_1}^*{z_2}^*....{z_n}^*$ for all $n \in \mathbb{Z}^+$ and complex $z_1, z_2, ..., z_n$

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, $(z_1)^* = {z_1}^*$ $\therefore$ P_1 is true

(2) If P_k is true, then $(z_1z_2....z_k)^* = {z_1}^*{z_2}^*....{z_k}^*$

Now $(z_1z_2....z_kz_{k+1})^* = ((z_1z_2....z_k)z_{k+1})^*$

$= (z_1z_2....z_k)^*z_{k+1}^*$ {using $(z_1z_2)^* = {z_1}^*{z_2}^*$}

$= {z_1}^*{z_2}^*....{z_k}^*z_{k+1}^*$ {using P_k}

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

c P_n is: $(z_n)^* = (z^*)^n$ for all $n \in \mathbb{Z}^+$ and complex z

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, $(z^1)^* = z^* = (z^*)^1$ $\therefore$ P_1 is true

(2) If P_k is true, then $(z^k)^* = (z^*)^k$

Now $(z^{k+1})^* = (zz^k)^*$

$= z^*(z^k)^*$ {using $(z_1z_2)^* = {z_1}^*{z_2}^*$}

$= z^*(z^*)^k$ {using P_k}

$= (z^*)^{k+1}$

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

REVIEW SET 9A

1 P_n is: $1 + 3 + 5 + 7 + ... + (2n-1) = n^2$ for all $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, LHS $= 1$ and RHS $= 1^2 = 1$ $\therefore$ P_1 is true

(2) If P_k is true, then $1 + 3 + 5 + 7 + ... + (2k-1) = k^2$

$\therefore$ $1 + 3 + 5 + 7 + ... + (2k-1) + (2[k+1]-1)$

$= k^2 + 2k + 2 - 1$ {using P_k}

$= k^2 + 2k + 1$

$= (k+1)^2$

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

2 P_n is: $7^n + 2$ is divisible by 3 for all $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, $7^1 + 2 = 9$ which is divisible by 3 $\therefore$ P_1 is true

(2) If P_k is true, then $7^k + 2 = 3A$ where $A \in \mathbb{Z}$

$\therefore$ $7^{k+1} + 2 = 7 \times 7^k + 2$

$= 7(3A - 2) + 2$ {using P_k}

$= 21A - 14 + 2$

$= 21A - 12$

$= 3(7A - 4)$ where $7A - 4$ is an integer as A is an integer

$\therefore$ $7^{k+1} + 2$ is divisible by 3

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

3 P_n is: $1 \times 2 \times 3 + 2 \times 3 \times 4 + 3 \times 4 \times 5 + ... + n(n+1)(n+2) = \dfrac{n(n+1)(n+2)(n+3)}{4}$ for all $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, LHS $= 1 \times 2 \times 3 = 6$, RHS $= \dfrac{1 \times 2 \times 3 \times 4}{4} = 6$ $\therefore$ P_1 is true

(2) If P_k is true, then

$$1 \times 2 \times 3 + 2 \times 3 \times 4 + ... + k(k+1)(k+2) = \frac{k(k+1)(k+2)(k+3)}{4}$$

$$\begin{aligned}
\therefore\quad & 1 \times 2 \times 3 + 2 \times 3 \times 4 + ... + k(k+1)(k+2) + (k+1)(k+2)(k+3) \\
&= \frac{k(k+1)(k+2)(k+3)}{4} + (k+1)(k+2)(k+3) \qquad \{\text{using } P_k\} \\
&= \frac{k(k+1)(k+2)(k+3)}{4} + \frac{4(k+1)(k+2)(k+3)}{4} \qquad \{\text{equalising denominators}\} \\
&= \frac{(k+1)(k+2)(k+3)(k+4)}{4}
\end{aligned}$$

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

4 P_n is: $1 + r + r^2 + r^3 + ... + r^{n-1} = \dfrac{1 - r^n}{1 - r}$ for all $n \in \mathbb{Z}^+$, $r \neq 1$

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, LHS $= 1$ and RHS $= \dfrac{1 - r}{1 - r} = 1$ as $r \neq 1$ $\therefore$ P_1 is true

(2) If P_k is true, then $1 + r + r^2 + r^3 + ... + r^{k-1} = \dfrac{1 - r^k}{1 - r}$

$$\begin{aligned}
\text{Now}\quad 1 + r + r^2 + r^3 + ... + r^{k-1} + r^k &= \frac{1 - r^k}{1 - r} + r^k \qquad \{\text{using } P_k\} \\
&= \frac{1 - r^k}{1 - r} + r^k\left(\frac{1 - r}{1 - r}\right) \qquad \{\text{equalising denominators}\} \\
&= \frac{1 - r^k + r^k - r^{k+1}}{1 - r} \\
&= \frac{1 - r^{k+1}}{1 - r}
\end{aligned}$$

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

5 P_n is: $5^{2n} - 1$ is divisible by 24 for all $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, $5^2 - 1 = 25 - 1 = 24$ is divisible by 24 $\therefore$ P_1 is true

(2) If P_k is true, then $5^{2k} - 1 = 24A$ where $A \in \mathbb{Z}$

$$\begin{aligned}
\text{Now}\quad 5^{2(k+1)} - 1 &= 5^{2k}5^2 - 1 \\
&= 25[24A + 1] - 1 \qquad \{\text{using } P_k\} \\
&= 25 \times 24A + 25 - 1 \\
&= 25 \times 24A + 24 \\
&= 24(25A + 1) \quad \text{where } 25A + 1 \text{ is an integer}
\end{aligned}$$

$\therefore$ $5^{2(k+1)} - 1$ is divisible by 24

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

6 P_n is: $5^n \geqslant 1+4n$ for all $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n=1$, we have $5^1 \geqslant 1+4(1)$
$\therefore\ 5 \geqslant 5$ which is true, so P_1 is true

(2) If P_k is true, then $5^k \geqslant 1+4k$

$$\begin{aligned}\text{Now}\quad 5^{k+1} = 5\times 5^k &\geqslant 5\times(1+4k) \quad \{\text{using } P_k\}\\ &\geqslant 5+20k\\ &\geqslant 5+4k \qquad \{k\geqslant 0\}\\ &\geqslant 1+4(k+1)\end{aligned}$$

$\therefore$ P_{k+1} is true.

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

7 P_n is: if $u_1 = 1$ and $u_{n+1} = 3u_n + 2^n$ for all $n \in \mathbb{Z}^+$, then $u_n = 3^n - 2^n$

Proof: (By the principle of mathematical induction)

(1) If $n=1$, $u_1 = 1 = 3^1 - 2^1$, so P_1 is true

(2) If P_k is true, then $u_k = 3^k - 2^k$ and

$$\begin{aligned}u_{k+1} &= 3u_k + 2^k\\ &= 3(3^k-2^k)+2^k \qquad \{\text{using } P_k\}\\ &= 3^{k+1} - 3\times 2^k + 2^k\\ &= 3^{k+1} - 2\times 2^k\\ &= 3^{k+1} - 2^{k+1}\end{aligned}$$

$\therefore$ P_{k+1} is true.

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

REVIEW SET 9B

1 P_n is: $1^2+3^2+5^2+7^2+ .... +(2n-1)^2 = \dfrac{n(2n+1)(2n-1)}{3}$ for all $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n=1$, LHS $= 1^2 = 1$, RHS $= \dfrac{1\times 3\times 1}{3} = 1$ $\therefore$ P_1 is true

(2) If P_k is true, then $1^2+3^2+5^2+ .... +(2k-1)^2 = \dfrac{k(2k+1)(2k-1)}{3}$

$$\begin{aligned}\therefore\quad & 1^2+3^2+5^2+ .... +(2k-1)^2+(2k+1)^2\\ &= \frac{k(2k+1)(2k-1)}{3} + (2k+1)^2 \qquad \{\text{using } P_k\}\\ &= \frac{k(2k+1)(2k-1)}{3} + \frac{3(2k+1)^2}{3} \qquad \{\text{equalising denominators}\}\\ &= \frac{(2k+1)[k(2k-1)+3(2k+1)]}{3}\\ &= \frac{(2k+1)[2k^2-k+6k+3]}{3}\\ &= \frac{(2k+1)(2k^2+5k+3)}{3}\\ &= \frac{(2k+1)(k+1)(2k+3)}{3}\\ &= \frac{(k+1)(2[k+1]+1)(2[k+1]-1)}{3}\end{aligned}$$

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

2 P_n is: $3^{2n+2} - 8n - 9$ is divisible by 64 for all $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, $3^4 - 8 - 9 = 81 - 17 = 64$ which is divisible by 64 $\therefore$ P_1 is true

(2) If P_k is true, then $3^{2k+2} - 8k - 9 = 64A$ where $A \in \mathbb{Z}$

Now $3^{2(k+1)+2} - 8(k+1) - 9$

$$\begin{aligned}
&= 3^{2k+2} \times 3^2 - 8k - 8 - 9 \\
&= 9[64A + 8k + 9] - 8k - 17 \quad \{\text{using } P_k\} \\
&= 9 \times 64A + 72k + 81 - 8k - 17 \\
&= 9 \times 64A + 64k + 64 \\
&= 64(9A + k + 1) \text{ where } 9A + k + 1 \text{ is} \in \mathbb{Z} \text{ as } A, k \in \mathbb{Z}
\end{aligned}$$

$\therefore$ $3^{2(k+1)+2} - 8(k+1) - 9$ is divisible by 64

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

3 P_n is: $3 + 5 \times 2 + 7 \times 2^2 + 9 \times 2^3 + ... + (2n+1)2^{n-1} = 1 + (2n-1) \times 2^n$ for all $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, LHS $= 3$ and RHS $= 1 + 1 \times 2^1 = 1 + 2 = 3$ $\therefore$ P_1 is true

(2) If P_k is true, then $3 + 5 \times 2 + 7 \times 2^2 + 9 \times 2^3 + ... + (2k+1)2^{k-1} = 1 + (2k-1) \times 2^k$

$\therefore$ $3 + 5 \times 2 + 7 \times 2^2 + 9 \times 2^3 + ... + (2k+1)2^{k-1} + (2k+3)2^k$

$$\begin{aligned}
&= 1 + (2k-1)2^k + (2k+3)2^k \quad \{\text{using } P_k\} \\
&= 1 + 2^k(2k - 1 + 2k + 3) \\
&= 1 + 2^k(4k+2) \\
&= 1 + 2^k(2)(2k+1) \\
&= 1 + (2k+1)2^{k+1} \\
&= 1 + (2[k+1] - 1)2^{[k+1]}
\end{aligned}$$

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

4 P_n is: $5^n + 3$ is divisible by 4 for all $n \in \mathbb{Z}$, $n \geqslant 0$

Proof: (By the principle of mathematical induction)

(1) If $n = 0$, $5^0 + 3 = 4$ which is divisible by 4 $\therefore$ P_0 is true

(2) If P_k is true, then $5^k + 3 = 4A$ where $A \in \mathbb{Z}$

Now $5^{k+1} + 3 = 5 \times 5^k + 3$

$$\begin{aligned}
&= 5[4A - 3] + 3 \quad \{\text{using } P_k\} \\
&= 20A - 15 + 3 \\
&= 20A - 12 \\
&= 4(5A - 3) \quad \text{where } 5A - 3 \in \mathbb{Z}, \text{ as } A \in \mathbb{Z}
\end{aligned}$$

So, $5^{k+1} + 3$ is divisible by 4

Since P_0 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z}$, $n \geqslant 0$ {Principle of mathematical induction}

5 P_n is: $1 \times 2^2 + 2 \times 3^2 + 3 \times 4^2 + ... + n(n+1)^2 = \dfrac{n(n+1)(n+2)(3n+5)}{12}$ for all $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, LHS $= 1 \times 2^2 = 4$ and RHS $= \dfrac{1 \times 2 \times 3 \times 8}{12} = \dfrac{48}{12} = 4$ $\therefore$ P_1 is true

(2) If P_k is true, then $1 \times 2^2 + 2 \times 3^2 + 3 \times 4^2 + ... + k(k+1)^2 = \dfrac{k(k+1)(k+2)(3k+5)}{12}$

$$\begin{aligned}
\therefore\quad & 1\times 2^2+2\times 3^2+3\times 4^2+\dots+k(k+1)^2+(k+1)(k+2)^2\\
&=\frac{k(k+1)(k+2)(3k+5)}{12}+(k+1)(k+2)^2 \qquad \{\text{using } P_k\}\\
&=\frac{k(k+1)(k+2)(3k+5)}{12}+\frac{12(k+1)(k+2)^2}{12} \qquad \{\text{equalising denominators}\}\\
&=\frac{(k+1)(k+2)[k(3k+5)+12(k+2)]}{12}\\
&=\frac{(k+1)(k+2)[3k^2+5k+12k+24]}{12}\\
&=\frac{(k+1)(k+2)(3k^2+17k+24)}{12}\\
&=\frac{(k+1)(k+2)(k+3)(3k+8)}{12}\\
&=\frac{(k+1)([k+1]+1)([k+1]+2)(3[k+1]+5)}{12}
\end{aligned}$$

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n\in\mathbb{Z}^+$ {Principle of mathematical induction}

6 P_n is: $5^n+3^n\geqslant 2^{2n+1}$ for all $n\in\mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n=1$, we have $5^1+3^1\geqslant 2^3$

$\therefore\ 8\geqslant 8$ which is true, so P_1 is true

(2) If P_k is true, then $5^k+3^k\geqslant 2^{2k+1}$, so $5^k\geqslant 2^{2k+1}-3^k$

$$\begin{aligned}
\text{Now}\quad 5^{k+1}+3^{k+1}&=5\times 5^k+3\times 3^k\\
&\geqslant 5(2^{2k+1}-3^k)+3\times 3^k \qquad \{\text{using } P_k\}\\
\therefore\quad 5^{k+1}+3^{k+1}&\geqslant 5\times 2^{2k+1}-5\times 3^k+3\times 3^k\\
\therefore\quad 5^{k+1}+3^{k+1}&\geqslant 5\times 2^{2k+1}-2\times 3^k\\
&\geqslant 5\times 2^{2k+1}-2\times 4^k \qquad \{4^k\geqslant 3^k \text{ as } k\geqslant 0\}\\
\therefore\quad 5^{k+1}+3^{k+1}&\geqslant 5\times 2^{2k+1}-2^{2k+1}\\
\therefore\quad 5^{k+1}+3^{k+1}&\geqslant 4\times 2^{2k+1}\\
\therefore\quad 5^{k+1}+3^{k+1}&\geqslant 2^{2k+3}\\
\therefore\quad 5^{k+1}+3^{k+1}&\geqslant 2^{2(k+1)+1} \qquad \therefore\ P_{k+1} \text{ is true.}
\end{aligned}$$

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n\in\mathbb{Z}^+$ {Principle of mathematical induction}

7 P_n is: if $u_1=9$ and $u_{n+1}=2u_n+3(5^n)$ then $u_n=2^{n+1}+5^n$ for all $n\in\mathbb{Z}^+$.

Proof: (By the principle of mathematical induction)

(1) If $n=1$, $u_1=2^2+5^1=9$ ✓ so P_1 is true

(2) If P_k is true, then $u_k=2^{k+1}+5^k$ and
$$\begin{aligned}
u_{k+1}&=2u_k+3(5^k)\\
&=2(2^{k+1}+5^k)+3(5^k) \qquad \{\text{using } P_k\}\\
&=2^{k+2}+2(5^k)+3(5^k)\\
&=2^{k+2}+5(5^k)\\
&=2^{k+2}+5^{k+1} \qquad \therefore\ P_{k+1} \text{ is true.}
\end{aligned}$$

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n\in\mathbb{Z}^+$ {Principle of mathematical induction}

REVIEW SET 9C

1 P_n is: $1 \times 3 + 2 \times 4 + 3 \times 5 + 4 \times 6 + ... + n(n+2) = \dfrac{n(n+1)(2n+7)}{6}$ for all $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, LHS $= 1 \times 3 = 3$ and RHS $= \dfrac{1 \times 2 \times 9}{6} = \dfrac{18}{6} = 3$ $\therefore$ P_1 is true.

(2) If P_k is true, then $1 \times 3 + 2 \times 4 + 3 \times 5 + ... + k(k+2) = \dfrac{k(k+1)(2k+7)}{6}$

$$\begin{aligned}
\therefore \quad & 1 \times 3 + 2 \times 4 + 3 \times 5 + ... + k(k+2) + (k+1)(k+3) \\
&= \frac{k(k+1)(2k+7)}{6} + (k+1)(k+3) \qquad \{\text{using } P_k\} \\
&= \frac{k(k+1)(2k+7)}{6} + \frac{6(k+1)(k+3)}{6} \qquad \{\text{equalising denominators}\} \\
&= \frac{(k+1)[k(2k+7) + 6(k+3)]}{6} \\
&= \frac{(k+1)[2k^2 + 13k + 18]}{6} \\
&= \frac{(k+1)(k+2)(2k+9)}{6} \\
&= \frac{(k+1)([k+1]+1)(2[k+1]+7)}{6} \qquad \therefore \; P_{k+1} \text{ is true.}
\end{aligned}$$

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

2 P_n is: $7^n - 1$ is divisible by 6 for all $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, $7^1 - 1 = 6$ which is divisible by 6 $\therefore$ P_1 is true.

(2) If P_k is true, then $7^k - 1 = 6A$ where $A \in \mathbb{Z}$

$$\begin{aligned}
\text{Now } \; 7^{k+1} - 1 &= 7 \times 7^k - 1 \\
&= 7[6A + 1] - 1 \qquad \{\text{using } P_k\} \\
&= 42A + 7 - 1 \\
&= 42A + 6 \\
&= 6(7A + 1) \quad \text{where } 7A + 1 \in \mathbb{Z}
\end{aligned}$$

Thus $7^{k+1} - 1$ is divisible by 6.

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

3 P_n is: $1^3 + 3^3 + 5^3 + ... + (2n-1)^3 = n^2(2n^2 - 1)$ for all $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, LHS $= 1^3 = 1$ and RHS $= 1^2(2-1) = 1 \times 1 = 1$ $\therefore$ P_1 is true.

(2) If P_k is true, then $1^3 + 3^3 + 5^3 + ... + (2k-1)^3 = k^2(2k^2 - 1)$

$$\begin{aligned}
\therefore \quad & 1^3 + 3^3 + 5^3 + ... + (2k-1)^3 + (2k+1)^3 \\
&= k^2(2k^2 - 1) + (2k+1)^3 \qquad \{\text{using } P_k\} \\
&= 2k^4 - k^2 + (2k)^3 + 3(2k)^2 1 + 3(2k)1^2 + 1^3 \\
&= 2k^4 - k^2 + 8k^3 + 12k^2 + 6k + 1 \\
&= 2k^4 + 8k^3 + 11k^2 + 6k + 1 \\
&= (k+1)^2(2k^2 + 4k + 1) \\
&= (k+1)^2(2[k^2 + 2k + 1] - 1) \\
&= (k+1)^2(2[k+1]^2 - 1)
\end{aligned}$$

-1	2	8	11	6	1
	0	-2	-6	-5	-1
-1	2	6	5	1	0
	0	-2	-4	-1	
	2	4	1	0	

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

4 P_n is: $3^n - 1 - 2n$ is divisible by 4 for all $n \in \mathbb{N}$.

Proof: (By the principle of mathematical induction)

(1) If $n = 0$, $3^0 - 1 - 2(0) = 1 - 1 - 0 = 0$ which is divisible by 4 $\therefore$ P_0 is true

(2) If P_k is true, then $3^k - 1 - 2k = 4A$ where $A \in \mathbb{Z}$

$$\begin{aligned}\text{Now } 3^{k+1} - 1 - 2(k+1) &= 3 \times 3^k - 1 - 2k - 2 \\ &= 3[4A + 1 + 2k] - 2k - 3 \quad \{\text{using } P_k\} \\ &= 12A + 3 + 6k - 2k - 3 \\ &= 12A + 4k \\ &= 4(3A + k) \text{ where } 3A + k \in \mathbb{Z}\end{aligned}$$

$\therefore$ $3^{k+1} - 1 - 2(k+1)$ is divisible by 4

Since P_0 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{N}$ {Principle of mathematical induction}

5 P_n is: $\dfrac{1}{1 \times 3} + \dfrac{1}{3 \times 5} + \dfrac{1}{5 \times 7} + ... + \dfrac{1}{(2n-1)(2n+1)} = \dfrac{n}{2n+1}$ for all $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, LHS $= \dfrac{1}{1 \times 3} = \dfrac{1}{3}$, RHS $= \dfrac{1}{2+1} = \dfrac{1}{3}$ $\therefore$ P_1 is true.

(2) If P_k is true, then $\dfrac{1}{1 \times 3} + \dfrac{1}{3 \times 5} + \dfrac{1}{5 \times 7} + ... + \dfrac{1}{(2k-1)(2k+1)} = \dfrac{k}{2k+1}$

$$\begin{aligned}\therefore \quad & \frac{1}{1 \times 3} + \frac{1}{3 \times 5} + \frac{1}{5 \times 7} + ... + \frac{1}{(2k-1)(2k+1)} + \frac{1}{(2k+1)(2k+3)} \\ &= \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)} \quad \{\text{using } P_k\} \\ &= \frac{k}{2k+1}\left(\frac{2k+3}{2k+3}\right) + \frac{1}{(2k+1)(2k+3)} \quad \{\text{equalising denominators}\} \\ &= \frac{2k^2 + 3k + 1}{(2k+1)(2k+3)} \\ &= \frac{(k+1)(2k+1)}{(2k+1)(2k+3)} \\ &= \frac{(k+1)}{2(k+1)+1}\end{aligned}$$

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

6 P_n is: if $u_1 = 5$ and $u_{n+1} = 2u_n - 3(-1)^n$ for $n \in \mathbb{Z}^+$, then $u_n = 3(2^n) + (-1)^n$

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, $3(2^1) + (-1)^1 = 6 - 1 = 5 = u_1$, so P_1 is true.

(2) If P_k is true, then $u_k = 3(2^k) + (-1)^k$,

$$\begin{aligned}\text{and } u_{k+1} &= 2u_k - 3(-1)^k \\ &= 2[3(2^k) + (-1)^k] - 3(-1)^k \quad \{\text{using } P_k\} \\ &= 6(2^k) + 2(-1)^k - 3(-1)^k \\ &= 3(2^{k+1}) - (-1)^k \\ &= 3(2^{k+1}) + (-1)^{k+1}\end{aligned}$$

$\therefore$ P_{k+1} is true.

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

7 $\sqrt[n]{n!} \leqslant \dfrac{n+1}{2} \quad \Leftrightarrow \quad n! \leqslant \left(\dfrac{n+1}{2}\right)^n$

$\therefore$ P_n is: $n! \leqslant \left(\dfrac{n+1}{2}\right)^n$ for $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, we have $1! \leqslant \left(\dfrac{1+1}{2}\right)^1$

$\therefore$ $1 \leqslant 1$ which is true, so P_1 is true

(2) If P_k is true, then $k! \leqslant \left(\dfrac{k+1}{2}\right)^k$

Now
$$\begin{aligned}(k+1)! &= (k+1)k! \\ &\leqslant (k+1)\left(\frac{k+1}{2}\right)^k \\ &\leqslant \frac{(k+1)^{k+1}}{2^k} \quad \text{.... (1)}\end{aligned}$$

Also,
$$\begin{aligned}(k+1)^k &= k^k + \tbinom{k}{1} k^{k-1} + ... \\ &= k^k + k^k + ... \\ &= 2k^k + ...\end{aligned}$$

$\therefore$ $(k+1)^k \geqslant 2k^k$ for $k \geqslant 1$

$\therefore$ $\left(\dfrac{k+1}{k}\right)^k \geqslant 2$

$\therefore$ $\frac{1}{2}\left(\dfrac{k+1}{k}\right)^k \geqslant 1$

$\therefore$ $\frac{1}{2}\left(\dfrac{k+2}{k+1}\right)^{k+1} \geqslant 1$ .... (2)

{replacing k with $k+1$}

Using (1) and (2),
$$\begin{aligned}(k+1)! &\leqslant \frac{(k+1)^{k+1}}{2^k} \times \tfrac{1}{2}\left(\frac{k+2}{k+1}\right)^{k+1} \\ &\leqslant \frac{(k+1)^{k+1}}{2^{k+1}} \times \frac{(k+2)^{k+1}}{(k+1)^{k+1}} \\ &\leqslant \frac{(k+2)^{k+1}}{2^{k+1}} \\ &\leqslant \left(\frac{[k+1]+1}{2}\right)^{k+1}\end{aligned}$$

Since P_1 is true, and P_{k+1} is true whenever P_k is true, then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

Chapter 10

THE UNIT CIRCLE AND RADIAN MEASURE

EXERCISE 10A

1 **a** $180^\circ = \pi$ radians
$\therefore\ 90^\circ = \frac{\pi}{2}$ radians

b $180^\circ = \pi$ radians
$\therefore\ 60^\circ = \frac{\pi}{3}$ radians

c $180^\circ = \pi$ radians
$\therefore\ 30^\circ = \frac{\pi}{6}$ radians

d $180^\circ = \pi$ radians
$\therefore\ 18^\circ = \frac{\pi}{10}$ radians

e $180^\circ = \pi$ radians
$\therefore\ 9^\circ = \frac{\pi}{20}$ radians

f $180^\circ = \pi$ radians
$\therefore\ 45^\circ = \frac{\pi}{4}$ radians
$\therefore\ 135^\circ = \frac{3\pi}{4}$ radians

g $180^\circ = \pi$ radians
$\therefore\ 45^\circ = \frac{\pi}{4}$ radians
$\therefore\ 225^\circ = \frac{5\pi}{4}$ radians

h $180^\circ = \pi$ radians
$\therefore\ 90^\circ = \frac{\pi}{2}$ radians
$\therefore\ 270^\circ = \frac{3\pi}{2}$ radians

i $360^\circ = 2 \times 180^\circ$
$= 2\pi$ radians

j $720^\circ = 4 \times 180^\circ$
$= 4\pi$ radians

k $180^\circ = \pi$ radians
$\therefore\ 45^\circ = \frac{\pi}{4}$ radians
$\therefore\ 315^\circ = \frac{7\pi}{4}$ radians

l $180^\circ = \pi$ radians
$\therefore\ 540^\circ = 3\pi$ radians

m $180^\circ = \pi$ radians
$\therefore\ 36^\circ = \frac{\pi}{5}$ radians

n $180^\circ = \pi$ radians
$\therefore\ 10^\circ = \frac{\pi}{18}$ radians
$\therefore\ 80^\circ = \frac{8\pi}{18}$ radians
$= \frac{4\pi}{9}$ radians

o $180^\circ = \pi$ radians
$\therefore\ 10^\circ = \frac{\pi}{18}$ radians
$\therefore\ 230^\circ = \frac{23\pi}{18}$ radians

2 **a** 36.7°
$= 36.7 \times \frac{\pi}{180}$ radians
≈ 0.641 radians

b 137.2°
$= 137.2 \times \frac{\pi}{180}$ radians
≈ 2.39 radians

c 317.9°
$= 317.9 \times \frac{\pi}{180}$ radians
≈ 5.55 radians

d 219.6°
$= 219.6 \times \frac{\pi}{180}$ radians
≈ 3.83 radians

e 396.7°
$= 396.7 \times \frac{\pi}{180}$ radians
≈ 6.92 radians

3 **a** $\frac{\pi}{5}$
$= \frac{180^\circ}{5}$
$= 36^\circ$

b $\frac{3\pi}{5}$
$= \frac{3\times180^\circ}{5}$
$= 108^\circ$

c $\frac{3\pi}{4}$
$= \frac{3\times180^\circ}{4}$
$= 135^\circ$

d $\frac{\pi}{18}$
$= \frac{180^\circ}{18}$
$= 10^\circ$

e $\frac{\pi}{9}$
$= \frac{180^\circ}{9}$
$= 20^\circ$

f $\frac{7\pi}{9}$
$= \frac{7\times180^\circ}{9}$
$= 140^\circ$

g $\frac{\pi}{10}$
$= \frac{180^\circ}{10}$
$= 18^\circ$

h $\frac{3\pi}{20}$
$= \frac{3\times180^\circ}{20}$
$= 27^\circ$

i $\frac{7\pi}{6}$
$= \frac{7\times180^\circ}{6}$
$= 210^\circ$

j $\frac{\pi}{8}$
$= \frac{180^\circ}{8}$
$= 22.5^\circ$

4 **a** 2^c
$= 2 \times \frac{180}{\pi}$ degrees
$\approx 114.59^\circ$

b 1.53^c
$= 1.53 \times \frac{180}{\pi}$ degrees
$\approx 87.66^\circ$

c 0.867^c
$= 0.867 \times \frac{180}{\pi}$ degrees
$\approx 49.68^\circ$

d 3.179^c
$= 3.179 \times \frac{180}{\pi}$ degrees
$\approx 182.14^\circ$

e 5.267^c
$= 5.267 \times \frac{180}{\pi}$ degrees
$\approx 301.78^\circ$

5 **a**

Degrees	0	45	90	135	180	225	270	315	360
Radians	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π

b

Degrees	0	30	60	90	120	150	180	210	240	270	300	330	360
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π

EXERCISE 10B

1 **a** arc length $= \frac{7\pi}{4} \times 9$
≈ 49.5 cm

area $= \frac{1}{2} \times \frac{7\pi}{4} \times 9^2$
≈ 223 cm^2

b arc length $= 4.67 \times 4.93$
≈ 23.0 cm

area $= \frac{1}{2}(4.67) \times 4.93^2$
≈ 56.8 cm^2

2 **a** $\theta = 107.9°, \quad l = 5.92$

$\therefore \quad \left(\frac{107.9}{360}\right) \times 2\pi \times r = 5.92$

$$\therefore \quad r = \frac{5.92 \times 360}{107.9 \times 2 \times \pi}$$

$\therefore \quad r \approx 3.14$ m

b area $= \left(\frac{107.9}{360}\right) \times \pi \times (3.1436)^2$
≈ 9.30 m^2

3 **a** area $= \frac{1}{2}\theta r^2$

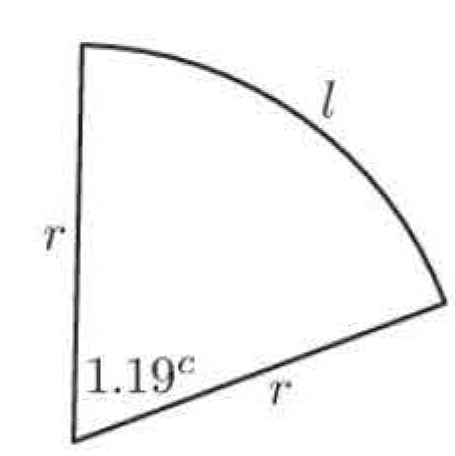
$\therefore \quad 20.8 = \frac{1}{2}(1.19) \times r^2$

$$\therefore \quad \frac{20.8 \times 2}{1.19} = r^2$$

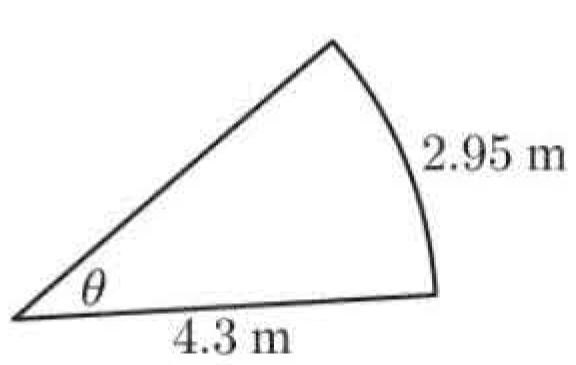
$$\therefore \quad r = \sqrt{\frac{20.8 \times 2}{1.19}}$$

$\therefore \quad r \approx 5.91$ cm

b perimeter
$= l + 2r$
$\approx 1.19 \times 5.912 + 2 \times 5.912$
≈ 18.9 cm



4 **a**



$l = \theta \times r$
$\therefore \quad 2.95 = \theta \times 4.3$
$\therefore \quad \theta \approx 0.686^c$

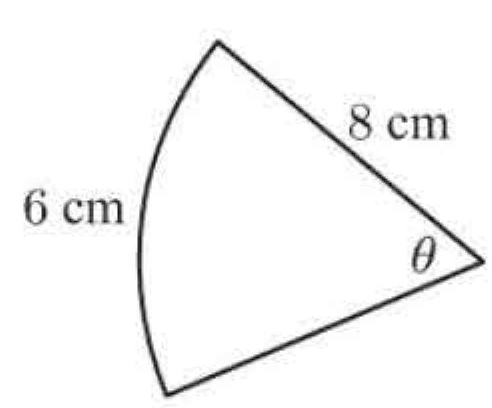
b area $= \frac{1}{2}\theta r^2$

$\therefore \quad 30 = \frac{1}{2} \times \theta \times 10^2$

$$\therefore \quad \frac{30 \times 2}{100} = \theta$$

$\therefore \quad \theta = 0.6^c$

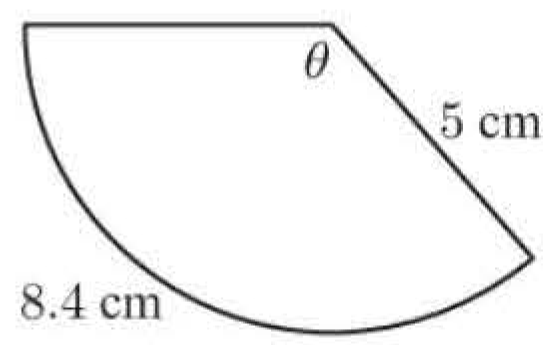
5 **a**



$l = \theta r$
$\therefore \quad 6 = \theta \times 8$
$\therefore \quad \theta = \frac{6}{8}$
$\therefore \quad \theta = 0.75^c$

area $= \frac{1}{2}\theta r^2$
$= \frac{1}{2}(0.75) \times 8^2$
$= 24$ cm^2

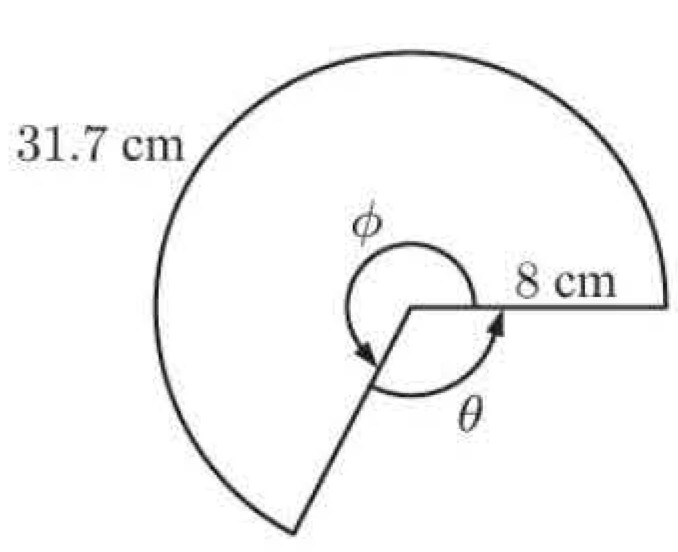
b



$l = \theta r$
$\therefore \quad 8.4 = \theta \times 5$
$\therefore \quad \theta = \frac{8.4}{5}$
$\therefore \quad \theta = 1.68^c$

area $= \frac{1}{2}\theta r^2$
$= \frac{1}{2}(1.68) \times 5^2$
$= 21$ cm^2

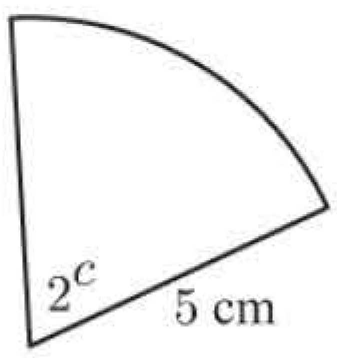
c



$l = \phi r$
$\therefore \quad 31.7 = \phi \times 8$
$\therefore \quad \phi = \frac{31.7}{8}$
$\therefore \quad \phi \approx 3.96^c$
But $\quad \theta = 2\pi - \phi$
$\therefore \quad \theta \approx 2.32^c$

area $= \frac{1}{2}\phi r^2$
$= \frac{1}{2} \times \frac{31.7}{8} \times 8^2$
$= 126.8$ cm^2

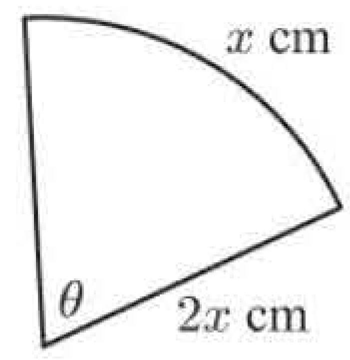
6



$$\begin{aligned}\text{arc length} &= \theta r\\ &= 2\times 5\\ &= 10\text{ cm}\end{aligned}$$

$$\begin{aligned}\text{area} &= \tfrac{1}{2}\theta r^2\\ &= \tfrac{1}{2}\times 2\times 5^2\\ &= 25\text{ cm}^2\end{aligned}$$

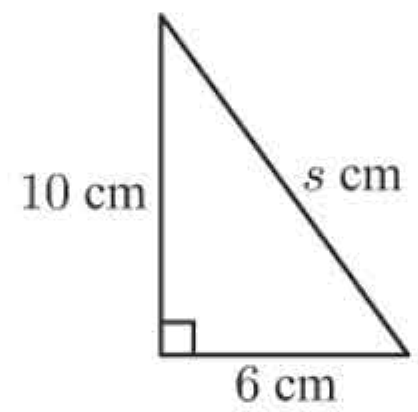
7



$$\begin{aligned}\text{arc length} &= \theta r\\ \therefore\quad x &= \theta(2x)\\ \therefore\quad \theta &= \tfrac{1}{2}\end{aligned}$$

$$\begin{aligned}\text{area} &= \tfrac{1}{2}\theta r^2\\ &= \tfrac{1}{2}\times(\tfrac{1}{2})\times(2x)^2\\ &= x^2\text{ cm}^2\end{aligned}$$

8 **a**



$$\begin{aligned}s^2 &= 6^2+10^2 \quad \{\text{Pythagoras}\}\\ \therefore\quad s &= \sqrt{6^2+10^2}\\ \therefore\quad s &\approx 11.6619\\ \therefore\quad s &\approx 11.7\end{aligned}$$

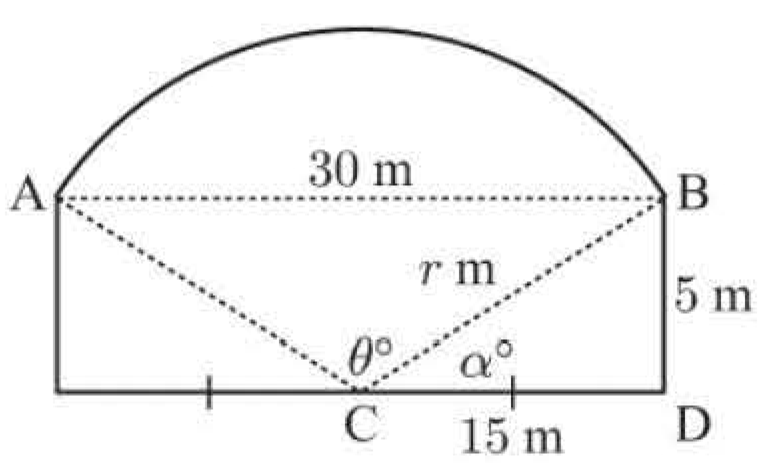
$\therefore$ slant length is 11.7 cm.

b $r = s \approx 11.7$

c
$$\begin{aligned}\text{arc length} &= \text{circumference of cone base}\\ &= 2\pi\times 6\\ &\approx 37.6991\\ &\approx 37.7\text{ cm}\end{aligned}$$

d
$$\begin{aligned}\text{arc length} &= \theta r\\ \therefore\quad 37.6991 &\approx \theta\times 11.6619\\ \therefore\quad \theta &\approx \frac{37.6991}{11.6619}\\ \therefore\quad \theta &\approx 3.23\text{ radians}\end{aligned}$$

9



a
$$\begin{aligned}\tan\alpha &= \tfrac{5}{15}\\ \therefore\quad \alpha &= \tan^{-1}\left(\tfrac{1}{3}\right)\\ \therefore\quad \alpha &\approx 18.43\end{aligned}$$

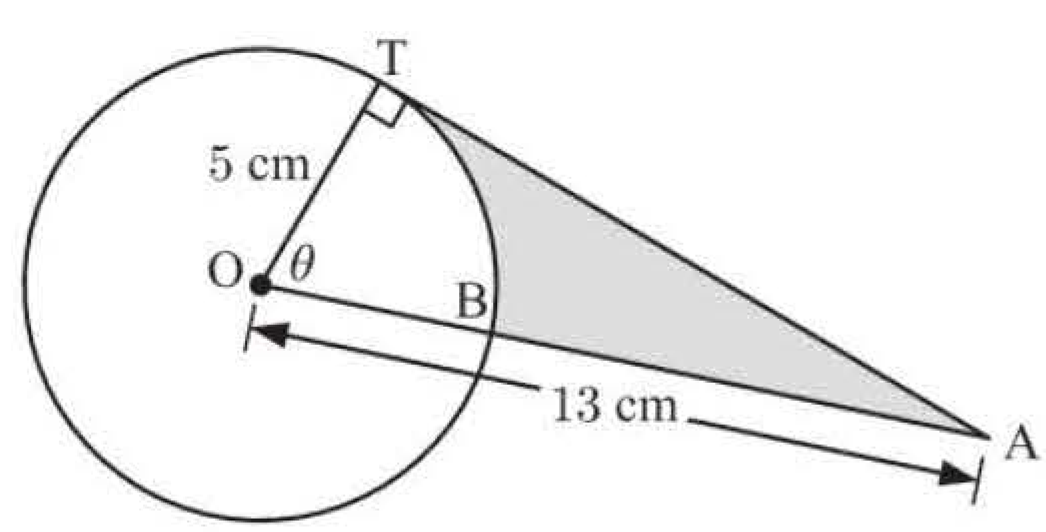
b
$$\begin{aligned}\theta+2\alpha &= 180 \quad \{\text{angles on a line}\}\\ \therefore\quad \theta &\approx 180-2\times 18.43\\ \therefore\quad \theta &\approx 143.1\end{aligned}$$

c
$$\begin{aligned}\text{area} &= 2\times\text{area of }\triangle\text{CDB}+\text{area of sector}\\ &= 2\times\tfrac{1}{2}\times\text{CD}\times\text{BD}+\left(\tfrac{\theta}{360}\right)\times\pi\times r^2\end{aligned}$$

Now $r^2 = 5^2+15^2 = 250$

$$\begin{aligned}\therefore\quad \text{area} &\approx 2\times\tfrac{1}{2}\times 15\times 5+\left(\tfrac{143.1}{360}\right)\times\pi\times 250\\ &\approx 387\text{ m}^2\end{aligned}$$

10 Since [AT] is a tangent, $\mathrm{O\widehat{T}A}$ is a right angle.



$$\begin{aligned}\therefore\quad \cos\theta &= \tfrac{5}{13}\\ \therefore\quad \theta &\approx 67.38°\end{aligned}$$

$$\begin{aligned}\text{arc length BT} &= \left(\tfrac{\theta}{360}\right)\times 2\pi r\\ &\approx \tfrac{67.38}{360}\times 2\times\pi\times 5\\ &\approx 5.88\text{ cm}\end{aligned}$$

$$\begin{aligned}\text{AT}^2+\text{OT}^2 &= \text{OA}^2 \quad \{\text{Pythagoras}\}\\ \therefore\quad \text{AT}^2 &= 13^2-5^2\\ \therefore\quad \text{AT} &= 12\text{ cm}\end{aligned}$$

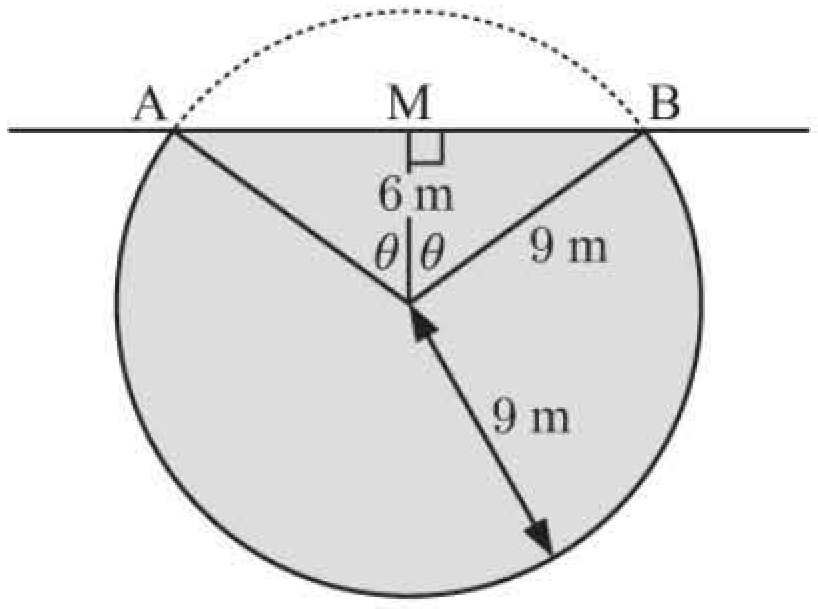
$$\begin{aligned}\therefore\quad \text{perimeter} &= \text{AT}+\text{arc length BT}+\text{AB}\\ &\approx 12+5.88+(13-5)\\ &\approx 25.9\text{ cm}\end{aligned}$$

11 **a**
$$\begin{aligned}l &= \left(\tfrac{\theta}{360}\right)\times 2\pi r\\ &= \frac{\frac{1}{60}}{360}\times 2\times\pi\times 6370\text{ km}\\ &\approx 1.853\text{ km}\end{aligned}$$

b $\text{speed} = \dfrac{\text{distance}}{\text{time}}$

$$\begin{aligned}\therefore\quad \text{time} &= \frac{\text{distance}}{\text{speed}}\\ &= \frac{2130\text{ km}}{480\text{ n miles h}^{-1}}\\ &= \frac{2130\text{ km}}{480\times 1.853\text{ km h}^{-1}}\\ &\approx 2.395\text{ hours}\\ &\approx 2\text{ hours }24\text{ min}\end{aligned}$$

12



$\cos\theta = \frac{6}{9} = \frac{2}{3}$

$\therefore \quad \theta = \cos^{-1}\left(\frac{2}{3}\right)$

$\therefore \quad \theta \approx 48.19°$

So, $360 - 2\theta \approx 263.62°$

Now $\text{MB} = \sqrt{9^2 - 6^2}$

$= \sqrt{45}$

$\therefore$ available feeding area

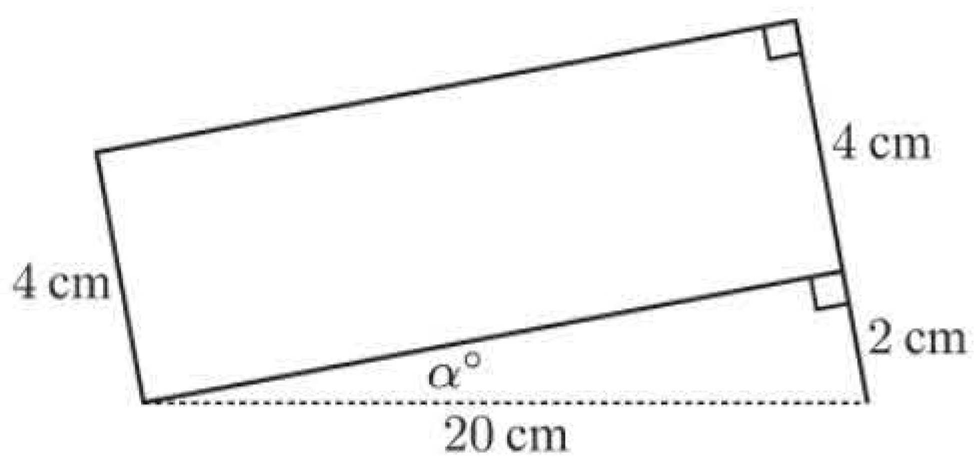
$=$ area of $\triangle$ + area of sector

$\approx \frac{1}{2} \times 2 \times \sqrt{45} \times 6$

$+ \left(\frac{263.62}{360}\right) \times \pi \times 9^2$

$\approx 227 \text{ m}^2$

13



a $\sin\alpha = \frac{2}{20} = 0.1$

$\therefore \quad \alpha = \sin^{-1}(0.1)$

$\therefore \quad \alpha \approx 5.739$

b $\theta + 90 + 90 + 2\alpha = 360$

$\therefore \quad \theta = 180 - 2\alpha$

$\approx 180 - 2 \times 5.739$

≈ 168.5

c $\phi + \theta = 360$

$\therefore \quad \phi \approx 360 - 168.5$

$\therefore \quad \phi \approx 191.5$

d length of belt

$= 2 \times \sqrt{20^2 - 2^2}$

$+ \frac{\theta}{360} \times 2\pi \times 4$

$+ \frac{\phi}{360} \times 2\pi \times 6$

≈ 71.62 cm

EXERCISE 10C

1 **a** **i** A$(\cos 26°, \sin 26°)$, B$(\cos 146°, \sin 146°)$, C$(\cos 199°, \sin 199°)$

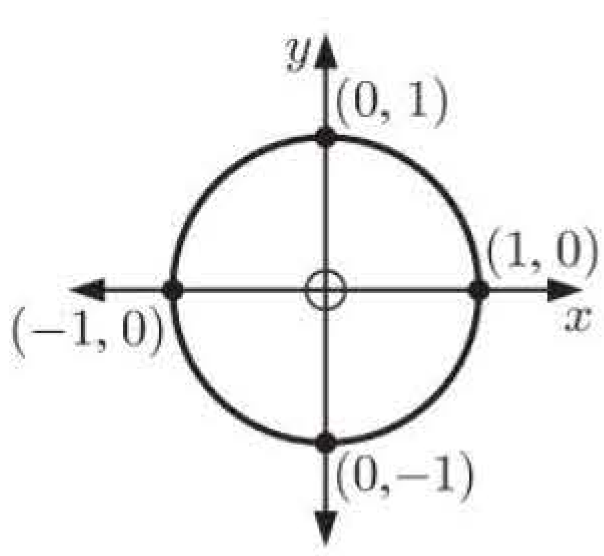
ii A$(0.899, 0.438)$, B$(-0.829, 0.559)$, C$(-0.946, -0.326)$

b **i** A$(\cos 123°, \sin 123°)$, B$(\cos 251°, \sin 251°)$, C$(\cos(-35°), \sin(-35°))$

ii A$(-0.545, 0.839)$, B$(-0.326, -0.946)$, C$(0.819, -0.574)$

2

θ (degrees)	0°	90°	180°	270°	360°	450°
θ (radians)	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$
sine	0	1	0	-1	0	1
cosine	1	0	-1	0	1	0
tangent	0	undef.	0	undef.	0	undef.



3 **a** **i** $\frac{1}{\sqrt{2}} \approx 0.707$

ii $\frac{\sqrt{3}}{2} \approx 0.866$

b

θ (degrees)	30°	45°	60°	135°	150°	240°	315°
θ (radians)	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	$\frac{4\pi}{3}$	$\frac{7\pi}{4}$
sine	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{2}}$
cosine	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\frac{1}{\sqrt{2}}$
tangent	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	$\sqrt{3}$	-1

4 **a**

Quadrant	*Degree measure*	*Radian measure*	$\cos\theta$	$\sin\theta$	$\tan\theta$
1	$0° < \theta < 90°$	$0 < \theta < \frac{\pi}{2}$	+ve	+ve	+ve
2	$90° < \theta < 180°$	$\frac{\pi}{2} < \theta < \pi$	−ve	+ve	−ve
3	$180° < \theta < 270°$	$\pi < \theta < \frac{3\pi}{2}$	−ve	−ve	+ve
4	$270° < \theta < 360°$	$\frac{3\pi}{2} < \theta < 2\pi$	+ve	−ve	−ve

b **i** 1 and 4

ii 2 and 3

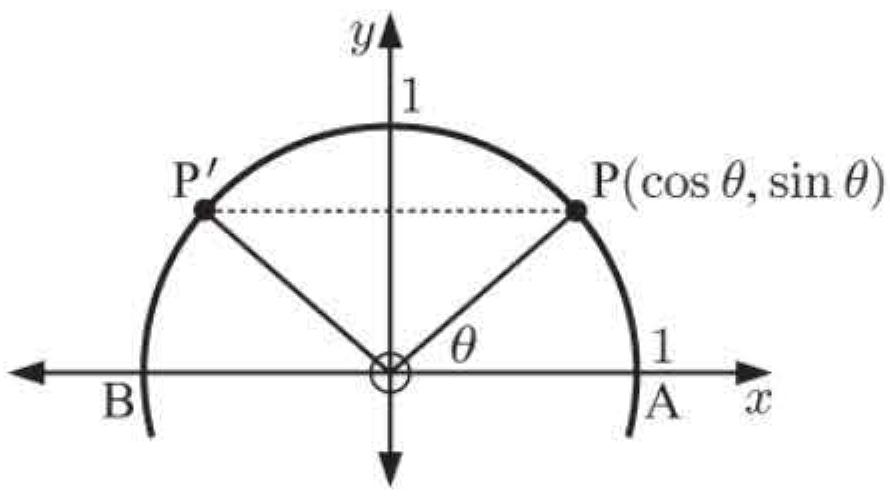
iii 3

iv 2

5 **a** **i** 0.985 **ii** 0.985 **iii** 0.866 **iv** 0.866 **v** 0.5 **vi** 0.5
vii 0.707 **viii** 0.707

b $\sin(180° - \theta) = \sin\theta$ as the points have the same y-coordinate.

c



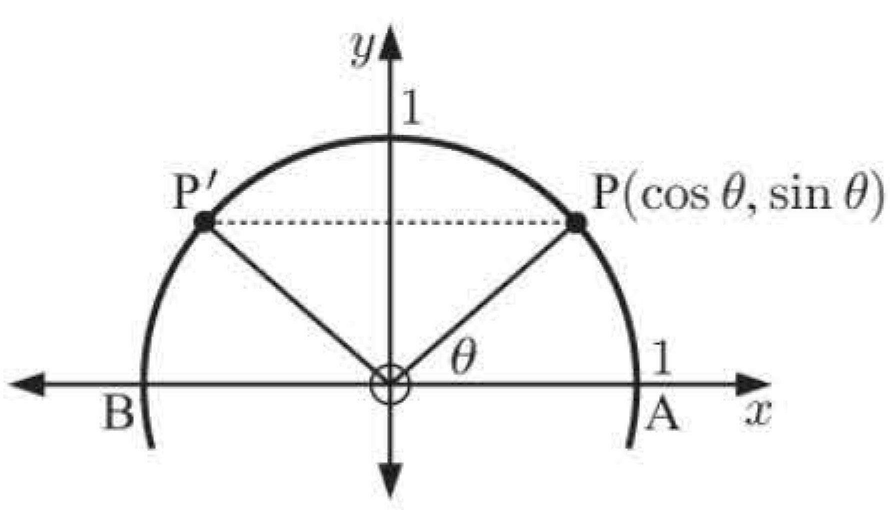
The diagram shows P reflected in the y-axis to P′, so $\text{P}'\widehat{\text{O}}\text{B} = \text{P}\widehat{\text{O}}\text{A} = \theta$, and P′ has coordinates $(-\cos\theta, \sin\theta)$.
But $\text{A}\widehat{\text{O}}\text{P}' = 180° - \theta$ $\{\text{A}\widehat{\text{O}}\text{P}' + \text{P}'\widehat{\text{O}}\text{B} = 180°\}$,
so P′ has coordinates $(\cos(180° - \theta), \sin(180° - \theta))$.
$\therefore$ $\sin(180° - \theta) = \sin\theta$ {equating y-coordinates of P′}

d **i** $180° - 45° = 135°$ **ii** $180° - 51° = 129°$ **iii** $\pi - \frac{\pi}{3} = \frac{2\pi}{3}$
iv $\pi - \frac{\pi}{6} = \frac{5\pi}{6}$ {using $\sin(180° - \theta) = \theta$}

6 **a** **i** 0.342 **ii** -0.342 **iii** 0.5 **iv** -0.5 **v** 0.906 **vi** -0.906
vii 0.174 **viii** -0.174

b $\cos(180° - \theta) = -\cos\theta$

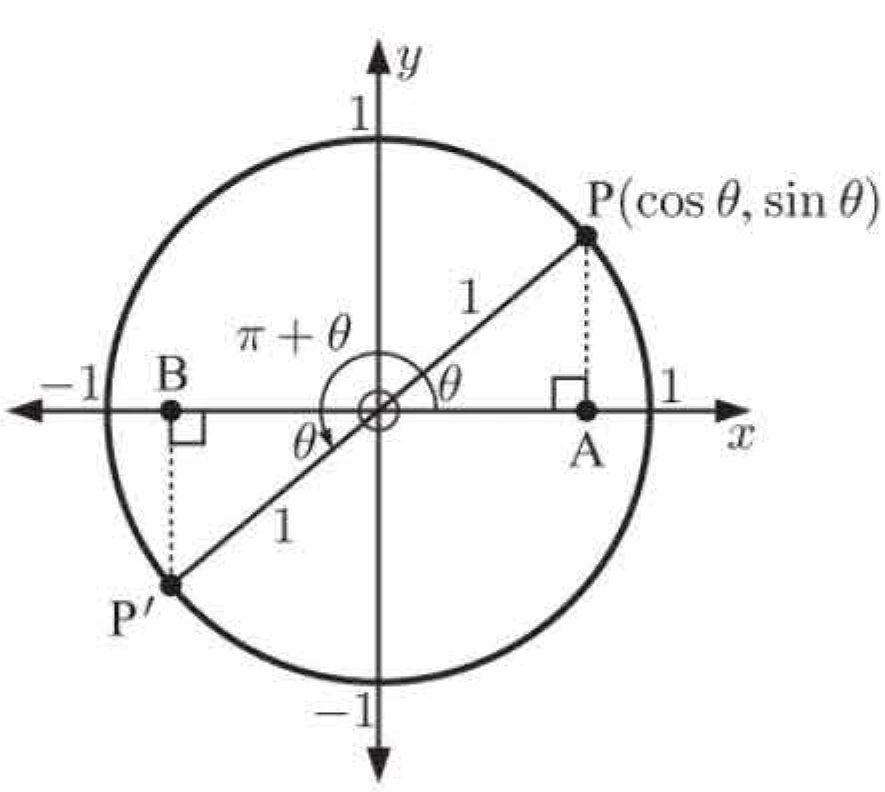
c



The diagram shows P reflected in the y-axis to P′, so $\text{P}'\widehat{\text{O}}\text{B} = \text{P}\widehat{\text{O}}\text{A} = \theta$, and P′ has coordinates $(-\cos\theta, \sin\theta)$.
But $\text{A}\widehat{\text{O}}\text{P}' = 180° - \theta$ $\{\text{A}\widehat{\text{O}}\text{P}' + \text{P}'\widehat{\text{O}}\text{B} = 180°\}$,
so P′ has coordinates $(\cos(180° - \theta), \sin(180° - \theta))$.
$\therefore$ $\cos(180° - \theta) = -\cos\theta$ {equating x-coordinates of P′}

d **i** $180° - 40° = 140°$ **ii** $180° - 19° = 161°$ **iii** $\pi - \frac{\pi}{5} = \frac{4\pi}{5}$
iv $\pi - \frac{2\pi}{5} = \frac{3\pi}{5}$ {using $\cos(180° - \theta) = -\cos\theta$}

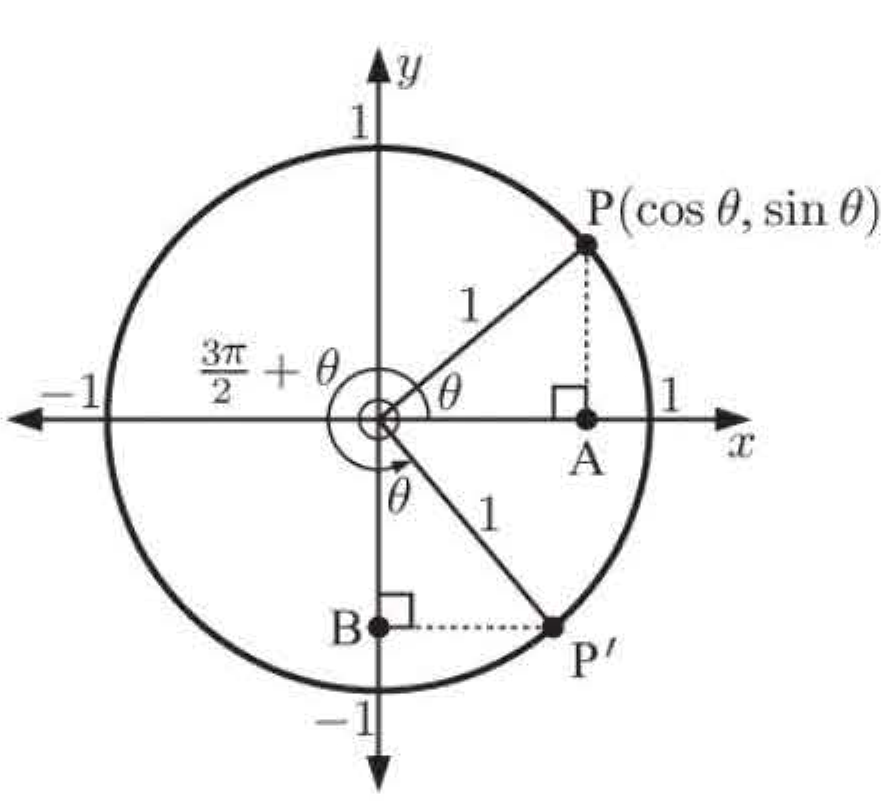
7 **a**



For $0 < \theta < \frac{\pi}{2}$:
The diagram shows P rotated through π to P′, so OP′ makes an angle of $\pi + \theta$ with the positive x-axis,
and $\text{P}'\widehat{\text{O}}\text{B} = \text{P}\widehat{\text{O}}\text{A} = \theta$ {vertically opposite angles}.
In $\triangle$s P′OB and POA:
- $\text{OP}' = \text{OP}$
- $\text{P}'\widehat{\text{O}}\text{B} = \text{P}\widehat{\text{O}}\text{A}$
- $\text{P}'\widehat{\text{B}}\text{O} = \text{P}\widehat{\text{A}}\text{O}$

$\therefore$ $\triangle$s P′OB and POA are congruent {AAcorS}
$\therefore$ $\text{OB} = \text{OA} = \cos\theta$
and $\text{BP}' = \text{AP} = \sin\theta$
$\therefore$ P′ has coordinates $(-\cos\theta, -\sin\theta)$
But P′ has coordinates $(\cos(\pi + \theta), \sin(\pi + \theta))$
$\therefore$ $\cos(\pi + \theta) = -\cos\theta$ and $\sin(\pi + \theta) = -\sin\theta$

b



For $0 < \theta < \frac{\pi}{2}$:
The diagram shows P rotated through $\frac{3\pi}{2}$ to P′,
so OP′ makes an angle of $\frac{3\pi}{2} + \theta$ with the positive x-axis.

reflex $\text{A}\widehat{\text{O}}\text{B} = \frac{3\pi}{2}$ $\therefore$ $\text{B}\widehat{\text{O}}\text{P}' = \text{reflex } \text{A}\widehat{\text{O}}\text{P}' - \text{reflex } \text{A}\widehat{\text{O}}\text{B}$
$= \frac{3\pi}{2} + \theta - \frac{3\pi}{2}$
$= \theta$

In $\triangle$s P′OB and POA:
- $\text{OP}' = \text{OP}$
- $\text{B}\widehat{\text{O}}\text{P}' = \text{A}\widehat{\text{O}}\text{P}$
- $\text{P}'\widehat{\text{B}}\text{O} = \text{P}\widehat{\text{A}}\text{O}$

$\therefore$ $\triangle$s P′OB and POA are congruent {AAcorS}

$\therefore$ $P'B = PA = \sin\theta$

and $OB = OA = \cos\theta$

$\therefore$ P′ has coordinates $(\sin\theta, -\cos\theta)$

But P′ has coordinates $(\cos(\frac{3\pi}{2}+\theta), \sin(\frac{3\pi}{2}+\theta))$

$\therefore$ $\cos(\frac{3\pi}{2}+\theta) = \sin\theta$ and $\sin(\frac{3\pi}{2}+\theta) = -\cos\theta$

8 **a** $\sin 137°$
$= \sin(180-137)°$
$= \sin 43°$
≈ 0.6820

b $\sin 59°$
$= \sin(180-59)°$
$= \sin 121°$
≈ 0.8572

c $\cos 143°$
$= -\cos(180-143)°$
$= -\cos 37°$
≈ -0.7986

d $\cos 24°$
$= -\cos(180-24)°$
$= -\cos 156°$
≈ 0.9135

e $\sin 115°$
$= \sin(180-115)°$
$= \sin 65°$
≈ 0.9063

f $\cos 132°$
$= -\cos(180-132)°$
$= -\cos 48°$
≈ -0.6691

9 **a** $A\widehat{O}Q = 180° - \theta$ or $\pi - \theta$ radians

b [OQ] is a reflection of [OP] in the y-axis and so Q has coordinates $(-\cos\theta, \sin\theta)$.

c $\cos(180° - \theta) = -\cos\theta$, $\sin(180° - \theta) = \sin\theta$

10 **a**

θ^c	$\sin\theta$	$\sin(-\theta)$	$\cos\theta$	$\cos(-\theta)$
0.75	0.682	−0.682	0.732	0.732
1.772	0.980	−0.980	−0.200	−0.200
3.414	−0.269	0.269	−0.963	−0.963
6.25	−0.0332	0.0332	0.999	0.999
−1.17	−0.921	0.921	0.390	0.390

b Suspect that $\sin(-\theta) = -\sin\theta$ and $\cos(-\theta) = \cos\theta$.

c **i** P is reflected in the x-axis to Q, so Q has coordinates $(\cos\theta, -\sin\theta)$.
But Q has coordinates $(\cos(-\theta), \sin(-\theta))$.
$\therefore$ $Q(\cos(-\theta), \sin(-\theta)) = Q(\cos\theta, -\sin\theta)$.
So the suspicion is correct.

ii The point Q on the unit circle corresponds to the angle $(2\pi - \theta)$ and the angle $(-\theta)$.
$\therefore$ $\cos(2\pi - \theta) = \cos(-\theta)$
But $\cos(-\theta) = \cos\theta$ {from **c i**}
$\therefore$ $\cos(2\pi - \theta) = \cos\theta$

EXERCISE 10D.1

1 **a** $\cos^2\theta + \sin^2\theta = 1$
$\therefore$ $\cos^2\theta + (\frac{1}{2})^2 = 1$
$\therefore$ $\cos^2\theta = \frac{3}{4}$
$\therefore$ $\cos\theta = \pm\frac{\sqrt{3}}{2}$

b $\cos^2\theta + \sin^2\theta = 1$
$\therefore$ $\cos^2\theta + (-\frac{1}{3})^2 = 1$
$\therefore$ $\cos^2\theta = \frac{8}{9}$
$\therefore$ $\cos\theta = \pm\frac{\sqrt{8}}{3}$
$\therefore$ $\cos\theta = \pm\frac{2\sqrt{2}}{3}$

c $\cos^2\theta + \sin^2\theta = 1$
$\therefore$ $\cos^2\theta + 0^2 = 1$
$\therefore$ $\cos\theta = \pm 1$

d $\cos^2\theta + \sin^2\theta = 1$
$\therefore$ $\cos^2\theta + (-1)^2 = 1$
$\therefore$ $\cos\theta = 0$

2 **a** $\cos^2\theta + \sin^2\theta = 1$
$\therefore\ (\frac{4}{5})^2 + \sin^2\theta = 1$
$\therefore\ \sin^2\theta = \frac{9}{25}$
$\therefore\ \sin\theta = \pm\frac{3}{5}$

b $\cos^2\theta + \sin^2\theta = 1$
$\therefore\ (-\frac{3}{4})^2 + \sin^2\theta = 1$
$\therefore\ \sin^2\theta = \frac{7}{16}$
$\therefore\ \sin\theta = \pm\frac{\sqrt{7}}{4}$

c $\cos^2\theta + \sin^2\theta = 1$
$\therefore\ 1^2 + \sin^2\theta = 1$
$\therefore\ \sin^2\theta = 0$
$\therefore\ \sin\theta = 0$

d $\cos^2\theta + \sin^2\theta = 1$
$\therefore\ 0^2 + \sin^2\theta = 1$
$\therefore\ \sin\theta = \pm 1$

3 **a** $\cos^2\theta + \sin^2\theta = 1$
$\therefore\ \frac{4}{9} + \sin^2\theta = 1$
$\therefore\ \sin^2\theta = \frac{5}{9}$
$\therefore\ \sin\theta = \pm\frac{\sqrt{5}}{3}$
But θ is in quadrant 1
where $\sin\theta > 0$
$\therefore\ \sin\theta = \frac{\sqrt{5}}{3}$

b $\cos^2\theta + \sin^2\theta = 1$
$\therefore\ \cos^2\theta + \frac{4}{25} = 1$
$\therefore\ \cos^2\theta = \frac{21}{25}$
$\therefore\ \cos\theta = \pm\frac{\sqrt{21}}{5}$
But θ is in quadrant 2
where $\cos\theta < 0$
$\therefore\ \cos\theta = -\frac{\sqrt{21}}{5}$

c $\cos^2\theta + \sin^2\theta = 1$
$\therefore\ \cos^2\theta + \frac{9}{25} = 1$
$\therefore\ \cos^2\theta = \frac{16}{25}$
$\therefore\ \cos\theta = \pm\frac{4}{5}$
But θ is in quadrant 4
where $\cos\theta > 0$
$\therefore\ \cos\theta = \frac{4}{5}$

d $\cos^2\theta + \sin^2\theta = 1$
$\therefore\ \frac{25}{169} + \sin^2\theta = 1$
$\therefore\ \sin^2\theta = \frac{144}{169}$
$\therefore\ \sin\theta = \pm\frac{12}{13}$
But θ is in quadrant 3
where $\sin\theta < 0$
$\therefore\ \sin\theta = -\frac{12}{13}$

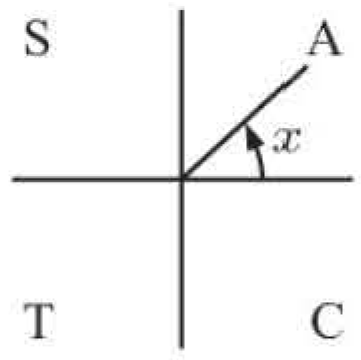
4 **a** $\cos^2 x + \sin^2 x = 1$
$\therefore\ \cos^2 x + \frac{1}{9} = 1$
$\therefore\ \cos^2 x = \frac{8}{9}$
$\therefore\ \cos x = \pm\frac{2\sqrt{2}}{3}$
But x is in quadrant 2
where $\cos x < 0$
$\therefore\ \cos x = -\frac{2\sqrt{2}}{3}$
and so $\tan x = \dfrac{\sin x}{\cos x} = \dfrac{\frac{1}{3}}{-\frac{2\sqrt{2}}{3}} = -\frac{1}{2\sqrt{2}}$

b $\cos^2 x + \sin^2 x = 1$
$\therefore\ \frac{1}{25} + \sin^2 x = 1$
$\therefore\ \sin^2 x = \frac{24}{25}$
$\therefore\ \sin x = \pm\frac{2\sqrt{6}}{5}$
But x is in quadrant 4
where $\sin x < 0$
$\therefore\ \sin x = -\frac{2\sqrt{6}}{5}$
and so $\tan x = \dfrac{\sin x}{\cos x} = \dfrac{-\frac{2\sqrt{6}}{5}}{\frac{1}{5}} = -2\sqrt{6}$

c $\cos^2 x + \sin^2 x = 1$
$\therefore\ \cos^2 x + \frac{1}{3} = 1$
$\therefore\ \cos^2 x = \frac{2}{3}$
$\therefore\ \cos x = \pm\frac{\sqrt{2}}{\sqrt{3}}$
But x is in quadrant 3
where $\cos x < 0$
$\therefore\ \cos x = -\frac{\sqrt{2}}{\sqrt{3}}$
and so $\tan x = \dfrac{\sin x}{\cos x} = \dfrac{-\frac{1}{\sqrt{3}}}{-\frac{\sqrt{2}}{\sqrt{3}}} = \frac{1}{\sqrt{2}}$

d $\cos^2 x + \sin^2 x = 1$
$\therefore\ \frac{9}{16} + \sin^2 x = 1$
$\therefore\ \sin^2 x = \frac{7}{16}$
$\therefore\ \sin x = \pm\frac{\sqrt{7}}{4}$
But x is in quadrant 2
where $\sin x > 0$
$\therefore\ \sin x = \frac{\sqrt{7}}{4}$
and so $\tan x = \dfrac{\sin x}{\cos x} = \dfrac{\frac{\sqrt{7}}{4}}{-\frac{3}{4}} = -\frac{\sqrt{7}}{3}$

5 **a** $\dfrac{\sin x}{\cos x} = \frac{2}{3}$



$\therefore \sin x = \frac{2}{3}\cos x$

Now $\cos^2 x + \sin^2 x = 1$

$\therefore \cos^2 x + \frac{4}{9}\cos^2 x = 1$

$\therefore \frac{13}{9}\cos^2 x = 1$

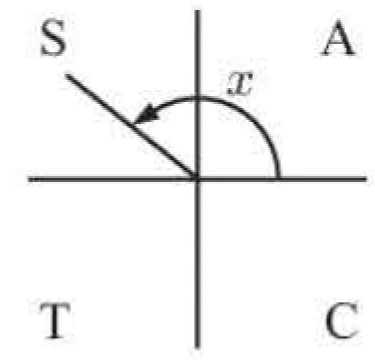
$\therefore \cos x = \pm\frac{3}{\sqrt{13}}$

But x is in quadrant 1

$\therefore \cos x$ and $\sin x$ are positive.

$\therefore \cos x = \frac{3}{\sqrt{13}}, \quad \sin x = \frac{2}{\sqrt{13}}$

b $\dfrac{\sin x}{\cos x} = -\frac{4}{3}$



$\therefore \sin x = -\frac{4}{3}\cos x$

Now $\cos^2 x + \sin^2 x = 1$

$\therefore \cos^2 x + \frac{16}{9}\cos^2 x = 1$

$\therefore \frac{25}{9}\cos^2 x = 1$

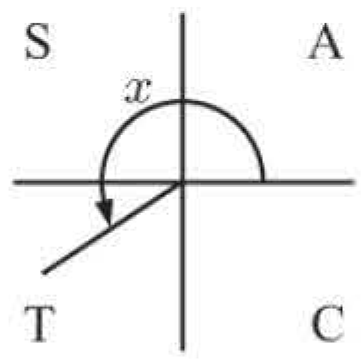
$\therefore \cos x = \pm\frac{3}{5}$

But x is in quadrant 2

$\therefore \cos x$ is negative and $\sin x$ is positive.

$\therefore \cos x = -\frac{3}{5}, \quad \sin x = \frac{4}{5}$

c $\dfrac{\sin x}{\cos x} = \frac{\sqrt{5}}{3}$



$\therefore \sin x = \frac{\sqrt{5}}{3}\cos x$

Now $\cos^2 x + \sin^2 x = 1$

$\therefore \cos^2 x + \frac{5}{9}\cos^2 x = 1$

$\therefore \frac{14}{9}\cos^2 x = 1$

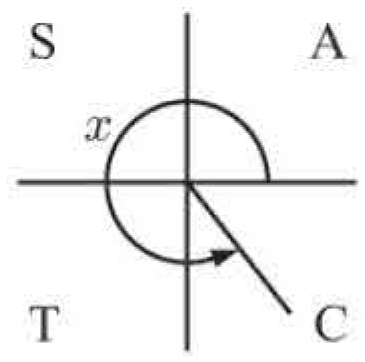
$\therefore \cos x = \pm\frac{3}{\sqrt{14}}$

But x is in quadrant 3

$\therefore \cos x$ and $\sin x$ are both negative.

$\therefore \cos x = -\frac{3}{\sqrt{14}}, \quad \sin x = -\frac{\sqrt{5}}{\sqrt{14}}$

d $\dfrac{\sin x}{\cos x} = -\frac{12}{5}$



$\therefore \sin x = -\frac{12}{5}\cos x$

Now $\cos^2 x + \sin^2 x = 1$

$\therefore \cos^2 x + \frac{144}{25}\cos^2 x = 1$

$\therefore \frac{169}{25}\cos^2 x = 1$

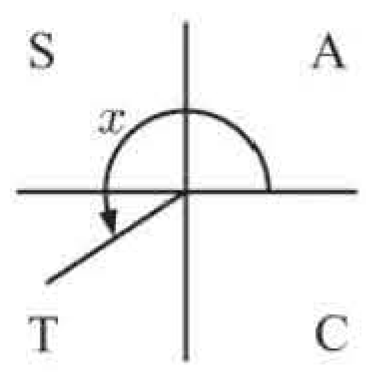
$\therefore \cos x = \pm\frac{5}{13}$

But x is in quadrant 4

$\therefore \cos x$ is positive and $\sin x$ is negative.

$\therefore \cos x = \frac{5}{13}, \quad \sin x = -\frac{12}{13}$

6 $\dfrac{\sin x}{\cos x} = k$



$\therefore \sin x = k\cos x$

Now $\cos^2 x + \sin^2 x = 1$

$\therefore \cos^2 x + k^2\cos^2 x = 1$

$\therefore (k^2+1)\cos^2 x = 1$

$\therefore \cos x = \dfrac{\pm 1}{\sqrt{k^2+1}}$

But x is in quadrant 3, $\therefore \cos x$ and $\sin x$ are both negative.

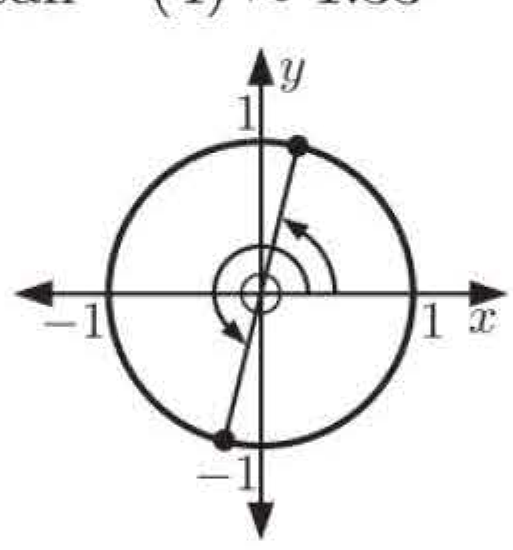
$\therefore \cos x = \dfrac{-1}{\sqrt{k^2+1}}, \quad \sin x = \dfrac{-k}{\sqrt{k^2+1}}$

EXERCISE 10D.2

1 **a** $\tan\theta = 4$

Using technology,

$\tan^{-1}(4) \approx 1.33$



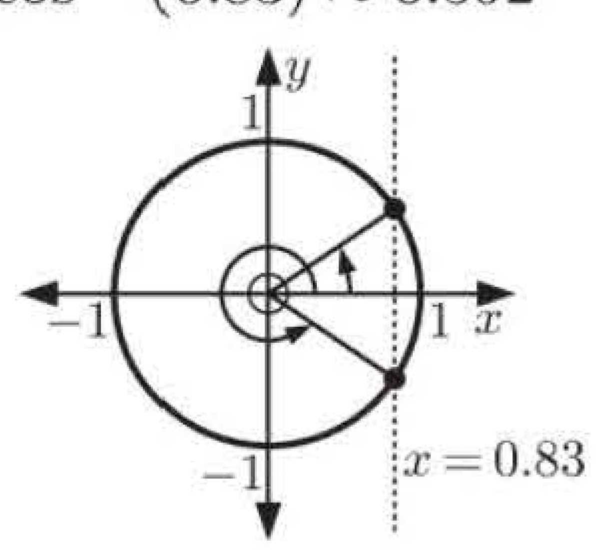
$\therefore \theta \approx 1.33$ or $\pi + 1.33$

$\therefore \theta \approx 1.33$ or 4.47

b $\cos\theta = 0.83$

Using technology,

$\cos^{-1}(0.83) \approx 0.592$



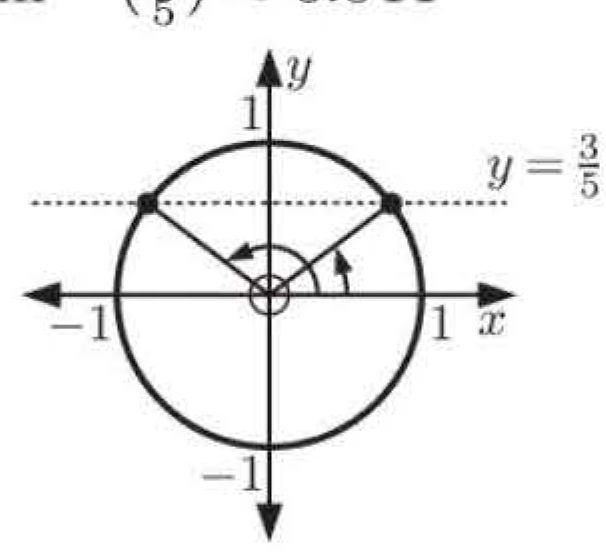
$\therefore \theta \approx 0.592$ or $2\pi - 0.592$

$\therefore \theta \approx 0.592$ or 5.69

c $\sin\theta = \frac{3}{5}$

Using technology,

$\sin^{-1}(\frac{3}{5}) \approx 0.644$

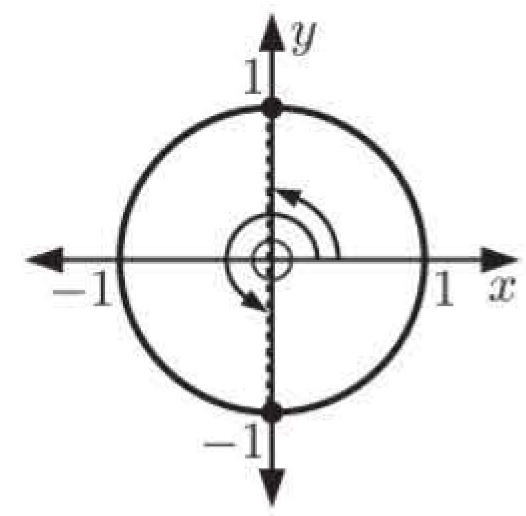


$\therefore \theta \approx 0.644$ or $\pi - 0.644$

$\therefore \theta \approx 0.644$ or 2.50

d $\cos\theta = 0$

$\therefore \ \cos^{-1}(0) = \frac{\pi}{2}$



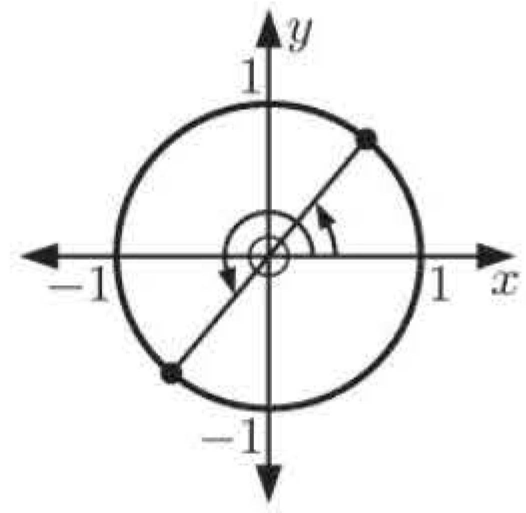
$\therefore \ \theta = \frac{\pi}{2}$ or $2\pi - \frac{\pi}{2}$

$\therefore \ \theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$

e $\tan\theta = 1.2$

Using technology,

$\tan^{-1}(1.2) \approx 0.876$



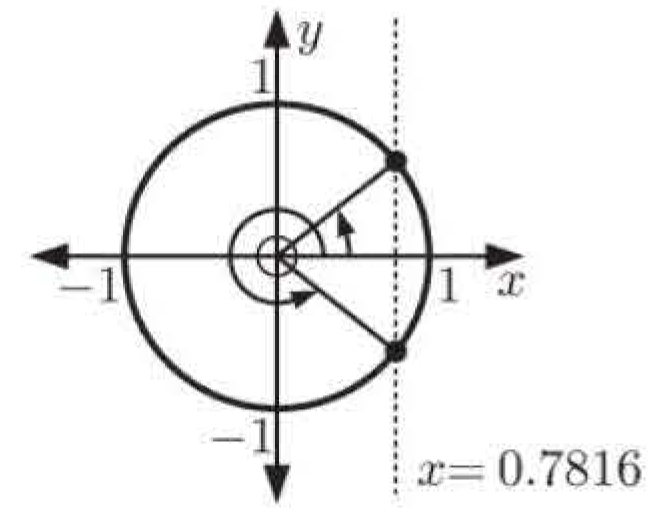
$\therefore \ \theta \approx 0.876$ or $\pi + 0.876$

$\therefore \ \theta \approx 0.876$ or 4.02

f $\cos\theta = 0.7816$

Using technology,

$\cos^{-1}(0.7816) \approx 0.674$



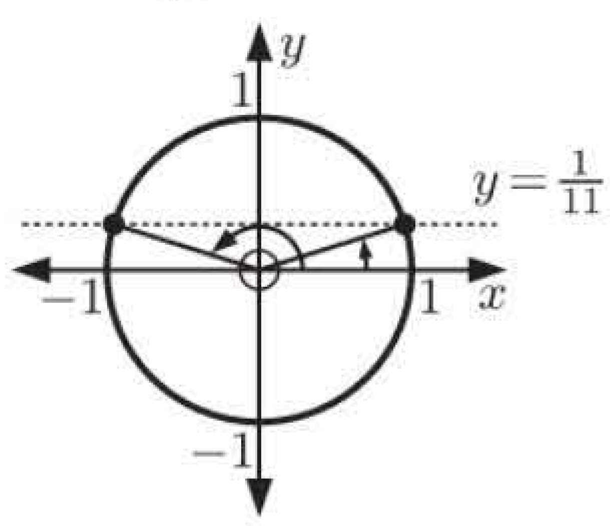
$\therefore \ \theta \approx 0.674$ or $2\pi - 0.674$

$\therefore \ \theta \approx 0.674$ or 5.61

g $\sin\theta = \frac{1}{11}$

Using technology,

$\sin^{-1}(\frac{1}{11}) \approx 0.0910$



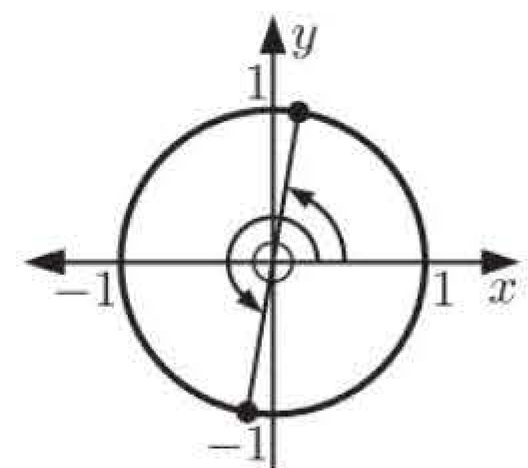
$\therefore \ \theta \approx 0.0910$ or $\pi - 0.0910$

$\therefore \ \theta \approx 0.0910$ or 3.05

h $\tan\theta = 20.2$

Using technology,

$\tan^{-1}(20.2) \approx 1.52$



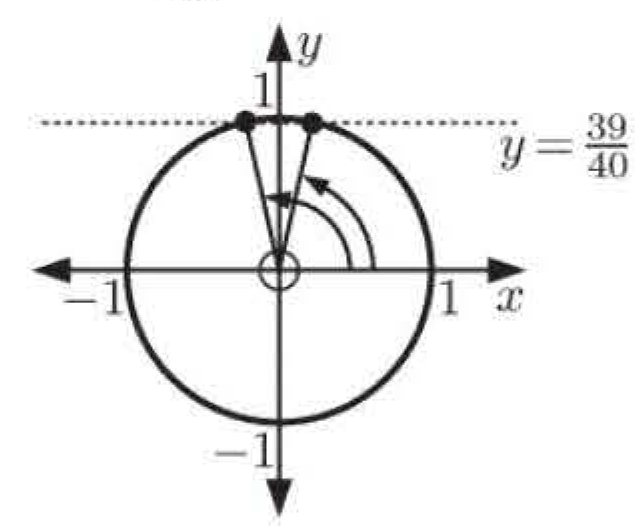
$\therefore \ \theta \approx 1.52$ or $\pi + 1.52$

$\therefore \ \theta \approx 1.52$ or 4.66

i $\sin\theta = \frac{39}{40}$

Using technology,

$\sin^{-1}(\frac{39}{40}) \approx 1.35$



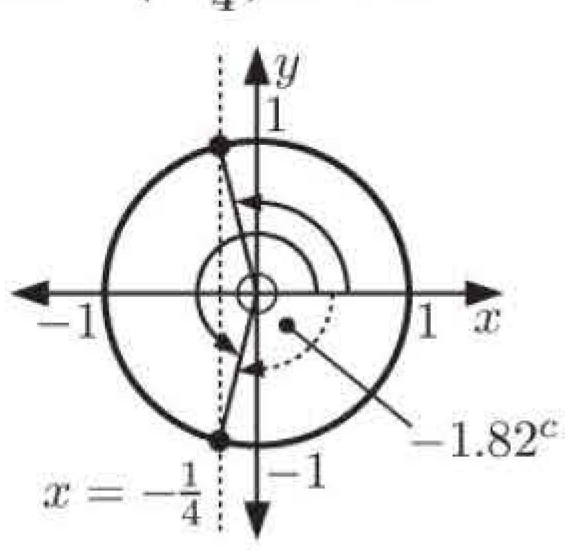
$\therefore \ \theta \approx 1.35$ or $\pi - 1.35$

$\therefore \ \theta \approx 1.35$ or 1.79

2 **a** $\cos\theta = -\frac{1}{4}$

Using technology,

$\cos^{-1}(-\frac{1}{4}) \approx 1.82$

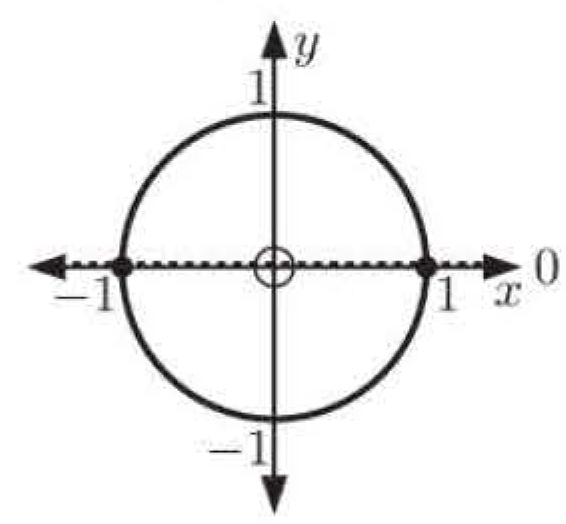


$\therefore \ \theta \approx 1.82$ or $2\pi - 1.82$

$\therefore \ \theta \approx 1.82$ or 4.46

b $\sin\theta = 0$

$\therefore \ \sin^{-1}(0) = 0$



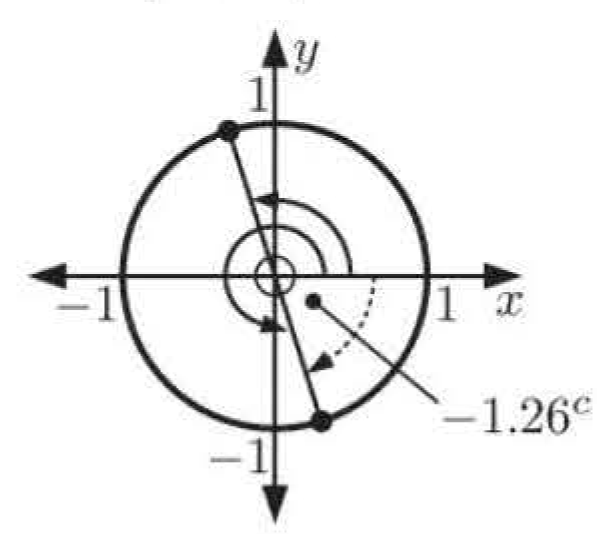
$\therefore \ \theta = 0$ or $\pi - 0$ or 2π

$\therefore \ \theta = 0, \pi,$ or 2π

c $\tan\theta = -3.1$

Using technology,

$\tan^{-1}(-3.1) \approx -1.26$



But $0 \leqslant \theta \leqslant 2\pi$

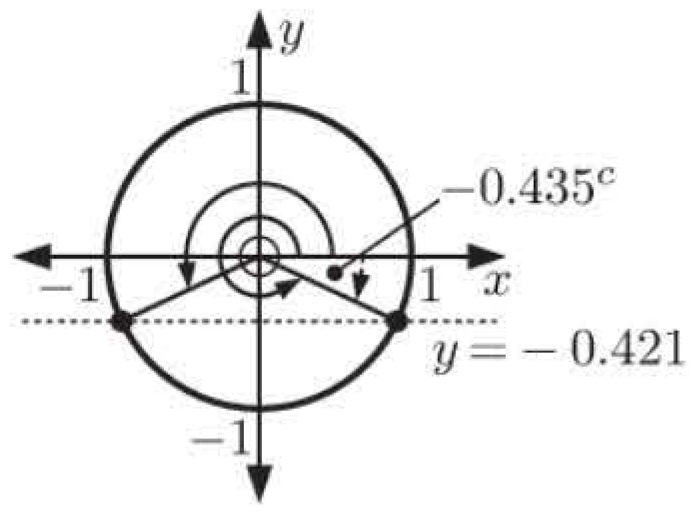
$\therefore \ \theta \approx \pi - 1.26$ or $2\pi - 1.26$

$\therefore \ \theta \approx 1.88$ or 5.02

d $\sin\theta = -0.421$

Using technology,

$\sin^{-1}(-0.421) \approx -0.435$



But $0 \leqslant \theta \leqslant 2\pi$

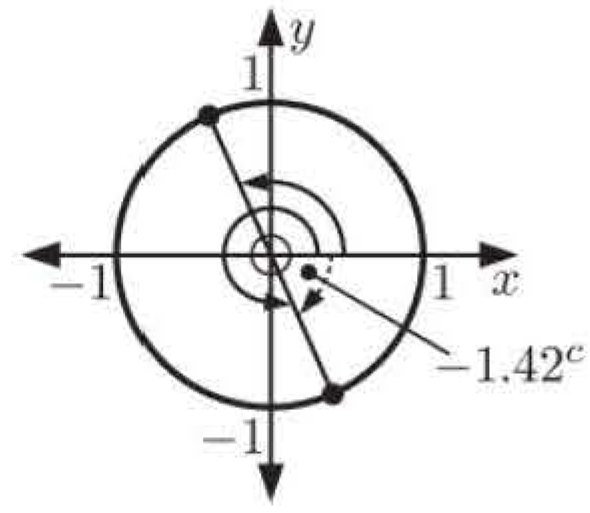
$\therefore\ \theta \approx \pi + 0.435$ or $2\pi - 0.435$

$\therefore\ \theta \approx 3.58$ or 5.85

e $\tan\theta = -6.67$

Using technology,

$\tan^{-1}(-6.67) \approx -1.42$



But $0 \leqslant \theta \leqslant 2\pi$

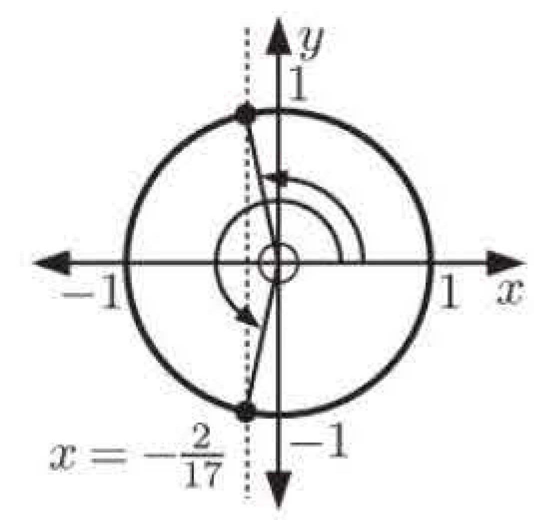
$\therefore\ \theta \approx \pi - 1.42$ or $2\pi - 1.42$

$\therefore\ \theta \approx 1.72$ or 4.86

f $\cos\theta = -\frac{2}{17}$

Using technology,

$\cos^{-1}(-\frac{2}{17}) \approx 1.69$



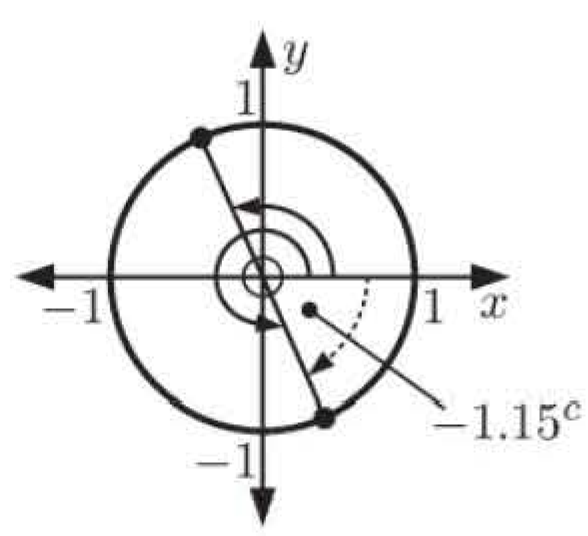
$\therefore\ \theta \approx 1.69$ or $2\pi - 1.69$

$\therefore\ \theta \approx 1.69$ or 4.59

g $\tan\theta = -\sqrt{5}$

Using technology,

$\tan^{-1}(-\sqrt{5}) \approx -1.15$



But $0 \leqslant \theta \leqslant 2\pi$

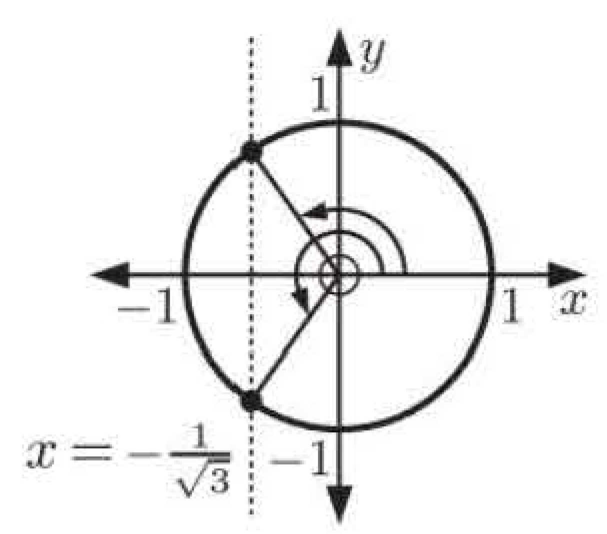
$\therefore\ \theta \approx \pi - 1.15$ or $2\pi - 1.15$

$\therefore\ \theta \approx 1.99$ or 5.13

h $\cos\theta = -\frac{1}{\sqrt{3}}$

Using technology,

$\cos^{-1}(-\frac{1}{\sqrt{3}}) \approx 2.19$



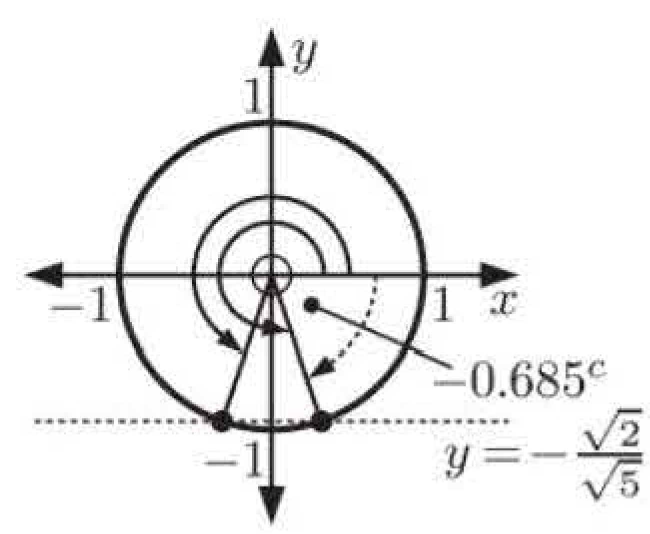
$\therefore\ \theta \approx 2.19$ or $2\pi - 2.19$

$\therefore\ \theta \approx 2.19$ or 4.10

i $\sin\theta = -\frac{\sqrt{2}}{\sqrt{5}}$

Using technology,

$\sin^{-1}(-\frac{\sqrt{2}}{\sqrt{5}}) \approx -0.685$



But $0 \leqslant \theta \leqslant 2\pi$

$\therefore\ \theta \approx \pi + 0.685$ or $2\pi - 0.685$

$\therefore\ \theta \approx 3.83$ or 5.60

EXERCISE 10E

1 **a** $\sin\theta + \sin(-\theta)$
$= \sin\theta - \sin\theta$
$= 0$

b $\tan(-\theta) - \tan\theta$
$= -\tan\theta - \tan\theta$
$= -2\tan\theta$

c $2\cos\theta + \cos(-\theta)$
$= 2\cos\theta + \cos\theta$
$= 3\cos\theta$

d $3\sin\theta - \sin(-\theta)$
$= 3\sin\theta - -\sin\theta$
$= 3\sin\theta + \sin\theta$
$= 4\sin\theta$

e $\cos^2(-\alpha)$
$= \cos(-\alpha) \times \cos(-\alpha)$
$= \cos\alpha \times \cos\alpha$
$= \cos^2\alpha$

f $\sin^2(-\alpha)$
$= \sin(-\alpha) \times \sin(-\alpha)$
$= -\sin\alpha \times -\sin\alpha$
$= \sin^2\alpha$

g $\cos(-\alpha)\cos\alpha - \sin(-\alpha)\sin\alpha$
$= \cos\alpha\cos\alpha - -\sin\alpha\sin\alpha$
$= \cos^2\alpha + \sin^2\alpha$
$= 1$

2 **a** $2\sin\theta - \cos(90^\circ - \theta)$
$= 2\sin\theta - \sin\theta$
$= \sin\theta$

b $\sin(-\theta) - \cos(90^\circ - \theta)$
$= -\sin\theta - \sin\theta$
$= -2\sin\theta$

c $\sin(90^\circ - \theta) - \cos\theta$
$= \cos\theta - \cos\theta$
$= 0$

d $3\cos(-\theta) - 4\sin(\frac{\pi}{2} - \theta)$
$= 3\cos\theta - 4\cos\theta$
$= -\cos\theta$

e $3\cos\theta + \sin(\frac{\pi}{2} - \theta)$
$= 3\cos\theta + \cos\theta$
$= 4\cos\theta$

f $\cos(\frac{\pi}{2} - \theta) + 4\sin\theta$
$= \sin\theta + 4\sin\theta$
$= 5\sin\theta$

3 $\sin(\theta - \phi) = \sin(-(\phi - \theta))$
$= -\sin(\phi - \theta)$

and $\cos(\theta - \phi) = \cos(-(\phi - \theta))$
$= \cos(\phi - \theta)$

4 **a** $\dfrac{\sin\theta}{\cos\theta} = \tan\theta$

b $\dfrac{\sin(-\theta)}{\cos(-\theta)} = \dfrac{-\sin\theta}{\cos\theta}$
$= -\tan\theta$

c $\dfrac{\sin(\frac{\pi}{2} - \theta)}{\cos\theta} = \dfrac{\cos\theta}{\cos\theta}$
$= 1$

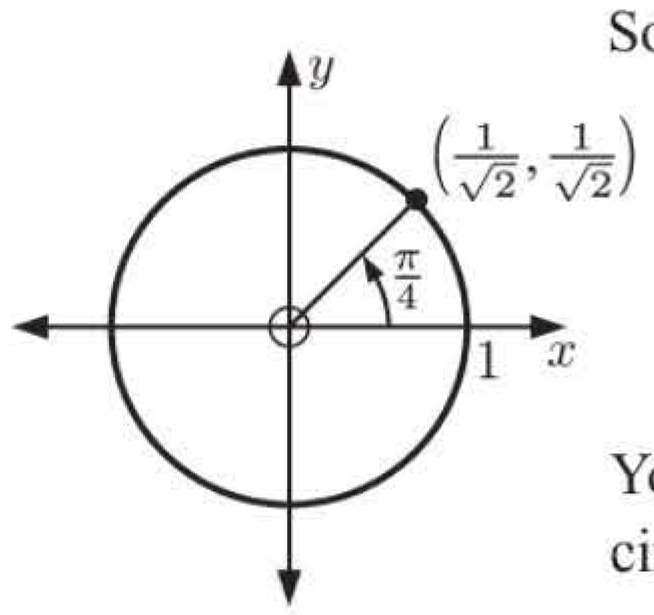
d $\dfrac{-\sin(-\theta)}{\cos\theta} = \dfrac{\sin\theta}{\cos\theta}$
$= \tan\theta$

e $\dfrac{\cos(\frac{\pi}{2} - \theta)}{\sin(\frac{\pi}{2} - \theta)} = \dfrac{\sin\theta}{\cos\theta}$
$= \tan\theta$

f $\dfrac{\cos(\frac{\pi}{2} - \theta)}{\cos\theta} = \dfrac{\sin\theta}{\cos\theta}$
$= \tan\theta$

EXERCISE 10F

1



So, $\cos(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$

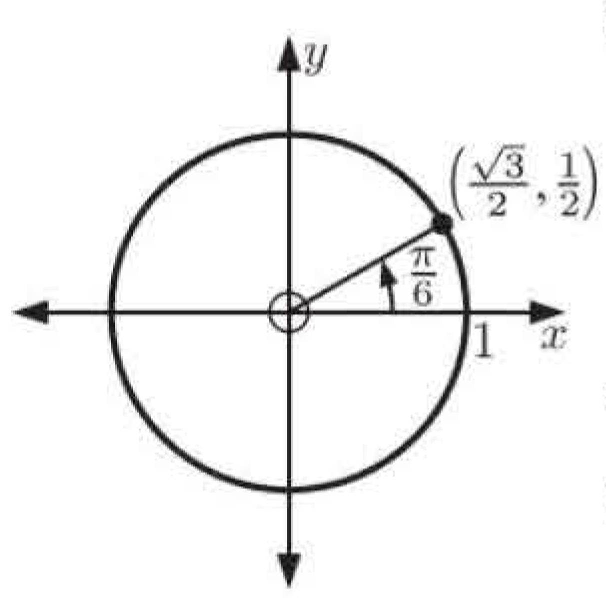
$\sin(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$

$\tan(\frac{\pi}{4}) = \dfrac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} = 1$

You should draw separate unit circle diagrams for each case.

	a	**b**	**c**	**d**	**e**
$\sin\theta$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	0	$-\frac{1}{\sqrt{2}}$
$\cos\theta$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	-1	$-\frac{1}{\sqrt{2}}$
$\tan\theta$	1	-1	-1	0	1

2



So, $\cos(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$

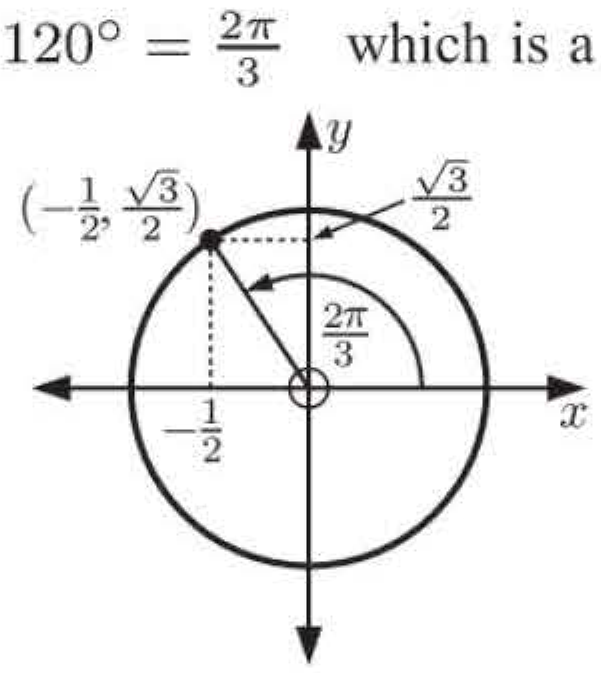
$\sin(\frac{\pi}{6}) = \frac{1}{2}$

$\tan(\frac{\pi}{6}) = \dfrac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}$

You should draw separate unit circle diagrams for each case.

	a	**b**	**c**	**d**	**e**
$\sin\beta$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$
$\cos\beta$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\tan\beta$	$\frac{1}{\sqrt{3}}$	$-\sqrt{3}$	$\frac{1}{\sqrt{3}}$	$-\sqrt{3}$	$-\frac{1}{\sqrt{3}}$

3 **a** $120° = \frac{2\pi}{3}$ which is a multiple of $\frac{\pi}{6}$

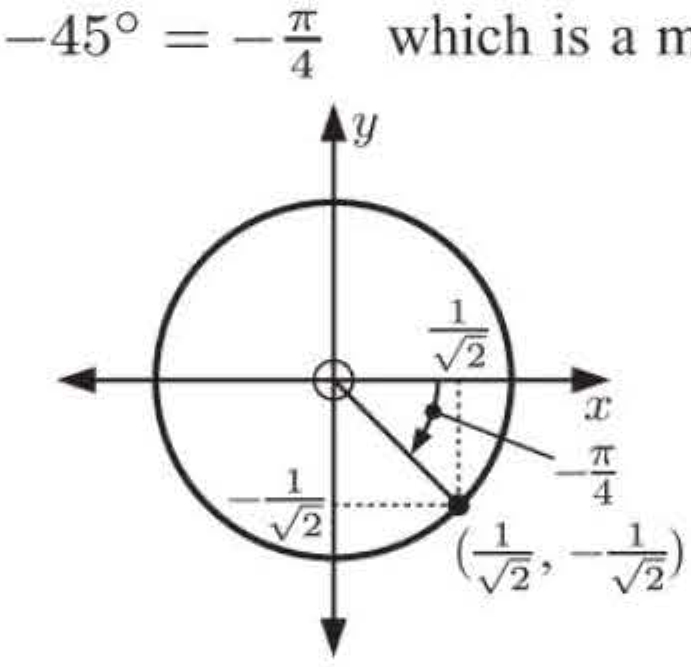


So, $\cos 120° = -\frac{1}{2}$

$\sin 120° = \frac{\sqrt{3}}{2}$

$\tan 120° = -\sqrt{3}$

b $-45° = -\frac{\pi}{4}$ which is a multiple of $\frac{\pi}{4}$

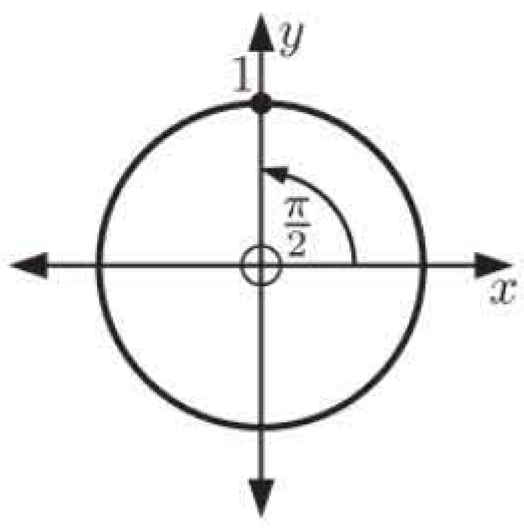


So, $\cos(-45°) = \frac{1}{\sqrt{2}}$

$\sin(-45°) = -\frac{1}{\sqrt{2}}$

$\tan(-45°) = -1$

4 **a** $90° = \frac{\pi}{2}$

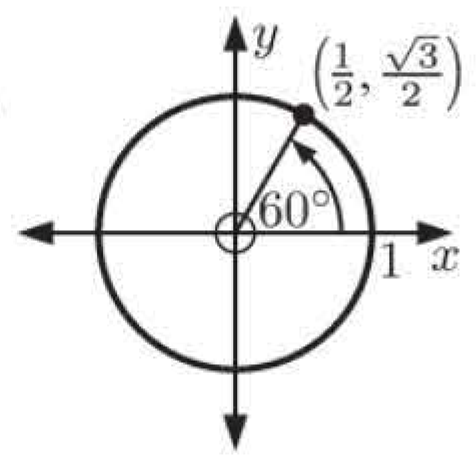


$\cos 90° = 0, \quad \sin 90° = 1$

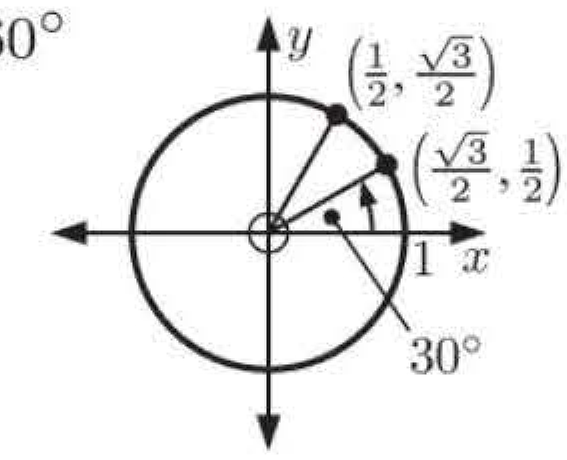
b $\tan 90° = \dfrac{\sin 90°}{\cos 90°} = \dfrac{1}{0}$

$\tan 90°$ is undefined

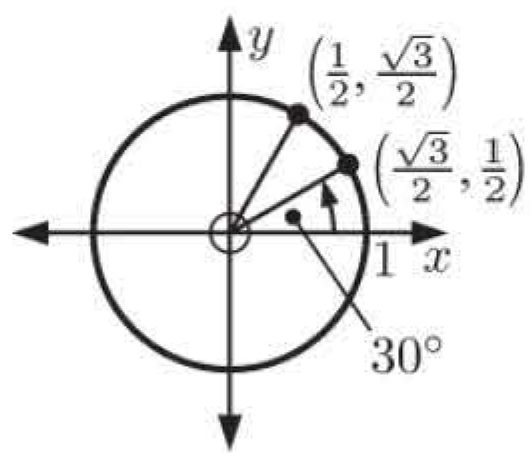
5 **a** $\sin^2 60°$
$= \sin 60° \times \sin 60°$
$= \frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2}$
$= \frac{3}{4}$



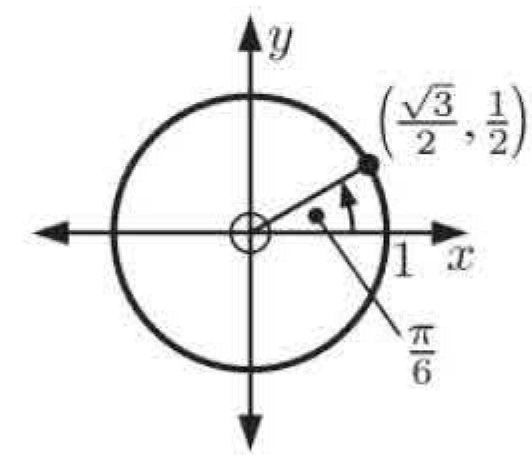
b $\sin 30° \cos 60°$
$= \frac{1}{2} \times \frac{1}{2}$
$= \frac{1}{4}$



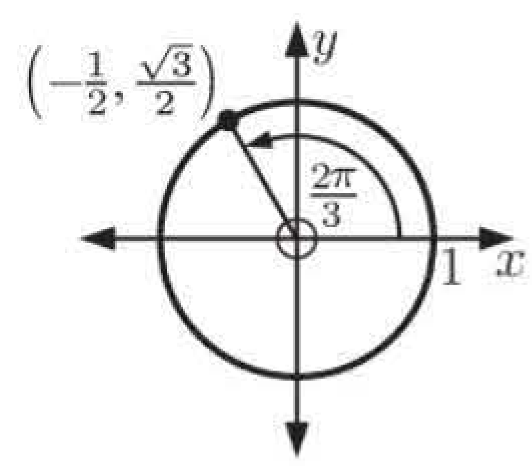
c $4 \sin 60° \cos 30°$
$= 4\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)$
$= 3$



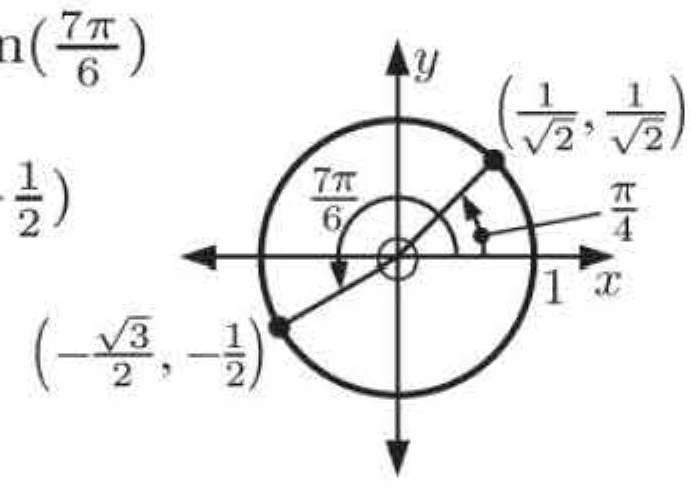
d $1 - \cos^2(\frac{\pi}{6})$
$= 1 - \left(\frac{\sqrt{3}}{2}\right)^2$
$= 1 - \frac{3}{4}$
$= \frac{1}{4}$



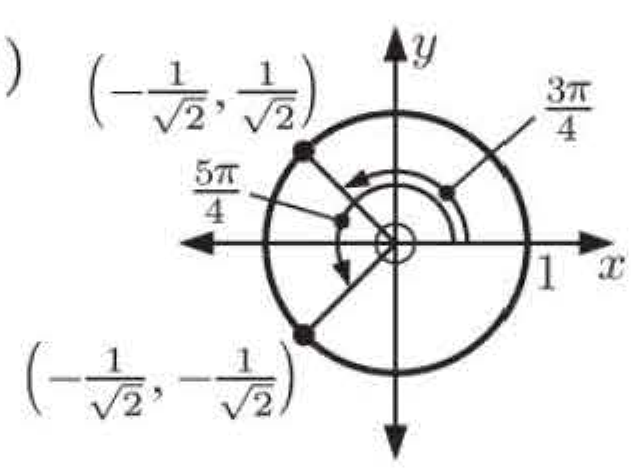
e $\sin^2(\frac{2\pi}{3}) - 1$
$= \left(\frac{\sqrt{3}}{2}\right)^2 - 1$
$= \frac{3}{4} - 1$
$= -\frac{1}{4}$



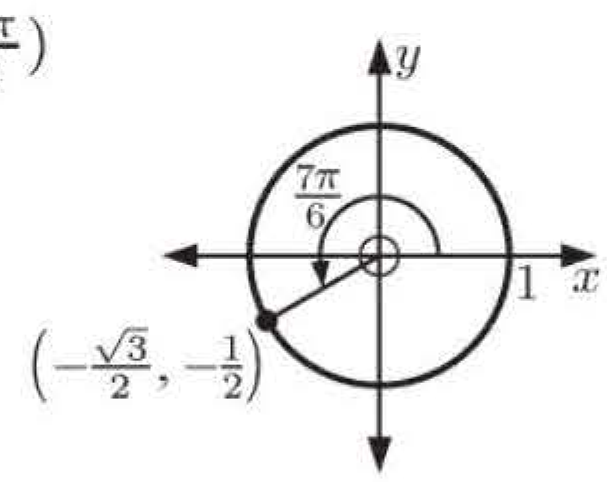
f $\cos^2(\frac{\pi}{4}) - \sin(\frac{7\pi}{6})$
$= \left(\frac{1}{\sqrt{2}}\right)^2 - (-\frac{1}{2})$
$= \frac{1}{2} + \frac{1}{2}$
$= 1$



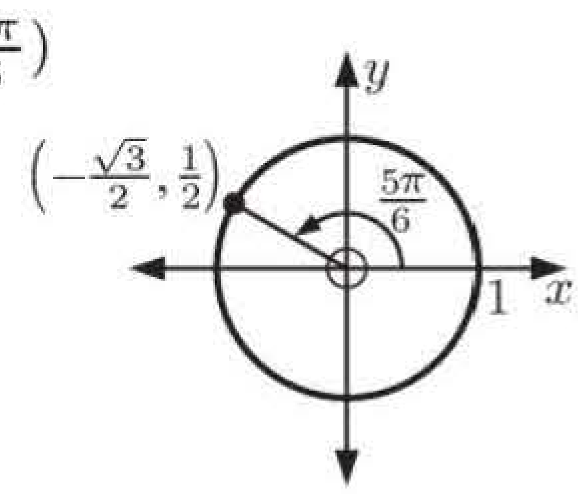
g $\sin(\frac{3\pi}{4}) - \cos(\frac{5\pi}{4})$
$= \frac{1}{\sqrt{2}} - \left(-\frac{1}{\sqrt{2}}\right)$
$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}$
$= \frac{2}{\sqrt{2}}$ or $\sqrt{2}$



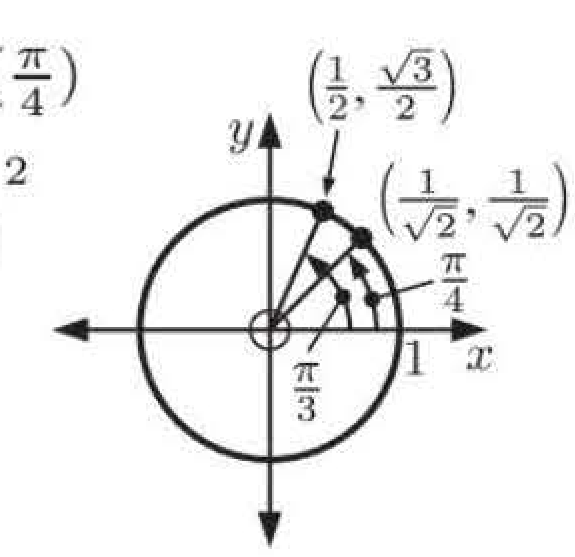
h $1 - 2\sin^2(\frac{7\pi}{6})$
$= 1 - 2(-\frac{1}{2})^2$
$= 1 - 2 \times \frac{1}{4}$
$= \frac{1}{2}$



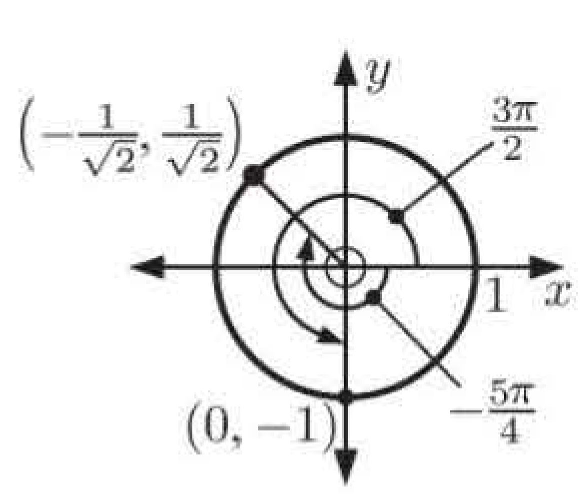
i $\cos^2(\frac{5\pi}{6}) - \sin^2(\frac{5\pi}{6})$
$= \left(-\frac{\sqrt{3}}{2}\right)^2 - (\frac{1}{2})^2$
$= \frac{3}{4} - \frac{1}{4}$
$= \frac{1}{2}$



j $\tan^2(\frac{\pi}{3}) - 2\sin^2(\frac{\pi}{4})$
$= \left(\sqrt{3}\right)^2 - 2\left(\frac{1}{\sqrt{2}}\right)^2$
$= 3 - 2(\frac{1}{2})$
$= 2$



k $2\tan(-\frac{5\pi}{4}) - \sin(\frac{3\pi}{2})$
$= 2(-1) - (-1)$
$= -1$

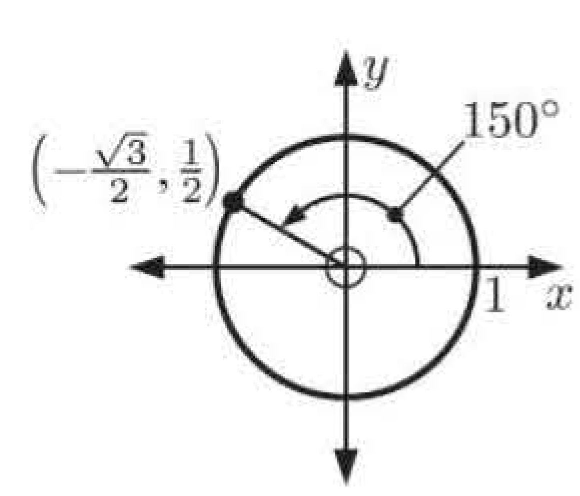


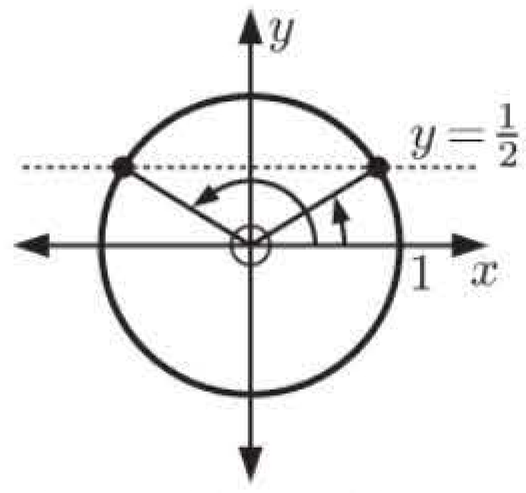
l $\dfrac{2\tan 150°}{1 - \tan^2 150°}$

$= \dfrac{2(-\frac{1}{\sqrt{3}})}{1 - (-\frac{1}{\sqrt{3}})^2}$

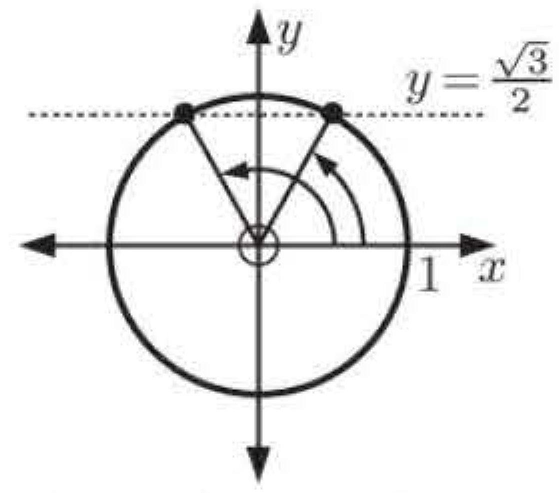
$= \dfrac{-\frac{2}{\sqrt{3}}}{1 - \frac{1}{3}}$

$= \dfrac{-\frac{2}{\sqrt{3}}}{\frac{2}{3}} = -\frac{3}{\sqrt{3}} = -\sqrt{3}$

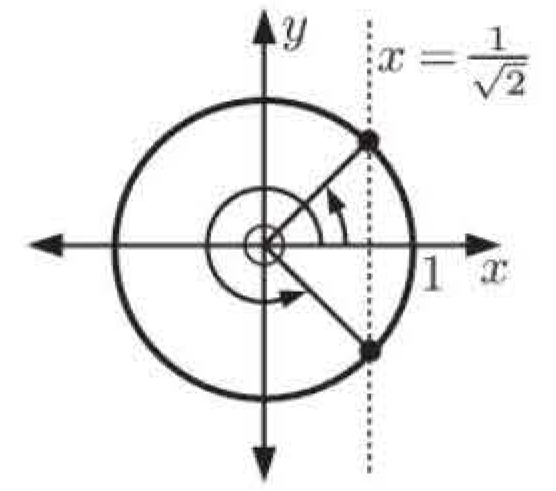


6 **a**


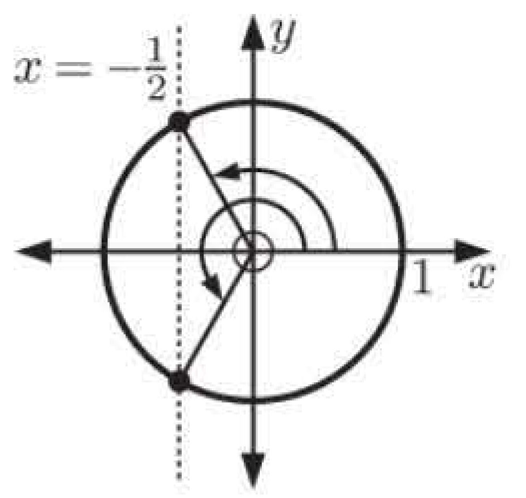
$\theta = 30°,\ 150°$

b


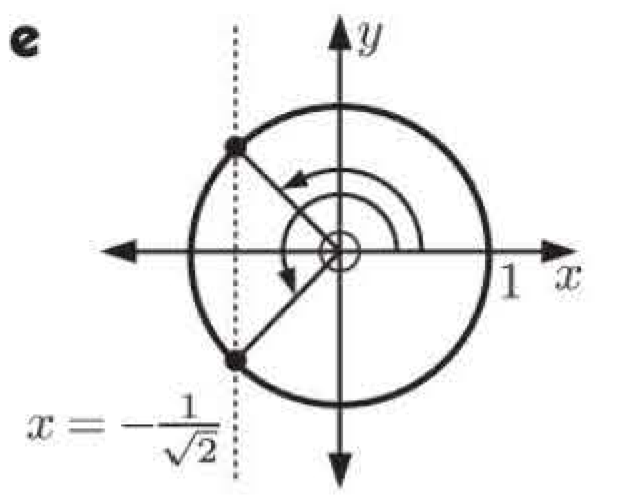
$\theta = 60°,\ 120°$

c


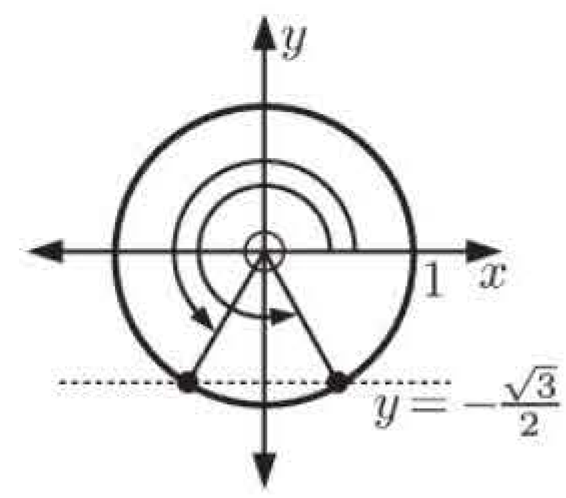
$\theta = 45°,\ 315°$

d


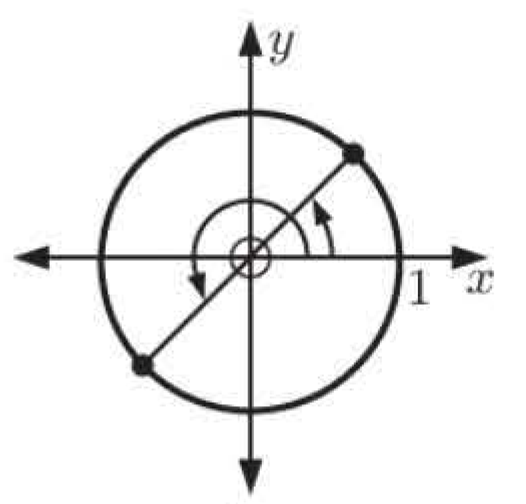
$\theta = 120°,\ 240°$

e


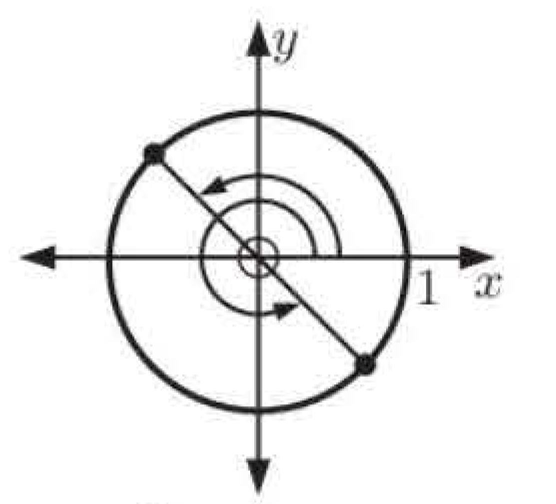
$\theta = 135°,\ 225°$

f


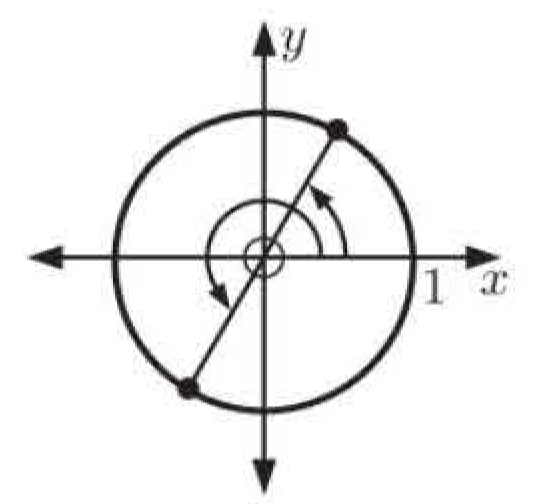
$\theta = 240°,\ 300°$

7 **a**


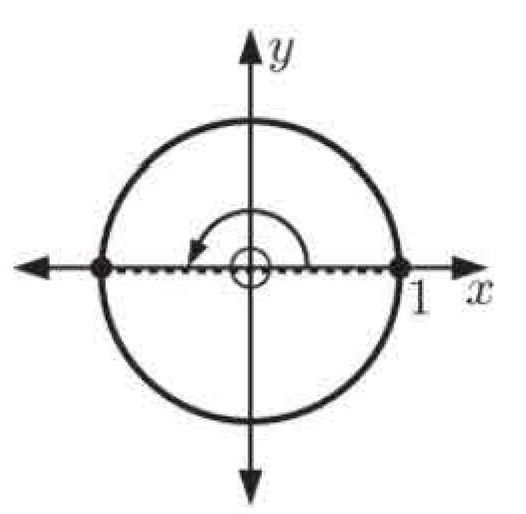
$\theta = \frac{\pi}{4},\ \frac{5\pi}{4}$

b


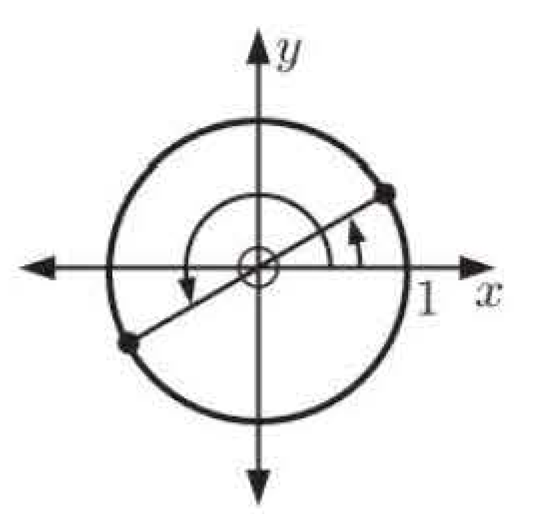
$\theta = \frac{3\pi}{4},\ \frac{7\pi}{4}$

c


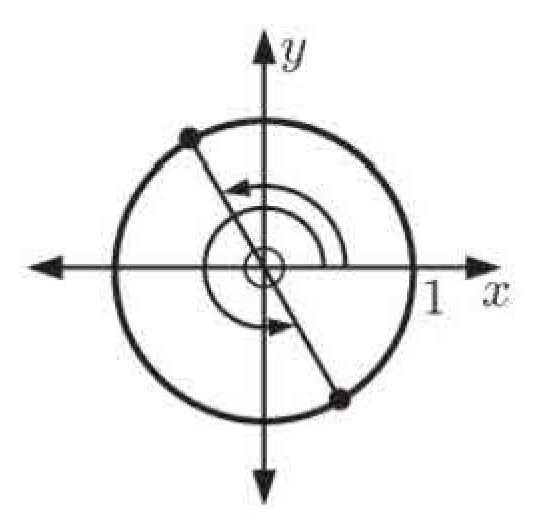
$\theta = \frac{\pi}{3},\ \frac{4\pi}{3}$

d


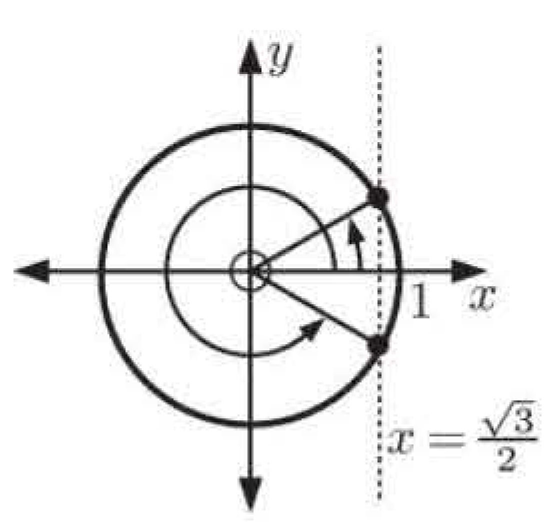
$\theta = 0,\ \pi,\ 2\pi$

e


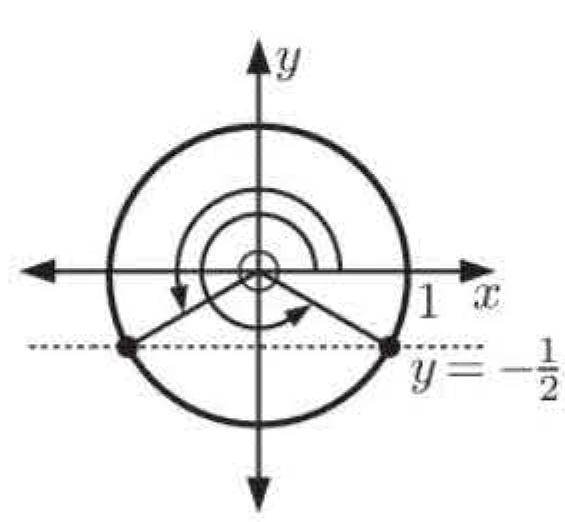
$\theta = \frac{\pi}{6},\ \frac{7\pi}{6}$

f


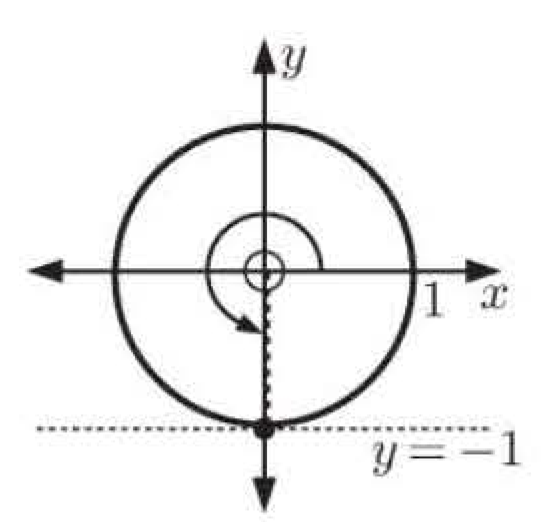
$\theta = \frac{2\pi}{3},\ \frac{5\pi}{3}$

8 **a**


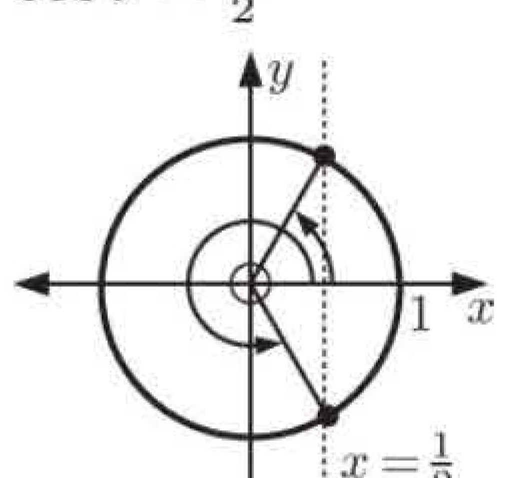
$\theta = \frac{\pi}{6},\ \frac{11\pi}{6},\ \frac{13\pi}{6},\ \frac{23\pi}{6}$

b


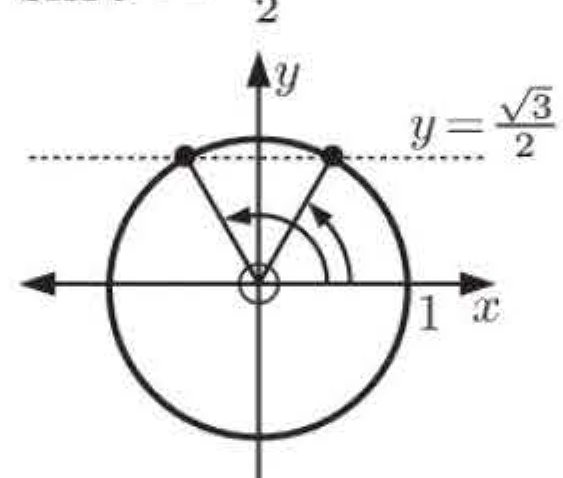
$\theta = \frac{7\pi}{6},\ \frac{11\pi}{6},\ \frac{19\pi}{6},\ \frac{23\pi}{6}$

c


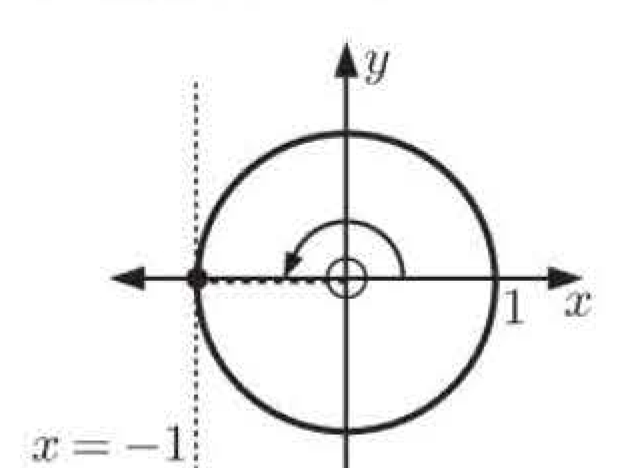
$\theta = \frac{3\pi}{2},\ \frac{7\pi}{2}$

9 **a** $\cos\theta = \frac{1}{2}$


$\therefore\ \theta = \frac{\pi}{3},\ \frac{5\pi}{3}$

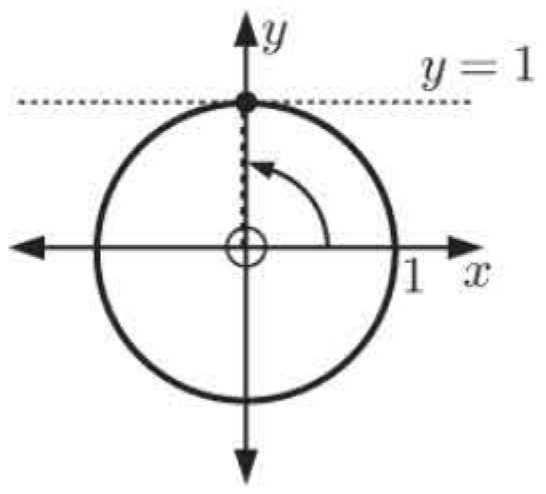
b $\sin\theta = \frac{\sqrt{3}}{2}$


$\therefore\ \theta = \frac{\pi}{3},\ \frac{2\pi}{3}$

c $\cos\theta = -1$


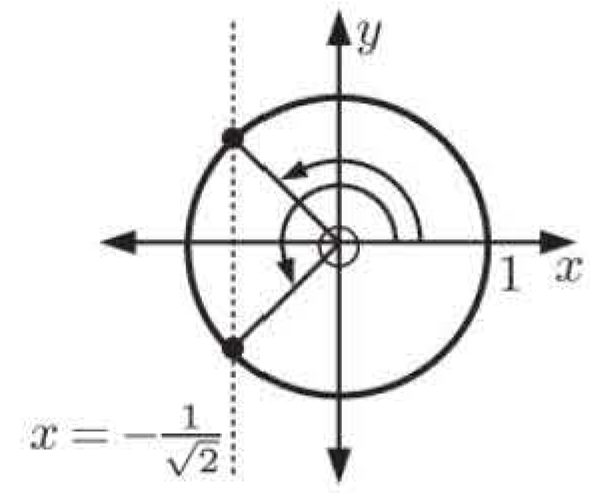
$\therefore\ \theta = \pi$

d $\sin\theta = 1$



$\therefore \quad \theta = \frac{\pi}{2}$

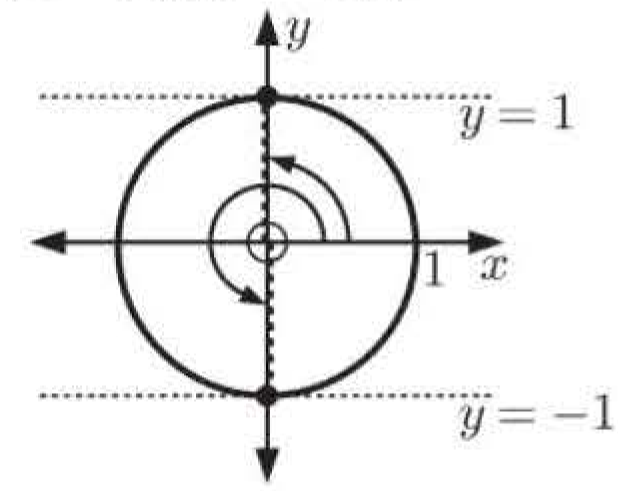
e $\cos\theta = -\frac{1}{\sqrt{2}}$



$\therefore \quad \theta = \frac{3\pi}{4},\ \frac{5\pi}{4}$

f $\sin^2\theta = 1$

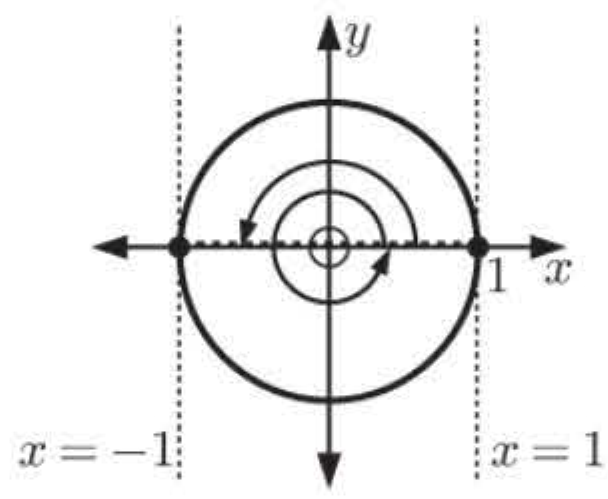
$\therefore \quad \sin\theta = \pm 1$



$\therefore \quad \theta = \frac{\pi}{2},\ \frac{3\pi}{2}$

g $\cos^2\theta = 1$

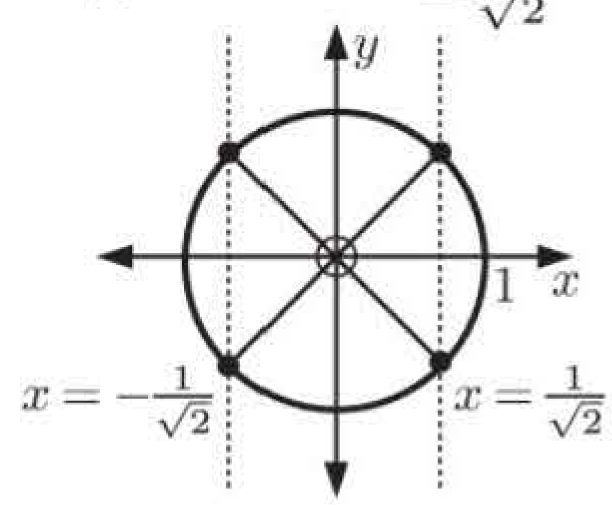
$\therefore \quad \cos\theta = \pm 1$



$\therefore \quad \theta = 0,\ \pi,\ 2\pi$

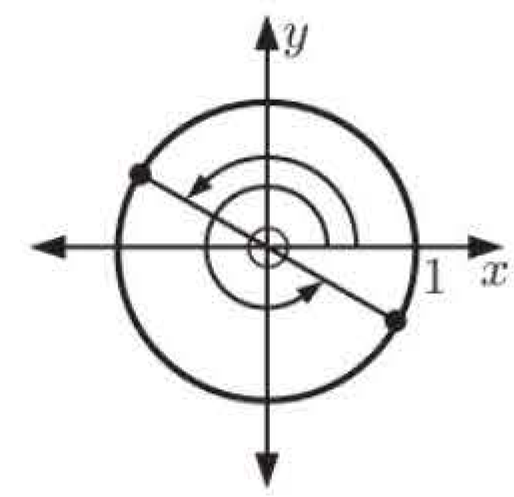
h $\cos^2\theta = \frac{1}{2}$

$\therefore \quad \cos\theta = \pm\frac{1}{\sqrt{2}}$



$\therefore \quad \theta = \frac{\pi}{4},\ \frac{3\pi}{4},\ \frac{5\pi}{4},\ \frac{7\pi}{4}$

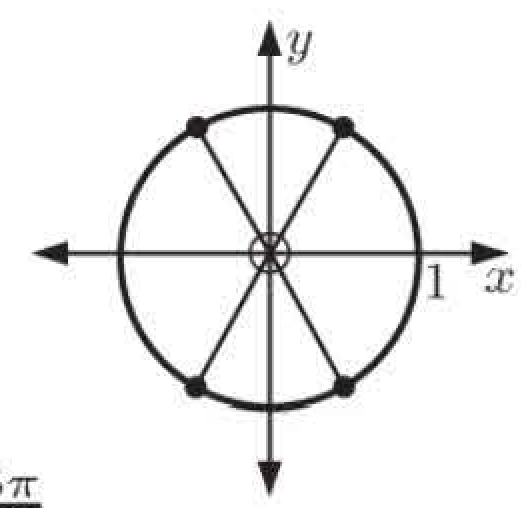
i $\tan\theta = -\frac{1}{\sqrt{3}}$



$\therefore \quad \theta = \frac{5\pi}{6},\ \frac{11\pi}{6}$

j $\tan^2\theta = 3$

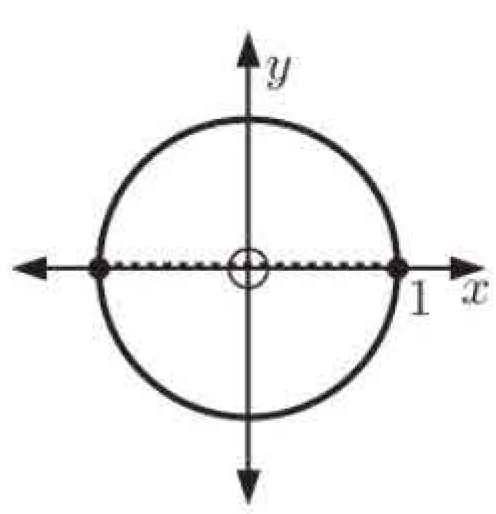
$\therefore \quad \tan\theta = \pm\sqrt{3}$



$\therefore \quad \theta = \frac{\pi}{3},\ \frac{2\pi}{3},\ \frac{4\pi}{3},\ \frac{5\pi}{3}$

10 **a** $\tan\theta$ is zero when $\dfrac{\sin\theta}{\cos\theta} = \dfrac{0}{\cos\theta}$

$\therefore$ when $\sin\theta = 0$

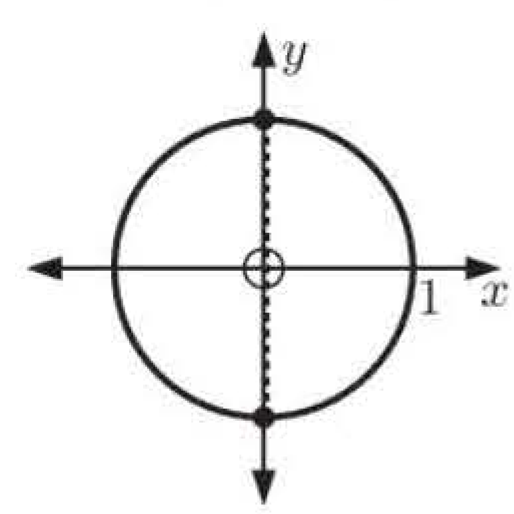


$\therefore \quad \theta = ...,\ -\pi,\ 0,\ \pi,\ 2\pi,\ ....$

$\therefore \quad \theta = k\pi,$ for $k \in \mathbb{Z}$

b $\tan\theta$ is undefined when $\dfrac{\sin\theta}{\cos\theta} = \dfrac{\sin\theta}{0}$

$\therefore$ when $\cos\theta = 0$



$\therefore \quad \theta = ...,\ -\frac{\pi}{2},\ \frac{\pi}{2},\ \frac{3\pi}{2},\ ....$

$\therefore \quad \theta = \frac{\pi}{2} + k\pi,$ for $k \in \mathbb{Z}$

REVIEW SET 10A

1 **a** 120°
$= \left(120 \times \frac{\pi}{180}\right)^c$
$= \frac{2\pi}{3}^c$

b 225°
$= 5 \times 45^\circ$
$= 5 \times \frac{\pi}{4}^c$
$= \frac{5\pi}{4}^c$

c 150°
$= 5 \times 30^\circ$
$= 5 \times \frac{\pi}{6}^c$
$= \frac{5\pi}{6}^c$

d 540°
$= 3 \times 180^\circ$
$= 3\pi^c$

2 **a** $\sin\frac{2\pi}{3} = \sin(\pi - \frac{2\pi}{3}) = \sin\frac{\pi}{3}$

$\therefore \quad \theta = \frac{\pi}{3}$

b $\sin 165^\circ = \sin(180 - 165)^\circ = \sin 15^\circ$

$\therefore \quad \theta = 15^\circ$

c $\cos 276^\circ = \cos(360 - 276)^\circ = \cos 84^\circ$

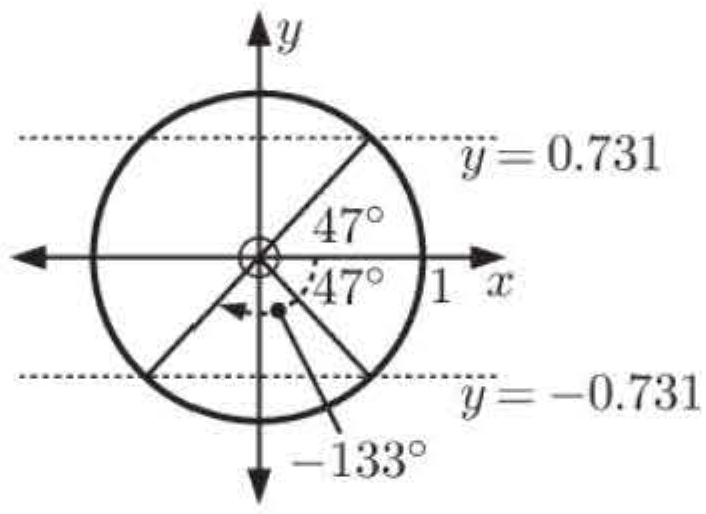
$\therefore \quad \theta = 84^\circ$

3 **a** $\sin 159^\circ$
$= \sin(180 - 159)^\circ$
$= \sin 21^\circ$
≈ 0.358

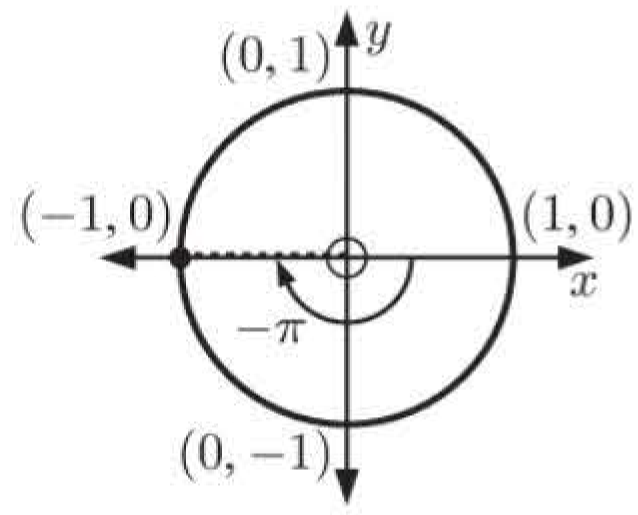
b $\cos 92^\circ$
$= -\cos(180 - 92)^\circ$
$= -\cos 88^\circ$
≈ -0.035

c $\cos 75^\circ$
$= -\cos(180 - 75)^\circ$
$= -\cos 105^\circ$
≈ 0.259

d $\sin(-133^\circ) = \sin(-47)^\circ$
$= -\sin 47^\circ$
≈ -0.731



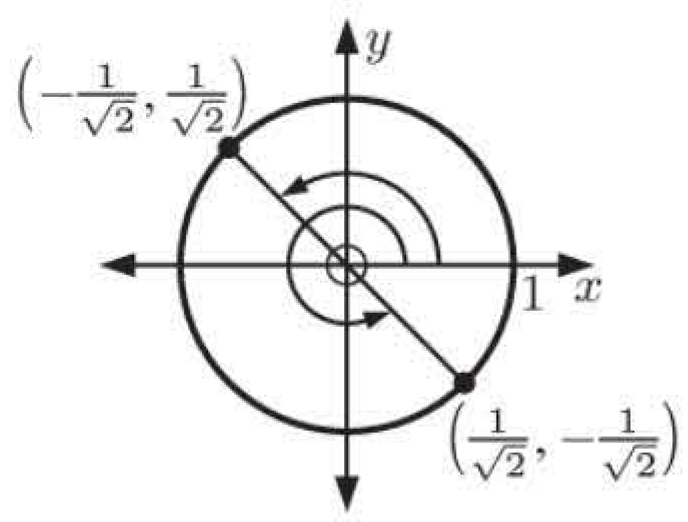
4



a $\cos 360^\circ = 1, \quad \sin 360^\circ = 0$

b $\cos(-\pi) = -1, \quad \sin(-\pi) = 0$

5



When $\cos\theta = -\sin\theta$,

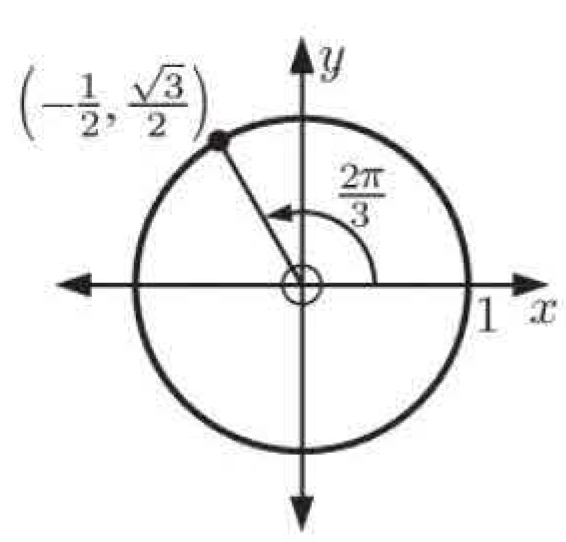
$\dfrac{\sin\theta}{\cos\theta} = -1$

$\therefore \tan\theta = -1$

and this only occurs at the two points shown.

So, $\theta = \frac{3\pi}{4}, \frac{7\pi}{4}$

6 **a**

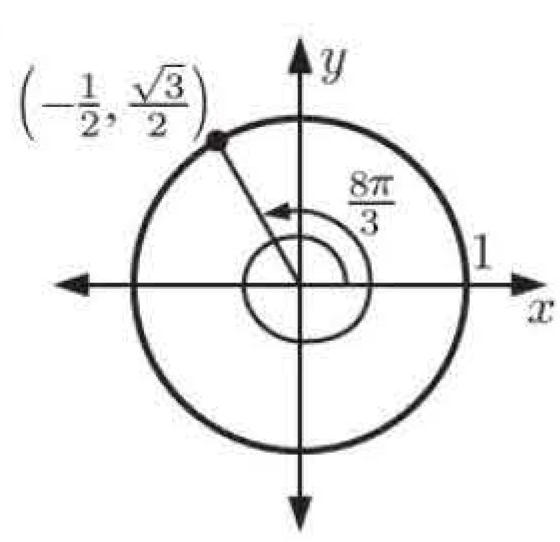


$\sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$

$\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$

$\tan\left(\frac{2\pi}{3}\right) = \dfrac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}}$
$= -\sqrt{3}$

b



$\sin\left(\frac{8\pi}{3}\right) = \frac{\sqrt{3}}{2}$

$\cos\left(\frac{8\pi}{3}\right) = -\frac{1}{2}$

$\tan\left(\frac{8\pi}{3}\right) = \dfrac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}}$
$= -\sqrt{3}$

7 $\cos^2 x + \sin^2 x = 1$
$\therefore \cos^2 x + \frac{1}{16} = 1$
$\therefore \cos^2 x = \frac{15}{16}$
$\therefore \cos x = \pm\frac{\sqrt{15}}{4}$

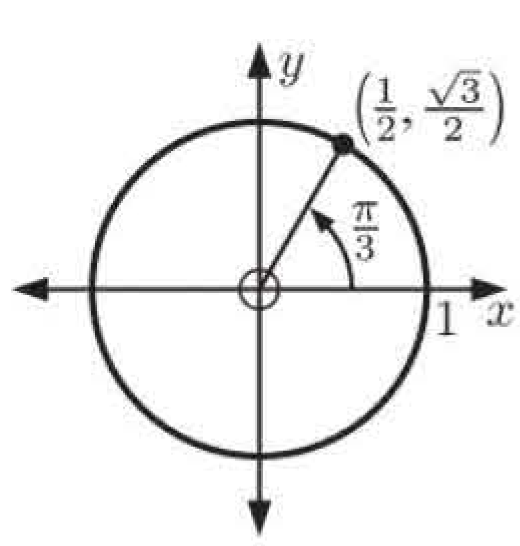
But x is in quadrant 3 where $\cos x < 0$

$\therefore \cos x = -\frac{\sqrt{15}}{4}$

and so $\tan x = \dfrac{\sin x}{\cos x} = \dfrac{-\frac{1}{4}}{-\frac{\sqrt{15}}{4}} = \frac{1}{\sqrt{15}}$

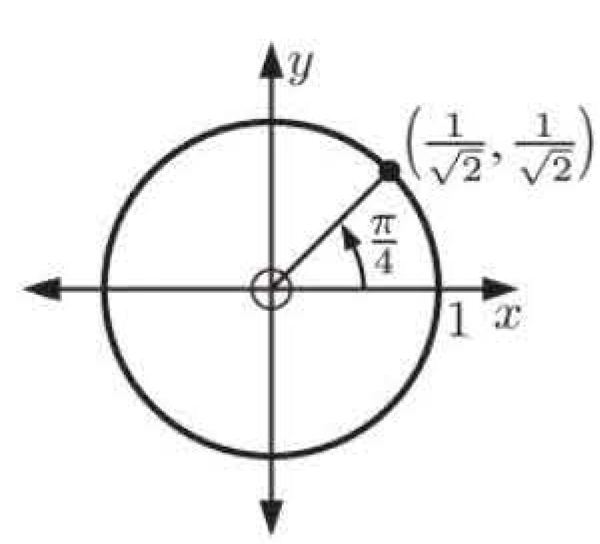
8 $\cos^2\theta + \sin^2\theta = 1$
$\therefore \frac{9}{16} + \sin^2\theta = 1$
$\therefore \sin^2\theta = \frac{7}{16}$
$\therefore \sin\theta = \pm\frac{\sqrt{7}}{4}$

9 **a**



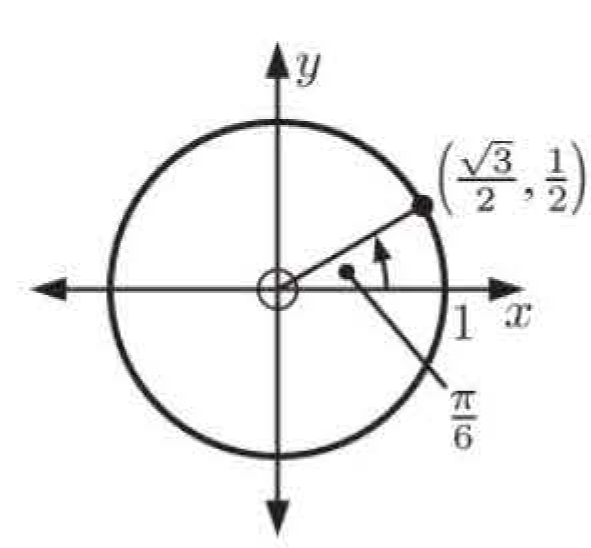
$2\sin(\frac{\pi}{3})\cos(\frac{\pi}{3})$
$= 2(\frac{\sqrt{3}}{2})(\frac{1}{2})$
$= \frac{\sqrt{3}}{2}$

b



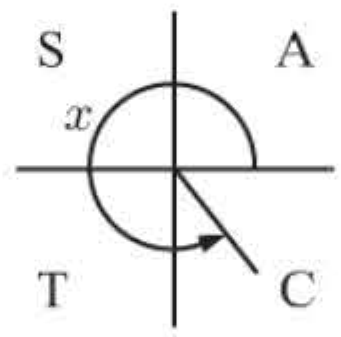
$\tan^2(\frac{\pi}{4}) - 1$
$= 1^2 - 1$
$= 0$

c



$\cos^2(\frac{\pi}{6}) - \sin^2(\frac{\pi}{6})$
$= \left(\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{1}{2}\right)^2$
$= \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$

10 $\dfrac{\sin x}{\cos x} = -\frac{3}{2}$



$\therefore\ \sin x = -\frac{3}{2}\cos x$

Now $\cos^2 x + \sin^2 x = 1$

$\therefore\ \cos^2 x + \frac{9}{4}\cos^2 x = 1$

$\therefore\ \frac{13}{4}\cos^2 x = 1$

$\therefore\ \cos x = \pm\frac{2}{\sqrt{13}}$

But x is in quadrant 4, so $\cos x$ is positive and $\sin x$ is negative.

$\therefore\ \cos x = \frac{2}{\sqrt{13}},\quad \sin x = -\frac{3}{\sqrt{13}}$

So, **a** $\sin x = -\frac{3}{\sqrt{13}}$ **b** $\cos x = \frac{2}{\sqrt{13}}$

11
$$\begin{aligned} \text{arc length} &= \theta r \\ &= 1 \times 4 \\ &= 4 \text{ units} \end{aligned}$$
$$\begin{aligned} \therefore\ \text{perimeter} &= 2 \times 4 + 4 \\ &= 12 \text{ units} \end{aligned}$$
$$\begin{aligned} \text{area} &= \tfrac{1}{2}\theta r^2 \\ &= \tfrac{1}{2} \times 1 \times 4^2 \\ &= 8 \text{ units}^2 \end{aligned}$$

12
$$\cos^2\theta + \sin^2\theta = 1$$
$$\therefore\ \left(\frac{\sqrt{11}}{\sqrt{17}}\right)^2 + \sin^2\theta = 1$$
$$\therefore\ \sin^2\theta = \tfrac{6}{17}$$
$$\therefore\ \sin\theta = \pm\frac{\sqrt{6}}{\sqrt{17}}$$
But θ is acute, $\therefore\ \sin\theta = \frac{\sqrt{6}}{\sqrt{17}}$
$$\tan\theta = \frac{\frac{\sqrt{6}}{\sqrt{17}}}{\frac{\sqrt{11}}{\sqrt{17}}} = \frac{\sqrt{6}}{\sqrt{11}}$$

13 **a**
$$\begin{aligned} &\cos(\tfrac{\pi}{2} - \theta) - \sin\theta \\ &= \sin\theta - \sin\theta \quad \{\text{complementary angle formula}\} \\ &= 0 \end{aligned}$$

b
$$\begin{aligned} &\cos(-\theta)\tan\theta \\ &= \cos\theta\,\frac{\sin\theta}{\cos\theta} \quad \{\text{negative angle formula}\} \\ &= \sin\theta \end{aligned}$$

c
$$\begin{aligned} &\sin(-\alpha)\cos(\alpha - \tfrac{\pi}{2}) \\ &= -\sin\alpha\cos(-(\tfrac{\pi}{2} - \alpha)) \quad \{\text{negative angle formula}\} \\ &= -\sin\alpha\cos(\tfrac{\pi}{2} - \alpha) \quad \{\text{negative angle formula}\} \\ &= -\sin\alpha\sin\alpha \quad \{\text{complementary angle formula}\} \\ &= -\sin^2\alpha \end{aligned}$$

REVIEW SET 10B

1 **a** The point is $(\cos 320°, \sin 320°) \approx (0.766, -0.643)$.

b The point is $(\cos 163°, \sin 163°) \approx (-0.956, 0.292)$.

2 **a**
$$\begin{aligned} &71° \\ &= \left(71 \times \tfrac{\pi}{180}\right)^c \\ &\approx 1.239^c \end{aligned}$$

b
$$\begin{aligned} &124.6° \\ &= \left(124.6 \times \tfrac{\pi}{180}\right)^c \\ &\approx 2.175^c \end{aligned}$$

c
$$\begin{aligned} &-142° \\ &= \left(-142 \times \tfrac{\pi}{180}\right)^c \\ &\approx -2.478^c \end{aligned}$$

3 **a**
$$\begin{aligned} &3^c \\ &= \left(3 \times \tfrac{180}{\pi}\right)^{\circ} \\ &\approx 171.89° \end{aligned}$$

b
$$\begin{aligned} &1.46^c \\ &= \left(1.46 \times \tfrac{180}{\pi}\right)^{\circ} \\ &\approx 83.65° \end{aligned}$$

c
$$\begin{aligned} &0.435^c \\ &= \left(0.435 \times \tfrac{180}{\pi}\right)^{\circ} \\ &\approx 24.92° \end{aligned}$$

d
$$\begin{aligned} &-5.271^c \\ &= \left(-5.271 \times \tfrac{180}{\pi}\right)^{\circ} \\ &\approx -302.01° \end{aligned}$$

4
$$\begin{aligned} \text{area} &= \tfrac{1}{2} \times \tfrac{5\pi}{12} \times 13^2 \\ &\approx 111 \text{ cm}^2 \end{aligned}$$

5 $M(\cos 73°, \sin 73°) \approx (0.292, 0.956)$,
$N(\cos 190°, \sin 190°) \approx (-0.985, -0.174)$,
$P(\cos(-53°), \sin(-53°)) = P(\cos 307°, \sin 307°) \approx (0.602, -0.799)$

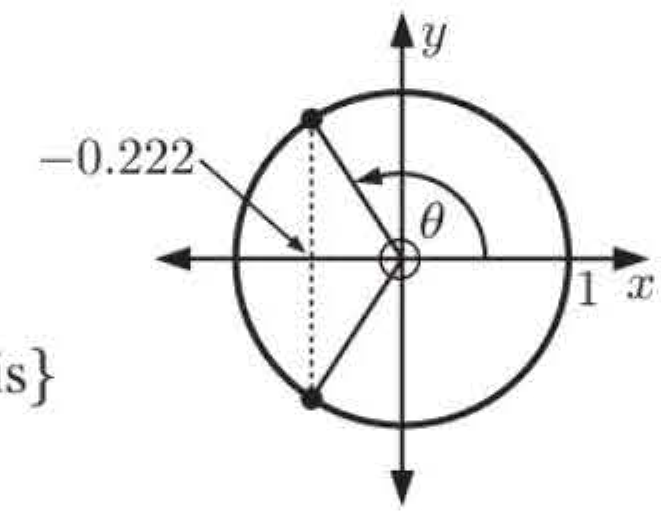
6 The x-coordinate of A $= -0.222$

$\therefore \quad \cos\theta = -0.222$

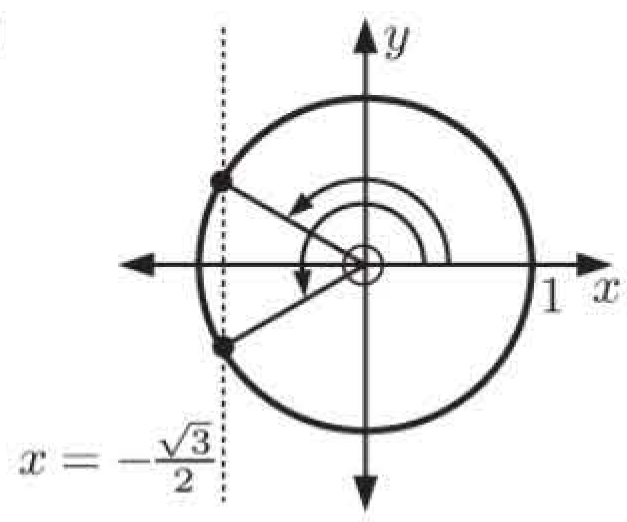
$\therefore \quad \theta = \cos^{-1}(-0.222)$

$\therefore \quad \theta \approx 102.8°, 257.2°$

$\therefore \quad \theta \approx 103°$ {taking angle to positive x-axis}

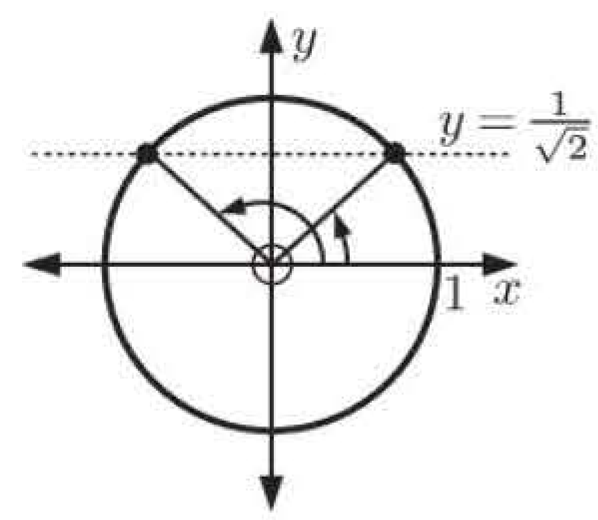


7 **a**



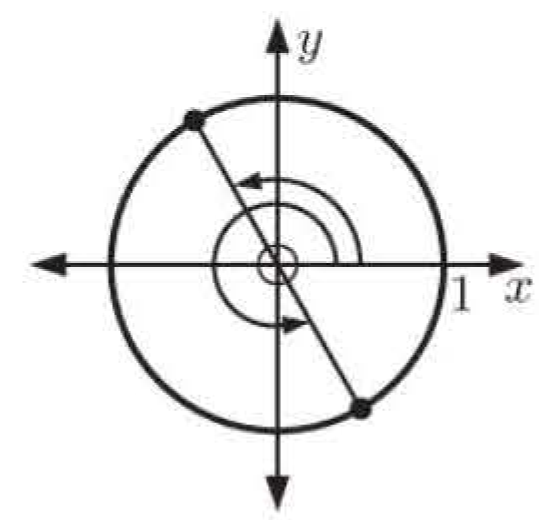
$\therefore \quad \theta = 150°$ or $210°$

b



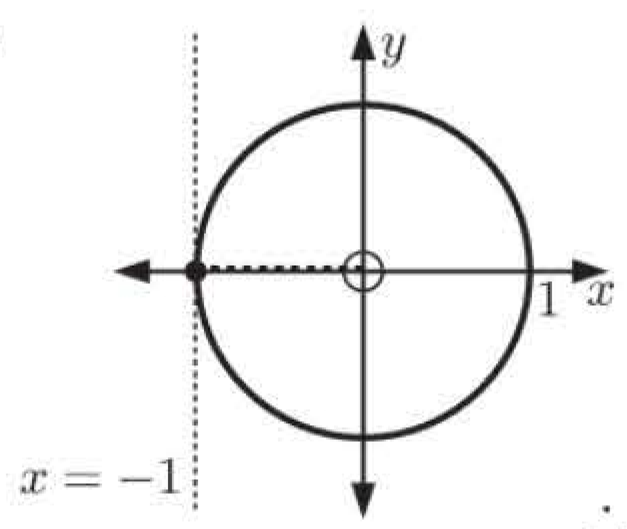
$\therefore \quad \theta = 45°$ or $135°$

c



$\therefore \quad \theta = 120°$ or $300°$

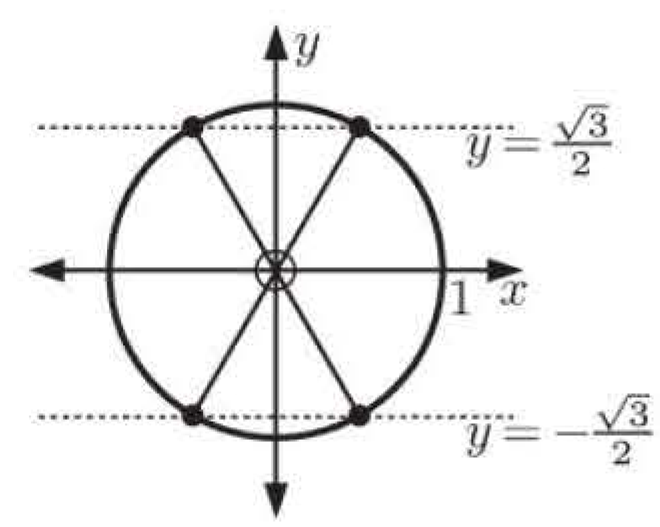
8 **a**



$\therefore \quad \theta = \pi$

b $\sin^2\theta = \frac{3}{4}$

$\therefore \quad \sin\theta = \pm\frac{\sqrt{3}}{2}$



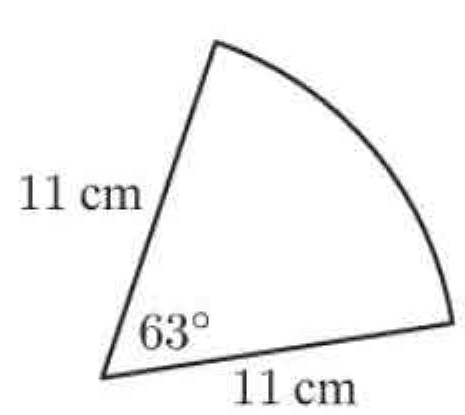
$\therefore \quad \theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

9 **a** $\sin 47° = \sin(180 - 47)°$
$= \sin 133°$
$\therefore \quad \theta = 133°$

b $\sin(\frac{\pi}{15}) = \sin(\pi - \frac{\pi}{15})$
$= \sin(\frac{14\pi}{15})$
$\therefore \quad \theta = \frac{14\pi}{15}$

c $\cos 186° = \cos(360 - 186)°$
$= \cos 174°$
$\therefore \quad \theta = 174°$

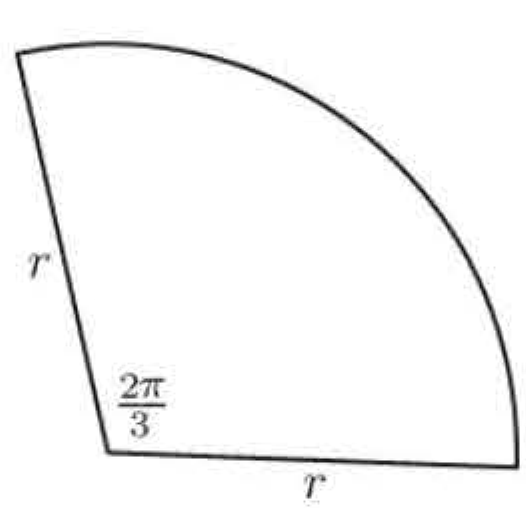
10



perimeter $= 2 \times 11 + \left(\frac{63}{360}\right) \times 2\pi \times 11$
≈ 34.1 cm

area $= \left(\frac{63}{360}\right) \times \pi \times 11^2$
$\approx 66.5 \text{ cm}^2$

11



perimeter $= 2r + (\frac{2\pi}{3})r$

$\therefore \quad 36 = r\left(2 + \frac{2\pi}{3}\right)$

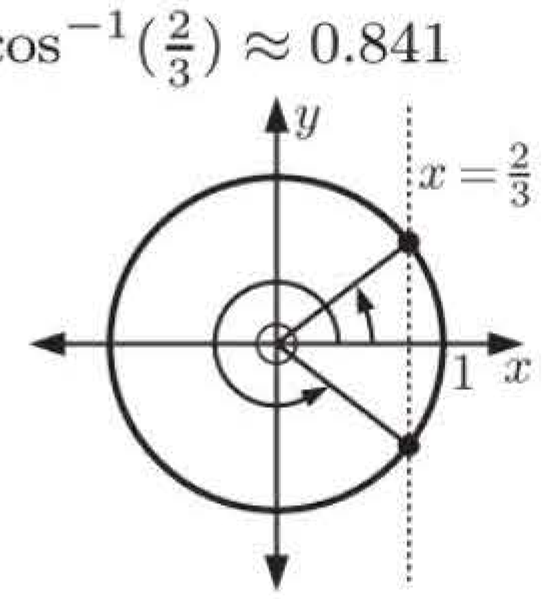
$\therefore \quad r = \dfrac{36}{2 + \frac{2\pi}{3}}$ cm

$\therefore \quad r \approx 8.79$ cm

area $\approx \frac{1}{2}(\frac{2\pi}{3}) \times (8.7925)^2$
$\approx 81.0 \text{ cm}^2$

12 **a** $\cos\theta = \frac{2}{3}$

Using technology,
$\cos^{-1}(\frac{2}{3}) \approx 0.841$

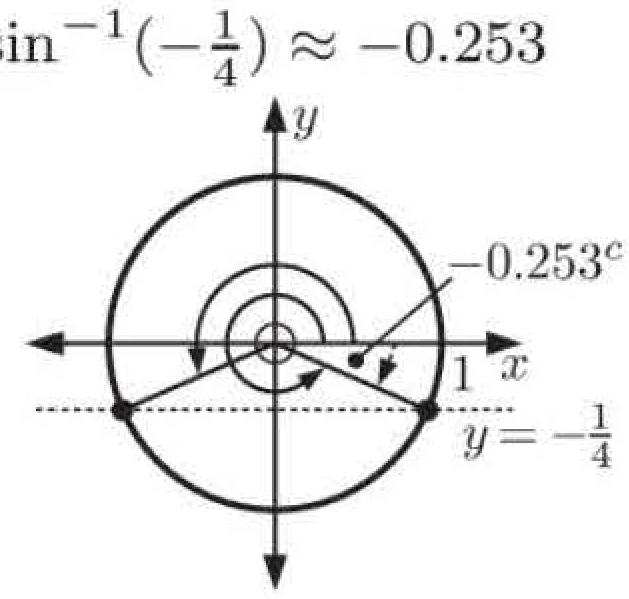


$\therefore \quad \theta \approx 0.841$ or $2\pi - 0.841$

$\therefore \quad \theta \approx 0.841$ or 5.44

b $\sin\theta = -\frac{1}{4}$

Using technology,
$\sin^{-1}(-\frac{1}{4}) \approx -0.253$



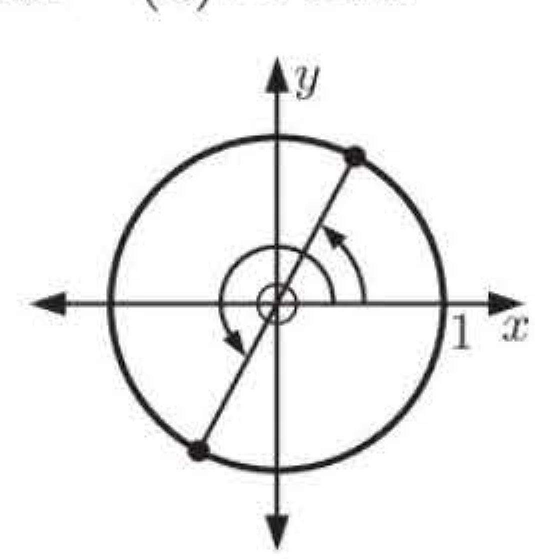
But $0 \leqslant \theta \leqslant 2\pi$

$\therefore \quad \theta \approx \pi + 0.253$ or $2\pi + (-0.253)$

$\therefore \quad \theta \approx 3.39$ or 6.03

c $\tan\theta = 3$

Using technology,
$\tan^{-1}(3) \approx 1.25$



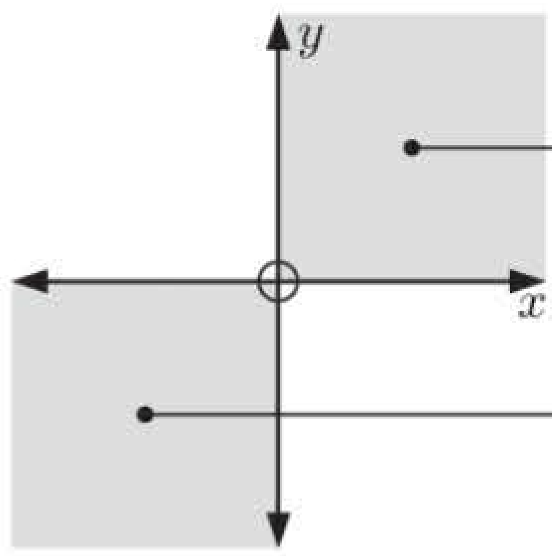
$\therefore \quad \theta \approx 1.25$ or $\pi + 1.25$

$\therefore \quad \theta \approx 1.25$ or 4.39

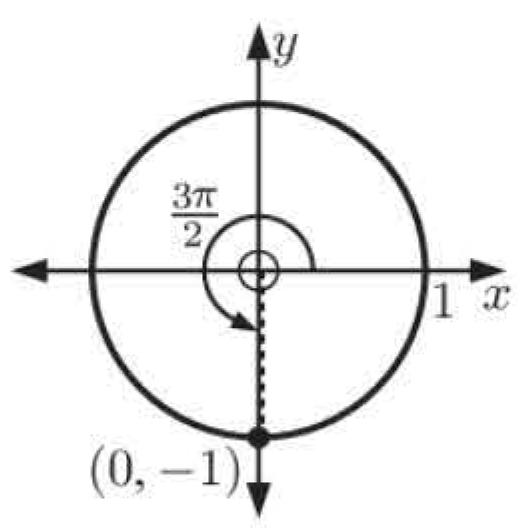
REVIEW SET 10C

1 **a** $\frac{2\pi}{5} = \frac{2\times180^\circ}{5} = 72^\circ$ **b** $\frac{5\pi}{4} = \frac{5\times180^\circ}{4} = 225^\circ$ **c** $\frac{7\pi}{9} = \frac{7\times180^\circ}{9} = 140^\circ$ **d** $\frac{11\pi}{6} = \frac{11\times180^\circ}{6} = 330^\circ$

2



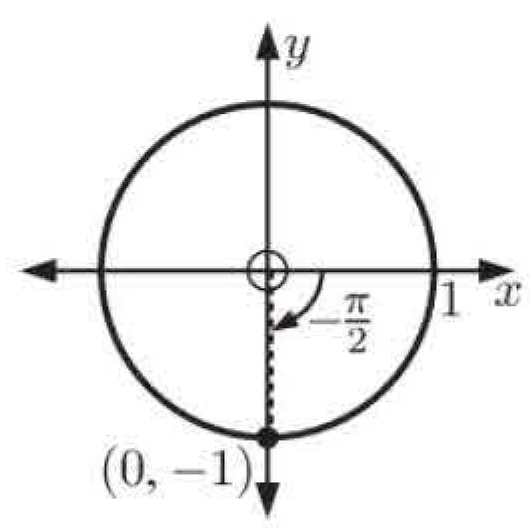
3 **a**



$\therefore \cos\left(\frac{3\pi}{2}\right) = 0$

$\sin\left(\frac{3\pi}{2}\right) = -1$

b



$\therefore \cos\left(-\frac{\pi}{2}\right) = 0$

$\sin\left(-\frac{\pi}{2}\right) = -1$

4 **a**
$$\sin(\pi-\theta) = \sin\theta$$
$$\therefore \sin(\pi-p) = \sin p = m$$

b
$$\sin(\theta+2\pi) = \sin\theta$$
$$\therefore \sin(p+2\pi) = \sin p = m$$

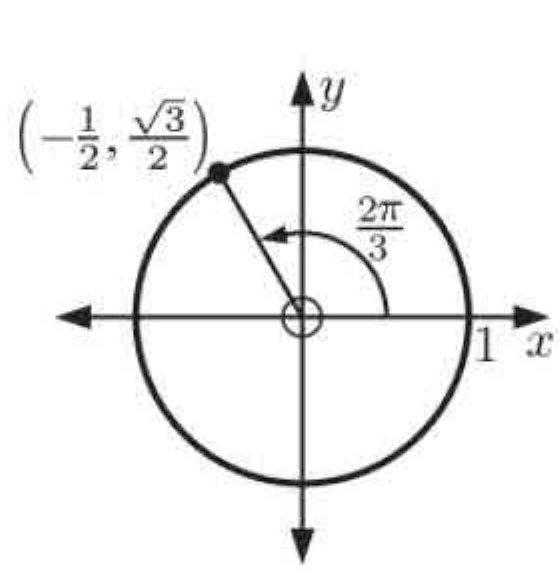
c
$$\cos^2 p + \sin^2 p = 1$$
$$\therefore \cos^2 p + m^2 = 1$$
$$\therefore \cos^2 p = 1 - m^2$$
$$\therefore \cos p = \pm\sqrt{1-m^2}$$
But p is acute, $\therefore \cos p = \sqrt{1-m^2}$

d
$$\tan p = \frac{\sin p}{\cos p} = \frac{m}{\sqrt{1-m^2}}$$

5 **a** **i** $\theta = 60^\circ$ {equilateral triangle} **ii** $\theta = \frac{\pi}{3}$ radians

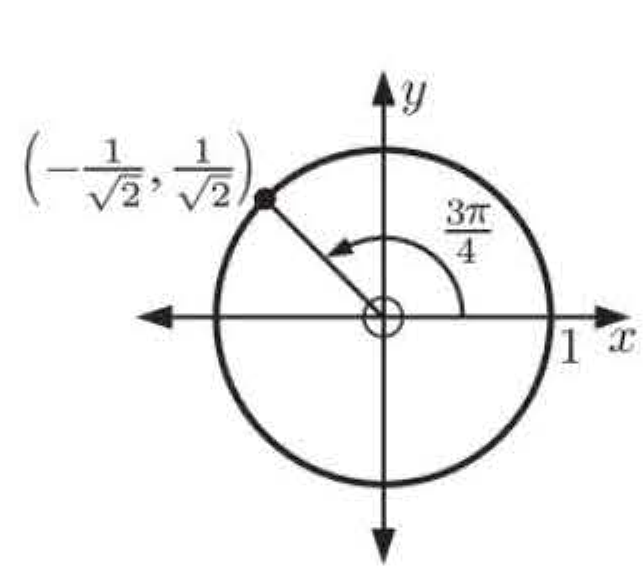
b arc length $= \theta r = \frac{\pi}{3}$ units **c** sector area $= \frac{1}{2}\theta r^2 = \frac{\pi}{6}$ units^2

6



$\tan^2(\frac{2\pi}{3})$
$= (-\sqrt{3})^2$
$= 3$

7

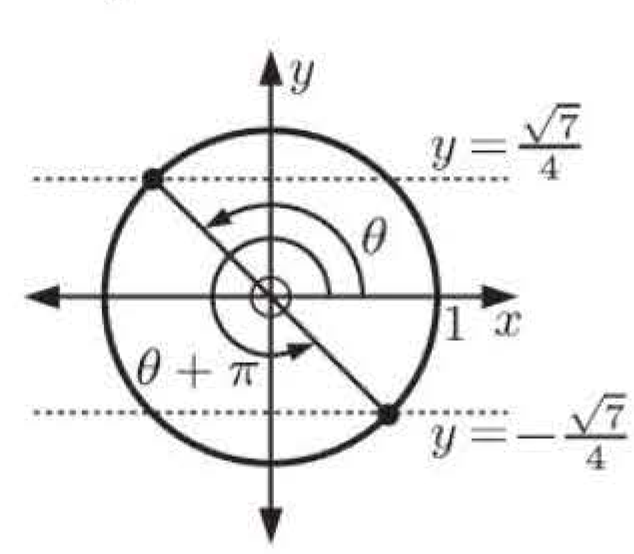


$\cos(\frac{3\pi}{4}) - \sin(\frac{3\pi}{4})$
$= -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}$
$= -\frac{2}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$
$= -\sqrt{2}$

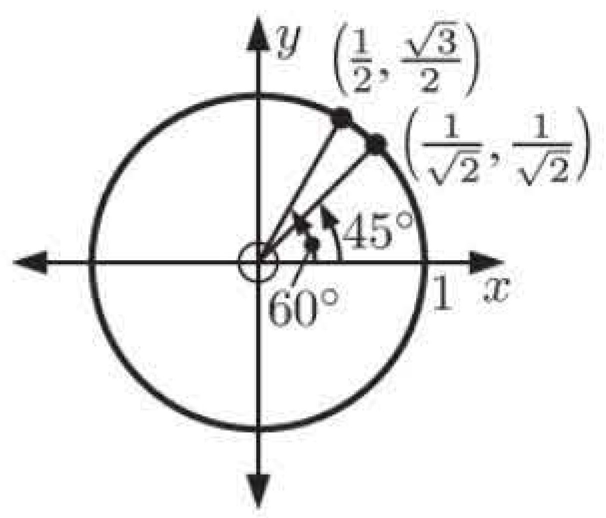
8 **a**
$$\cos^2\theta + \sin^2\theta = 1$$
$$\therefore \frac{9}{16} + \sin^2\theta = 1$$
$$\therefore \sin^2\theta = \frac{7}{16}$$
$$\therefore \sin\theta = \pm\frac{\sqrt{7}}{4}$$
But θ is in quadrant 2 where $\sin\theta > 0$
$$\therefore \sin\theta = \frac{\sqrt{7}}{4}$$

b $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{\sqrt{7}}{4}}{-\frac{3}{4}} = -\frac{\sqrt{7}}{3}$

c
$$\sin(\theta+\pi)$$
$$= -\sin\theta$$
$$= -\frac{\sqrt{7}}{4}$$

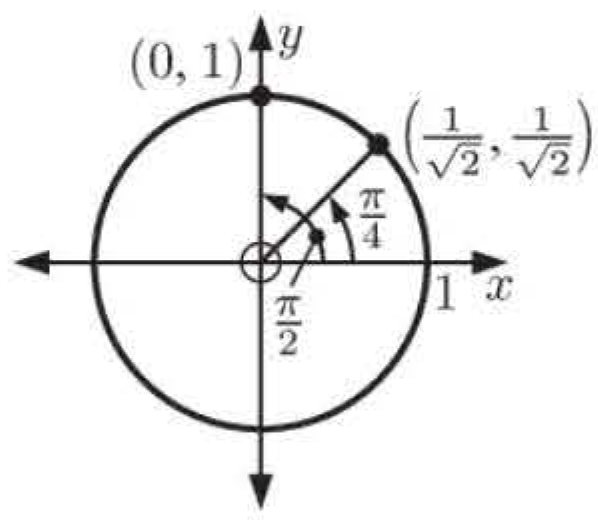


9 **a**



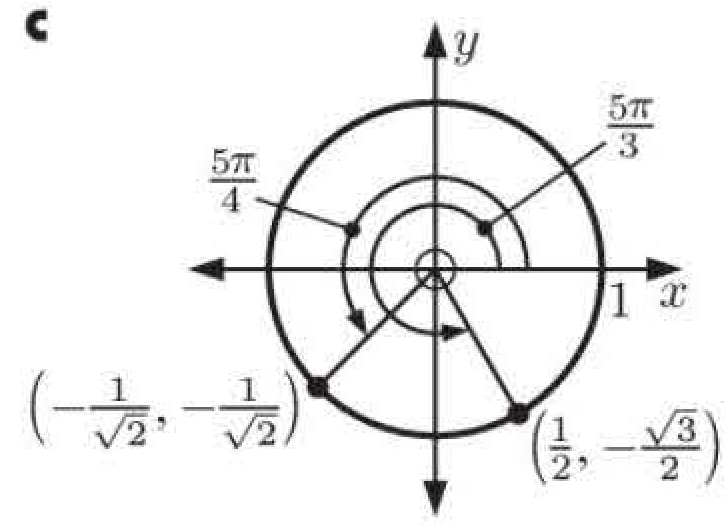
$\tan^2 60° - \sin^2 45°$
$= (\sqrt{3})^2 - \left(\frac{1}{\sqrt{2}}\right)^2$
$= 3 - \frac{1}{2}$
$= 2\frac{1}{2}$

b



$\cos^2(\frac{\pi}{4}) + \sin(\frac{\pi}{2})$
$= \left(\frac{1}{\sqrt{2}}\right)^2 + 1$
$= \frac{1}{2} + 1$
$= 1\frac{1}{2}$

c

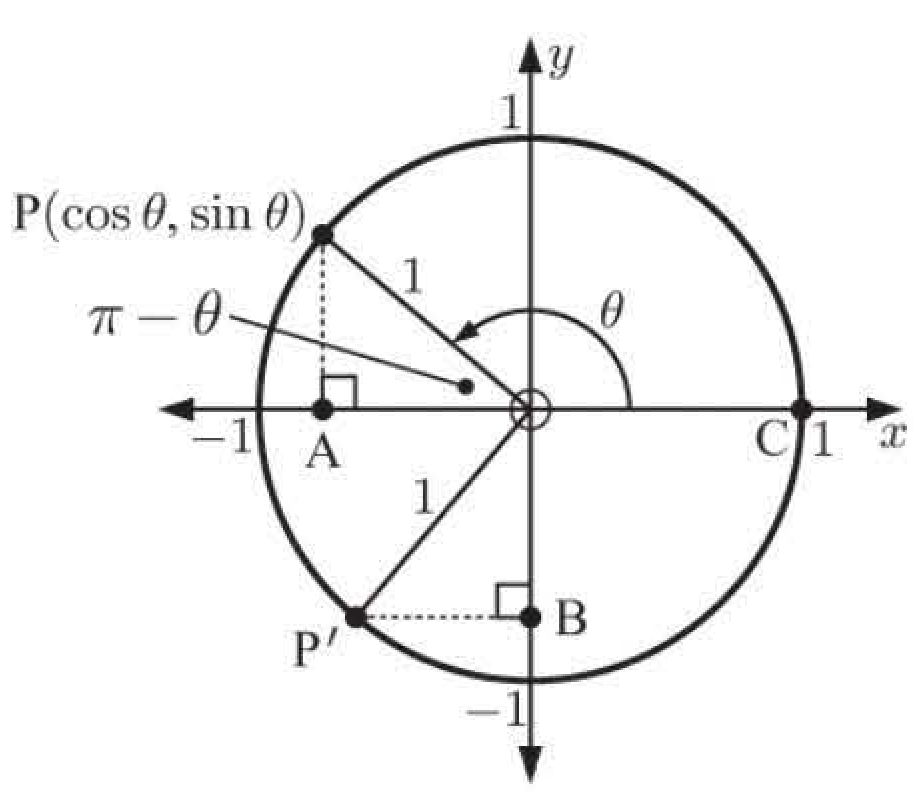


$\cos(\frac{5\pi}{3}) - \tan(\frac{5\pi}{4})$
$= \frac{1}{2} - 1$
$= -\frac{1}{2}$

10 **a** $\sin(\pi - \theta) - \sin\theta = \sin\theta - \sin\theta$
$= 0$

b $\sin(\frac{\pi}{2} - \theta) - 2\cos\theta = \cos\theta - 2\cos\theta$
$= -\cos\theta$

11



For $\frac{\pi}{2} < \theta < \pi$:

The diagram shows P rotated through $\frac{\pi}{2}$ to P′, so OP′ makes an angle of $\frac{\pi}{2} + \theta$ with the positive x-axis.

$\mathrm{P\widehat{O}A} = \pi - \theta$ and $\mathrm{P'\widehat{O}B} = \text{reflex } \mathrm{C\widehat{O}B} - \text{reflex } \mathrm{C\widehat{O}P'}$
$= \frac{3\pi}{2} - (\frac{\pi}{2} + \theta)$
$= \pi - \theta$

In △s P′OB and POA:
- $\mathrm{OP'} = \mathrm{OP}$
- $\mathrm{P'\widehat{O}B} = \mathrm{P\widehat{O}A}$
- $\mathrm{P'\widehat{B}O} = \mathrm{P\widehat{A}O}$

∴ △s P′OB and POA are congruent {AAcorS}

∴ $\mathrm{P'B} = \mathrm{PA} = \sin\theta$

So P′ has x-coordinate $-\sin\theta$

But P′ has x-coordinate $\cos(\frac{\pi}{2} + \theta)$

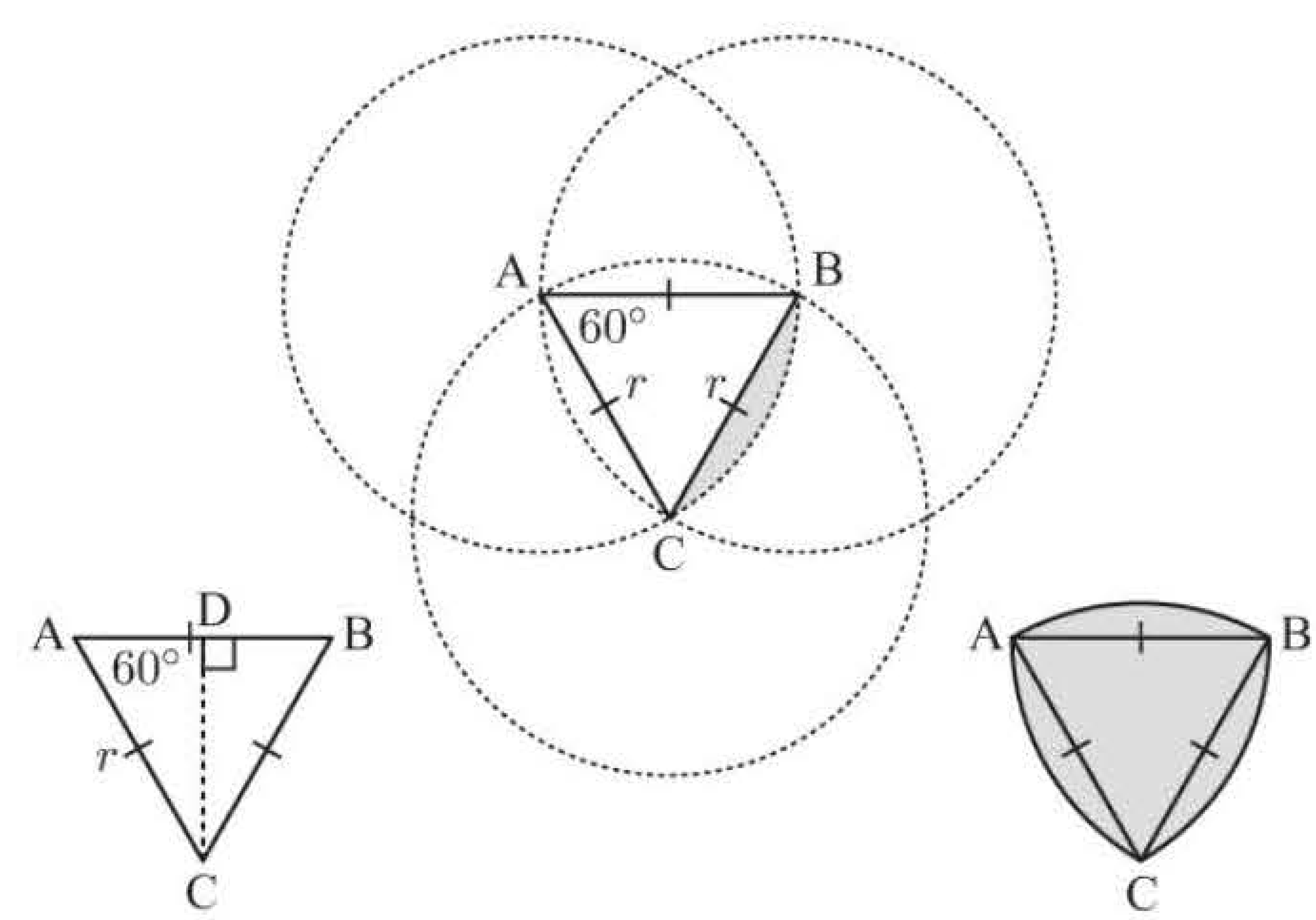
∴ $\cos(\frac{\pi}{2} + \theta) = -\sin\theta$

12 [AB], [AC], and [BC] are all radii,
so $\mathrm{AB} = \mathrm{AC} = \mathrm{BC} = r$.
Hence △ABC is equilateral
and so $\mathrm{C\widehat{A}B} = 60°$.

$\therefore \sin 60° = \dfrac{\mathrm{CD}}{\mathrm{AC}}$

$\therefore \mathrm{CD} = \sin 60° \times \mathrm{AC} = \frac{\sqrt{3}}{2}r$

$\therefore$ area of $\triangle = \frac{1}{2}(r)(\frac{\sqrt{3}}{2}r)$
$= \frac{\sqrt{3}}{4}r^2$



shaded area of sector
$=$ area of sector $-$ area of △
$= \frac{60}{360}\pi r^2 - \frac{\sqrt{3}}{4}r^2$
$= \frac{\pi}{6}r^2 - \frac{\sqrt{3}}{4}r^2$

$\therefore$ shaded area of figure $= 3\left[\frac{\pi}{6}r^2 - \frac{\sqrt{3}}{4}r^2\right] + \frac{\sqrt{3}}{4}r^2$
$= \frac{\pi}{2}r^2 - \frac{3\sqrt{3}}{4}r^2 + \frac{\sqrt{3}}{4}r^2$
$= \frac{\pi}{2}r^2 - \frac{2}{4}\sqrt{3}r^2$
$= \frac{r^2}{2}\left(\pi - \sqrt{3}\right)$

Chapter 11

NON-RIGHT ANGLED TRIANGLE TRIGONOMETRY

EXERCISE 11A

1 **a** area
$= \frac{1}{2} \times 9 \times 10 \times \sin 40^\circ$
$\approx 28.9 \text{ cm}^2$

b area
$= \frac{1}{2} \times 25 \times 31 \times \sin 82^\circ$
$\approx 384 \text{ km}^2$

c area
$= \frac{1}{2} \times 10.2 \times 6.4 \times \sin(\frac{2\pi}{3})$
$\approx 28.3 \text{ cm}^2$

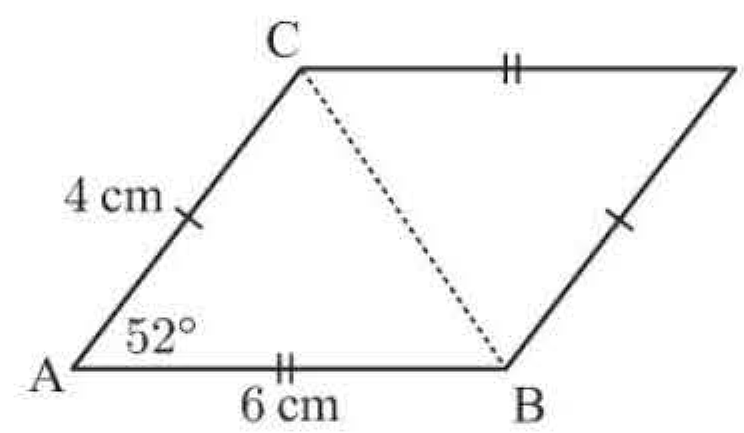
2 area $= 150 \text{ cm}^2$

$\therefore\ \frac{1}{2} \times 17 \times x \times \sin 68^\circ = 150$

$\therefore\ x = \dfrac{2 \times 150}{17 \times \sin 68^\circ}$

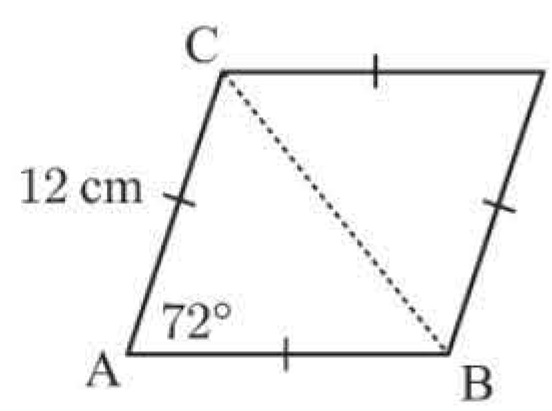
$\therefore\ x \approx 19.0$

3



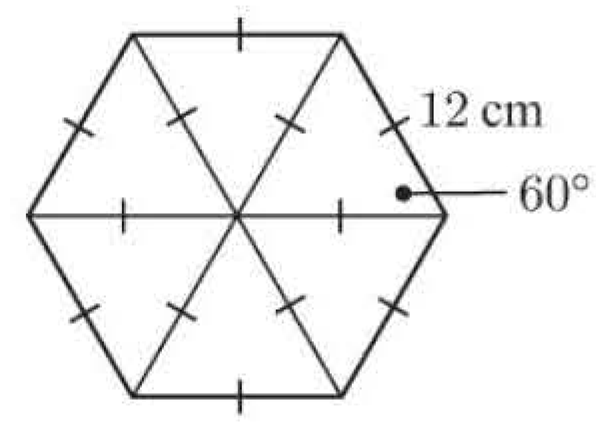
area $= 2 \times$ area $\triangle$ABC
$= 2 \times \frac{1}{2} \times 4 \times 6 \times \sin 52^\circ$
$\approx 18.9 \text{ cm}^2$

4



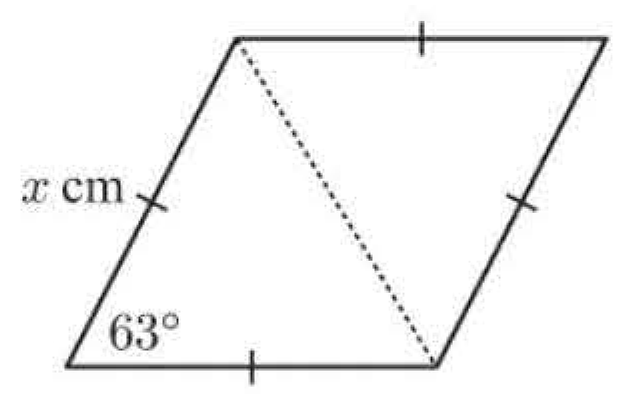
area $= 2 \times$ area $\triangle$ABC
$= 2 \times \frac{1}{2} \times 12^2 \times \sin 72^\circ$
$\approx 137 \text{ cm}^2$

5



area $= 6 \times$ area of $\triangle$
$= 6 \times \frac{1}{2} \times 12^2 \times \sin 60^\circ$
$\approx 374 \text{ cm}^2$

6



area $= 2 \times \frac{1}{2}x^2 \sin 63^\circ$

$\therefore\ x^2 \sin 63^\circ = 50$

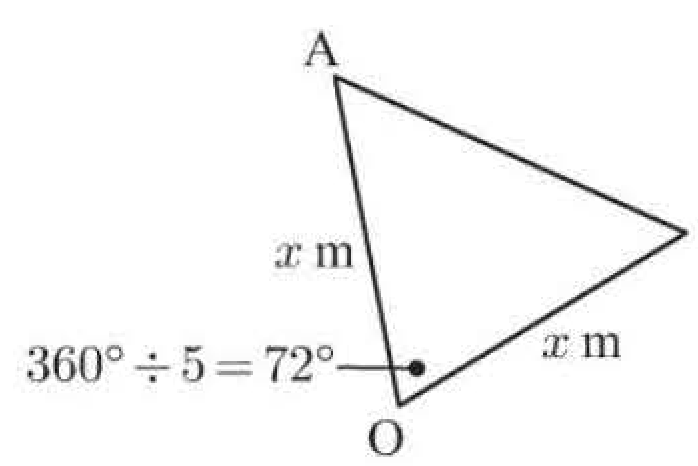
$\therefore\ x^2 = \dfrac{50}{\sin 63^\circ}$

$\therefore\ x = \sqrt{\dfrac{50}{\sin 63^\circ}} \quad \{x > 0\}$

$\therefore\ x \approx 7.49$

So, sides are 7.49 cm long.

7



area of $\triangle = \frac{338}{5}$

$\therefore\ \frac{1}{2}x^2 \sin 72^\circ = \frac{338}{5}$

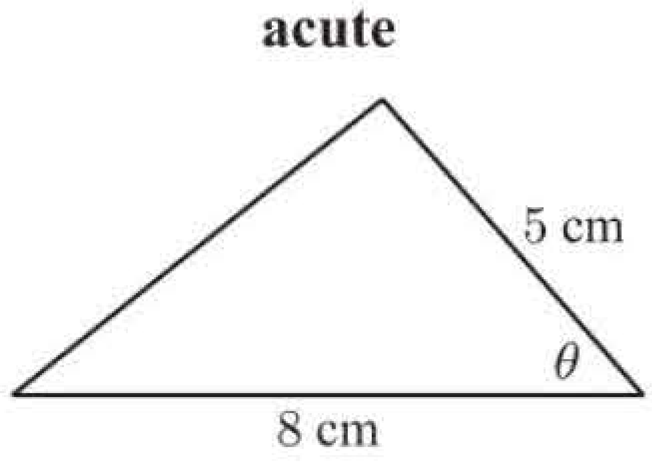
$\therefore\ x^2 = \dfrac{2 \times 338}{5 \times \sin 72^\circ}$

$\therefore\ x = \sqrt{\dfrac{2 \times 338}{5 \times \sin 72^\circ}} \quad \{x > 0\}$

$\therefore\ x \approx 11.9$

So, $\text{OA} \approx 11.9$ m

8 **a** **acute**



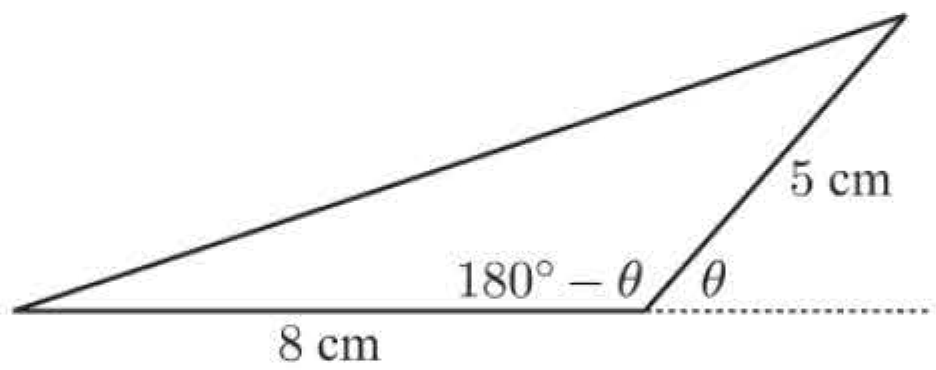
area $= \frac{1}{2} \times 5 \times 8 \times \sin\theta$

$\therefore\ 15 = 20 \sin\theta$

$\therefore\ \sin\theta = \frac{3}{4}$

$\therefore\ \theta = \sin^{-1}(\frac{3}{4})$
$\approx 48.6^\circ$

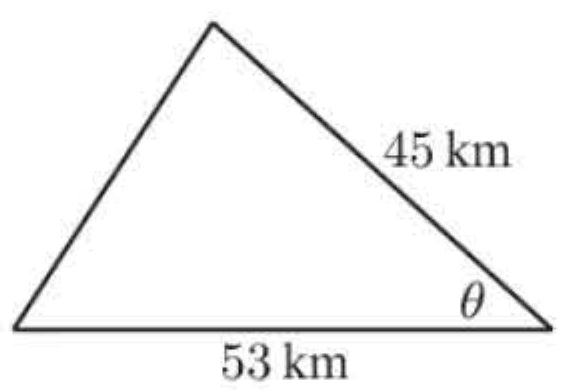
or **obtuse**



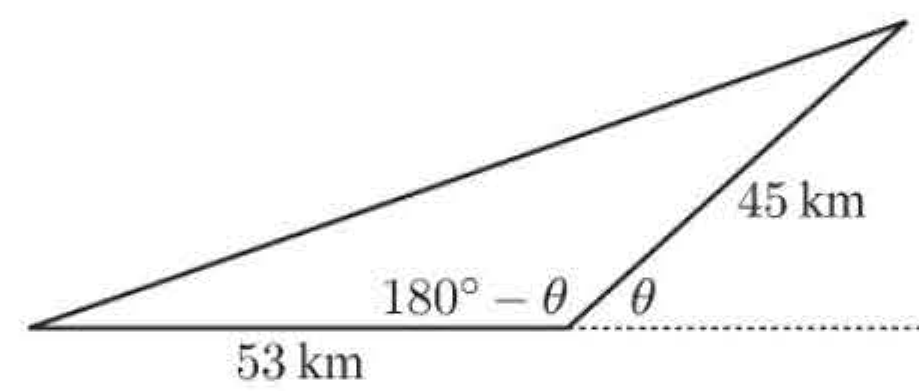
also $180° - \theta \approx 180° - 48.6° \approx 131.4°$

So, if the included angle is acute then its value is $\approx 48.6°$, otherwise if the included angle is obtuse then its value is $\approx 131.4°$.

b **acute**



or **obtuse**



$$\text{area} = \tfrac{1}{2} \times 45 \times 53 \times \sin\theta$$

$$\therefore\ 800 = \frac{45 \times 53}{2} \times \sin\theta$$

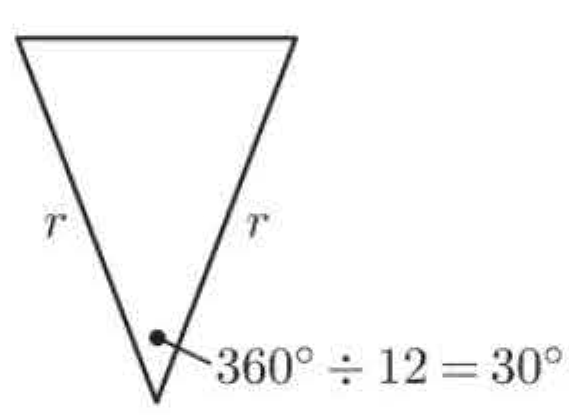
$$\therefore\ \sin\theta = \frac{2 \times 800}{45 \times 53}$$

$$\therefore\ \theta = \sin^{-1}\left(\frac{2 \times 800}{45 \times 53}\right) \approx 42.1°$$

also $180° - \theta \approx 180° - 42.1° \approx 137.9°$

So, if the included angle is acute then its value is $\approx 42.1°$, otherwise if the included angle is obtuse then its value is $\approx 137.9°$.

9



total area of 8 coins
$= 8 \times 12 \times \frac{1}{2} r^2 \sin 30°$
$= 48r^2(\frac{1}{2})$
$= 24r^2$

area of \$10 note
$= 8r \times 4r$
$= 32r^2$

fraction covered
$= \dfrac{24r^2}{32r^2}$
$= \frac{3}{4}$ $\therefore$ $\frac{1}{4}$ is not covered

10 **a** shaded area
$= \text{area of sector} - \text{area of triangle}$
$= \frac{1}{2} \times 1.5 \times 12^2 - \frac{1}{2} \times 12 \times 12 \times \sin(1.5^c)$
$\approx 36.2\ \text{cm}^2$

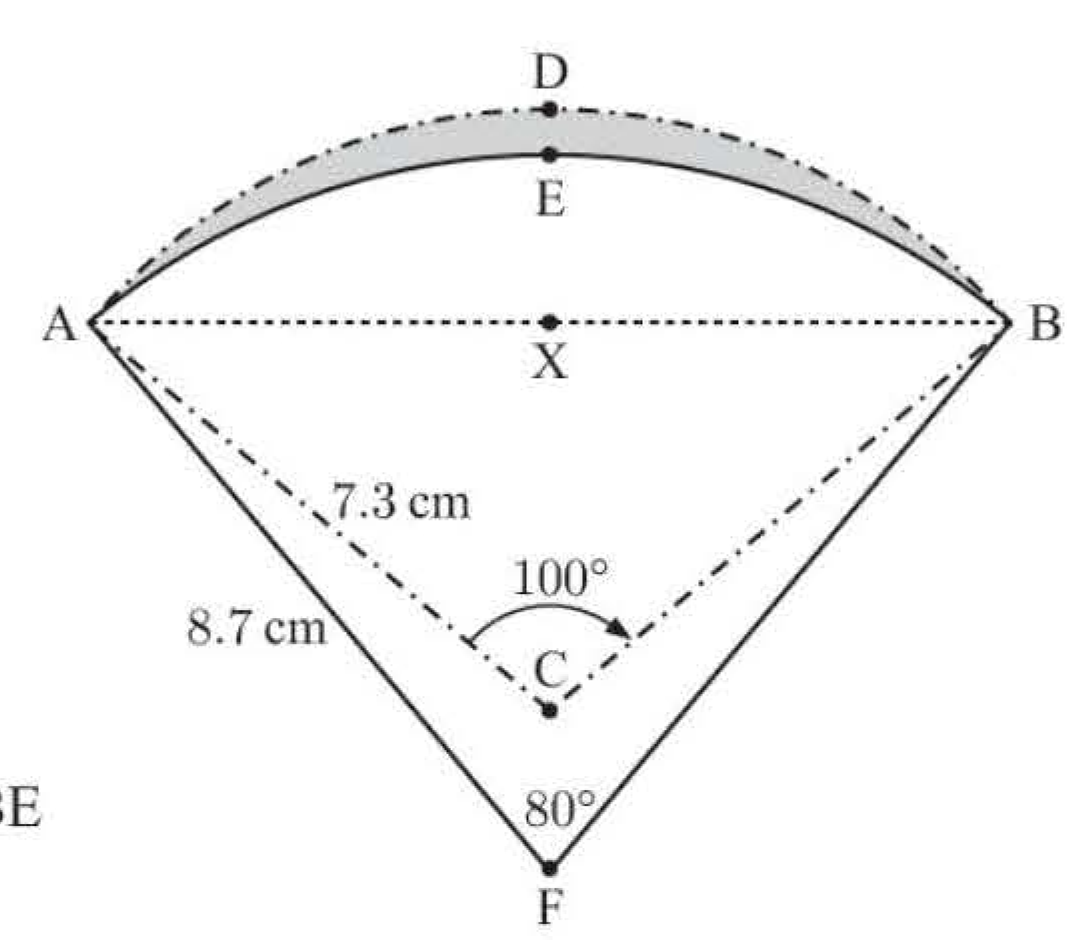
b shaded area
$= \text{area of triangle} - \text{area of sector}$
$= \frac{1}{2} \times 12 \times 30 \times \sin(0.66^c) - \frac{1}{2} \times 0.66 \times 12^2$
$\approx 62.8\ \text{cm}^2$

c shaded area
$= \text{area of sector} - \text{area of triangle}$
$= \left(\frac{135}{360}\right) \times \pi \times 7^2 - \frac{1}{2} \times 7 \times 7 \times \sin 135°$
$\approx 40.4\ \text{mm}^2$

11 area segment AXBD
$= \text{area sector ACBD} - \text{area } \triangle\text{ACB}$
$= \left(\frac{100}{360}\right) \times \pi \times 7.3^2 - \frac{1}{2} \times 7.3 \times 7.3 \times \sin 100°$
$\approx 20.264\ \text{cm}^2$

area segment AXBE
$= \text{area sector AFBE} - \text{area } \triangle\text{AFB}$
$= \left(\frac{80}{360}\right) \times \pi \times 8.7^2 - \frac{1}{2} \times 8.7 \times 8.7 \times \sin 80°$
$\approx 15.572\ \text{cm}^2$

$\therefore$ shaded area $=$ area segment AXBD $-$ area segment AXBE
$\approx 20.264 - 15.572 \approx 4.69\ \text{cm}^2$



EXERCISE 11B

1 **a** $\text{BC}^2 = 21^2 + 15^2 - 2 \times 21 \times 15 \times \cos 105°$

$\therefore \quad \text{BC} = \sqrt{21^2 + 15^2 - 2 \times 21 \times 15 \times \cos 105°} \approx 28.8 \text{ cm}$

b $\text{PQ}^2 = 6.3^2 + 4.8^2 - 2 \times 6.3 \times 4.8 \times \cos 32°$

$\therefore \quad \text{PQ} = \sqrt{6.3^2 + 4.8^2 - 2 \times 6.3 \times 4.8 \times \cos 32°} \approx 3.38 \text{ km}$

c $\text{KM}^2 = 6.2^2 + 14.8^2 - 2 \times 6.2 \times 14.8 \times \cos 72°$

$\therefore \quad \text{KM} = \sqrt{6.2^2 + 14.8^2 - 2 \times 6.2 \times 14.8 \times \cos 72°} \approx 14.2 \text{ m}$

2 $\cos \text{B}\widehat{\text{A}}\text{C} = \dfrac{12^2 + 13^2 - 11^2}{2 \times 12 \times 13}$

$\therefore \quad \text{B}\widehat{\text{A}}\text{C} = \cos^{-1}\left(\frac{192}{312}\right)$

$\therefore \quad \text{B}\widehat{\text{A}}\text{C} \approx 52.0°$

$\cos \text{A}\widehat{\text{B}}\text{C} = \dfrac{13^2 + 11^2 - 12^2}{2 \times 13 \times 11}$

$\therefore \quad \text{A}\widehat{\text{B}}\text{C} = \cos^{-1}\left(\frac{146}{286}\right)$

$\therefore \quad \text{A}\widehat{\text{B}}\text{C} \approx 59.3°$

$\text{A}\widehat{\text{C}}\text{B} = 180° - \text{B}\widehat{\text{A}}\text{C} - \text{A}\widehat{\text{B}}\text{C}$

$\approx 180° - 52.0° - 59.3°$

$\approx 68.7°$

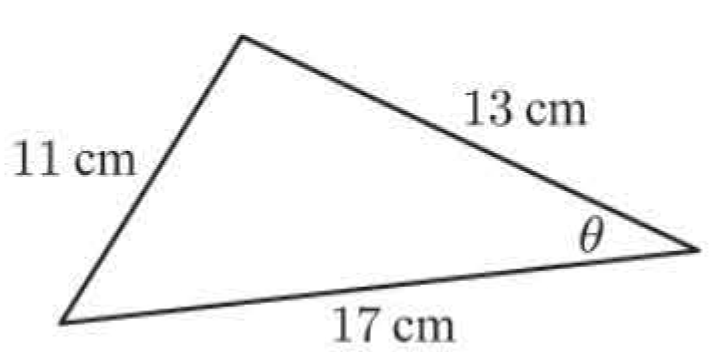
3 **a** $\cos \text{P}\widehat{\text{Q}}\text{R} = \dfrac{5^2 + 7^2 - 10^2}{2 \times 5 \times 7}$

$\therefore \quad \text{P}\widehat{\text{Q}}\text{R} = \cos^{-1}\left(\frac{-26}{70}\right) \approx 112°$

b area $\approx \frac{1}{2} \times 5 \times 7 \times \sin 112°$

$\approx 16.2 \text{ cm}^2$

4 **a**



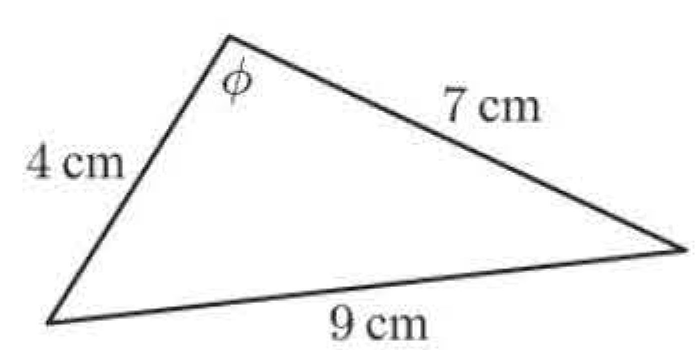
The smallest angle is opposite the shortest side.

$\cos \theta = \dfrac{13^2 + 17^2 - 11^2}{2 \times 13 \times 17}$

$\therefore \quad \theta = \cos^{-1}\left(\frac{337}{442}\right) \approx 40.3°$

So, the smallest angle measures $40.3°$.

b



The largest angle is opposite the longest side.

$\cos \phi = \dfrac{4^2 + 7^2 - 9^2}{2 \times 4 \times 7}$

$\therefore \quad \phi = \cos^{-1}\left(-\frac{16}{56}\right) \approx 106.60°$

So, the largest angle measures about $107°$.

5 **a** $\cos \theta = \dfrac{2^2 + 5^2 - 4^2}{2 \times 2 \times 5}$

$= \frac{13}{20}$

$= 0.65$

b $x^2 = 5^2 + 3^2 - 2 \times 5 \times 3 \times \cos \theta$

$\therefore \quad x = \sqrt{5^2 + 3^2 - 2 \times 5 \times 3 \times 0.65}$

$\therefore \quad x \approx 3.81$

6 **a** $7^2 = x^2 + 6^2 - 2 \times x \times 6 \times \cos 60°$

$\therefore \quad 49 = x^2 + 36 - 12x \times \left(\frac{1}{2}\right)$

$\therefore \quad x^2 - 6x - 13 = 0$

$\therefore \quad x = \dfrac{6 \pm \sqrt{36 - 4(1)(-13)}}{2}$

$= \dfrac{6 \pm \sqrt{88}}{2}$

$= 3 \pm \sqrt{22}$

But $x > 0$, so $x = 3 + \sqrt{22}$

b $5^2 = x^2 + 3^2 - 2 \times x \times 3 \times \cos 120°$

$\therefore \quad 25 = x^2 + 9 - 6x \times \left(-\frac{1}{2}\right)$

$\therefore \quad x^2 + 3x - 16 = 0$

$\therefore \quad x = \dfrac{-3 \pm \sqrt{9 - 4(1)(-16)}}{2}$

$= \dfrac{-3 \pm \sqrt{73}}{2}$

But $x > 0$, so $x = \dfrac{-3 + \sqrt{73}}{2}$

c $5^2 = (2x)^2 + x^2 - 2 \times (2x) \times x \times \cos 60°$

$\therefore \quad 25 = 4x^2 + x^2 - 4x^2\left(\frac{1}{2}\right)$

$\therefore \quad 3x^2 = 25$

$\therefore \quad x^2 = \frac{25}{3}$

$\therefore \quad x = \pm\frac{5}{\sqrt{3}}$

But $x > 0$, so $x = \frac{5}{\sqrt{3}}$

7 **a** $\text{Area} = 11.6\ \text{m}^2$

$\therefore\ 11.6 = \frac{1}{2} \times 6 \times 4 \times \sin\theta$

$\therefore\ \sin\theta = \frac{29}{30}$

$\therefore\ \theta = \sin^{-1}\left(\frac{29}{30}\right)$

$\therefore\ \theta \approx 75.2^\circ$

b Let the third side have length x m.
By the cosine rule,

$x^2 = 6^2 + 4^2 - 2 \times 6 \times 4 \times \cos 75.2^\circ$

$\therefore\ x = \sqrt{6^2 + 4^2 - 2 \times 6 \times 4 \times \cos 75.2^\circ}$

$\therefore\ x \approx 6.30$

The third side has length 6.30 m.

8 **a** $11^2 = x^2 + 8^2 - 2 \times x \times 8 \times \cos 70^\circ$

$\therefore\ 121 = x^2 + 64 - 16x \cos 70^\circ$

$\therefore\ x^2 - (16\cos 70^\circ)x - 57 = 0$

Using the quadratic formula or technology,
$x \approx -5.29$ or 10.8.
But $x > 0$, so $x \approx 10.8$.

b $13^2 = x^2 + 5^2 - 2 \times x \times 5 \times \cos 130^\circ$

$\therefore\ 169 = x^2 + 25 - 10x \cos 130^\circ$

$\therefore\ x^2 - (10\cos 130^\circ)x - 144 = 0$

Using the quadratic formula or technology,
$x \approx -15.6$ or 9.21.
But $x > 0$, so $x \approx 9.21$.

9 $5^2 = x^2 + 6^2 - 2 \times x \times 6 \times \cos 40^\circ$

$\therefore\ 25 = x^2 + 36 - 12x \cos 40^\circ$

$\therefore\ x^2 - (12\cos 40^\circ)x + 11 = 0$

Using the quadratic formula or technology, $x \approx 1.41$ or 7.78.

10 **a**

$$(3x+1)^2 = (x+2)^2 + (x+3)^2 - 2(x+2)(x+3)\cos\theta$$

$$\therefore\ 9x^2 + 6x + 1 = x^2 + 4x + 4 + x^2 + 6x + 9 - 2(x^2 + 5x + 6)(-\tfrac{1}{5})$$

$$\therefore\ 9x^2 + 6x + 1 = 2x^2 + 10x + 13 + \tfrac{2}{5}x^2 + 2x + \tfrac{12}{5}$$

$$\therefore\ \tfrac{33}{5}x^2 - 6x - \tfrac{72}{5} = 0$$

$$\therefore\ 33x^2 - 30x - 72 = 0$$

$$\therefore\ 3(11x + 12)(x - 2) = 0$$

$$\therefore\ x = -\tfrac{12}{11} \text{ or } 2$$

But $3x + 1 > 0$, so $x = 2$

b $\cos^2\theta + \sin^2\theta = 1$

$\therefore\ \frac{1}{25} + \sin^2\theta = 1$

$\therefore\ \sin^2\theta = \frac{24}{25}$

$\therefore\ \sin\theta = \pm\frac{\sqrt{24}}{5}$

But $0^\circ < \theta < 180^\circ$, so $\sin\theta > 0$

$\therefore\ \sin\theta = \frac{\sqrt{24}}{5}$

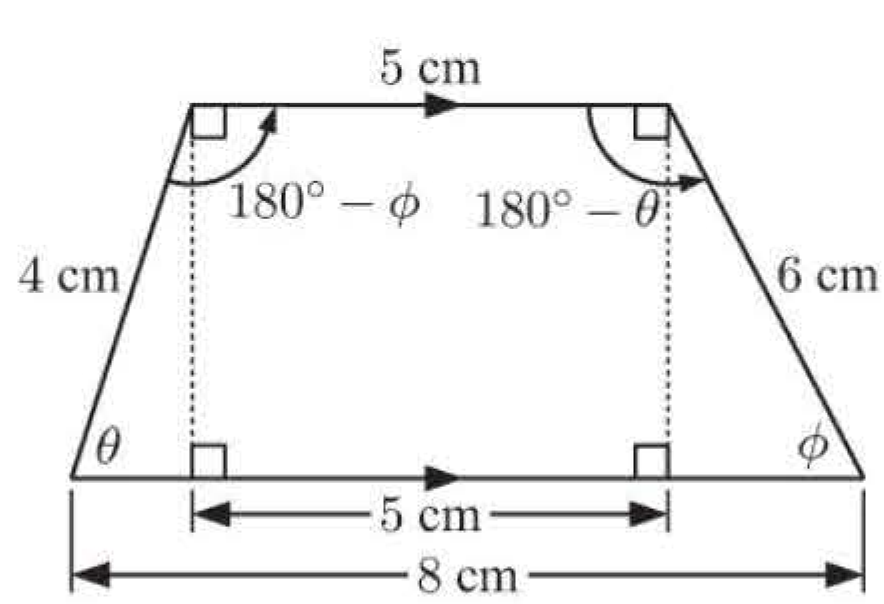
$\therefore\ \text{area} = \frac{1}{2} \times (x+2) \times (x+3) \times \sin\theta$

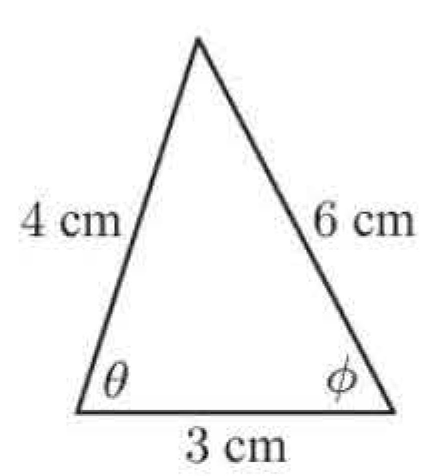
$= \frac{1}{2} \times 4 \times 5 \times \frac{\sqrt{24}}{5}$

$= 2\sqrt{24}$

$= 4\sqrt{6}\ \text{cm}^2$

11





$\cos\theta = \dfrac{4^2 + 3^2 - 6^2}{2 \times 3 \times 4}$

$\therefore\ \theta = \cos^{-1}\left(\dfrac{16 + 9 - 36}{24}\right)$

$= \cos^{-1}(-\frac{11}{24})$

$\approx 117.3^\circ$

$\cos\phi = \dfrac{3^2 + 6^2 - 4^2}{2 \times 3 \times 6}$

$\therefore\ \phi = \cos^{-1}\left(\dfrac{9 + 36 - 16}{36}\right)$

$= \cos^{-1}(\frac{29}{36})$

$\approx 36.3^\circ$

$\therefore\ 180^\circ - \theta \approx 180^\circ - 117.3^\circ \approx 62.7^\circ$ and $180^\circ - \phi \approx 180^\circ - 36.3^\circ \approx 143.7^\circ$

So, the angles are 36.3°, 62.7°, 117.3°, and 143.7°.

EXERCISE 11C.1

1 **a** By the sine rule,

$$\frac{x}{\sin 48°} = \frac{23}{\sin 37°}$$

$$\therefore \ x = \frac{23 \times \sin 48°}{\sin 37°}$$

$$\therefore \ x \approx 28.4$$

b By the sine rule,

$$\frac{x}{\sin 115°} = \frac{11}{\sin 48°}$$

$$\therefore \ x = \frac{11 \times \sin 115°}{\sin 48°}$$

$$\therefore \ x \approx 13.4$$

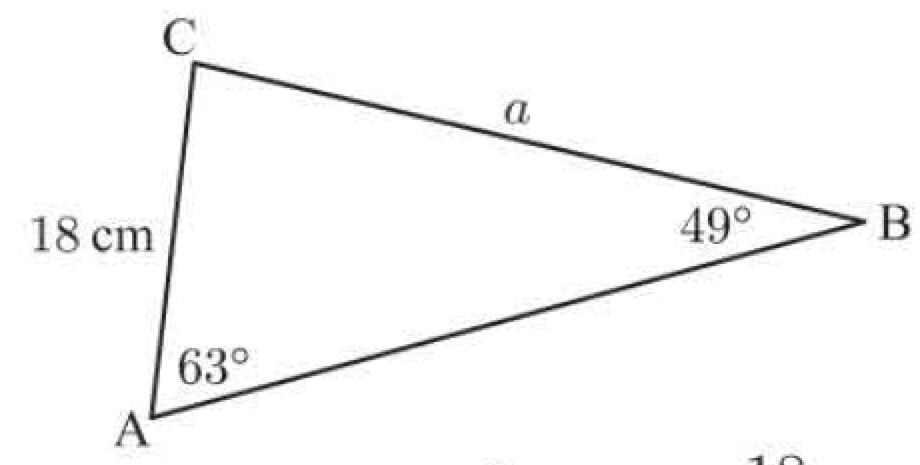
c By the sine rule,

$$\frac{x}{\sin 51°} = \frac{4.8}{\sin 80°}$$

$$\therefore \ x = \frac{4.8 \times \sin 51°}{\sin 80°}$$

$$\therefore \ x \approx 3.79$$

2 **a**

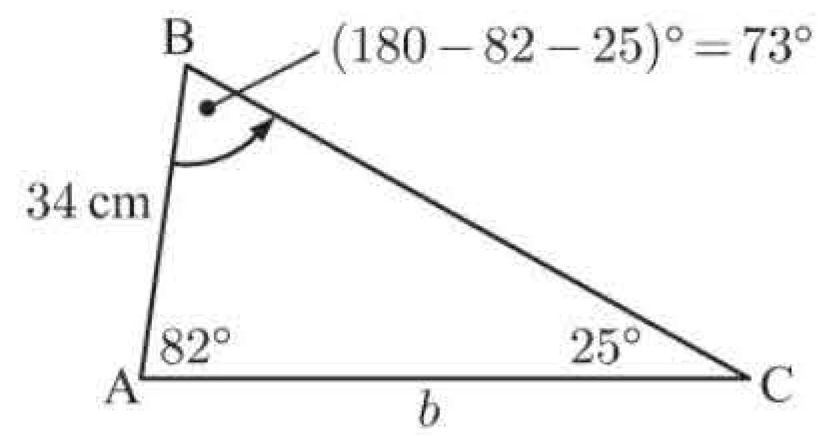


By the sine rule, $\dfrac{a}{\sin 63°} = \dfrac{18}{\sin 49°}$

$$\therefore \ a = \frac{18 \times \sin 63°}{\sin 49°}$$

$$\therefore \ a \approx 21.3 \text{ cm}$$

b

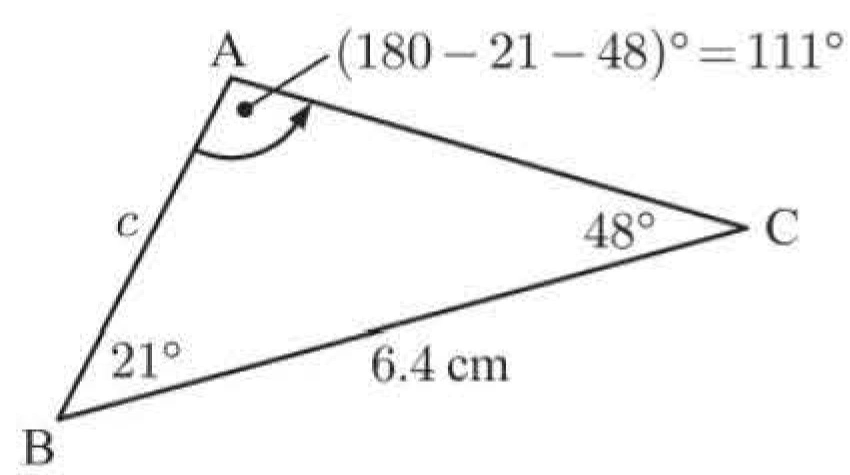


By the sine rule, $\dfrac{b}{\sin 73°} = \dfrac{34}{\sin 25°}$

$$\therefore \ b = \frac{34 \times \sin 73°}{\sin 25°}$$

$$\therefore \ b \approx 76.9 \text{ cm}$$

c



By the sine rule, $\dfrac{c}{\sin 48°} = \dfrac{6.4}{\sin 111°}$

$$\therefore \ c = \frac{6.4 \times \sin 48°}{\sin 111°}$$

$$\therefore \ c \approx 5.09 \text{ cm}$$

EXERCISE 11C.2

1 By the sine rule, $\dfrac{\sin C}{11} = \dfrac{\sin 40°}{8}$

$$\therefore \ \sin C = \frac{11 \times \sin 40°}{8}$$

$$\therefore \ C = \sin^{-1}\left(\frac{11 \times \sin 40°}{8}\right) \text{ or its supplement}$$

$$\therefore \ C \approx 62.1° \text{ or } (180 - 62.1)°$$

$$\therefore \ C \approx 62.1° \text{ or } 117.9°$$

2 **a**

$$\frac{\sin \widehat{BAC}}{a} = \frac{\sin \widehat{ABC}}{b}$$

$$\therefore \ \sin \widehat{BAC} = \frac{14.6 \times \sin 65°}{17.4}$$

$$\therefore \ \widehat{BAC} = \sin^{-1}\left(\frac{14.6 \times \sin 65°}{17.4}\right)$$

or its supplement

$$\therefore \ \widehat{BAC} \approx 49.5° \text{ or } 180° - 49.5°$$

$$\therefore \ \widehat{BAC} \approx 49.5° \text{ or } 130.5°$$

Check: $\widehat{BAC} = 130.5°$ is impossible as $\widehat{BAC} + \widehat{ABC} = 130.5° + 65°$ is already over $180°$. $\therefore \ \widehat{BAC} \approx 49.5°$

b

$$\frac{\sin \widehat{ABC}}{43.8} = \frac{\sin 43°}{31.4}$$

$$\therefore \ \sin \widehat{ABC} = \frac{43.8 \times \sin 43°}{31.4}$$

$$\therefore \ \widehat{ABC} = \sin^{-1}\left(\frac{43.8 \times \sin 43°}{31.4}\right)$$

or its supplement

$$\therefore \ \widehat{ABC} \approx 72.0° \text{ or } 108°$$

both of which are possible as $108 + 43 = 151$ which is < 180.

c $\dfrac{\sin A\widehat{C}B}{4.8} = \dfrac{\sin 71^\circ}{6.5}$

$\therefore \sin A\widehat{C}B = \dfrac{4.8 \times \sin 71^\circ}{6.5}$

$\therefore A\widehat{C}B = \sin^{-1}\left(\dfrac{4.8 \times \sin 71^\circ}{6.5}\right)$ or its supplement

$\therefore A\widehat{C}B \approx 44.3^\circ$ or 135.7°

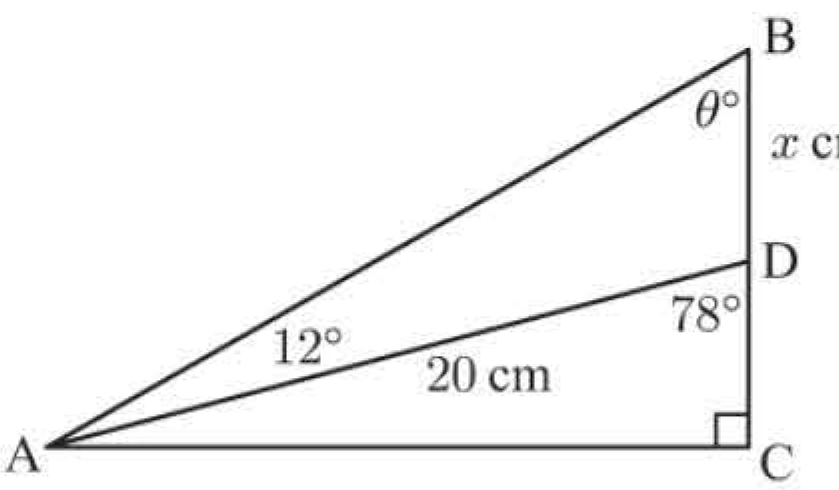
But $135.7 + 71 > 180$, so this case is impossible. $\therefore A\widehat{C}B \approx 44.3^\circ$

3 The third angle is $180^\circ - 85^\circ - 68^\circ = 27^\circ$

Now $\dfrac{\sin 85^\circ}{11.4} \approx 0.087\,38$ and $\dfrac{\sin 27^\circ}{9.8} \approx 0.046\,32$

This is not possible since $\dfrac{\sin 85^\circ}{11.4} \neq \dfrac{\sin 27^\circ}{9.8}$ violates the sine rule.

4



In $\triangle ABD$,

$\theta = 78 - 12$

$\therefore A\widehat{B}C = 66^\circ$

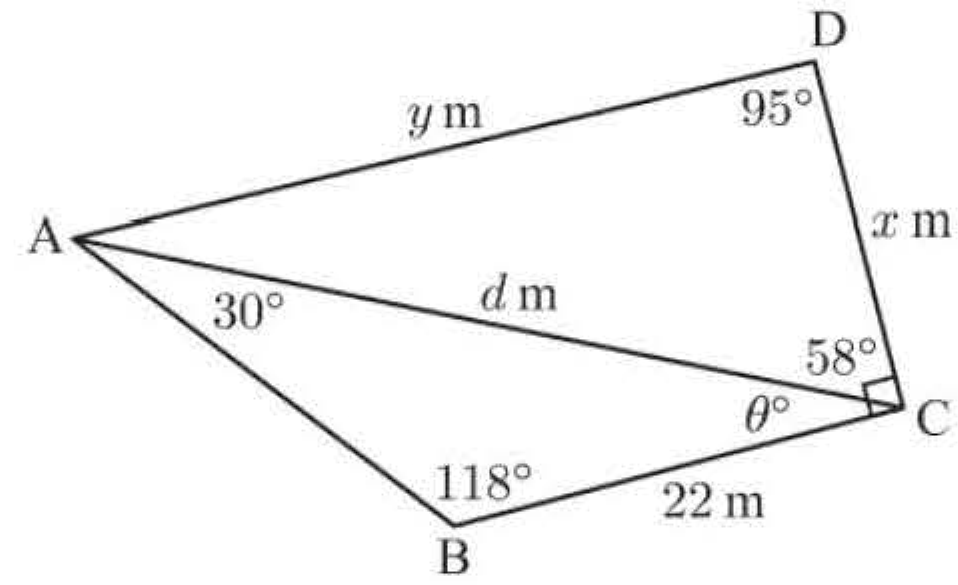
Now $\dfrac{x}{\sin 12^\circ} = \dfrac{20}{\sin 66^\circ}$

$\therefore x = \dfrac{20 \times \sin 12^\circ}{\sin 66^\circ}$

$\therefore x \approx 4.55$

$\therefore BD \approx 4.55$ cm

5 First we find the length of the diagonal, d m.



$\dfrac{d}{\sin 118^\circ} = \dfrac{22}{\sin 30^\circ}$

$\therefore d = \dfrac{22 \times \sin 118^\circ}{\sin 30^\circ}$

$\therefore d \approx 38.85$

Now $\theta = 180 - 30 - 118 = 32$

$\therefore A\widehat{C}D = 90^\circ - 32^\circ = 58^\circ$

Using the sine rule, $\dfrac{y}{\sin 58^\circ} = \dfrac{38.85}{\sin 95^\circ}$ and $\dfrac{x}{\sin(180 - 95 - 58)^\circ} \approx \dfrac{38.85}{\sin 95^\circ}$

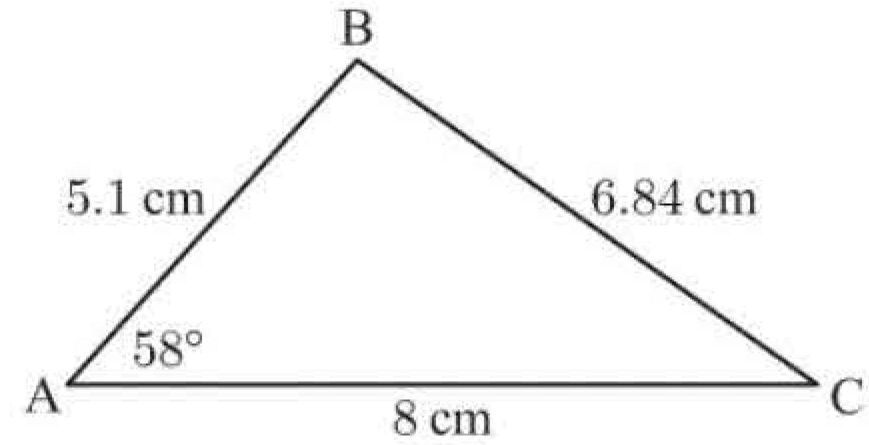
$\therefore y \approx \dfrac{38.85 \times \sin 58^\circ}{\sin 95^\circ}$

$\therefore y \approx 33.1$

$\therefore x \approx \dfrac{38.85 \times \sin 27^\circ}{\sin 95^\circ}$

$\therefore x \approx 17.7$

6



a $\dfrac{\sin \widehat{B}}{8} = \dfrac{\sin 58^\circ}{6.84}$

$\therefore \sin \widehat{B} = \dfrac{8 \sin 58^\circ}{6.84}$

$\therefore \widehat{B} = \sin^{-1}\left(\dfrac{8 \sin 58^\circ}{6.84}\right)$ or its supplement

$\therefore \widehat{B} \approx 83^\circ$ or $(180 - 83)^\circ$

$\therefore \widehat{B} \approx 83^\circ$ or 97°

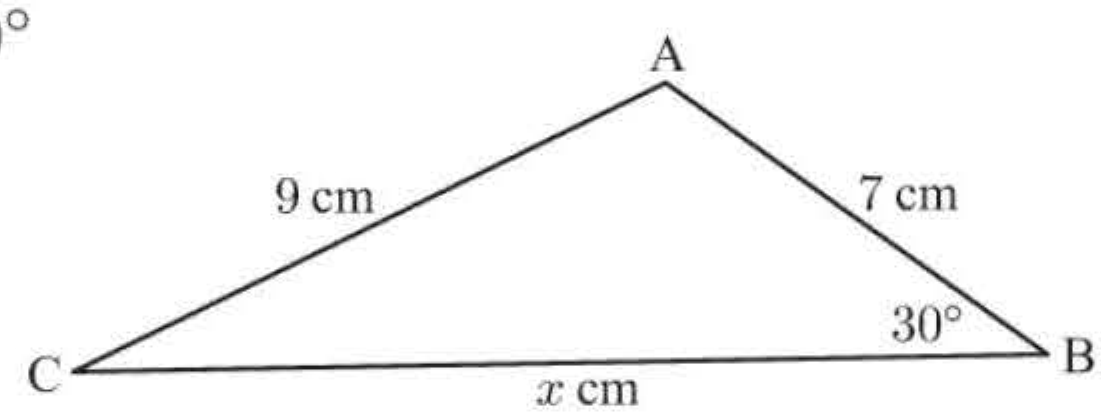
b $\cos \widehat{B} = \dfrac{5.1^2 + 6.84^2 - 8^2}{2 \times 5.1 \times 6.84}$

$\therefore \widehat{B} = \cos^{-1}\left(\dfrac{5.1^2 + 6.84^2 - 8^2}{2 \times 5.1 \times 6.84}\right)$

$\therefore \widehat{B} \approx 83^\circ$

c When faced with using either the sine rule or the cosine rule, it is better to use the cosine rule as it avoids the ambiguous case.

7



$$9^2 = x^2 + 7^2 - 2 \times x \times 7 \times \cos 30^\circ$$
$$\therefore\ 81 = x^2 + 49 - 14x(\tfrac{\sqrt{3}}{2})$$
$$\therefore\ x^2 - \tfrac{14\sqrt{3}}{2}x - 32 = 0$$

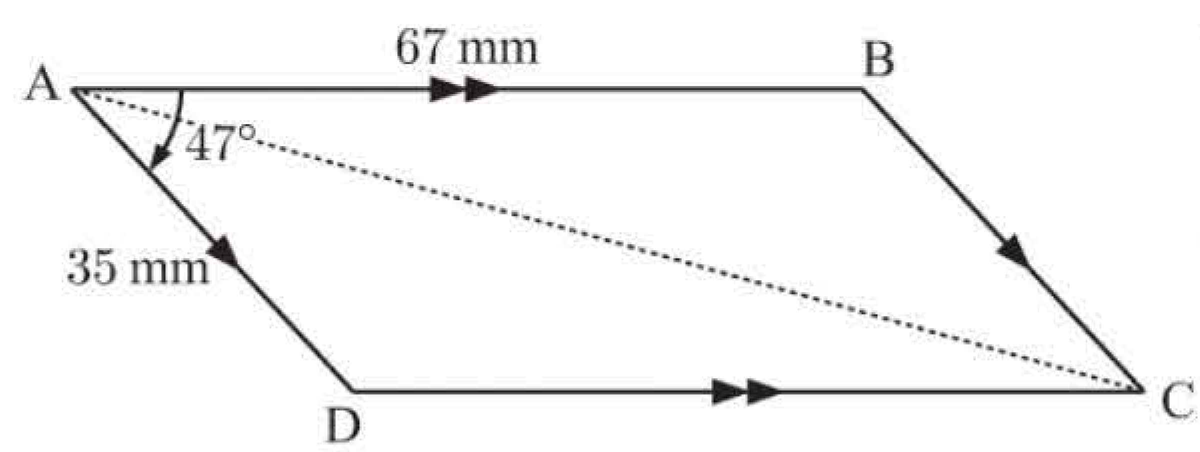
Using the quadratic formula or technology,

$$x \approx -2.23 \text{ or } 14.35$$

but $x > 0$, so $x \approx 14.35$

$$\therefore \text{ area of triangle} \approx \tfrac{1}{2} \times 7 \times 14.35 \times \sin 30^\circ$$
$$\approx 25.1 \text{ cm}^2$$

8



$$\begin{aligned} \text{A}\widehat{\text{B}}\text{C} &= 180^\circ - 47^\circ \\ &= 133^\circ \end{aligned}$$

$$\cos 133^\circ = \frac{67^2 + 35^2 - \text{AC}^2}{2 \times 67 \times 35} \quad \{\text{cosine rule}\}$$
$$\text{AC}^2 = 5714 - 4690 \cos 133^\circ$$
$$\text{AC} = 94.41 \text{ mm} \quad \{\text{as AC} > 0\}$$

$$\frac{\sin 133^\circ}{94.41} = \frac{\sin \text{B}\widehat{\text{A}}\text{C}}{35}$$
$$\text{B}\widehat{\text{A}}\text{C} = \sin^{-1}\left(\frac{35 \sin 133^\circ}{94.41}\right)$$
$$\approx 15.7^\circ$$

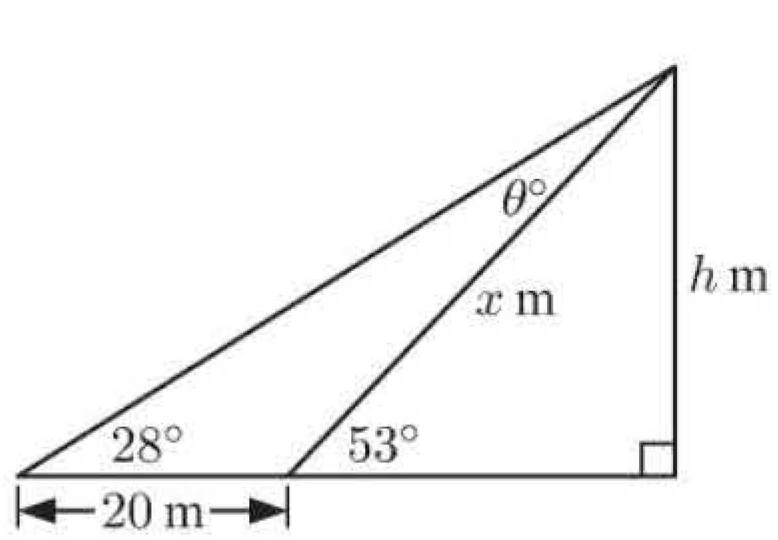
9

$$\frac{2x-5}{\sin 45^\circ} = \frac{x+3}{\sin 30^\circ}$$
$$\therefore\ (2x-5)\sin 30^\circ = (x+3)\sin 45^\circ$$
$$\therefore\ \frac{2x-5}{2} = \frac{x+3}{\sqrt{2}}$$
$$\therefore\ 2\sqrt{2}x - 5\sqrt{2} = 2x + 6$$
$$\therefore\ -6 - 5\sqrt{2} = x(2 - 2\sqrt{2})$$

$$\begin{aligned} \therefore\ x &= \left(\frac{-6-5\sqrt{2}}{2-2\sqrt{2}}\right)\left(\frac{2+2\sqrt{2}}{2+2\sqrt{2}}\right) \\ &= \frac{-12 - 12\sqrt{2} - 10\sqrt{2} - 10(2)}{4 - 4(2)} \\ &= \frac{-32 - 22\sqrt{2}}{-4} \\ &= 8 + \tfrac{11}{2}\sqrt{2} \end{aligned}$$

EXERCISE 11D

1



$$\theta^\circ + 28^\circ = 53^\circ$$
{exterior angle of a $\triangle$ theorem}
$$\therefore\ \theta = 25$$

By the sine rule,

$$\frac{x}{\sin 28^\circ} = \frac{20}{\sin 25^\circ}$$
$$\therefore\ x \approx \frac{20 \times \sin 28^\circ}{\sin 25^\circ}$$
$$\therefore\ x \approx 22.22$$

and $\sin 53^\circ = \dfrac{h}{x}$

$$\begin{aligned} \therefore\ h &= x \sin 53^\circ \\ &\approx 22.22 \times \sin 53^\circ \\ &\approx 17.7 \text{ m} \end{aligned}$$

$\therefore$ the pole is 17.7 m high.

2 $\text{PR}^2 = 63^2 + 175^2 - 2 \times 63 \times 175 \times \cos 112^\circ$

$\therefore\ \text{PR} = \sqrt{63^2 + 175^2 - 2 \times 63 \times 175 \times \cos 112^\circ}$

$\therefore\ \text{PR} \approx 207$ m

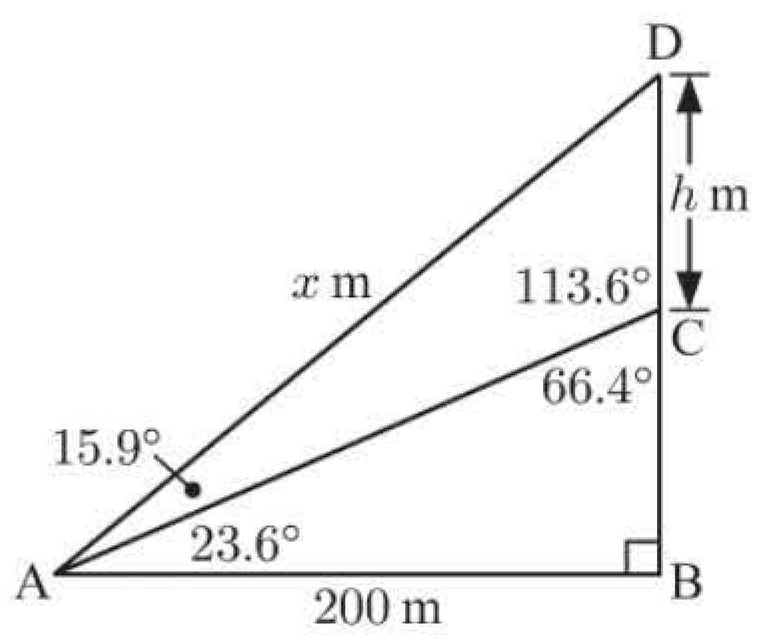
3 $\cos T = \dfrac{220^2 + 340^2 - 165^2}{2 \times 220 \times 340}$

$\therefore\ T = \cos^{-1}\left(\dfrac{136\,775}{149\,600}\right)$

$\therefore\ T \approx 23.9$

$\therefore$ the tee shot was 23.9° off line.

4



In $\triangle$ABD,

$$\cos(23.6 + 15.9)^\circ = \frac{200}{x}$$

$$\therefore\ x = \frac{200}{\cos 39.5^\circ}$$

$$\therefore\ x \approx 259.2$$

In $\triangle$ACD,

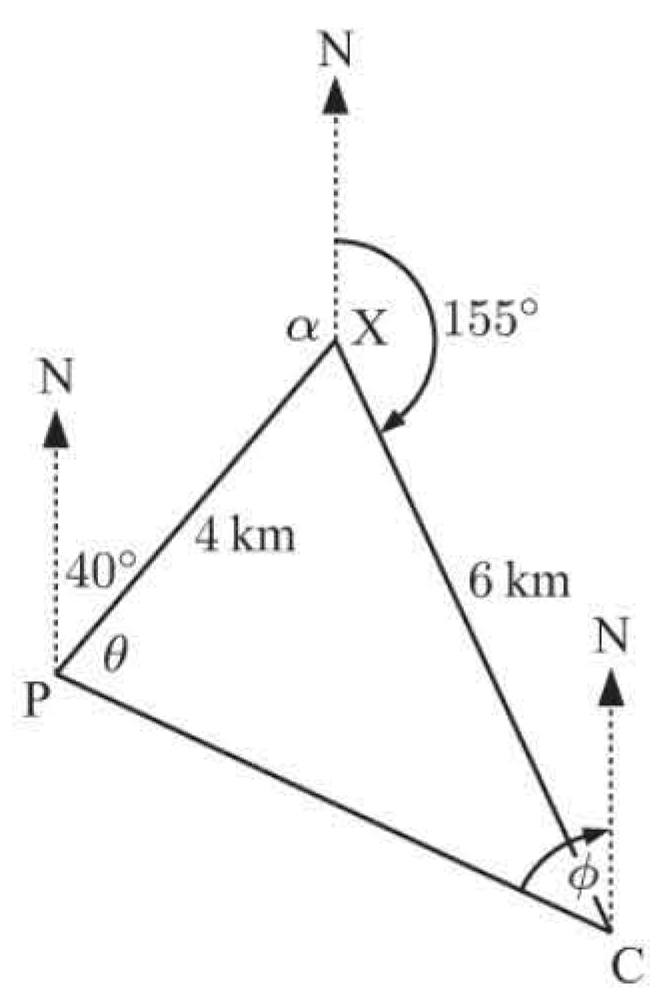
$$\frac{h}{\sin 15.9^\circ} = \frac{x}{\sin 113.6^\circ}$$

$$\therefore\ h \approx \frac{259.2 \times \sin 15.9^\circ}{\sin 113.6^\circ}$$

$$\therefore\ h \approx 77.5$$

$\therefore$ the tower is 77.5 m high.

5



a

$$\alpha = 140^\circ \quad \{\text{co-interior angles}\}$$

$$\therefore\ \text{P}\widehat{\text{X}}\text{C} = 360^\circ - 140^\circ - 155^\circ \quad \{\text{angles at a point}\}$$

$$= 65^\circ$$

So, $\text{PC}^2 = 4^2 + 6^2 - 2 \times 4 \times 6 \cos 65^\circ$

$$\therefore\ \text{PC} = \sqrt{16 + 36 - 48\cos 65^\circ}$$

$$\approx 5.6315 \text{ km}$$

$\therefore$ Esko hikes 5.63 km.

b

$$\cos\theta \approx \frac{4^2 + 5.6315^2 - 6^2}{2 \times 4 \times 5.6315}$$

$$\therefore\ \theta \approx 74.9^\circ$$

$\therefore$ bearing $= 40^\circ + \theta$

$$\approx 114.9^\circ$$

$\therefore$ Esko hikes on a bearing of 115°.

c **i**

$$\text{speed} = \frac{\text{distance}}{\text{time}} \quad \Rightarrow \quad \text{time} = \frac{\text{distance}}{\text{speed}}$$

$$\therefore\ \text{time}_{\text{Ritva}} = \frac{4+6}{10} = 1 \text{ hour} \quad \text{and} \quad \text{time}_{\text{Esko}} \approx \frac{5.6315}{6} \approx 0.9386 \text{ hours}$$

$$\approx 56.32 \text{ min}$$

So Esko arrives at the campsite first.

ii $60 - 56.32 = 3.68$

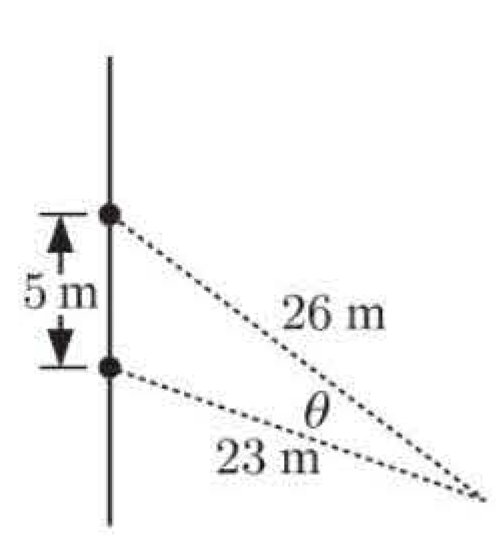
Esko needs to wait about 3.68 minutes before Ritva arrives.

d $\phi \approx 180^\circ - 114.9^\circ \approx 65.1^\circ \quad \{\text{co-interior angles}\}$

$\therefore\ 360^\circ - \phi \approx 295^\circ$

The return bearing is 295°.

6



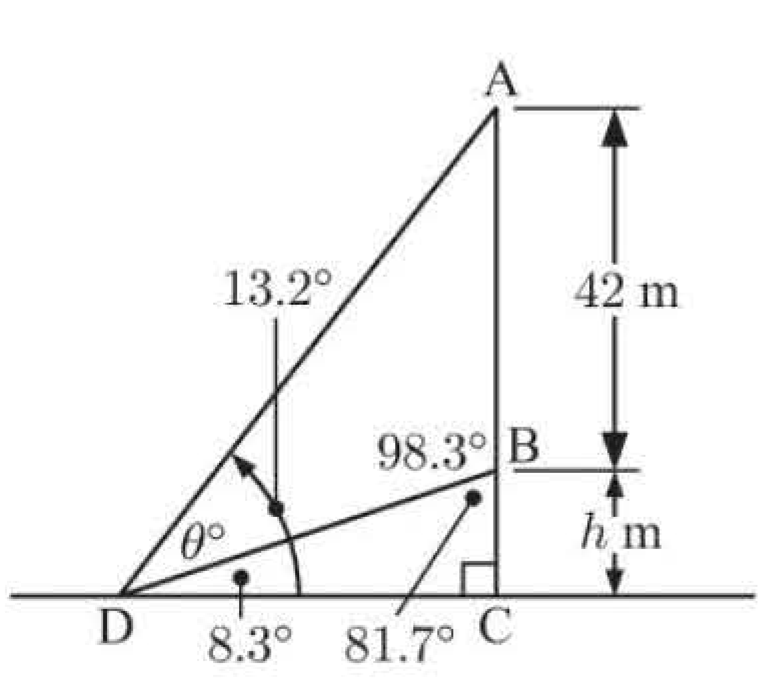
$$\cos\theta = \frac{23^2 + 26^2 - 5^2}{2 \times 23 \times 26}$$

$$\therefore\ \theta = \cos^{-1}\left(\tfrac{1180}{1196}\right)$$

$$\therefore\ \theta \approx 9.38^\circ$$

$\therefore$ the angle of view is 9.38°.

7



$\theta = 13.2^\circ - 8.3^\circ = 4.9^\circ$

In $\triangle$ABD,

$$\frac{\text{AD}}{\sin 98.3^\circ} = \frac{42}{\sin 4.9^\circ}$$

$$\therefore\ \text{AD} = \frac{42 \times \sin 98.3^\circ}{\sin 4.9^\circ}$$

$$\therefore\ \text{AD} \approx 486.56 \text{ m}$$

In $\triangle$ADC,

$$\sin 13.2^\circ = \frac{h + 42}{\text{AD}}$$

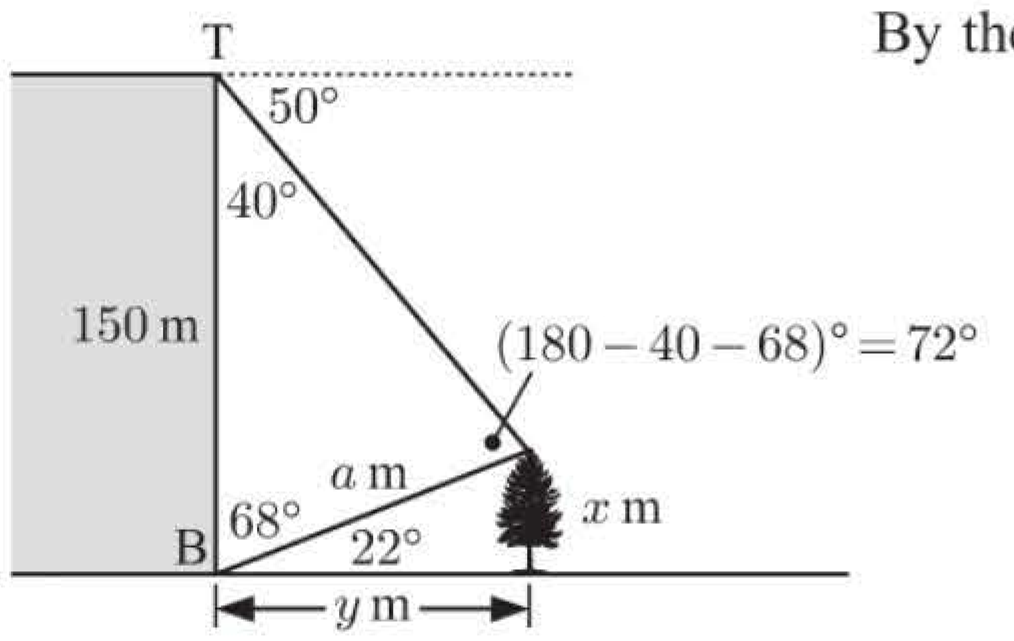
$$\therefore\ h + 42 \approx 486.56 \times \sin 13.2^\circ$$

$$\therefore\ h + 42 \approx 111.1$$

$$\therefore\ h \approx 69.1$$

$\therefore$ the hill is 69.1 m high.

8



By the sine rule, $\dfrac{a}{\sin 40°} = \dfrac{150}{\sin 72°}$

$\therefore \quad a = \dfrac{150 \times \sin 40°}{\sin 72°}$

$\therefore \quad a \approx 101.38$

a $\sin 22° \approx \dfrac{x}{101.38}$

$\therefore \quad x \approx 101.38 \times \sin 22°$

$\therefore \quad x \approx 38.0$

$\therefore$ the tree is 38.0 m high.

b $\cos 22° \approx \dfrac{y}{101.38}$

$\therefore \quad y \approx 101.38 \times \cos 22°$

$\therefore \quad y \approx 94.0$

$\therefore$ the tree is 94.0 m from the building.

9 Using Pythagoras' theorem

$\text{RQ} = \sqrt{4^2 + 7^2} = \sqrt{65}$ cm

$\text{PQ} = \sqrt{8^2 + 7^2} = \sqrt{113}$ cm

$\text{PR} = \sqrt{8^2 + 4^2} = \sqrt{80}$ cm

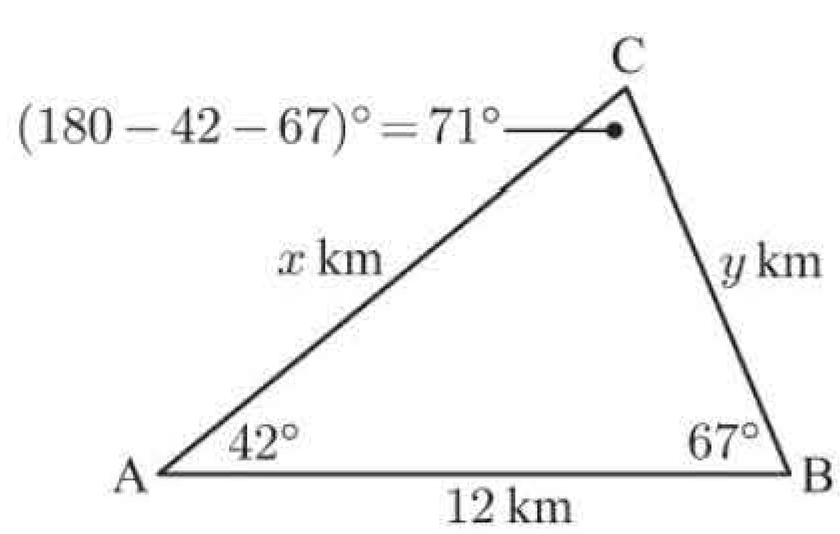
Now $\cos \text{P}\widehat{\text{Q}}\text{R} = \dfrac{(\sqrt{113})^2 + (\sqrt{65})^2 - (\sqrt{80})^2}{2 \times \sqrt{113} \times \sqrt{65}}$

$\therefore \quad \cos \text{P}\widehat{\text{Q}}\text{R} \approx \left(\frac{98}{171.4}\right)$

$\therefore \quad \text{P}\widehat{\text{Q}}\text{R} \approx \cos^{-1}\left(\frac{98}{171.4}\right)$

$\therefore \quad \text{P}\widehat{\text{Q}}\text{R} \approx 55.1$ So, $\text{P}\widehat{\text{Q}}\text{R}$ measures $55.1°$.

10



$\dfrac{x}{\sin 67°} = \dfrac{12}{\sin 71°} = \dfrac{y}{\sin 42°}$

$\therefore \quad x = \dfrac{12 \times \sin 67°}{\sin 71°}$ and $y = \dfrac{12 \times \sin 42°}{\sin 71°}$

$\therefore \quad x \approx 11.7$ $\therefore \quad y \approx 8.49$

So, C is 11.7 km from A and 8.49 km from B.

11 **a** $\text{QS} = \sqrt{8^2 + 12^2 - 2 \times 8 \times 12 \times \cos 70°}$

≈ 11.93

$\therefore$ area $\approx \frac{1}{2} \times 8 \times 12 \times \sin 70° + \frac{1}{2} \times 10 \times 11.93 \times \sin 30°$

$\approx 74.9 \text{ km}^2$

b 1 ha is 100 m $\times$ 100 m

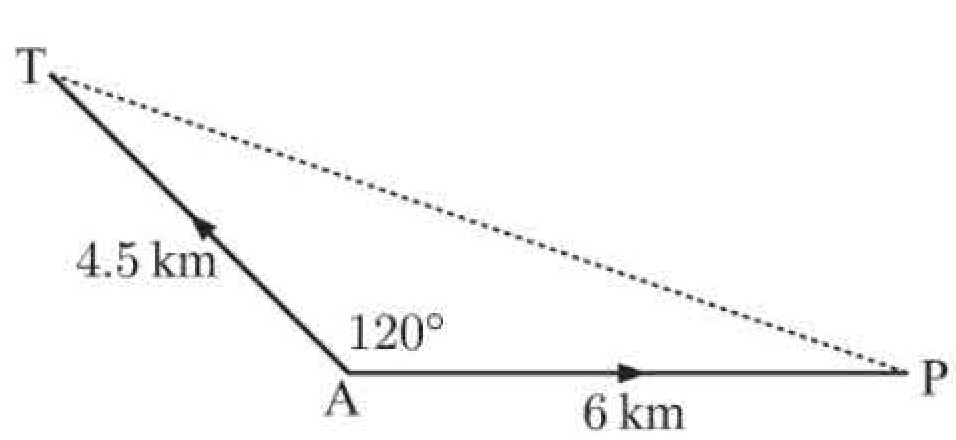
$= 0.1 \text{ km} \times 0.1 \text{ km}$

$= 0.01 \text{ km}^2$

$\therefore \quad 1 \text{ km}^2 = 100$ ha

$\therefore$ area ≈ 7490 ha

12



Distance = speed $\times$ time

So, after 45 min = 0.75 h, $\text{AT} = 6 \times 0.75 = 4.5$ km

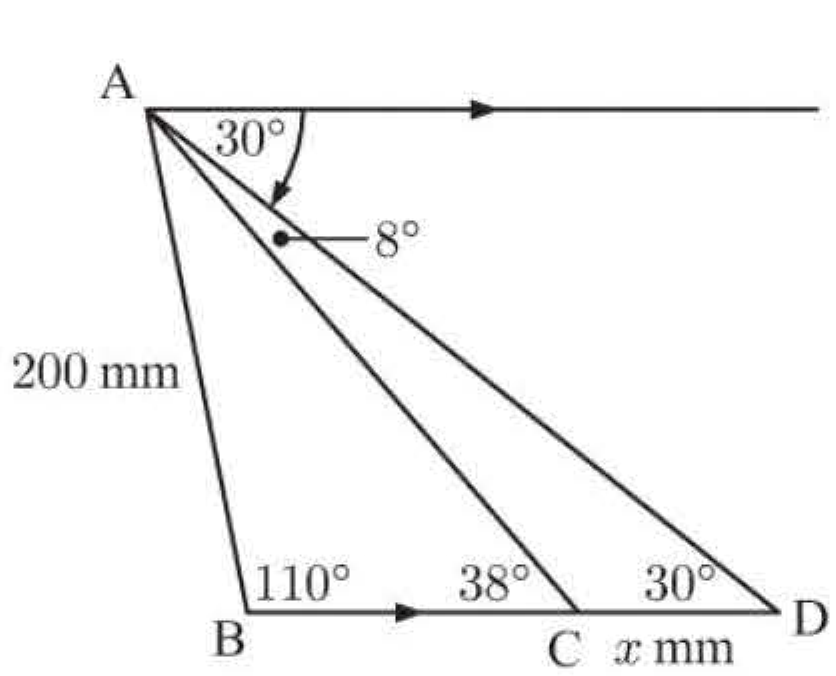
$\text{AP} = 8 \times 0.75 = 6$ km

Now $\text{PT} = \sqrt{4.5^2 + 6^2 - 2 \times 4.5 \times 6 \times \cos 120°}$

$\therefore \quad \text{PT} \approx 9.12$

So, they are 9.12 km apart.

13



In $\triangle$ABC, $\dfrac{\text{AC}}{\sin 110°} = \dfrac{200}{\sin 38°}$

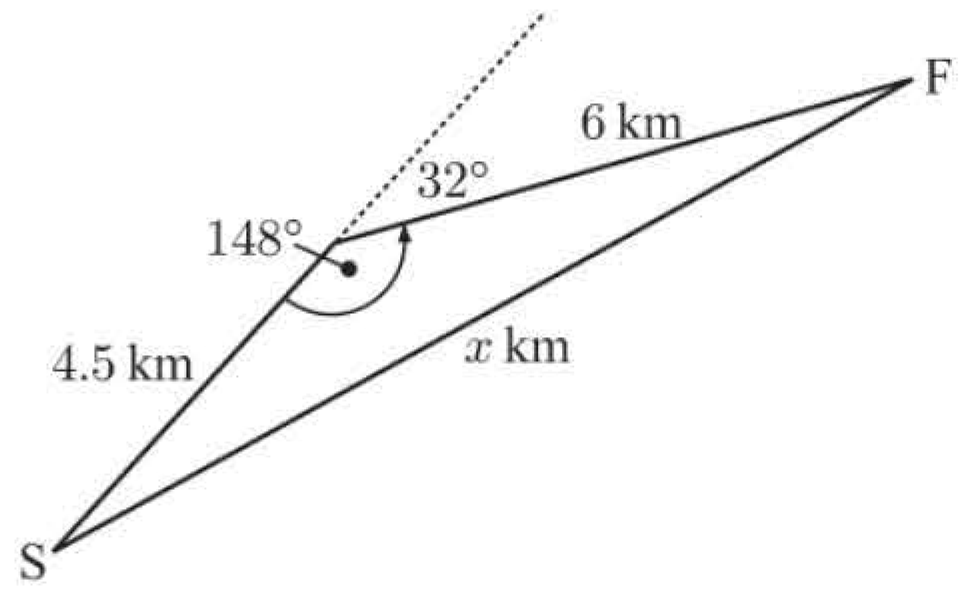
$\therefore \quad \text{AC} = \dfrac{200 \times \sin 110°}{\sin 38°} \approx 305.26$

and in $\triangle$ACD, $\dfrac{x}{\sin 8°} \approx \dfrac{305.26}{\sin 30°}$

$\therefore \quad x \approx \dfrac{305.26 \times \sin 8°}{\sin 30°} \approx 84.968$

$\therefore$ the metal strip is 85.0 mm wide.

14

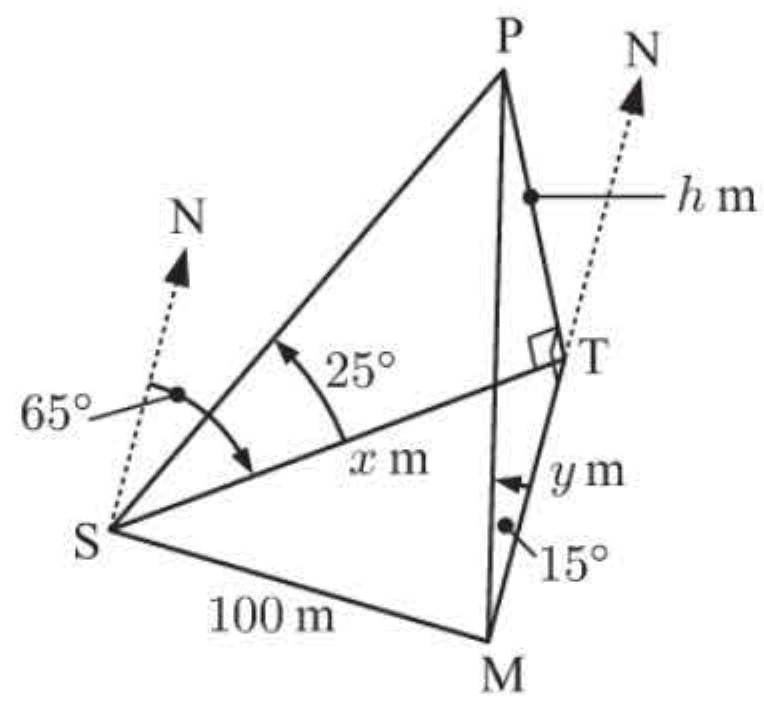


$$x = \sqrt{6^2 + (4.5)^2 - 2 \times 6 \times 4.5 \times \cos 148^\circ}$$

$\therefore \quad x \approx 10.1$

$\therefore$ the orienteer is 10.1 km from the start.

15



In $\triangle$PST, $\tan 25^\circ = \dfrac{h}{x}$

$\therefore \quad x = \dfrac{h}{\tan 25^\circ}$

$\approx 2.145h$

In $\triangle$PMT, $\tan 15^\circ = \dfrac{h}{y}$

$\therefore \quad y = \dfrac{h}{\tan 15^\circ}$

$\approx 3.732h$

But $\widehat{\text{STM}} = 65^\circ$ {equal alternate angles}

and $100^2 = x^2 + y^2 - 2xy\cos 65^\circ$

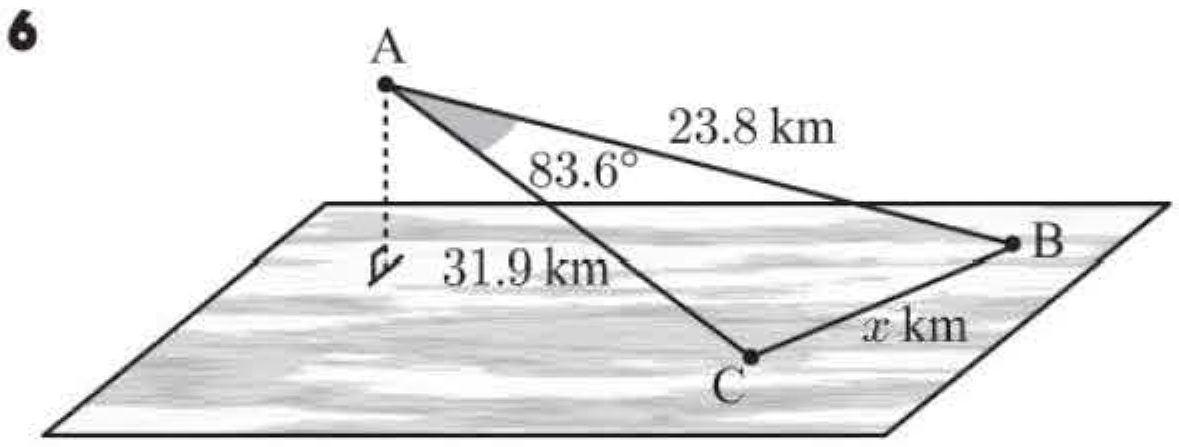
$\therefore \quad 10\,000 \approx (2.145h)^2 + (3.732h)^2 - 2 \times (2.145)(3.732)h^2 \cos 65^\circ$

$\therefore \quad 10\,000 \approx 11.762\,h^2$

$\therefore \quad h^2 \approx 850.17$

$\therefore \quad h \approx 29.2$ So, the tree is 29.2 m high.

16



By the cosine rule

$x^2 = 23.8^2 + 31.9^2 - 2 \times 23.8 \times 31.9 \times \cos 83.6^\circ$

$\therefore \quad x = \sqrt{23.8^2 + 31.9^2 - 2 \times 23.8 \times 31.9 \times \cos 83.6^\circ}$

$\therefore \quad x \approx 37.6$

$\therefore$ B and C are 37.6 km apart.

REVIEW SET 11A

1 area $= \frac{1}{2} \times 7 \times 8 \times \sin 30^\circ$

$= 28 \times \frac{1}{2}$

$= 14 \text{ km}^2$

2 If the unknown is an angle, use the cosine rule to avoid the ambiguous case.

3 **a** By the cosine rule, $7^2 = 8^2 + x^2 - 2 \times 8 \times x \times \cos 60^\circ$

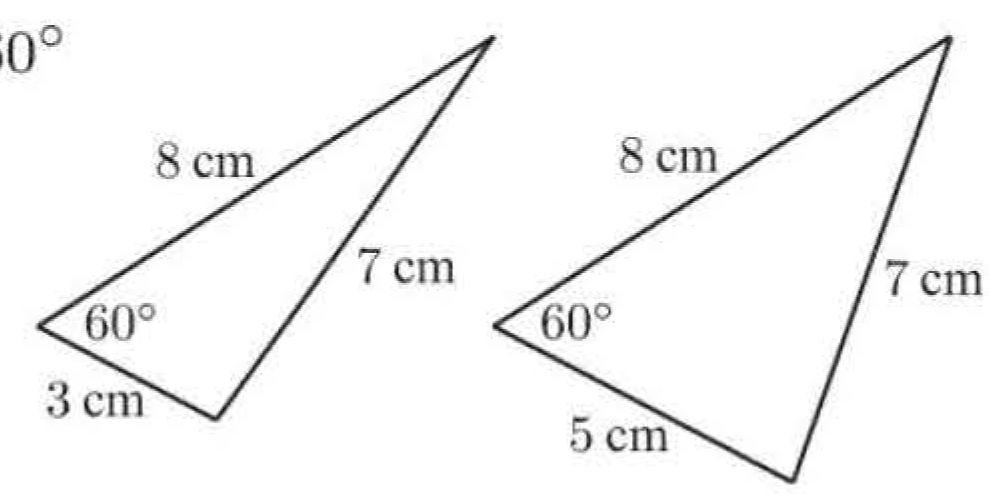
$\therefore \quad 49 = 64 + x^2 - 16x\left(\frac{1}{2}\right)$

$\therefore \quad 49 = 64 + x^2 - 8x$

$\therefore \quad x^2 - 8x + 15 = 0$

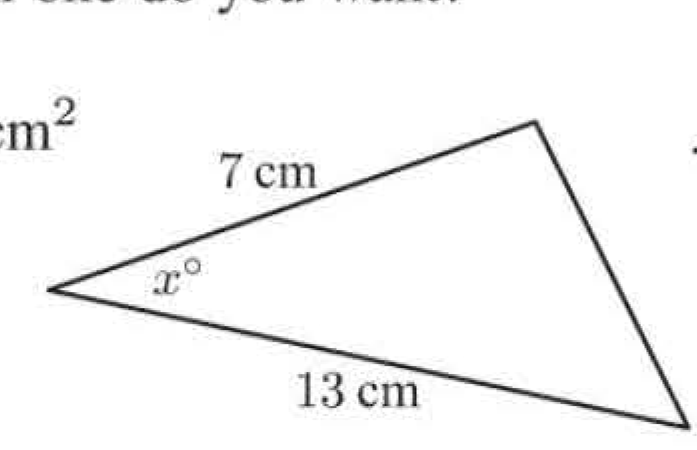
$\therefore \quad (x-3)(x-5) = 0$

$\therefore \quad x = 3$ or 5



b Kady's response should be "Please supply me with additional information as there are two possibilities. Which one do you want?"

4 area $= 42 \text{ cm}^2$

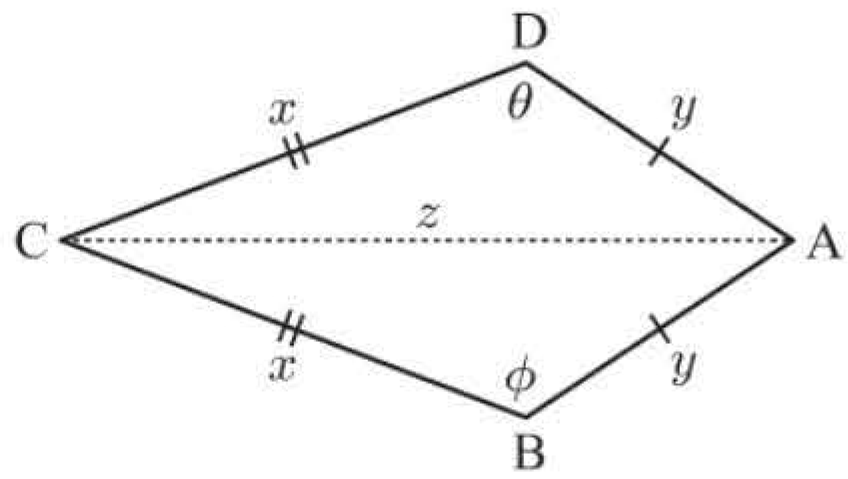


$\therefore \quad \frac{1}{2} \times 7 \times 13 \times \sin x^\circ = 42$

$\therefore \quad \sin x^\circ = \dfrac{42 \times 2}{7 \times 13}$

$= \frac{12}{13}$

5 **a**



Using $\triangle$ADC,

$z^2 = x^2 + y^2 - 2xy\cos\theta$ .... (1)

Using $\triangle$ABC,

$z^2 = x^2 + y^2 - 2xy\cos\phi$ .... (2)

Equating (1) and (2),

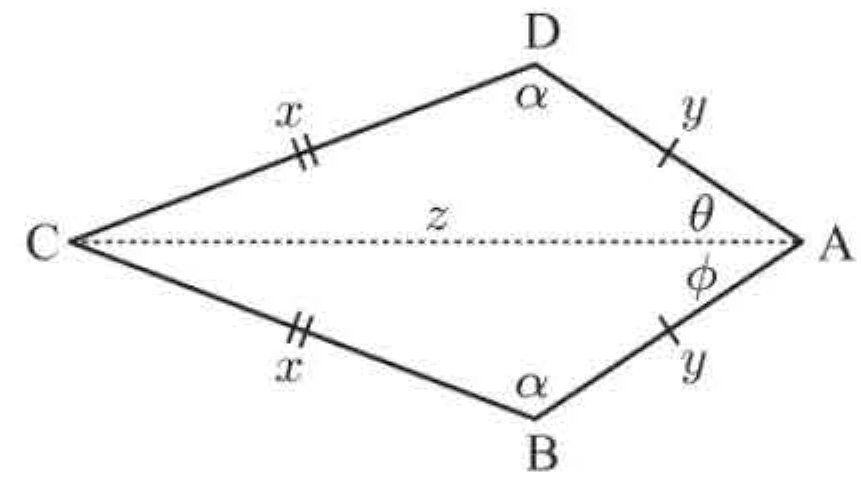
$\cos\theta = \cos\phi$

and since $0 < \theta,\ \phi < 180$,

$\theta = \phi$

$\therefore\ \widehat{ADC} = \widehat{ABC}$

b



Using $\triangle$DAC,

$\dfrac{\sin\theta}{x} = \dfrac{\sin\alpha}{z}$ .... (1)

Using $\triangle$BAC,

$\dfrac{\sin\phi}{x} = \dfrac{\sin\alpha}{z}$ .... (2)

$\{\widehat{ADC} = \widehat{ABC}$ from **a**$\}$

Equating (1) and (2),

$\sin\theta = \sin\phi$

$\therefore\ \theta = \phi$ or $\theta = 180 - \phi$

but $\widehat{DAB} = \theta + \phi < 180°$

$\therefore\ \theta = \phi$

$\therefore\ \widehat{DAC} = \widehat{BAC}$

6 Total distance travelled $= x + 10$ km

$\therefore$ AB $= (x + 10) - 4 = x + 6$ km

Now $(x+6)^2 = x^2 + 10^2 - 2 \times x \times 10 \times \cos 120°$

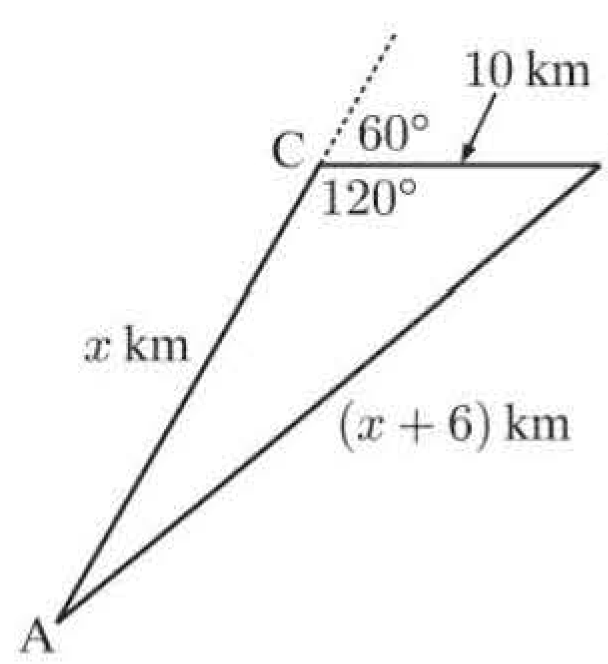
$\therefore\ x^2 + 12x + 36 = x^2 + 100 - 20x(-\frac{1}{2})$

$\therefore\ 12x + 36 = 100 + 10x$

$\therefore\ 2x = 64$

$\therefore\ x = 32$

$\therefore$ the boat travelled $x + 10 = 42$ km.



7 shaded area = area of sector $-$ area of $\triangle$

$= \frac{1}{2} \times \frac{13\pi}{18} \times 7^2 - \frac{1}{2} \times 7 \times 7 \times \sin\left(\frac{13\pi}{18}\right)$

$= \frac{49}{2}\left(\frac{13\pi}{18} - \sin(\frac{13\pi}{18})\right)$

8 **a** $d^2 = x^2 + 5^2 - 2 \times x \times 5 \times \cos 20°$

$\therefore\ d^2 = x^2 - (10\cos 20°)x + 25$

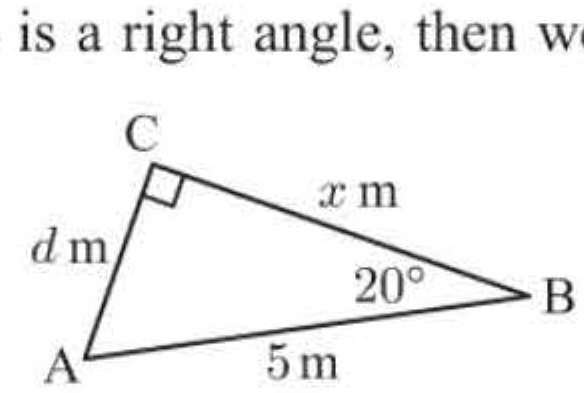
b d^2 is minimised when $x = \dfrac{-b}{2a}$

$\therefore\ x = \dfrac{10\cos 20°}{2}$

$\therefore\ x = 5\cos 20°$

$\therefore\ d$ is minimised when $x = 5\cos 20°$

c If $\widehat{BCA}$ is a right angle, then we have



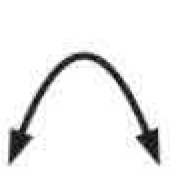
Now $\cos 20° = \dfrac{x}{5}$

$\therefore\ x = 5\cos 20°$

and from **b**, d is minimised when $x = 5\cos 20°$

$\therefore\ d$ is minimised when $\widehat{BCA}$ is a right angle.

9 **a** $y = -x^2 + 12x - 20$ has $a = -1 < 0$

$\therefore$ its shape is  , and y is maximised when $x = \dfrac{-b}{2a} = \dfrac{-12}{-2} = 6$.

When $x = 6$, $y = -6^2 + 12(6) - 20 = -36 + 72 - 20 = 16$

$\therefore$ the maximum value of $y = -x^2 + 12x - 20$ is 16, which occurs when $x = 6$.

b **i** The perimeter is 20

$\therefore\ x+y+8=20$

$\therefore\ y=12-x$

ii $y^2=x^2+8^2-2\times x\times 8\times\cos\theta$

$\therefore\ y^2=x^2+64-16x\cos\theta$

iii Since $y=12-x$, $(12-x)^2=x^2+64-16x\cos\theta$

$$\therefore\ 144-24x+\cancel{x^2}=\cancel{x^2}+64-16x\cos\theta$$

$$\therefore\ 16x\cos\theta=24x-80$$

$$\therefore\ \cos\theta=\frac{24x-80}{16x}=\frac{3x-10}{2x}$$

c Area $A=\frac{1}{2}\times x\times 8\times\sin\theta$

$$=4x\sin\theta$$

$$\therefore\ A^2=16x^2\sin^2\theta$$

$$=16x^2(1-\cos^2\theta)$$

$$=16x^2\left[1-\left(\frac{3x-10}{2x}\right)^2\right]$$

$$=16x^2\left[1-\frac{9x^2-60x+100}{4x^2}\right]$$

$$=16x^2-4(9x^2-60x+100)$$

$$=16x^2-36x^2+240x-400$$

$$=-20x^2+240x-400$$

$$=20(-x^2+12x-20)$$

d A is maximised when A^2 is maximised since $A>0$.

From **a**, $-x^2+12x-20$ has a maximum value of 16 when $x=6$.

When $x=6$, $A^2=20(16)$

$$=320$$

$$\therefore\ A=\sqrt{320}\quad\{A>0\}$$

$$=8\sqrt{5}$$

Also, when $x=6$, $y=12-6$

$$=6$$

$\therefore$ the maximum area of the triangle is $8\sqrt{5}$ units^2, and the triangle is isosceles when this occurs.

REVIEW SET 11B

1 **a** $\cos x^\circ=\dfrac{13^2+19^2-11^2}{2\times 13\times 19}$

$\therefore\ \cos x^\circ=\frac{409}{494}$

$\therefore\ x^\circ=\cos^{-1}\left(\frac{409}{494}\right)$

$\therefore\ x\approx 34.1$

b $x^2=15^2+17^2-2\times 15\times 17\times\cos 72^\circ$

$\therefore\ x=\sqrt{15^2+17^2-2\times 15\times 17\times\cos 72^\circ}$

$\therefore\ x\approx 18.9$

2 $\text{AC}^2=11^2+9.8^2-2\times 11\times 9.8\times\cos 74^\circ$

$\therefore\ \text{AC}=\sqrt{11^2+9.8^2-2\times 11\times 9.8\times\cos 74^\circ}$

$\therefore\ \text{AC}\approx 12.554$ cm

$\therefore\ \text{AC}\approx 12.6$ cm

Now $\dfrac{\sin C}{11}=\dfrac{\sin 74^\circ}{\text{AC}}$

$\therefore\ \sin C\approx\dfrac{11\times\sin 74^\circ}{12.554}$

$\therefore\ C\approx\sin^{-1}\left(\dfrac{11\times\sin 74^\circ}{12.554}\right)$ or its supplement

$\therefore\ C\approx 57.4^\circ$ or 122.6°

↑ impossible as $122.6+74>180$

$\therefore\ C$ measures 57.4°

$\therefore\ A$ measures $180^\circ-74^\circ-57.4^\circ=48.6^\circ$.

3 $\text{DB}^2=7^2+11^2-2\times 7\times 11\times\cos 110^\circ$

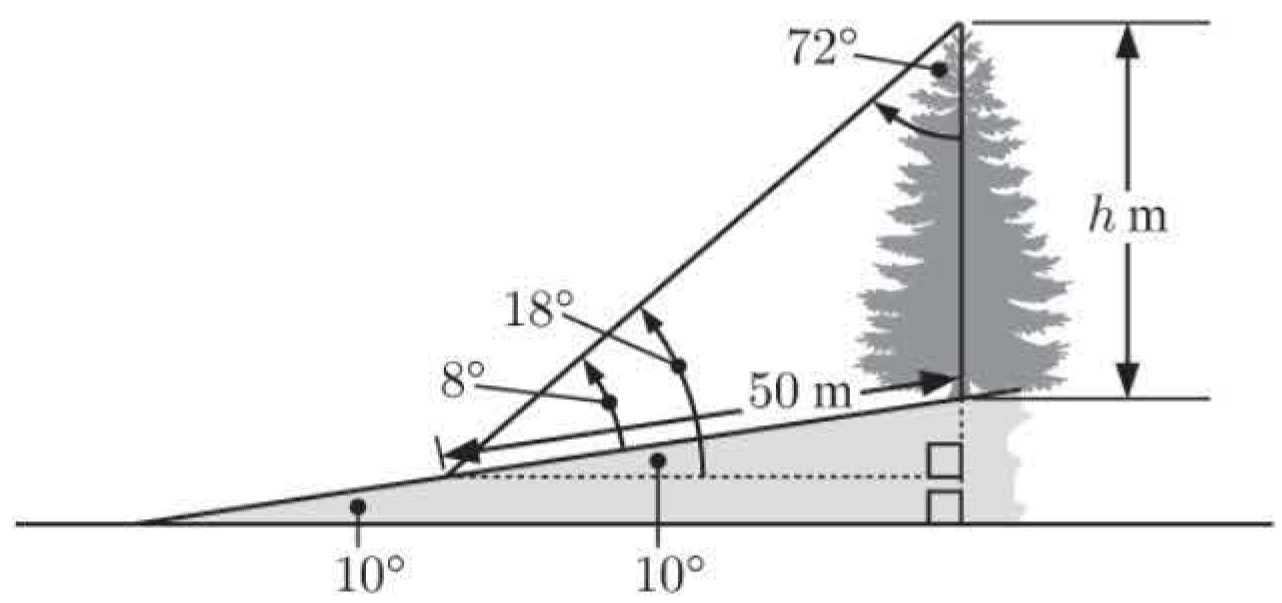
$\therefore\ \text{DB}=\sqrt{7^2+11^2-2\times 7\times 11\times\cos 110^\circ}\approx 14.922$ cm

$\therefore$ total area = area $\triangle$ABD + area $\triangle$BCD

$$\approx\tfrac{1}{2}\times 7\times 11\times\sin 110^\circ+\tfrac{1}{2}\times 16\times 14.922\times\sin 40^\circ$$

$$\approx 113\text{ cm}^2$$

4



$$\frac{h}{\sin 8°} = \frac{50}{\sin 72°}$$

$$\therefore \; h = \frac{50 \times \sin 8°}{\sin 72°}$$

$$\therefore \; h \approx 7.32$$

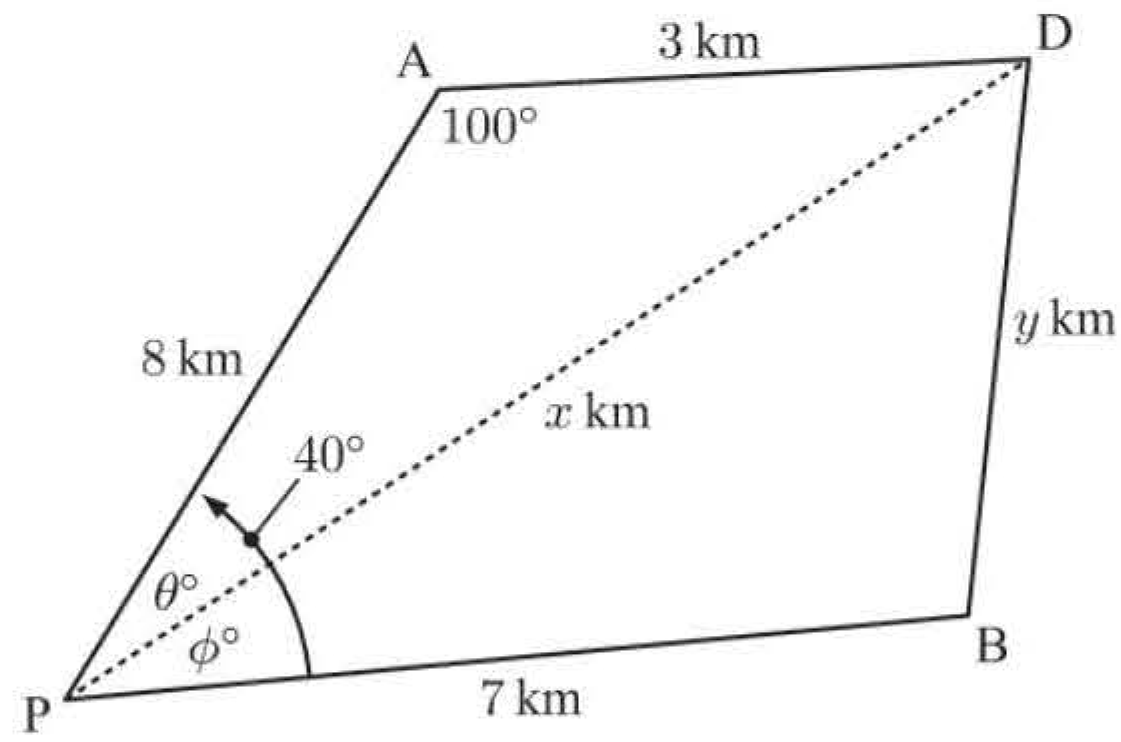
So, the tree is 7.32 m high.

5

$$x^2 = 8^2 + 3^2 - 2 \times 8 \times 3 \times \cos 100°$$

$$\therefore \; x = \sqrt{8^2 + 3^2 - 48 \cos 100°}$$

$$\therefore \; x \approx 9.0186$$



Now $\dfrac{\sin \theta°}{3} \approx \dfrac{\sin 100°}{9.0186}$

$$\therefore \; \sin \theta° \approx \frac{3 \times \sin 100°}{9.0186}$$

$$\therefore \; \theta \approx \sin^{-1}\left(\frac{3 \times \sin 100°}{9.0186}\right)$$

or its supplement

$$\therefore \; \theta \approx 19.1 \;\text{ or }\; \underbrace{160.9}_{\text{impossible}}$$

$$\therefore \; \theta \approx 19.1$$

$$\therefore \; \phi \approx 40 - 19.1 \approx 20.9$$

$$\therefore \; y^2 = x^2 + 7^2 - 2 \times x \times 7 \times \cos \phi°$$

$$\therefore \; y \approx \sqrt{(9.0186)^2 + 7^2 - 2 \times (9.0186) \times 7 \times \cos 20.9°}$$

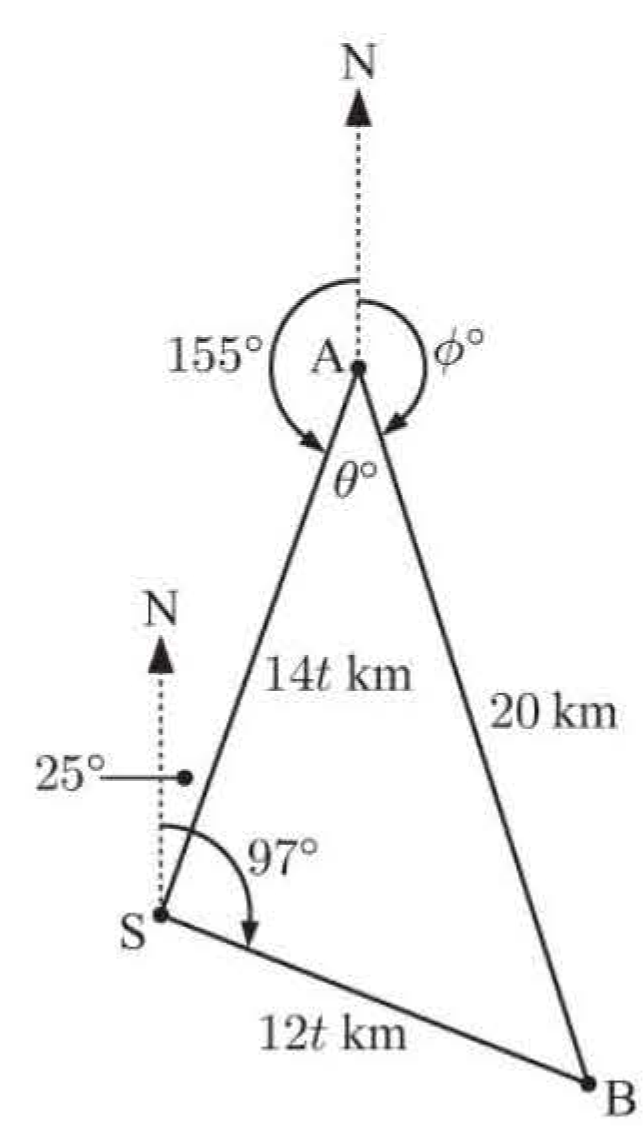
$$\therefore \; y \approx 3.52$$

So, Brett still has to walk 3.52 km.

6 **a** speed $= \dfrac{\text{distance}}{\text{time}}$ $\quad \therefore$ distance $=$ speed $\times$ time

$\therefore$ in t hours, runner A travels $14 \times t = 14t$ km

and runner B travels $12 \times t = 12t$ km



Now $\widehat{ASB} = 97° - 25° = 72°$

$$\therefore \; 20^2 = (14t)^2 + (12t)^2 - 2(14t)(12t)\cos 72°$$

$$\therefore \; 400 = 196t^2 + 144t^2 - 336t^2 \cos 72°$$

$$\therefore \; 400 \approx 236.2t^2$$

$$\therefore \; t^2 \approx 1.69$$

$$\therefore \; t \approx 1.30 \quad \{t > 0\}$$

$\therefore$ A and B are 20 km apart after 1 hour 18 minutes, at 2:18 pm.

b When $t \approx 1.30$, SA $\approx 14 \times 1.30 \approx 18.22$ km

and SB $\approx 12 \times 1.30 \approx 15.62$ km

$$\therefore \; \cos \theta° \approx \frac{18.22^2 + 20^2 - 15.62^2}{2 \times 18.22 \times 20}$$

$$\therefore \; \theta° \approx \cos^{-1}\left(\tfrac{488.1}{728.8}\right)$$

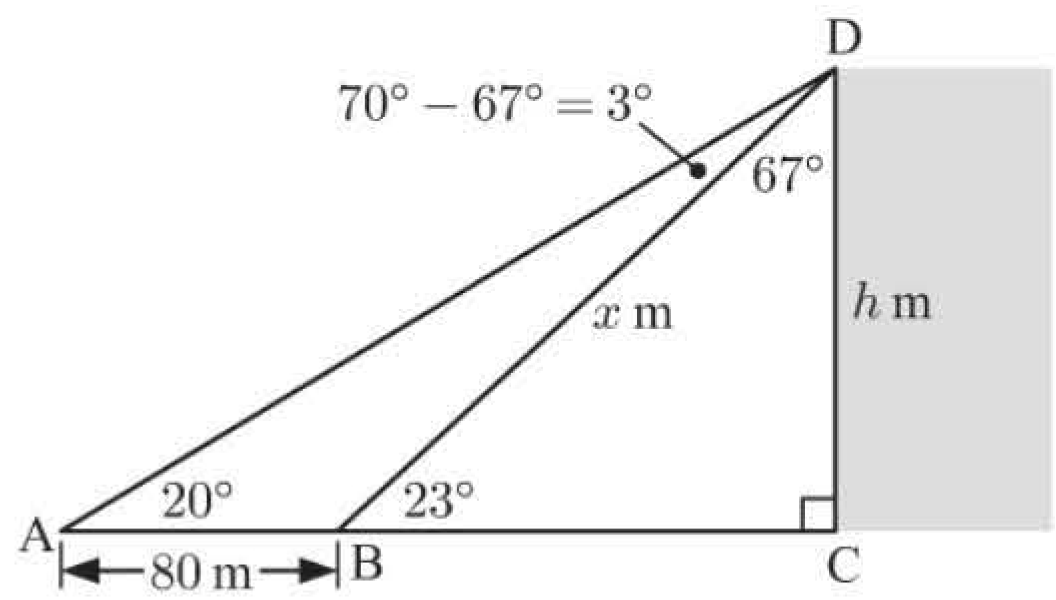
$$\therefore \; \theta \approx 48.0$$

$$\therefore \; \phi \approx 360 - 155 - 48 \qquad \{180° - 25° = 155°, \; \text{co-interior angles}\}$$

$$\approx 157$$

$\therefore$ B is on a bearing of 157° from A.

7



In $\triangle$ABD, $\dfrac{x}{\sin 20^\circ} = \dfrac{80}{\sin 3^\circ}$

$$\therefore\ x = \frac{80 \times \sin 20^\circ}{\sin 3^\circ} \approx 522.8$$

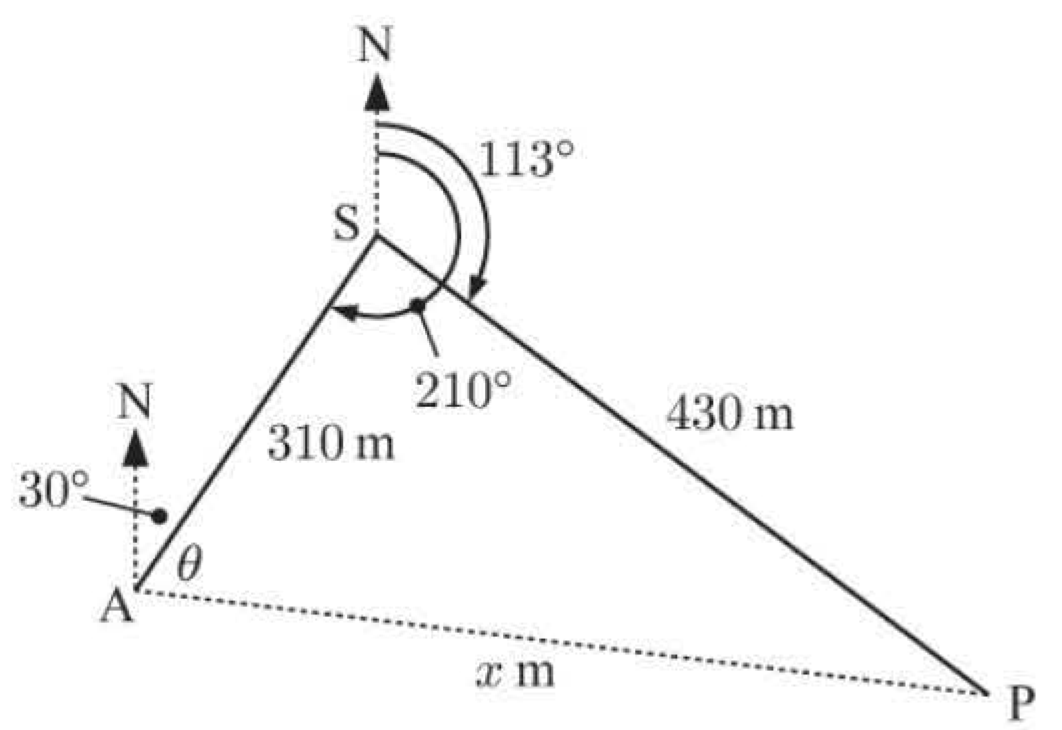
Now $\sin 23^\circ = \dfrac{h}{x}$

$$\therefore\ h \approx 522.8 \times \sin 23^\circ$$

$$\therefore\ h \approx 204$$

So the building is 204 m tall.

8



$$\widehat{\text{ASP}} = 210^\circ - 113^\circ = 97^\circ$$

$$\therefore\ x^2 = 310^2 + 430^2 - 2 \times 310 \times 430 \times \cos 97^\circ$$

$$\therefore\ x = \sqrt{310^2 + 430^2 - 2 \times 310 \times 430 \times \cos 97^\circ}$$

$$\therefore\ x \approx 559.9$$

$\therefore$ Peter and Alix are 560 m apart.

and $\cos\theta \approx \dfrac{310^2 + 559.9^2 - 430^2}{2 \times 310 \times 559.9}$

$$\therefore\ \theta \approx 49.7$$

and $30 + \theta \approx 79.7$

$\therefore$ the bearing of Peter from Alix is 079.7°.

REVIEW SET 11C

1 **a** $\cos x^\circ = \dfrac{11^2 + 19^2 - 13^2}{2 \times 11 \times 19}$

$$\therefore\ \cos x^\circ = \tfrac{313}{418}$$

$$\therefore\ x^\circ = \cos^{-1}\left(\tfrac{313}{418}\right)$$

$$\therefore\ x \approx 41.5$$

b $x^2 = 14^2 + 21^2 - 2 \times 14 \times 21 \times \cos 47^\circ$

$$\therefore\ x = \sqrt{14^2 + 21^2 - 2 \times 14 \times 21 \times \cos 47^\circ}$$

$$\therefore\ x \approx 15.4$$

2 **a** area $= 80\ \text{cm}^2$

$$\therefore\ \tfrac{1}{2} \times 11.3 \times 19.2 \times \sin x^\circ = 80$$

$$\therefore\ \sin x^\circ = \frac{2 \times 80}{11.3 \times 19.2}$$

$$\therefore\ x^\circ = \sin^{-1}\left(\frac{160}{216.96}\right)$$

$$\approx 47.5^\circ$$

$$\therefore\ x \approx 47.5 \text{ or } 180 - 47.5$$

$$\therefore\ x \approx 47.5 \text{ or } 132.5$$

b $\text{AC}^2 = 19.2^2 + 11.3^2 - 2 \times 19.2 \times 11.3 \times \cos x^\circ$

$$\text{AC} = \sqrt{368.64 + 127.69 - 433.92\cos x^\circ}$$

But $x \approx 47.5$ or 132.5

$\therefore\ \text{AC} = \sqrt{496.33 - 433.92\cos 47.5^\circ}$ or $\text{AC} = \sqrt{496.33 - 433.92\cos 132.5^\circ}$

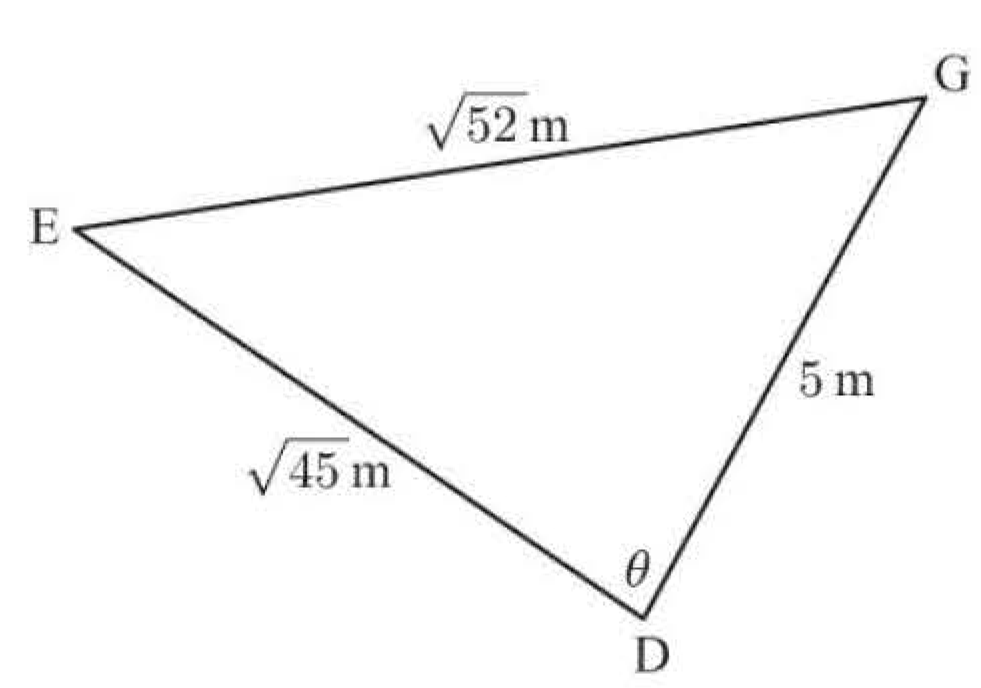
≈ 14.3 cm or ≈ 28.1 cm

3 Using Pythagoras,

$$\text{ED} = \sqrt{6^2 + 3^2} = \sqrt{45}\ \text{m}$$

$$\text{DG} = \sqrt{4^2 + 3^2} = \sqrt{25} = 5\ \text{m}$$

$$\text{EG} = \sqrt{6^2 + 4^2} = \sqrt{52}\ \text{m}$$



Using the cosine rule, $\cos\theta = \dfrac{(\sqrt{45})^2 + 5^2 - (\sqrt{52})^2}{2 \times \sqrt{45} \times 5}$

$$\therefore\ \theta = \cos^{-1}\left(\tfrac{18}{10\sqrt{45}}\right)$$

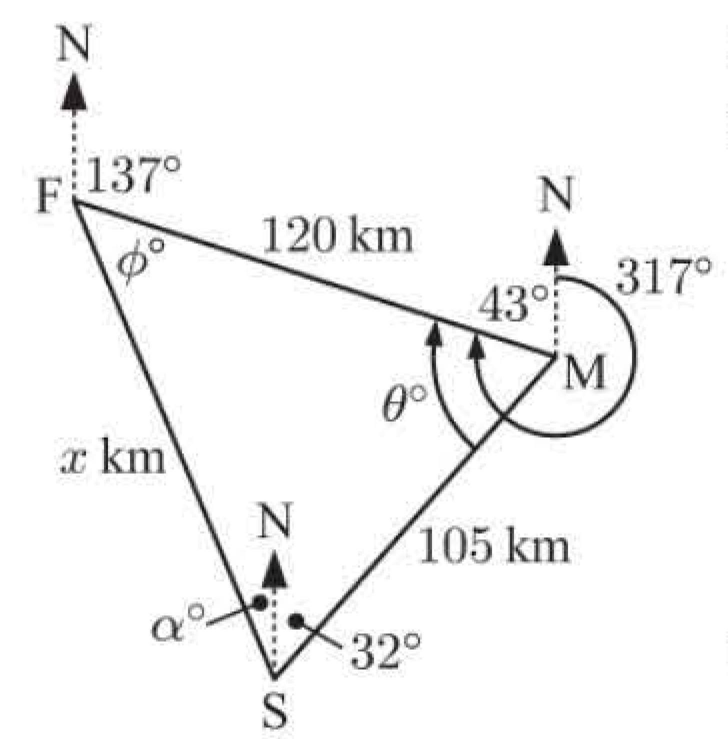
$\therefore\ \theta \approx 74.4^\circ$ Thus $\widehat{\text{EDG}}$ measures 74.4°.

4 **a** $$\text{BD}^2 = 120^2 + 125^2 - 2 \times 120 \times 125 \cos 75^\circ$$
$$\therefore \ \text{BD} = \sqrt{120^2 + 125^2 - 2 \times 120 \times 125 \cos 75^\circ}$$
$$\approx 149.2 \text{ m}$$

The area of the block = area of $\triangle$ABD + area of $\triangle$BCD
$$\approx \tfrac{1}{2} \times 120 \times 125 \times \sin 75^\circ + \tfrac{1}{2} \times 149.2 \times 90 \times \sin 30^\circ$$
$$\approx 10\,600 \text{ m}^2$$

b ≈ 1.06 ha $\{10\,000 \text{ m}^2 = 1 \text{ ha}\}$

5



distance = speed × time

So, in 45 minutes, $140 \times \frac{3}{4} = 105$ km is travelled. $\{45 \text{ minutes} = \frac{3}{4} \text{ hour}\}$

In 40 minutes, $180 \times \frac{2}{3} = 120$ km is travelled. $\{40 \text{ minutes} = \frac{2}{3} \text{ hour}\}$

We notice that $\theta + 43 + 32 = 180$ {co-interior angles add to 180°}
$$\therefore \ \theta = 105$$

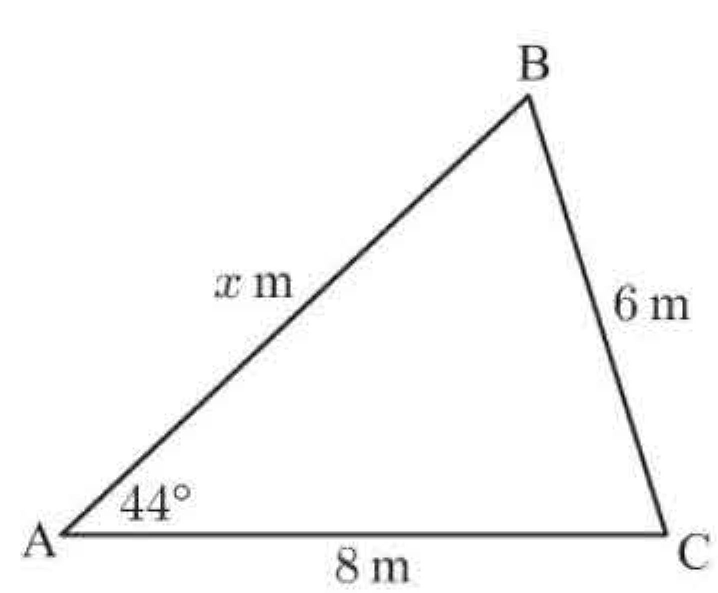
Using the cosine rule, $x^2 = 120^2 + 105^2 - 2 \times 120 \times 105 \times \cos 105^\circ$
$$\therefore \ x = \sqrt{120^2 + 105^2 - 2 \times 120 \times 105 \times \cos 105^\circ}$$
$$\therefore \ x \approx 178.74$$

So, the car is 179 km from the start.

$$\text{Now} \quad \frac{\sin \phi^\circ}{105} \approx \frac{\sin 105^\circ}{178.74}$$
$$\therefore \ \sin \phi^\circ \approx \frac{105 \times \sin 105^\circ}{178.74}$$
$$\therefore \ \phi \approx 34.6$$
$$\therefore \ \alpha \approx 180 - 105 - 34.6 - 32 \approx 8.4 \approx 8$$

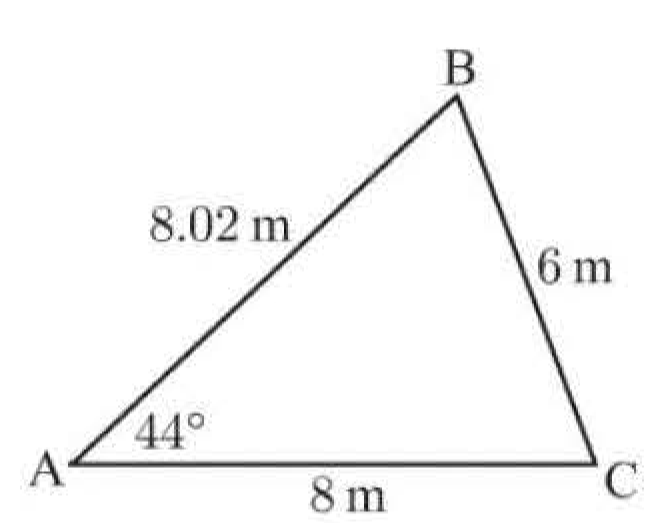
So, the bearing from its starting point is $360^\circ - 8^\circ = 352^\circ$.

6 **a**

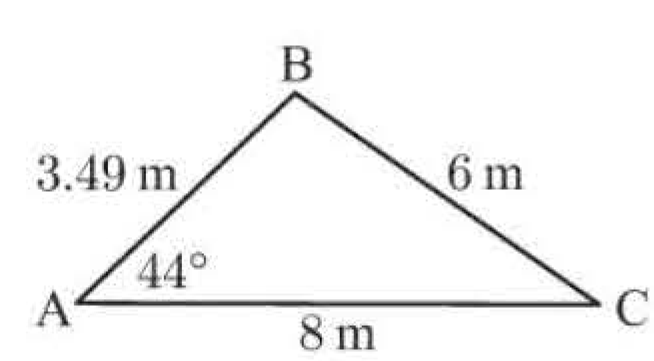


By the cosine rule, $6^2 = x^2 + 8^2 - 2 \times x \times 8 \times \cos 44^\circ$
$$\therefore \ 36 = x^2 + 64 - 16x \times \cos 44^\circ$$
$$\therefore \ x^2 - 11.51x + 28 \approx 0$$
$$\therefore \ x \approx \frac{11.51 \pm \sqrt{11.51^2 - 4(1)(28)}}{2}$$
$$\therefore \ x \approx \frac{11.51 \pm 4.524}{2}$$
$$\therefore \ x \approx 8.02 \text{ or } 3.49$$

Frank needs additional information as there are two possible cases:
(1) when AB ≈ 8.02 m and
(2) when AB ≈ 3.49 m



Case (1)

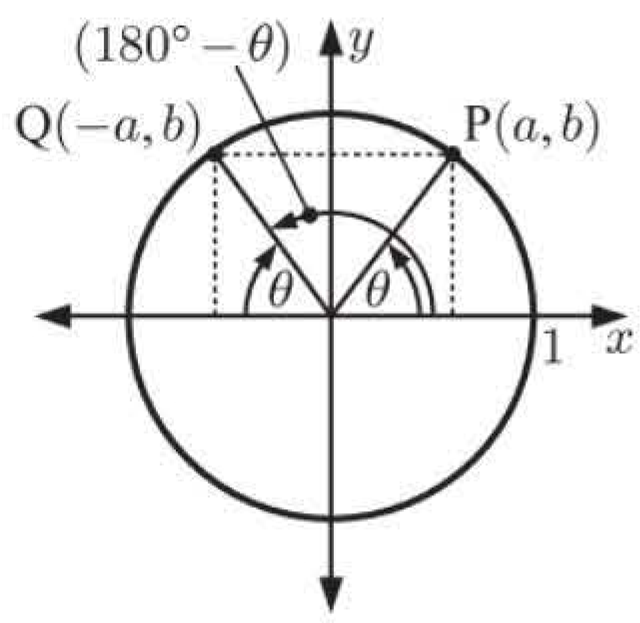


Case (2)

b Volume = area × depth
$$= \tfrac{1}{2} \times 8 \times x \times \sin 44^\circ \times 0.1 \quad \text{and is a maximum when } x \approx 8.02 \text{ m}$$
$$\approx 4 \times 8.02 \times \sin 44^\circ \times 0.1$$
$$\approx 2.23 \text{ m}^3$$

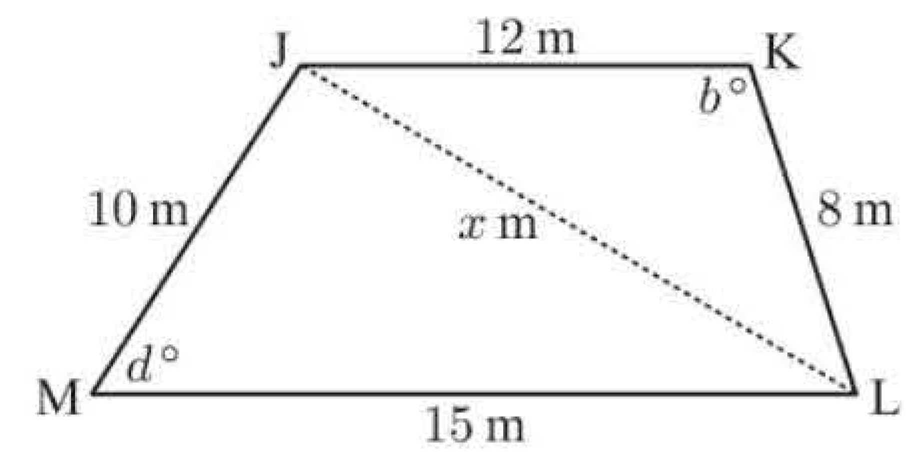
So, the maximum volume of soil needed is 2.23 m^3.

7 **a**



$$\therefore \quad \cos(180° - \theta) = -a$$
$$= -\cos\theta \quad \{\cos\theta = a\}$$

b **i** Using $\triangle$JLM,



$$x^2 = 10^2 + 15^2 - 2 \times 10 \times 15 \cos d°$$
$$\therefore \quad x^2 = 325 - 300 \cos d° \quad .... (1)$$

Using $\triangle$JLK,

$$x^2 = 12^2 + 8^2 - 2 \times 12 \times 8 \cos b°$$
$$\therefore \quad x^2 = 208 - 192 \cos b° \quad .... (2)$$

Equating (1) and (2), $\quad 325 - 300 \cos d° = 208 - 192 \cos b°$

$$\therefore \quad 300 \cos d° - 192 \cos b° = 117$$

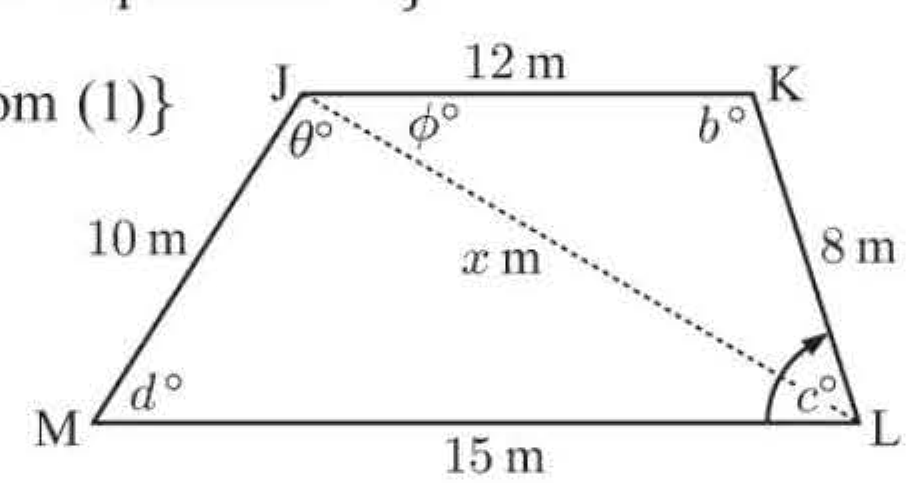
ii If $b + d = 180$, then $b = 180 - d$

$$\therefore \quad 300 \cos d° - 192 \cos(180 - d)° = 117$$
$$\therefore \quad 300 \cos d° + 192 \cos d° = 117 \quad \{\text{from } \mathbf{a}\}$$
$$\therefore \quad 492 \cos d° = 117$$
$$\therefore \quad d = \cos^{-1}\left(\tfrac{117}{492}\right)$$
$$\therefore \quad d \approx 76.2$$
$$\text{and} \quad b = 180 - d \approx 103.8$$

iii If $b + d = 180$, then $a + c = 180$ also $\{$angles in a quadrilateral$\}$

If $d \approx 76.2$, then $x \approx \sqrt{325 - 300\cos(76.2)°}$ $\{$from (1)$\}$

$$\therefore \quad x \approx 15.93$$



In $\triangle$JLM, $\cos\theta° \approx \dfrac{10^2 + 15.93^2 - 15^2}{2 \times 10 \times 15.93}$

$$\therefore \quad \theta° \approx \cos^{-1}\left(\tfrac{128.7}{318.5}\right)$$
$$\therefore \quad \theta \approx 66.2$$

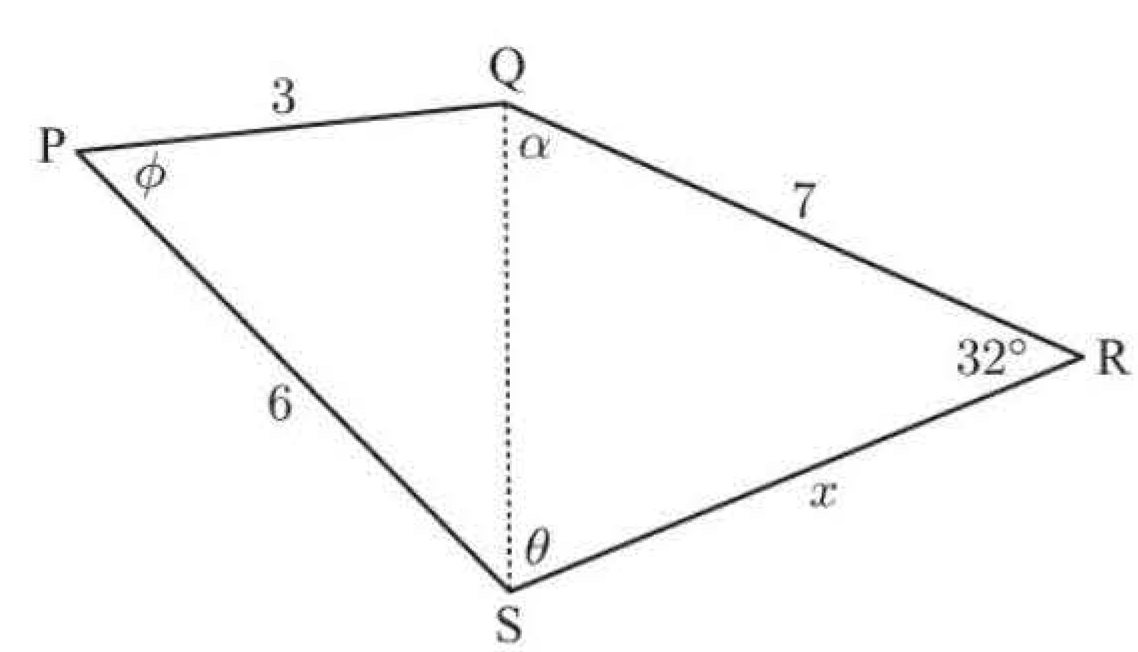
In $\triangle$JLK, $\cos\phi° \approx \dfrac{12^2 + 15.93^2 - 8^2}{2 \times 12 \times 15.93}$

$$\therefore \quad \phi° \approx \cos^{-1}\left(\tfrac{333.7}{382.2}\right)$$
$$\therefore \quad \phi \approx 29.2$$

$\therefore \quad a = \theta + \phi \approx 95.4$ and $c = 180 - a \approx 84.6$

8 **a** $\quad QS^2 = 6^2 + 3^2 - 2 \times 6 \times 3 \times \cos\phi$

$$\therefore \quad QS = \sqrt{45 - 36\cos\phi}$$



b **i** If $\phi = 50°$, $QS = \sqrt{45 - 36\cos 50°}$

$$\approx 4.675$$
$$\therefore \quad \frac{\sin\theta}{7} \approx \frac{\sin 32°}{4.675}$$
$$\therefore \quad \sin\theta \approx \frac{7 \times \sin 32°}{4.675}$$
$$\therefore \quad \theta \approx \sin^{-1}\left(\frac{7 \times \sin 32°}{4.675}\right) \quad \text{or its supplement}$$
$$\therefore \quad \theta \approx 52.5° \quad \text{or} \quad (180 - 52.5)°$$
$$\therefore \quad \widehat{RSQ} \approx 52.5° \quad \text{or} \quad 127.5°$$

but since $\widehat{RSQ}$ is acute it must be $\approx 52.5°$.

ii $\alpha = 180° - 32° - \theta \quad \{\theta \approx 52.5°\}$

$\approx 95.5°$

$$\therefore \quad \frac{x}{\sin 95.5°} \approx \frac{7}{\sin 52.5°}$$

$$\therefore \quad x \approx \frac{7 \times \sin 95.5°}{\sin 52.5°}$$

$$\therefore \quad x \approx 8.78$$

$$\therefore \quad \text{perimeter} \approx 6 + 3 + 7 + 8.78$$

$$\approx 24.8 \text{ units}$$

iii area of PQRS = area of $\triangle$PQS + area of $\triangle$QRS

$$= \tfrac{1}{2} \times 3 \times 6 \times \sin 50° + \tfrac{1}{2} \times 7 \times 8.78 \times \sin 32°$$

$$\approx 23.2 \text{ units}^2$$

or if θ is obtuse we can similarly calculate that the area of PQRS ≈ 12.6 units^2.

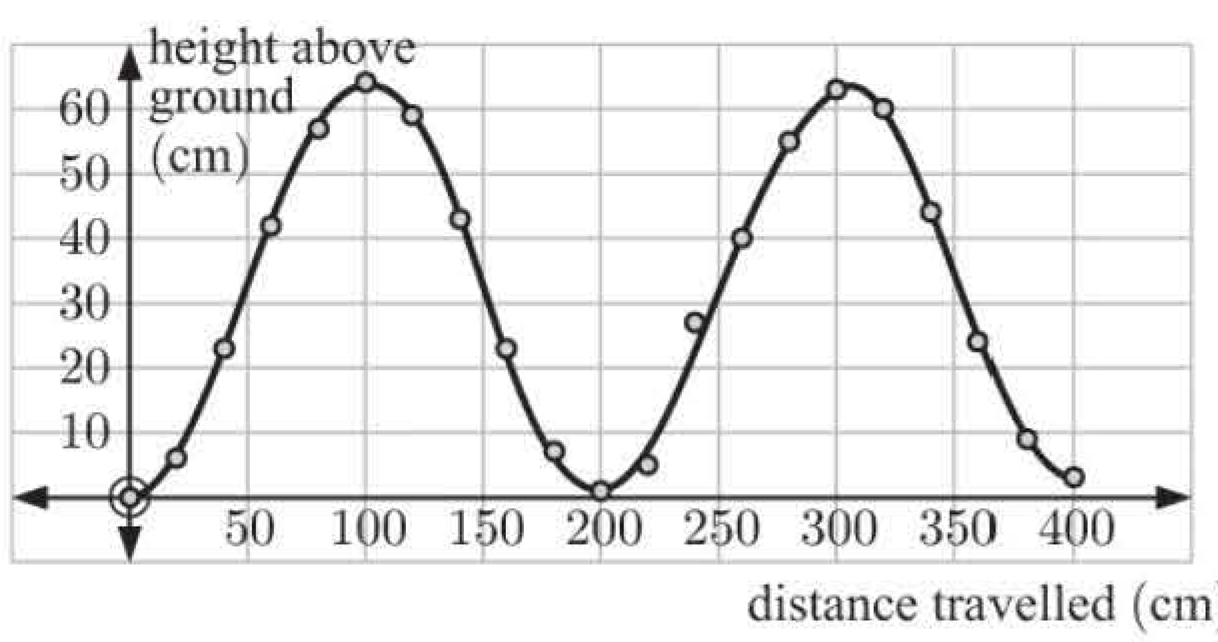
Chapter 12

TRIGONOMETRIC FUNCTIONS

EXERCISE 12A

1 **a** periodic **b** periodic **c** periodic **d** not periodic **e** periodic
f periodic **g** not periodic **h** not periodic

2 **a**



c The data is periodic.

i The minimum value from the table is 0 and the maximum value is 64.

So, the principal axis is $y \approx \frac{0+64}{2}$

$\therefore\ y \approx 32$

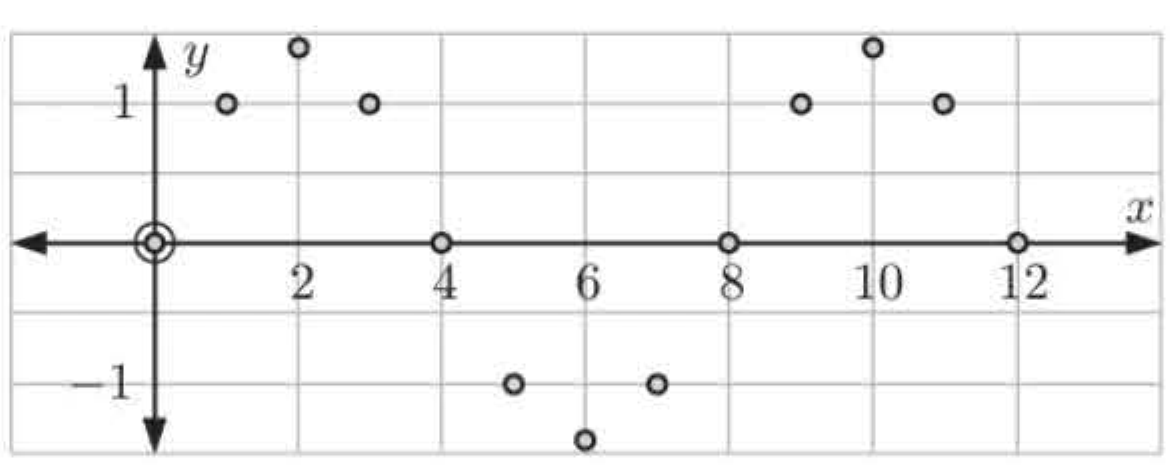
ii The maximum value is ≈ 64 cm.

iii The period is ≈ 200 cm.

iv The amplitude is ≈ 32 cm.

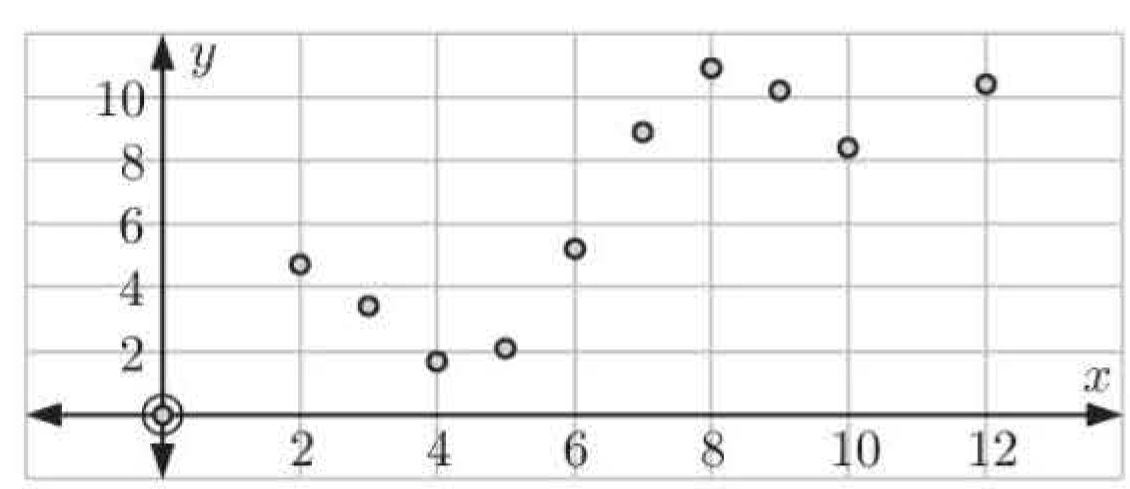
b A curve can be fitted to the data as the distance travelled is continuous.

3 **a**



Data exhibits periodic behaviour.

b



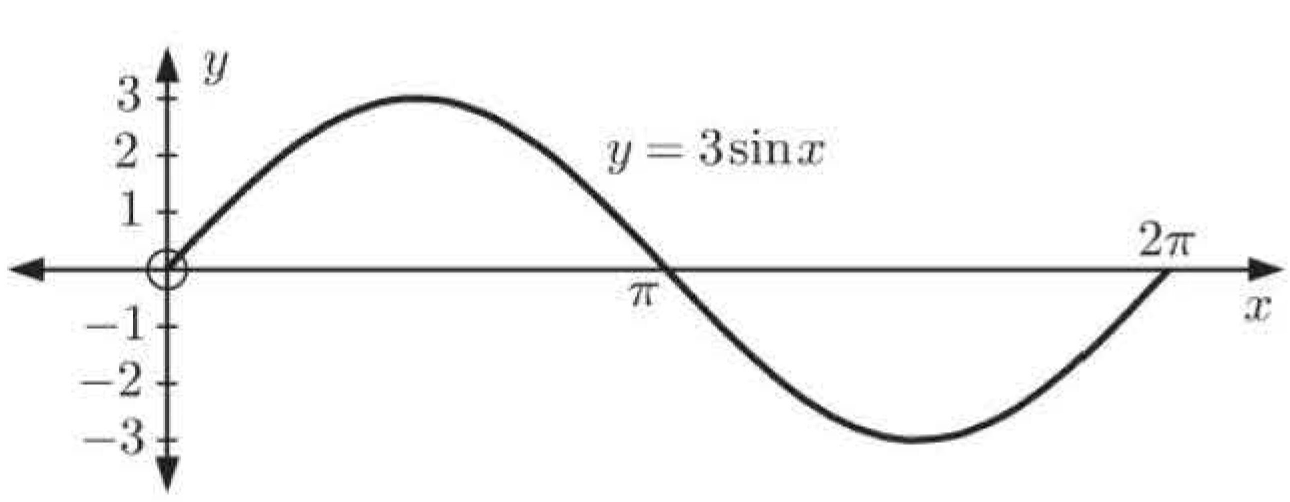
Not enough information to say data is periodic.

EXERCISE 12B.1

1 **a** $y = 3\sin x$

has amplitude 3 and period $\frac{2\pi}{1} = 2\pi$

When $x = 0,\ y = 0$.

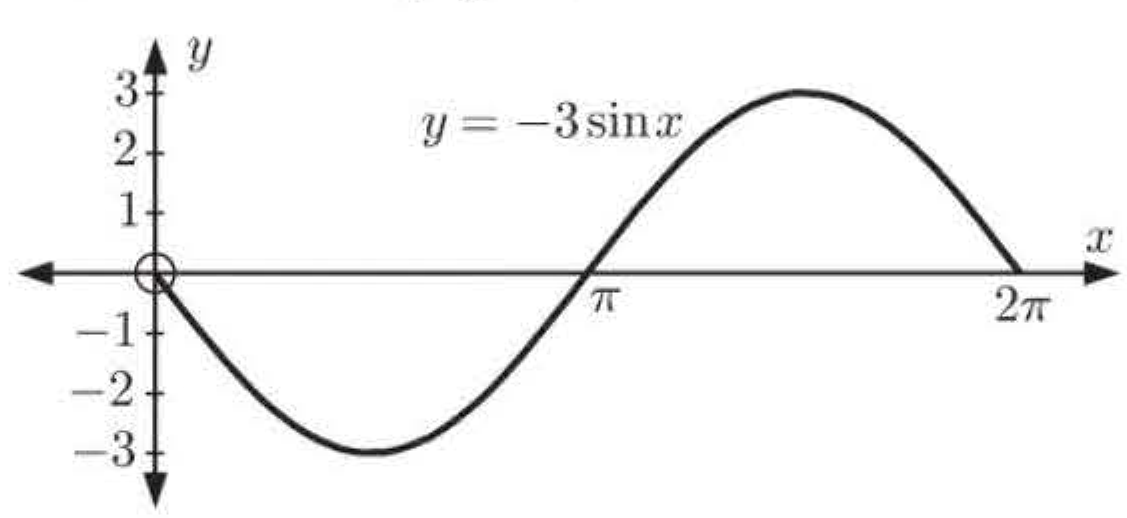


b $y = -3\sin x$

has amplitude $|-3| = 3$

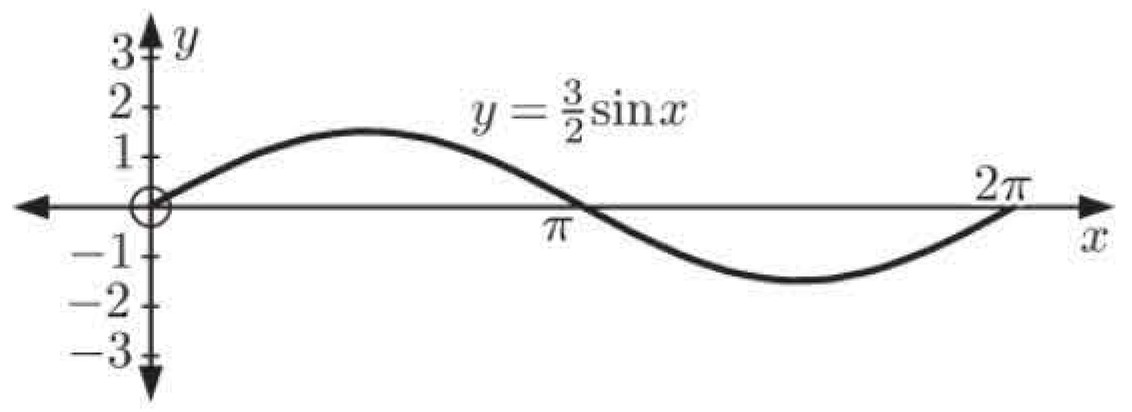
and period $\frac{2\pi}{1} = 2\pi$.

When $x = 0,\ y = 0$.

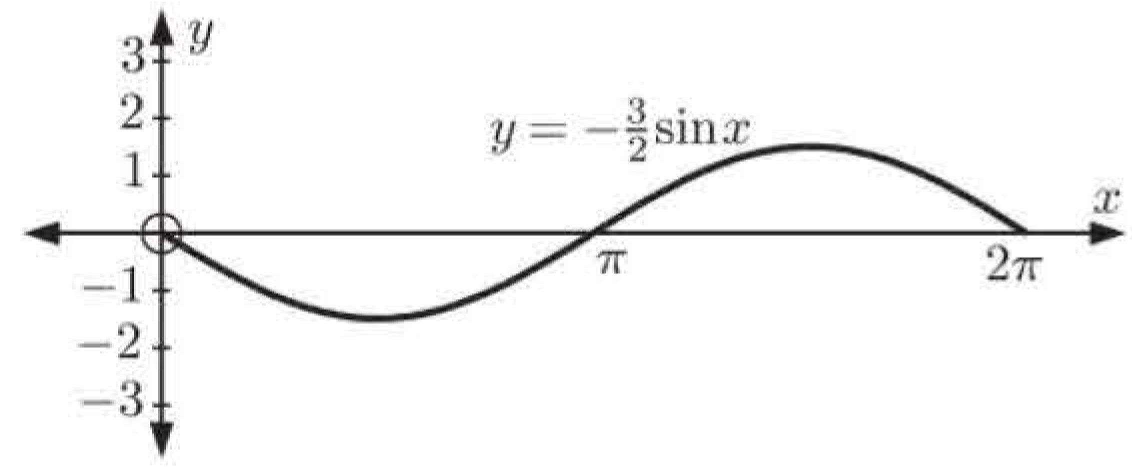


It is the reflection of $y = 3\sin x$ in the x-axis.

c $y = \frac{3}{2}\sin x$
has amplitude $\frac{3}{2}$ and period $\frac{2\pi}{1} = 2\pi$.
When $x = 0$, $y = 0$.

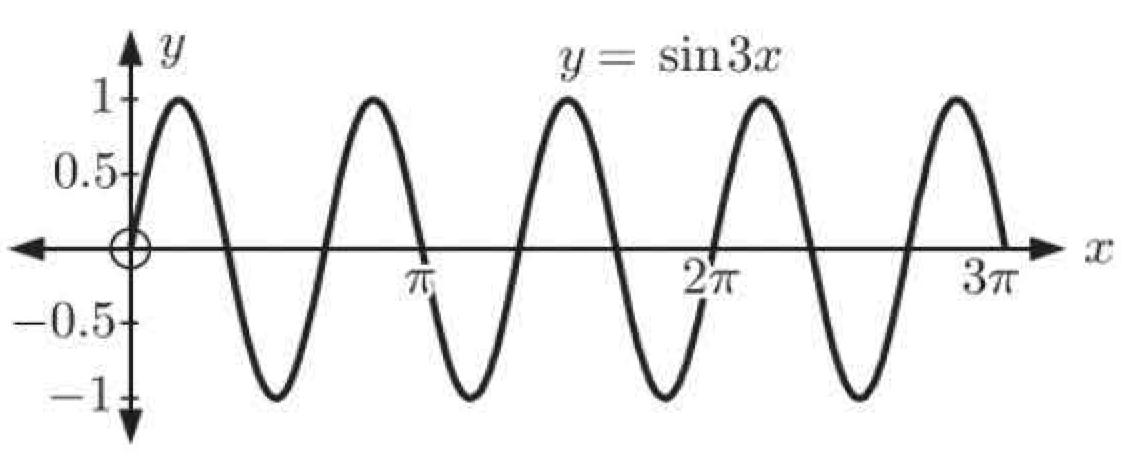


d $y = -\frac{3}{2}\sin x$
has amplitude $\left|-\frac{3}{2}\right| = \frac{3}{2}$
and period $\frac{2\pi}{1} = 2\pi$.

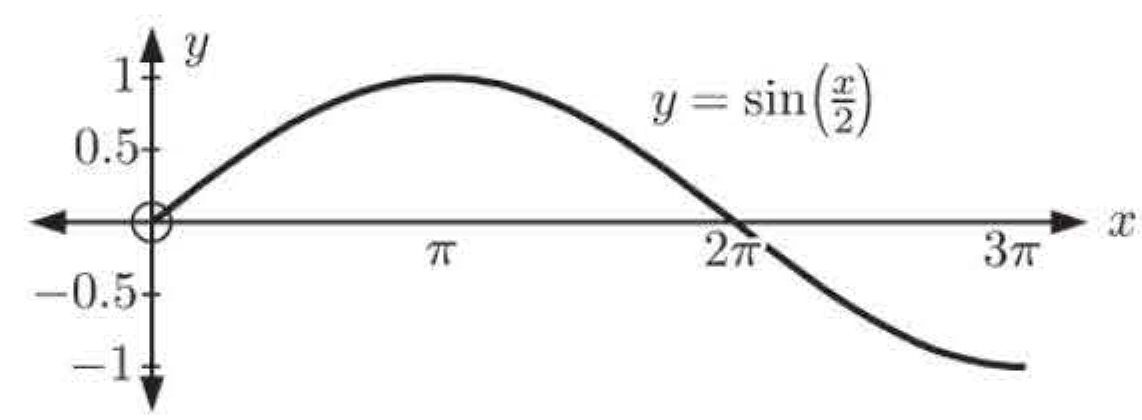


It is the reflection of $y = \frac{3}{2}\sin x$ in the x-axis.

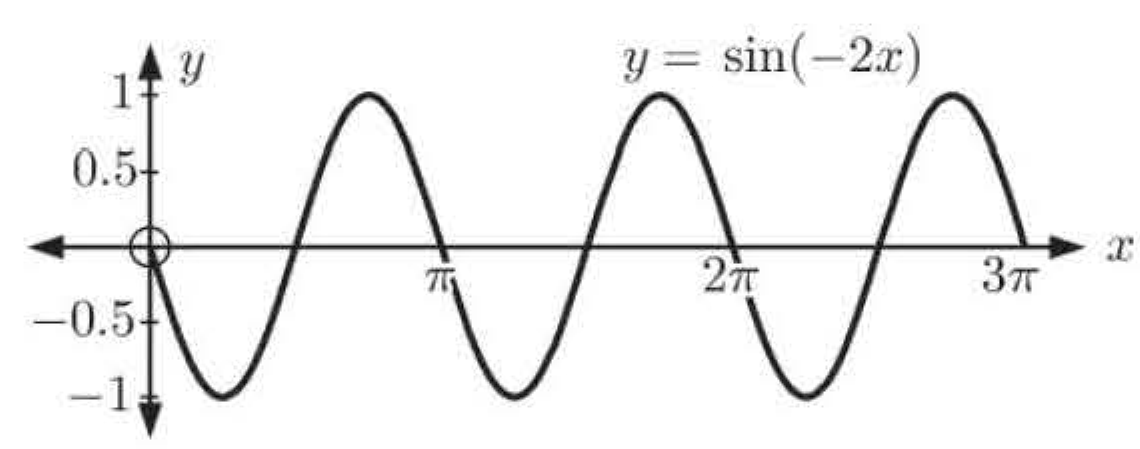
2 **a** $y = \sin 3x$
has amplitude 1 and period $\frac{2\pi}{3}$.
When $x = 0$, $y = 0$.



b $y = \sin\left(\frac{x}{2}\right)$
has amplitude 1 and period $\dfrac{2\pi}{\frac{1}{2}} = 4\pi$.
When $x = 0$, $y = 0$.



c $y = \sin(-2x)$
has amplitude 1 and period $\dfrac{2\pi}{|-2|} = \pi$.
When $x = 0$, $y = 0$.



It is the reflection of $y = \sin 2x$ in the y-axis.

3 **a** period $= \dfrac{2\pi}{4}$
$= \frac{\pi}{2}$

b period $= \dfrac{2\pi}{|-4|}$
$= \frac{\pi}{2}$

c period $= \dfrac{2\pi}{\left(\frac{1}{3}\right)}$
$= 6\pi$

d period $= \dfrac{2\pi}{0.6}$
$= \frac{20\pi}{6} = \frac{10\pi}{3}$

4 **a** $\dfrac{2\pi}{b} = 5\pi$
$\therefore\ b = \frac{2}{5}$

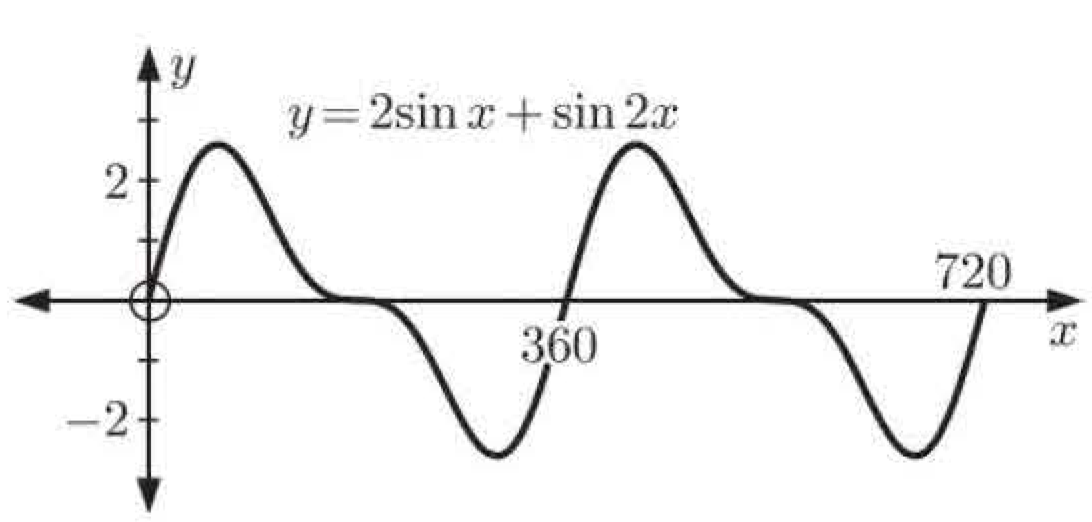
b $\dfrac{2\pi}{b} = \dfrac{2\pi}{3}$
$\therefore\ b = 3$

c $\dfrac{2\pi}{b} = 12\pi$
$\therefore\ b = \frac{1}{6}$

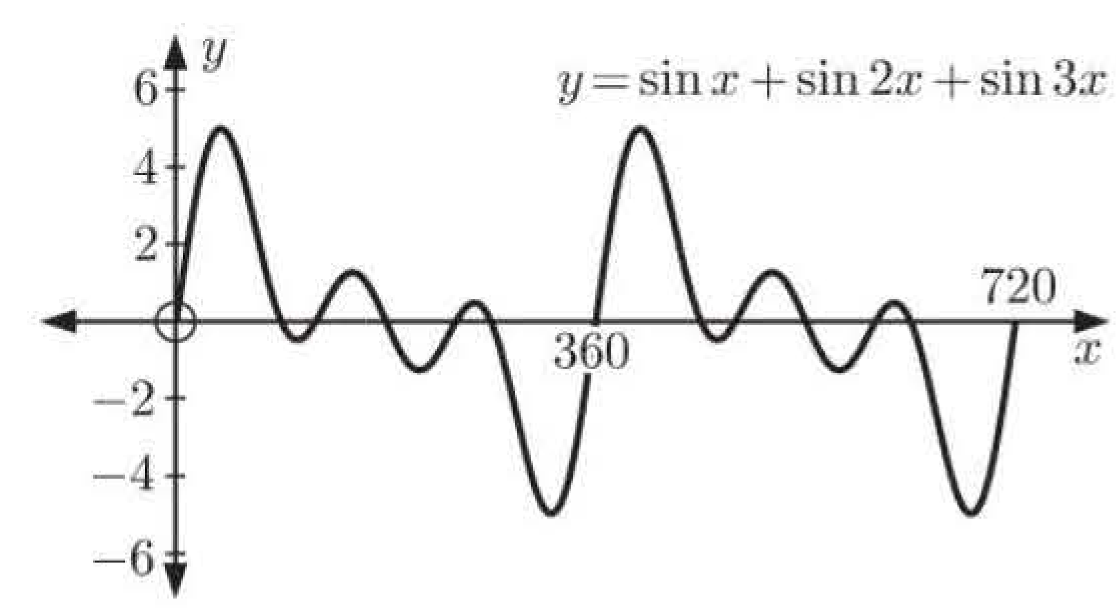
d $\dfrac{2\pi}{b} = 4$
$\therefore\ b = \frac{\pi}{2}$

e $\dfrac{2\pi}{b} = 100$
$\therefore\ b = \frac{2\pi}{100} = \frac{\pi}{50}$

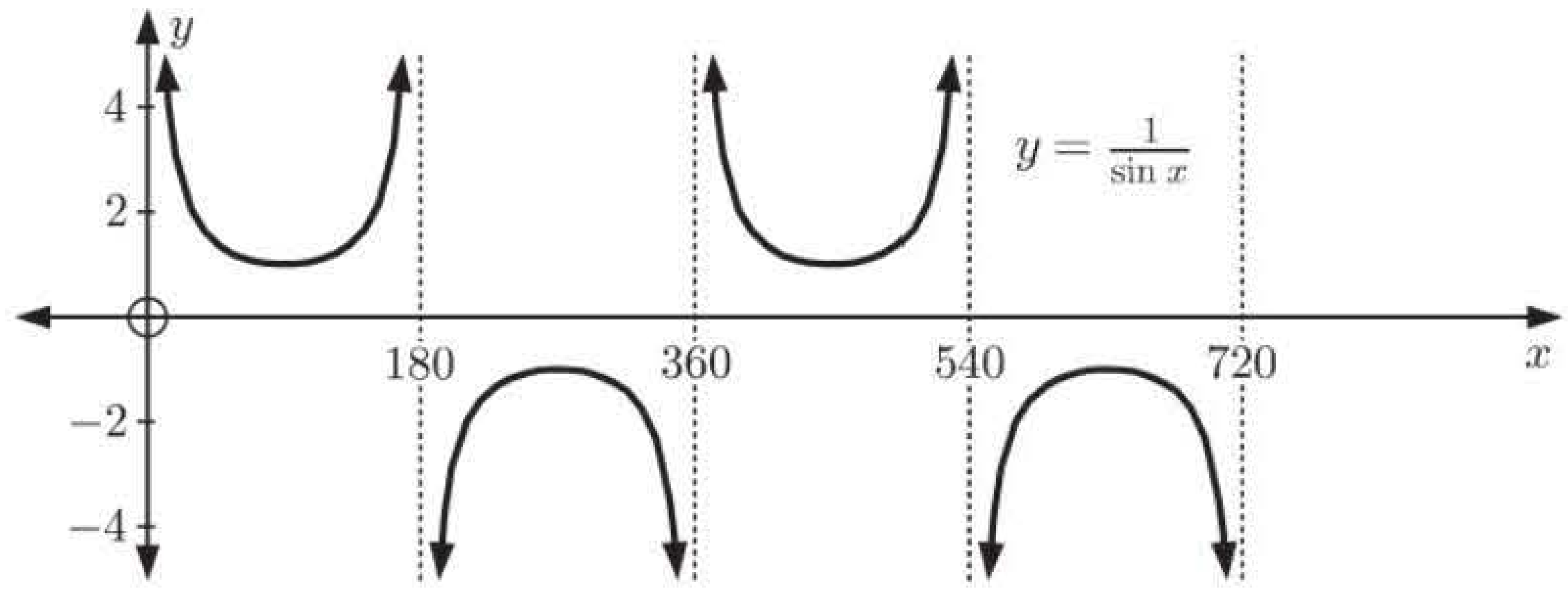
5 **a**



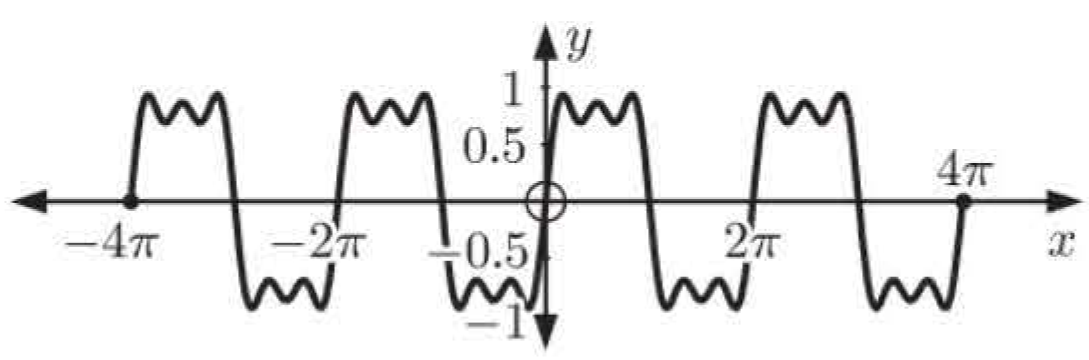
b



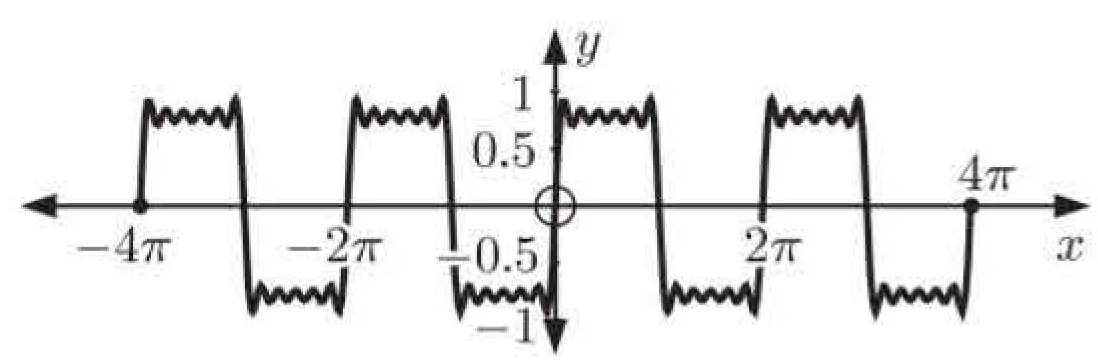
c



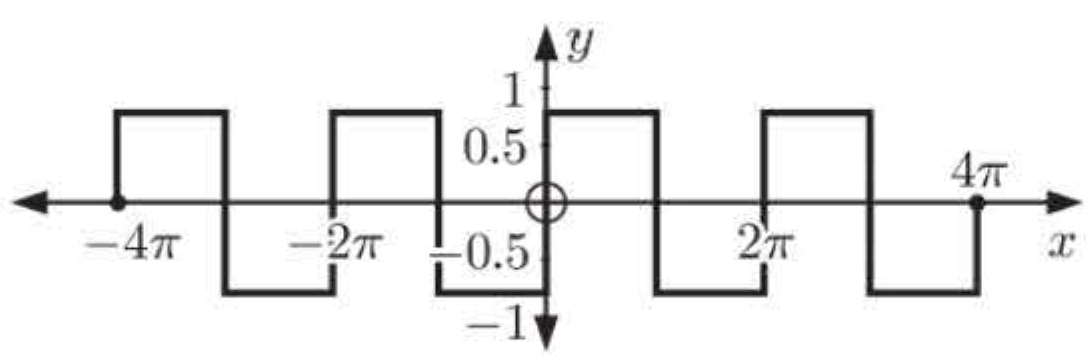
6 **a** **i**



ii

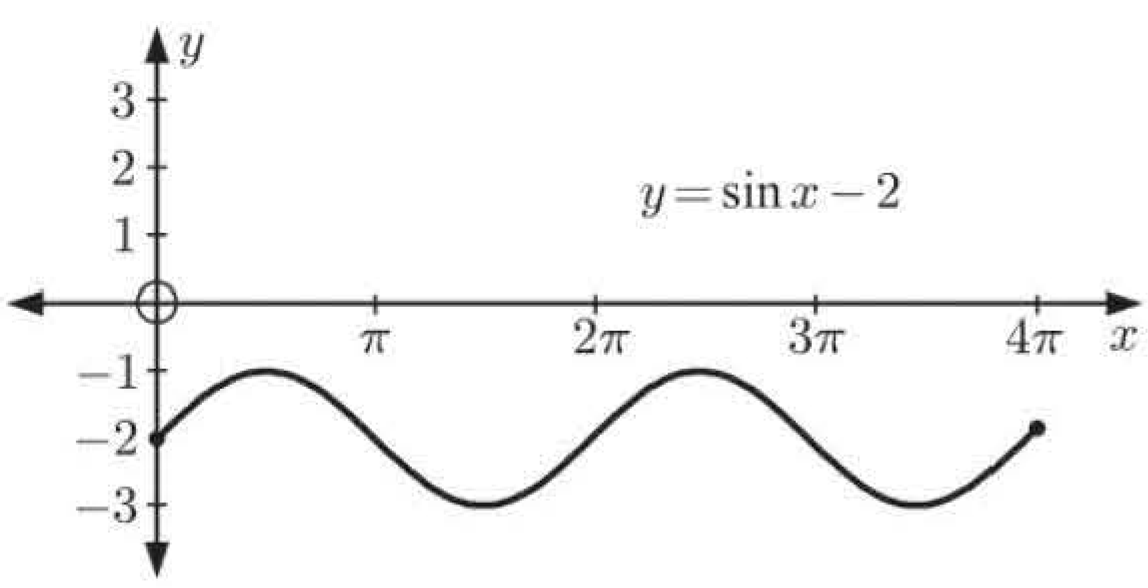


b Prediction:



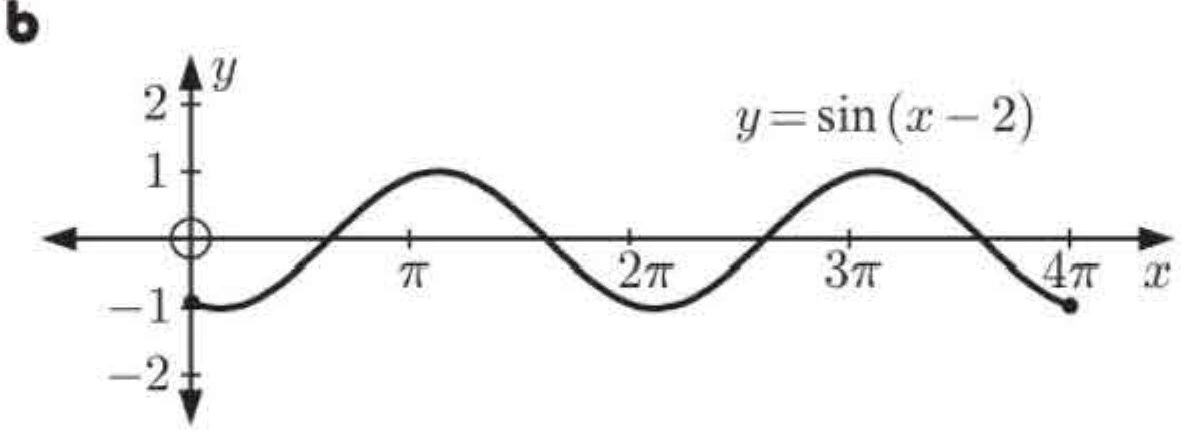
EXERCISE 12B.2

1 **a**



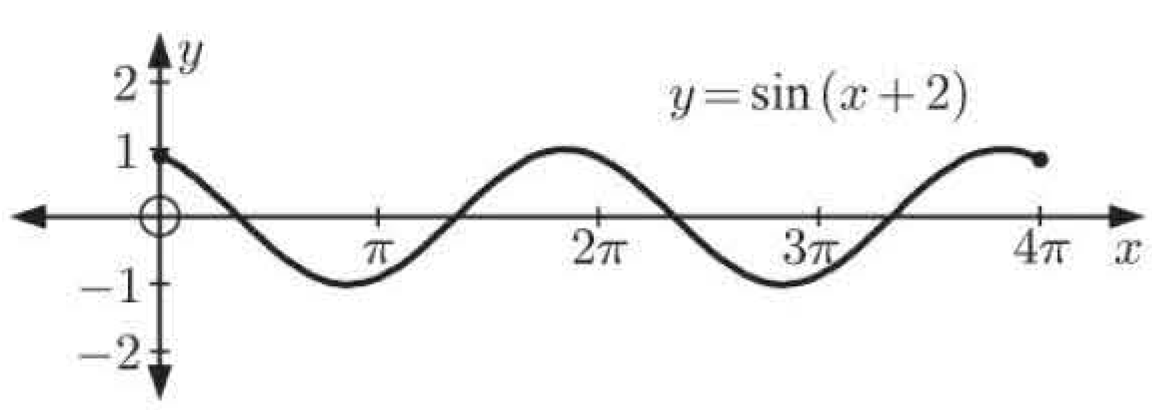
This is the graph of $y = \sin x$ translated by $\binom{0}{-2}$.

b



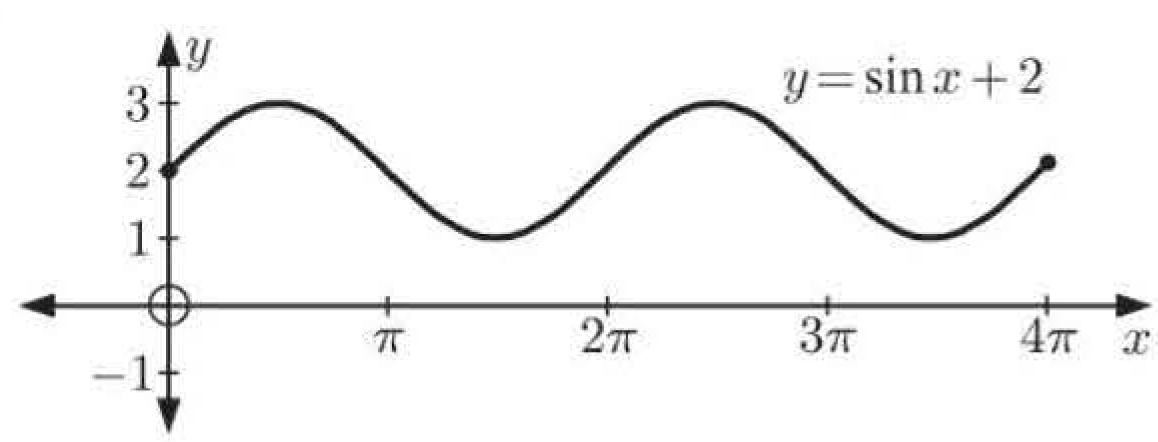
This is the graph of $y = \sin x$ translated by $\binom{2}{0}$.

c



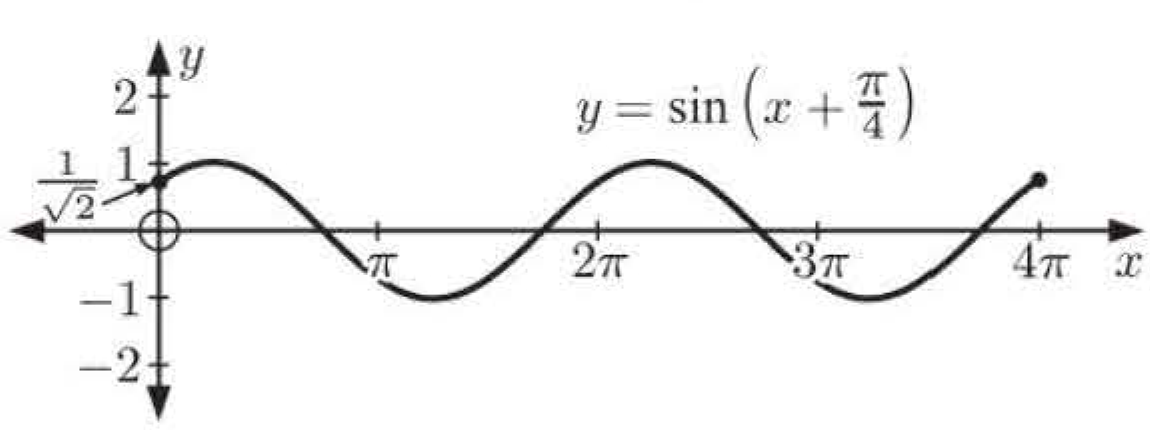
This is the graph of $y = \sin x$ translated by $\binom{-2}{0}$.

d



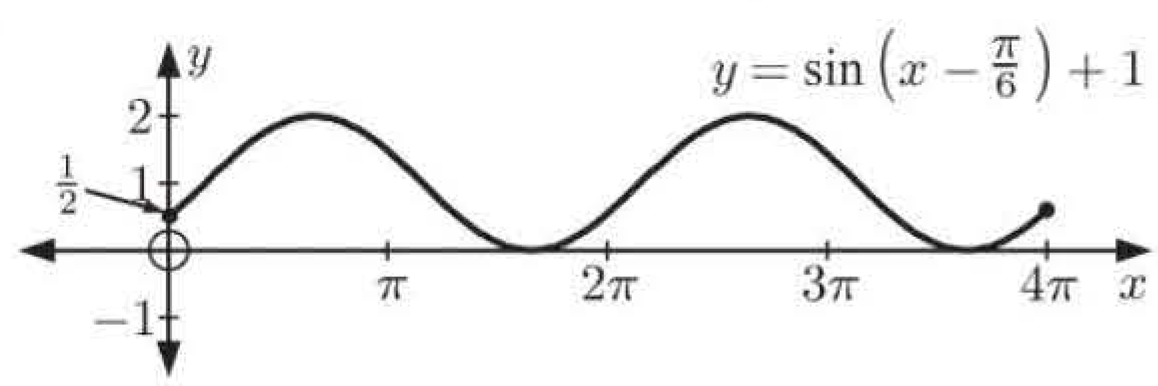
This is the graph of $y = \sin x$ translated by $\binom{0}{2}$.

e



This is the graph of $y = \sin x$ translated by $\begin{pmatrix} -\frac{\pi}{4} \\ 0 \end{pmatrix}$.

f



This is the graph of $y = \sin x$ translated by $\begin{pmatrix} \frac{\pi}{6} \\ 1 \end{pmatrix}$.

2 **a** period $= \dfrac{2\pi}{5} = \frac{2\pi}{5}$ **b** period $= \dfrac{2\pi}{(\frac{1}{4})} = 8\pi$ **c** period $= \dfrac{2\pi}{|-2|} = \pi$

3 **a** $\dfrac{2\pi}{b} = 3\pi$ $\therefore\ b = \frac{2}{3}$

b $\dfrac{2\pi}{b} = \dfrac{\pi}{10}$ $\therefore\ b = 20$

c $\dfrac{2\pi}{b} = 100\pi$ $\therefore\ b = \frac{2}{100} = \frac{1}{50}$

d $\dfrac{2\pi}{b} = 50$ $\therefore\ b = \frac{2\pi}{50} = \frac{\pi}{25}$

4 **a** A translation of $\binom{0}{-1}$, or vertically down 1 unit.

b A translation of $\begin{pmatrix} \frac{\pi}{4} \\ 0 \end{pmatrix}$, or horizontally $\frac{\pi}{4}$ units right.

c A vertical stretch of factor 2.

d A horizontal stretch of factor $\frac{1}{4}$.

e A vertical stretch of factor $\frac{1}{2}$.

f A horizontal stretch of factor 4.

g A reflection in the x-axis.

h A translation of $\binom{-2}{-3}$.

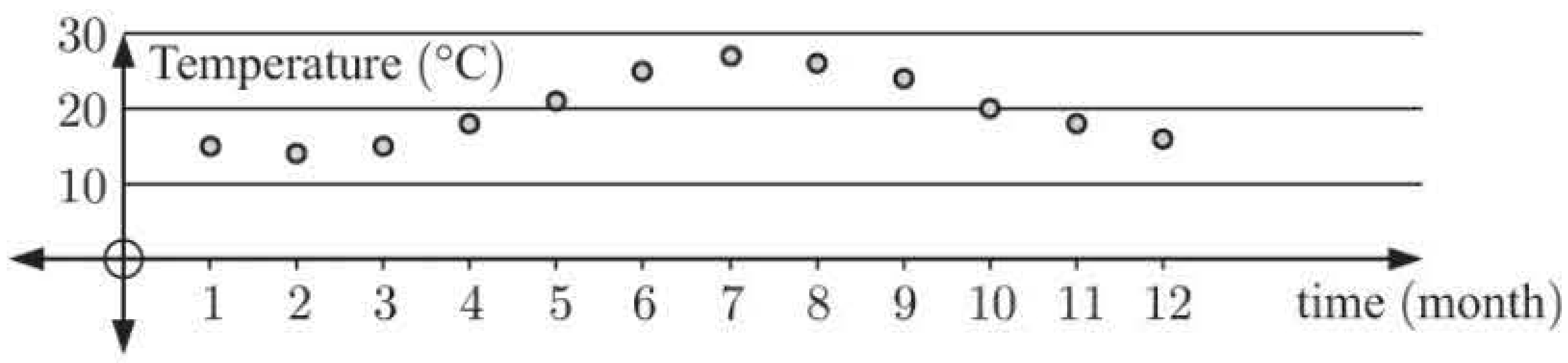
i A vertical stretch of factor 2 followed by a horizontal stretch of factor $\frac{1}{3}$.

j A translation of $\begin{pmatrix} \frac{\pi}{3} \\ 2 \end{pmatrix}$.

EXERCISE 12C

1 **a**

Month, t	1	2	3	4	5	6	7	8	9	10	11	12
Temp, T	15	14	15	18	21	25	27	26	24	20	18	16



The period is 12 months so $\dfrac{2\pi}{b} = 12$

$\therefore\ b = \frac{\pi}{6}$ {assuming $b > 0$}.

Amplitude, $a \approx \dfrac{\text{max.} - \text{min.}}{2} \approx \dfrac{27-14}{2} \approx 6.5$

As the principal axis is midway between min. and max., then $d \approx \dfrac{27+14}{2} \approx 20.5$

When T is 20.5 (midway between min. and max.), $c \approx \dfrac{2+7}{2} \approx 4.5$ {average of t values}

$\therefore\ T \approx 6.5\sin(\frac{\pi}{6}(t-4.5)) + 20.5$ where $\frac{\pi}{6} \approx 0.524$

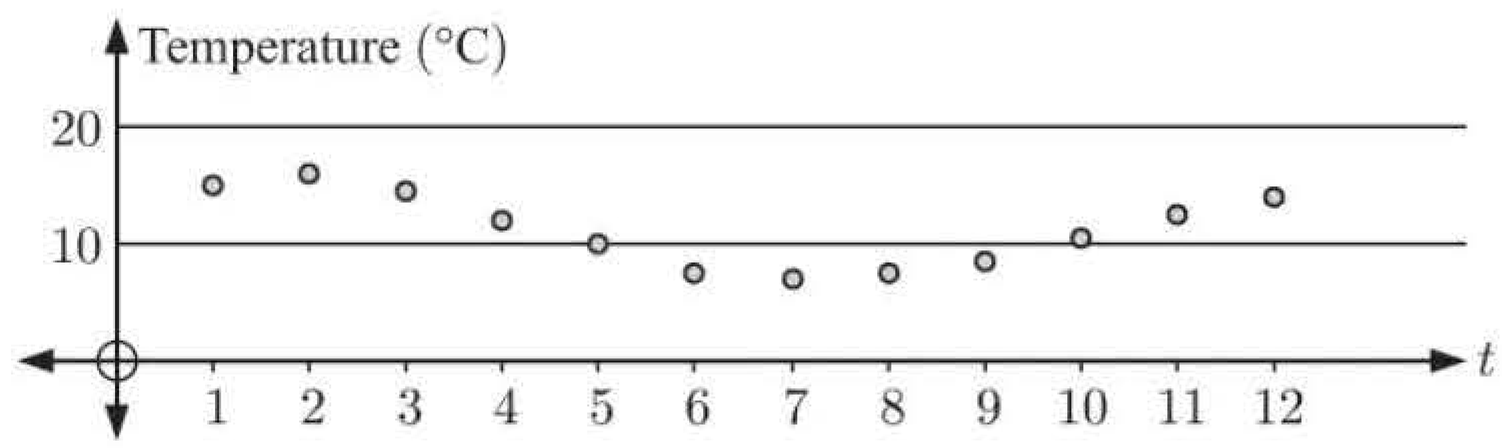
b Using technology, $T \approx 6.14\sin(0.575t - 2.70) + 20.4$

$\therefore\ T \approx 6.14\sin(0.575(t - 4.70)) + 20.4$

The model fits reasonably well.

2 **a**

Month, t	1	2	3	4	5	6	7	8	9	10	11	12
Temp, T	15	16	$14\frac{1}{2}$	12	10	$7\frac{1}{2}$	7	$7\frac{1}{2}$	$8\frac{1}{2}$	$10\frac{1}{2}$	$12\frac{1}{2}$	14



The period is $\dfrac{2\pi}{b} = 12$ $\therefore\ b = \frac{\pi}{6}$ $\{b > 0\}$

Amplitude, $a \approx \dfrac{\text{max.} - \text{min.}}{2} \approx \dfrac{16-7}{2} \approx 4.5$

As the principal axis is midway between min. and max. then $d \approx \dfrac{16+7}{2} \approx 11.5$

At min., $t = 7$ and at max., $t = 2 + 12 = 14$ $\therefore$ $c \approx \dfrac{7+14}{2} \approx 10.5$

So, $T \approx 4.5\sin(\frac{\pi}{6}(t - 10.5)) + 11.5$

b Using technology, $T \approx 4.29\sin(0.533t + 0.769) + 11.2$

$\therefore$ $T \approx 4.29\sin(0.533(t + 1.44)) + 11.2$

Note: (1) $\frac{\pi}{6} \approx 0.524$ ✓

(2) $1.44 - (-10.5)$
$= 11.94 \approx 12$

3 **a** For the model $H = a\sin(b(t - c)) + d$

$$\text{period} = \frac{2\pi}{b} = 12.4 \text{ hours} \quad \therefore \quad b = \frac{2\pi}{12.4} \approx 0.507$$

We let the principal axis be 0, so $d = 0$

$\therefore$ the amplitude $a = 7$, so the min. is -7, and the max. is $+7$

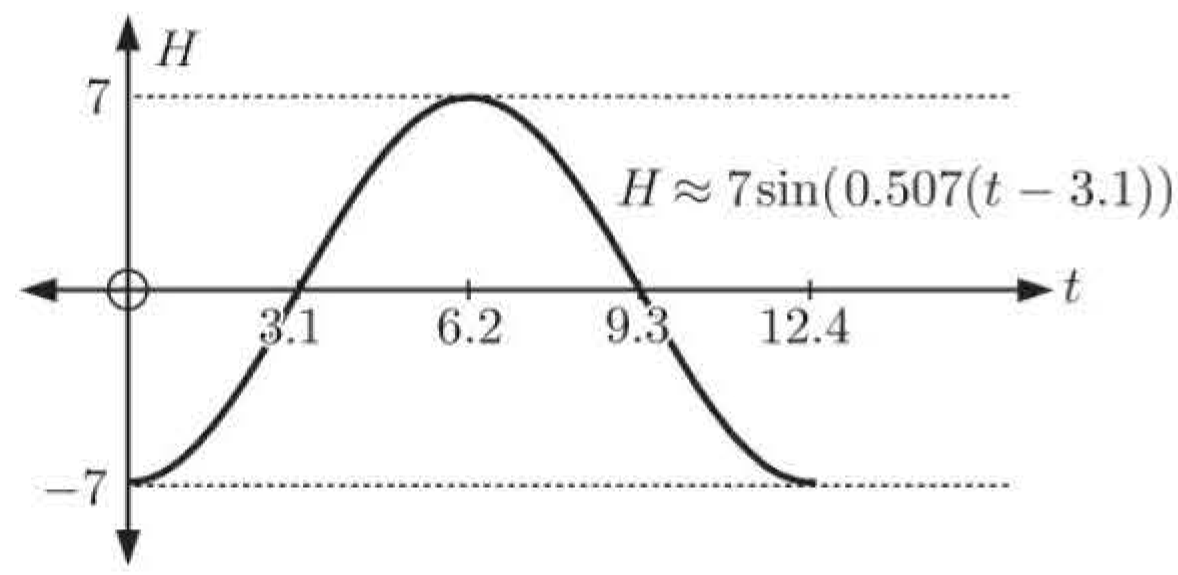
Let $t = 0$ correspond to 'low tide' $\therefore$ $t = 6.2$ corresponds to 'high tide'

$\therefore$ $c = \dfrac{0 + 6.2}{2} = 3.1$

So, $H \approx 7\sin(0.507(t - 3.1)) + 0$

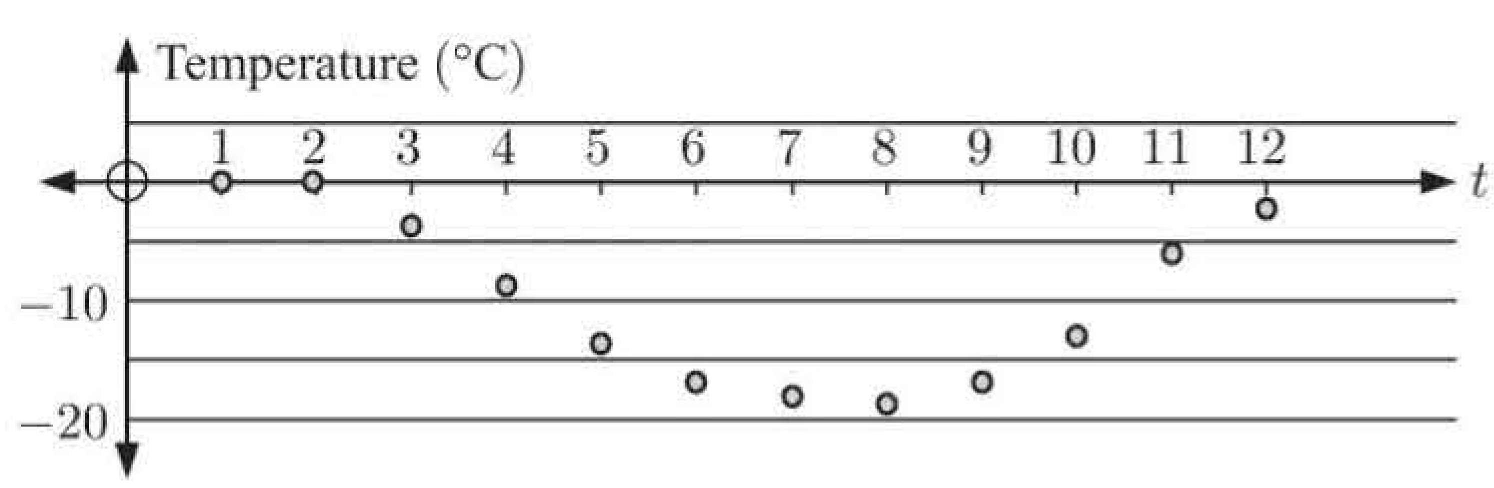
$\therefore$ $H \approx 7\sin(0.507(t - 3.1))$

b



4 **a**

Month, t	1	2	3	4	5	6	7	8	9	10	11	12
Temp, T	0	0	-4	-9	-14	-17	-18	-19	-17	-13	-6	-2



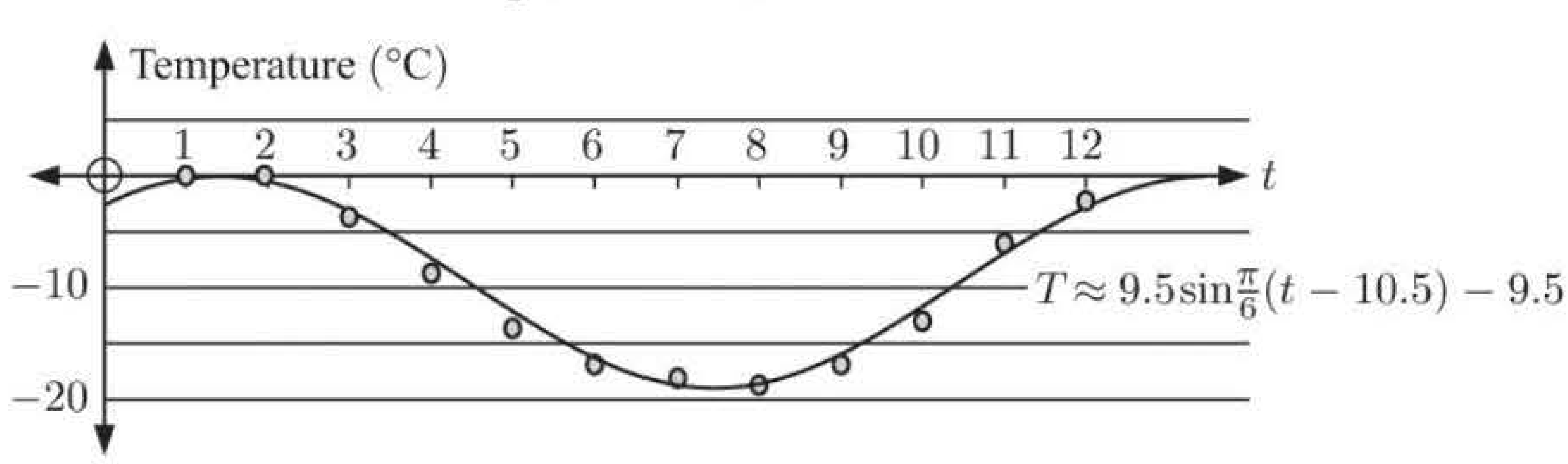
The period is $\dfrac{2\pi}{b} = 12$ $\therefore$ $b = \frac{\pi}{6}$ $\{b > 0\}$

$$\text{Amplitude,} \quad a \approx \frac{\text{max.} - \text{min.}}{2} \approx \frac{0 - (-19)}{2} \approx 9.5$$

$$d \approx \frac{\text{max.} + \text{min.}}{2} \approx \frac{0 + (-19)}{2} \approx -9.5$$

At min., $t = 8$ and at max., $t = 1 + 12 = 13$ $\therefore$ $c \approx \dfrac{8+13}{2} \approx 10.5$

So, $T \approx 9.5\sin(\frac{\pi}{6}(t - 10.5)) - 9.5$



b The model is reasonably appropriate.

5 Let the model be $H = a\sin(b(t-c)) + d$ metres

When $t = 0$, $H = \underset{\text{min.}}{2}$ and when $t = 50$, $H = \underset{\text{max.}}{22}$

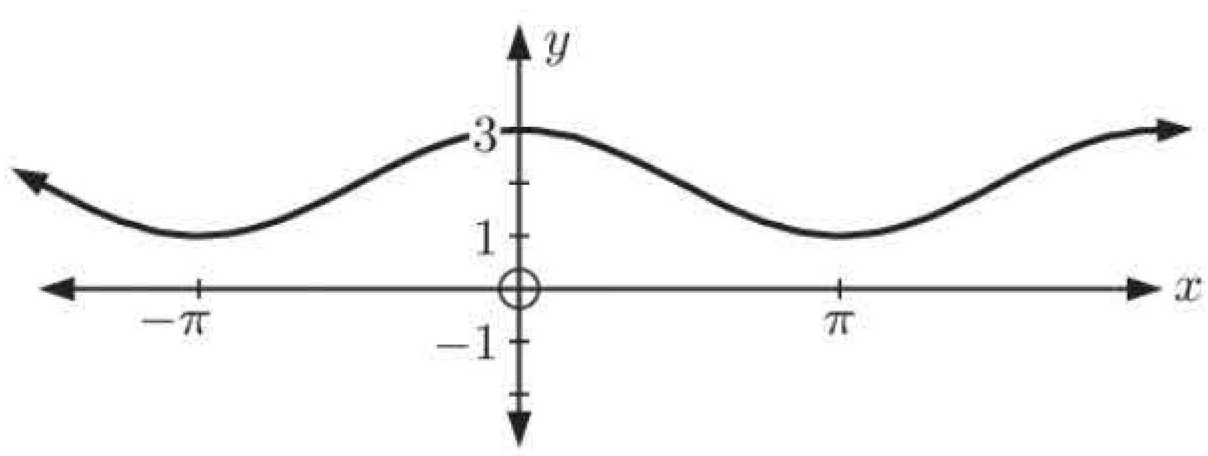
period $= \dfrac{2\pi}{b} = 100 \quad \therefore \quad b = \dfrac{2\pi}{100} = \dfrac{\pi}{50}$

$a = 10$ {from the diagram}, $\quad d = \dfrac{\text{max.} + \text{min.}}{2} = \dfrac{22+2}{2} = 12$

$c = \dfrac{0+50}{2} = 25$ {values of t at min. and max.} $\quad \therefore \quad H = 10\sin(\frac{\pi}{50}(t-25)) + 12$

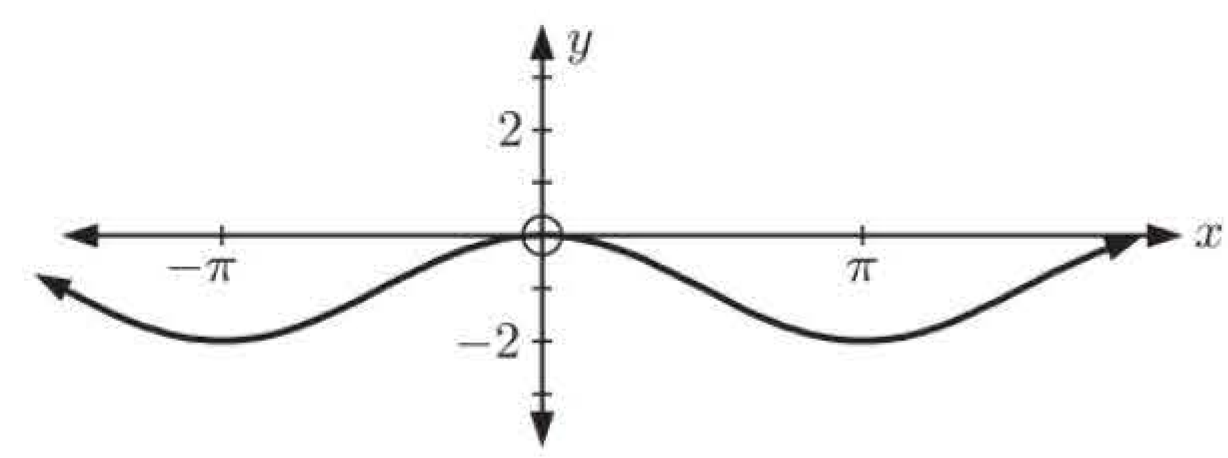
EXERCISE 12D

1 **a** $y = \cos x + 2$



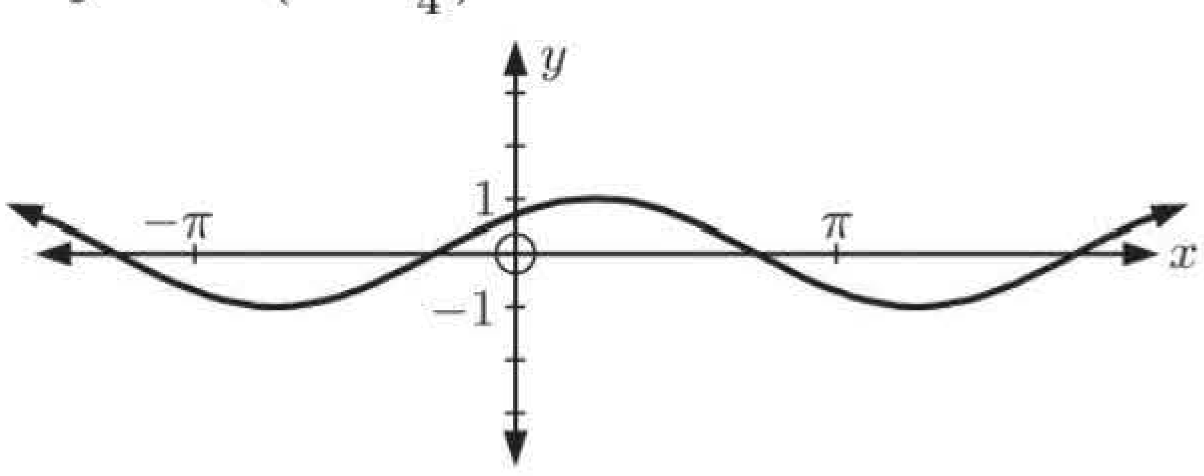
This is a vertical translation of $y = \cos x$ through $\binom{0}{2}$.

b $y = \cos x - 1$



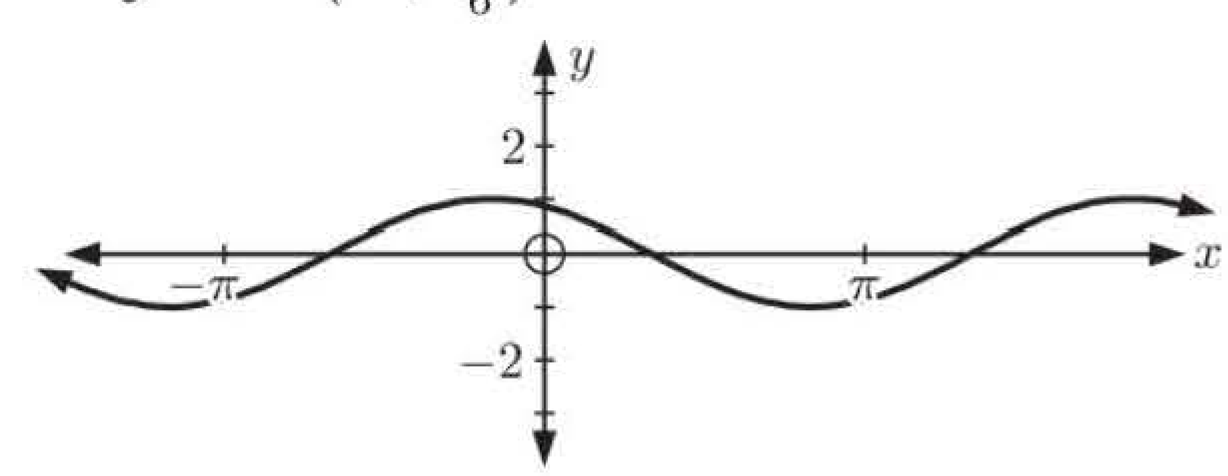
This is a vertical translation of $y = \cos x$ through $\binom{0}{-1}$.

c $y = \cos(x - \frac{\pi}{4})$



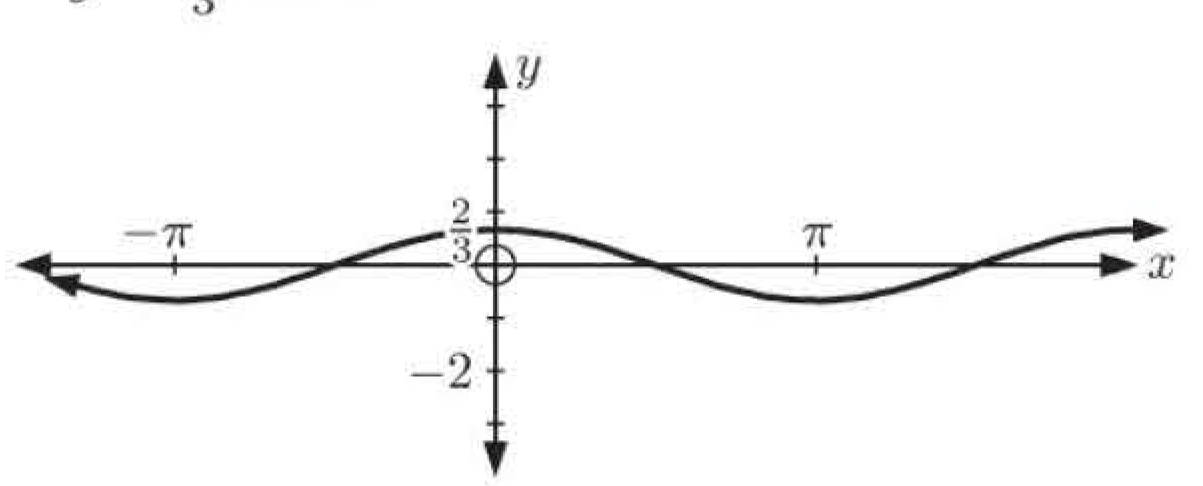
This is a horizontal translation of $y = \cos x$ through $\begin{pmatrix} \frac{\pi}{4} \\ 0 \end{pmatrix}$.

d $y = \cos(x + \frac{\pi}{6})$



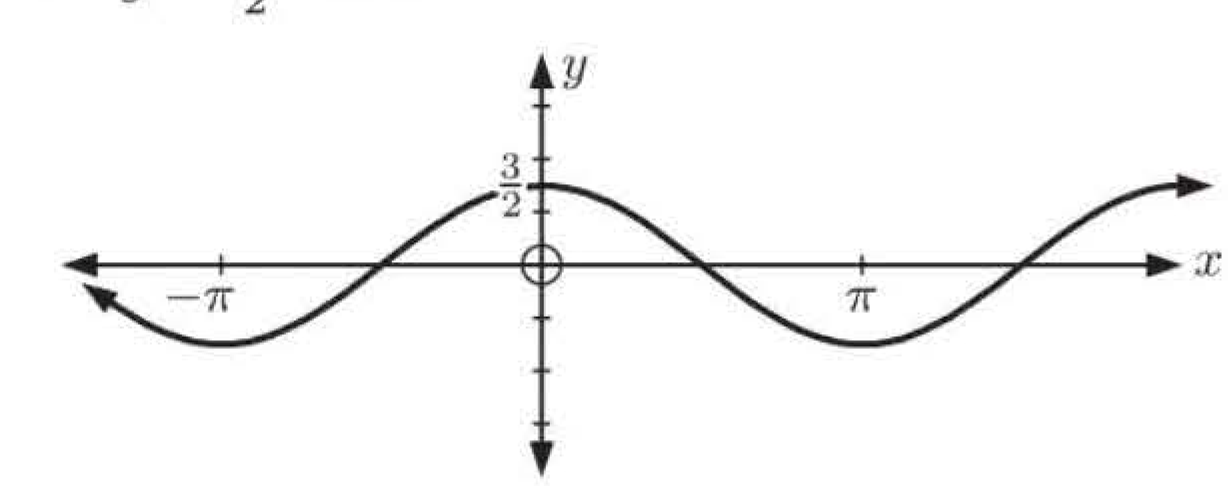
This is a horizontal translation of $y = \cos x$ through $\begin{pmatrix} -\frac{\pi}{6} \\ 0 \end{pmatrix}$.

e $y = \frac{2}{3}\cos x$



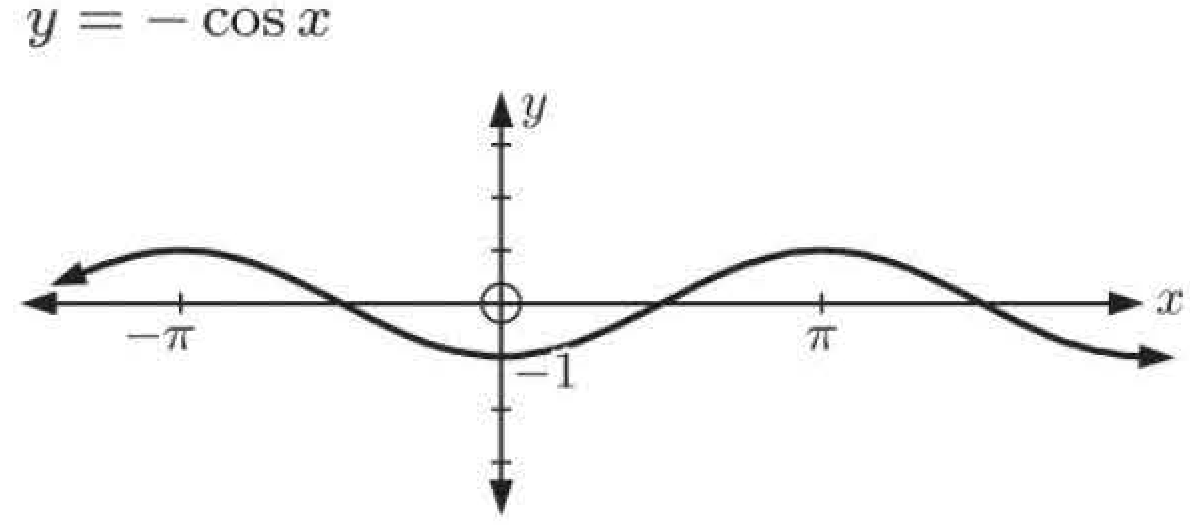
This is a vertical stretch of $y = \cos x$ with factor $\frac{2}{3}$.

f $y = \frac{3}{2}\cos x$



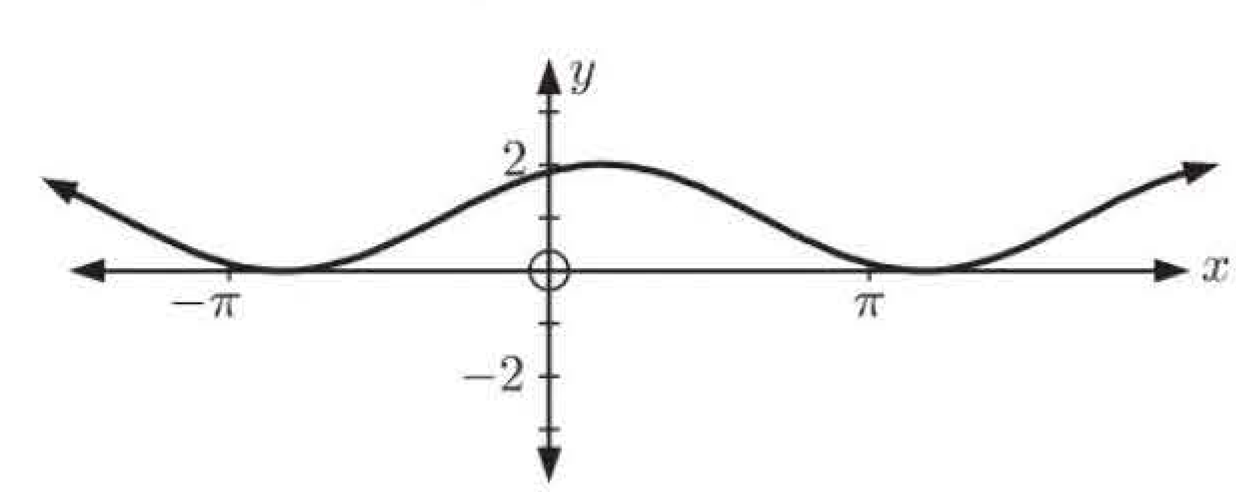
This is a vertical stretch of $y = \cos x$ with factor $\frac{3}{2}$.

g $y = -\cos x$



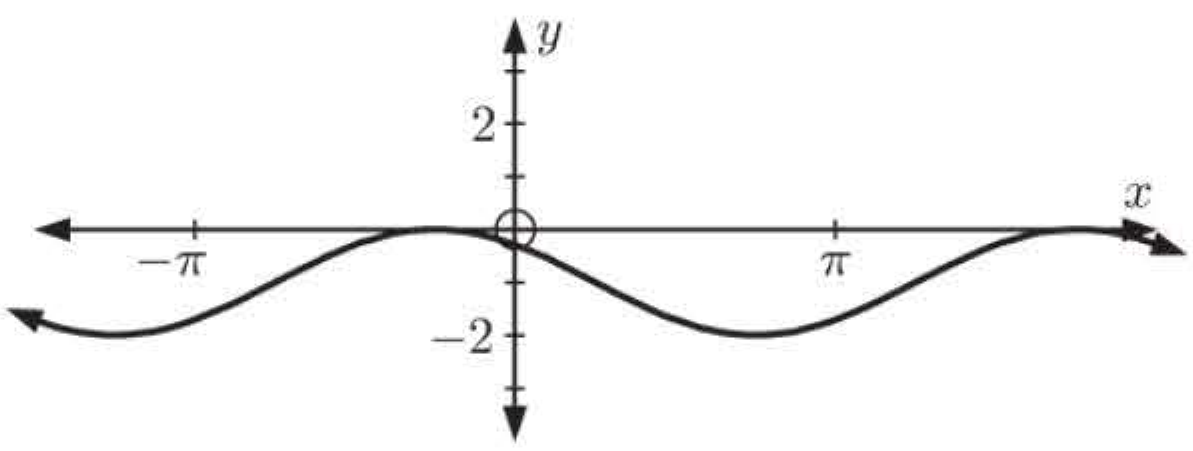
This is a reflection of $y = \cos x$ in the x-axis.

h $y = \cos(x - \frac{\pi}{6}) + 1$



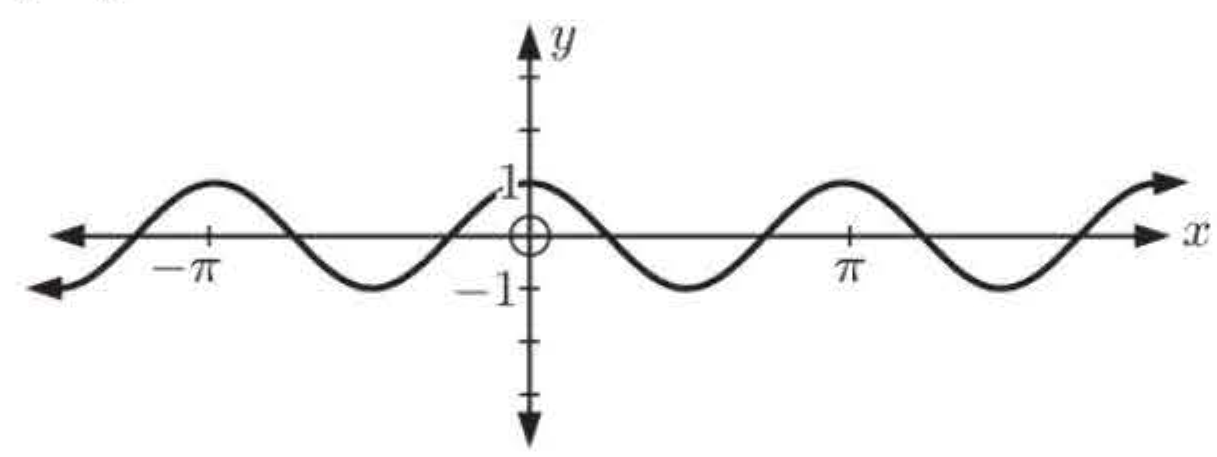
This is a translation of $\begin{pmatrix} \frac{\pi}{6} \\ 1 \end{pmatrix}$.

i $y = \cos(x + \frac{\pi}{4}) - 1$



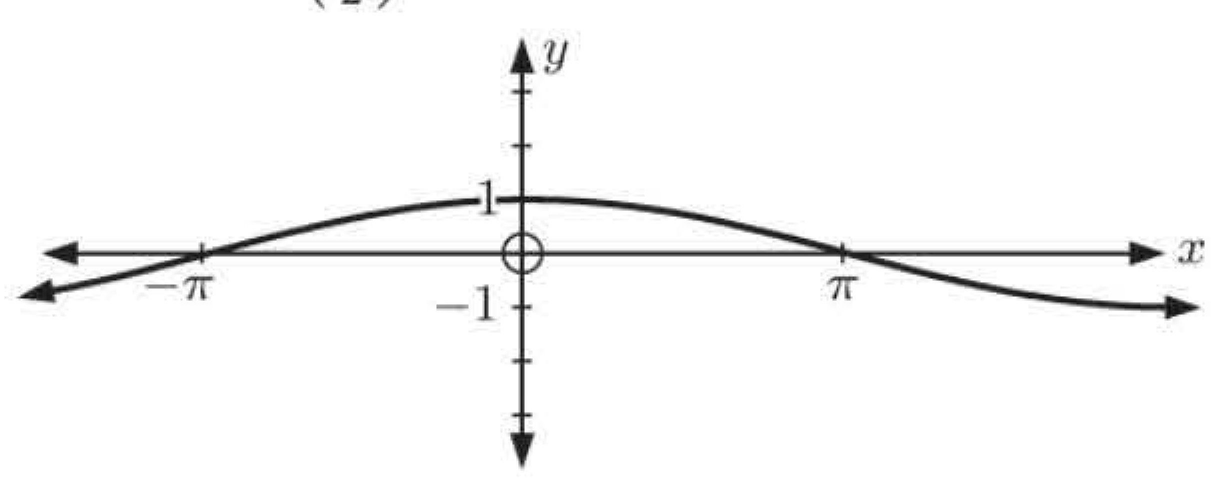
This is a translation of $\begin{pmatrix} -\frac{\pi}{4} \\ -1 \end{pmatrix}$.

j $y = \cos 2x$



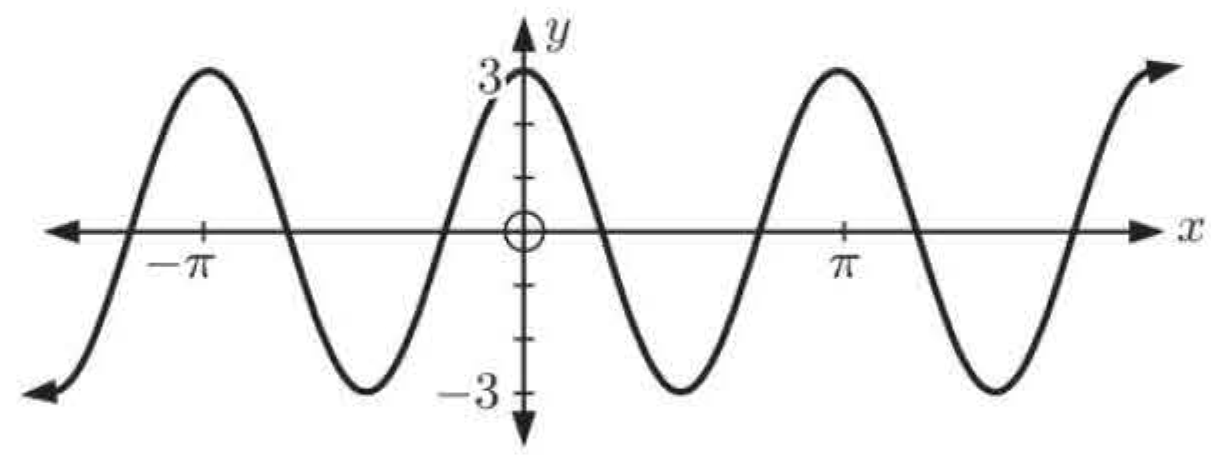
This is a horizontal stretch of factor $\frac{1}{2}$.

k $y = \cos\left(\frac{x}{2}\right)$



This is a horizontal stretch of factor 2.

l $y = 3\cos 2x$



This is a horizontal stretch of factor $\frac{1}{2}$ followed by a vertical stretch of factor 3.

2 **a** period $= \frac{2\pi}{3}$ **b** period $= \dfrac{2\pi}{\frac{1}{3}} = 6\pi$ **c** period $= \dfrac{2\pi}{\frac{\pi}{50}} = 100$

3 a controls the amplitude {amplitude $= |a|$}. b controls the period {period $= \dfrac{2\pi}{|b|}$}.

c controls the horizontal translation. d controls the vertical translation.

4 **a** If $y = a\cos(b(x - c)) + d$, then $a = 2$, $\pi = \dfrac{2\pi}{b}$ $\therefore$ $b = 2$

c and d are 0 as there is no horizontal or vertical shift. $\therefore$ $y = 2\cos(2x)$

b If $y = a\cos(b(x - c)) + d$, then $a = 1$, $4\pi = \dfrac{2\pi}{b}$ $\therefore$ $b = \frac{1}{2}$

A vertical shift of 2 units, no horizontal shift $\therefore$ $d = 2$, $c = 0$.

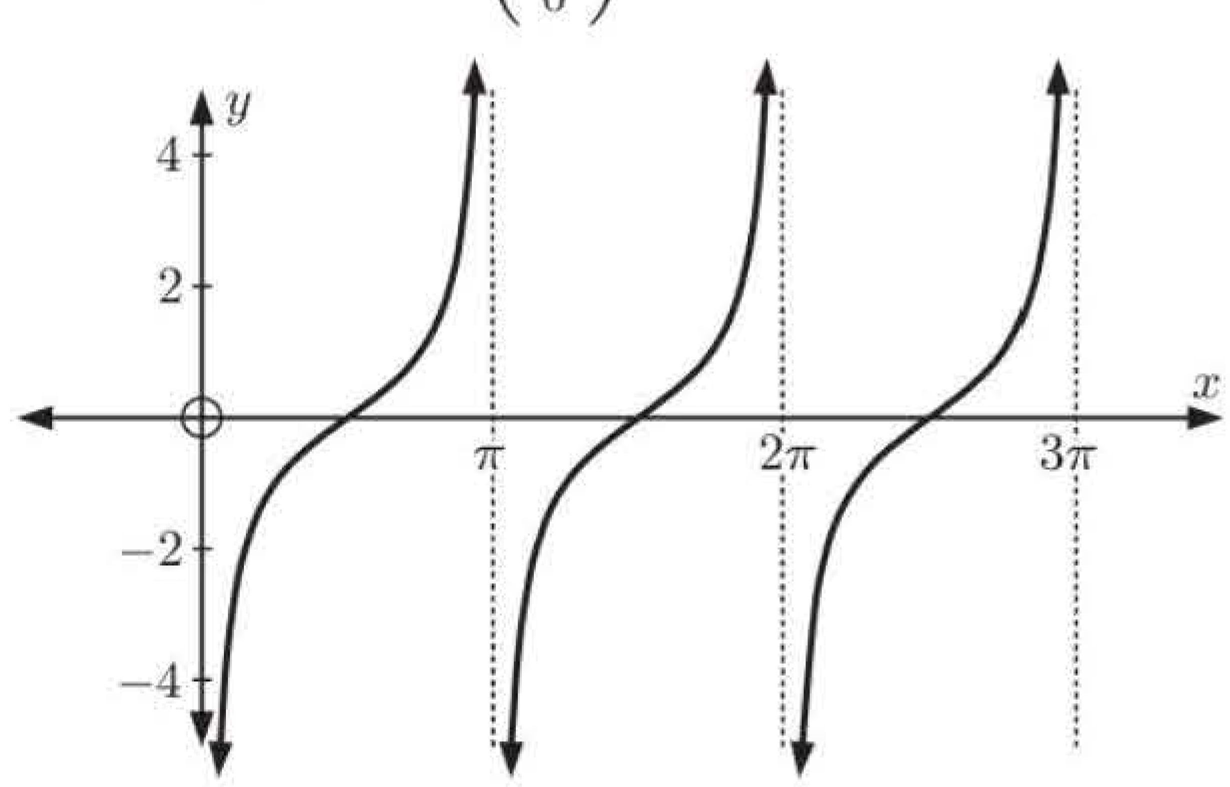
So, $y = \cos(\frac{1}{2}x) + 2$ or $y = \cos\left(\frac{x}{2}\right) + 2$.

c If $y = a\cos(b(x - c)) + d$, then $a = -5$, $6 = \dfrac{2\pi}{b}$ $\therefore$ $b = \frac{\pi}{3}$

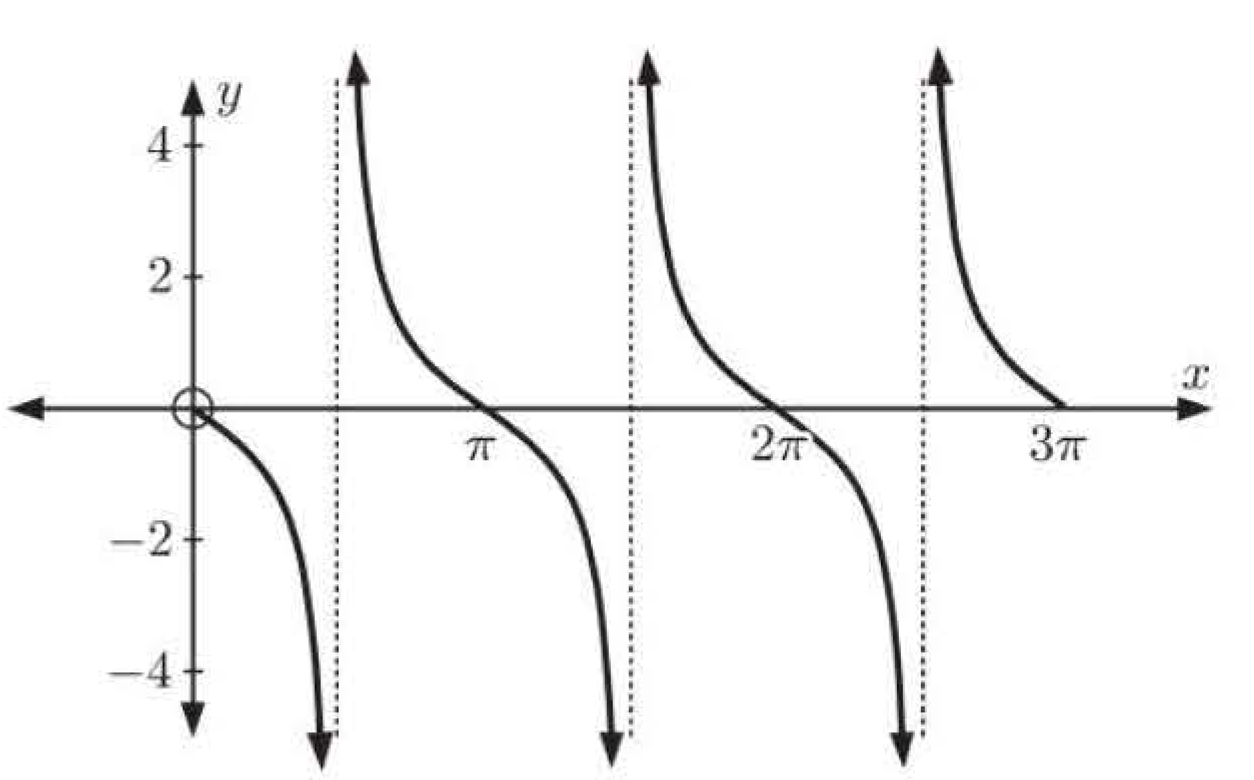
$c = d = 0$ {as there is no translation} $\therefore$ $y = -5\cos\left(\frac{\pi}{3}x\right)$

EXERCISE 12E

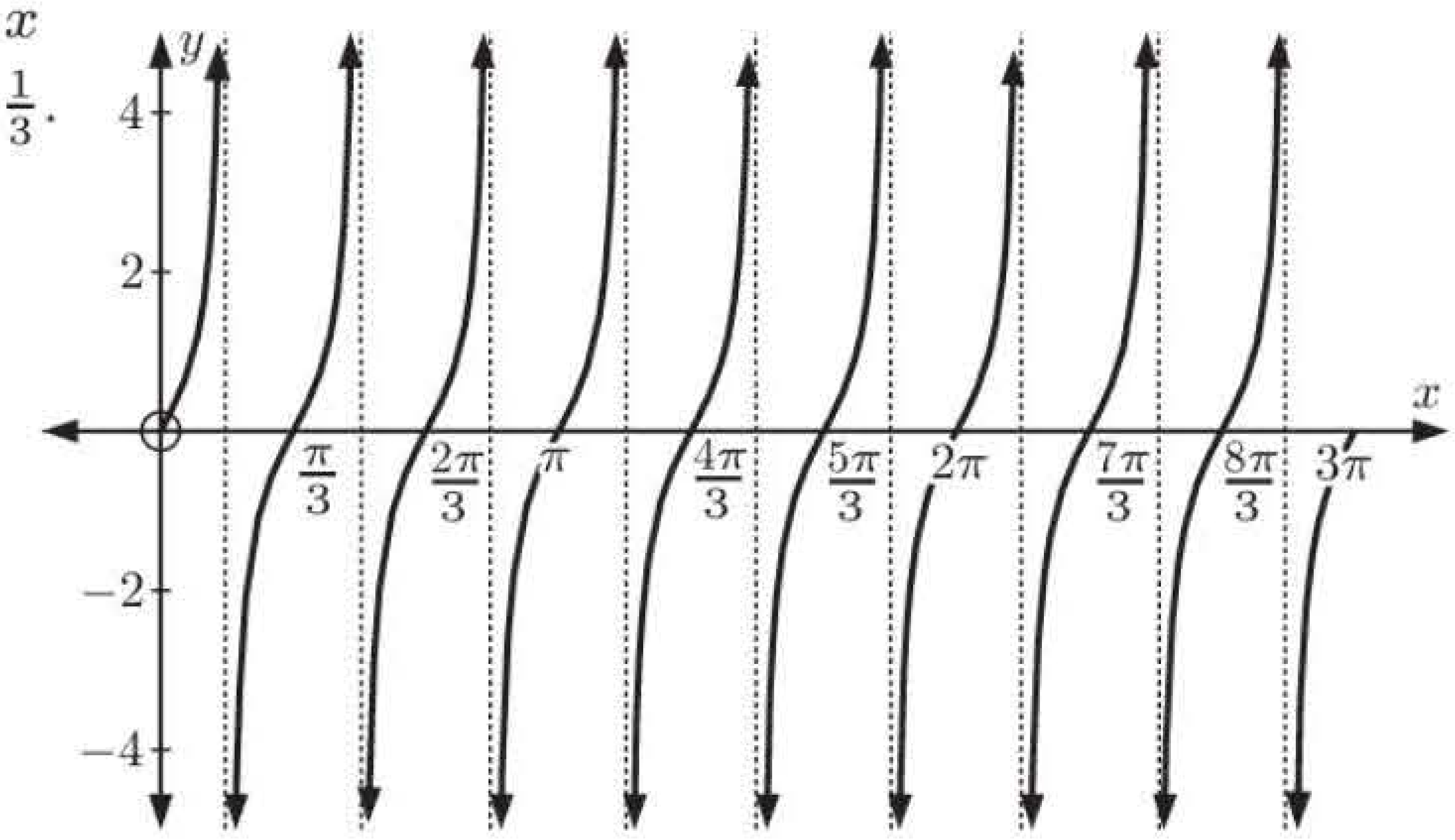
1 **a** **i** $y = \tan(x - \frac{\pi}{2})$ is $y = \tan x$ translated $\begin{pmatrix} \frac{\pi}{2} \\ 0 \end{pmatrix}$.



ii $y = -\tan x$ is $y = \tan x$ reflected in the x-axis.



iii $y = \tan 3x$ comes from $y = \tan x$ under a horizontal stretch of factor $\frac{1}{3}$.



2 **a** translation through $\binom{1}{2}$ **b** reflection in x-axis

c horizontal stretch, factor 2; vertical stretch, factor 2

3 **a** period $= \frac{\pi}{1} = \pi$ **b** period $= \frac{\pi}{3}$ **c** period $= \frac{\pi}{n}$

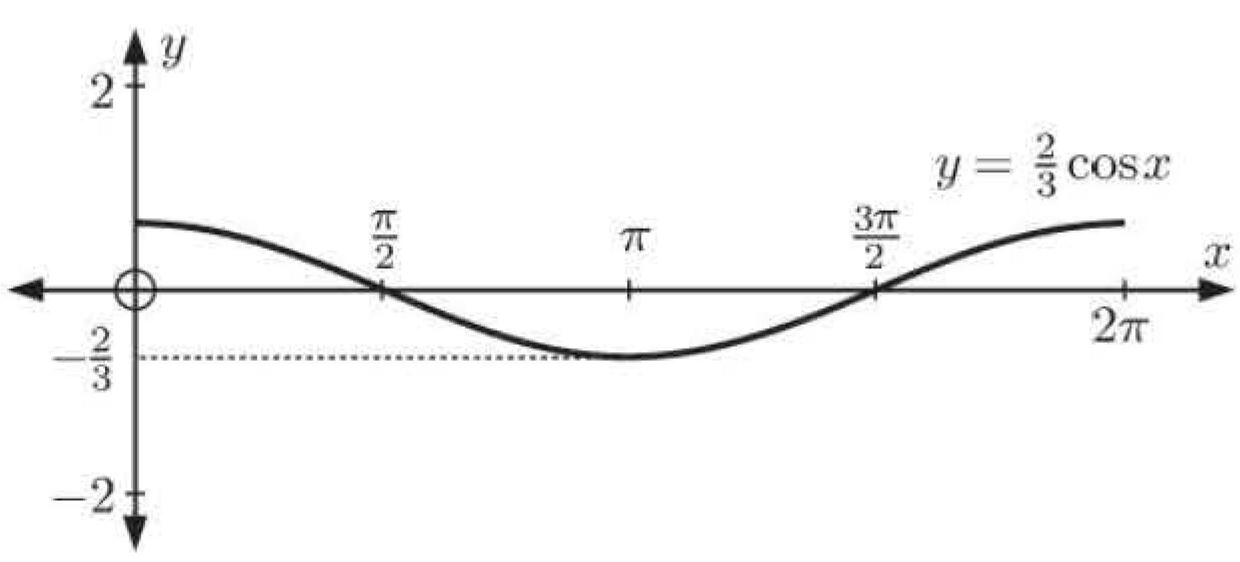
EXERCISE 12F

1 **a** amplitude $= |1| = 1$ **b** amplitude undefined **c** amplitude $= |-1| = 1$

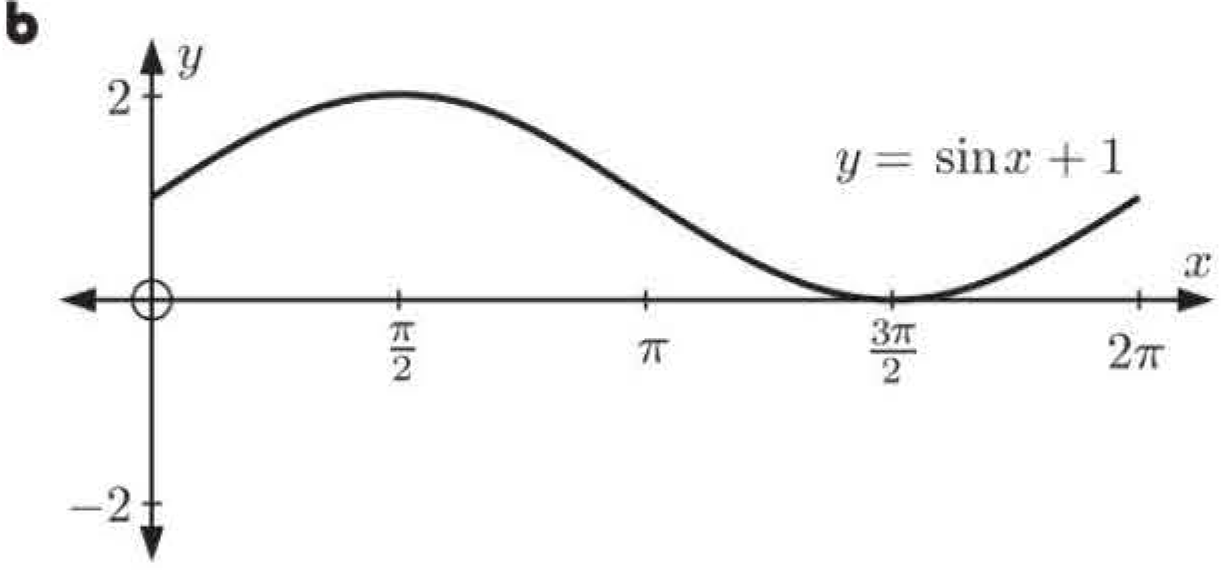
2 **a** period $= \dfrac{\pi}{1} = \pi$ **b** period $= \dfrac{2\pi}{\frac{1}{3}} = 6\pi$ **c** period $= \dfrac{2\pi}{2} = \pi$

3 **a** $\dfrac{2\pi}{b} = 2\pi$ $\therefore\ b = 1$ **b** $\dfrac{2\pi}{b} = \dfrac{2\pi}{3}$ $\therefore\ b = 3$ **c** $\dfrac{\pi}{b} = \dfrac{\pi}{2}$ $\therefore\ b = 2$ **d** $\dfrac{2\pi}{b} = 4$ $\therefore\ b = \frac{\pi}{2}$

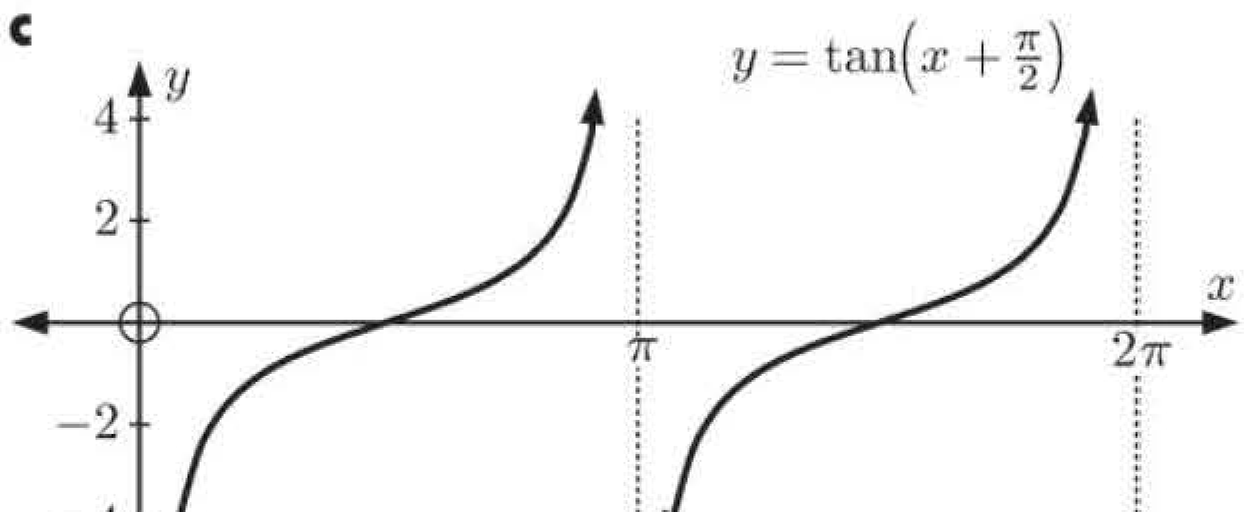
4 **a**



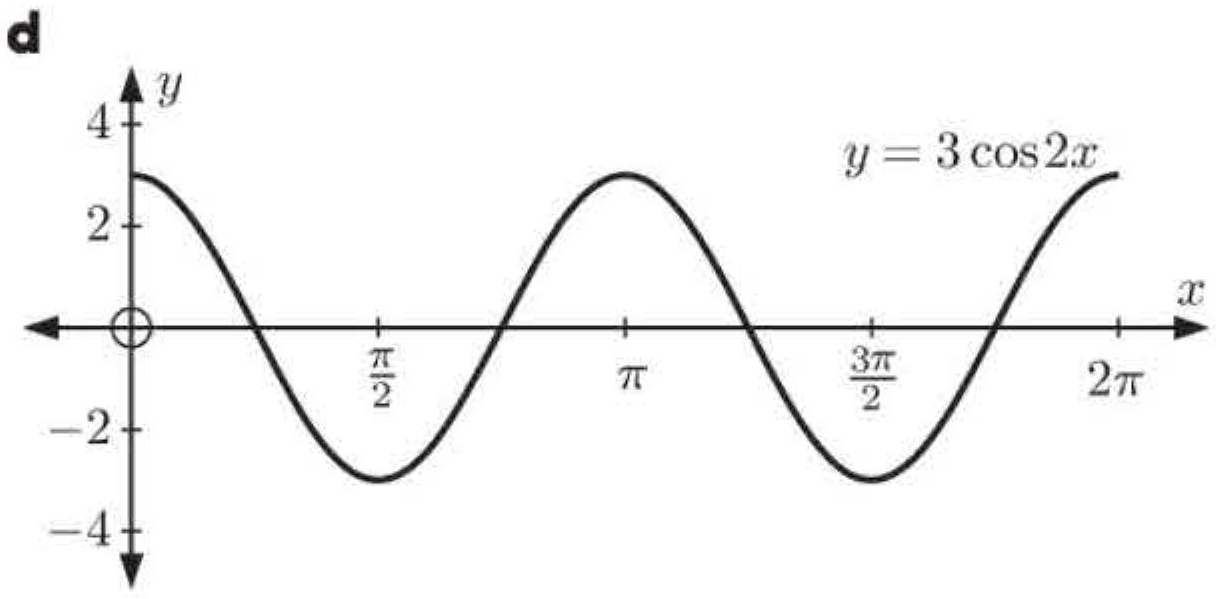
b



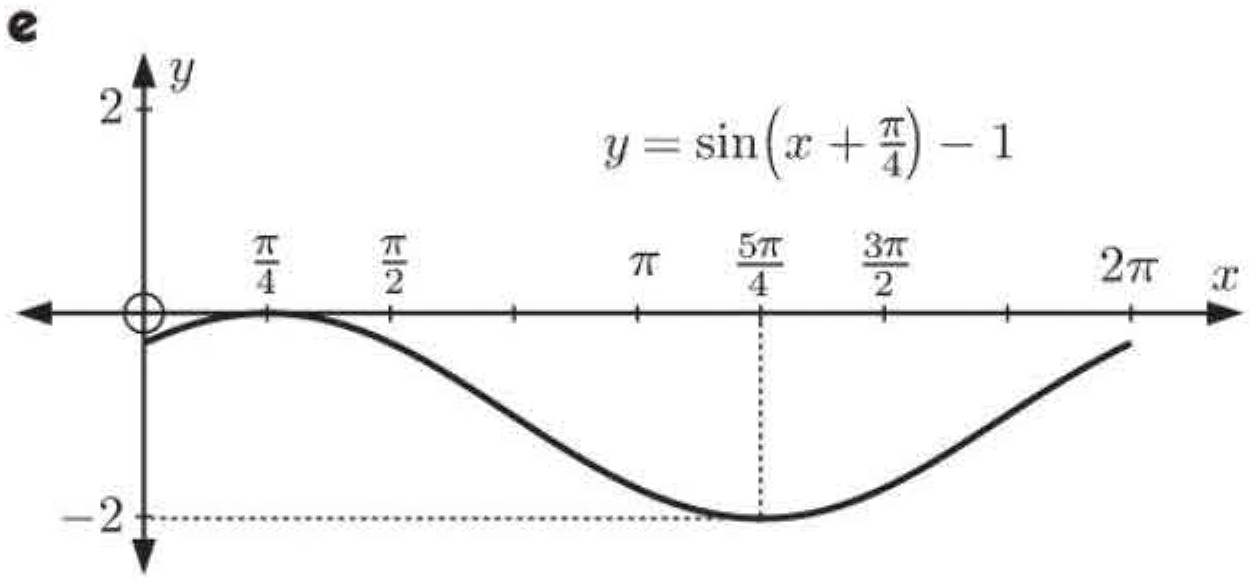
c



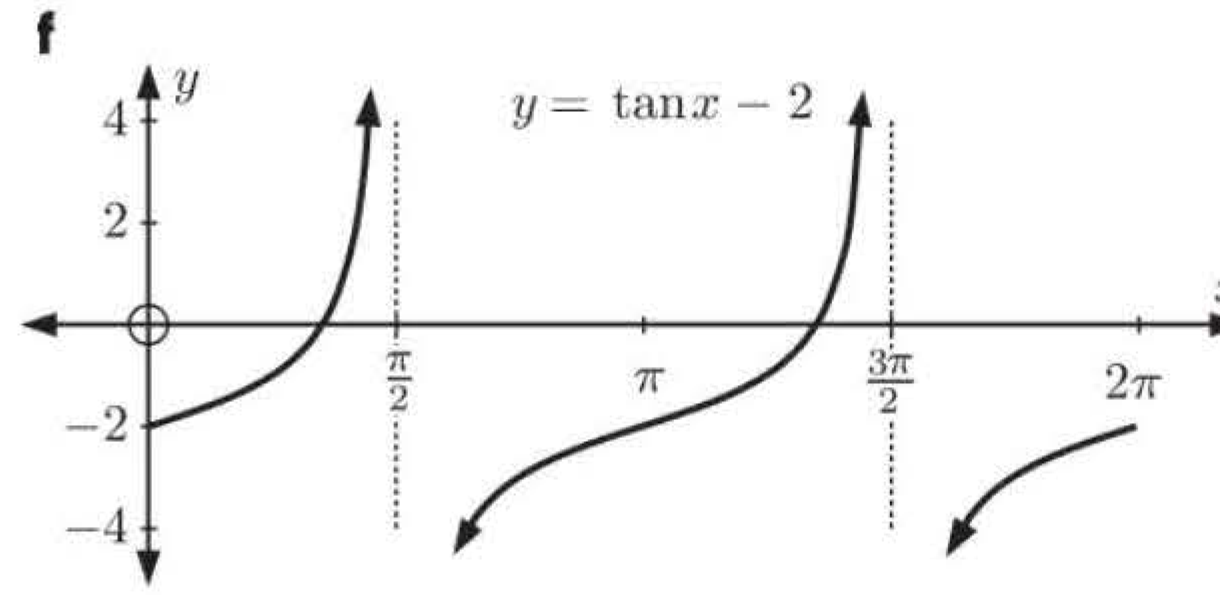
d



e



f



5 **a** $y = -\sin 5x$ has maximum value $-(-1) = 1$ $\{$when $\sin 5x = -1\}$
and minimum value $-(1) = -1$ $\{$when $\sin 5x = 1\}$

b $y = 3\cos x$ has maximum value $3(1) = 3$ $\{$when $\cos x = 1\}$
and minimum value $3(-1) = -3$ $\{$when $\cos x = -1\}$

c $y = 2\tan x$ has no maximum or minimum values.

d $y = -\cos 2x + 3$ has maximum value $-(-1) + 3 = 4$ $\{$when $\cos 2x = -1\}$
and minimum value $-(1) + 3 = 2$ $\{$when $\cos 2x = 1\}$

e $y = 1 + 2\sin x$ has maximum value $1 + 2(1) = 3$ $\{$when $\sin x = 1\}$
and minimum value $1 + 2(-1) = -1$ $\{$when $\sin x = -1\}$

f $y = \sin\left(x - \frac{\pi}{2}\right) - 3$ has maximum value $1 - 3 = -2$ $\{$when $\sin\left(x - \frac{\pi}{2}\right) = 1\}$
and minimum value $-1 - 3 = -4$ $\{$when $\sin\left(x - \frac{\pi}{2}\right) = -1\}$

6 **a** vertical stretch, factor $\frac{1}{2}$
b horizontal stretch, factor 4
c reflection in the x-axis
d vertical translation down 2 units
e horizontal translation $\frac{\pi}{4}$ units to the left
f reflection in the y-axis

7 The amplitude is 2, so $m = 2$.
The principal axis is $y = -3$, so $n = -3$.

8 The period is 2π, so $\dfrac{\pi}{p} = 2\pi$

$\therefore\ p = \frac{1}{2}$

The graph has undergone a vertical translation of 1 unit, so $q = 1$.

EXERCISE 12G

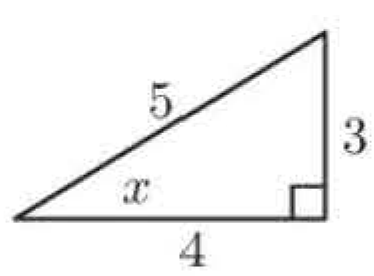
1 **a** $\sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$
$\therefore\ \csc(\frac{\pi}{3}) = \frac{2}{\sqrt{3}}$

b $\tan(\frac{2\pi}{3}) = -\sqrt{3}$
$\therefore\ \cot(\frac{2\pi}{3}) = -\frac{1}{\sqrt{3}}$

c $\cos(\frac{5\pi}{6}) = -\frac{\sqrt{3}}{2}$
$\therefore\ \sec(\frac{5\pi}{6}) = -\frac{2}{\sqrt{3}}$

d $\tan(\pi) = 0$
$\therefore$ $\cot(\pi)$ is undefined.

2 **a** $\sin x = \frac{3}{5},\ 0 \leqslant x \leqslant \frac{\pi}{2}$

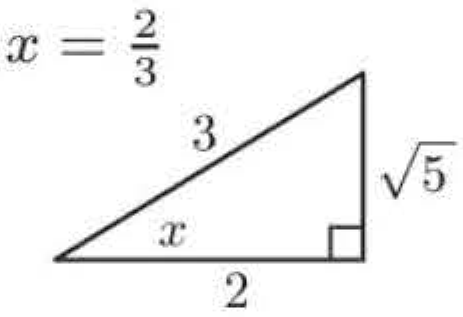


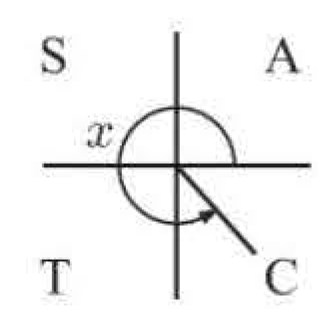
$\therefore\ \csc x = \dfrac{1}{\sin x} = \frac{5}{3}$

$\sec x = \dfrac{1}{\cos x} = \frac{5}{4}$

$\cot x = \dfrac{1}{\tan x} = \frac{4}{3}$

b $\cos x = \frac{2}{3}$





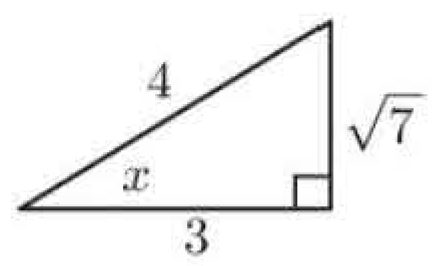
$\therefore\ \sin x = -\frac{\sqrt{5}}{3}$ and $\tan x = -\frac{\sqrt{5}}{2}$

$\therefore\ \csc x = -\frac{3}{\sqrt{5}}$

$\sec x = \frac{3}{2}$

$\cot x = -\frac{2}{\sqrt{5}}$

3 **a** $\cos x = \frac{3}{4}$



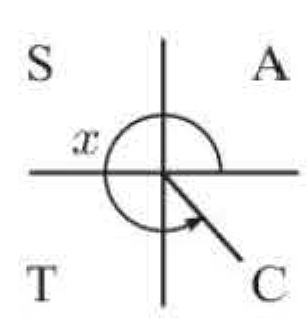
$\therefore\ \sin x = -\frac{\sqrt{7}}{4}$

$\tan x = -\frac{\sqrt{7}}{3}$

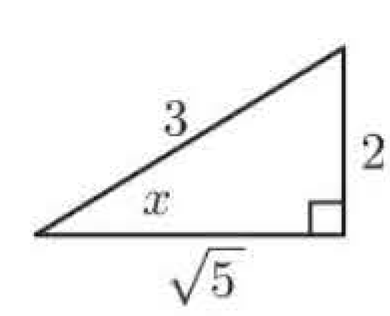
$\csc x = -\frac{4}{\sqrt{7}}$

$\sec x = \frac{4}{3}$

$\cot x = -\frac{3}{\sqrt{7}}$



b $\sin x = -\frac{2}{3}$



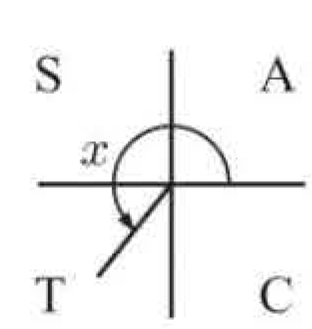
$\therefore\ \cos x = -\frac{\sqrt{5}}{3}$

$\tan x = \frac{2}{\sqrt{5}}$

$\csc x = -\frac{3}{2}$

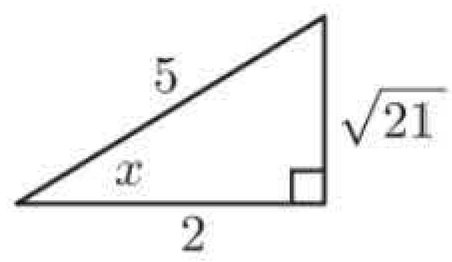
$\sec x = -\frac{3}{\sqrt{5}}$

$\cot x = \frac{\sqrt{5}}{2}$



c $\sec x = \frac{5}{2}$

$\therefore \quad \cos x = \frac{2}{5}$

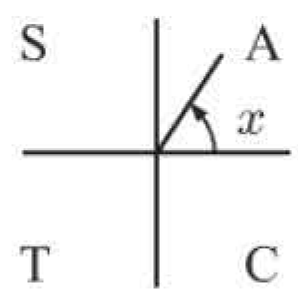


$\therefore \quad \sin x = \frac{\sqrt{21}}{5}$

$\tan x = \frac{\sqrt{21}}{2}$

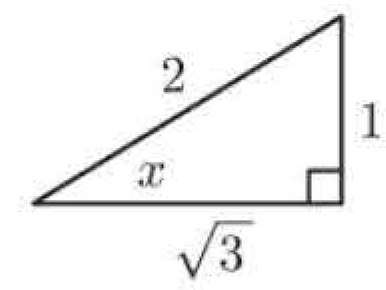
$\csc x = \frac{5}{\sqrt{21}}$

$\cot x = \frac{2}{\sqrt{21}}$



d $\csc x = 2$

$\therefore \quad \sin x = \frac{1}{2}$

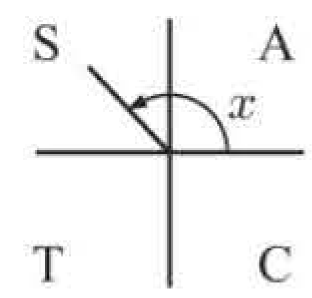


$\therefore \quad \cos x = -\frac{\sqrt{3}}{2}$

$\tan x = -\frac{1}{\sqrt{3}}$

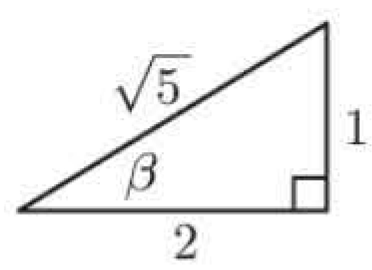
$\sec x = -\frac{2}{\sqrt{3}}$

$\cot x = -\sqrt{3}$



e $\tan \beta = \frac{1}{2}$

$\therefore \quad \cot \beta = 2$

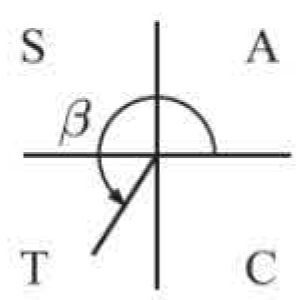


$\therefore \quad \sin \beta = -\frac{1}{\sqrt{5}}$

$\cos \beta = -\frac{2}{\sqrt{5}}$

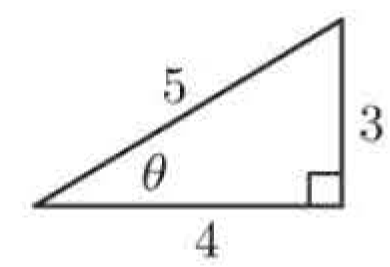
$\csc \beta = -\sqrt{5}$

$\sec \beta = -\frac{\sqrt{5}}{2}$



f $\cot \theta = \frac{4}{3}$

$\therefore \quad \tan \theta = \frac{3}{4}$

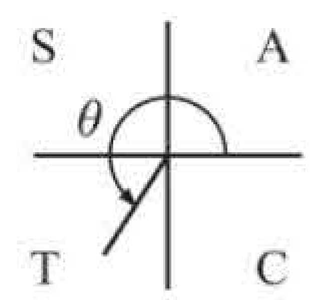


$\therefore \quad \sin \theta = -\frac{3}{5}$

$\cos \theta = -\frac{4}{5}$

$\csc \theta = -\frac{5}{3}$

$\sec \theta = -\frac{5}{4}$



4 **a** $\tan x \cot x$

$= \dfrac{\sin x}{\cos x} \times \dfrac{\cos x}{\sin x}$

$= 1$

b $\sin x \csc x$

$= \sin x \times \dfrac{1}{\sin x}$

$= 1$

c $\csc x \cot x$

$= \dfrac{1}{\sin x} \times \dfrac{\cos x}{\sin x}$

$= \dfrac{\cos x}{\sin^2 x}$

d $\sin x \cot x$

$= \sin x \times \dfrac{\cos x}{\sin x}$

$= \cos x$

e $\dfrac{\cot x}{\csc x}$

$= \dfrac{\cos x}{\sin x} \div \dfrac{1}{\sin x}$

$= \dfrac{\cos x}{\sin x} \times \dfrac{\sin x}{1}$

$= \cos x$

f $\dfrac{2 \sin x \cot x + 3 \cos x}{\cot x}$

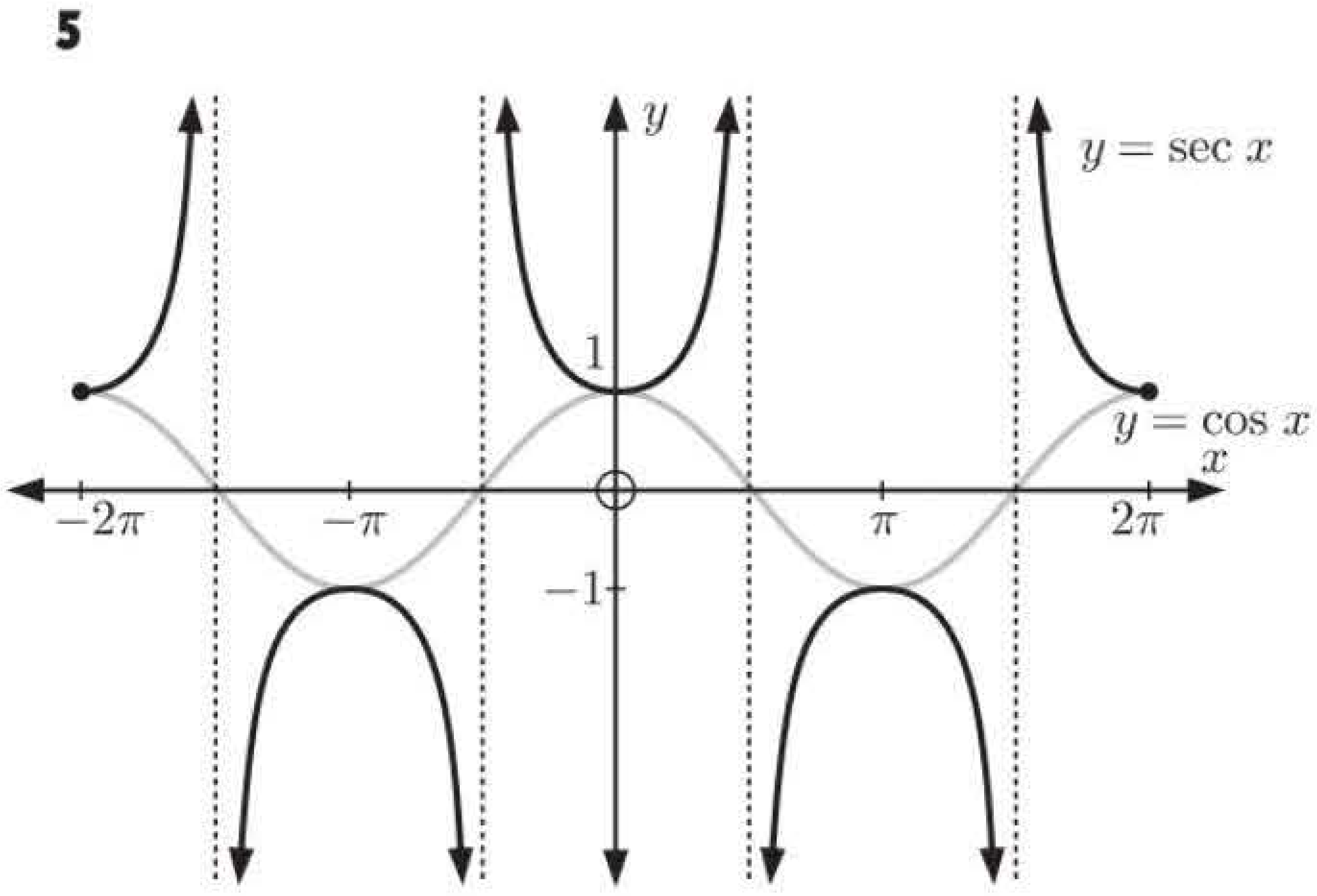
$= \dfrac{2 \sin x \times \dfrac{\cos x}{\sin x} + 3 \cos x}{\dfrac{\cos x}{\sin x}}$

$= (2 \cos x + 3 \cos x) \times \dfrac{\sin x}{\cos x}$

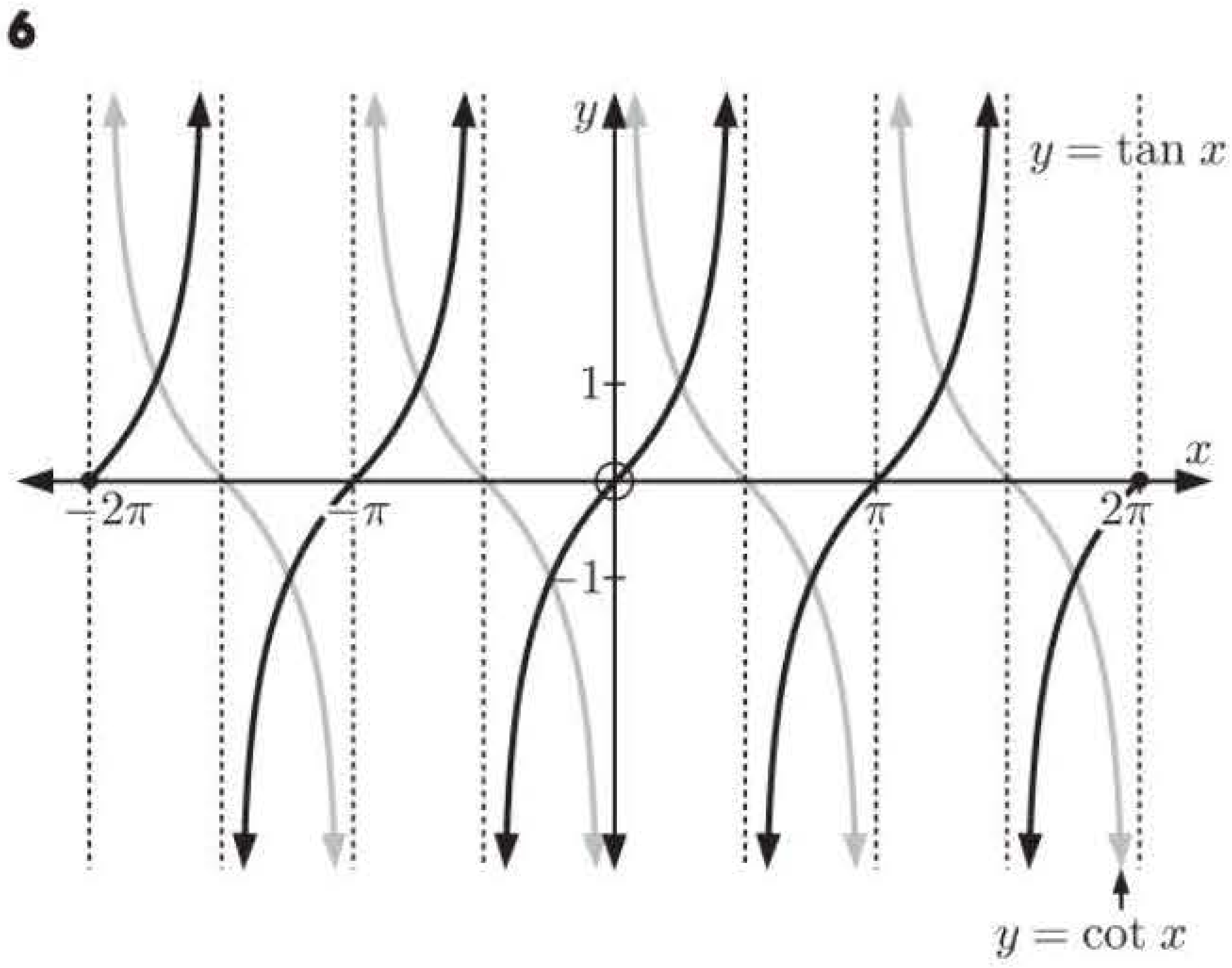
$= 5 \cos x \times \dfrac{\sin x}{\cos x}$

$= 5 \sin x$

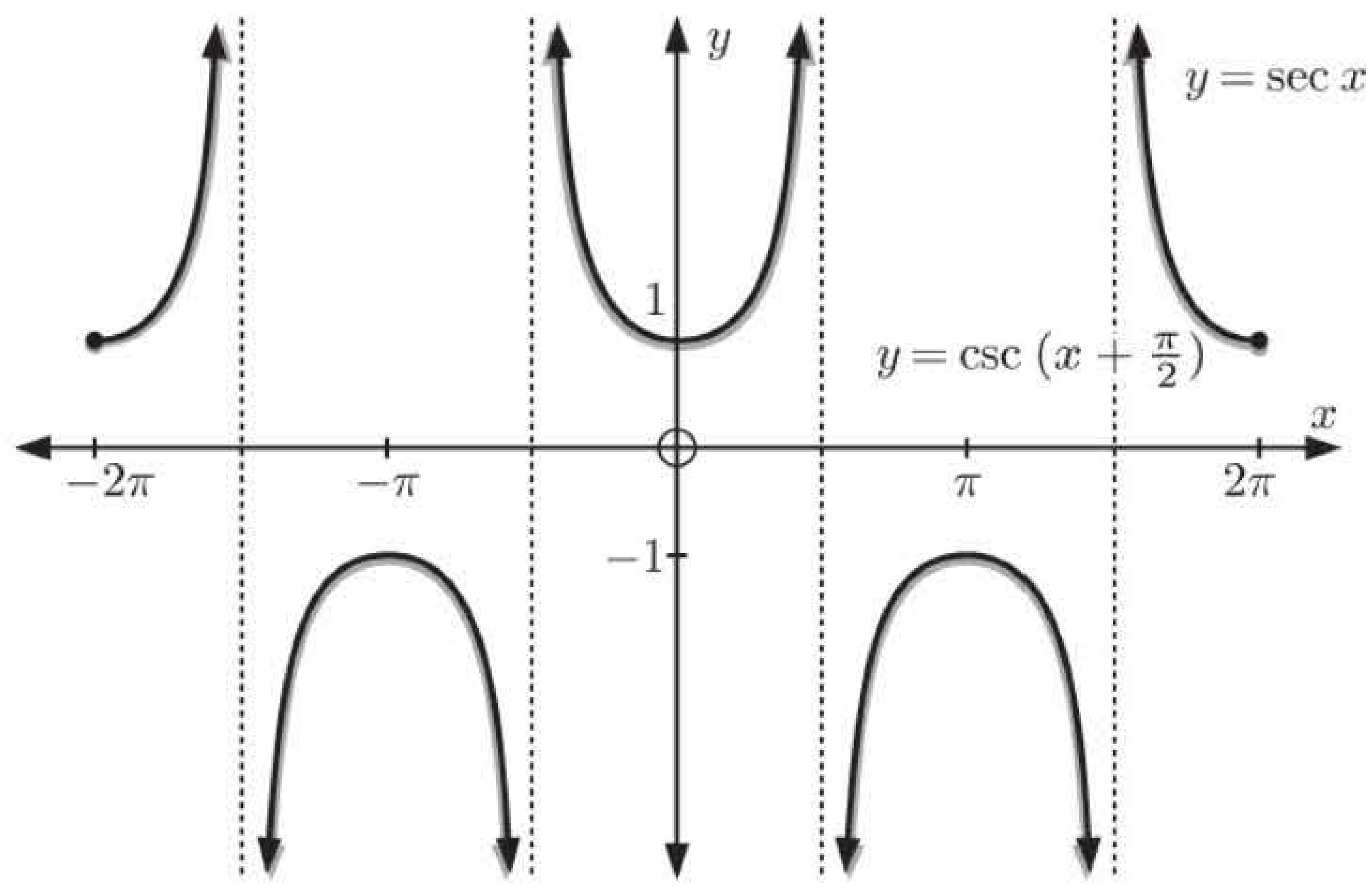
5



6



7



$\sin(x+\frac{\pi}{2}) = \cos x \quad \therefore \quad \csc(x+\frac{\pi}{2}) = \dfrac{1}{\sin(x+\frac{\pi}{2})} = \dfrac{1}{\cos x} = \sec x$

EXERCISE 12H

1

Function	*Restricted domain*	*Restricted range*	*Inverse function*	*Domain*	*Range*
$y = \sin x$	$-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$	$-1 \leqslant y \leqslant 1$	$y = \arcsin x$	$-1 \leqslant x \leqslant 1$	$-\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$
$y = \cos x$	$0 \leqslant x \leqslant \pi$	$-1 \leqslant y \leqslant 1$	$y = \arccos x$	$-1 \leqslant x \leqslant 1$	$0 \leqslant y \leqslant \pi$
$y = \tan x$	$-\frac{\pi}{2} < x < \frac{\pi}{2}$	$y \in \mathbb{R}$	$y = \arctan x$	$x \in \mathbb{R}$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$

2 **a** $\arccos(1) = 0$ **b** $\arcsin(-1) = -\frac{\pi}{2}$ **c** $\arctan(1) = \frac{\pi}{4}$

d $\arctan(-1) = -\frac{\pi}{4}$ **e** $\arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$ **f** $\arccos\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$

g $\arctan(\sqrt{3}) = \frac{\pi}{3}$ **h** $\arccos\left(-\frac{1}{\sqrt{2}}\right) = \frac{3\pi}{4}$ **i** $\arctan\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$

j $\sin^{-1}(-0.767) \approx -0.874$ **k** $\cos^{-1}(0.327) \approx 1.24$ **l** $\tan^{-1}(-50) \approx -1.55$

3 **a** The inverse transformation from $y = \sin x$ to $y = \arcsin x$ has an invariant point where $\sin x = \arcsin x \quad \therefore$ at $(0, 0)$.

b The inverse transformation from $y = \tan x$ to $y = \arctan x$ has an invariant point where $\tan x = \arctan x \quad \therefore$ at $(0, 0)$.

c The inverse transformation from $y = \cos x$ to $y = \arccos x$ has an invariant point where $\cos x = \arccos x \quad \therefore$ at $(0.739, 0.739)$.

4 **a** $y = \arctan x$ has horizontal asymptotes $y = -\frac{\pi}{2}$ and $y = \frac{\pi}{2}$.

b The functions $y = \arcsin x$ and $y = \arccos x$ each have points on the lines $x = -1$ and $x = 1$. So, these functions do not have vertical asymptotes.

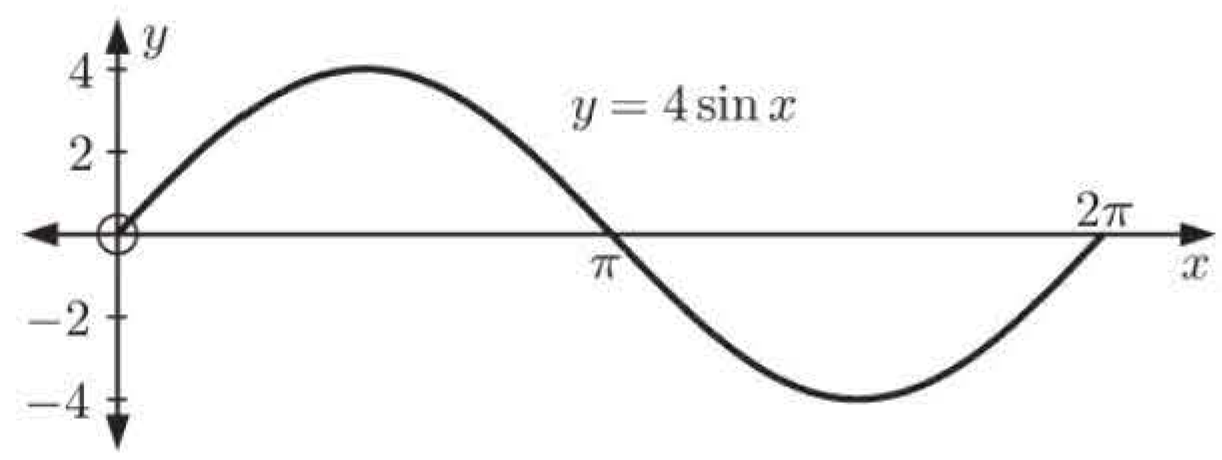
5 **a** $\arcsin(\sin\frac{\pi}{3}) = \frac{\pi}{3}$ **b** $\arccos(\cos(-\frac{\pi}{6})) = \frac{\pi}{6}$ **c** $\tan(\arctan(0.3)) = 0.3$

d $\cos(\arccos(-\frac{1}{2})) = -\frac{1}{2}$ **e** $\arctan(\tan\pi) = 0$ **f** $\arcsin(\sin\frac{4\pi}{3}) = -\frac{\pi}{3}$

REVIEW SET 12A

1 **a** not periodic **b** periodic

2 $y = 4\sin x$ has amplitude 4.



3 **a** $-1 \leqslant \sin x \leqslant 1$

$\therefore$ $1 + \sin x$ has minimum $1 + (-1) = 0$ and maximum $1 + 1 = 2$.

b $-1 \leqslant \cos 3x \leqslant 1$

$\therefore$ $-2\cos 3x$ has minimum $-2(1) = -2$ and maximum $-2(-1) = 2$.

4 **a** period $= \dfrac{2\pi}{\frac{1}{5}} = 10\pi$

b period $= \dfrac{2\pi}{4} = \frac{\pi}{2}$

c period $= \dfrac{2\pi}{\frac{1}{2}} = 4\pi$

d period $= \frac{\pi}{3}$

5

Function	*Period*	*Amplitude*	*Domain*	*Range*
$y = -3\sin(\frac{x}{4}) + 1$	8π	3	$x \in \mathbb{R}$	$-2 \leqslant y \leqslant 4$
$y = \tan 2x$	$\frac{\pi}{2}$	undefined	$x \neq \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}, ...$	$y \in \mathbb{R}$
$y = 3\cos \pi x$	2	3	$x \in \mathbb{R}$	$-3 \leqslant y \leqslant 3$

6 **a** If $y = a\cos(b(x - c)) + d$

then $a = -4$, $\dfrac{2\pi}{b} = \pi$

$\therefore$ $b = 2$

$c = d = 0$

$\therefore$ $y = -4\cos 2x$

b If $y = a\cos(b(x - c)) + d$

then $a = 1$, $\dfrac{2\pi}{b} = 8$ $\therefore$ $b = \frac{\pi}{4}$

$d = \dfrac{\text{max.} + \text{min.}}{2} = \dfrac{3 + 1}{2} = 2$

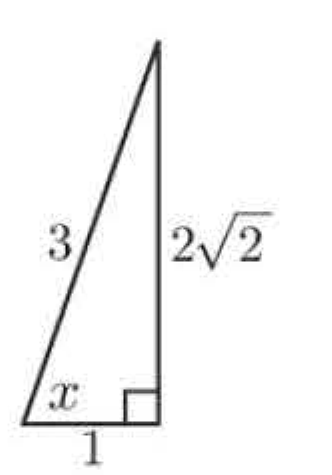
$c = 0$

So, $y = \cos\left(\frac{\pi}{4}x\right) + 2$

7 **a** A vertical stretch with scale factor 3 followed by a horizontal stretch with scale factor $\frac{1}{2}$.

b A translation of $\begin{pmatrix} \frac{\pi}{3} \\ -1 \end{pmatrix}$.

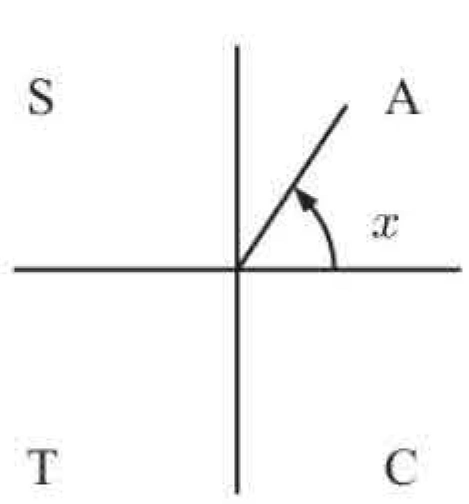
8 **a** $\cos x = \frac{1}{3}$



$0 < x < \pi$

but $\cos x > 0$

$\therefore$ $0 < x < \frac{\pi}{2}$



$\therefore$ $\sin x = \frac{2\sqrt{2}}{3}$

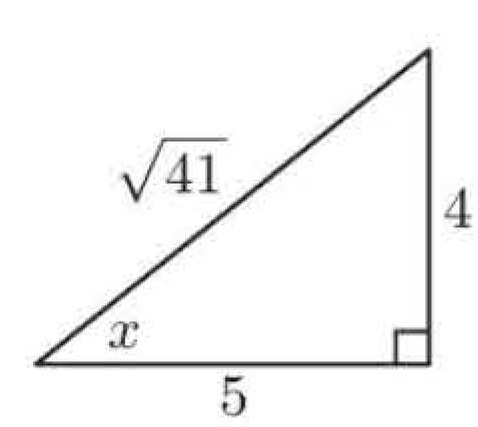
$\tan x = 2\sqrt{2}$

$\csc x = \frac{3}{2\sqrt{2}}$

$\sec x = 3$

$\cot x = \frac{1}{2\sqrt{2}}$

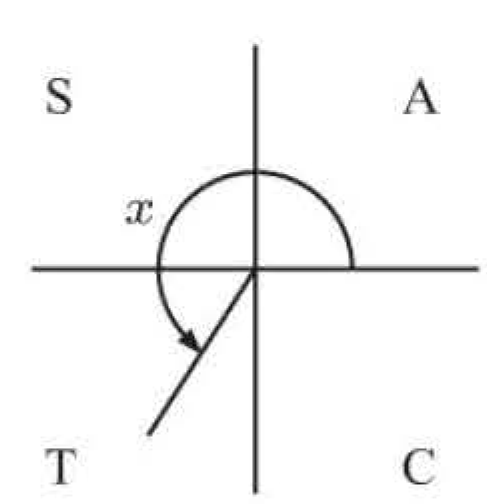
b $\tan x = \frac{4}{5}$



$\pi < x < 2\pi$

but $\tan x > 0$

$\therefore$ $\pi < x < \frac{3\pi}{2}$



$\therefore$ $\sin x = -\frac{4}{\sqrt{41}}$

$\cos x = -\frac{5}{\sqrt{41}}$

$\csc x = -\frac{\sqrt{41}}{4}$

$\sec x = -\frac{\sqrt{41}}{5}$

$\cot x = \frac{5}{4}$

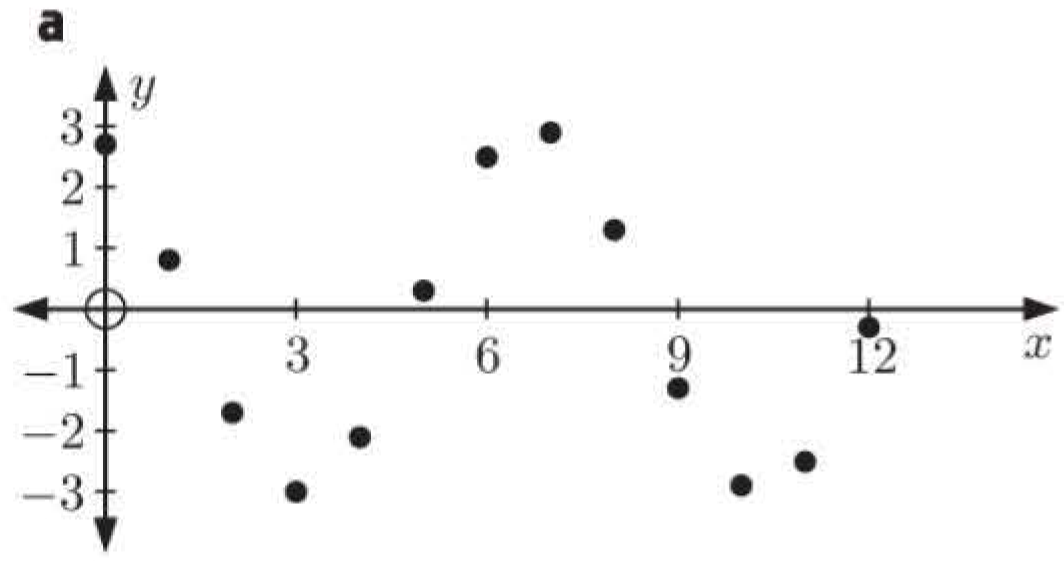
9 **a** $\arctan(\tan(-0.5)) = -0.5$

b $\arcsin(\sin(-\frac{\pi}{6})) = -\frac{\pi}{6}$

c $\arccos(\cos 2\pi) = 0$

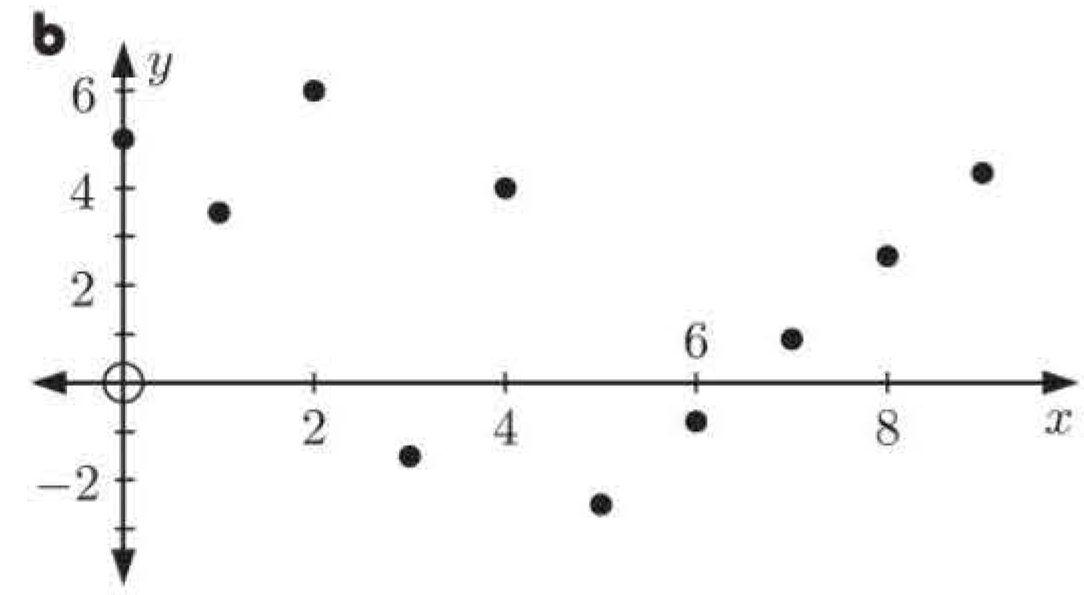
REVIEW SET 12B

1 **a**



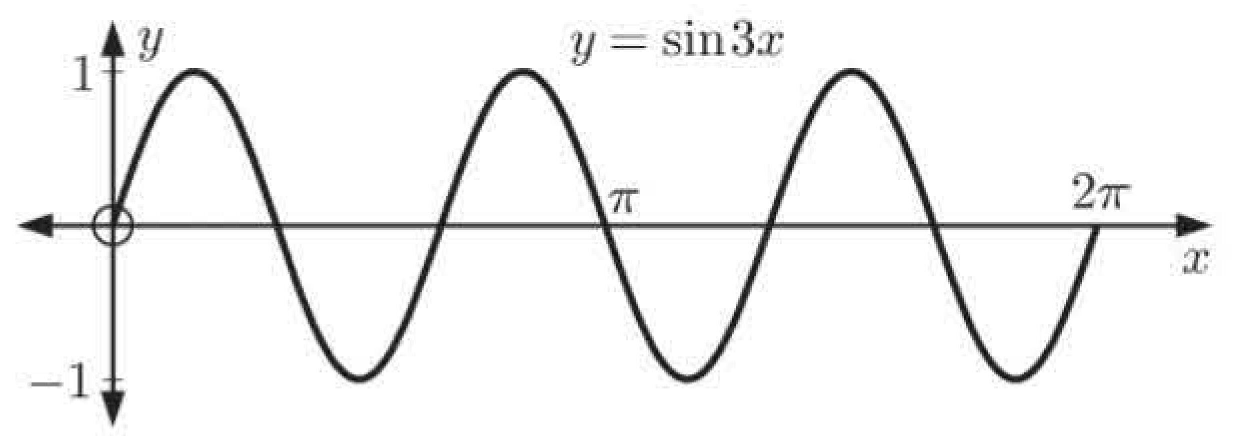
approximately periodic

b



not periodic

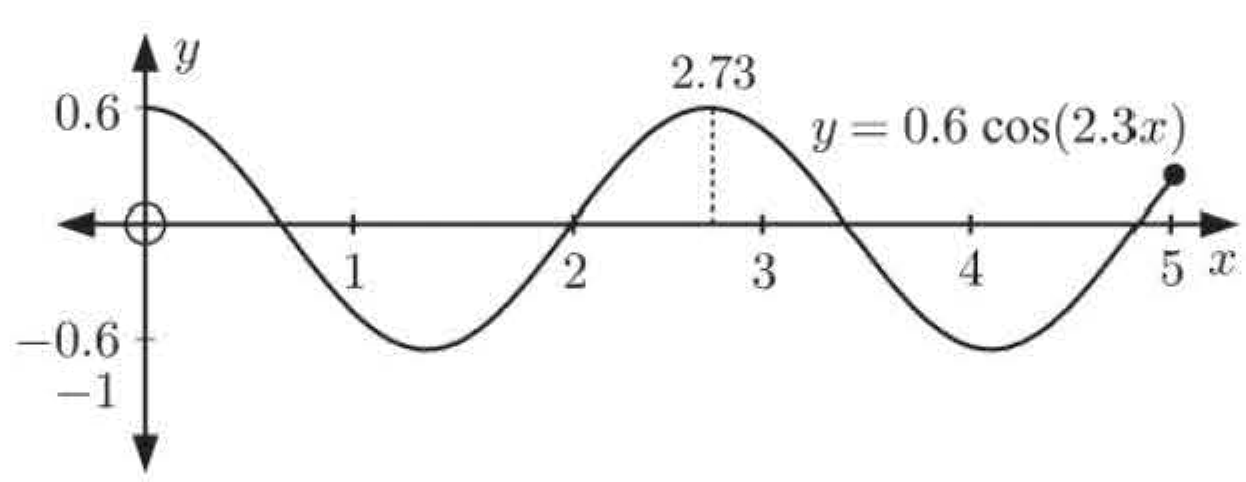
2 $y = \sin 3x$ has period $\frac{2\pi}{3}$.



3 **a** period $= \dfrac{2\pi}{\frac{1}{3}} = 6\pi$

b period $= \frac{\pi}{4}$

4 $y = 0.6\cos(2.3x)$ has period $\frac{2\pi}{2.3} \approx 2.73$



5 **a** maximum $= -5°C$, minimum $= -79°C$

b amplitude $= \dfrac{-5 - -79}{2} = 37°C$, so $a = 37$

principal axis is $y = \dfrac{-5 + -79}{2} = -42$, so $c = -42$

Now, we see that the temperature is $-68°C$ and rising on days 600 and 1300, so we estimate the period to be 700 days.

$\therefore\ b \approx \frac{2\pi}{700} \approx 0.008\,98$

So, $T \approx 37\sin(0.008\,98n) - 42$ °C

c A Mars year is equivalent to one period of the temperature pattern, so 1 Mars year $\approx$ 700 Mars days.

6 Minimum = mean value − amplitude $= c - |a|$, maximum = mean value + amplitude $= c + |a|$.

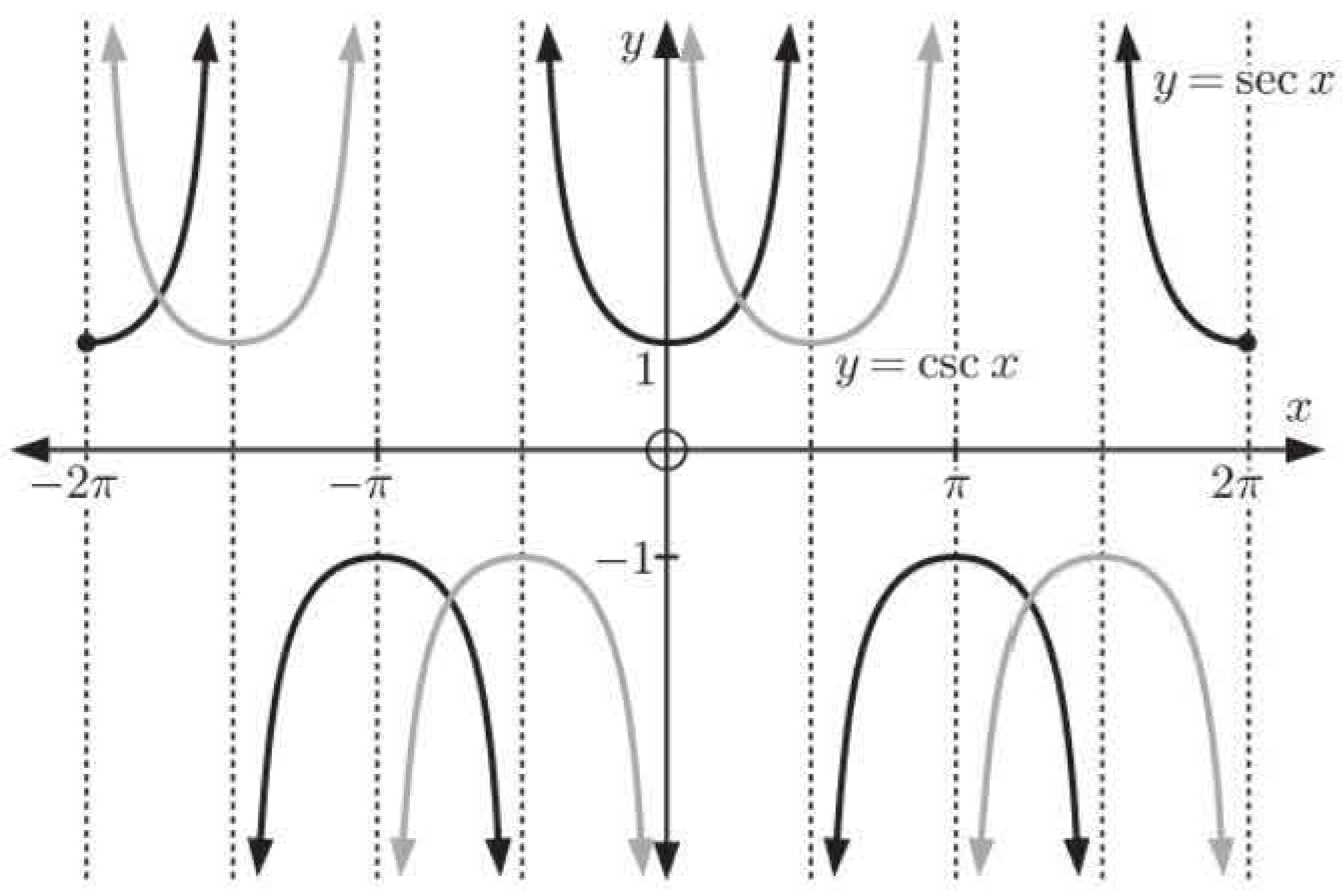
a $y = 5\sin x - 3$ has $a = 5$, $c = -3$
so $\min = -3 - 5 = -8$
and $\max = -3 + 5 = 2$

b $y = \frac{1}{3}\cos x + 1$ has $a = \frac{1}{3}$, $c = 1$
so $\min = 1 - \frac{1}{3} = \frac{2}{3}$
and $\max = 1 + \frac{1}{3} = 1\frac{1}{3}$

7 **a** A reflection in the x-axis followed by a horizontal stretch with scale factor $\frac{1}{2}$.

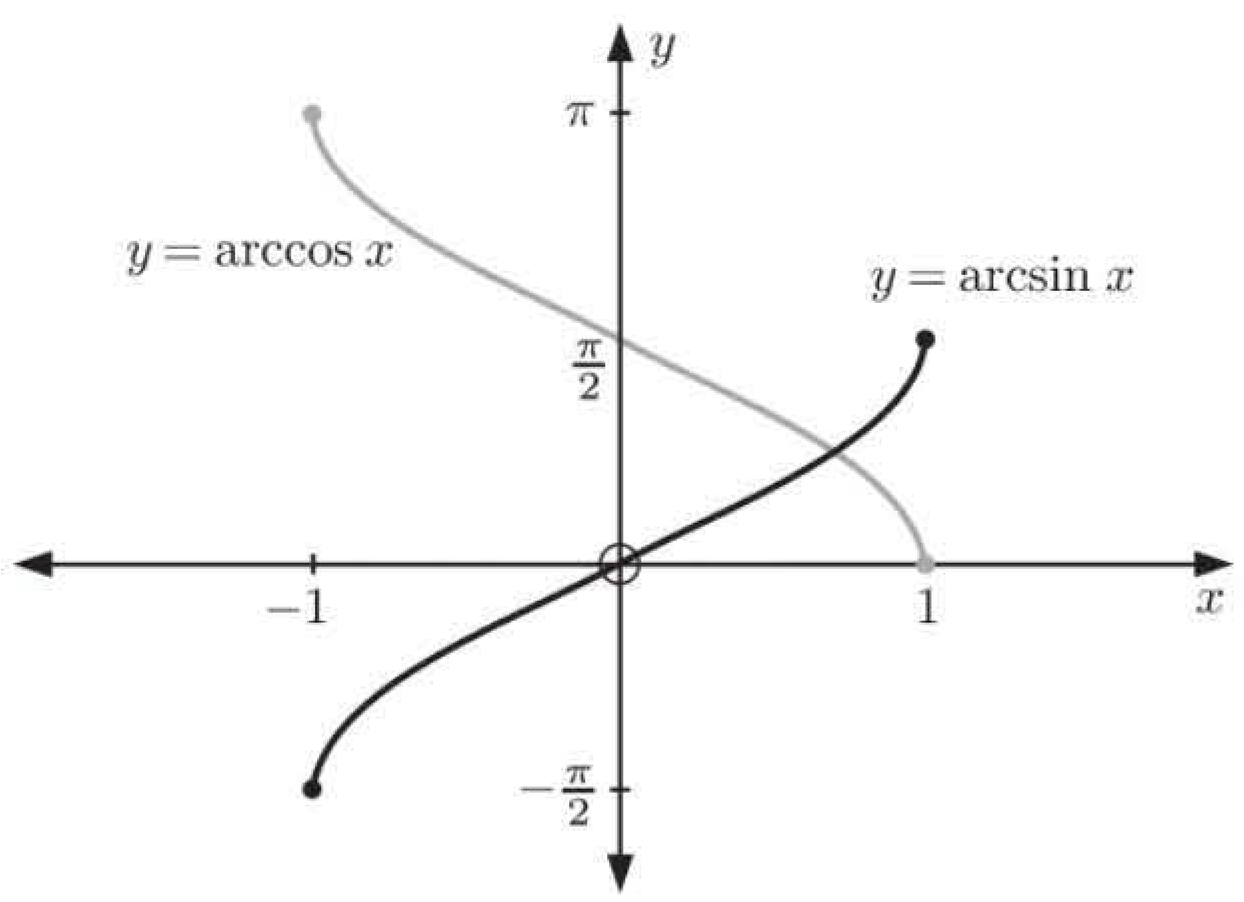
b A vertical stretch with scale factor 2, followed by a translation of $\begin{pmatrix} \frac{\pi}{4} \\ \frac{1}{2} \end{pmatrix}$, followed by a horizontal stretch with scale factor 2.

8 **a**



b a translation of $\begin{pmatrix} \frac{\pi}{2} \\ 0 \end{pmatrix}$

9 **a**



b $y = \arcsin x$:

Domain $= \{x \mid -1 \leqslant x \leqslant 1\}$

Range $= \{y \mid -\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}\}$

$y = \arccos x$:

Domain $= \{x \mid -1 \leqslant x \leqslant 1\}$

Range $= \{y \mid 0 \leqslant y \leqslant \pi\}$

c A reflection in the y-axis (*or* a reflection in the x-axis), followed by a translation of $\begin{pmatrix} 0 \\ \frac{\pi}{2} \end{pmatrix}$.

REVIEW SET 12C

1 **a** period $= \dfrac{2\pi}{b} = 6\pi$

$\therefore \; b = \frac{1}{3}$

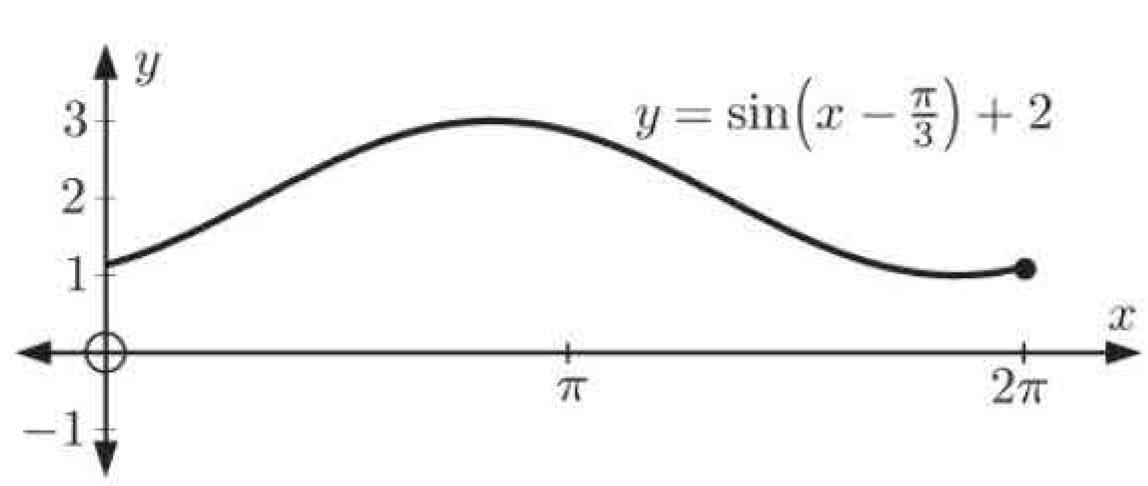
b period $= \dfrac{2\pi}{b} = \frac{\pi}{12}$

$\therefore \; b = 24$

c period $= \dfrac{2\pi}{b} = 9$

$\therefore \; b = \frac{2\pi}{9}$

2 **a**



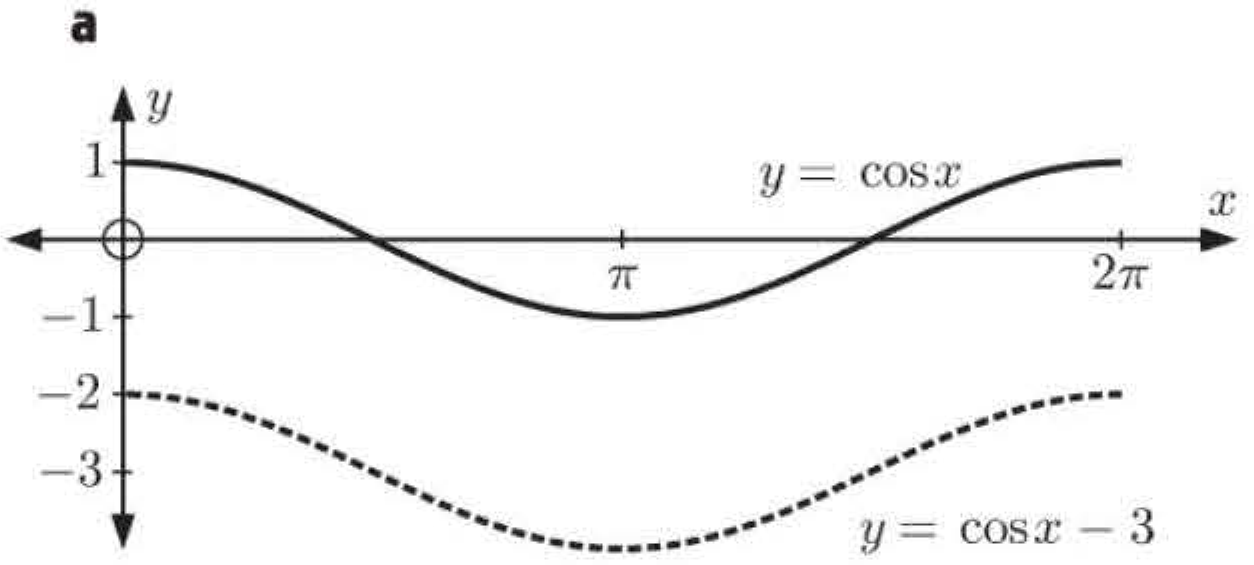
b $f(x)$ has minimum value $-1 + 2 = 1$ and maximum value $1 + 2 = 3$

$\therefore \; f(x) = k$ will have solutions for $1 \leqslant k \leqslant 3$

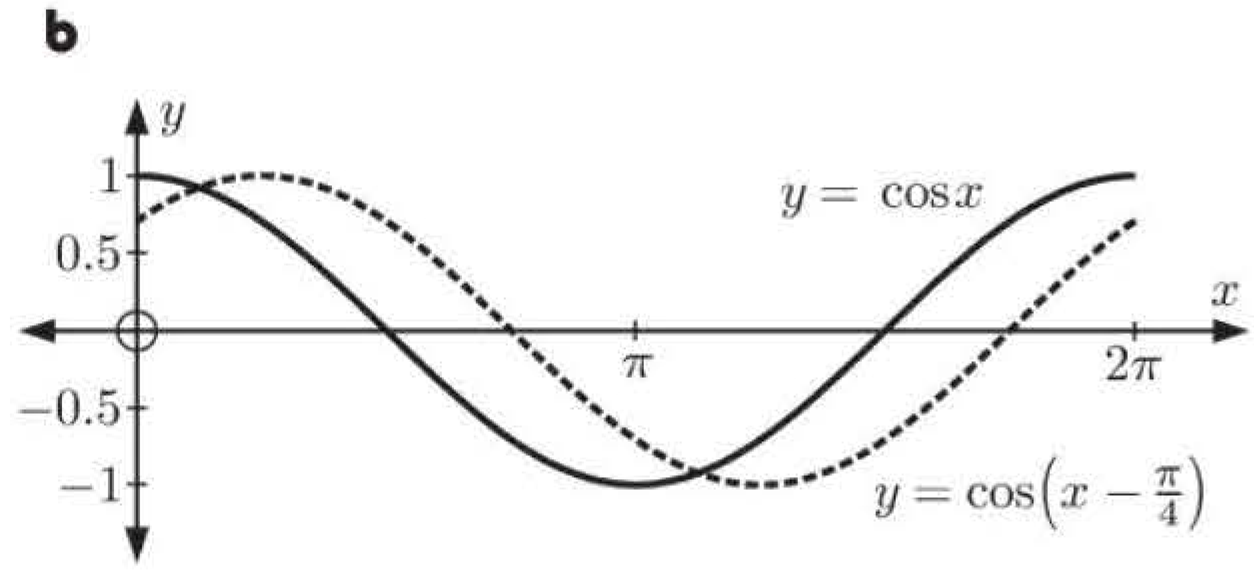
3 **a** The graph is periodic because it repeats itself over and over in a horizontal direction in intervals of the same length.

b **i** period $= 8$ **ii** maximum value $= 5$ **iii** minimum value $= -1$

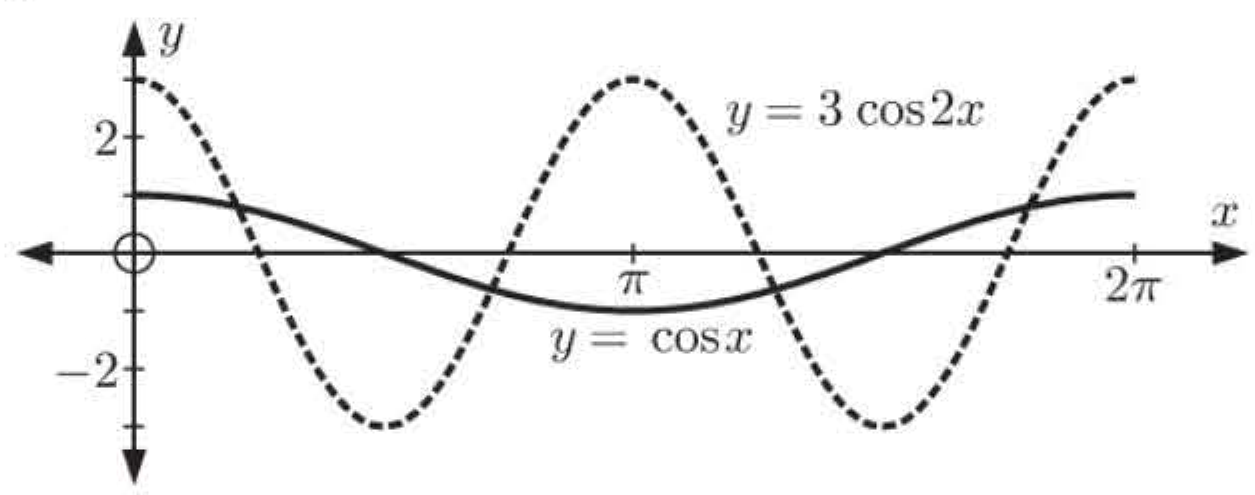
4 **a**



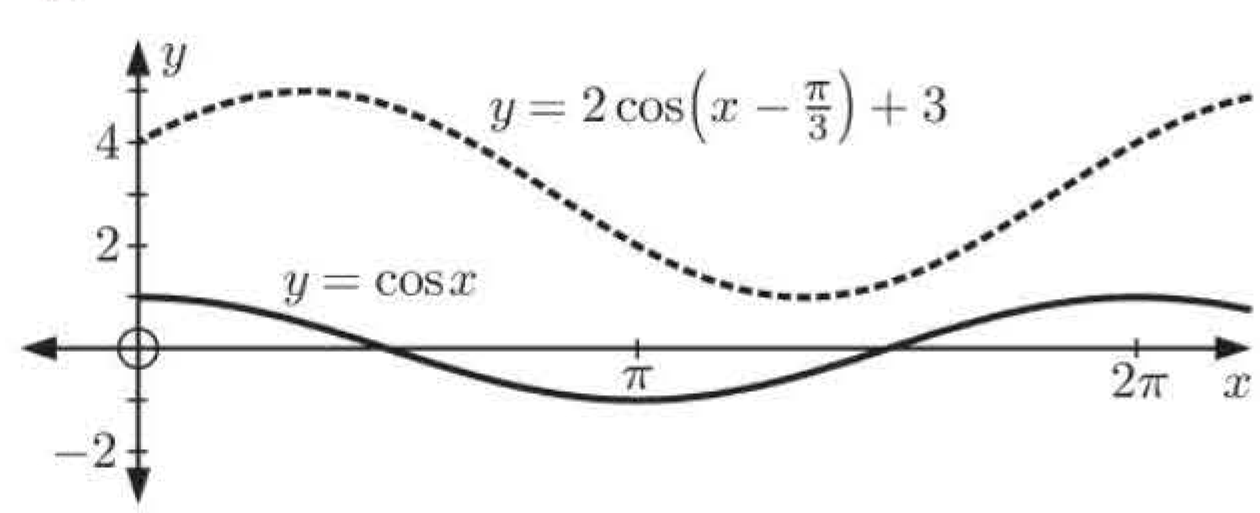
b



c

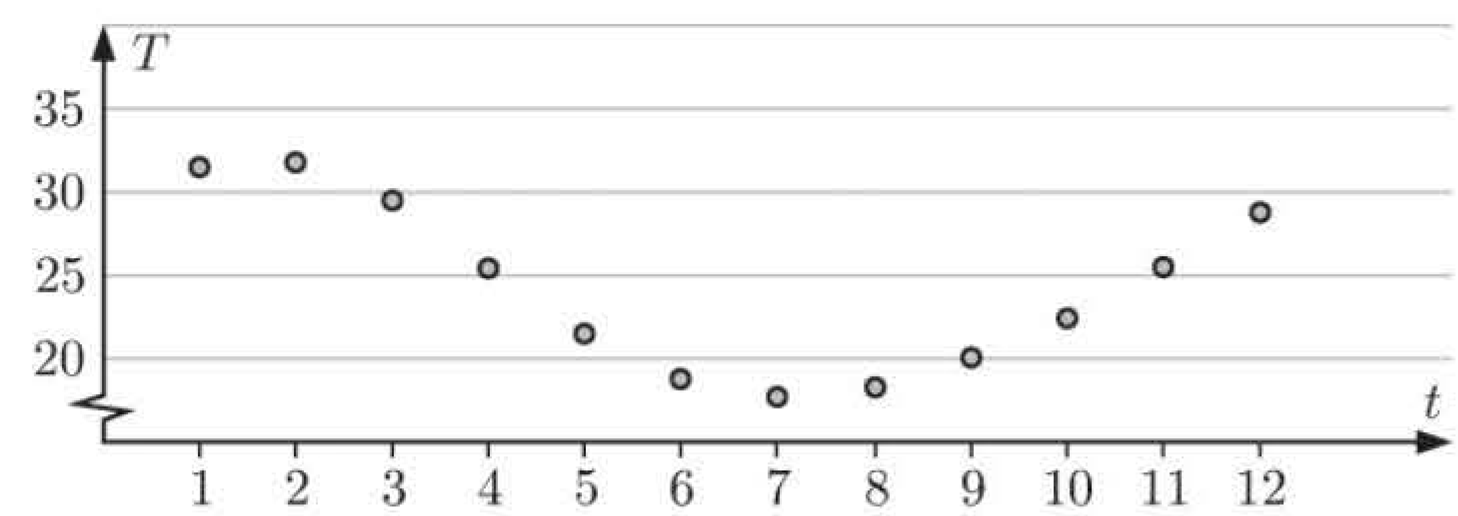


d



5

Month	1	2	3	4	5	6	7	8	9	10	11	12
Temp	31.5	31.8	29.5	25.4	21.5	18.8	17.7	18.3	20.1	22.4	25.5	28.8



a $T = a\sin b(t-c)+d$ period $= \dfrac{2\pi}{b} = 12, \quad \therefore \quad b = \dfrac{2\pi}{12} = \dfrac{\pi}{6}$

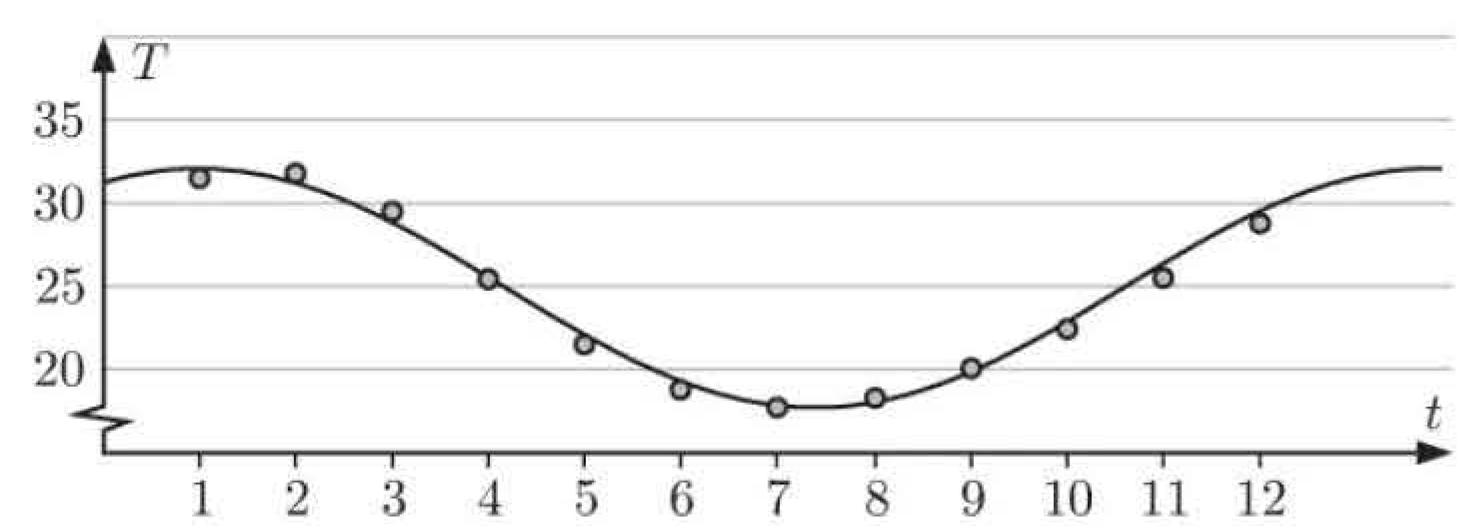
max. $= 31.8$
min. $= 17.7$

$$\therefore \quad a = \frac{\text{max.} - \text{min.}}{2} \approx \frac{31.8-17.7}{2} \approx 7.05$$

$$d = \frac{\text{max.} + \text{min.}}{2} \approx \frac{31.8+17.7}{2} \approx 24.75$$

$$c = \frac{7+14}{2} = 10.5 \quad \{\text{values of } t \text{ at min. and max.}\}$$

So, $T \approx 7.05\sin(\frac{\pi}{6}(t-10.5)) + 24.75$



b From technology, $T \approx 7.21\sin(0.488t + 1.082) + 24.75$
$\approx 7.21\sin(0.488(t+2.22)) + 24.75$

The model fits reasonably well.

6 **a** translation through $\begin{pmatrix} \frac{\pi}{3} \\ 1 \end{pmatrix}$

b vertical stretch with scale factor 2, followed by a reflection in the x-axis

c horizontal stretch with scale factor $\frac{1}{3}$

7 **a** If $y = a\sin(bx - c) + d$

then $a = \dfrac{\text{max.} - \text{min.}}{2} = \dfrac{1 - -\frac{1}{2}}{2} = \frac{3}{4}$,

$\dfrac{2\pi}{b} = \frac{\pi}{2} \quad \therefore \quad b = 4,$

$d = \dfrac{\text{max.} + \text{min.}}{2} = \dfrac{1 - \frac{1}{2}}{2} = \frac{1}{4}$

So, $y = \frac{3}{4}\sin(4x - c) + \frac{1}{4}$

and passes through $(0, 0)$

$\therefore \quad \frac{3}{4}\sin(0 - c) + \frac{1}{4} = 0$

$\therefore \quad \sin(-c) = -\frac{1}{3}$

$\therefore \quad c = \arcsin(\frac{1}{3})$

$\therefore \quad c \approx 0.340$

So, $y = \frac{3}{4}\sin(4x - 0.340) + \frac{1}{4}$.

b If $y = a\tan(b(x - c)) + d$

then principal axis $= 0 \quad \therefore \quad d = 0,$

$\dfrac{\pi}{b} = \pi \quad \therefore \quad b = 1, \; c = \frac{\pi}{2}$

So, $y = a\tan(x - \frac{\pi}{2})$

and passes through $(\frac{\pi}{4}, -1)$

$\therefore \quad a\tan(\frac{\pi}{4} - \frac{\pi}{2}) = -1$

$\therefore \quad a\tan(-\frac{\pi}{4}) = -1$

$\therefore \quad a(-1) = -1$

$\therefore \quad a = 1$

So, $y = \tan(x - \frac{\pi}{2})$.

8 **a**

$$\begin{aligned} &\csc x \tan x \\ &= \frac{1}{\sin x}\frac{\sin x}{\cos x} \\ &= \sec x \end{aligned}$$

b

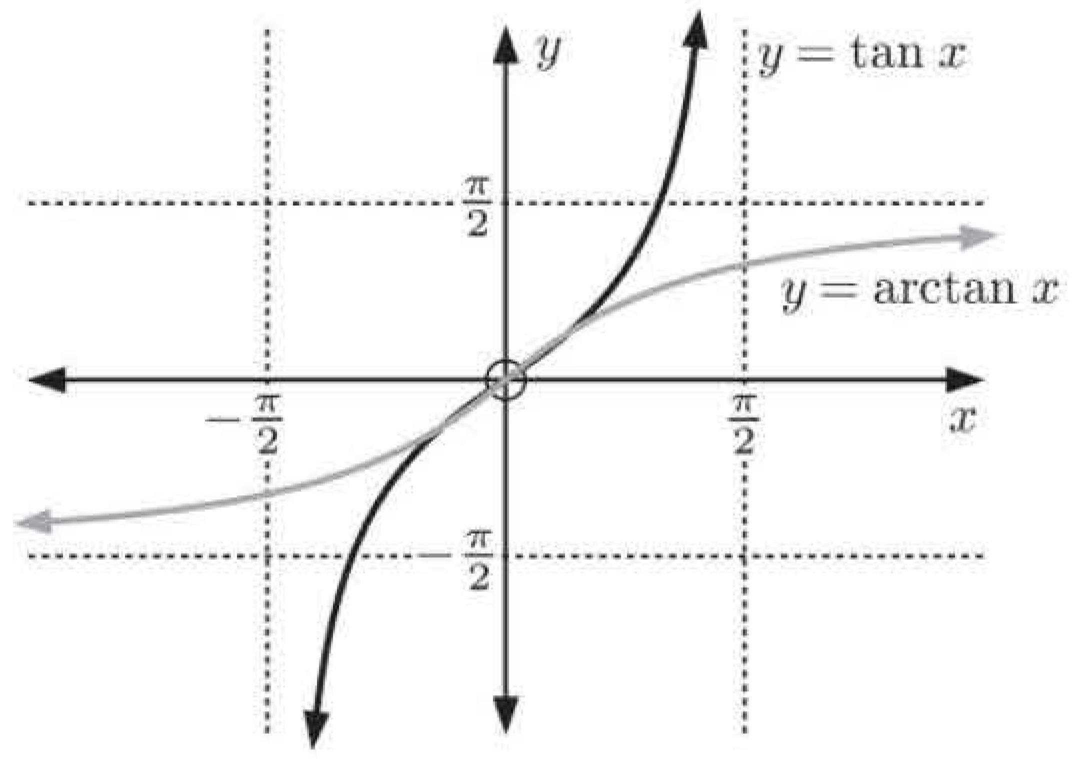
$$\begin{aligned} \frac{\tan x}{\sec x} &= \frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos x}} \\ &= \sin x \end{aligned}$$

c

$$\begin{aligned} &\sec x - \tan x \sin x \\ &= \frac{1}{\cos x} - \frac{\sin x}{\cos x}\sin x \\ &= \frac{1 - \sin^2 x}{\cos x} \\ &= \frac{\cos^2 x}{\cos x} \\ &= \cos x \end{aligned}$$

9 **a** $y = \arctan x$ is the inverse function of $y = \tan x$ for the restricted domain $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

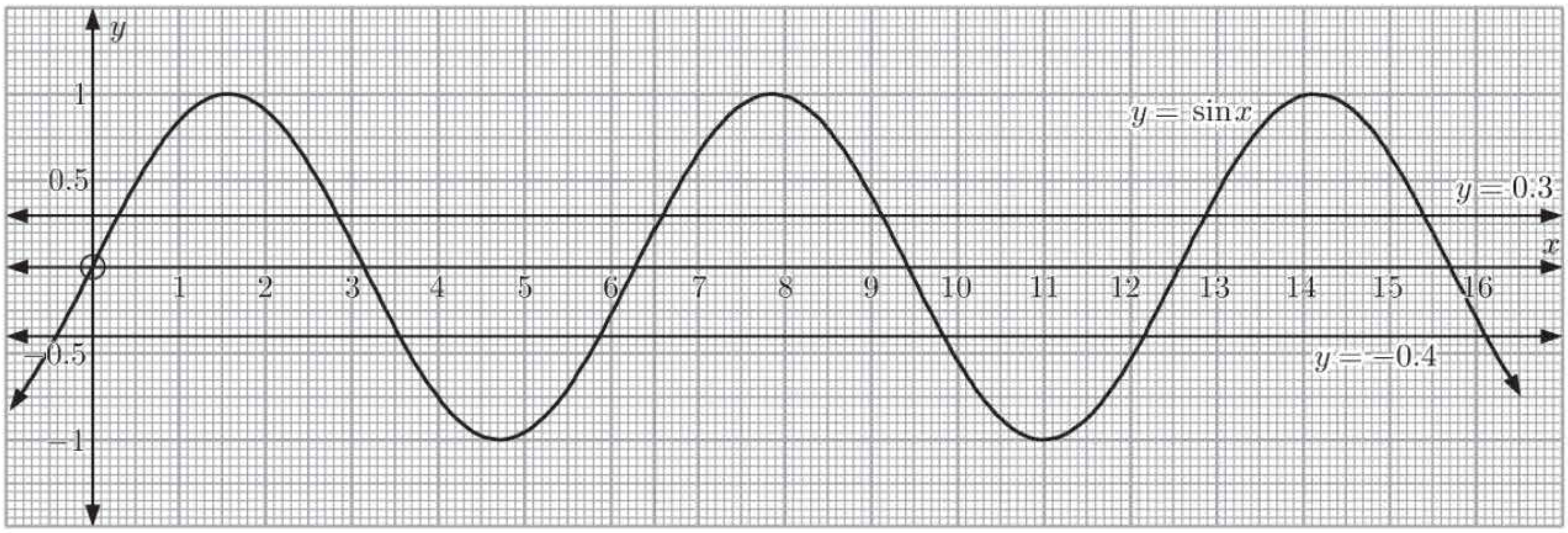
b



Chapter 13
TRIGONOMETRIC EQUATIONS AND IDENTITIES

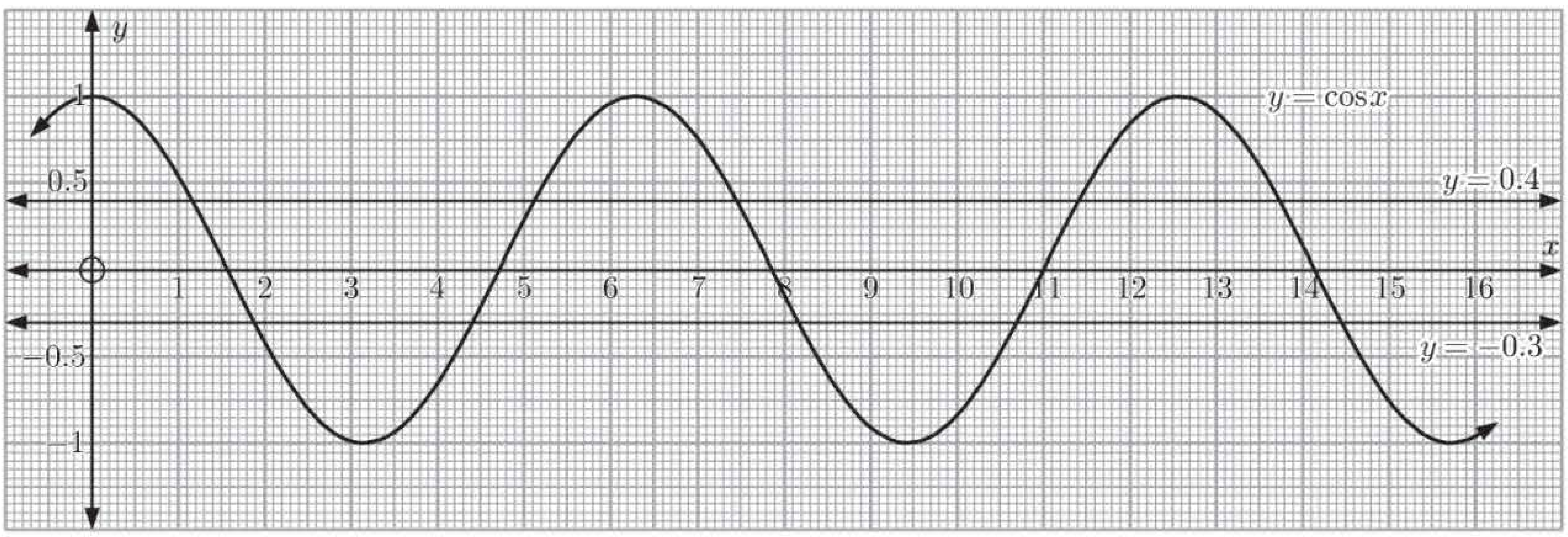
EXERCISE 13A.1

1



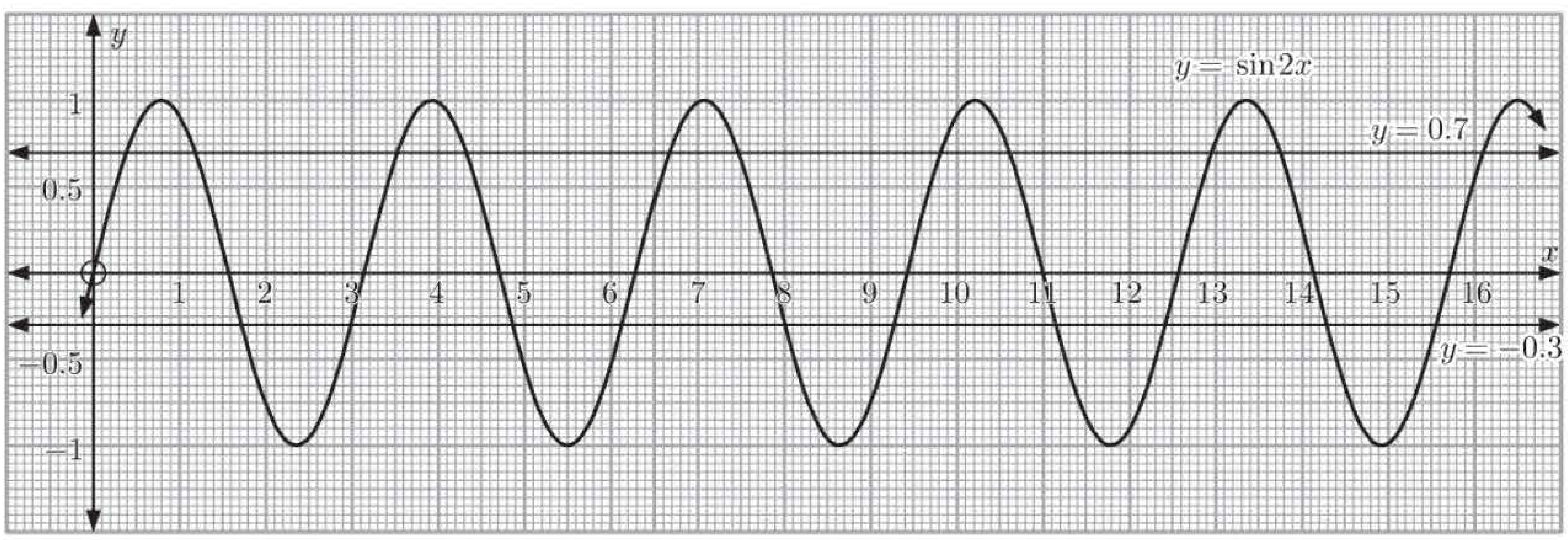
a When $\sin x = 0.3$, $x \approx 0.3, 2.8, 6.6, 9.1, 12.9$ **b** When $\sin x = -0.4$, $x \approx 5.9, 9.8, 12.2$

2



a When $\cos x = 0.4$, $x \approx 1.2, 5.1, 7.4$ **b** When $\cos x = -0.3$, $x \approx 4.4, 8.2, 10.7$

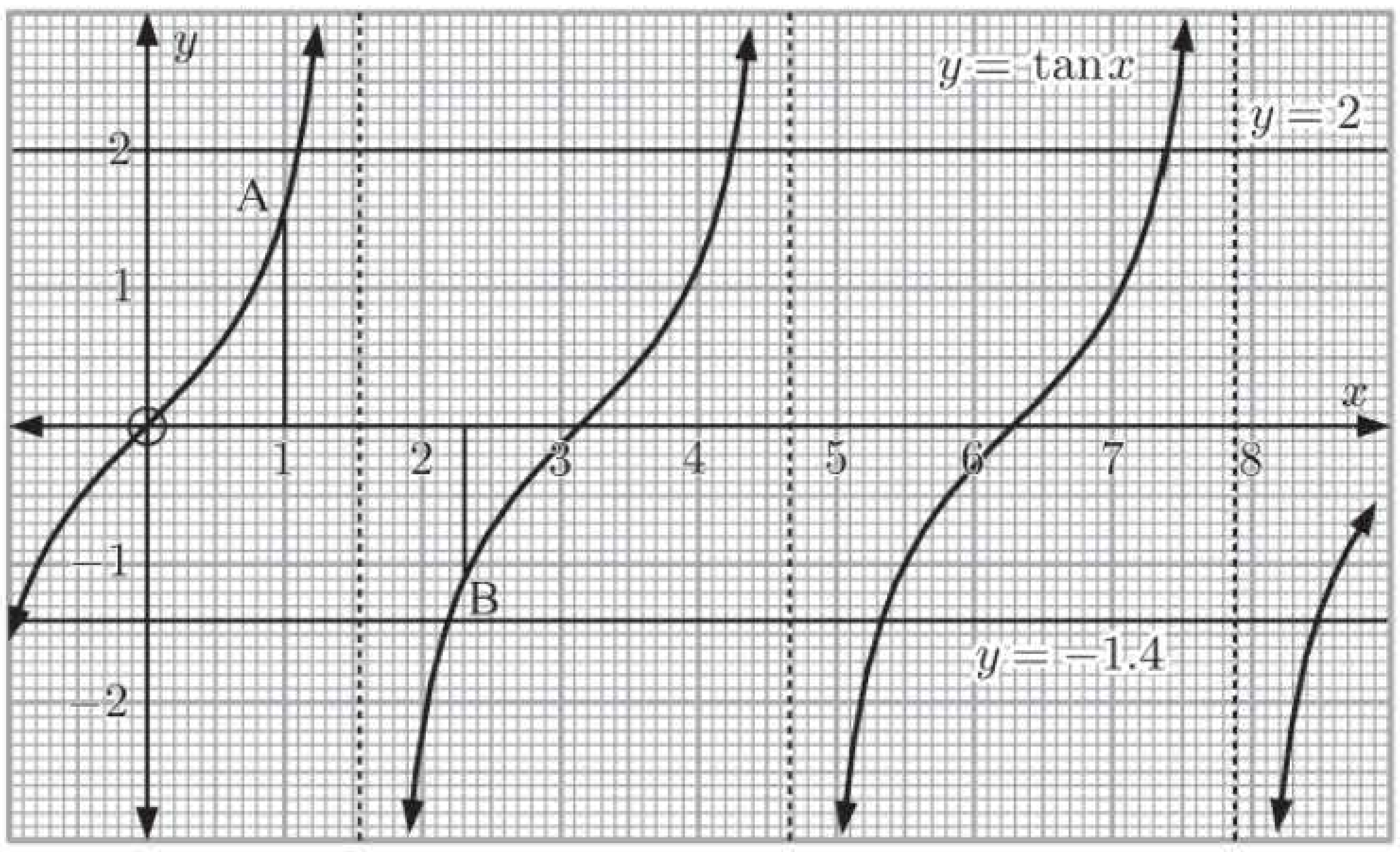
3



a When $\sin 2x = 0.7$, $x \approx 0.4, 1.2, 3.5, 4.3, 6.7, 7.5, 9.8, 10.6, 13.0, 13.7$

b When $\sin 2x = -0.3$, $x \approx 1.7, 3.0, 4.9, 6.1, 8.0, 9.3, 11.1, 12.4, 14.3, 15.6$

4



a **i** $\tan 1 \approx 1.6$ {point A}

ii $\tan 2.3 \approx -1.1$ {point B}

Using technology, we see that $\tan 1 \approx 1.557$ and $\tan 2.3 \approx -1.119$.

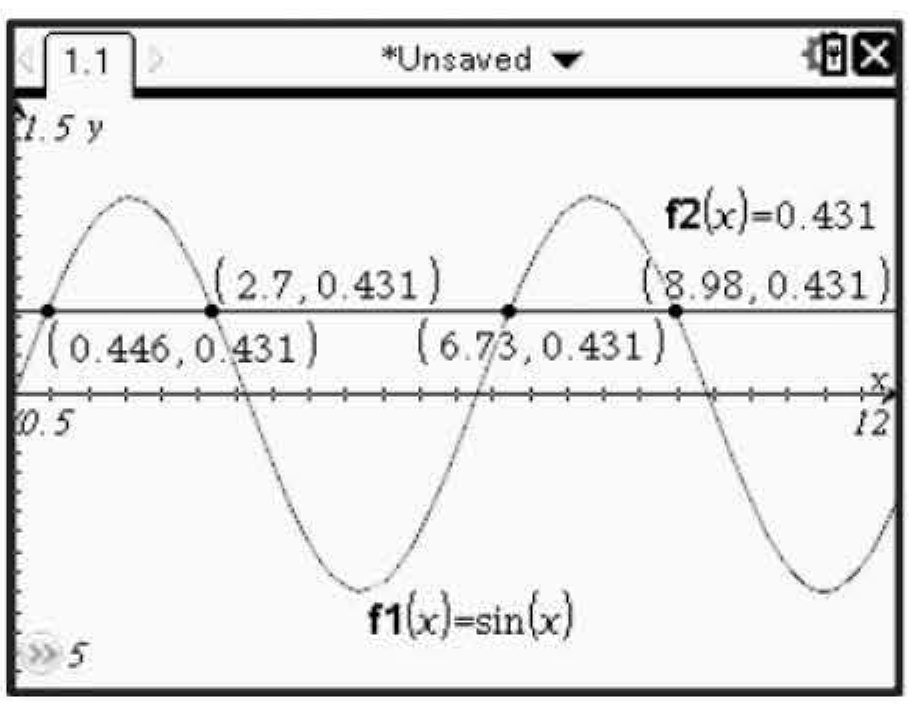
b **i** When $\tan x = 2$,
$x \approx 1.1,\ 4.2,\ 7.4$

ii When $\tan x = -1.4$,
$x \approx 2.2,\ 5.3$

EXERCISE 13A.2

1 **a** $\sin x = 0.431$, $0 < x < 12$

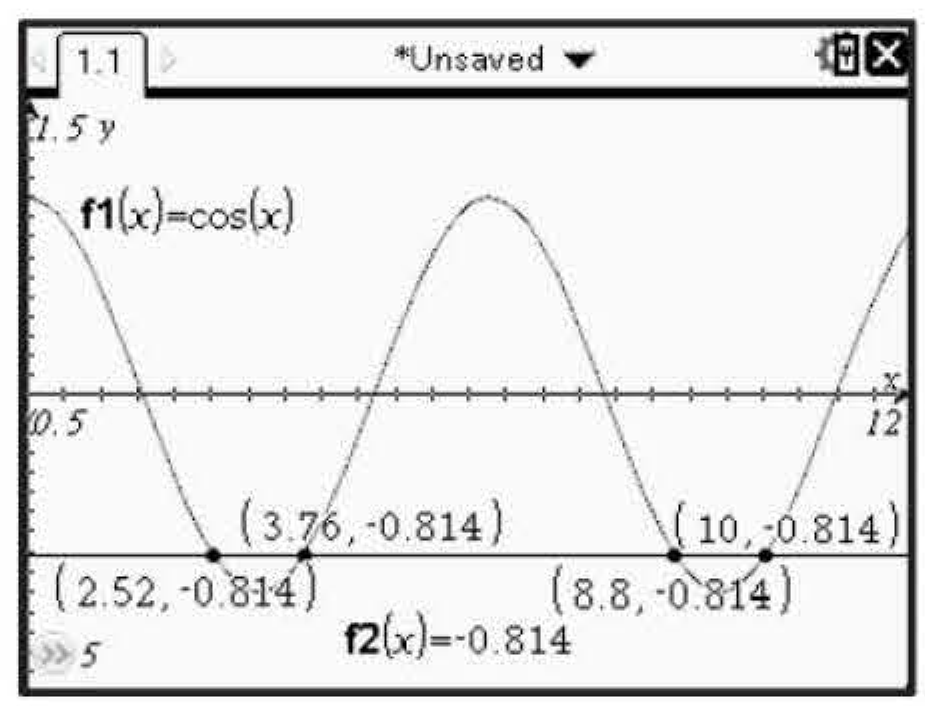
We graph $Y_1 = \sin x$ and $Y_2 = 0.431$ on the same set of axes.



$\therefore\ x \approx 0.446,\ 2.70,\ 6.73,\ 8.98$

b $\cos x = -0.814$, $0 < x < 12$

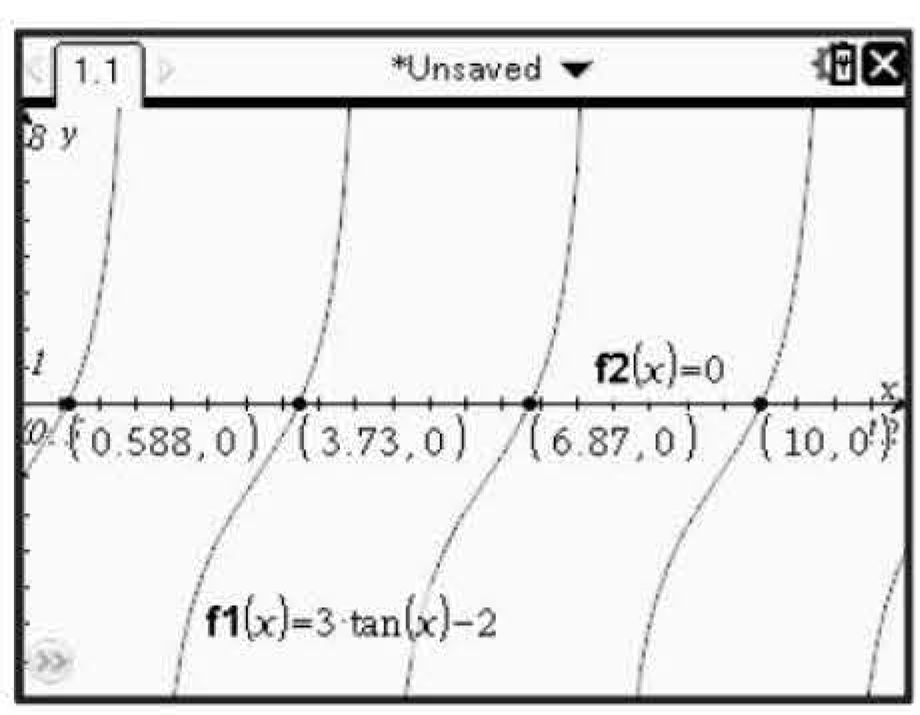
We graph $Y_1 = \cos x$ and $Y_2 = -0.814$ on the same set of axes.



$\therefore\ x \approx 2.52,\ 3.76,\ 8.80,\ 10.0$

c $3\tan x - 2 = 0$, $0 < x < 12$

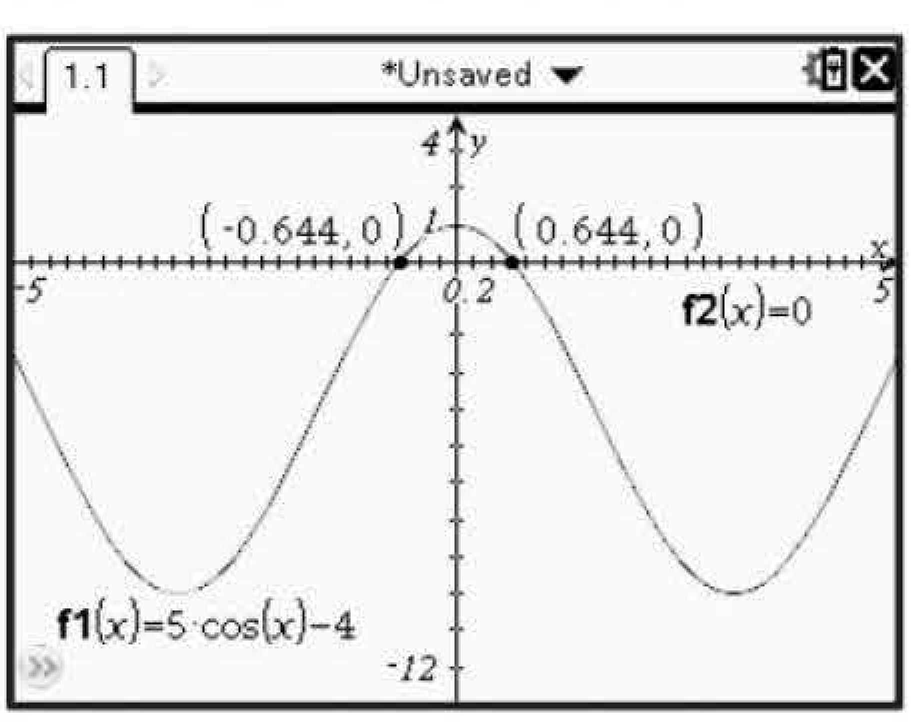
We graph $Y_1 = 3\tan x - 2$ and $Y_2 = 0$ on the same set of axes.



$\therefore\ x \approx 0.588,\ 3.73,\ 6.87,\ 10.0$

2 **a** $5\cos x - 4 = 0$, $-5 \leqslant x \leqslant 5$

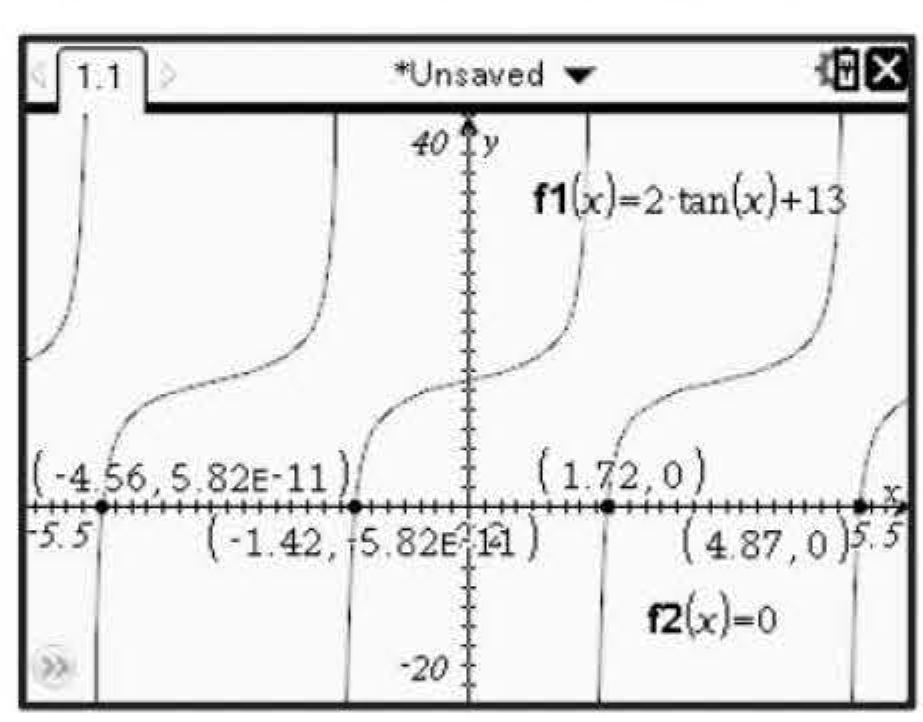
We graph $Y_1 = 5\cos x - 4$ and $Y_2 = 0$ on the same set of axes.



$\therefore\ x \approx -0.644,\ 0.644$

b $2\tan x + 13 = 0$, $-5 \leqslant x \leqslant 5$

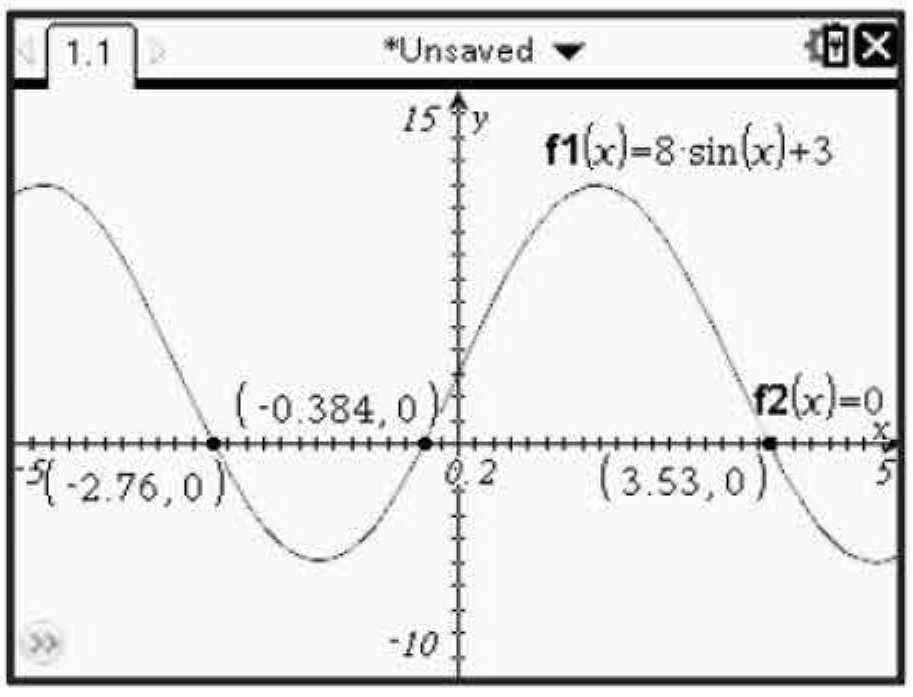
We graph $Y_1 = 2\tan x + 13$ and $Y_2 = 0$ on the same set of axes.



$\therefore\ x \approx -4.56,\ -1.42,\ 1.72,\ 4.87$

c $8\sin x + 3 = 0, \quad -5 \leqslant x \leqslant 5$

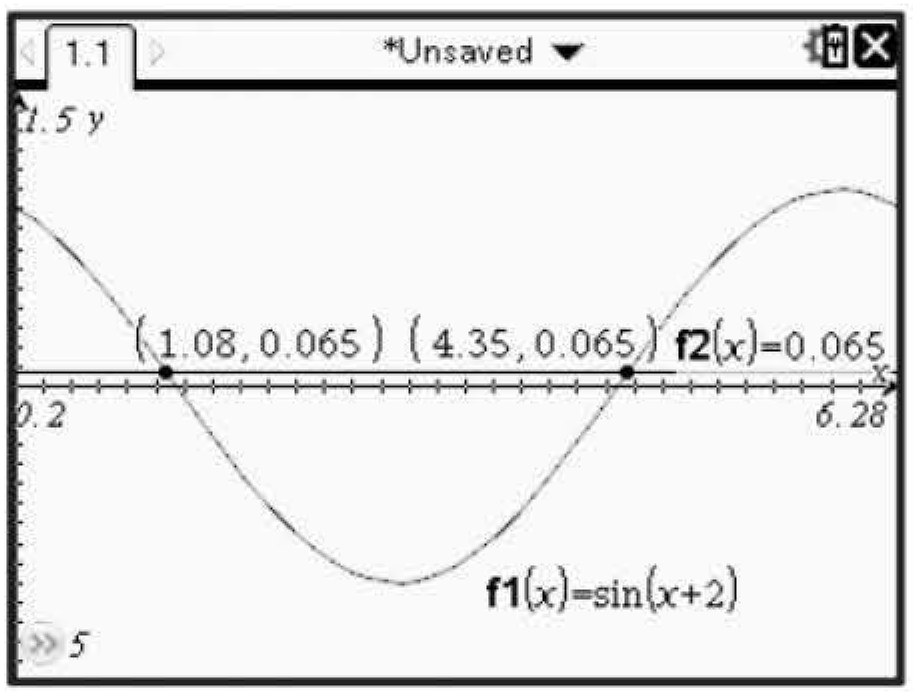
We graph $\mathrm{Y}_1 = 8\sin x + 3$ and $\mathrm{Y}_2 = 0$ on the same set of axes.



$\therefore \quad x \approx -2.76, -0.384, 3.53$

3 **a** $\sin(x+2) = 0.0652, \quad 0 \leqslant x \leqslant 2\pi$

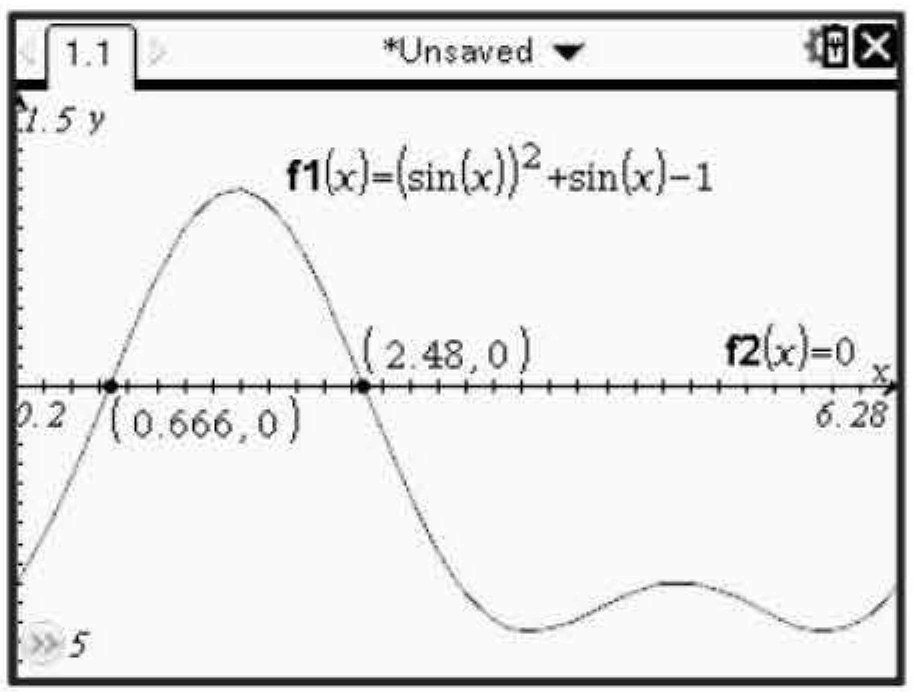
We graph $\mathrm{Y}_1 = \sin(x+2)$ and $\mathrm{Y}_2 = 0.0652$ on the same set of axes.



$\therefore \quad x \approx 1.08, 4.35$

b $\sin^2 x + \sin x - 1 = 0, \quad 0 \leqslant x \leqslant 2\pi$

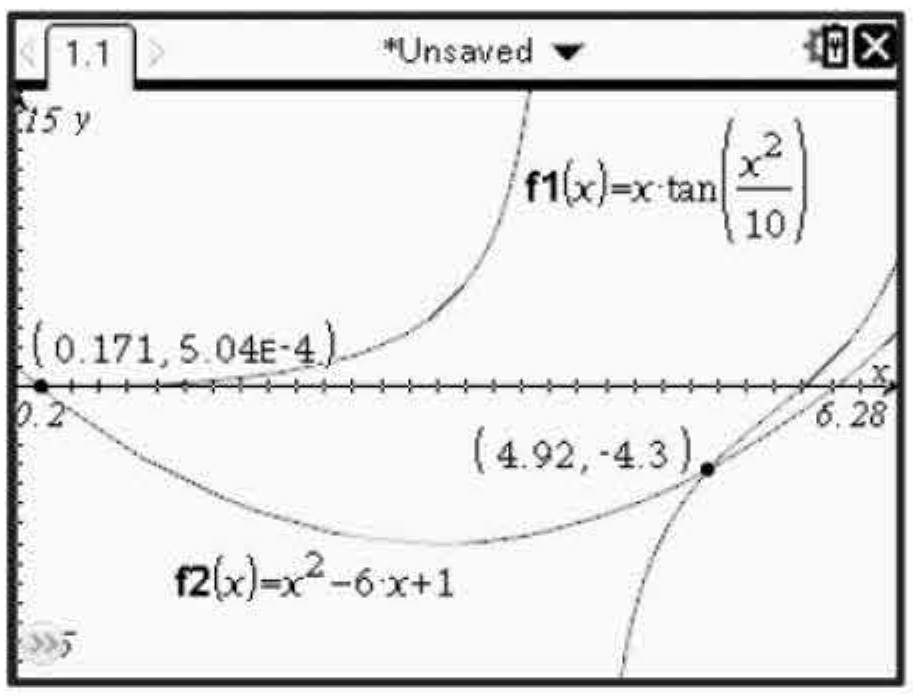
We graph $\mathrm{Y}_1 = \sin^2 x + \sin x - 1$ and $\mathrm{Y}_2 = 0$ on the same set of axes.



$\therefore \quad x \approx 0.666, 2.48$

c $x\tan\left(\frac{x^2}{10}\right) = x^2 - 6x + 1, \quad 0 \leqslant x \leqslant 2\pi$

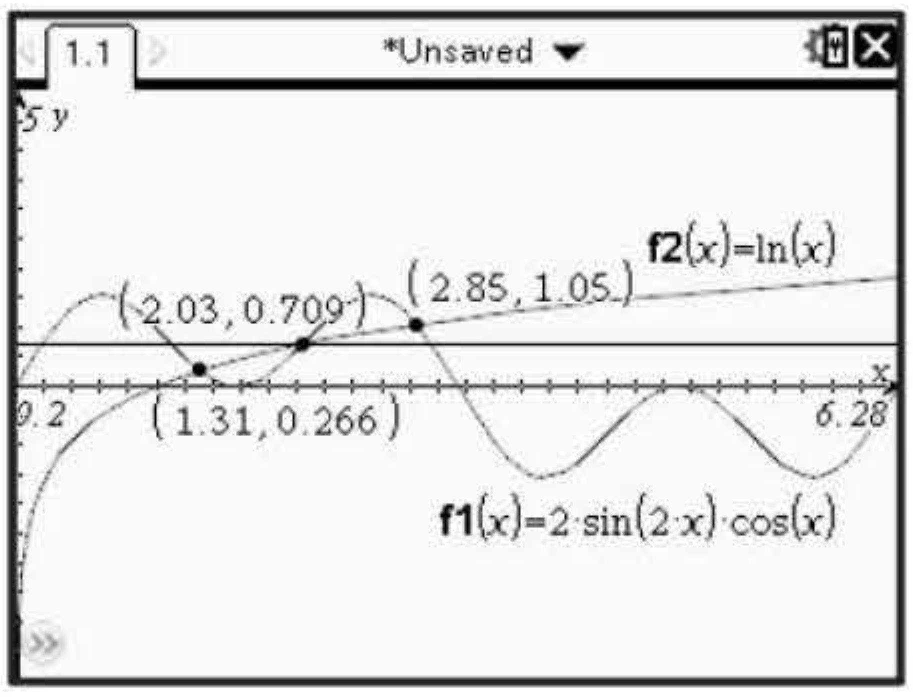
We graph $\mathrm{Y}_1 = x\tan\left(\frac{x^2}{10}\right)$ and $\mathrm{Y}_2 = x^2 - 6x + 1$ on the same set of axes.



$\therefore \quad x \approx 0.171, 4.92$

d $2\sin(2x)\cos x = \ln x, \quad 0 \leqslant x \leqslant 2\pi$

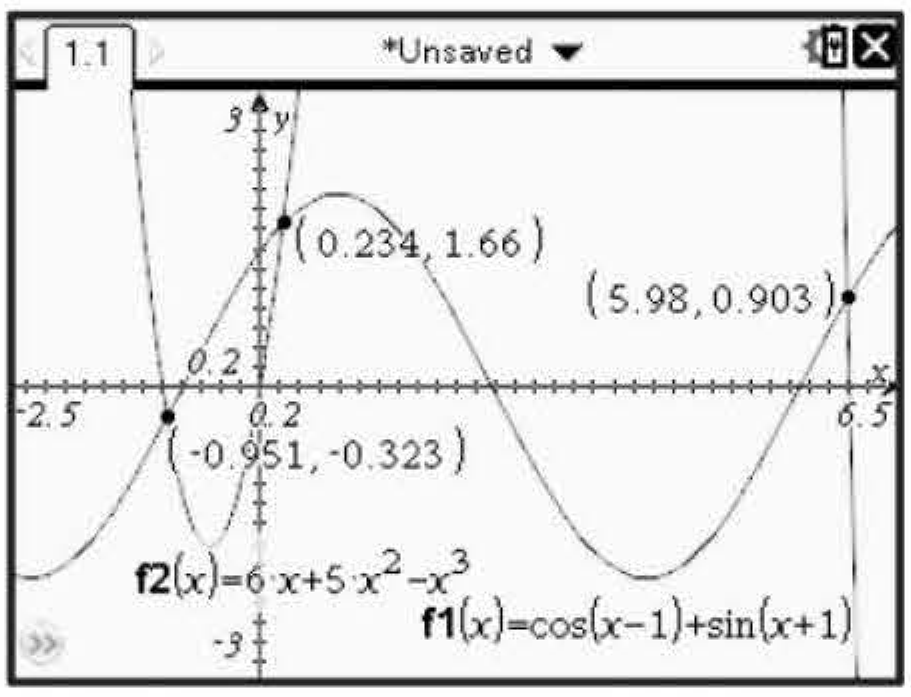
We graph $\mathrm{Y}_1 = 2\sin(2x)\cos x$ and $\mathrm{Y}_2 = \ln x$ on the same set of axes.



$\therefore \quad x \approx 1.31, 2.03, 2.85$

4 $\cos(x-1) + \sin(x+1) = 6x + 5x^2 - x^3, \quad -2 \leqslant x \leqslant 6$

We graph $\mathrm{Y}_1 = \cos(x-1) + \sin(x+1)$ and $\mathrm{Y}_2 = 6x + 5x^2 - x^3$ on the same set of axes.



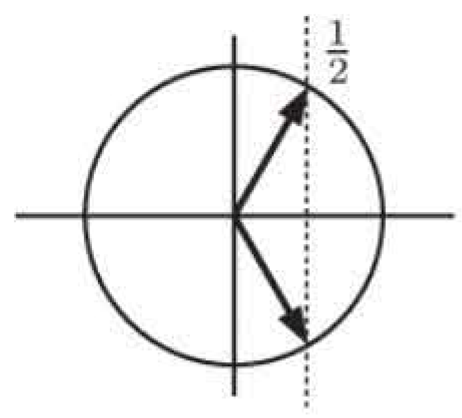
$\therefore \quad x \approx -0.951, 0.234, 5.98$

EXERCISE 13A.3

1 **a** $2\cos x - 1 = 0$

$\therefore\ \cos x = \frac{1}{2}$

There are two points on the unit circle with cosine $\frac{1}{2}$.



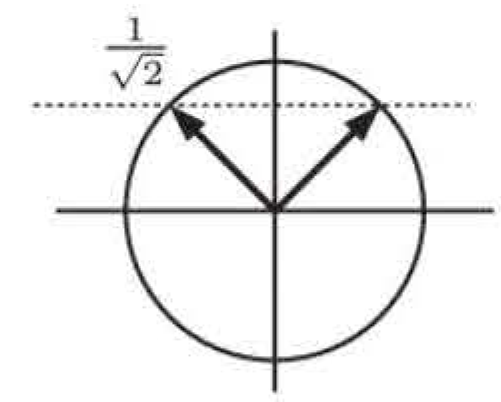
They correspond to angles $\frac{\pi}{3}$ and $\frac{5\pi}{3}$.

For the domain $0 \leqslant x \leqslant 4\pi$ we have 4 solutions: $x = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}$, or $\frac{11\pi}{3}$.

b $\sqrt{2}\sin x = 1$

$\therefore\ \sin x = \frac{1}{\sqrt{2}}$

There are two points on the unit circle with sine $\frac{1}{\sqrt{2}}$.



They correspond to angles $\frac{\pi}{4}$ and $\frac{3\pi}{4}$.

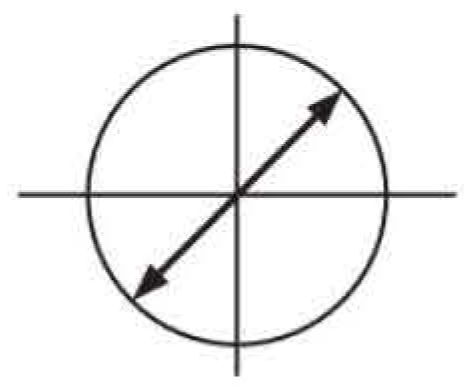
For the domain $0 \leqslant x \leqslant 4\pi$ we have 4 solutions: $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}$, or $\frac{11\pi}{4}$.

c $\tan x = 1$

There are two points on the unit circle with tangent 1.

They correspond to angles $\frac{\pi}{4}$ and $\frac{5\pi}{4}$.

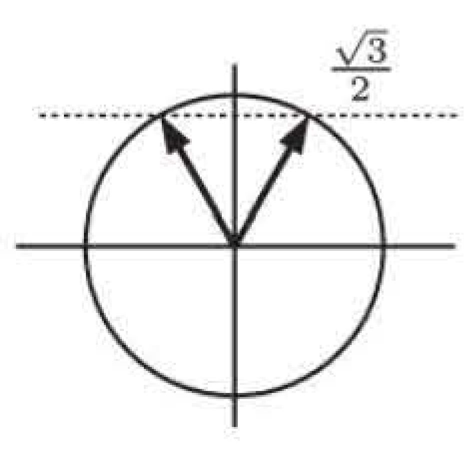
For the domain $0 \leqslant x \leqslant 4\pi$ we have 4 solutions: $x = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}$, or $\frac{13\pi}{4}$.



2 **a** $2\sin x - \sqrt{3} = 0$

$\therefore\ \sin x = \frac{\sqrt{3}}{2}$

There are two points on the unit circle with sine $\frac{\sqrt{3}}{2}$.



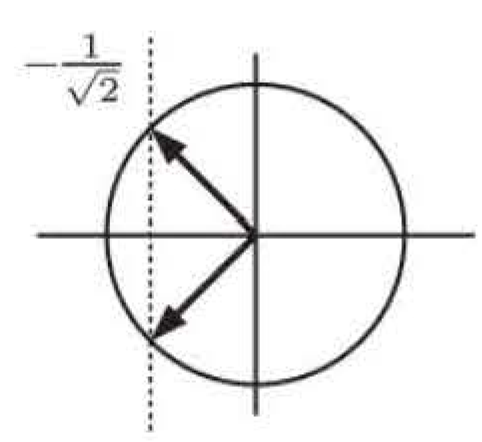
They correspond to angles $\frac{\pi}{3}$ and $\frac{2\pi}{3}$.

For the domain $-2\pi \leqslant x \leqslant 2\pi$ we have 4 solutions: $x = -\frac{5\pi}{3}, -\frac{4\pi}{3}, \frac{\pi}{3}$, or $\frac{2\pi}{3}$.

b $\sqrt{2}\cos x + 1 = 0$

$\therefore\ \cos x = -\frac{1}{\sqrt{2}}$

There are two points on the unit circle with cosine $-\frac{1}{\sqrt{2}}$.



They correspond to angles $\frac{3\pi}{4}$ and $\frac{5\pi}{4}$.

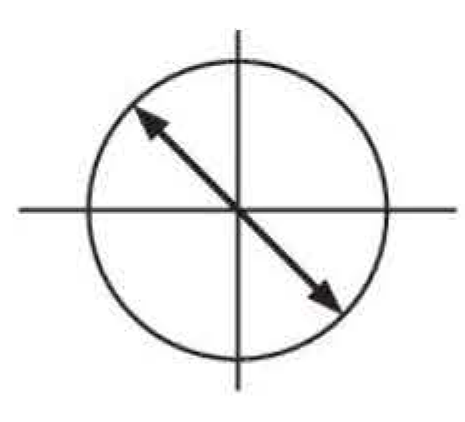
For the domain $-2\pi \leqslant x \leqslant 2\pi$ we have 4 solutions: $x = -\frac{5\pi}{4}, -\frac{3\pi}{4}, \frac{3\pi}{4}$, or $\frac{5\pi}{4}$.

c $\tan x = -1$

There are two points on the unit circle with tangent -1.

They correspond to angles $\frac{3\pi}{4}$ and $\frac{7\pi}{4}$.

For the domain $-2\pi \leqslant x \leqslant 2\pi$ we have 4 solutions: $x = -\frac{5\pi}{4}, -\frac{\pi}{4}, \frac{3\pi}{4}$, or $\frac{7\pi}{4}$.



3 **a** If $0 \leqslant x \leqslant 2\pi$
then $0 \leqslant 2x \leqslant 4\pi$

b If $0 \leqslant x \leqslant 2\pi$
then $0 \leqslant \dfrac{x}{3} \leqslant \frac{2\pi}{3}$

c If $0 \leqslant x \leqslant 2\pi$
then $\frac{\pi}{2} \leqslant x + \frac{\pi}{2} \leqslant \frac{5\pi}{2}$

d If $0 \leqslant x \leqslant 2\pi$
then $-\frac{\pi}{6} \leqslant x - \frac{\pi}{6} \leqslant \frac{11\pi}{6}$

e If $0 \leqslant x \leqslant 2\pi$
then $-\frac{\pi}{4} \leqslant (x - \frac{\pi}{4}) \leqslant \frac{7\pi}{4}$
and so $-\frac{\pi}{2} \leqslant 2(x - \frac{\pi}{4}) \leqslant \frac{7\pi}{2}$

f If $0 \leqslant x \leqslant 2\pi$
then $-2\pi \leqslant -x \leqslant 0$

4 **a** If $-\pi \leqslant x \leqslant \pi$
then $-3\pi \leqslant 3x \leqslant 3\pi$

b If $-\pi \leqslant x \leqslant \pi$
then $-\frac{\pi}{4} \leqslant \dfrac{x}{4} \leqslant \frac{\pi}{4}$

c If $-\pi \leqslant x \leqslant \pi$
then $-\frac{3\pi}{2} \leqslant x - \frac{\pi}{2} \leqslant \frac{\pi}{2}$

d If $-\pi \leqslant x \leqslant \pi$
then $-2\pi \leqslant 2x \leqslant 2\pi$
and so $-\frac{3\pi}{2} \leqslant 2x + \frac{\pi}{2} \leqslant \frac{5\pi}{2}$

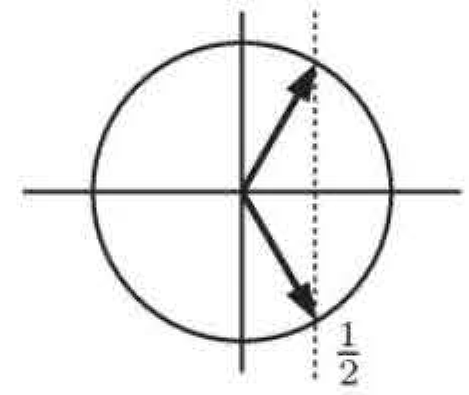
e If $-\pi \leqslant x \leqslant \pi$
then $2\pi \geqslant -2x \geqslant -2\pi$
and so $-2\pi \leqslant -2x \leqslant 2\pi$

f If $-\pi \leqslant x \leqslant \pi$
then $\pi - -\pi \geqslant \pi - x \geqslant \pi - \pi$
and so $0 \leqslant \pi - x \leqslant 2\pi$

5 The three equations all have the form $\cos\theta = \frac{1}{2}$.

There are two points on the unit circle with cosine $\frac{1}{2}$.

They correspond to angles $\frac{\pi}{3}$ and $\frac{5\pi}{3}$.



a In this case θ is simply x, so we have the domain $0 \leqslant x \leqslant 3\pi$.

The solutions for this domain are $x = \frac{\pi}{3}, \frac{5\pi}{3}$, or $\frac{7\pi}{3}$.

b In this case θ is $2x$.

If $0 \leqslant x \leqslant 3\pi$ then $0 \leqslant 2x \leqslant 6\pi$.

$\therefore\ 2x = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}, \frac{13\pi}{3}$, or $\frac{17\pi}{3}$

$\therefore\ x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{13\pi}{6}$, or $\frac{17\pi}{6}$

c In this case θ is $x + \frac{\pi}{3}$.

If $0 \leqslant x \leqslant 3\pi$ then $\frac{\pi}{3} \leqslant x + \frac{\pi}{3} \leqslant \frac{10\pi}{3}$.

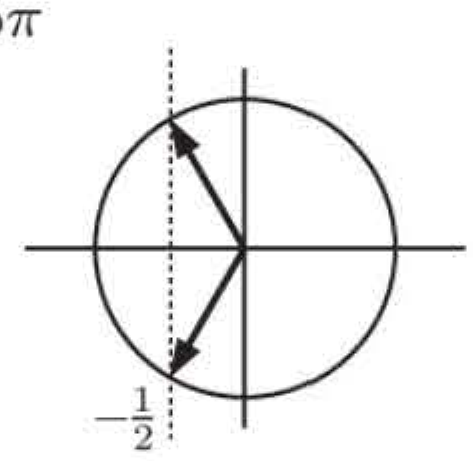
$\therefore\ x + \frac{\pi}{3} = \frac{\pi}{3}, \frac{5\pi}{3}$, or $\frac{7\pi}{3}$

$\therefore\ x = 0, \frac{4\pi}{3}$, or 2π

6 **a** $\cos x = -\frac{1}{2}$, $\quad 0 \leqslant x \leqslant 5\pi$

There are two points on the unit circle with cosine $-\frac{1}{2}$.

They correspond to angles $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$.



For the domain $0 \leqslant x \leqslant 5\pi$:

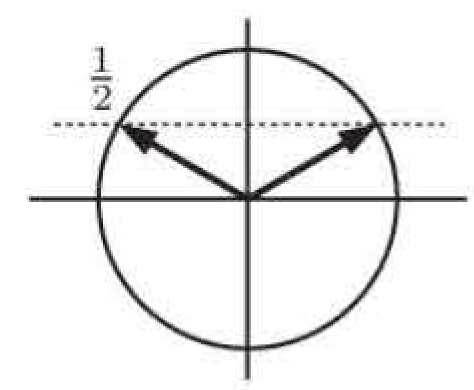
$x = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}, \frac{10\pi}{3}, \frac{14\pi}{3}$

b $2\sin x - 1 = 0$, $\quad -360° \leqslant x \leqslant 360°$

$\therefore\ \sin x = \frac{1}{2}$

There are two points on the unit circle with sine $\frac{1}{2}$.

They correspond to angles $30°$ and $150°$.



For the domain $-360° \leqslant x \leqslant 360°$:

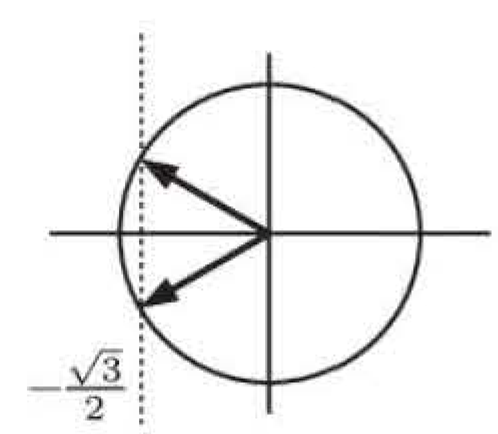
$x = -330°, -210°, 30°, 150°$

c $2\cos x + \sqrt{3} = 0$, $\quad 0 \leqslant x \leqslant 3\pi$

$\therefore\ \cos x = -\frac{\sqrt{3}}{2}$

There are two points on the unit circle with cosine $-\frac{\sqrt{3}}{2}$.

They correspond to angles $\frac{5\pi}{6}$ and $\frac{7\pi}{6}$.



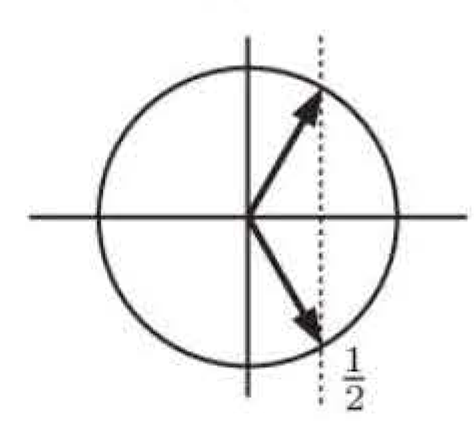
For the domain $0 \leqslant x \leqslant 3\pi$:

$x = \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{17\pi}{6}$

d $\cos(x - \frac{2\pi}{3}) = \frac{1}{2}$, $\quad -2\pi \leqslant x \leqslant 2\pi$

There are two points on the unit circle with cosine $\frac{1}{2}$.

They correspond to angles $\frac{\pi}{3}$ and $\frac{5\pi}{3}$.



$-2\pi \leqslant x \leqslant 2\pi$

$\therefore\ -\frac{8\pi}{3} \leqslant x - \frac{2\pi}{3} \leqslant \frac{4\pi}{3}$

So, $x - \frac{2\pi}{3} = -\frac{7\pi}{3}, -\frac{5\pi}{3}, -\frac{\pi}{3}, \frac{\pi}{3}$

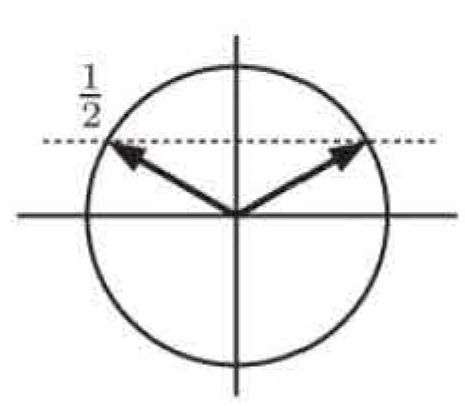
$\therefore\ x = -\frac{5\pi}{3}, -\pi, \frac{\pi}{3}, \pi$

e $2\sin(x + \frac{\pi}{3}) = 1$, $\quad -3\pi \leqslant x \leqslant 3\pi$

$\therefore\ \sin(x + \frac{\pi}{3}) = \frac{1}{2}$

There are two points on the unit circle with sine $\frac{1}{2}$.

They correspond to angles $\frac{\pi}{6}$ and $\frac{5\pi}{6}$.



$-3\pi \leqslant x \leqslant 3\pi$

$\therefore\ -\frac{8\pi}{3} \leqslant x + \frac{\pi}{3} \leqslant \frac{10\pi}{3}$

So, $x + \frac{\pi}{3} = -\frac{11\pi}{6}, -\frac{7\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$

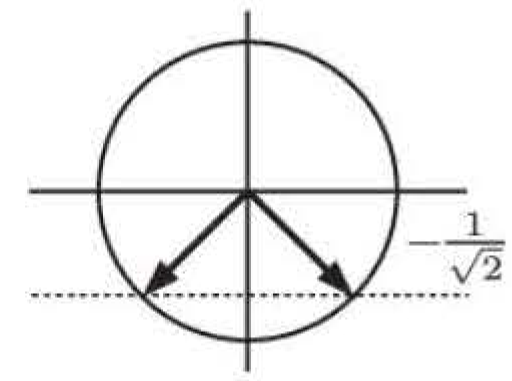
$\therefore\ x = -\frac{13\pi}{6}, -\frac{3\pi}{2}, -\frac{\pi}{6}, \frac{\pi}{2}, \frac{11\pi}{6}, \frac{5\pi}{2}$

f $\sqrt{2}\sin(x - \frac{\pi}{4}) + 1 = 0$, $\quad 0 \leqslant x \leqslant 3\pi$

$\therefore\ \sin(x - \frac{\pi}{4}) = -\frac{1}{\sqrt{2}}$

There are two points on the unit circle with sine $-\frac{1}{\sqrt{2}}$.

They correspond to angles $\frac{5\pi}{4}$ and $\frac{7\pi}{4}$.



$0 \leqslant x \leqslant 3\pi$

$\therefore\ -\frac{\pi}{4} \leqslant x - \frac{\pi}{4} \leqslant \frac{11\pi}{4}$

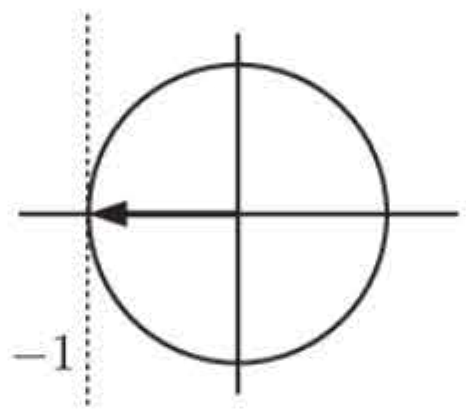
So, $x - \frac{\pi}{4} = -\frac{\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

$\therefore\ x = 0, \frac{3\pi}{2}, 2\pi$

g $3\cos 2x + 3 = 0, \quad 0 \leqslant x \leqslant 3\pi$

$\therefore \quad \cos 2x = -1$

The one point on the unit circle with cosine -1 corresponds to angle π.



$0 \leqslant x \leqslant 3\pi$

$\therefore \quad 0 \leqslant 2x \leqslant 6\pi$

So, $2x = \pi, 3\pi, 5\pi$

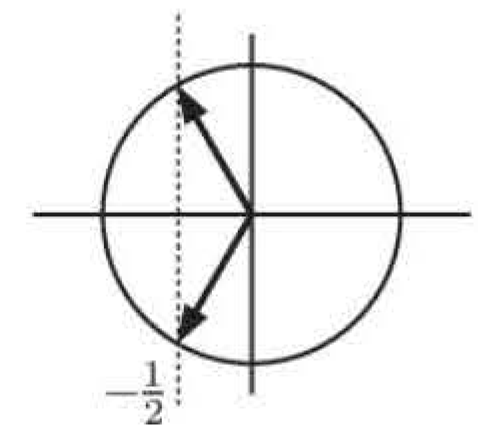
$\therefore \quad x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$

h $4\cos 3x + 2 = 0, \quad -\pi \leqslant x \leqslant \pi$

$\therefore \quad \cos 3x = -\frac{1}{2}$

There are two points on the unit circle with cosine $-\frac{1}{2}$.

They correspond to angles $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$.



$-\pi \leqslant x \leqslant \pi$

$\therefore \quad -3\pi \leqslant 3x \leqslant 3\pi$

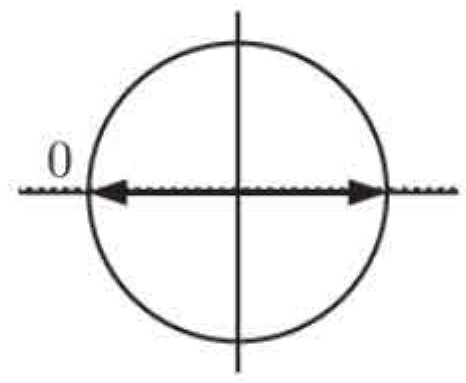
So, $3x = -\frac{8\pi}{3}, -\frac{4\pi}{3}, -\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}$

$\therefore \quad x = -\frac{8\pi}{9}, -\frac{4\pi}{9}, -\frac{2\pi}{9}, \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{8\pi}{9}$

i $\sin\left(4(x - \frac{\pi}{4})\right) = 0, \quad 0 \leqslant x \leqslant \pi$

There are two points on the unit circle with sine 0.

They correspond to angles 0 and π.



$0 \leqslant x \leqslant \pi$

$\therefore \quad -\frac{\pi}{4} \leqslant x - \frac{\pi}{4} \leqslant \frac{3\pi}{4}$

$\therefore \quad -\pi \leqslant 4(x - \frac{\pi}{4}) \leqslant 3\pi$

So, $4(x - \frac{\pi}{4}) = -\pi, 0, \pi, 2\pi, 3\pi$

$\therefore \quad x - \frac{\pi}{4} = -\frac{\pi}{4}, 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$

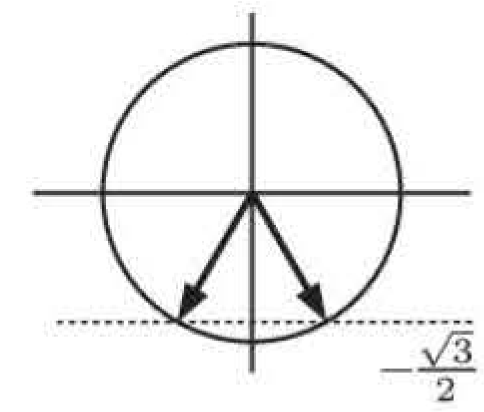
$\therefore \quad x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi$

j $2\sin\left(2(x - \frac{\pi}{3})\right) = -\sqrt{3}, \quad 0 \leqslant x \leqslant 2\pi$

$\therefore \quad \sin\left(2(x - \frac{\pi}{3})\right) = -\frac{\sqrt{3}}{2}$

There are two points on the unit circle with sine $-\frac{\sqrt{3}}{2}$.

They correspond to angles $\frac{4\pi}{3}$ and $\frac{5\pi}{3}$.



$0 \leqslant x \leqslant 2\pi$

$\therefore \quad -\frac{\pi}{3} \leqslant x - \frac{\pi}{3} \leqslant \frac{5\pi}{3}$

$\therefore \quad -\frac{2\pi}{3} \leqslant 2(x - \frac{\pi}{3}) \leqslant \frac{10\pi}{3}$

So, $2(x - \frac{\pi}{3}) = -\frac{2\pi}{3}, -\frac{\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{10\pi}{3}$

$\therefore \quad x - \frac{\pi}{3} = -\frac{\pi}{3}, -\frac{\pi}{6}, \frac{2\pi}{3}, \frac{5\pi}{6}, \frac{5\pi}{3}$

$\therefore \quad x = 0, \frac{\pi}{6}, \pi, \frac{7\pi}{6}, 2\pi$

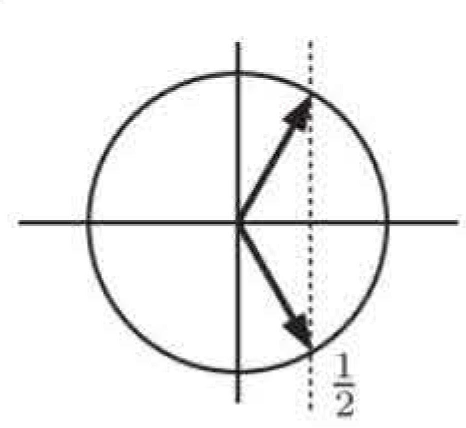
7 **a** $\sec x = 2, \quad x \in [0, 2\pi]$

$\therefore \quad \dfrac{1}{\cos x} = 2$

$\therefore \quad \cos x = \frac{1}{2}, \quad 0 \leqslant x \leqslant 2\pi$

There are two points on the unit circle with cosine $\frac{1}{2}$.

They correspond to angles $\frac{\pi}{3}$ and $\frac{5\pi}{3}$.



For the domain $0 \leqslant x \leqslant 2\pi$:

$x = \frac{\pi}{3}, \frac{5\pi}{3}$

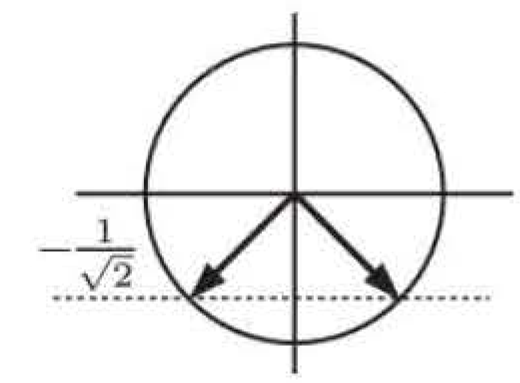
b $\csc x = -\sqrt{2}, \quad x \in [0, 2\pi]$

$\therefore \quad \dfrac{1}{\sin x} = -\sqrt{2}$

$\therefore \quad \sin x = -\frac{1}{\sqrt{2}}, \quad 0 \leqslant x \leqslant 2\pi$

There are two points on the unit circle with sine $-\frac{1}{\sqrt{2}}$.

They correspond to angles $\frac{5\pi}{4}$ and $\frac{7\pi}{4}$.



For the domain $0 \leqslant x \leqslant 2\pi$:

$x = \frac{5\pi}{4}, \frac{7\pi}{4}$

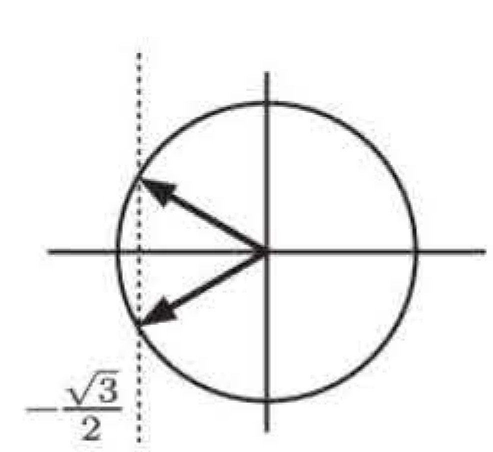
c $\sqrt{3}\sec 2x = -2, \quad x \in [0, 2\pi]$

$\therefore \quad \dfrac{\sqrt{3}}{\cos 2x} = -2$

$\therefore \quad \cos 2x = -\frac{\sqrt{3}}{2}, \quad 0 \leqslant x \leqslant 2\pi$

There are two points on the unit circle with cosine $-\frac{\sqrt{3}}{2}$.

They correspond to angles $\frac{5\pi}{6}$ and $\frac{7\pi}{6}$.



$0 \leqslant x \leqslant 2\pi$

$\therefore \quad 0 \leqslant 2x \leqslant 4\pi$

So, $2x = \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{17\pi}{6}, \frac{19\pi}{6}$

$\therefore \quad x = \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{17\pi}{12}, \frac{19\pi}{12}$

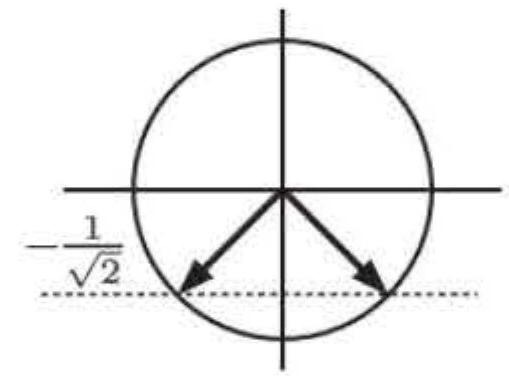
d $\csc(x+\frac{\pi}{6})+\sqrt{2}=0, \quad x\in[0,\,2\pi]$

$\therefore \quad \dfrac{1}{\sin(x+\frac{\pi}{6})}=-\sqrt{2}$

$\therefore \quad \sin(x+\frac{\pi}{6})=-\frac{1}{\sqrt{2}}, \quad 0\leqslant x\leqslant 2\pi$

There are two points on the unit circle with sine $-\frac{1}{\sqrt{2}}$.

They correspond to angles $\frac{5\pi}{4}$ and $\frac{7\pi}{4}$.



$0\leqslant x\leqslant 2\pi$

$\therefore \quad \frac{\pi}{6}\leqslant x+\frac{\pi}{6}\leqslant\frac{13\pi}{6}$

So, $x+\frac{\pi}{6}=\frac{5\pi}{4},\,\frac{7\pi}{4}$

$\therefore \quad x=\frac{13\pi}{12},\,\frac{19\pi}{12}$

8 $\tan x=\sqrt{3}, \quad 0\leqslant x\leqslant 2\pi$

There are two points on the unit circle with tangent $\sqrt{3}$. They correspond to angles $\frac{\pi}{3}$ and $\frac{4\pi}{3}$.

a $0\leqslant x\leqslant 2\pi$

$\therefore \quad -\frac{\pi}{6}\leqslant x-\frac{\pi}{6}\leqslant\frac{11\pi}{6}$

$\therefore \quad x-\frac{\pi}{6}=\frac{\pi}{3},\,\frac{4\pi}{3}$

$\therefore \quad x=\frac{\pi}{2},\,\frac{3\pi}{2}$

b $0\leqslant x\leqslant 2\pi$

$\therefore \quad 0\leqslant 4x\leqslant 8\pi$

$\therefore \quad 4x=\frac{\pi}{3},\,\frac{4\pi}{3},\,\frac{7\pi}{3},\,\frac{10\pi}{3},\,\frac{13\pi}{3},\,\frac{16\pi}{3},\,\frac{19\pi}{3},\,\frac{22\pi}{3}$

$\therefore \quad x=\frac{\pi}{12},\,\frac{\pi}{3},\,\frac{7\pi}{12},\,\frac{5\pi}{6},\,\frac{13\pi}{12},\,\frac{4\pi}{3},\,\frac{19\pi}{12},\,\frac{11\pi}{6}$

c $\tan^2 x=3, \quad 0\leqslant x\leqslant 2\pi$

$\therefore \quad \tan x=\pm\sqrt{3}$

$\therefore \quad x=\frac{\pi}{3},\,\frac{2\pi}{3},\,\frac{4\pi}{3},\,\frac{5\pi}{3}$

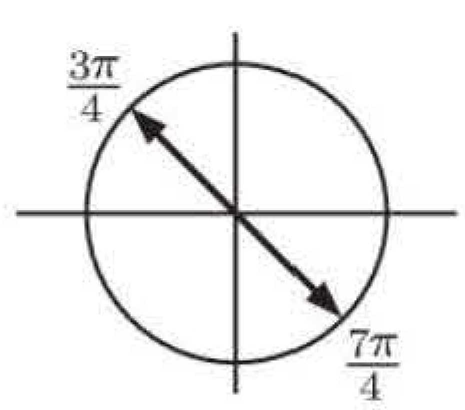
9 **a** $\cot x+1=0, \quad 0\leqslant x\leqslant 2\pi$

$\therefore \quad \dfrac{1}{\tan x}=-1$

$\therefore \quad \tan x=-1$

There are two points on the unit circle with tangent -1.

They correspond to angles $\frac{3\pi}{4}$ and $\frac{7\pi}{4}$.



For the domain $0\leqslant x\leqslant 2\pi$:

$x=\frac{3\pi}{4},\,\frac{7\pi}{4}$

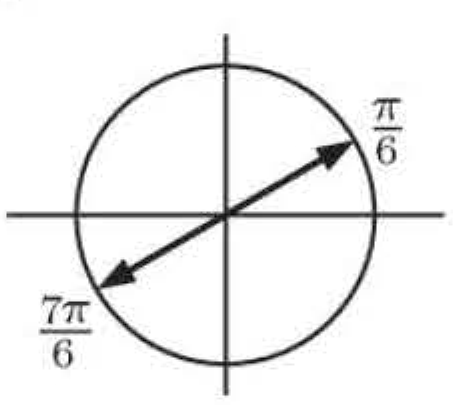
b $\cot(2x-\frac{\pi}{4})-\sqrt{3}=0, \quad 0\leqslant x\leqslant 2\pi$

$\therefore \quad \dfrac{1}{\tan(2x-\frac{\pi}{4})}=\sqrt{3}$

$\therefore \quad \tan(2x-\frac{\pi}{4})=\frac{1}{\sqrt{3}}$

There are two points on the unit circle with tangent $\frac{1}{\sqrt{3}}$.

They correspond to angles $\frac{\pi}{6}$ and $\frac{7\pi}{6}$.



$0\leqslant x\leqslant 2\pi$

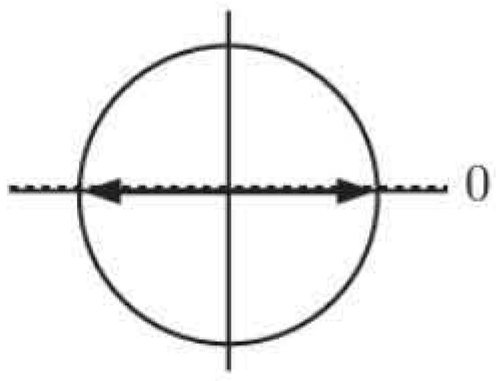
$\therefore \quad -\frac{\pi}{4}\leqslant 2x-\frac{\pi}{4}\leqslant\frac{15\pi}{4}$

So, $2x-\frac{\pi}{4}=\frac{\pi}{6},\,\frac{7\pi}{6},\,\frac{13\pi}{6},\,\frac{19\pi}{6}$

$\therefore \quad 2x=\frac{5\pi}{12},\,\frac{17\pi}{12},\,\frac{29\pi}{12},\,\frac{41\pi}{12}$

$\therefore \quad x=\frac{5\pi}{24},\,\frac{17\pi}{24},\,\frac{29\pi}{24},\,\frac{41\pi}{24}$

10 **a** The zeros of $y = \sin 2x$ are the solutions of $\sin 2x = 0$ $\{0° \leqslant x \leqslant 180°\}$

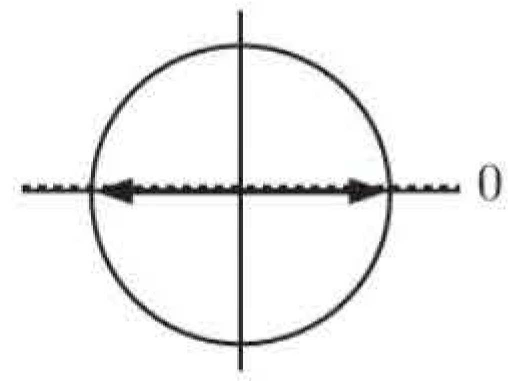


$0° \leqslant x \leqslant 180°$ $\therefore$ $0 \leqslant 2x \leqslant 360°$

So, $2x = 0°, 180°, 360°$

$\therefore$ $x = 0°, 90°, 180°$

b The zeros of $y = \sin(x - \frac{\pi}{4})$ are the solutions of $\sin(x - \frac{\pi}{4}) = 0$ $\{0 \leqslant x \leqslant 3\pi\}$

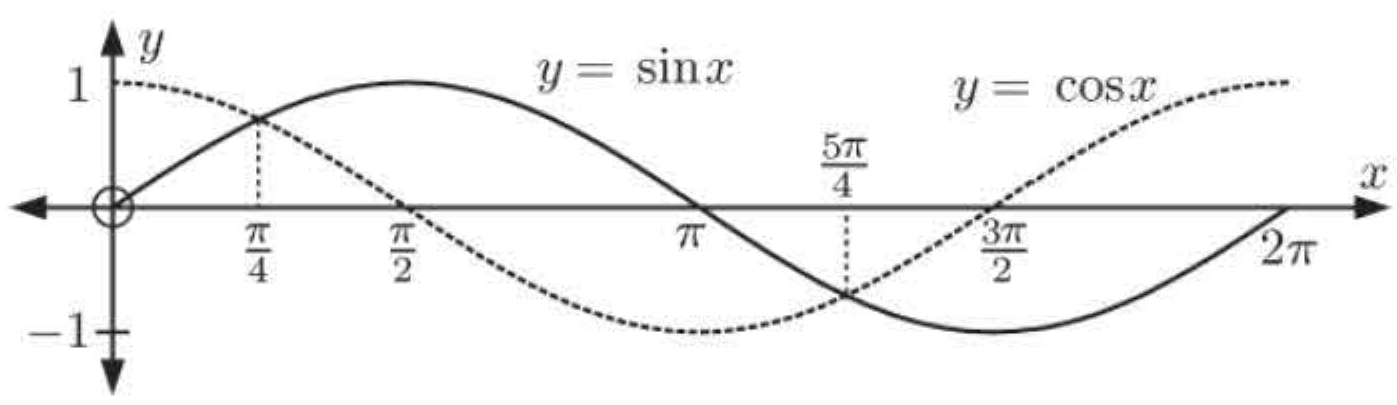


$0 \leqslant x \leqslant 3\pi$ $\therefore$ $-\frac{\pi}{4} \leqslant x - \frac{\pi}{4} \leqslant \frac{11\pi}{4}$

So, $x - \frac{\pi}{4} = 0, \pi, 2\pi$

$\therefore$ $x = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}$

11 **a**



b $x = \frac{\pi}{4}$ or $\frac{5\pi}{4}$

12 **a**

$$\sin x = -\cos x, \quad 0 \leqslant x \leqslant 2\pi$$

$$\therefore \quad \frac{\sin x}{\cos x} = -1$$

$$\therefore \quad \tan x = -1$$

On the domain $0 \leqslant x \leqslant 2\pi$,

$\therefore$ $x = \frac{3\pi}{4}, \frac{7\pi}{4}$

b

$$\sin(3x) = \cos(3x), \quad 0 \leqslant x \leqslant 2\pi$$

$$\therefore \quad \frac{\sin(3x)}{\cos(3x)} = 1$$

$$\therefore \quad \tan(3x) = 1$$

The two points on the unit circle with tangent 1 correspond to angles $\frac{\pi}{4}$ and $\frac{5\pi}{4}$.

$0 \leqslant x \leqslant 2\pi$ $\therefore$ $0 \leqslant 3x \leqslant 6\pi$

So, $3x = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}, \frac{17\pi}{4}, \frac{21\pi}{4}$

$\therefore$ $x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{3\pi}{4}, \frac{13\pi}{12}, \frac{17\pi}{12}, \frac{7\pi}{4}$

c

$$\sin(2x) = \sqrt{3}\cos(2x), \quad 0 \leqslant x \leqslant 2\pi$$

$$\therefore \quad \frac{\sin(2x)}{\cos(2x)} = \sqrt{3}$$

$$\therefore \quad \tan(2x) = \sqrt{3}$$

The two points on the unit circle with tangent $\sqrt{3}$ correspond to angles $\frac{\pi}{3}$ and $\frac{4\pi}{3}$.

$0 \leqslant x \leqslant 2\pi$ $\therefore$ $0 \leqslant 2x \leqslant 4\pi$

So, $2x = \frac{\pi}{3}, \frac{4\pi}{3}, \frac{7\pi}{3}, \frac{10\pi}{3}$

$\therefore$ $x = \frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}, \frac{5\pi}{3}$

13 **a** The range of $y = \arctan x$ is $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

$\frac{\pi}{4}$ is within the range.

$\therefore$ $x = \tan(\frac{\pi}{4}) = 1$

b The range of $y = \arcsin x$ is $-\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$.

$-\frac{\pi}{3}$ is within the range.

$\therefore$ $x = \sin(-\frac{\pi}{3}) = -\frac{\sqrt{3}}{2}$

c The range of $y = \arccos x$ is $0 \leqslant y \leqslant \pi$.

$\frac{3\pi}{4}$ is within the range.

$\therefore$ $x = \cos(\frac{3\pi}{4}) = -\frac{1}{\sqrt{2}}$

d The range of $y = \arcsin(x + 1)$ is $-\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$.

$\frac{\pi}{6}$ is within the range.

$\therefore$ $x + 1 = \sin(\frac{\pi}{6})$

$\therefore$ $x + 1 = \frac{1}{2}$

$\therefore$ $x = -\frac{1}{2}$

e The range of $y = \arccos x$ is $0 \leqslant y \leqslant \pi$.
$-\frac{\pi}{4}$ is outside this range.
$\therefore$ no solution exists, even though we can find $\cos(-\frac{\pi}{4})$.

f The range of $y = \arctan(x - \sqrt{3})$ is $-\frac{\pi}{2} < y < \frac{\pi}{2}$.
$-\frac{\pi}{3}$ is within the range.
$\therefore\ x - \sqrt{3} = \tan(-\frac{\pi}{3})$
$\therefore\ x - \sqrt{3} = -\sqrt{3}$
$\therefore\ x = 0$

EXERCISE 13B

1 **a** $H(t) = 10\sin\left(\frac{\pi}{50}(t-25)\right) + 12$ metres
After 50 seconds, $t = 50$ and $H(50) = 10\sin\left(\frac{\pi}{50}(50-25)\right) + 12 = 22$ metres.
So, the green light will be 22 metres high.

b A full circle is completed in one period.
period $= \dfrac{2\pi}{\frac{\pi}{50}} = 2\pi \times \frac{50}{\pi} = 100$ seconds
So, it takes 100 seconds for the wheel to complete a full circle.

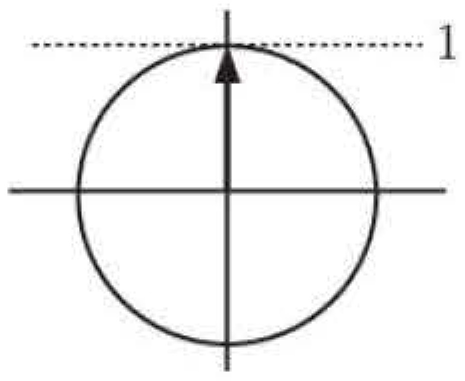
c We need to solve $H(t) = 16$ so $10\sin\left(\frac{\pi}{50}(t-25)\right) + 12 = 16$.
3 minutes $= 3 \times 60 = 180$ seconds
Using technology to find the intersection points of $Y_1 = 10\sin\left(\frac{\pi}{50}(t-25)\right) + 12$ and $Y_2 = 16$ for $0 \leqslant t \leqslant 180$, $t \approx 31.5, 68.5, 132, 168$
So, in the first three minutes, the green light is 16 metres above the ground at 31.5, 68.5, 132, and 168 seconds.

2 **a** $P(t) = 7500 + 3000\sin(\frac{\pi t}{8})$, $0 \leqslant t \leqslant 12$

i $P(0) = 7500 + 3000\sin 0$
$= 7500 + 0$
$= 7500$ grasshoppers

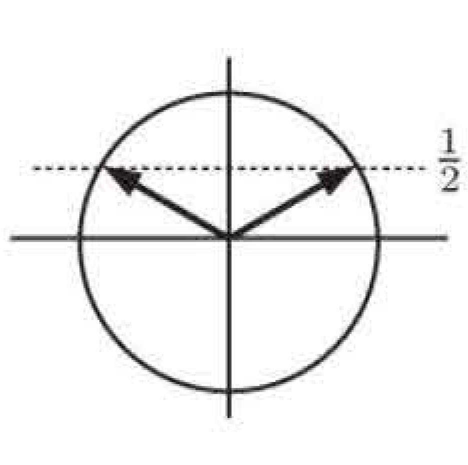
ii $P(5) = 7500 + 3000\sin(\frac{5\pi}{8})$
$\approx 10\,271.63$
$\approx 10\,300$ grasshoppers

b The greatest value of $P(t)$ occurs when $\sin\left(\frac{\pi t}{8}\right) = 1$, so the greatest population is $7500 + 3000 = 10\,500$ grasshoppers.



The point on the unit circle with sine 1 corresponds to angle $\frac{\pi}{2}$.
$0 \leqslant t \leqslant 12 \quad \therefore\ 0 \leqslant \frac{\pi t}{8} \leqslant \frac{3\pi}{2}$
So $\frac{\pi t}{8} = \frac{\pi}{2}$
$\therefore\ t = 4$
So the greatest population occurs after 4 weeks.

c **i** When $P(t) = 9000$,
$7500 + 3000\sin\left(\frac{\pi t}{8}\right) = 9000$
$\therefore\ 3000\sin\left(\frac{\pi t}{8}\right) = 1500$
$\therefore\ \sin\left(\frac{\pi t}{8}\right) = \frac{1}{2}$



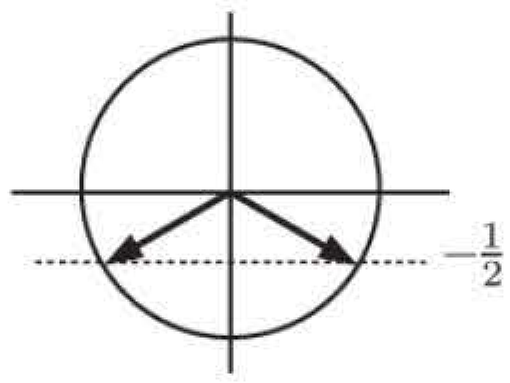
The points on the unit circle with sine $\frac{1}{2}$ correspond to angles $\frac{\pi}{6}$ and $\frac{5\pi}{6}$.
$0 \leqslant t \leqslant 12 \quad \therefore\ 0 \leqslant \frac{\pi t}{8} \leqslant \frac{3\pi}{2}$
So $\frac{\pi t}{8} = \frac{\pi}{6}, \frac{5\pi}{6}$
$\therefore\ t = \frac{8}{6}, \frac{40}{6}$
$\therefore\ t = 1\frac{1}{3}$ or $6\frac{2}{3}$
So, the population is 9000 at $1\frac{1}{3}$ weeks and $6\frac{2}{3}$ weeks.

ii When $P(t) = 6000$,

$7500 + 3000\sin\left(\frac{\pi t}{8}\right) = 6000$

$\therefore \quad 3000\sin\left(\frac{\pi t}{8}\right) = -1500$

$\therefore \quad \sin\left(\frac{\pi t}{8}\right) = -\frac{1}{2}$



The points on the unit circle with sine $-\frac{1}{2}$ correspond to angles $\frac{7\pi}{6}$ and $\frac{11\pi}{6}$.

$0 \leqslant t \leqslant 12 \quad \therefore \quad 0 \leqslant \frac{\pi t}{8} \leqslant \frac{3\pi}{2}$

So $\frac{\pi t}{8} = \frac{7\pi}{6}$

$\therefore \quad t = \frac{56}{6}$

$\therefore \quad t = 9\frac{1}{3}$

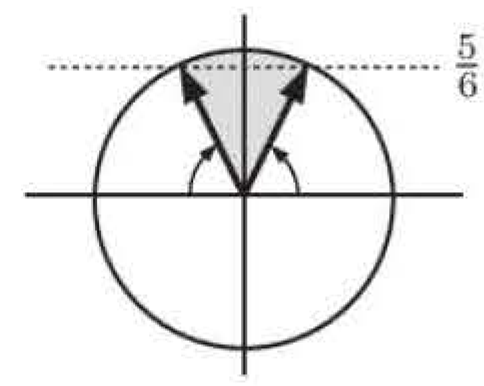
So, the population is 6000 at $9\frac{1}{3}$ weeks.

d If $P(t) > 10\,000$, then

$7500 + 3000\sin\left(\frac{\pi t}{8}\right) > 10\,000$

$\therefore \quad 3000\sin\left(\frac{\pi t}{8}\right) > 2500$

$\therefore \quad \sin\left(\frac{\pi t}{8}\right) > \frac{5}{6}$



Solving $\sin\left(\frac{\pi t}{8}\right) = \frac{5}{6}$ using technology

$t \approx 2.51$ or 5.49 So, $2.51 \leqslant t \leqslant 5.49$ weeks.

3 $H(t) = 20 - 19\sin\left(\frac{2\pi t}{3}\right)$

a $H(0) = 20 - 19(0)$

$= 20$ m

So, at time $t = 0$, the light is 20 m above the ground.

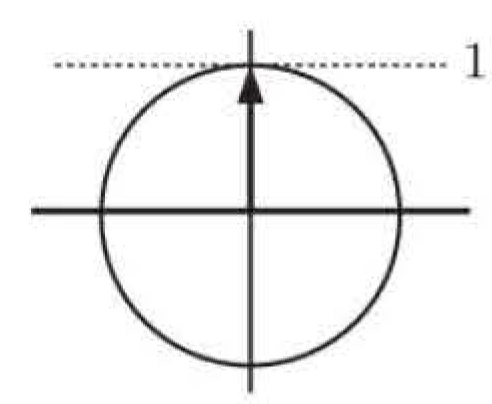
b H is lowest when $\sin\left(\frac{2\pi t}{3}\right) = 1$

$\therefore \quad \dfrac{2\pi t}{3} = \frac{\pi}{2} + k2\pi$

$\therefore \quad \dfrac{2t}{3} = \frac{1}{2} + k\,2$

$\therefore \quad t = \frac{3}{4} + k\,3$

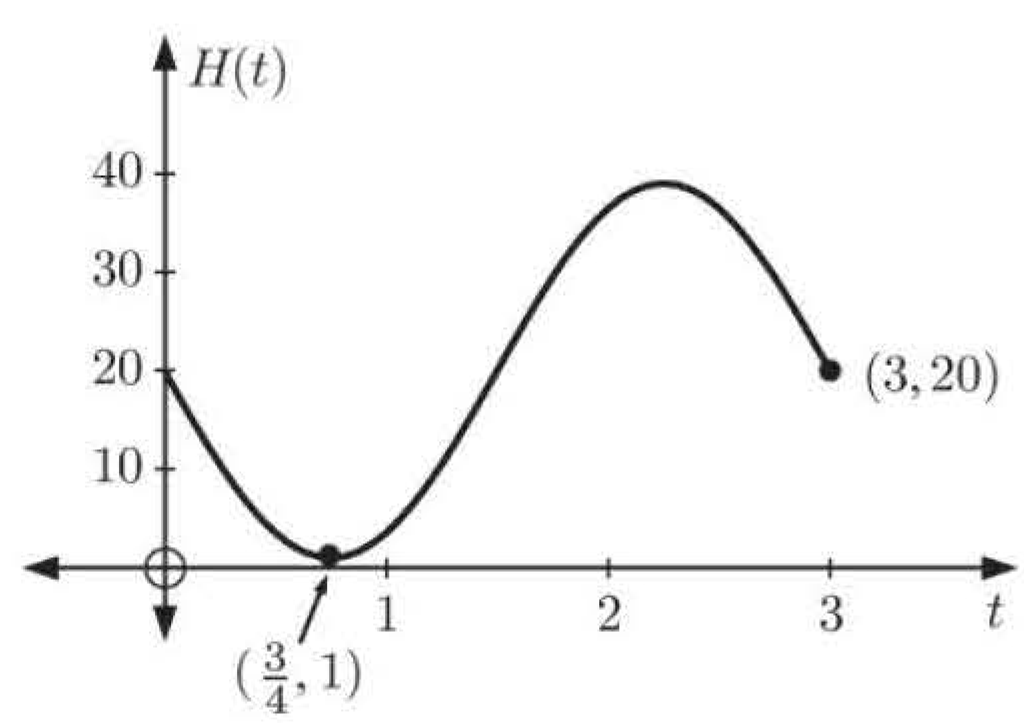
$\therefore \quad t = \frac{3}{4}$ min $\{$as $k = 0\}$



c period $= \dfrac{2\pi}{\frac{2\pi}{3}} = 3$ min

$\therefore$ one revolution takes 3 minutes

d



4 $P(t) = 400 + 250\sin\left(\frac{\pi t}{2}\right)$

a $P(0) = 400 + 250(0)$

$= 400$ water buffalo

b **i** $P(\frac{1}{2}) = 400 + 250\sin\left(\dfrac{\pi(\frac{1}{2})}{2}\right)$

$= 400 + 250\sin(\frac{\pi}{4})$

$= 400 + 250 \times \frac{1}{\sqrt{2}}$

≈ 577 water buffalo

ii $P(2) = 400 + 250\sin\pi$

$= 400 + 250(0)$

$= 400$ water buffalo

c $P(1) = 400 + 250\sin(\frac{\pi}{2})$

$= 400 + 250 \times 1$

$= 650$ water buffalo

This is the maximum herd size.

d $P(t)$ is smallest when $\sin\left(\frac{\pi t}{2}\right) = -1$ and is $400 - 250 = 150$ water buffalo.

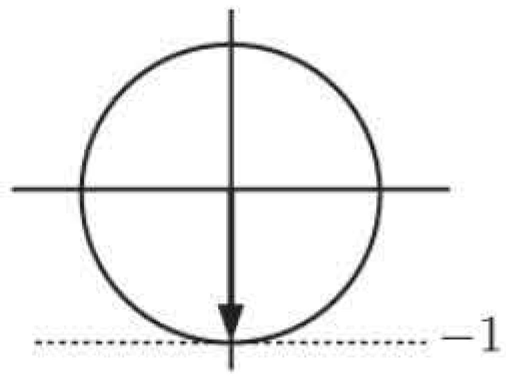
It occurs when

$$\frac{\pi t}{2} = \frac{3\pi}{2} + k2\pi$$

$$\therefore \quad \frac{t}{2} = \tfrac{3}{2} + k\,2$$

$$\therefore \quad t = 3 + 4k$$

So, the first time is after 3 years.

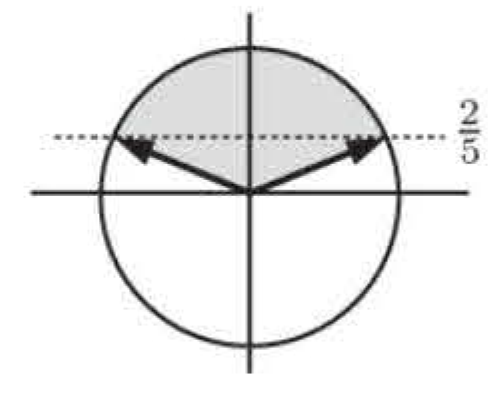


e If $P(t) > 500$ then

$$400 + 250\sin\left(\tfrac{\pi t}{2}\right) > 500$$

$$\therefore \quad 250\sin\left(\tfrac{\pi t}{2}\right) > 100$$

$$\therefore \quad \sin\left(\tfrac{\pi t}{2}\right) > \tfrac{2}{5}$$



Using technology:

$\sin\left(\frac{\pi t}{2}\right) = \frac{2}{5}$ when

$$\frac{\pi t}{2} \approx 0.4115 \text{ or } \pi - 0.4115$$

$$\therefore \quad t \approx 0.262 \text{ or } 1.74$$

So, for $\sin\left(\frac{\pi t}{2}\right) > \frac{2}{5}$, $0.262 < t < 1.74$

$\therefore$ the herd first exceeded 500 when $t \approx 0.262$ years.

5 **a** The period is 4 seconds.

$$\therefore \quad \frac{2\pi}{b} = 4$$

$$\therefore \quad b = \tfrac{\pi}{2}$$

Amplitude is 3 $\therefore \ a = 3$

$d = 1 + 3 = 4, \quad c = 0$

$\therefore \ H(t) = 3\cos(\frac{\pi}{2}(t-0)) + 4$ metres

$\therefore \ H(t) = 3\cos(\frac{\pi}{2}t) + 4$ metres

Check: When $t = 0$, $H(0) = 3\cos 0 + 4 = 7$ ✓

b X enters the water when $H(t) = 2$

$$\therefore \quad 3\cos\left(\tfrac{\pi t}{2}\right) + 4 = 2$$

$$\therefore \quad \cos\left(\tfrac{\pi t}{2}\right) = -\tfrac{2}{3}$$

Using technology, $t \approx 1.46$ seconds

6 $C(t) = 9.2\sin(\frac{\pi}{7}(t-4)) + 107.8$ cents L^{-1}

a **i** 107.8 is the median value. Values are between $107.8 - 9.2$ and $107.8 + 9.2$

$= 98.6$ cents L^{-1} (min.) and 117.0 cents L^{-1} (max.)

$\therefore$ the statement is true.

ii period $= \dfrac{2\pi}{\frac{\pi}{7}} = 14$ days $\therefore$ true

b $C(7) = 9.2\sin(\frac{\pi}{7}(3)) + 107.8 \approx 116.8$ cents L^{-1}

c When $C(t) = \$1.10\ \text{L}^{-1}$ then $9.2\sin(\frac{\pi}{7}(t-4)) + 107.8 = 110$

$$\therefore \quad \sin(\tfrac{\pi}{7}(t-4)) = \frac{2.2}{9.2} \approx 0.239\,13$$

$$\therefore \quad \tfrac{\pi}{7}(t-4) \approx 0.2415,\ \pi - 0.2415,\ 2\pi + 0.2415,\ 3\pi - 0.2415$$

$$\approx 0.2415,\ 2.9001,\ 6.5247,\ 9.1833$$

$$\therefore \quad t - 4 \approx 0.538,\ 6.462,\ 14.538,\ 20.462$$

$$\therefore \quad t \approx 4.538,\ 10.462,\ 18.538,\ 24.462$$

So, the price is \$1.10 per litre on the 5th, 11th, 19th, and 25th days.

d The minimum cost per litre is $-9.2 + 107.8 = 98.6$ cents L^{-1}

when $\sin(\frac{\pi}{7}(t-4)) = -1$

$$\therefore \quad \tfrac{\pi}{7}(t-4) = \tfrac{3\pi}{2}$$

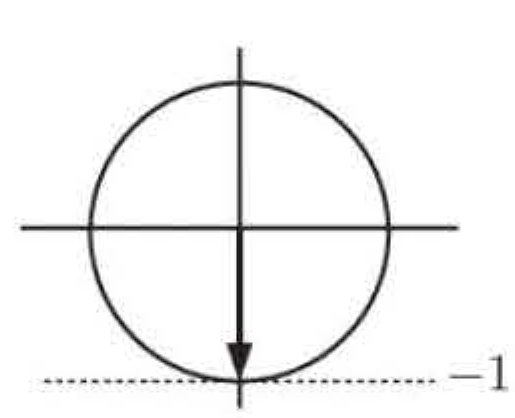
$$\therefore \quad \frac{t-4}{7} = \tfrac{3}{2}$$

$$\therefore \quad 2t - 8 = 21$$

$$\therefore \quad 2t = 29$$

$$\therefore \quad t = 14.5 \pm 14k$$

{period is 14 days}



So, the minimum occurred on the 1st day and the 15th day.

EXERCISE 13C.1

1 **a** $\sin\theta + \sin\theta$
$= 2\sin\theta$

b $2\cos\theta + \cos\theta$
$= 3\cos\theta$

c $3\sin\theta - \sin\theta$
$= 2\sin\theta$

d $3\sin\theta - 2\sin\theta$
$= \sin\theta$

e $\tan\theta - 3\tan\theta$
$= -2\tan\theta$

f $2\cos^2\theta - 5\cos^2\theta$
$= -3\cos^2\theta$

2 **a** $3\sin^2\theta + 3\cos^2\theta$
$= 3(\sin^2\theta + \cos^2\theta)$
$= 3(1)$
$= 3$

b $-2\sin^2\theta - 2\cos^2\theta$
$= -2(\sin^2\theta + \cos^2\theta)$
$= -2(1)$
$= -2$

c $-\cos^2\theta - \sin^2\theta$
$= -(\cos^2\theta + \sin^2\theta)$
$= -(1)$
$= -1$

d $3 - 3\sin^2\theta$
$= 3(1 - \sin^2\theta)$
$= 3\cos^2\theta$

e $4 - 4\cos^2\theta$
$= 4(1 - \cos^2\theta)$
$= 4\sin^2\theta$

f $\cos^3\theta + \cos\theta\sin^2\theta$
$= \cos\theta(\cos^2\theta + \sin^2\theta)$
$= \cos\theta(1)$
$= \cos\theta$

g $\cos^2\theta - 1$
$= 1 - \sin^2\theta - 1$
$= -\sin^2\theta$

h $\sin^2\theta - 1$
$= 1 - \cos^2\theta - 1$
$= -\cos^2\theta$

i $2\cos^2\theta - 2$
$= -2(1 - \cos^2\theta)$
$= -2\sin^2\theta$

j $\dfrac{1 - \sin^2\theta}{\cos^2\theta}$
$= \dfrac{\cos^2\theta}{\cos^2\theta}$
$= 1$

k $\dfrac{1 - \cos^2\theta}{\sin\theta}$
$= \dfrac{\sin^2\theta}{\sin\theta}$
$= \sin\theta$

l $\dfrac{\cos^2\theta - 1}{-\sin\theta}$
$= \dfrac{1 - \sin^2\theta - 1}{-\sin\theta}$
$= \dfrac{-\sin^2\theta}{-\sin\theta}$
$= \sin\theta$

3 **a** $3\tan x - \dfrac{\sin x}{\cos x}$
$= 3\tan x - \tan x$
$= 2\tan x$

b $\dfrac{\sin^2 x}{\cos^2 x}$
$= \left(\dfrac{\sin x}{\cos x}\right)^2$
$= \tan^2 x$

c $\tan x \cos x$
$= \dfrac{\sin x}{\cos x} \times \cos x$
$= \sin x$

d $\dfrac{\sin x}{\tan x}$
$= \sin x \div \dfrac{\sin x}{\cos x}$
$= \sin x \times \dfrac{\cos x}{\sin x}$
$= \cos x$

e $3\sin x + 2\cos x \tan x$
$= 3\sin x + 2\cos x \dfrac{\sin x}{\cos x}$
$= 3\sin x + 2\sin x$
$= 5\sin x$

f $\dfrac{2\tan x}{\sin x}$
$= 2\left(\dfrac{\cancel{\sin x}}{\cos x}\right) \times \dfrac{1}{\cancel{\sin x}_1}$
$= \dfrac{2}{\cos x}$

4 **a** $(1 + \sin\theta)^2$
$= 1 + 2\sin\theta + \sin^2\theta$

b $(\sin\alpha - 2)^2$
$= \sin^2\alpha - 4\sin\alpha + 4$

c $(\tan\alpha - 1)^2$
$= \tan^2\alpha - 2\tan\alpha + 1$

d $(\sin\alpha + \cos\alpha)^2$
$= \sin^2\alpha + 2\sin\alpha\cos\alpha + \cos^2\alpha$
$= 1 + 2\sin\alpha\cos\alpha$

e $(\sin\beta - \cos\beta)^2$
$= \sin^2\beta - 2\sin\beta\cos\beta + \cos^2\beta$
$= 1 - 2\sin\beta\cos\beta$

f $-(2 - \cos\alpha)^2$
$= -[4 - 4\cos\alpha + \cos^2\alpha]$
$= -4 + 4\cos\alpha - \cos^2\alpha$

5 **a**
$$\begin{aligned}&1-\sec^2\beta\\&=1-\frac{1}{\cos^2\beta}\\&=\frac{\cos^2\beta-1}{\cos^2\beta}\\&=\frac{-\sin^2\beta}{\cos^2\beta}\\&=-\tan^2\beta\end{aligned}$$

b
$$\begin{aligned}&\frac{\tan^2\theta(\cot^2\theta+1)}{\tan^2\theta+1}\\&=\frac{\tan^2\theta\cot^2\theta+\tan^2\theta}{\tan^2\theta+1}\\&=\frac{1+\tan^2\theta}{\tan^2\theta+1}\\&=1\end{aligned}$$

c
$$\begin{aligned}&\cos^2\alpha(\sec^2\alpha-1)\\&=\cos^2\alpha\sec^2\alpha-\cos^2\alpha\\&=1-\cos^2\alpha\\&=\sin^2\alpha\end{aligned}$$

d
$$\begin{aligned}&(\sin x+\tan x)(\sin x-\tan x)\\&=\sin^2x-\tan^2x\quad\{\text{or }-\sin^2x\tan^2x\}\end{aligned}$$

e
$$\begin{aligned}&(2\sin\theta+3\cos\theta)^2+(3\sin\theta-2\cos\theta)^2\\&=4\sin^2\theta+12\sin\theta\cos\theta+9\cos^2\theta\\&\qquad+9\sin^2\theta-12\sin\theta\cos\theta+4\cos^2\theta\\&=13\sin^2\theta+13\cos^2\theta\\&=13(\sin^2\theta+\cos^2\theta)\\&=13\end{aligned}$$

f
$$\begin{aligned}&(1+\csc\theta)(\sin\theta-\sin^2\theta)\\&=\sin\theta-\sin^2\theta+\csc\theta\sin\theta-\csc\theta\sin^2\theta\\&=\sin\theta-\sin^2\theta+1-\sin\theta\\&=1-\sin^2\theta\\&=\cos^2\theta\end{aligned}$$

g
$$\begin{aligned}&\sec A-\sin A\tan A-\cos A\\&=\frac{1}{\cos A}-\frac{\sin A\sin A}{\cos A}-\frac{\cos^2A}{\cos A}\\&=\frac{1-\sin^2A-\cos^2A}{\cos A}\\&=\frac{1-(\sin^2A+\cos^2A)}{\cos A}\\&=0\end{aligned}$$

EXERCISE 13C.2

1 **a**
$$\begin{aligned}&1-\sin^2\theta\\&=(1+\sin\theta)(1-\sin\theta)\end{aligned}$$

b
$$\begin{aligned}&\sin^2\alpha-\cos^2\alpha\\&=(\sin\alpha+\cos\alpha)(\sin\alpha-\cos\alpha)\end{aligned}$$

c
$$\begin{aligned}&\tan^2\alpha-1\\&=(\tan\alpha+1)(\tan\alpha-1)\end{aligned}$$

d
$$\begin{aligned}&2\sin^2\beta-\sin\beta\\&=\sin\beta(2\sin\beta-1)\end{aligned}$$

e
$$\begin{aligned}&2\cos\phi+3\cos^2\phi\\&=\cos\phi(2+3\cos\phi)\end{aligned}$$

f
$$\begin{aligned}&3\sin^2\theta-6\sin\theta\\&=3\sin\theta(\sin\theta-2)\end{aligned}$$

g
$$\begin{aligned}&\tan^2\theta+5\tan\theta+6\\&=(\tan\theta+2)(\tan\theta+3)\end{aligned}$$

h
$$\begin{aligned}&2\cos^2\theta+7\cos\theta+3\\&=(2\cos\theta+1)(\cos\theta+3)\end{aligned}$$

i
$$\begin{aligned}&6\cos^2\alpha-\cos\alpha-1\\&=(3\cos\alpha+1)(2\cos\alpha-1)\end{aligned}$$

j
$$\begin{aligned}&3\tan^2\alpha-2\tan\alpha\\&=\tan\alpha(3\tan\alpha-2)\end{aligned}$$

k
$$\begin{aligned}&\sec^2\beta-\csc^2\beta\\&=(\sec\beta+\csc\beta)(\sec\beta-\csc\beta)\end{aligned}$$

l
$$\begin{aligned}&2\cot^2x-3\cot x+1\\&=(2\cot x-1)(\cot x-1)\end{aligned}$$

m
$$\begin{aligned}&2\sin^2x+7\sin x\cos x+3\cos^2x\\&=(2\sin x+\cos x)(\sin x+3\cos x)\end{aligned}$$

2 **a**
$$\begin{aligned}&\frac{1-\sin^2\alpha}{1-\sin\alpha}\\&=\frac{(1+\sin\alpha)\cancel{(1-\sin\alpha)}}{\cancel{1-\sin\alpha}_{\,1}}\\&=1+\sin\alpha\end{aligned}$$

b
$$\begin{aligned}&\frac{\tan^2\beta-1}{\tan\beta+1}\\&=\frac{\cancel{(\tan\beta+1)}(\tan\beta-1)}{\cancel{\tan\beta+1}_{\,1}}\\&=\tan\beta-1\end{aligned}$$

c $\dfrac{\cos^2\phi - \sin^2\phi}{\cos\phi + \sin\phi}$

$= \dfrac{\cancel{(\cos\phi + \sin\phi)}(\cos\phi - \sin\phi)}{\cancel{\cos\phi + \sin\phi}_{\,1}}$

$= \cos\phi - \sin\phi$

d $\dfrac{\cos^2\phi - \sin^2\phi}{\cos\phi - \sin\phi}$

$= \dfrac{(\cos\phi + \sin\phi)\cancel{(\cos\phi - \sin\phi)}}{\cancel{\cos\phi - \sin\phi}_{\,1}}$

$= \cos\phi + \sin\phi$

e $\dfrac{\sin\alpha + \cos\alpha}{\sin^2\alpha - \cos^2\alpha}$

$= \dfrac{\cancel{\sin\alpha + \cos\alpha}^{\,1}}{\cancel{(\sin\alpha + \cos\alpha)}(\sin\alpha - \cos\alpha)}$

$= \dfrac{1}{\sin\alpha - \cos\alpha}$

f $\dfrac{3 - 3\sin^2\theta}{6\cos\theta} = \dfrac{3(1 - \sin^2\theta)}{6\cos\theta}$

$= \dfrac{{}^{1}\cancel{3}\cos^{\cancel{2}}\theta}{{}_{2}\cancel{6}\cancel{\cos\theta}}$

$= \dfrac{\cos\theta}{2}$

g $1 - \dfrac{\cos^2\theta}{1 + \sin\theta}$

$= \dfrac{(1 + \sin\theta) - (1 - \sin^2\theta)}{1 + \sin\theta}$

$= \dfrac{1 + \sin\theta - 1 + \sin^2\theta}{1 + \sin\theta}$

$= \dfrac{\sin\theta\cancel{(1 + \sin\theta)}}{\cancel{1 + \sin\theta}_{\,1}}$

$= \sin\theta$

h $\dfrac{1 + \cot\theta}{\csc\theta} - \dfrac{\sec\theta}{\tan\theta + \cot\theta}$

$= \sin\theta\left(1 + \dfrac{\cos\theta}{\sin\theta}\right) - \dfrac{1}{\cos\theta\left(\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\right)}$

$= \sin\theta + \cos\theta - \dfrac{1}{\sin\theta + \frac{\cos^2\theta}{\sin\theta}}$

$= \sin\theta + \cos\theta - \dfrac{\sin\theta}{\sin^2\theta + \cos^2\theta}$

$= \sin\theta + \cos\theta - \sin\theta$

$= \cos\theta$

i $\dfrac{\tan^2\theta}{\sec\theta - 1} = \dfrac{\sec^2\theta - 1}{\sec\theta - 1}$

$= \dfrac{(\sec\theta + 1)\cancel{(\sec\theta - 1)}}{\cancel{\sec\theta - 1}_{\,1}}$

$= \sec\theta + 1$

3 **a** $(\cos\theta + \sin\theta)^2 + (\cos\theta - \sin\theta)^2$

$= \cos^2\theta + \cancel{2\cos\theta\sin\theta} + \sin^2\theta$
$\quad + \cos^2\theta - \cancel{2\cos\theta\sin\theta} + \sin^2\theta$

$= 2\cos^2\theta + 2\sin^2\theta$

$= 2(\cos^2\theta + \sin^2\theta)$

$= 2$

b $(2\sin\theta + 3\cos\theta)^2 + (3\sin\theta - 2\cos\theta)^2$

$= 4\sin^2\theta + \cancel{12\sin\theta\cos\theta} + 9\cos^2\theta$
$\quad + 9\sin^2\theta - \cancel{12\sin\theta\cos\theta} + 4\cos^2\theta$

$= 13\sin^2\theta + 13\cos^2\theta$

$= 13(\sin^2\theta + \cos^2\theta)$

$= 13$

c $(1 - \cos\theta)\left(1 + \dfrac{1}{\cos\theta}\right)$

$= 1 + \dfrac{1}{\cos\theta} - \cos\theta - 1$

$= \dfrac{1}{\cos\theta} - \cos\theta$

$= \dfrac{1 - \cos^2\theta}{\cos\theta}$

$= \dfrac{\sin^2\theta}{\cos\theta}$

$= \tan\theta\sin\theta$

d $\left(1 + \dfrac{1}{\sin\theta}\right)(\sin\theta - \sin^2\theta)$

$= \cancel{\sin\theta} - \sin^2\theta + 1 - \cancel{\sin\theta}$

$= 1 - \sin^2\theta$

$= \cos^2\theta$

e $$\begin{aligned}\sec A - \cos A &= \frac{1}{\cos A} - \cos A\\ &= \frac{1-\cos^2 A}{\cos A}\\ &= \frac{\sin^2 A}{\cos A}\\ &= \tan A \sin A\end{aligned}$$

f $$\begin{aligned}\frac{\cos\theta}{1-\sin\theta} &= \frac{\cos\theta(1+\sin\theta)}{(1-\sin\theta)(1+\sin\theta)}\\ &= \frac{\cos\theta+\cos\theta\sin\theta}{1-\sin^2\theta}\\ &= \frac{\cos\theta+\cos\theta\sin\theta}{\cos^2\theta}\\ &= \frac{1}{\cos\theta}+\frac{\sin\theta}{\cos\theta}\\ &= \sec\theta+\tan\theta\end{aligned}$$

g $$\begin{aligned}&\frac{\cos\alpha}{1-\tan\alpha}+\frac{\sin\alpha}{1-\cot\alpha}\\ &= \frac{\cos\alpha}{1-\frac{\sin\alpha}{\cos\alpha}}+\frac{\sin\alpha}{1-\frac{\cos\alpha}{\sin\alpha}}\\ &= \frac{\cos\alpha}{\frac{\cos\alpha-\sin\alpha}{\cos\alpha}}+\frac{\sin\alpha}{\frac{\sin\alpha-\cos\alpha}{\sin\alpha}}\\ &= \frac{\cos^2\alpha}{\cos\alpha-\sin\alpha}+\frac{\sin^2\alpha}{\sin\alpha-\cos\alpha}\\ &= \frac{\cos^2\alpha-\sin^2\alpha}{\cos\alpha-\sin\alpha}\\ &= \frac{(\cos\alpha+\sin\alpha)\cancel{(\cos\alpha-\sin\alpha)}}{\cancel{\cos\alpha-\sin\alpha}_{\,1}}\\ &= \sin\alpha+\cos\alpha\end{aligned}$$

h $$\begin{aligned}&\frac{\sin\theta}{1+\cos\theta}+\frac{1+\cos\theta}{\sin\theta}\\ &= \frac{\sin^2\theta+(1+\cos\theta)(1+\cos\theta)}{\sin\theta(1+\cos\theta)}\\ &= \frac{\sin^2\theta+1+2\cos\theta+\cos^2\theta}{\sin\theta(1+\cos\theta)}\\ &= \frac{1+1+2\cos\theta}{\sin\theta(1+\cos\theta)}\\ &= \frac{2\cancel{(1+\cos\theta)}}{\sin\theta\cancel{(1+\cos\theta)}}\\ &= \frac{2}{\sin\theta}\\ &= 2\csc\theta\end{aligned}$$

i $$\begin{aligned}&\frac{\sin\theta}{1-\cos\theta}-\frac{\sin\theta}{1+\cos\theta}\\ &= \frac{\sin\theta(1+\cos\theta)-\sin\theta(1-\cos\theta)}{(1-\cos\theta)(1+\cos\theta)}\\ &= \frac{\cancel{\sin\theta}+\sin\theta\cos\theta-\cancel{\sin\theta}+\sin\theta\cos\theta}{1-\cos^2\theta}\\ &= \frac{2\cancel{\sin\theta}\cos\theta}{\sin^{\cancel{2}}\theta}\\ &= \frac{2\cos\theta}{\sin\theta} = 2\cot\theta\end{aligned}$$

j $$\begin{aligned}&\frac{1}{1-\sin\theta}+\frac{1}{1+\sin\theta}\\ &= \frac{1+\sin\theta+1-\sin\theta}{(1-\sin\theta)(1+\sin\theta)}\\ &= \frac{2}{1-\sin^2\theta}\\ &= \frac{2}{\cos^2\theta}\\ &= 2\sec^2\theta\end{aligned}$$

EXERCISE 13D

1 **a** $$\begin{aligned}\sin 2\theta &= 2\sin\theta\cos\theta\\ &= 2(\tfrac{4}{5})(\tfrac{3}{5})\\ &= \tfrac{24}{25}\end{aligned}$$

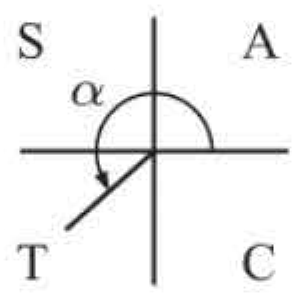
b $$\begin{aligned}\cos 2\theta &= \cos^2\theta-\sin^2\theta\\ &= \tfrac{9}{25}-\tfrac{16}{25}\\ &= -\tfrac{7}{25}\end{aligned}$$

c $$\begin{aligned}\tan 2\theta &= \frac{\sin 2\theta}{\cos 2\theta}\\ &= \frac{\frac{24}{25}}{-\frac{7}{25}}\\ &= -\tfrac{24}{7}\end{aligned}$$

2 **a** $$\begin{aligned}\cos 2A &= 2\cos^2 A-1\\ &= 2(\tfrac{1}{3})^2-1\\ &= 2\times\tfrac{1}{9}-1\\ &= \tfrac{2}{9}-1\\ &= -\tfrac{7}{9}\end{aligned}$$

b $$\begin{aligned}\cos 2\phi &= 1-2\sin^2\phi\\ &= 1-2(-\tfrac{2}{3})^2\\ &= 1-2(\tfrac{4}{9})\\ &= 1-\tfrac{8}{9}\\ &= \tfrac{1}{9}\end{aligned}$$

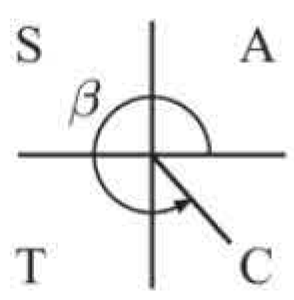
3 **a** $\sin\alpha = -\frac{2}{3}$
α is in Q3
$\therefore\ \cos\alpha < 0$



$$\cos^2\alpha + \sin^2\alpha = 1$$
$$\therefore\ \cos^2\alpha + \tfrac{4}{9} = 1$$
$$\therefore\ \cos^2\alpha = \tfrac{5}{9}$$
$$\therefore\ \cos\alpha = -\tfrac{\sqrt{5}}{3} \quad \{\text{since } \cos\alpha < 0\}$$

b $$\sin 2\alpha = 2\sin\alpha\cos\alpha = 2\left(-\tfrac{2}{3}\right)\left(-\tfrac{\sqrt{5}}{3}\right) = \tfrac{4\sqrt{5}}{9}$$

4 **a** $\cos\beta = \frac{2}{5}$
β is in Q4
$\therefore\ \sin\beta < 0$



$$\cos^2\beta + \sin^2\beta = 1$$
$$\therefore\ \tfrac{4}{25} + \sin^2\beta = 1$$
$$\therefore\ \sin^2\beta = \tfrac{21}{25}$$
$$\therefore\ \sin\beta = -\tfrac{\sqrt{21}}{5} \quad \{\text{since } \sin\beta < 0\}$$

b $$\sin 2\beta = 2\sin\beta\cos\beta = 2\left(-\tfrac{\sqrt{21}}{5}\right)\left(\tfrac{2}{5}\right) = -\tfrac{4\sqrt{21}}{25}$$

5 α is acute $\therefore$ $\cos\alpha$ and $\sin\alpha$ are positive

a $$\cos 2\alpha = 2\cos^2\alpha - 1$$
$$\therefore\ -\tfrac{7}{9} = 2\cos^2\alpha - 1$$
$$\therefore\ 2\cos^2\alpha = \tfrac{2}{9}$$
$$\therefore\ \cos^2\alpha = \tfrac{1}{9}$$
$$\therefore\ \cos\alpha = \tfrac{1}{3} \quad \{\text{since } \cos\alpha > 0\}$$

b $$\sin^2\alpha = 1 - \cos^2\alpha$$
$$\therefore\ \sin\alpha = \sqrt{1-\cos^2\alpha} \quad \{\text{since } \sin\alpha > 0\}$$
$$= \sqrt{1-\tfrac{1}{9}} = \sqrt{\tfrac{8}{9}} = \tfrac{2\sqrt{2}}{3}$$

6 **a** $$\tan 2A = \tfrac{21}{20}$$
$$\therefore\ \frac{2\tan A}{1-\tan^2 A} = \tfrac{21}{20}$$
$$\therefore\ 40\tan A = 21 - 21\tan^2 A$$
$$\therefore\ 21\tan^2 A + 40\tan A - 21 = 0$$
$$\therefore\ (7\tan A - 3)(3\tan A + 7) = 0$$
$$\therefore\ \tan A = \tfrac{3}{7} \text{ or } -\tfrac{7}{3}$$
but A is obtuse $\therefore$ $\tan A$ is negative
$$\therefore\ \tan A = -\tfrac{7}{3}$$

b $$\tan 2A = -\tfrac{12}{5}$$
$$\therefore\ \frac{2\tan A}{1-\tan^2 A} = -\tfrac{12}{5}$$
$$\therefore\ 10\tan A = -12 + 12\tan^2 A$$
$$\therefore\ 12\tan^2 A - 10\tan A - 12 = 0$$
$$\therefore\ 2(6\tan^2 A - 5\tan A - 6) = 0$$
$$\therefore\ 2(3\tan A + 2)(2\tan A - 3) = 0$$
$$\therefore\ \tan A = -\tfrac{2}{3} \text{ or } \tfrac{3}{2}$$
but A is acute $\therefore$ $\tan A$ is positive
$$\therefore\ \tan A = \tfrac{3}{2}$$

7 $$\tan(\tfrac{\pi}{4}) = 1$$
$$\therefore\ \tan(2\times\tfrac{\pi}{8}) = 1$$
$$\therefore\ \frac{2\tan\frac{\pi}{8}}{1-\tan^2(\frac{\pi}{8})} = 1$$
$$\therefore\ 2\tan(\tfrac{\pi}{8}) = 1 - \tan^2(\tfrac{\pi}{8})$$
$$\therefore\ \tan^2(\tfrac{\pi}{8}) + 2\tan(\tfrac{\pi}{8}) - 1 = 0$$
$$\therefore\ \tan(\tfrac{\pi}{8}) = \frac{-2\pm\sqrt{2^2-4(1)(-1)}}{2} = \frac{-2\pm 2\sqrt{2}}{2} = -1\pm\sqrt{2}$$
but $\frac{\pi}{8}$ is in Q1 $\therefore$ $\tan(\frac{\pi}{8})$ is positive
$$\therefore\ \tan(\tfrac{\pi}{8}) = \sqrt{2}-1$$

8 $$\left[\cos(\tfrac{\pi}{12}) + \sin(\tfrac{\pi}{12})\right]^2$$
$$= \cos^2(\tfrac{\pi}{12}) + 2\cos(\tfrac{\pi}{12})\sin(\tfrac{\pi}{12}) + \sin^2(\tfrac{\pi}{12})$$
$$= 1 + 2\cos(\tfrac{\pi}{12})\sin(\tfrac{\pi}{12})$$
$$= 1 + \sin(\tfrac{\pi}{6}) \quad \{\sin 2A = 2\cos A\sin A\}$$
$$= 1 + \tfrac{1}{2}$$
$$= \tfrac{3}{2}$$

9 **a** $2\sin\alpha\cos\alpha$
$= \sin 2\alpha$

b $4\cos\alpha\sin\alpha$
$= 2(2\sin\alpha\cos\alpha)$
$= 2\sin 2\alpha$

c $\sin\alpha\cos\alpha$
$= \frac{1}{2}(2\sin\alpha\cos\alpha)$
$= \frac{1}{2}\sin 2\alpha$

d $2\cos^2\beta - 1$
$= \cos 2\beta$

e $1 - 2\cos^2\phi$
$= -(2\cos^2\phi - 1)$
$= -\cos 2\phi$

f $1 - 2\sin^2 N$
$= \cos 2N$

g $2\sin^2 M - 1$
$= -(1 - 2\sin^2 M)$
$= -\cos 2M$

h $\cos^2\alpha - \sin^2\alpha$
$= \cos 2\alpha$

i $\sin^2\alpha - \cos^2\alpha$
$= -(\cos^2\alpha - \sin^2\alpha)$
$= -\cos 2\alpha$

j $2\sin 2A\cos 2A$
$= \sin 2(2A)$
$= \sin 4A$

k $2\cos 3\alpha\sin 3\alpha$
$= \sin 2(3\alpha)$
$= \sin 6\alpha$

l $2\cos^2 4\theta - 1$
$= \cos 2(4\theta)$
$= \cos 8\theta$

m $1 - 2\cos^2 3\beta$
$= -(2\cos^2 3\beta - 1)$
$= -\cos 2(3\beta)$
$= -\cos 6\beta$

n $1 - 2\sin^2 5\alpha$
$= \cos 2(5\alpha)$
$= \cos 10\alpha$

o $2\sin^2 3D - 1$
$= -(1 - 2\sin^2 3D)$
$= -\cos 2(3D)$
$= -\cos 6D$

p $\cos^2 2A - \sin^2 2A$
$= \cos 2(2A)$
$= \cos 4A$

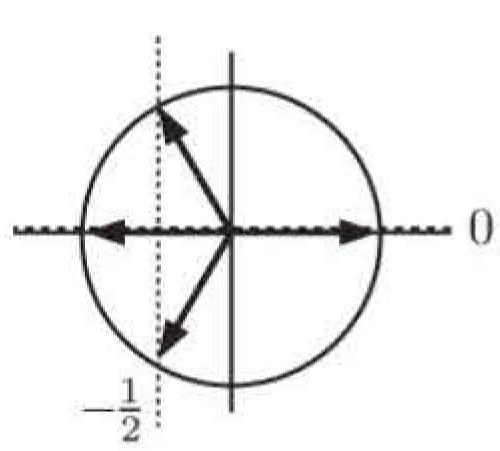
q $\cos^2(\frac{\alpha}{2}) - \sin^2(\frac{\alpha}{2})$
$= \cos 2(\frac{\alpha}{2})$
$= \cos\alpha$

r $2\sin^2 3P - 2\cos^2 3P$
$= -2[\cos^2 3P - \sin^2 3P]$
$= -2\cos 2(3P)$
$= -2\cos 6P$

10 **a** $(\sin\theta + \cos\theta)^2$
$= \sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta$
$= \sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta$
$= 1 + \sin 2\theta$

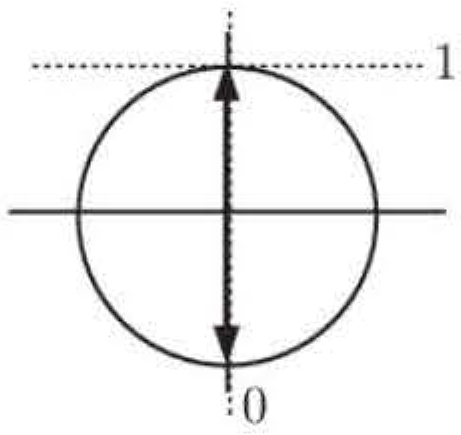
b $\cos^4\theta - \sin^4\theta$
$= (\cos^2\theta + \sin^2\theta)(\cos^2\theta - \sin^2\theta)$
$= 1 \times \cos 2\theta$
$= \cos 2\theta$

11 **a** $\sin 2x + \sin x = 0$
$\therefore\ 2\sin x\cos x + \sin x = 0$
$\therefore\ \sin x(2\cos x + 1) = 0$
$\therefore\ \sin x = 0 \quad \text{or} \quad \cos x = -\frac{1}{2}$



$\therefore\ x = 0, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, 2\pi$

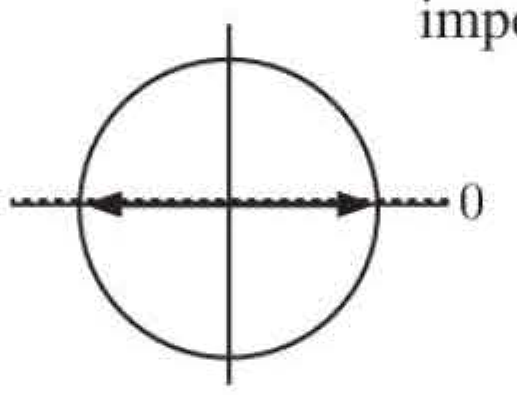
b $\sin 2x - 2\cos x = 0$
$\therefore\ 2\sin x\cos x - 2\cos x = 0$
$\therefore\ 2\cos x(\sin x - 1) = 0$
$\therefore\ \cos x = 0 \quad \text{or} \quad \sin x = 1$



$\therefore\ x = \frac{\pi}{2}, \frac{3\pi}{2}$

c $\sin 2x + 3\sin x = 0$
$\therefore\ 2\sin x\cos x + 3\sin x = 0$
$\therefore\ \sin x(2\cos x + 3) = 0$
$\therefore\ \sin x = 0 \quad \text{or} \quad \cos x = -\frac{3}{2}$

impossible

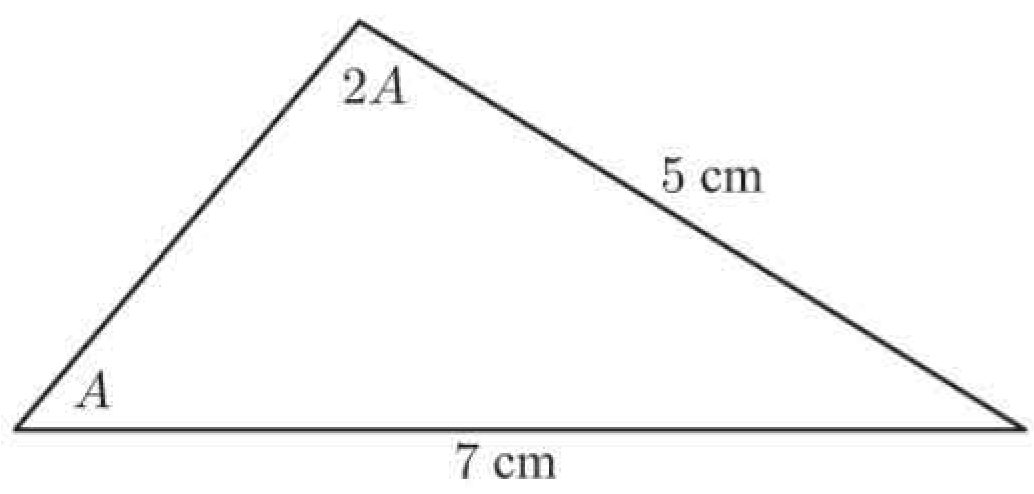


$\therefore\ x = 0, \pi, 2\pi$

12 **a**
$$\begin{aligned}&\tfrac{1}{2} - \tfrac{1}{2}\cos 2\theta\\ &= \tfrac{1}{2} - \tfrac{1}{2}(1 - 2\sin^2\theta)\\ &= \tfrac{1}{2} - \tfrac{1}{2} + \sin^2\theta\\ &= \sin^2\theta\end{aligned}$$

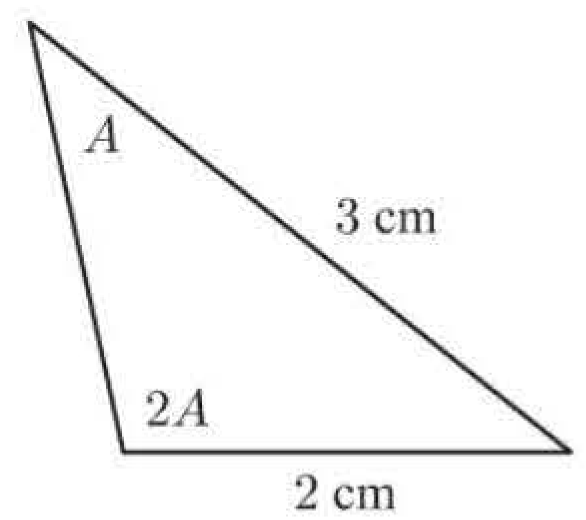
b
$$\begin{aligned}&\tfrac{1}{2} + \tfrac{1}{2}\cos 2\theta\\ &= \tfrac{1}{2} + \tfrac{1}{2}(2\cos^2\theta - 1)\\ &= \tfrac{1}{2} + \cos^2\theta - \tfrac{1}{2}\\ &= \cos^2\theta\end{aligned}$$

13 **a**



Using the sine rule,
$$\frac{\sin 2A}{7} = \frac{\sin A}{5}$$
$$\therefore \quad \frac{2\cancel{\sin A}\cos A}{7} = \frac{\cancel{\sin A}}{5}$$
$$\therefore \quad \cos A = \tfrac{7}{10}$$

b



Using the sine rule,
$$\frac{\sin 2A}{3} = \frac{\sin A}{2}$$
$$\therefore \quad \frac{2\cancel{\sin A}\cos A}{3} = \frac{\cancel{\sin A}}{2}$$
$$\therefore \quad \cos A = \tfrac{3}{4}$$

14 **a**
$$\begin{aligned}\frac{\sin 2\theta}{1-\cos 2\theta} &= \frac{2\sin\theta\cos\theta}{1-(1-2\sin^2\theta)}\\ &= \frac{2\sin\theta\cos\theta}{1-1+2\sin^2\theta}\\ &= \frac{\cancel{2}\cancel{\sin\theta}\cos\theta}{\cancel{2}\sin^{\cancel{2}}\theta}\\ &= \frac{\cos\theta}{\sin\theta}\\ &= \cot\theta\end{aligned}$$

b
$$\begin{aligned}&\frac{\sin\theta + \sin 2\theta}{1+\cos\theta+\cos 2\theta}\\ &= \frac{\sin\theta + 2\sin\theta\cos\theta}{\cancel{1}+\cos\theta+2\cos^2\theta-\cancel{1}}\\ &= \frac{\sin\theta\cancel{(1+2\cos\theta)}}{\cos\theta\cancel{(1+2\cos\theta)}}\\ &= \frac{\sin\theta}{\cos\theta}\\ &= \tan\theta\end{aligned}$$

15 **a**
$$\begin{aligned}&\frac{\sin 2\theta}{1+\cos 2\theta} - \tan\theta\\ &= \frac{2\sin\theta\cos\theta}{\cancel{1}+(2\cos^2\theta-\cancel{1})} - \frac{\sin\theta}{\cos\theta}\\ &= \frac{\cancel{2}\sin\theta\cancel{\cos\theta}}{\cancel{2}\cos^{\cancel{2}}\theta} - \frac{\sin\theta}{\cos\theta}\\ &= \frac{\sin\theta}{\cos\theta} - \frac{\sin\theta}{\cos\theta}\\ &= 0\end{aligned}$$

b
$$\begin{aligned}\tan\theta + \cot 2\theta &= \frac{\sin\theta}{\cos\theta} + \frac{\cos 2\theta}{\sin 2\theta}\\ &= \frac{\sin\theta}{\cos\theta} + \frac{1-2\sin^2\theta}{2\sin\theta\cos\theta}\\ &= \frac{\cancel{2\sin^2\theta}+1-\cancel{2\sin^2\theta}}{2\sin\theta\cos\theta}\\ &= \frac{1}{2\sin\theta\cos\theta}\\ &= \frac{1}{\sin 2\theta}\\ &= \csc 2\theta\end{aligned}$$

c
$$\begin{aligned}\frac{\sin 2\theta}{\sin\theta} - \frac{\cos 2\theta}{\cos\theta} &= \frac{2\cancel{\sin\theta}\cos\theta}{\cancel{\sin\theta}_1} - \frac{2\cos^2\theta-1}{\cos\theta}\\ &= 2\cos\theta - \frac{2\cos^2\theta-1}{\cos\theta}\\ &= \frac{\cancel{2\cos^2\theta}-\cancel{2\cos^2\theta}+1}{\cos\theta}\\ &= \frac{1}{\cos\theta} = \sec\theta\end{aligned}$$

EXERCISE 13E

1 **a** $\sin(90^\circ + \theta)$
$= \sin 90^\circ \cos\theta + \cos 90^\circ \sin\theta$
$= (1)\cos\theta + (0)\sin\theta$
$= \cos\theta$

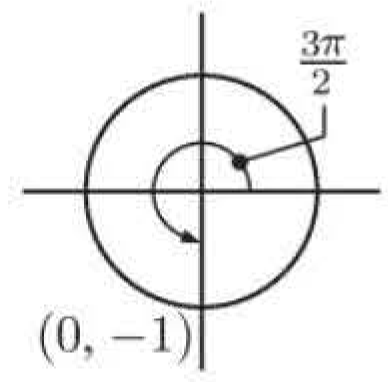
b $\cos(90^\circ + \theta)$
$= \cos 90^\circ \cos\theta - \sin 90^\circ \sin\theta$
$= (0)\cos\theta - (1)\sin\theta$
$= -\sin\theta$

c $\sin(180^\circ - \theta)$
$= \sin 180^\circ \cos\theta - \cos 180^\circ \sin\theta$
$= (0)\cos\theta - (-1)\sin\theta$
$= \sin\theta$

d $\cos(\pi + \alpha)$
$= \cos\pi\cos\alpha - \sin\pi\sin\alpha$
$= (-1)\cos\alpha - (0)\sin\alpha$
$= -\cos\alpha$

e $\sin(2\pi - A)$
$= \sin 2\pi \cos A - \cos 2\pi \sin A$
$= (0)\cos A - (1)\sin A$
$= -\sin A$

f $\cos\left(\frac{3\pi}{2} - \theta\right)$
$= \cos\left(\frac{3\pi}{2}\right)\cos\theta + \sin\left(\frac{3\pi}{2}\right)\sin\theta$
$= (0)\cos\theta + (-1)\sin\theta$
$= -\sin\theta$

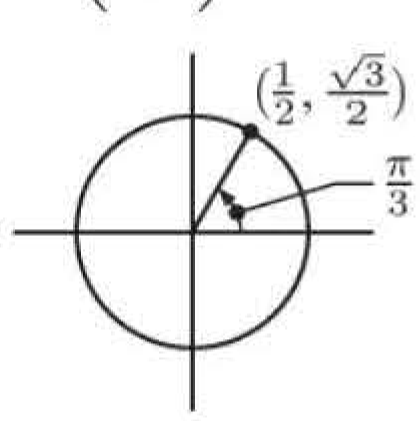


g $\tan(\frac{\pi}{4} + \theta)$
$= \dfrac{\tan\frac{\pi}{4} + \tan\theta}{1 - \tan\frac{\pi}{4}\tan\theta}$
$= \dfrac{1 + \tan\theta}{1 - \tan\theta}$

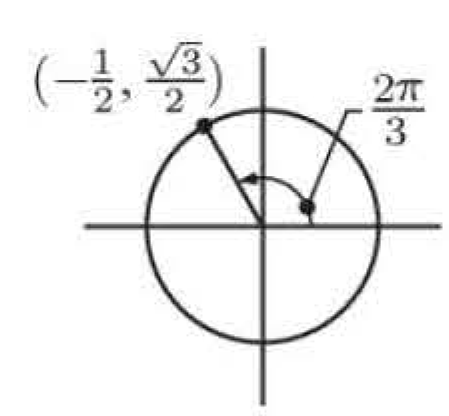
h $\tan(\theta - \frac{3\pi}{4})$
$= \dfrac{\tan\theta - \tan\frac{3\pi}{4}}{1 + \tan\theta\tan\frac{3\pi}{4}}$
$= \dfrac{\tan\theta - (-1)}{1 + \tan\theta(-1)}$
$= \dfrac{1 + \tan\theta}{1 - \tan\theta}$

i $\tan(\pi + \theta)$
$= \dfrac{\tan\pi + \tan\theta}{1 - \tan\pi\tan\theta}$
$= \dfrac{0 + \tan\theta}{1 - (0)\tan\theta}$
$= \tan\theta$

2 **a** $\sin\left(\theta + \frac{\pi}{3}\right)$
$= \sin\theta\cos\left(\frac{\pi}{3}\right) + \cos\theta\sin\left(\frac{\pi}{3}\right)$
$= \sin\theta \times \left(\frac{1}{2}\right) + \cos\theta \times \left(\frac{\sqrt{3}}{2}\right)$
$= \frac{1}{2}\sin\theta + \frac{\sqrt{3}}{2}\cos\theta$



b $\cos\left(\frac{2\pi}{3} - \theta\right)$
$= \cos\left(\frac{2\pi}{3}\right)\cos\theta + \sin\left(\frac{2\pi}{3}\right)\sin\theta$
$= \left(-\frac{1}{2}\right)\cos\theta + \left(\frac{\sqrt{3}}{2}\right)\sin\theta$
$= -\frac{1}{2}\cos\theta + \frac{\sqrt{3}}{2}\sin\theta$
$= \frac{\sqrt{3}}{2}\sin\theta - \frac{1}{2}\cos\theta$



c $\cos\left(\theta + \frac{\pi}{4}\right)$
$= \cos\theta\cos\left(\frac{\pi}{4}\right) - \sin\theta\sin\left(\frac{\pi}{4}\right)$
$= \frac{1}{\sqrt{2}}\cos\theta - \frac{1}{\sqrt{2}}\sin\theta$
$= -\frac{1}{\sqrt{2}}\sin\theta + \frac{1}{\sqrt{2}}\cos\theta$

d $\sin\left(\frac{\pi}{6} - \theta\right)$
$= \sin\left(\frac{\pi}{6}\right)\cos\theta - \cos\left(\frac{\pi}{6}\right)\sin\theta$
$= \frac{1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta$
$= -\frac{\sqrt{3}}{2}\sin\theta + \frac{1}{2}\cos\theta$

3 **a** $\cos 2\theta\cos\theta + \sin 2\theta\sin\theta$
$= \cos(2\theta - \theta)$
$= \cos\theta$

b $\sin 2A\cos A + \cos 2A\sin A$
$= \sin(2A + A)$
$= \sin 3A$

c $\cos A\sin B - \sin A\cos B$
$= \sin B\cos A - \cos B\sin A$
$= \sin(B - A)$

d $\sin\alpha\sin\beta + \cos\alpha\cos\beta$
$= \cos\alpha\cos\beta + \sin\alpha\sin\beta$
$= \cos(\alpha - \beta)$

e $\sin\phi\sin\theta - \cos\phi\cos\theta$
$= -[\cos\phi\cos\theta - \sin\phi\sin\theta]$
$= -\cos(\phi + \theta)$

f $2\sin\alpha\cos\beta - 2\cos\alpha\sin\beta$
$= 2\left[\sin\alpha\cos\beta - \cos\alpha\sin\beta\right]$
$= 2\sin(\alpha - \beta)$

g $\dfrac{\tan 2\theta - \tan\theta}{1+\tan 2\theta\tan\theta} = \tan(2\theta-\theta)$
$= \tan\theta$

h $\dfrac{\tan 2A + \tan A}{1-\tan 2A\tan A} = \tan(2A+A)$
$= \tan 3A$

4 **a** $\sin 2\theta = \sin(\theta+\theta)$
$= \sin\theta\cos\theta + \cos\theta\sin\theta$
$= 2\sin\theta\cos\theta$

b $\cos 2\theta = \cos(\theta+\theta)$
$= \cos\theta\cos\theta - \sin\theta\sin\theta$
$= \cos^2\theta - \sin^2\theta$

c $\tan 2\theta = \tan(\theta+\theta)$
$= \dfrac{\tan\theta+\tan\theta}{1-\tan\theta\tan\theta}$
$= \dfrac{2\tan\theta}{1-\tan^2\theta}$

5 **a** $\cos(\alpha+\beta)\cos(\alpha-\beta) - \sin(\alpha+\beta)\sin(\alpha-\beta)$
$= \cos\left[(\alpha+\beta)+(\alpha-\beta)\right]$
$= \cos 2\alpha$

b $\sin(\theta-2\phi)\cos(\theta+\phi) - \cos(\theta-2\phi)\sin(\theta+\phi)$
$= \sin\left[(\theta-2\phi)-(\theta+\phi)\right]$
$= \sin(-3\phi)$
$= -\sin 3\phi$

c $\cos\alpha\cos(\beta-\alpha)$
$- \sin\alpha\sin(\beta-\alpha)$
$= \cos\left[\alpha+(\beta-\alpha)\right]$
$= \cos\beta$

6 **a** $\cos 75^\circ$
$= \cos(45^\circ+30^\circ)$
$= \cos 45^\circ\cos 30^\circ - \sin 45^\circ\sin 30^\circ$
$= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right)$
$= \left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)\frac{\sqrt{2}}{\sqrt{2}}$
$= \frac{\sqrt{6}-\sqrt{2}}{4}$

b $\sin 105^\circ$
$= \sin(60^\circ+45^\circ)$
$= \sin 60^\circ\cos 45^\circ + \cos 60^\circ\sin 45^\circ$
$= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right)$
$= \left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)\frac{\sqrt{2}}{\sqrt{2}}$
$= \frac{\sqrt{6}+\sqrt{2}}{4}$

c $\cos\left(\frac{13\pi}{12}\right)$
$= \cos\left(\frac{10\pi}{12}+\frac{3\pi}{12}\right)$
$= \cos\left(\frac{5\pi}{6}+\frac{\pi}{4}\right)$
$= \cos\left(\frac{5\pi}{6}\right)\cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{5\pi}{6}\right)\sin\left(\frac{\pi}{4}\right)$
$= \left(-\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right)$
$= \left(\frac{-\sqrt{3}-1}{2\sqrt{2}}\right)\frac{\sqrt{2}}{\sqrt{2}}$
$= \frac{-\sqrt{6}-\sqrt{2}}{4}$

7 **a** $\tan\left(\frac{5\pi}{12}\right)$
$= \tan\left(\frac{5\times 180^\circ}{12}\right)$
$= \tan 75^\circ$
$= \tan(45^\circ+30^\circ)$
$= \dfrac{\tan 45^\circ + \tan 30^\circ}{1-\tan 45^\circ\tan 30^\circ}$
$= \dfrac{1+\frac{1}{\sqrt{3}}}{1-(1)\left(\frac{1}{\sqrt{3}}\right)}$
$= \left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)\left(\frac{\sqrt{3}+1}{\sqrt{3}+1}\right)$
$= \dfrac{3+\sqrt{3}+\sqrt{3}+1}{3-1}$
$= \dfrac{4+2\sqrt{3}}{2}$
$= 2+\sqrt{3}$

b $\tan 105^\circ$
$= \tan(180^\circ - 75^\circ)$
$= \dfrac{\tan 180^\circ - \tan 75^\circ}{1+\tan 180^\circ\tan 75^\circ}$
$= \dfrac{0-(2+\sqrt{3})}{1+(0)(2+\sqrt{3})}$ {using **a**}
$= -2-\sqrt{3}$

8 $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

$= \dfrac{\frac{2}{3} - \frac{1}{5}}{1 - (\frac{2}{3})(-\frac{1}{5})}$

$= \dfrac{\frac{10}{15} - \frac{3}{15}}{1 + \frac{2}{15}} = \dfrac{\frac{7}{15}}{\frac{17}{15}} = \frac{7}{17}$

9 $\tan(A + \frac{\pi}{4}) = \dfrac{\tan A + \tan\frac{\pi}{4}}{1 - \tan A \tan\frac{\pi}{4}}$

$= \dfrac{\frac{3}{4} + 1}{1 - (\frac{3}{4})(1)}$

$= \dfrac{\frac{7}{4}}{\frac{1}{4}} = 7$

10 **a** $\tan(A + \frac{\pi}{4})\tan(A - \frac{\pi}{4})$

$= \dfrac{\tan A + \tan\frac{\pi}{4}}{1 - \tan A \tan\frac{\pi}{4}} \times \dfrac{\tan A - \tan\frac{\pi}{4}}{1 + \tan A \tan\frac{\pi}{4}}$

$= \left(\dfrac{\tan A + 1}{1 - \tan A}\right)\left(\dfrac{\tan A - 1}{1 + \tan A}\right)$

$= \dfrac{\tan^2 A - 1}{1 - \tan^2 A}$

$= -1$

b $\dfrac{\tan(A+B) + \tan(A-B)}{1 - \tan(A+B)\tan(A-B)}$

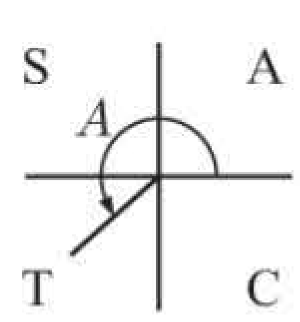
$= \tan[(A+B) + (A-B)]$

$= \tan 2A$

11 $\sin A = -\frac{1}{3}$

A is in Q3

$\therefore \quad \cos A < 0$



$\cos^2 A + \sin^2 A = 1$

$\therefore \quad \cos^2 A + \frac{1}{9} = 1$

$\therefore \quad \cos^2 A = \frac{8}{9}$

$\therefore \quad \cos A = -\frac{2\sqrt{2}}{3}$

$\therefore \quad \tan A = \dfrac{\sin A}{\cos A}$

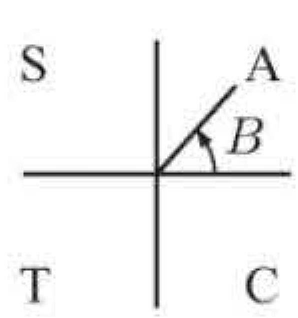
$= \dfrac{-\frac{1}{3}}{-\frac{2\sqrt{2}}{3}}$

$= \frac{1}{2\sqrt{2}}$

$\cos B = \frac{1}{\sqrt{5}}$

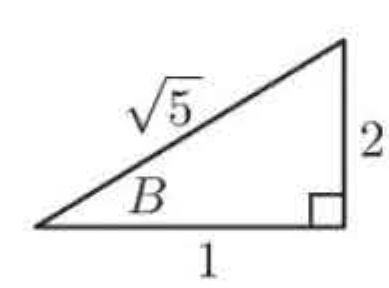
B is in Q1

$\therefore \quad \sin B > 0$



$\therefore \quad \sin B = \frac{2}{\sqrt{5}}$

$\tan B = 2$



a $\tan(A+B)$

$= \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

$= \dfrac{\frac{1}{2\sqrt{2}} + 2}{1 - (\frac{1}{2\sqrt{2}})(2)} \times \left(\dfrac{2\sqrt{2}}{2\sqrt{2}}\right)$

$= \dfrac{1 + 4\sqrt{2}}{2\sqrt{2} - 2} \times \left(\dfrac{2\sqrt{2} + 2}{2\sqrt{2} + 2}\right)$

$= \dfrac{2\sqrt{2} + 2 + 16 + 8\sqrt{2}}{8 - 4}$

$= \dfrac{18 + 10\sqrt{2}}{4}$

$= \dfrac{9 + 5\sqrt{2}}{2}$

b $\tan 2A = \dfrac{2\tan A}{1 - \tan^2 A}$

$= \dfrac{2(\frac{1}{2\sqrt{2}})}{1 - (\frac{1}{2\sqrt{2}})^2}$

$= \dfrac{\frac{1}{\sqrt{2}}}{1 - \frac{1}{8}}$

$= \dfrac{\frac{1}{\sqrt{2}}}{\frac{7}{8}}$

$= \frac{8}{7\sqrt{2}} \times \left(\frac{\sqrt{2}}{\sqrt{2}}\right)$

$= \frac{8\sqrt{2}}{14}$

$= \frac{4\sqrt{2}}{7}$

12 $\dfrac{\tan 80^\circ - \tan 20^\circ}{1 + \tan 80^\circ \tan 20^\circ} = \tan(80^\circ - 20^\circ)$

$= \tan 60^\circ$

$= \sqrt{3}$

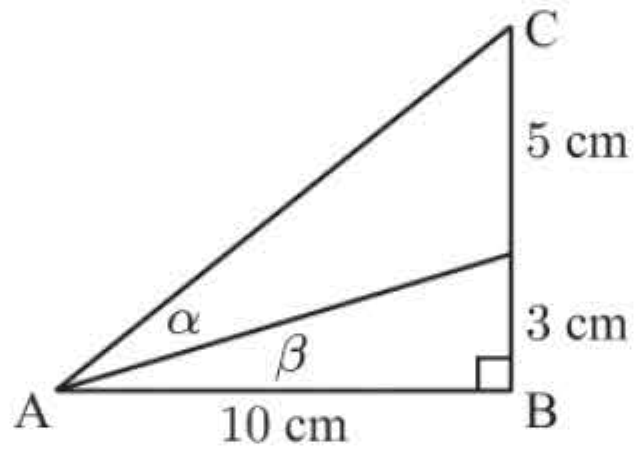
13

$$\tan(A+B) = \tfrac{3}{5}$$
$$\therefore \frac{\tan A + \tan B}{1 - \tan A \tan B} = \tfrac{3}{5}$$
$$\therefore \frac{\tan A + \frac{2}{3}}{1 - \tan A(\frac{2}{3})} = \tfrac{3}{5}$$
$$\therefore 5\tan A + \tfrac{10}{3} = 3 - 2\tan A$$
$$\therefore 7\tan A = -\tfrac{1}{3}$$
$$\therefore \tan A = -\tfrac{1}{21}$$

14

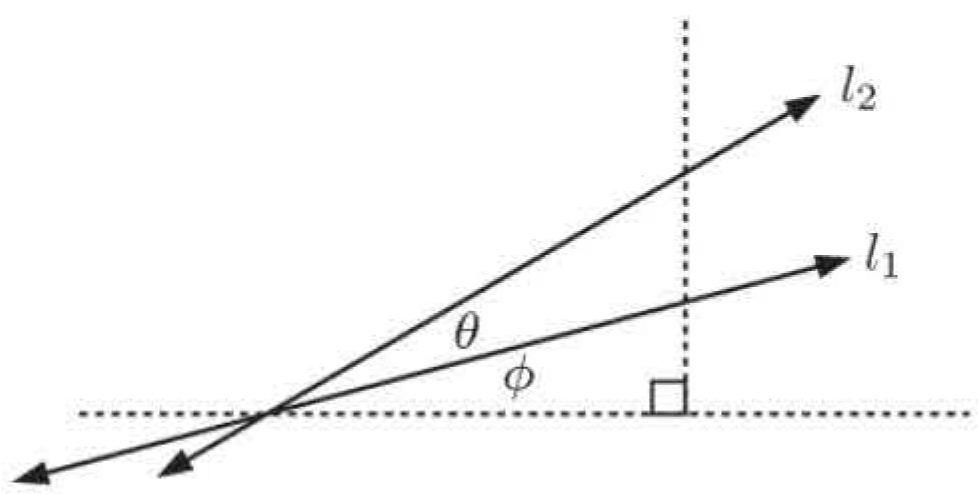
$$\tan(A-B)\tan(A+B) = 1$$
$$\therefore \left(\frac{\tan A - \tan B}{1 + \tan A \tan B}\right)\left(\frac{\tan A + \tan B}{1 - \tan A \tan B}\right) = 1$$
$$\therefore \frac{\tan^2 A - \tan^2 B}{1 - \tan^2 A \tan^2 B} = 1$$
$$\therefore \tan^2 A - \tan^2 B = 1 - \tan^2 A \tan^2 B$$
$$\therefore \tan^2 A(\tan^2 B + 1) = 1 + \tan^2 B$$
$$\therefore \tan^2 A = 1$$
$$\therefore \tan A = \pm 1$$

15



$$\tan\beta = \tfrac{3}{10}$$
$$\tan(\alpha+\beta) = \tfrac{8}{10}$$
$$\therefore \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \tfrac{8}{10}$$
$$\therefore \frac{\tan\alpha + \frac{3}{10}}{1 - \tan\alpha(\frac{3}{10})} = \tfrac{4}{5}$$
$$\therefore 5\tan\alpha + \tfrac{15}{10} = 4 - \tfrac{12}{10}\tan\alpha$$
$$\therefore \tfrac{31}{5}\tan\alpha = \tfrac{5}{2}$$
$$\therefore \tan\alpha = \tfrac{25}{62}$$

16



θ is the acute angle between the lines l_1 and l_2.

$$\tan\phi = \tfrac{1}{2} \quad \{\text{gradient of } l_1\}$$
$$\tan(\theta+\phi) = \tfrac{2}{3} \quad \{\text{gradient of } l_2\}$$
$$\therefore \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi} = \tfrac{2}{3}$$
$$\therefore \frac{\tan\theta + \frac{1}{2}}{1 - \tan\theta(\frac{1}{2})} = \tfrac{2}{3}$$
$$\therefore 3\tan\theta + \tfrac{3}{2} = 2 - \tan\theta$$
$$\therefore 4\tan\theta = \tfrac{1}{2}$$
$$\therefore \tan\theta = \tfrac{1}{8}$$

$\therefore$ the tangent of the acute angle is $\frac{1}{8}$.

17 $\tan(A+B+C) = \tan[(A+B)+C]$

$$= \frac{\tan(A+B) + \tan C}{1 - \tan(A+B)\tan C}$$
$$= \frac{\frac{\tan A + \tan B}{1 - \tan A \tan B} + \tan C}{1 - \frac{\tan A + \tan B}{1 - \tan A \tan B} \times \tan C}$$
$$= \frac{\tan A + \tan B + \tan C(1 - \tan A \tan B)}{1 - \tan A \tan B - (\tan A + \tan B)\times \tan C} \quad \{\times \text{ top and bottom by } 1 - \tan A \tan B\}$$
$$= \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan A \tan C - \tan B \tan C}$$

If A, B, and C are the angles of a triangle then $A + B + C = 180°$

$$\therefore \tan(A+B+C) = 0$$
$$\therefore \tan A + \tan B + \tan C - \tan A \tan B \tan C = 0$$
$$\therefore \tan A + \tan B + \tan C = \tan A \tan B \tan C$$

18 **a**
$$\begin{aligned}&\sqrt{2}\cos\left(\theta+\tfrac{\pi}{4}\right)\\ &=\sqrt{2}\left[\cos\theta\cos\left(\tfrac{\pi}{4}\right)-\sin\theta\sin\left(\tfrac{\pi}{4}\right)\right]\\ &=\sqrt{2}\left[\tfrac{1}{\sqrt{2}}\cos\theta-\tfrac{1}{\sqrt{2}}\sin\theta\right]\\ &=\cos\theta-\sin\theta\end{aligned}$$

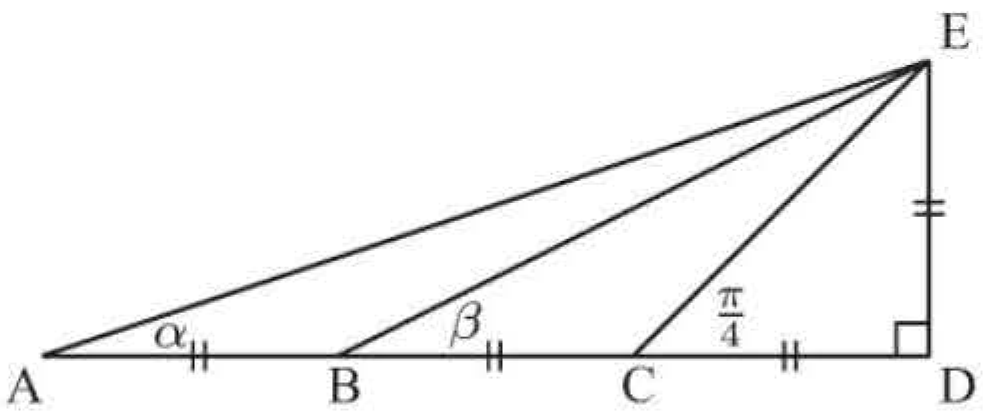
b
$$\begin{aligned}&2\cos\left(\theta-\tfrac{\pi}{3}\right)\\ &=2\left[\cos\theta\cos\left(\tfrac{\pi}{3}\right)+\sin\theta\sin\left(\tfrac{\pi}{3}\right)\right]\\ &=2\left[\tfrac{1}{2}\cos\theta+\tfrac{\sqrt{3}}{2}\sin\theta\right]\\ &=\cos\theta+\sqrt{3}\sin\theta\end{aligned}$$

c
$$\begin{aligned}&\cos(\alpha+\beta)-\cos(\alpha-\beta)\\ &=\cos\alpha\cos\beta-\sin\alpha\sin\beta-[\cos\alpha\cos\beta+\sin\alpha\sin\beta]\\ &=\cancel{\cos\alpha\cos\beta}-\sin\alpha\sin\beta-\cancel{\cos\alpha\cos\beta}-\sin\alpha\sin\beta\\ &=-2\sin\alpha\sin\beta\end{aligned}$$

d
$$\begin{aligned}&\cos(\alpha+\beta)\cos(\alpha-\beta)\\ &=[\cos\alpha\cos\beta-\sin\alpha\sin\beta]\,[\cos\alpha\cos\beta+\sin\alpha\sin\beta]\\ &=\cos^2\alpha\cos^2\beta-\sin^2\alpha\sin^2\beta\\ &=\cos^2\alpha\left[1-\sin^2\beta\right]-\left[1-\cos^2\alpha\right]\sin^2\beta\\ &=\cos^2\alpha-\cancel{\cos^2\alpha\sin^2\beta}-\sin^2\beta+\cancel{\cos^2\alpha\sin^2\beta}\\ &=\cos^2\alpha-\sin^2\beta\end{aligned}$$

19 $\tan\alpha=\frac{1}{3}$ and $\tan\beta=\frac{1}{2}$

$$\begin{aligned}\therefore\ \tan(\alpha+\beta)&=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}\\ &=\frac{\frac{1}{3}+\frac{1}{2}}{1-(\frac{1}{3})(\frac{1}{2})}\\ &=\frac{\frac{5}{6}}{\frac{5}{6}}=1\end{aligned}$$



$\therefore\ \alpha+\beta=\frac{\pi}{4}+k\pi,\ k\in\mathbb{Z}$ $\{$since $\tan\frac{\pi}{4}=1\}$

But clearly both α and $\beta<\frac{\pi}{2}$

$\therefore\ \alpha+\beta=\frac{\pi}{4}$

20
$$\begin{aligned}\sqrt{3}\sin x+\cos x&=k\sin(x+b)\\ &=k[\sin x\cos b+\cos x\sin b]\\ &=k\cos b\sin x+k\sin b\cos x\end{aligned}$$

Equating coefficients of $\sin x$ and $\cos x$,

$k\cos b=\sqrt{3}$ and $k\sin b=1$ ...... (1)

$\therefore\ k^2\cos^2 b=3$ and $k^2\sin^2 b=1$ {squaring both sides}

$\therefore\ k^2(\cos^2 b+\sin^2 b)=4$ {adding the 2 equations}

$\therefore\ k^2=4$

$\therefore\ k=2$ $\{k>0\}$

Substituting $k=2$ into (1) gives

$\cos b=\frac{\sqrt{3}}{2}$ and $\sin b=\frac{1}{2}$

$\therefore\ b=\frac{\pi}{6}$

21 **a**
$$\begin{aligned}&\cos 3\theta\\ &=\cos(2\theta+\theta)\\ &=\cos 2\theta\cos\theta-\sin 2\theta\sin\theta\\ &=(2\cos^2\theta-1)\cos\theta-(2\sin\theta\cos\theta)\sin\theta\\ &=2\cos^3\theta-\cos\theta-2\sin^2\theta\cos\theta\\ &=2\cos^3\theta-\cos\theta-2(1-\cos^2\theta)\cos\theta\\ &=2\cos^3\theta-\cos\theta-2\cos\theta+2\cos^3\theta\\ &=4\cos^3\theta-3\cos\theta\end{aligned}$$

b $8\cos^3\theta-6\cos\theta+1=0$

$\therefore\ 8\cos^3\theta-6\cos\theta=-1$

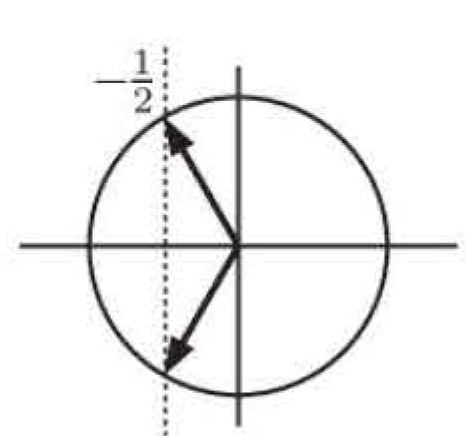
$\therefore\ 4\cos^3\theta-3\cos\theta=-\frac{1}{2}$

$\therefore\ \cos 3\theta=-\frac{1}{2}$

$\therefore\ 3\theta=-\frac{8\pi}{3},\ -\frac{4\pi}{3},\ -\frac{2\pi}{3},\ \frac{2\pi}{3},\ \frac{4\pi}{3},\ \frac{8\pi}{3}$

$\{3\theta\in[-3\pi,\ 3\pi]\}$

$\therefore\ \theta=-\frac{8\pi}{9},\ -\frac{4\pi}{9},\ -\frac{2\pi}{9},\ \frac{2\pi}{9},\ \frac{4\pi}{9},\ \frac{8\pi}{9}$

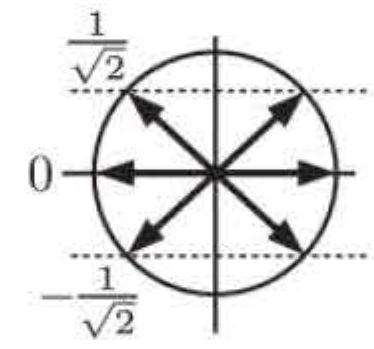


22 **a**
$$\begin{aligned}&\sin 3\theta\\&= \sin(2\theta+\theta)\\&= \sin 2\theta\cos\theta + \cos 2\theta \sin\theta\\&= (2\sin\theta\cos\theta)\cos\theta + (1-2\sin^2\theta)\sin\theta\\&= 2\sin\theta\cos^2\theta + \sin\theta - 2\sin^3\theta\\&= 2\sin\theta(1-\sin^2\theta) + \sin\theta - 2\sin^3\theta\\&= 2\sin\theta - 2\sin^3\theta + \sin\theta - 2\sin^3\theta\\&= -4\sin^3\theta + 3\sin\theta\end{aligned}$$

b
$$\begin{aligned}\sin 3\theta &= \sin\theta\\ \therefore\ -4\sin^3\theta + 3\sin\theta &= \sin\theta\\ \therefore\ 4\sin^3\theta - 2\sin\theta &= 0\\ \therefore\ 2\sin\theta(2\sin^2\theta - 1) &= 0\end{aligned}$$
$\therefore\ \sin\theta = 0$ or $\sin^2\theta = \frac{1}{2}$

$\therefore\ \sin\theta = 0$ or $\sin\theta = \pm\frac{1}{\sqrt{2}}$

$\therefore\ \theta = 0, \frac{\pi}{4}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{7\pi}{4}, 2\pi, \frac{9\pi}{4}, \frac{11\pi}{4}, 3\pi$



23 The period of a function $f(x)$ is the smallest $p > 0$ such that

$$f(x+p) = f(x) \quad \text{for all } x$$
$$\therefore\ \sin[n(x+p)] = \sin(nx)$$
$$\therefore\ \sin(nx+np) = \sin(nx)$$
$$\therefore\ \sin(nx)\cos(np) + \cos(nx)\sin(np) = \sin(nx)$$

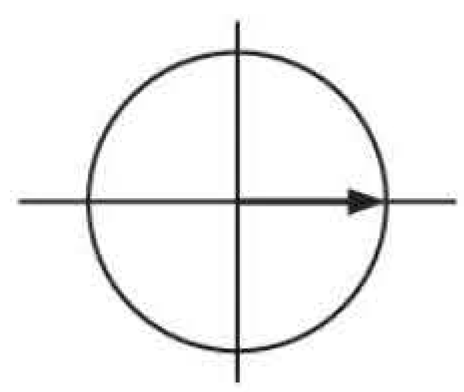
Equating coefficients of $\sin(nx)$ and $\cos(nx)$,

$\cos(np) = 1$ and $\sin(np) = 0$

$\therefore\ np = 2k\pi, \quad k\in\mathbb{Z}$

$\therefore\ p = \dfrac{2k\pi}{n}, \quad k \in \mathbb{Z}$

The smallest $p > 0$ occurs when $k = 1$ $\therefore\ p = \dfrac{2\pi}{n}$



24 **a** $2\cos x - 5\sin x = k\cos(x+b)$
$$\begin{aligned}&= k[\cos x\cos b - \sin x \sin b]\\&= k\cos b\cos x - k\sin b\sin x\end{aligned}$$

Equating coefficients of $\cos x$ and $\sin x$,

$k\cos b = 2$ and $k\sin b = 5$ ..... (1)

$\therefore\ k^2\cos^2 b = 4$ and $k^2\sin^2 b = 25$ {squaring both sides}

$\therefore\ k^2(\cos^2 b + \sin^2 b) = 29$ {adding the two equations}

$\therefore\ k^2 = 29 \quad \therefore\ k = \sqrt{29} \quad \{k > 0\}$

Substituting $k = \sqrt{29}$ into (1) gives

$\cos b = \frac{2}{\sqrt{29}}$ and $\sin b = \frac{5}{\sqrt{29}}$

$\therefore\ b = \cos^{-1}(\frac{2}{\sqrt{29}}) \approx 1.19$

$\therefore\ 2\cos x - 5\sin x \approx \sqrt{29}\cos(x+1.19)$

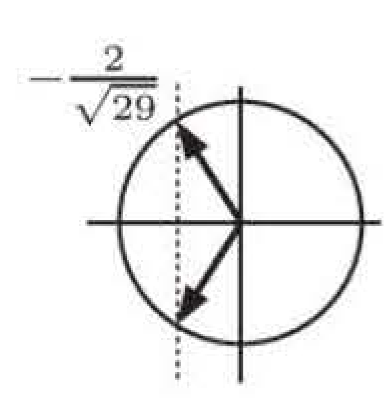
b $2\cos x - 5\sin x = -2$

$\therefore\ \sqrt{29}\cos(x+1.19) \approx -2$

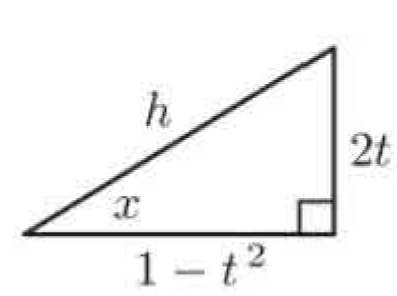
$\therefore\ \cos(x+1.19) \approx -\frac{2}{\sqrt{29}}$

$\therefore\ x + 1.19 \approx 1.951, 4.33$

$\therefore\ x \approx 0.761, 3.14 \leftarrow$ this solution is exactly π



c $\tan x = \tan\left(2\times\frac{x}{2}\right)$
$$= \frac{2\tan\left(\frac{x}{2}\right)}{1-\tan^2\left(\frac{x}{2}\right)}$$
$$= \frac{2t}{1-t^2}$$



Now
$$\begin{aligned}h^2 &= (1-t^2)^2 + (2t)^2\\&= 1 - 2t^2 + t^4 + 4t^2\\&= 1 + 2t^2 + t^4\\&= (1+t^2)^2\end{aligned}$$

$\therefore\ h = 1 + t^2 \quad$ {since $h > 0$}

$\therefore\ \sin x = \dfrac{2t}{1+t^2}$ and $\cos x = \dfrac{1-t^2}{1+t^2}$

d

$$2\cos x - 5\sin x = -2$$

$$\therefore\ 2\left(\frac{1-t^2}{1+t^2}\right) - 5\left(\frac{2t}{1+t^2}\right) = -2$$

$$\therefore\ \frac{2-2t^2-10t}{1+t^2} = -2$$

$$\therefore\ 2 - \cancel{2t^2} - 10t = -2 - \cancel{2t^2}$$

$$\therefore\ 4 = 10t$$

$$\therefore\ t = \tfrac{2}{5}$$

So $\tan\left(\frac{x}{2}\right) = \frac{2}{5}$

$$\therefore\ \frac{x}{2} \approx 0.3805 \quad \left\{0 \leqslant \frac{x}{2} \leqslant \frac{\pi}{2}\right\}$$

$$\therefore\ x \approx 0.761$$

The $x = \pi$ solution has been lost since t is undefined when $x = \pi$.

25 Let $\arctan(5) = \theta$ and $\arctan(\frac{2}{3}) = \phi$

$\therefore\ 5 = \tan\theta$ $\qquad$ $\therefore\ \frac{2}{3} = \tan\phi$

Now $\tan(\theta - \phi) = \dfrac{\tan\theta - \tan\phi}{1 + \tan\theta\tan\phi}$

$$\therefore\ \tan(\theta-\phi) = \frac{5 - \frac{2}{3}}{1 + 5\times\frac{2}{3}} = \frac{\frac{13}{3}}{\frac{13}{3}} = 1$$

$$\therefore\ \theta - \phi = \tfrac{\pi}{4} + k\pi, \quad k \in \mathbb{Z}$$

$$\therefore\ \theta - \phi = \tfrac{\pi}{4} \qquad \{\text{since } 0 < \theta,\ \phi < \tfrac{\pi}{2}\}$$

$$\therefore\ \arctan(5) - \arctan(\tfrac{2}{3}) = \tfrac{\pi}{4}$$

26 **a** Let $\arctan(\frac{1}{5}) = \theta$ and $\arctan(\frac{2}{3}) = \phi$

$\therefore\ \frac{1}{5} = \tan\theta$ $\qquad$ $\therefore\ \frac{2}{3} = \tan\phi$

Now $\tan(\theta + \phi) = \dfrac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}$

$$\therefore\ \tan(\theta+\phi) = \frac{\frac{1}{5} + \frac{2}{3}}{1 - \frac{1}{5}\times\frac{2}{3}} = \frac{\frac{13}{15}}{\frac{13}{15}} = 1$$

$$\therefore\ \theta + \phi = \tfrac{\pi}{4} + k\pi, \quad k \in \mathbb{Z}$$

$$\therefore\ \theta + \phi = \tfrac{\pi}{4} \qquad \{\text{since } 0 < \theta,\ \phi < \tfrac{\pi}{2}\}$$

$$\therefore\ \arctan(\tfrac{1}{5}) + \arctan(\tfrac{2}{3}) = \tfrac{\pi}{4}$$

b Let $\arctan(\frac{4}{3}) = \theta$ and $\arctan(\frac{1}{2}) = \phi$

$\therefore\ \frac{4}{3} = \tan\theta$ $\qquad$ $\therefore\ \frac{1}{2} = \tan\phi$

$$\therefore\ \tan 2\phi = \frac{2(\frac{1}{2})}{1 - (\frac{1}{2})^2} = \frac{1}{\frac{3}{4}} = \frac{4}{3}$$

$$\text{Now} \quad \tan(\theta - 2\phi) = \frac{\tan\theta - \tan 2\phi}{1 + \tan\theta \tan 2\phi}$$
$$= \frac{\frac{4}{3} - \frac{4}{3}}{1 + (\frac{4}{3})(\frac{4}{3})}$$
$$= 0$$
$$\therefore \quad \theta - 2\phi = 0 + k\pi, \quad k \in \mathbb{Z}$$
$$\therefore \quad \theta - 2\phi = 0 \quad \{\text{since } 0 < \theta,\ \phi < \tfrac{\pi}{2}\}$$
$$\therefore \quad \theta = 2\phi$$
$$\therefore \quad \arctan(\tfrac{4}{3}) = 2\arctan(\tfrac{1}{2})$$

27 Let $\arctan(\frac{1}{5}) = \theta$ and $\arctan(\frac{1}{239}) = \phi$

$\therefore \quad \frac{1}{5} = \tan\theta$ $\qquad \therefore \quad \frac{1}{239} = \tan\phi$

$$\therefore \quad \tan 2\theta = \frac{2(\frac{1}{5})}{1 - (\frac{1}{5})^2} = \frac{\frac{2}{5}}{\frac{24}{25}} = \frac{5}{12}$$

$$\text{and} \quad \tan 4\theta = \frac{\cancel{2}(\frac{5}{\cancel{12}_6})}{1 - (\frac{5}{12})^2} = \frac{\frac{5}{6}}{1 - \frac{25}{144}} = \frac{\frac{5}{6}}{\frac{119}{144}} = \frac{120}{119}$$

$$\therefore \quad \tan(4\theta - \phi) = \frac{\tan 4\theta - \tan\phi}{1 + \tan 4\theta \tan\phi}$$
$$= \frac{\frac{120}{119} - \frac{1}{239}}{1 + \frac{120}{119} \times \frac{1}{239}}$$
$$= \frac{\frac{28\,680 - 119}{28\,441}}{\frac{28\,441 + 120}{28\,441}}$$
$$= \cancel{\frac{28\,561}{28\,441}} \times \cancel{\frac{28\,441}{28\,561}}$$
$$= 1$$
$$\therefore \quad 4\theta - \phi = \tfrac{\pi}{4} + k\pi, \quad k \in \mathbb{Z}$$
$$\therefore \quad 4\theta - \phi = \tfrac{\pi}{4} \quad \{\text{since } 0 < \theta,\ \phi < \tfrac{\pi}{4}\}$$
$$\therefore \quad 4\arctan(\tfrac{1}{5}) - \arctan(\tfrac{1}{239}) = \tfrac{\pi}{4}$$

28 **a** $\sin(A + B) + \sin(A - B) = \sin A\cos B + \cancel{\cos A\sin B} + \sin A\cos B - \cancel{\cos A\sin B}$
$= 2\sin A\cos B$

b $2\sin A\cos B = \sin(A + B) + \sin(A - B)$
$\therefore \quad \sin A\cos B = \frac{1}{2}\sin(A + B) + \frac{1}{2}\sin(A - B)$ {dividing both sides by 2}

c **i** $\sin 3\theta\cos\theta$
$= \frac{1}{2}\sin(3\theta + \theta) + \frac{1}{2}\sin(3\theta - \theta)$
$= \frac{1}{2}\sin 4\theta + \frac{1}{2}\sin 2\theta$

ii $\sin 6\alpha\cos\alpha$
$= \frac{1}{2}\sin(6\alpha + \alpha) + \frac{1}{2}\sin(6\alpha - \alpha)$
$= \frac{1}{2}\sin 7\alpha + \frac{1}{2}\sin 5\alpha$

iii $2\sin 5\beta\cos\beta$
$= 2\left[\frac{1}{2}\sin(5\beta + \beta) + \frac{1}{2}\sin(5\beta - \beta)\right]$
$= \sin 6\beta + \sin 4\beta$

iv $4\cos\theta\sin 4\theta$
$= 4\left[\sin 4\theta\cos\theta\right]$
$= 4\left[\frac{1}{2}\sin 5\theta + \frac{1}{2}\sin 3\theta\right]$
$= 2\sin 5\theta + 2\sin 3\theta$

v $6\cos 4\alpha \sin 3\alpha$
$= 6\sin 3\alpha \cos 4\alpha$
$= 6\left[\frac{1}{2}\sin 7\alpha + \frac{1}{2}\sin(-\alpha)\right]$
$= 3\sin 7\alpha + 3\sin(-\alpha)$
$= 3\sin 7\alpha - 3\sin\alpha$

vi $\frac{1}{3}\cos 5A \sin 3A$
$= \frac{1}{3}\sin 3A \cos 5A$
$= \frac{1}{3}\left[\frac{1}{2}\sin 8A + \frac{1}{2}\sin(-2A)\right]$
$= \frac{1}{6}\sin 8A - \frac{1}{6}\sin 2A$

29 **a** $\cos(A+B) + \cos(A-B)$
$= \cos A\cos B - \cancel{\sin A\sin B} + \cos A\cos B + \cancel{\sin A\sin B}$
$= 2\cos A\cos B$

b $2\cos A\cos B = \cos(A+B) + \cos(A-B)$
$\therefore\ \cos A\cos B = \frac{1}{2}\cos(A+B) + \frac{1}{2}\cos(A-B)$ {dividing both sides by 2}

c **i** $\cos 4\theta\cos\theta$
$= \frac{1}{2}\cos 5\theta + \frac{1}{2}\cos 3\theta$

ii $\cos 7\alpha\cos\alpha$
$= \frac{1}{2}\cos 8\alpha + \frac{1}{2}\cos 6\alpha$

iii $2\cos 3\beta\cos\beta$
$= 2\left[\frac{1}{2}\cos 4\beta + \frac{1}{2}\cos 2\beta\right]$
$= \cos 4\beta + \cos 2\beta$

iv $6\cos x\cos 7x$
$= 6\cos 7x\cos x$
$= 6\left[\frac{1}{2}\cos 8x + \frac{1}{2}\cos 6x\right]$
$= 3\cos 8x + 3\cos 6x$

v $3\cos P\cos 4P$
$= 3\cos 4P\cos P$
$= 3\left[\frac{1}{2}\cos 5P + \frac{1}{2}\cos 3P\right]$
$= \frac{3}{2}\cos 5P + \frac{3}{2}\cos 3P$

vi $\frac{1}{4}\cos 4x\cos 2x$
$= \frac{1}{4}\left[\frac{1}{2}\cos 6x + \frac{1}{2}\cos 2x\right]$
$= \frac{1}{8}\cos 6x + \frac{1}{8}\cos 2x$

30 **a** $\cos(A-B) - \cos(A+B)$
$= \cancel{\cos A\cos B} + \sin A\sin B - [\cancel{\cos A\cos B} - \sin A\sin B]$
$= \sin A\sin B + \sin A\sin B$
$= 2\sin A\sin B$

b $2\sin A\sin B = \cos(A-B) - \cos(A+B)$
$\therefore\ \sin A\sin B = \frac{1}{2}\cos(A-B) - \frac{1}{2}\cos(A+B)$ {dividing both sides by 2}

c **i** $\sin 3\theta\sin\theta$
$= \frac{1}{2}\cos 2\theta - \frac{1}{2}\cos 4\theta$

ii $\sin 6\alpha\sin\alpha$
$= \frac{1}{2}\cos 5\alpha - \frac{1}{2}\cos 7\alpha$

iii $2\sin 5\beta\sin\beta$
$= 2\left[\frac{1}{2}\cos 4\beta - \frac{1}{2}\cos 6\beta\right]$
$= \cos 4\beta - \cos 6\beta$

iv $4\sin\theta\sin 4\theta$
$= 4\sin 4\theta\sin\theta$
$= 4\left[\frac{1}{2}\cos 3\theta - \frac{1}{2}\cos 5\theta\right]$
$= 2\cos 3\theta - 2\cos 5\theta$

v $10\sin 2A\sin 8A$
$= 10\sin 8A\sin 2A$
$= 10\left[\frac{1}{2}\cos 6A - \frac{1}{2}\cos 10A\right]$
$= 5\cos 6A - 5\cos 10A$

vi $\frac{1}{5}\sin 3M\sin 7M$
$= \frac{1}{5}\sin 7M\sin 3M$
$= \frac{1}{5}\left[\frac{1}{2}\cos 4M - \frac{1}{2}\cos 10M\right]$
$= \frac{1}{10}\cos 4M - \frac{1}{10}\cos 10M$

31 **a** (1) becomes $\sin A\cos A = \frac{1}{2}\sin(A+A) + \frac{1}{2}\sin(A-A)$
$= \frac{1}{2}\sin(2A) + \frac{1}{2}\sin(0)$
$= \frac{1}{2}\sin 2A$

(2) becomes $\cos A\cos A = \frac{1}{2}\cos(A+A) + \frac{1}{2}\cos(A-A)$
$\therefore\ \cos^2 A = \frac{1}{2}\cos(2A) + \frac{1}{2}\cos(0)$
$= \frac{1}{2}\cos 2A + \frac{1}{2}$

(3) becomes $\sin A\sin A = \frac{1}{2}\cos(A-A) - \frac{1}{2}\cos(A+A)$
$\therefore\ \sin^2 A = \frac{1}{2}\cos(0) - \frac{1}{2}\cos(2A)$
$= \frac{1}{2} - \frac{1}{2}\cos 2A$

b **i** $A+B=S$ $\quad\therefore\quad 2A=S+D \quad\therefore\quad A=\frac{S+D}{2}$

$A-B=D$ $\quad$ and $\quad B=S-A=S-\left(\frac{S+D}{2}\right)=\frac{2S}{2}-\left(\frac{S+D}{2}\right)=\frac{2S-S-D}{2}=\frac{S-D}{2}$

ii $\sin A\cos B=\frac{1}{2}\sin(A+B)+\frac{1}{2}\sin(A-B)$ becomes

$\sin\left(\frac{S+D}{2}\right)\cos\left(\frac{S-D}{2}\right)=\frac{1}{2}\sin S+\frac{1}{2}\sin D$

or $\quad \sin S+\sin D=2\sin\left(\frac{S+D}{2}\right)\cos\left(\frac{S-D}{2}\right)$ .... (4)

iii Replacing D by $(-D)$ in (4) gives

$\sin S+\sin(-D)=2\sin\left(\frac{S-D}{2}\right)\cos\left(\frac{S+D}{2}\right)$

or $\quad \sin S-\sin D=2\cos\left(\frac{S+D}{2}\right)\sin\left(\frac{S-D}{2}\right)$

c $\cos A\cos B=\frac{1}{2}\cos(A+B)+\frac{1}{2}\cos(A-B)$ becomes

$\cos\left(\frac{S+D}{2}\right)\cos\left(\frac{S-D}{2}\right)=\frac{1}{2}\cos S+\frac{1}{2}\cos D$

or $\quad \cos S+\cos D=2\cos\left(\frac{S+D}{2}\right)\cos\left(\frac{S-D}{2}\right)$

d $\sin A\sin B=\frac{1}{2}\cos(A-B)-\frac{1}{2}\cos(A+B)$ becomes

$\sin\left(\frac{S+D}{2}\right)\sin\left(\frac{S-D}{2}\right)=\frac{1}{2}\cos D-\frac{1}{2}\cos S$

or $\quad \cos D-\cos S=2\sin\left(\frac{S+D}{2}\right)\sin\left(\frac{S-D}{2}\right)$

or $\quad \cos S-\cos D=-2\sin\left(\frac{S+D}{2}\right)\sin\left(\frac{S-D}{2}\right)$

32 **a**
$$\begin{aligned}&\sin 5x+\sin x\\&=2\sin\left(\tfrac{5x+x}{2}\right)\cos\left(\tfrac{5x-x}{2}\right)\\&=2\sin 3x\cos 2x\end{aligned}$$

b
$$\begin{aligned}&\cos 8A+\cos 2A\\&=2\cos\left(\tfrac{8A+2A}{2}\right)\cos\left(\tfrac{8A-2A}{2}\right)\\&=2\cos 5A\cos 3A\end{aligned}$$

c
$$\begin{aligned}&\cos 3\alpha-\cos\alpha\\&=-2\sin\left(\tfrac{3\alpha+\alpha}{2}\right)\sin\left(\tfrac{3\alpha-\alpha}{2}\right)\\&=-2\sin 2\alpha\sin\alpha\end{aligned}$$

d
$$\begin{aligned}&\sin 5\theta-\sin 3\theta\\&=2\cos\left(\tfrac{5\theta+3\theta}{2}\right)\sin\left(\tfrac{5\theta-3\theta}{2}\right)\\&=2\cos 4\theta\sin\theta\end{aligned}$$

e
$$\begin{aligned}&\cos 7\alpha-\cos\alpha\\&=-2\sin\left(\tfrac{7\alpha+\alpha}{2}\right)\sin\left(\tfrac{7\alpha-\alpha}{2}\right)\\&=-2\sin 4\alpha\sin 3\alpha\end{aligned}$$

f
$$\begin{aligned}&\sin 3\alpha+\sin 7\alpha\\&=\sin 7\alpha+\sin 3\alpha\\&=2\sin\left(\tfrac{7\alpha+3\alpha}{2}\right)\cos\left(\tfrac{7\alpha-3\alpha}{2}\right)\\&=2\sin 5\alpha\cos 2\alpha\end{aligned}$$

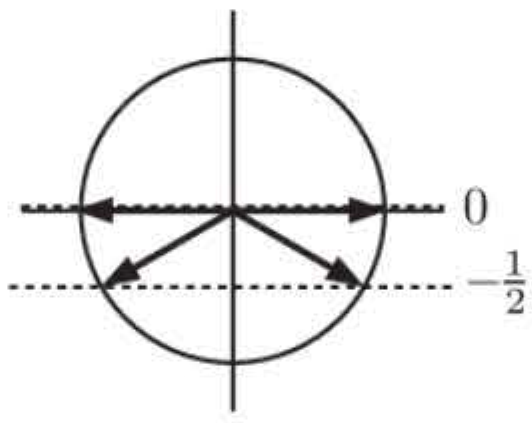
g
$$\begin{aligned}&\cos 2B-\cos 4B\\&=-\left[\cos 4B-\cos 2B\right]\\&=-\left[-2\sin\left(\tfrac{4B+2B}{2}\right)\sin\left(\tfrac{4B-2B}{2}\right)\right]\\&=2\sin 3B\sin B\end{aligned}$$

h
$$\begin{aligned}&\sin(x+h)-\sin x\\&=2\cos\left(\tfrac{x+h+x}{2}\right)\sin\left(\tfrac{x+h-x}{2}\right)\\&=2\cos\left(\tfrac{2x+h}{2}\right)\sin\left(\tfrac{h}{2}\right)\\&=2\cos\left(x+\tfrac{h}{2}\right)\sin\left(\tfrac{h}{2}\right)\end{aligned}$$

i
$$\begin{aligned}&\cos(x+h)-\cos x\\&=-2\sin\left(\tfrac{x+h+x}{2}\right)\sin\left(\tfrac{x+h-x}{2}\right)\\&=-2\sin\left(\tfrac{2x+h}{2}\right)\sin\left(\tfrac{h}{2}\right)\\&=-2\sin\left(x+\tfrac{h}{2}\right)\sin\left(\tfrac{h}{2}\right)\end{aligned}$$

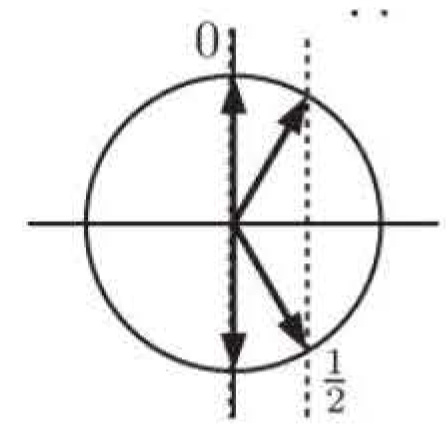
EXERCISE 13F

1 **a**
$$\begin{aligned} 2\sin^2 x + \sin x &= 0 \\ \therefore\ \sin x(2\sin x + 1) &= 0 \\ \therefore\ \sin x &= 0 \text{ or } -\tfrac{1}{2} \end{aligned}$$



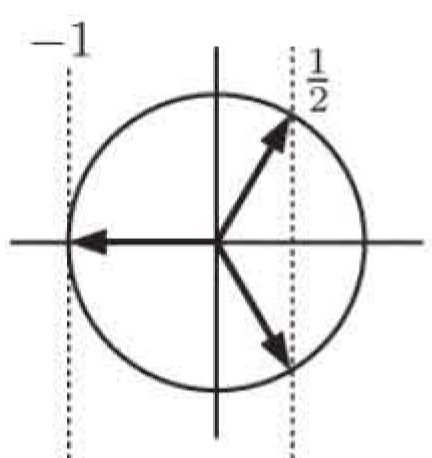
$\therefore\ x = 0,\ \pi,\ \frac{7\pi}{6},\ \frac{11\pi}{6},\ 2\pi$

b
$$\begin{aligned} 2\cos^2 x &= \cos x \\ \therefore\ 2\cos^2 x - \cos x &= 0 \\ \therefore\ \cos x(2\cos x - 1) &= 0 \\ \therefore\ \cos x &= 0 \text{ or } \tfrac{1}{2} \end{aligned}$$



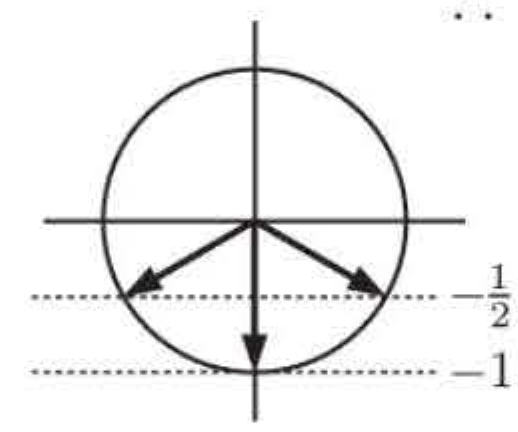
$\therefore\ x = \frac{\pi}{3},\ \frac{\pi}{2},\ \frac{3\pi}{2},\ \frac{5\pi}{3}$

c
$$\begin{aligned} 2\cos^2 x + \cos x - 1 &= 0 \\ \therefore\ (2\cos x - 1)(\cos x + 1) &= 0 \\ \therefore\ \cos x &= \tfrac{1}{2} \text{ or } -1 \end{aligned}$$



$\therefore\ x = \frac{\pi}{3},\ \pi,\ \frac{5\pi}{3}$

d
$$\begin{aligned} 2\sin^2 x + 3\sin x + 1 &= 0 \\ \therefore\ (2\sin x + 1)(\sin x + 1) &= 0 \\ \therefore\ \sin x &= -\tfrac{1}{2} \text{ or } -1 \end{aligned}$$

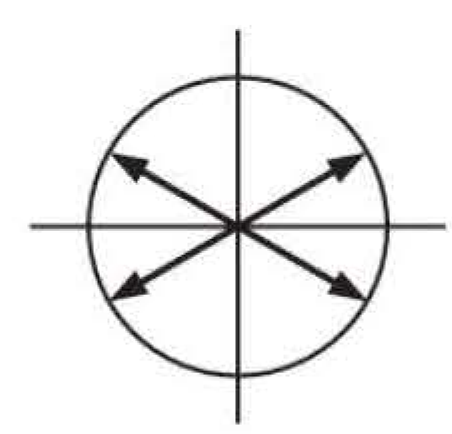


$\therefore\ x = \frac{7\pi}{6},\ \frac{3\pi}{2},\ \frac{11\pi}{6}$

e
$$\begin{aligned} \sin^2 x &= 2 - \cos x \\ \therefore\ 1 - \cos^2 x &= 2 - \cos x \\ \therefore\ \cos^2 x - \cos x + 1 &= 0 \\ \text{where } \Delta &= (-1)^2 - 4(1)(1) \\ &= 1 - 4 \\ &= -3 \end{aligned}$$

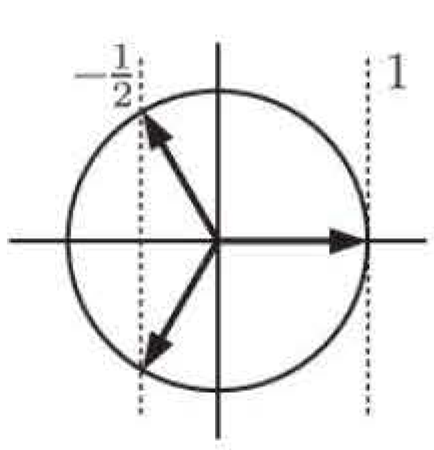
$\therefore$ no real solutions exist

f
$$\begin{aligned} 3\tan x &= \cot x \\ \therefore\ 3\tan x &= \frac{1}{\tan x} \\ \therefore\ 3\tan^2 x &= 1 \\ \therefore\ \tan^2 x &= \tfrac{1}{3} \\ \therefore\ \tan x &= \pm\tfrac{1}{\sqrt{3}} \end{aligned}$$



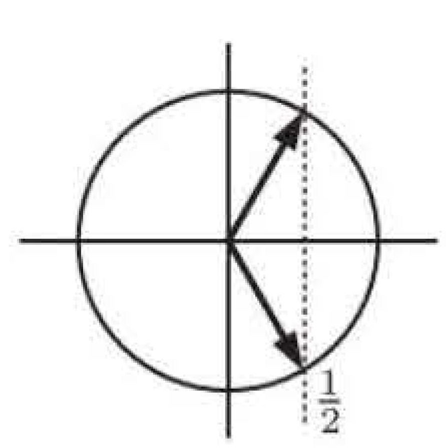
$\therefore\ x = \frac{\pi}{6},\ \frac{5\pi}{6},\ \frac{7\pi}{6},\ \frac{11\pi}{6}$

2 **a**
$$\begin{aligned} \cos 2x - \cos x &= 0 \\ \therefore\ (2\cos^2 x - 1) - \cos x &= 0 \\ \therefore\ 2\cos^2 x - \cos x - 1 &= 0 \\ \therefore\ (2\cos x + 1)(\cos x - 1) &= 0 \\ \therefore\ \cos x &= -\tfrac{1}{2} \text{ or } 1 \end{aligned}$$



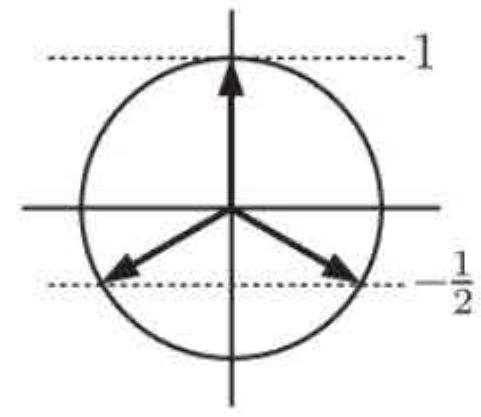
$\therefore\ x = 0,\ \frac{2\pi}{3},\ \frac{4\pi}{3},\ 2\pi$

b
$$\begin{aligned} \cos 2x + 3\cos x &= 1 \\ \therefore\ (2\cos^2 x - 1) + 3\cos x &= 1 \\ \therefore\ 2\cos^2 x + 3\cos x - 2 &= 0 \\ \therefore\ (2\cos x - 1)(\cos x + 2) &= 0 \\ \therefore\ \cos x &= \tfrac{1}{2} \\ & \{-1 \leqslant \cos x \leqslant 1\} \end{aligned}$$

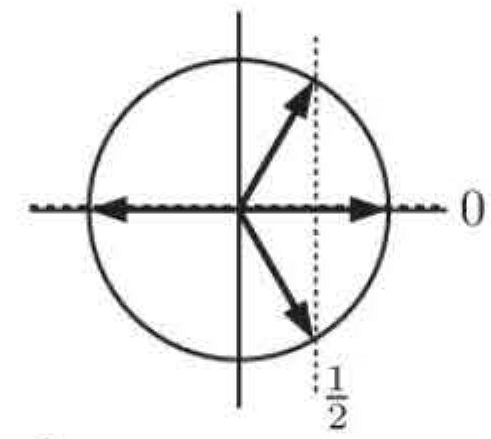


$\therefore\ x = \frac{\pi}{3},\ \frac{5\pi}{3}$

c
$$\begin{aligned} \cos 2x + \sin x &= 0 \\ \therefore\ (1 - 2\sin^2 x) + \sin x &= 0 \\ \therefore\ -2\sin^2 x + \sin x + 1 &= 0 \\ \therefore\ 2\sin^2 x - \sin x - 1 &= 0 \\ \therefore\ (2\sin x + 1)(\sin x - 1) &= 0 \\ \therefore\ \sin x &= -\tfrac{1}{2} \text{ or } 1 \\ \therefore\ x &= \tfrac{\pi}{2}, \tfrac{7\pi}{6}, \tfrac{11\pi}{6} \end{aligned}$$

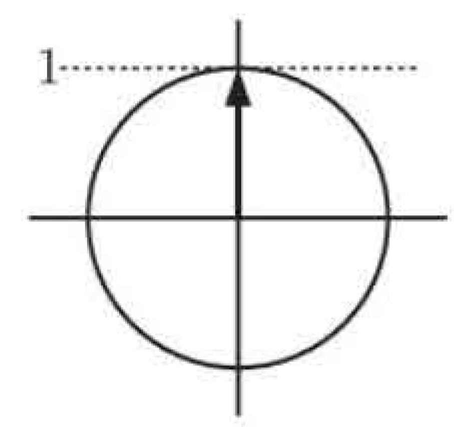


d
$$\begin{aligned} \sin 4x &= \sin 2x \\ \therefore\ 2\sin 2x \cos 2x &= \sin 2x \\ \therefore\ 2\sin 2x \cos 2x - \sin 2x &= 0 \\ \therefore\ \sin 2x(2\cos 2x - 1) &= 0 \\ \therefore\ \sin 2x = 0 \quad \text{or} \quad \cos 2x &= \tfrac{1}{2} \end{aligned}$$
$$\therefore\ 2x = 0, \tfrac{\pi}{3}, \pi, \tfrac{5\pi}{3}, 2\pi, \tfrac{7\pi}{3}, 3\pi, \tfrac{11\pi}{3}, 4\pi \quad \{0 \leqslant 2x \leqslant 4\pi\}$$
$$\therefore\ x = 0, \tfrac{\pi}{6}, \tfrac{\pi}{2}, \tfrac{5\pi}{6}, \pi, \tfrac{7\pi}{6}, \tfrac{3\pi}{2}, \tfrac{11\pi}{6}, 2\pi$$



e $\sin x + \cos x = \sqrt{2}$

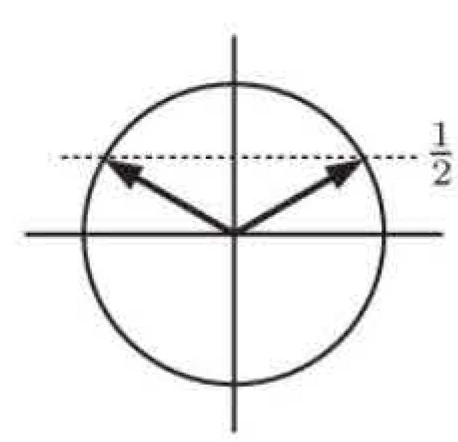
Squaring both sides we get:
$$\begin{aligned} \sin^2 x + 2\sin x\cos x + \cos^2 x &= 2 \\ \therefore\ \sin 2x + 1 &= 2 \\ \therefore\ \sin 2x &= 1 \\ \therefore\ 2x &= \tfrac{\pi}{2}, \tfrac{5\pi}{2} \quad \{0 \leqslant 2x \leqslant 4\pi\} \\ \therefore\ x &= \tfrac{\pi}{4}, \tfrac{5\pi}{4} \end{aligned}$$



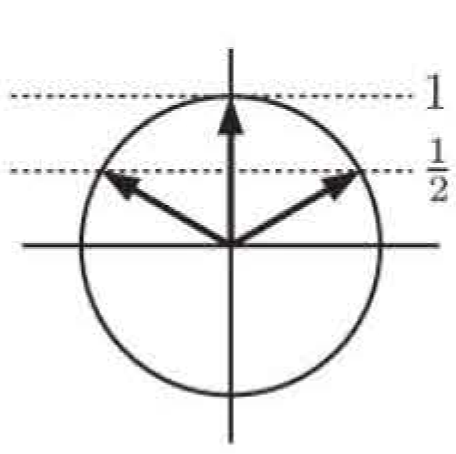
Since we squared the original equation, we must check our answers.

$$\sin\tfrac{\pi}{4} + \cos\tfrac{\pi}{4} = \tfrac{1}{\sqrt{2}} + \tfrac{1}{\sqrt{2}} = \tfrac{2}{\sqrt{2}} = \sqrt{2} \quad \checkmark$$
$$\sin\tfrac{5\pi}{4} + \cos\tfrac{5\pi}{4} = -\tfrac{1}{\sqrt{2}} + (-\tfrac{1}{\sqrt{2}}) = -\tfrac{2}{\sqrt{2}} = -\sqrt{2} \quad \times$$
$$\therefore\ x = \tfrac{\pi}{4} \text{ is the only solution}$$

f
$$\begin{aligned} 2\cos^2 x &= 3\sin x \\ \therefore\ 2(1 - \sin^2 x) &= 3\sin x \\ \therefore\ 2\sin^2 x + 3\sin x - 2 &= 0 \\ \therefore\ (2\sin x - 1)(\sin x + 2) &= 0 \\ \therefore\ \sin x &= \tfrac{1}{2} \quad \{-1 \leqslant \sin x \leqslant 1\} \\ \therefore\ x &= \tfrac{\pi}{6}, \tfrac{5\pi}{6} \end{aligned}$$

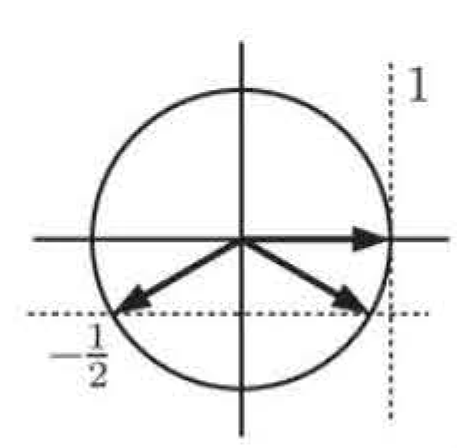


3 **a**
$$\begin{aligned} 2\sin x + \csc x &= 3 \\ \therefore\ 2\sin^2 x + 1 &= 3\sin x \\ \therefore\ 2\sin^2 x - 3\sin x + 1 &= 0 \\ \therefore\ (2\sin x - 1)(\sin x - 1) &= 0 \\ \therefore\ \sin x &= \tfrac{1}{2} \text{ or } 1 \end{aligned}$$



$$\therefore\ x = \tfrac{\pi}{6}, \tfrac{\pi}{2}, \tfrac{5\pi}{6}$$

b
$$\begin{aligned} \sin 2x + \cos x - 2\sin x - 1 &= 0 \\ \therefore\ 2\sin x\cos x + \cos x - 2\sin x - 1 &= 0 \\ \therefore\ (2\sin x + 1)(\cos x - 1) &= 0 \\ \therefore\ \sin x = -\tfrac{1}{2} \quad \text{or} \quad \cos x &= 1 \end{aligned}$$



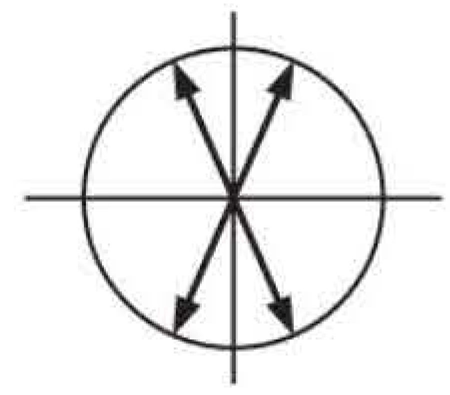
$$\therefore\ x = -\tfrac{5\pi}{6}, -\tfrac{\pi}{6}, 0$$

c $\tan^4 x - 2\tan^2 x - 3 = 0$

$\therefore \ (\tan^2 x - 3)(\tan^2 x + 1) = 0$

$\therefore \ \tan^2 x = 3 \text{ or } -1$

$\therefore \ \tan^2 x = \pm\sqrt{3}$



$\therefore \ x = -\frac{2\pi}{3}, -\frac{\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3}$

4 **a** $2\cos^2 x = \sin x$

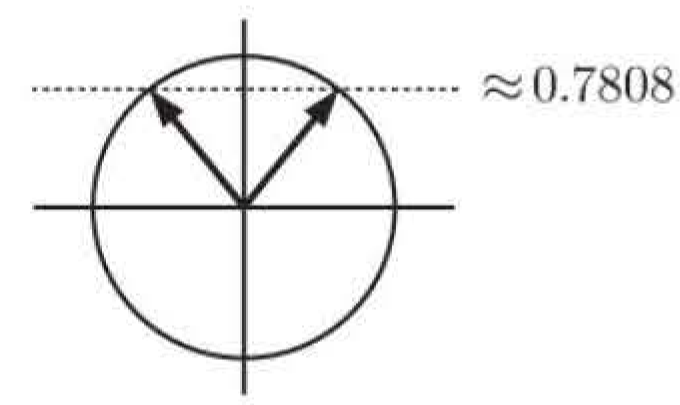
$\therefore \ 2(1 - \sin^2 x) - \sin x = 0$

$\therefore \ 2 - 2\sin^2 x - \sin x = 0$

$\therefore \ 2\sin^2 x + \sin x - 2 = 0$

$\therefore \ \sin x = \dfrac{-1 \pm \sqrt{1 - 4(2)(-2)}}{2(2)}$

$\therefore \ \sin x = \dfrac{-1 \pm \sqrt{17}}{4} \approx 0.7808$ or -1.281



$\therefore \ \sin x \approx 0.7808$ as $-1 \leqslant \sin x \leqslant 1$

$\therefore \ x \approx \arcsin(0.7808)$ or $\pi - \arcsin(0.7808)$

$\therefore \ x \approx 0.896$ or 2.25

b $\cos 2x + 5\sin x = 0$

$\therefore \ 1 - 2\sin^2 x + 5\sin x = 0$

$\therefore \ 2\sin^2 x - 5\sin x - 1 = 0$

$\therefore \ \sin x = \dfrac{5 \pm \sqrt{25 - 4(2)(-1)}}{2(2)}$

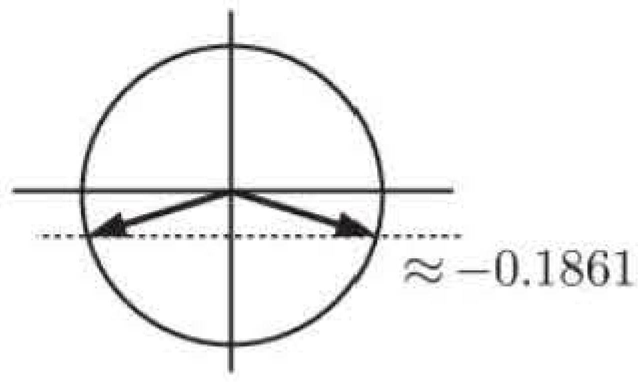
$= \dfrac{5 \pm \sqrt{33}}{4} \approx 2.6861$ or -0.1861

$\therefore \ \sin x \approx -0.1861 \quad \{\text{as } -1 \leqslant \sin x \leqslant 1\}$

Now $\arcsin(-0.1861) \approx -0.1872$

$\therefore \ x \approx \pi + 0.1872$ or $2\pi - 0.1872$

$\therefore \ x \approx 3.33$ or 6.10



c $2\tan^2 x + 3\sec^2 x = 7$

$\therefore \ 2\tan^2 x + 3(\tan^2 x + 1) = 7$

$\therefore \ 5\tan^2 x + 3 = 7$

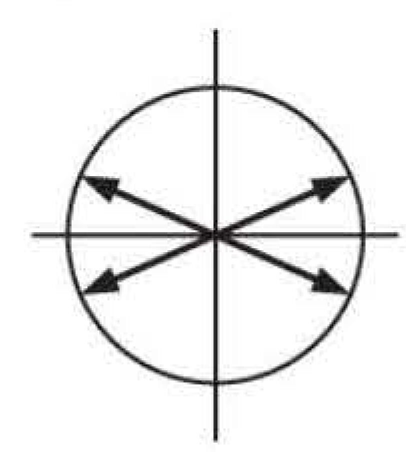
$\therefore \ 5\tan^2 x = 4$

$\therefore \ \tan^2 x = \frac{4}{5}$

$\therefore \ \tan x = \pm\frac{2}{\sqrt{5}}$

$\therefore \ x \approx 0.730, \pi - 0.730, \pi + 0.730, 2\pi - 0.730$

$\therefore \ x \approx 0.730, 2.41, 3.87, 5.55$



EXERCISE 13G

1 **a** $1 + \sin x + \sin^2 x + \sin^3 x + ... + \sin^{n-1} x$ is a geometric series with $u_1 = 1, \quad r = \sin x$

$\therefore \ \text{sum} = \dfrac{u_1(1 - r^n)}{1 - r}$

$= \dfrac{1(1 - \sin^n x)}{1 - \sin x}$

$= \dfrac{1 - \sin^n x}{1 - \sin x}$

b $S = \dfrac{u_1}{1 - r} = \dfrac{1}{1 - \sin x}$

for $-1 < \sin x < 1$, not convergent for $\sin x = \pm 1$

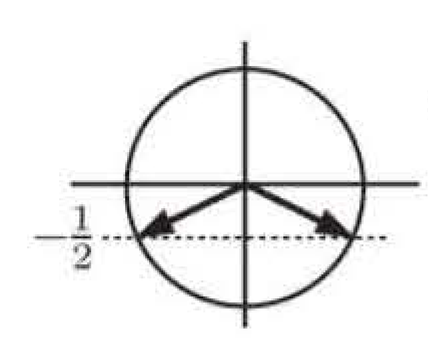
c If $S = \frac{2}{3}, \quad \dfrac{1}{1 - \sin x} = \frac{2}{3}$

$\therefore \ 3 = 2 - 2\sin x$

$\therefore \ 2\sin x = -1$

$\therefore \ \sin x = -\frac{1}{2}$

$\therefore \ x = \frac{7\pi}{6}$ or $\frac{11\pi}{6}$



2 **a** **i** $2\sin x(\cos x + \cos 3x)$

$= 2\sin x\cos x + 2\sin x\cos 3x$

$= \sin 2x + \sin 4x + \sin(-2x)$

$\{2\sin A\cos B = \sin(A + B) + \sin(A - B)\}$

$= \cancel{\sin 2x} + \sin 4x - \cancel{\sin 2x}$

$= \sin 4x$

ii $2\sin x(\cos x + \cos 3x + \cos 5x)$

$= 2\sin x(\cos x + \cos 3x) + 2\sin x\cos 5x$

$= \sin 4x + \sin 6x + \sin(-4x) \quad \{\text{from } \mathbf{i}\}$

$= \cancel{\sin 4x} + \sin 6x - \cancel{\sin 4x}$

$= \sin 6x$

b **i** $2\sin x(\cos x + \cos 3x + \cos 5x + \cos 7x) = \sin 8x$

ii $2\sin x(\cos x + \cos 3x + \cos 5x + ... + \cos 19x) = \sin 20x$

$$\therefore \quad \cos x + \cos 3x + \cos 5x + ... + \cos 19x = \frac{\sin 20x}{2\sin x}$$

iii In general, $\cos x + \cos 3x + \cos 5x + ... + \cos(2n-1)x = \dfrac{\sin 2nx}{2\sin x}$

c P_n is "$\cos\theta + \cos 3\theta + \cos 5\theta + ... + \cos(2n-1)\theta = \dfrac{\sin 2n\theta}{2\sin\theta}$", $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, LHS $= \cos\theta$ and RHS $= \dfrac{\sin 2\theta}{2\sin\theta} = \dfrac{2\sin\theta\cos\theta}{2\sin\theta} = \cos\theta$ $\therefore$ P_1 is true.

(2) If P_k is true, then

$$\cos\theta + \cos 3\theta + \cos 5\theta + ... + \cos(2k-1)\theta = \frac{\sin 2k\theta}{2\sin\theta}$$

$\therefore \quad \cos\theta + \cos 3\theta + \cos 5\theta + ... + \cos(2k-1)\theta + \cos(2k+1)\theta$

$$= \frac{\sin 2k\theta}{2\sin\theta} + \cos(2k+1)\theta$$

$$= \frac{\sin 2k\theta + 2\sin\theta\cos(2k+1)\theta}{2\sin\theta}$$

$$= \frac{\sin 2k\theta + \sin[\theta + (2k+1)\theta] + \sin[\theta - (2k+1)\theta]}{2\sin\theta}$$

$$= \frac{\sin 2k\theta + \sin(\theta + 2k\theta + \theta) + \sin(\theta - 2k\theta - \theta)}{2\sin\theta}$$

$$= \frac{\sin 2k\theta + \sin(2k\theta + 2\theta) + \sin(-2k\theta)}{2\sin\theta}$$

$$= \frac{\cancel{\sin 2k\theta} + \sin 2(k+1)\theta - \cancel{\sin 2k\theta}}{2\sin\theta}$$

$$= \frac{\sin 2(k+1)\theta}{2\sin\theta}$$

Thus P_{k+1} is true whenever P_k is true and P_1 is true.

$\therefore$ P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

3 **a** **i**

$$\begin{aligned}
&\sin x\cos x\cos 2x \\
&= \tfrac{1}{2}(2\sin x\cos x)\cos 2x \\
&= \tfrac{1}{2}\sin 2x\cos 2x \\
&= \tfrac{1}{4}(2\sin 2x\cos 2x) \\
&= \tfrac{1}{4}\sin 4x \quad .... (1) \\
&= \frac{\sin(2^2x)}{2^2}
\end{aligned}$$

ii

$$\begin{aligned}
&(\sin x\cos x\cos 2x)\cos 4x \\
&= \tfrac{1}{4}\sin 4x\cos 4x \quad \{\text{from (1)}\} \\
&= \tfrac{1}{8}(2\sin 4x\cos 4x) \\
&= \frac{\sin 8x}{8} \\
&= \frac{\sin(2^3x)}{2^3}
\end{aligned}$$

b **i** $\dfrac{\sin(2^4x)}{2^4}$

ii $\dfrac{\sin(2^6x)}{2^6}$

c $\sin x\cos x\cos 2x\cos 4x....\cos(2^n x) = \dfrac{\sin(2^{n+1}x)}{2^{n+1}}$

P_n is "$\sin x\cos x\cos 2x\cos 4x....\cos(2^n x) = \dfrac{\sin(2^{n+1}x)}{2^{n+1}}$", $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, LHS $= \sin x\cos x\cos 2x = \dfrac{\sin(2^2x)}{2^2}$ {from part **a i**}

and RHS $= \dfrac{\sin(2^{1+1}x)}{2^{1+1}} = \dfrac{\sin(2^2x)}{2^2}$

$\therefore$ P_1 is true.

(2) If P_k is true, then $\sin x \cos x \cos 2x \cos 4x .... \cos(2^k x) = \dfrac{\sin(2^{k+1}x)}{2^{k+1}}$

$$\therefore \quad \sin x \cos x \cos 2x \cos 4x .... \cos(2^k x)\cos(2^{k+1}x) = \frac{\sin(2^{k+1}x)}{2^{k+1}}\cos(2^{k+1}x)$$
$$= \tfrac{1}{2}\,\frac{2\sin(2^{k+1}x)\cos(2^{k+1}x)}{2^{k+1}}$$
$$= \frac{\sin 2(2^{k+1}x)}{2 \times 2^{k+1}}$$
$$= \frac{\sin(2^{k+2}x)}{2^{k+2}}$$

Thus P_{k+1} is true whenever P_k is true and P_1 is true.

$\therefore$ P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

4 **a** P_n is "$\sin\theta + \sin 3\theta + \sin 5\theta + ... + \sin(2n-1)\theta = \dfrac{1-\cos 2n\theta}{2\sin\theta}$", $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, LHS $= \sin\theta$ and RHS $= \dfrac{1-\cos 2\theta}{2\sin\theta} = \dfrac{1-(1-2\sin^2\theta)}{2\sin\theta} = \dfrac{2\sin^2\theta}{2\sin\theta} = \sin\theta$

$\therefore$ P_1 is true.

(2) If P_k is true, then

$$\sin\theta + \sin 3\theta + \sin 5\theta + ... + \sin(2k-1)\theta = \frac{1-\cos 2k\theta}{2\sin\theta}$$

$$\therefore \quad \sin\theta + \sin 3\theta + \sin 5\theta + ... + \sin(2k-1)\theta + \sin(2k+1)\theta$$
$$= \frac{1-\cos 2k\theta}{2\sin\theta} + \sin(2k+1)\theta$$
$$= \frac{1-\cos 2k\theta + 2\sin(2k+1)\theta\sin\theta}{2\sin\theta}$$
$$= \frac{1-\cos 2k\theta + \cos[(2k+1)\theta - \theta] - \cos[(2k+1)\theta + \theta]}{2\sin\theta}$$
$$= \frac{1-\cancel{\cos 2k\theta} + \cancel{\cos 2k\theta} - \cos[(2k+2)\theta]}{2\sin\theta}$$
$$= \frac{1-\cos 2(k+1)\theta}{2\sin\theta}$$

Thus P_{k+1} is true whenever P_k is true and P_1 is true.

$\therefore$ P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

b Thus $\sin\frac{\pi}{7} + \sin\frac{3\pi}{7} + \sin\frac{5\pi}{7} + ... + \sin\frac{13\pi}{7}$ has $2n - 1 = 13$ and $\theta = \frac{\pi}{7}$

$\therefore$ $n = 7$ and $\theta = \frac{\pi}{7}$

$\therefore$ the sum is $\dfrac{1-\cos(2\times 7\times\frac{\pi}{7})}{2\sin\frac{\pi}{7}} = \dfrac{1-\cos 2\pi}{2\sin\frac{\pi}{7}} = \dfrac{1-1}{2\sin\frac{\pi}{7}} = 0$

5 P_n is "$\cos x \times \cos 2x \times \cos 4x \times \cos 8x \times .... \times \cos(2^{n-1}x) = \dfrac{\sin(2^n x)}{2^n \times \sin x}$", $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, LHS $= \cos x$, RHS $= \dfrac{\sin 2x}{2\sin x} = \dfrac{2\sin x\cos x}{2\sin x} = \cos x$ $\therefore$ P_1 is true.

(2) If P_k is true, then $\cos x \times \cos 2x \times \cos 4x \times .... \times \cos(2^{k-1}x) = \dfrac{\sin(2^k x)}{2^k \sin x}$

$\therefore \quad \cos x \times \cos 2x \times \cos 4x \times .... \times \cos(2^{k-1}x) \times \cos(2^k x)$

$$= \frac{\sin(2^k x)}{2^k \sin x} \times \cos(2^k x)$$

$$= \frac{2\sin(2^k x)\cos(2^k x)}{2 \times 2^k \sin x}$$

$$= \frac{\sin(2 \times 2^k x)}{2^{k+1}\sin x} \qquad \{2\sin\theta\cos\theta = \sin 2\theta\}$$

$$= \frac{\sin(2^{k+1}x)}{2^{k+1}\sin x}$$

Thus P_{k+1} is true whenever P_k is true and P_1 is true.

$\therefore$ P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

6 P_n is "$\cos^2\theta + \cos^2 2\theta + \cos^2 3\theta + ... + \cos^2(n\theta) = \frac{1}{2}\left[n + \frac{\cos[(n+1)\theta]\sin n\theta}{\sin\theta}\right]$", $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, LHS $= \cos^2\theta$, RHS $= \frac{1}{2}\left[1 + \frac{\cos 2\theta \cancel{\sin\theta}}{\cancel{\sin\theta}}\right]$

$$= \tfrac{1}{2} + \tfrac{1}{2}(2\cos^2\theta - 1)$$

$$= \cos^2\theta \qquad \therefore \quad P_1 \text{ is true.}$$

(2) If P_k is true then $\cos^2\theta + \cos^2 2\theta + \cos^2 3\theta + ... + \cos^2(k\theta) = \frac{1}{2}\left[k + \frac{\cos[(k+1)\theta]\sin k\theta}{\sin\theta}\right]$

$\therefore \quad \cos^2\theta + \cos^2 2\theta + \cos^2 3\theta + ... + \cos^2(k\theta) + \cos^2[(k+1)\theta]$

$$= \tfrac{1}{2}\left[k + \frac{\cos[(k+1)\theta]\sin k\theta}{\sin\theta}\right] + \cos^2[(k+1)\theta]$$

$$= \tfrac{1}{2}\left[k + \frac{\cos[(k+1)\theta]\sin k\theta}{\sin\theta}\right] + \tfrac{1}{2} + \tfrac{1}{2}\cos[2(k+1)\theta] \qquad \{\cos^2\theta = \tfrac{1}{2} + \tfrac{1}{2}\cos 2\theta\}$$

$$= \tfrac{1}{2}(k+1) + \frac{\cos[(k+1)\theta]\sin k\theta + \cos[2(k+1)\theta]\sin\theta}{2\sin\theta}$$

$$= \tfrac{1}{2}(k+1) + \frac{\tfrac{1}{2}\sin[k\theta + (k+1)\theta] + \tfrac{1}{2}\sin[k\theta - (k+1)\theta] + \tfrac{1}{2}\sin[\theta + 2(k+1)\theta] + \tfrac{1}{2}\sin[\theta - 2(k+1)\theta]}{2\sin\theta}$$

{products to sums formula}

$$= \tfrac{1}{2}(k+1) + \frac{\tfrac{1}{2}\sin[(2k+1)\theta] + \tfrac{1}{2}\sin(-\theta) + \tfrac{1}{2}\sin[(2k+3)\theta] + \tfrac{1}{2}\sin[(-2k-1)\theta]}{2\sin\theta}$$

$$= \tfrac{1}{2}(k+1) + \frac{\cancel{\tfrac{1}{2}\sin[(2k+1)\theta]} - \tfrac{1}{2}\sin\theta + \tfrac{1}{2}\sin[(2k+3)\theta] - \cancel{\tfrac{1}{2}\sin[(2k+1)\theta]}}{2\sin\theta}$$

$$= \tfrac{1}{2}(k+1) + \frac{\tfrac{1}{2}\sin[(2k+3)\theta] - \tfrac{1}{2}\sin\theta}{2\sin\theta}$$

$$= \tfrac{1}{2}(k+1) + \frac{\cos\left[\frac{(2k+3)\theta+\theta}{2}\right]\sin\left[\frac{(2k+3)\theta-\theta}{2}\right]}{2\sin\theta} \qquad \{\text{factor formula}\}$$

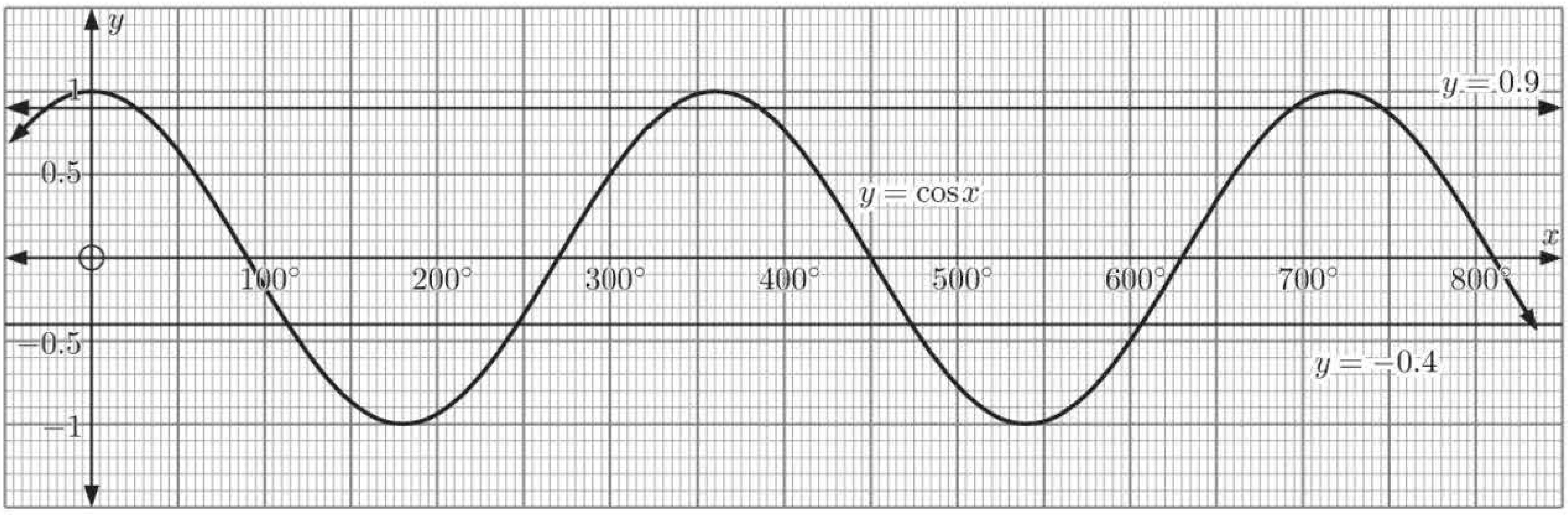
$$= \tfrac{1}{2}(k+1) + \frac{\cos[(k+2)\theta]\sin[(k+1)\theta]}{2\sin\theta} = \tfrac{1}{2}\left[(k+1) + \frac{\cos[(k+2)\theta]\sin[(k+1)\theta]}{\sin\theta}\right]$$

Thus P_{k+1} is true whenever P_k is true and P_1 is true.

$\therefore$ P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

REVIEW SET 13A

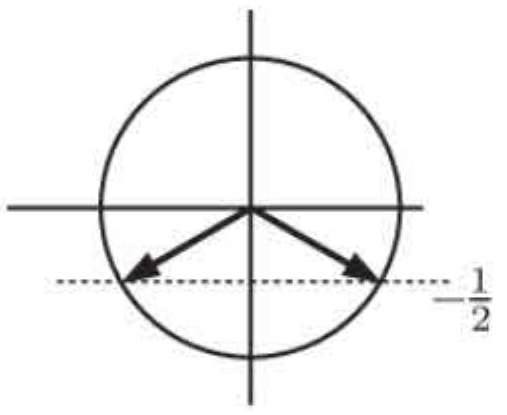
1



a When $\cos x = -0.4,\quad 0° \leqslant x \leqslant 800°,$
$x \approx 115°, 245°, 475°, 605°$

b When $\cos x = 0.9,\quad 0° \leqslant x \leqslant 600°,$
$x \approx 25°, 335°, 385°$

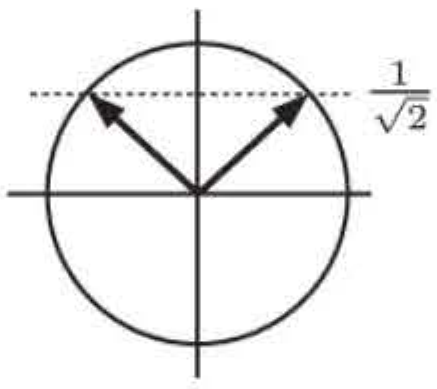
2 **a** $2\sin x = -1,\quad 0 \leqslant x \leqslant 4\pi$
$\therefore \quad \sin x = -\frac{1}{2}$



The points on the unit circle with sine $-\frac{1}{2}$ correspond to angles $\frac{7\pi}{6}$ and $\frac{11\pi}{6}$.

$\therefore \quad x = \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6}$

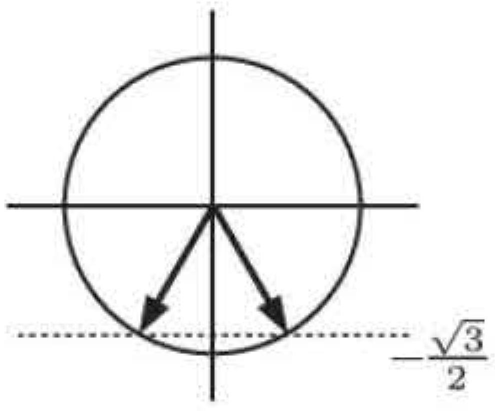
b $\sqrt{2}\sin x - 1 = 0,\quad -2\pi \leqslant x \leqslant 2\pi$
$\therefore \quad \sin x = \frac{1}{\sqrt{2}}$



The points on the unit circle with sine $\frac{1}{\sqrt{2}}$ correspond to angles $\frac{\pi}{4}$ and $\frac{3\pi}{4}$.

$\therefore \quad x = -\frac{7\pi}{4}, -\frac{5\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}$

3 **a** $2\sin 3x + \sqrt{3} = 0,\quad 0 \leqslant x \leqslant 2\pi$
$\therefore \quad \sin 3x = -\frac{\sqrt{3}}{2}$



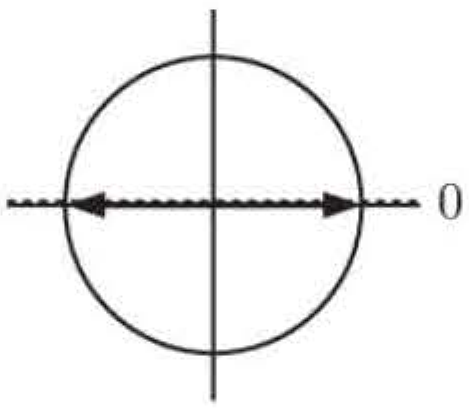
The points on the unit circle with sine $-\frac{\sqrt{3}}{2}$ correspond to angles $\frac{4\pi}{3}$ and $\frac{5\pi}{3}$.

$0 \leqslant x \leqslant 2\pi \quad \therefore \quad 0 \leqslant 3x \leqslant 6\pi$

So $\quad 3x = \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{10\pi}{3}, \frac{11\pi}{3}, \frac{16\pi}{3}, \frac{17\pi}{3}$

So, the x-intercepts are
$\frac{4\pi}{9}, \frac{5\pi}{9}, \frac{10\pi}{9}, \frac{11\pi}{9}, \frac{16\pi}{9}, \frac{17\pi}{9}$.

b $\sqrt{2}\sin(x + \frac{\pi}{4}) = 0,\quad 0 \leqslant x \leqslant 3\pi$
$\therefore \quad \sin(x + \frac{\pi}{4}) = 0$



The points on the unit circle with sine 0 correspond to angles 0 and π.

$0 \leqslant x \leqslant 3\pi \quad \therefore \quad \frac{\pi}{4} \leqslant x + \frac{\pi}{4} \leqslant \frac{13\pi}{4}$

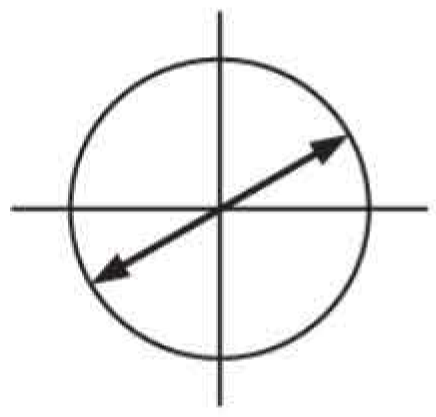
So $\quad x + \frac{\pi}{4} = \pi, 2\pi, 3\pi$

So, the x-intercepts are $\frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}$.

4 **a** $\cot x = \sqrt{3}, \quad x \in [0, 2\pi]$

$\therefore \dfrac{1}{\tan x} = \sqrt{3}$

$\therefore \tan x = \frac{1}{\sqrt{3}}$



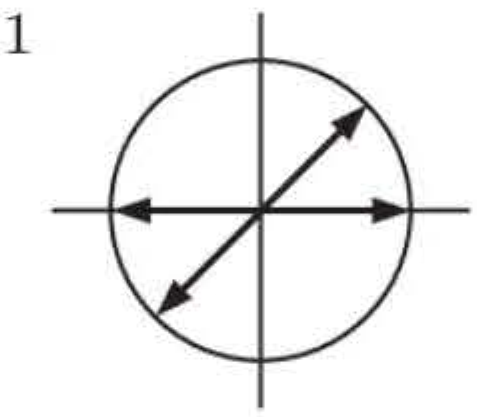
$\therefore x = \frac{\pi}{6}, \frac{7\pi}{6}$

b $\sec^2 x = \tan x + 1, \quad x \in [0, 2\pi]$

$\therefore \tan^2 x \cancel{+1} = \tan x \cancel{+1}$

$\therefore \tan x(\tan x - 1) = 0$

$\therefore \tan x = 0$ or 1



$\therefore x = 0, \frac{\pi}{4}, \pi, \frac{5\pi}{4}, 2\pi$

5 **a**
$$\begin{aligned}&\cos\left(\tfrac{3\pi}{2} - \theta\right)\\ &= \cos\left(\tfrac{3\pi}{2}\right)\cos\theta + \sin\left(\tfrac{3\pi}{2}\right)\sin\theta\\ &= (0)\cos\theta + (-1)\sin\theta\\ &= -\sin\theta\end{aligned}$$

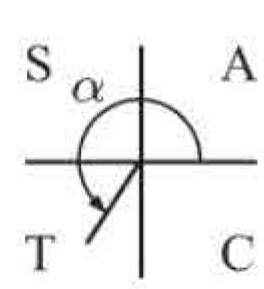
b
$$\begin{aligned}&\sin\left(\theta + \tfrac{\pi}{2}\right)\\ &= \sin\theta\cos\left(\tfrac{\pi}{2}\right) + \cos\theta\sin\left(\tfrac{\pi}{2}\right)\\ &= \sin\theta(0) + \cos\theta(1)\\ &= \cos\theta\end{aligned}$$

6 **a**
$$\begin{aligned}&\frac{1-\cos^2\theta}{1+\cos\theta}\\ &= \frac{\cancel{(1+\cos\theta)}(1-\cos\theta)}{\cancel{1+\cos\theta}\ {}_1}\\ &= 1-\cos\theta\end{aligned}$$

b
$$\begin{aligned}&\frac{\sin\alpha - \cos\alpha}{\sin^2\alpha - \cos^2\alpha}\\ &= \frac{\cancel{\sin\alpha - \cos\alpha}\ {}^1}{(\sin\alpha + \cos\alpha)\cancel{(\sin\alpha - \cos\alpha)}}\\ &= \frac{1}{\sin\alpha + \cos\alpha}\end{aligned}$$

c
$$\begin{aligned}&\frac{4\sin^2\alpha - 4}{8\cos\alpha}\\ &= \frac{-4(1-\sin^2\alpha)}{8\cos\alpha}\\ &= \frac{-4\cos^2\alpha}{8\cos\alpha}\\ &= \frac{-\cos\alpha}{2}\end{aligned}$$

7 **a**



$$\begin{aligned}\cos^2\alpha + \sin^2\alpha &= 1\\ \therefore \cos^2\alpha + \tfrac{9}{16} &= 1\\ \therefore \cos^2\alpha &= \tfrac{7}{16}\\ \therefore \cos\alpha &= \pm\tfrac{\sqrt{7}}{4}\end{aligned}$$

But in Q3, $\cos\alpha < 0$

$\therefore \cos\alpha = -\frac{\sqrt{7}}{4}$

b
$$\begin{aligned}\sin 2\alpha &= 2\sin\alpha\cos\alpha\\ &= 2(-\tfrac{3}{4})(-\tfrac{\sqrt{7}}{4})\\ &= \tfrac{3\sqrt{7}}{8}\end{aligned}$$

c
$$\begin{aligned}\cos 2\alpha &= \cos^2\alpha - \sin^2\alpha\\ &= \left(-\tfrac{\sqrt{7}}{4}\right)^2 - \left(-\tfrac{3}{4}\right)^2\\ &= \tfrac{7}{16} - \tfrac{9}{16}\\ &= -\tfrac{2}{16}\\ &= -\tfrac{1}{8}\end{aligned}$$

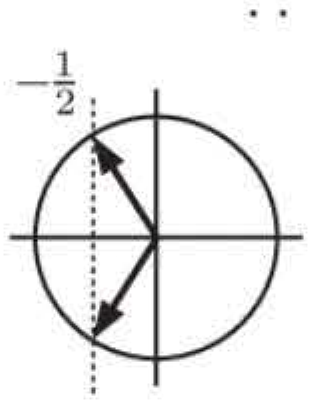
d
$$\begin{aligned}\tan 2\alpha &= \frac{\sin 2\alpha}{\cos 2\alpha}\\ &= \frac{\frac{3\sqrt{7}}{8}}{-\frac{1}{8}}\\ &= -3\sqrt{7}\end{aligned}$$

8
$$\begin{aligned}\frac{\sin 2\alpha - \sin\alpha}{\cos 2\alpha - \cos\alpha + 1} &= \frac{2\sin\alpha\cos\alpha - \sin\alpha}{2\cos^2\alpha - 1 - \cos\alpha + 1}\\ &= \frac{\sin\alpha\cancel{(2\cos\alpha - 1)}}{\cos\alpha\cancel{(2\cos\alpha - 1)}}\\ &= \frac{\sin\alpha}{\cos\alpha}\\ &= \tan\alpha\end{aligned}$$

9 **a**
$$\begin{aligned}&\cos(165^\circ)\\ &= \cos(120^\circ + 45^\circ)\\ &= \cos 120^\circ \cos 45^\circ - \sin 120^\circ \sin 45^\circ\\ &= \left(-\tfrac{1}{2}\right)\left(\tfrac{1}{\sqrt{2}}\right) - \left(\tfrac{\sqrt{3}}{2}\right)\left(\tfrac{1}{\sqrt{2}}\right)\\ &= \frac{-1-\sqrt{3}}{2\sqrt{2}}\end{aligned}$$

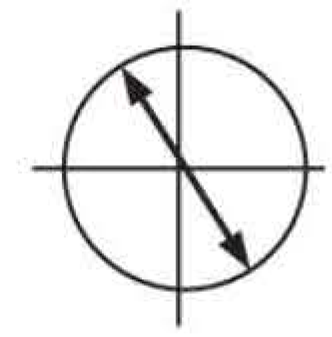
b
$$\begin{aligned}\tan(\tfrac{\pi}{12}) &= \tan(\tfrac{3\pi}{12} - \tfrac{2\pi}{12})\\ &= \frac{\tan(\frac{\pi}{4}) - \tan(\frac{\pi}{6})}{1+\tan(\frac{\pi}{4})\tan(\frac{\pi}{6})}\\ &= \frac{1-\frac{1}{\sqrt{3}}}{1+(1)(\frac{1}{\sqrt{3}})} \times \left(\tfrac{\sqrt{3}}{\sqrt{3}}\right)\\ &= \frac{\sqrt{3}-1}{\sqrt{3}+1} \times \left(\frac{\sqrt{3}-1}{\sqrt{3}-1}\right)\\ &= \frac{3-2\sqrt{3}+1}{3-1}\\ &= \frac{4-2\sqrt{3}}{2}\\ &= 2-\sqrt{3}\end{aligned}$$

10 **a**
$$2\cos 2x + 1 = 0$$
$$\therefore\ \cos 2x = -\tfrac{1}{2}$$



$$\therefore\ 2x = \tfrac{2\pi}{3}, \tfrac{4\pi}{3}, \tfrac{8\pi}{3}, \tfrac{10\pi}{3}$$
$$\{0 \leqslant 2x \leqslant 4\pi\}$$
$$\therefore\ x = \tfrac{\pi}{3}, \tfrac{2\pi}{3}, \tfrac{4\pi}{3}, \tfrac{5\pi}{3}$$

b
$$\sin 2x = -\sqrt{3}\cos 2x$$
$$\therefore\ \frac{\sin 2x}{\cos 2x} = -\sqrt{3}$$
$$\therefore\ \tan 2x = -\sqrt{3}$$

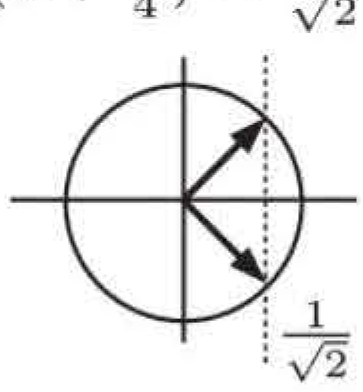


$$\therefore\ 2x = \tfrac{2\pi}{3}, \tfrac{5\pi}{3}, \tfrac{8\pi}{3}, \tfrac{11\pi}{3}$$
$$\{0 \leqslant 2x \leqslant 4\pi\}$$
$$\therefore\ x = \tfrac{\pi}{3}, \tfrac{5\pi}{6}, \tfrac{4\pi}{3}, \tfrac{11\pi}{6}$$

11 **a**
$$\begin{aligned}\sin 2\theta &= 2\sin\theta\cos\theta\\ &= 2\left(\frac{b}{c}\right)\left(\frac{a}{c}\right)\\ &= \frac{2ab}{c^2}\end{aligned}$$

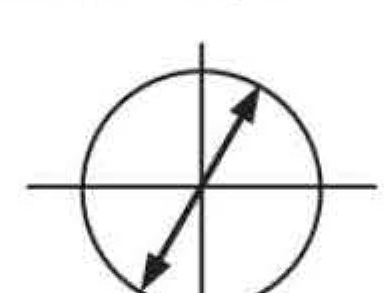
b
$$\begin{aligned}\cos 2\theta &= \cos^2\theta - \sin^2\theta\\ &= \left(\frac{a}{c}\right)^2 - \left(\frac{b}{c}\right)^2\\ &= \frac{a^2-b^2}{c^2}\end{aligned}$$

12 **a**
$$\sqrt{2}\cos(x+\tfrac{\pi}{4}) - 1 = 0, \quad x \in [0, 4\pi]$$
$$\therefore\ \cos(x+\tfrac{\pi}{4}) = \tfrac{1}{\sqrt{2}}$$



$$\therefore\ x+\tfrac{\pi}{4} = \tfrac{\pi}{4}, \tfrac{7\pi}{4}, \tfrac{9\pi}{4}, \tfrac{15\pi}{4}, \tfrac{17\pi}{4}$$
$$\{\tfrac{\pi}{4} \leqslant x+\tfrac{\pi}{4} \leqslant \tfrac{17\pi}{4}\}$$
$$\therefore\ x = 0, \tfrac{3\pi}{2}, 2\pi, \tfrac{7\pi}{2}, 4\pi$$

b
$$\tan 2x - \sqrt{3} = 0, \quad x \in [0, 2\pi]$$
$$\therefore\ \tan 2x = \sqrt{3}$$



$$\therefore\ 2x = \tfrac{\pi}{3}, \tfrac{4\pi}{3}, \tfrac{7\pi}{3}, \tfrac{10\pi}{3}$$
$$\{0 \leqslant 2x \leqslant 4\pi\}$$
$$\therefore\ x = \tfrac{\pi}{6}, \tfrac{2\pi}{3}, \tfrac{7\pi}{6}, \tfrac{5\pi}{3}$$

13 **a** By the sine rule, $\dfrac{\sin 2\alpha}{5} = \dfrac{\sin\alpha}{3}$
$$\therefore\ \frac{2\sin\alpha\cos\alpha}{\sin\alpha} = \tfrac{5}{3}$$
$$\therefore\ 2\cos\alpha = \tfrac{5}{3} \quad \{\sin\alpha \neq 0\}$$
$$\therefore\ \cos\alpha = \tfrac{5}{6}$$

b Using the cosine rule
$$3^2 = x^2 + 5^2 - 2\times x\times 5\times\cos\alpha$$
$$\therefore\ 9 = x^2 + 25 - 10x\left(\tfrac{5}{6}\right)$$
$$\therefore\ x^2 - \tfrac{25}{3}x + 16 = 0$$
$$\therefore\ 3x^2 - 25x + 48 = 0$$

c
$$(3x-16)(x-3) = 0$$
$$\therefore\ x = \tfrac{16}{3} \text{ or } 3$$

REVIEW SET 13B

1 **a** $\sin x = 0.382$
$\sin^{-1}(0.382) \approx 0.392$ {using technology}
$\therefore$ on $0 \leqslant x \leqslant 8$ the solutions are
$x \approx 0.392,\ \pi - 0.392,\ 2\pi + 0.392$
$\therefore\ x \approx 0.392, 2.75, 6.68$

b $\tan\left(\frac{x}{2}\right) = -0.458$
$\tan^{-1}(-0.458) \approx -0.429$
{using technology}
$\therefore$ on $0 \leqslant \dfrac{x}{2} \leqslant 4$ the solutions are
$\dfrac{x}{2} \approx -0.429 + \pi$
$\therefore\ \dfrac{x}{2} \approx 2.71$
$\therefore\ x \approx 5.42$

2 **a** $\cos x = 0.4379$
$\cos^{-1}(0.4379) \approx 1.12$ {using technology}
$\therefore$ on $0 \leqslant x \leqslant 10$ the solutions are
$x \approx 1.12,\ 2\pi - 1.12,\ 2\pi + 1.12$
$\therefore\ x \approx 1.12, 5.17, 7.40$

b $\cos(x - 2.4) = -0.6014$
$\cos^{-1}(-0.6014) \approx 2.216$
{using technology}
$\therefore$ on $-2.4 \leqslant x - 2.4 \leqslant 3.6$ the solutions are $x - 2.4 \approx -2.216, 2.216$
$\therefore\ x \approx 0.184, 4.62$

3 **a** $\sin 2A = 2\sin A\cos A$
$= 2(\frac{5}{13})(\frac{12}{13})$
$= \frac{120}{169}$

b $\cos 2A = \cos^2 A - \sin^2 A$
$= (\frac{12}{13})^2 - (\frac{5}{13})^2$
$= \frac{144-25}{169}$
$= \frac{119}{169}$

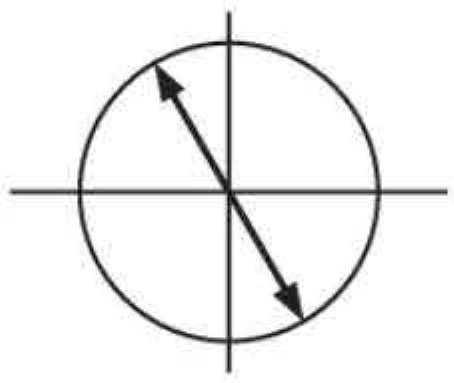
c $\tan 2A = \dfrac{\sin 2A}{\cos 2A}$
$= \dfrac{\frac{120}{169}}{\frac{119}{169}}$
$= \frac{120}{119}$

4 **a** **i** $\tan x = 4$
$\tan^{-1}(4) \approx 1.33$ {using technology}
$\therefore$ on $0 \leqslant x \leqslant 10$ the solutions are
$x \approx 1.33,\ 1.33 + \pi,\ 1.33 + 2\pi$
$\therefore\ x \approx 1.33, 4.47, 7.61$

ii $\tan\left(\frac{x}{4}\right) = 4$
$\tan^{-1}(4) \approx 1.33$ {using technology}
$\therefore$ on $0 \leqslant \dfrac{x}{4} \leqslant \frac{5}{2}$ the solutions are
$\dfrac{x}{4} \approx 1.33$
$\therefore\ x \approx 5.30$

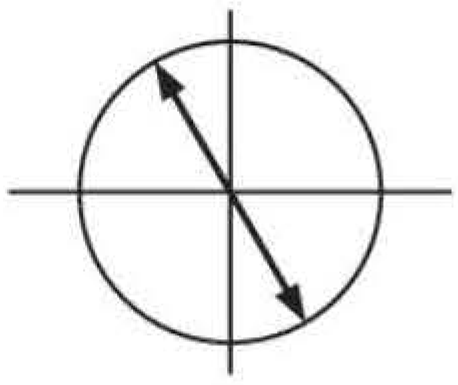
iii $\tan(x - 1.5) = 4$
$\tan^{-1}(4) \approx 1.33$ {using technology}
$\therefore$ on $-1.5 \leqslant x - 1.5 \leqslant 8.5$ the solutions are $x - 1.5 \approx 1.33,\ 1.33 + \pi,\ 1.33 + 2\pi$
$\therefore\ x - 1.5 \approx 1.33, 4.47, 7.61$
$\therefore\ x \approx 2.83, 5.97, 9.11$

b **i** $\tan(x + \frac{\pi}{6}) = -\sqrt{3},\quad -\pi \leqslant x \leqslant \pi$



The points on the unit circle with tangent $-\sqrt{3}$ correspond to angles $\frac{2\pi}{3}$ and $\frac{5\pi}{3}$.
$-\pi \leqslant x \leqslant \pi \quad \therefore\ -\frac{5\pi}{6} \leqslant x + \frac{\pi}{6} \leqslant \frac{7\pi}{6}$
So $x + \frac{\pi}{6} = -\frac{\pi}{3}, \frac{2\pi}{3}$
$\therefore\ x = -\frac{\pi}{2}, \frac{\pi}{2}$

ii $\tan 2x = -\sqrt{3},\quad -\pi \leqslant x \leqslant \pi$



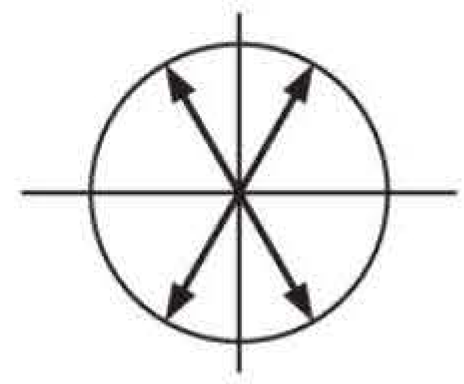
The points on the unit circle with tangent $-\sqrt{3}$ correspond to angles $\frac{2\pi}{3}$ and $\frac{5\pi}{3}$.
$-\pi \leqslant x \leqslant \pi \quad \therefore\ -2\pi \leqslant 2x \leqslant 2\pi$
So $2x = -\frac{4\pi}{3}, -\frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{3}$
$\therefore\ x = -\frac{2\pi}{3}, -\frac{\pi}{6}, \frac{\pi}{3}, \frac{5\pi}{6}$

iii $\tan^2 x - 3 = 0, \quad -\pi \leqslant x \leqslant \pi$

$\therefore\ \tan x = \pm\sqrt{3}$

The points on the unit circle with tangent $\pm\sqrt{3}$ correspond to angles $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$.

$\therefore\ x = -\frac{2\pi}{3}, -\frac{\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3}$



c $3\tan(x - 1.2) = -2$

$\therefore\ \tan(x - 1.2) = -\frac{2}{3}$

$\tan^{-1}(-\frac{2}{3}) \approx -0.588$ {using technology}

$\therefore$ on $-1.2 \leqslant x - 1.2 \leqslant 8.8$ the solutions are

$x - 1.2 \approx -0.588,\ -0.588 + \pi,\ -0.588 + 2\pi$

$\therefore\ x - 1.2 \approx -0.588, 2.55, 5.70$

$\therefore\ x \approx 0.612, 3.75, 6.90$

5 **a**

$$\begin{aligned}&\sqrt{2}\cos\left(\theta + \tfrac{\pi}{4}\right)\\ &= \sqrt{2}\left[\cos\theta\cos\left(\tfrac{\pi}{4}\right) - \sin\theta\sin\left(\tfrac{\pi}{4}\right)\right]\\ &= \sqrt{2}\left[\cos\theta \times \tfrac{1}{\sqrt{2}} - \sin\theta \times \tfrac{1}{\sqrt{2}}\right]\\ &= \cos\theta - \sin\theta\end{aligned}$$

b

$$\begin{aligned}&\cos\alpha\cos(\beta-\alpha) - \sin\alpha\sin(\beta-\alpha)\\ &= \cos\left[\alpha + (\beta - \alpha)\right]\\ &= \cos\beta\end{aligned}$$

6

$\cos 2A = 1 - 2\sin^2 A$

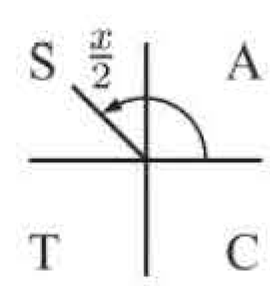
$\therefore\ \cos x = 1 - 2\sin^2\left(\frac{x}{2}\right)$ {letting $2A = x,\ A = \frac{x}{2}$}

$\therefore\ -\frac{3}{4} = 1 - 2\sin^2\left(\frac{x}{2}\right)$

$\therefore\ 2\sin^2\left(\frac{x}{2}\right) = \frac{7}{4}$

$\therefore\ \sin^2\left(\frac{x}{2}\right) = \frac{7}{8}$

$\therefore\ \sin\left(\frac{x}{2}\right) = \pm\frac{\sqrt{7}}{2\sqrt{2}}$



But $\frac{\pi}{2} < \frac{x}{2} < \frac{3\pi}{4}$ (in Q2) $\quad \therefore\ \sin\left(\frac{x}{2}\right) = \frac{\sqrt{7}}{2\sqrt{2}}$

7 **a**

$\cos x = 0.3$

$\cos^{-1}(0.3) \approx 1.27$ {using technology}

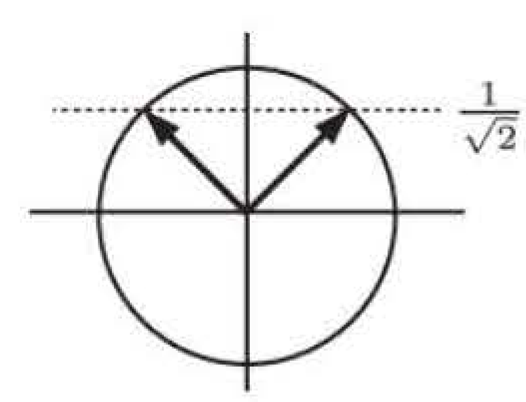
$\therefore$ on $0 \leqslant x \leqslant 2\pi$ the solutions are

$x \approx 1.27,\ 2\pi - 1.27$

$\therefore\ x \approx 1.27, 5.02$

b $2\sin(3x) = \sqrt{2}$

$\therefore\ \sin(3x) = \frac{1}{\sqrt{2}}$



on $0 \leqslant x \leqslant 2\pi \quad \therefore\ 0 \leqslant 3x \leqslant 6\pi$

so $3x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}, \frac{11\pi}{4}, \frac{17\pi}{4}, \frac{19\pi}{4}$

$\therefore\ x = \frac{\pi}{12}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{11\pi}{12}, \frac{17\pi}{12}, \frac{19\pi}{12}$

c $43 + 8\sin x = 50.1$

$\therefore\ 8\sin x = 7.1$

$\therefore\ \sin x = \frac{7.1}{8}$

$\sin^{-1}\left(\frac{7.1}{8}\right) \approx 1.09$

$\therefore$ on $0 \leqslant x \leqslant 2\pi$ the solutions

are $x \approx 1.09,\ \pi - 1.09$

$\therefore\ x \approx 1.09, 2.05$

8 $P(t) = 5 + 2\sin\left(\frac{\pi t}{3}\right),\ \ 0 \leqslant t \leqslant 8,$ where $P(t)$ is in thousands of water beetles.

a $P(0) = 5 + 2\sin 0$

$= 5$

So, 5000 water beetles.

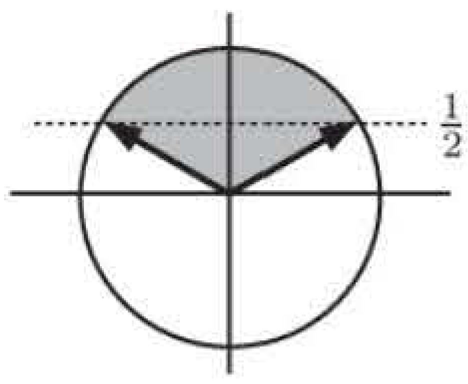
b Smallest $P = 5 + 2(-1) = 3$

Largest $P = 5 + 2(1) = 7$

$\therefore$ smallest is 3000 water beetles

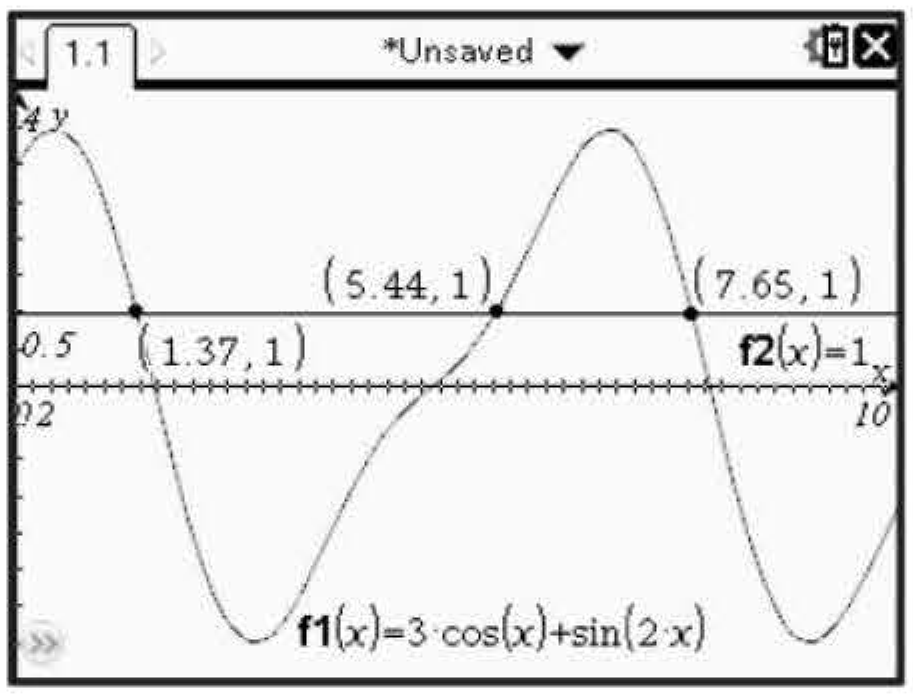
largest is 7000 water beetles

c If population is > 6000,
then $P(t) > 6$
$\therefore\ 5 + 2\sin\left(\frac{\pi t}{3}\right) > 6$
$\therefore\ 2\sin\left(\frac{\pi t}{3}\right) > 1$
$\therefore\ \sin\left(\frac{\pi t}{3}\right) > \frac{1}{2}$



The points on the unit circle with sine $\frac{1}{2}$ correspond to angles $\frac{\pi}{6}$ and $\frac{5\pi}{6}$.
$0 \leqslant t \leqslant 8 \quad \therefore\ 0 \leqslant \frac{\pi t}{3} \leqslant \frac{8\pi}{3}$
So $\frac{\pi}{6} < \frac{\pi t}{3} < \frac{5\pi}{6},\quad \frac{13\pi}{6} < \frac{\pi t}{3} \leqslant \frac{8\pi}{3}$
$\therefore\ \frac{1}{2} < t < \frac{5}{2},\quad \frac{13}{2} < t \leqslant 8$
$\therefore\ 0.5 < t < 2.5,\quad 6.5 < t \leqslant 8$

9 $3\cos x + \sin 2x = 1,\quad 0 \leqslant x \leqslant 10$
We graph $Y_1 = 3\cos x + \sin 2x$ and $Y_2 = 1$ on the same set of axes.

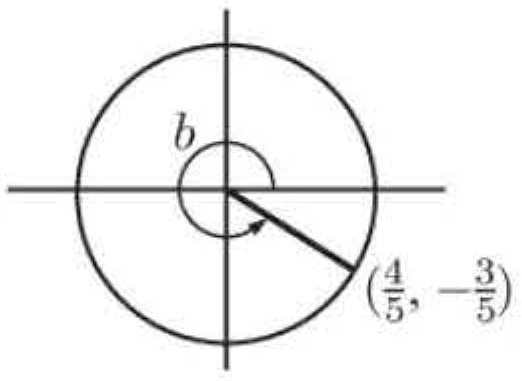


$\therefore\ x \approx 1.37, 5.44, 7.65$

10 $3\sin x + 4\cos x = k\cos(x+b)$
$= k[\cos x\cos b - \sin x\sin b]$
$= k\cos x\cos b - k\sin x\sin b$

Equating coefficients of $\cos x$ and $\sin x$,
$k\cos b = 4$ .... (1) and $-k\sin b = 3$ .... (2)
$\therefore\ k^2\cos^2 b = 16$ and $k^2\sin^2 b = 9$ {squaring both sides}
$\therefore\ k^2(\cos^2 b + \sin^2 b) = 25$ {adding the 2 equations}
$\therefore\ k^2 = 25$
$\therefore\ k = 5$ {since $k > 0$}

Substituting $k = 5$ into (1) gives $5\cos b = 4 \quad \therefore\ \cos b = \frac{4}{5}$
and into (2) gives $-5\sin b = 3 \quad \therefore\ \sin b = -\frac{3}{5}$
$\therefore\ b \approx 5.64$



So, $3\sin x + 4\cos x \approx 5\cos(x + 5.64)$

11 Let the shooter be x m from the wall.

$\therefore\ \tan\alpha = \dfrac{20}{x},\quad \tan 2\alpha = \dfrac{45}{x}$

$\therefore\ \dfrac{2\tan\alpha}{1-\tan^2\alpha} = \dfrac{45}{x}$

$\therefore\ 2x\tan\alpha = 45 - 45\tan^2\alpha$

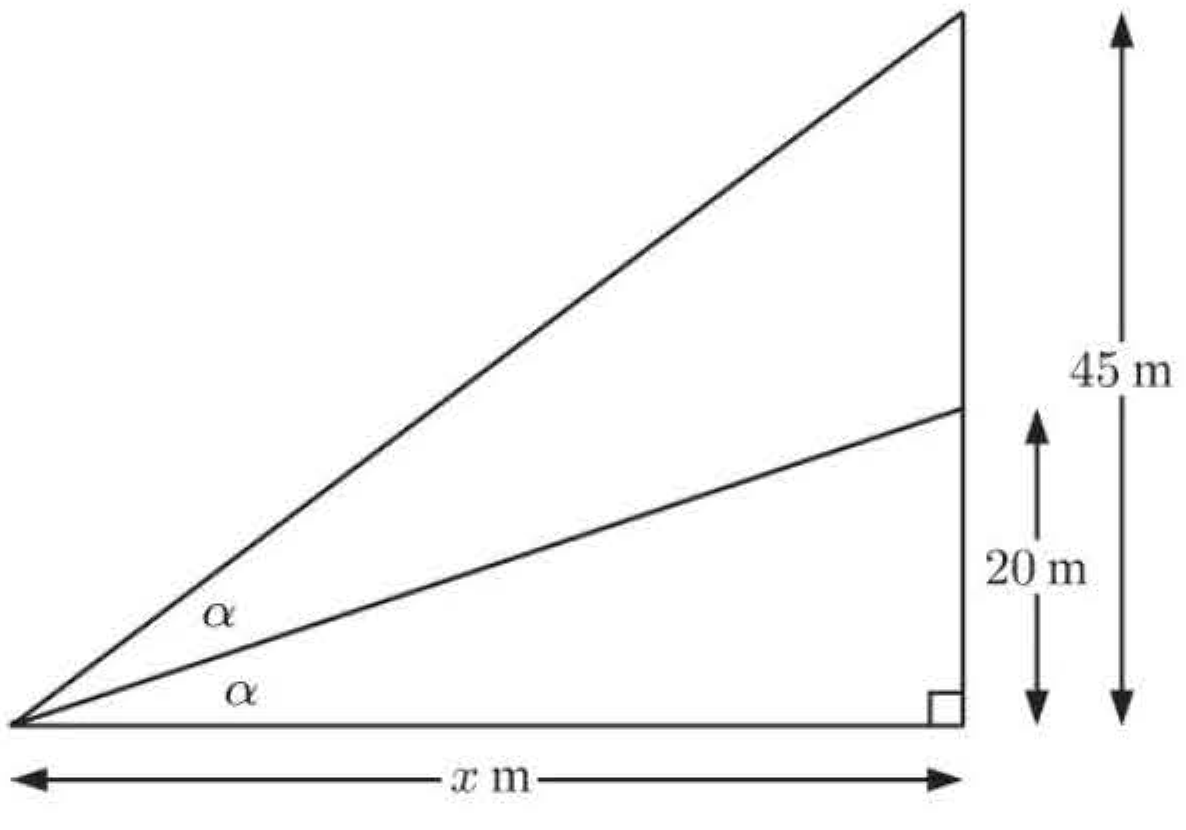
$\therefore\ 2x\left(\dfrac{20}{x}\right) = 45 - 45\left(\dfrac{20}{x}\right)^2$

$\therefore\ 40 = 45 - \dfrac{18\,000}{x^2}$

$\therefore\ \dfrac{18\,000}{x^2} = 5$

$\therefore\ x^2 = 3600$

$\therefore\ x = 60$ So, the shooter is 60 m from the wall.



12
$$\tan 2\theta = \tfrac{4}{3}$$
$$\therefore\ \frac{2\tan\theta}{1-\tan^2\theta} = \tfrac{4}{3}$$
$$\therefore\ 6\tan\theta = 4 - 4\tan^2\theta$$
$$\therefore\ 2(2\tan^2\theta + 3\tan\theta - 2) = 0$$
$$\therefore\ 2(2\tan\theta - 1)(\tan\theta + 2) = 0$$
$$\therefore\ \tan\theta = \tfrac{1}{2} \text{ or } -2$$

But θ is clearly acute,

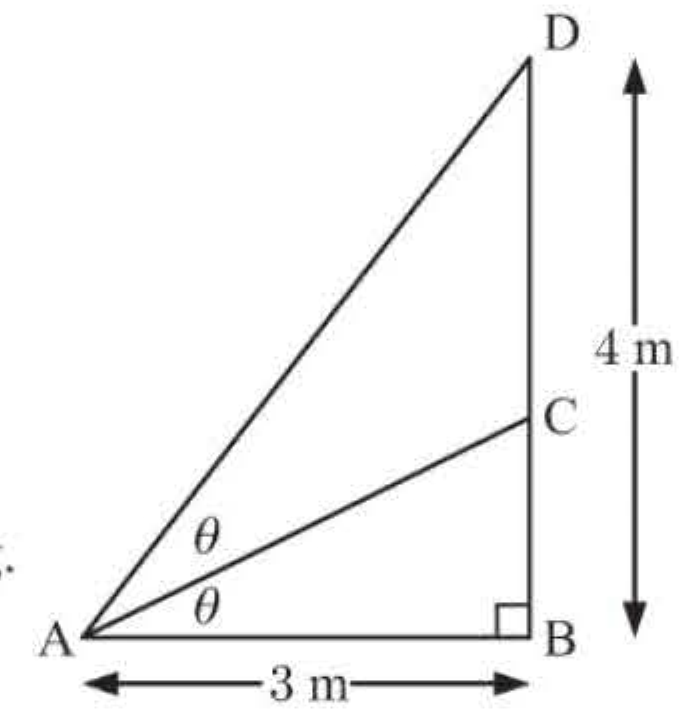
so $\tan\theta > 0$

$\therefore\ \tan\theta = \frac{1}{2}$

$\therefore\ \dfrac{\text{BC}}{3} = \frac{1}{2}$

$\therefore\ \text{BC} = 1.5$

$\therefore$ [BC] is 1.5 m long.



13 $P(t) = 40 + 12\sin\frac{2\pi}{7}\left(t - \frac{37}{12}\right)$ mg

a $P(t)$ has a minimum of $40 + 12(-1)$
$= 28$ mg per m^3

b When $\sin\frac{2\pi}{7}(t - \frac{37}{12}) = -1$

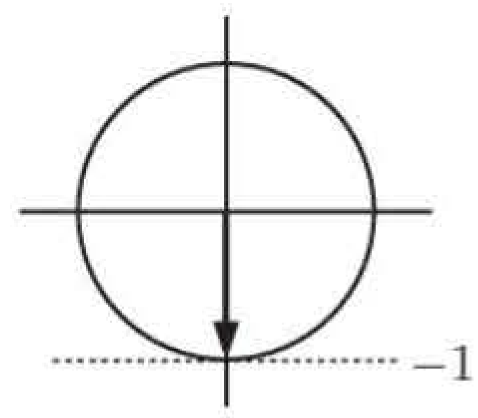
$\therefore\ \frac{2\pi}{7}(t - \frac{37}{12}) = \frac{3\pi}{2} + k2\pi$

$\therefore\ \frac{2}{7}(t - \frac{37}{12}) = \frac{3}{2} + k\,2$

So, $t - \frac{37}{12} = \frac{21}{4} + k\,7$

$\therefore\ t = 8\frac{1}{3} + k\,7$

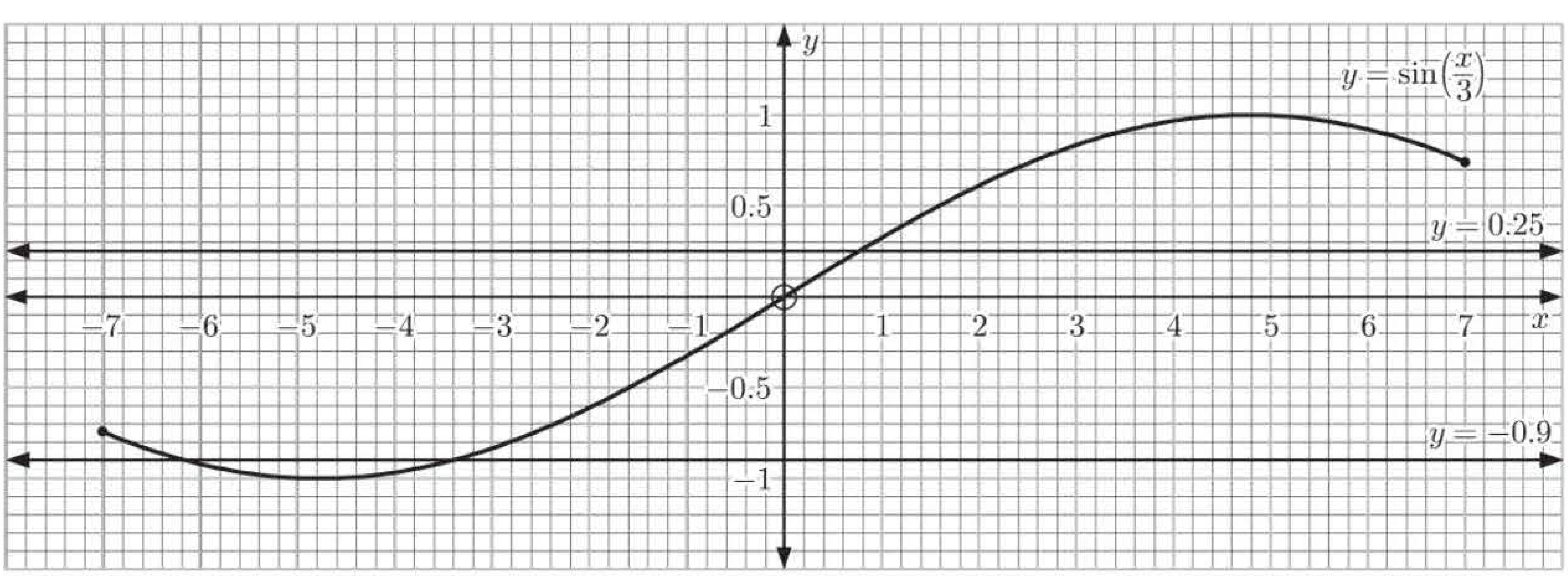
$\therefore\ t = 1\frac{1}{3}, 8\frac{1}{3}, 15\frac{1}{3}$, and so on.



$\therefore$ on Mondays at 8:00 am $\{1\frac{1}{3}$ days after midnight Saturday$\}$

REVIEW SET 13C

1



a $\sin\left(\frac{x}{3}\right) = -0.9, \quad -7 \leqslant x \leqslant 7$

$\therefore\ x \approx -6.1, -3.4$

b $\sin\left(\frac{x}{3}\right) = \frac{1}{4}, \quad -7 \leqslant x \leqslant 7$

$\therefore\ x \approx 0.8$

2 **a** $\sin^2 x - \sin x - 2 = 0, \ 0 \leqslant x \leqslant 2\pi$

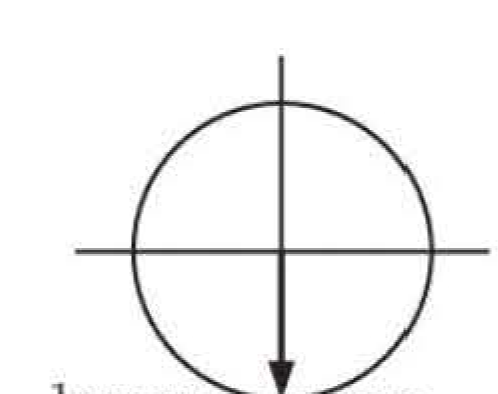
$\therefore\ (\sin x - 2)(\sin x + 1) = 0$

$\therefore\ \sin x = 2$ or -1

But $\sin x$ values lie between -1 and 1 inclusive.

$\therefore\ \sin x = -1$

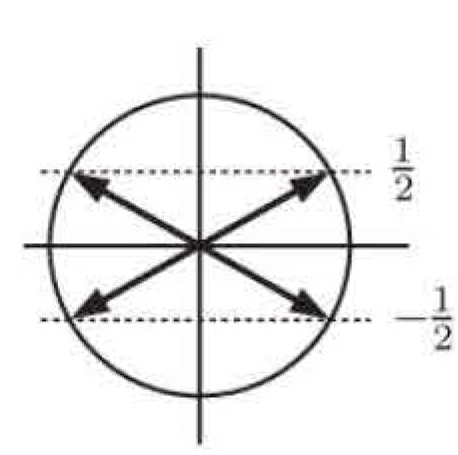
$\therefore\ x = \frac{3\pi}{2}$



b $4\sin^2 x = 1, \quad 0 \leqslant x \leqslant 2\pi$

$\therefore\ \sin^2 x = \frac{1}{4}$

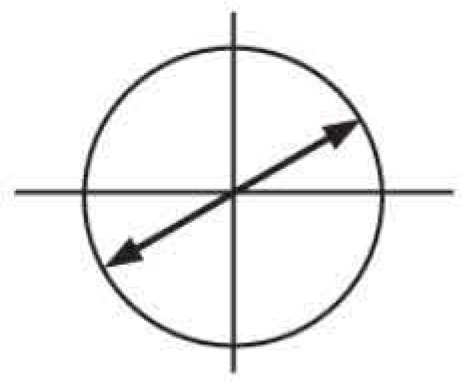
$\therefore\ \sin x = \pm\frac{1}{2}$



$\therefore\ x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

3 **a** $\tan(x - \frac{\pi}{3}) = \frac{1}{\sqrt{3}}, \quad 0 \leqslant x \leqslant 4\pi$

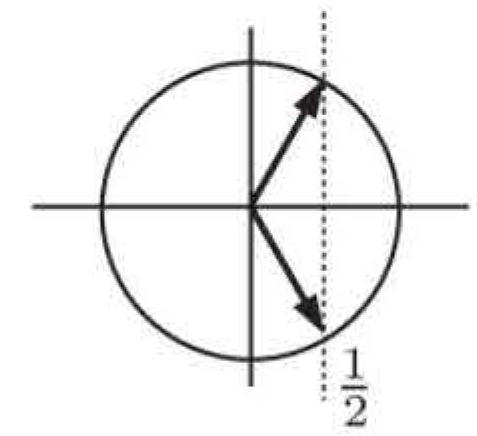
The points on the unit circle with tangent $\frac{1}{\sqrt{3}}$ correspond to angles $\frac{\pi}{6}$ and $\frac{7\pi}{6}$.



$$0 \leqslant x \leqslant 4\pi$$
$$\therefore \; -\tfrac{\pi}{3} \leqslant x - \tfrac{\pi}{3} \leqslant \tfrac{11\pi}{3}$$
So $x - \frac{\pi}{3} = \frac{\pi}{6}, \frac{7\pi}{6}, \frac{13\pi}{6}, \frac{19\pi}{6}$
$$\therefore \; x = \tfrac{\pi}{2}, \tfrac{3\pi}{2}, \tfrac{5\pi}{2}, \tfrac{7\pi}{2}$$

b $\cos(x + \frac{2\pi}{3}) = \frac{1}{2}, \quad -2\pi \leqslant x \leqslant 2\pi$

The points on the unit circle with cosine $\frac{1}{2}$ correspond to angles $\frac{\pi}{3}$ and $\frac{5\pi}{3}$.



$$-2\pi \leqslant x \leqslant 2\pi$$
$$\therefore \; -\tfrac{4\pi}{3} \leqslant x + \tfrac{2\pi}{3} \leqslant \tfrac{8\pi}{3}$$
So $x + \frac{2\pi}{3} = -\frac{\pi}{3}, \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}$
$$\therefore \; x = -\pi, -\tfrac{\pi}{3}, \pi, \tfrac{5\pi}{3}$$

4 **a**
$$\begin{aligned} &\cos^3\theta + \sin^2\theta\cos\theta \\ &= \cos\theta(\cos^2\theta + \sin^2\theta) \\ &= \cos\theta \end{aligned}$$

b
$$\begin{aligned} \frac{\cos^2\theta - 1}{\sin\theta} &= \frac{-(1-\cos^2\theta)}{\sin\theta} \\ &= -\frac{\sin^2\theta}{\sin\theta} \\ &= -\sin\theta \end{aligned}$$

c
$$\begin{aligned} &5 - 5\sin^2\theta \\ &= 5(1-\sin^2\theta) \\ &= 5\cos^2\theta \end{aligned}$$

d
$$\begin{aligned} \frac{\sin^2\theta - 1}{\cos\theta} &= -\frac{(1-\sin^2\theta)}{\cos\theta} \\ &= -\frac{\cos^2\theta}{\cos\theta} \\ &= -\cos\theta \end{aligned}$$

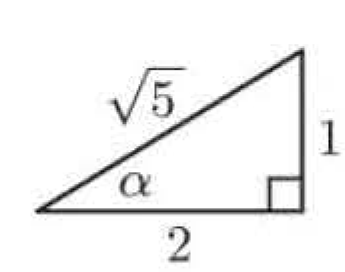
e
$$\begin{aligned} &\frac{\tan\theta + \cot\theta}{\sec\theta} \\ &= \frac{\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}}{\frac{1}{\cos\theta}} \\ &= \cos\theta\left(\frac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin\theta}\right) \\ &= \frac{\cancel{\cos\theta}}{\cancel{\cos\theta}\sin\theta} \qquad \{\text{as } \sin^2\theta + \cos^2\theta = 1\} \\ &= \frac{1}{\sin\theta} \\ &= \csc\theta \end{aligned}$$

f
$$\begin{aligned} &\cos^2\theta(\tan\theta + 1)^2 - 1 \\ &= \cos^2\theta(\tan^2\theta + 2\tan\theta + 1) - 1 \\ &= \cos^2\theta\left(\frac{\sin^2\theta}{\cos^2\theta} + \frac{2\sin\theta}{\cos\theta} + 1\right) - 1 \\ &= \sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta - 1 \\ &= 2\sin\theta\cos\theta \quad \{\text{as } \sin^2\theta + \cos^2\theta = 1\} \\ &= \sin 2\theta \end{aligned}$$

5
$$\tan 2\alpha = \tfrac{4}{3}, \quad 0 < \alpha < \tfrac{\pi}{2}$$
$$\therefore \; \frac{2\tan\alpha}{1-\tan^2\alpha} = \frac{4}{3}$$
$$\therefore \; 6\tan\alpha = 4 - 4\tan^2\alpha$$
$$\therefore \; 4\tan^2\alpha + 6\tan\alpha - 4 = 0$$
$$\therefore \; 2(2\tan^2\alpha + 3\tan\alpha - 2) = 0$$
$$\therefore \; 2(2\tan\alpha - 1)(\tan\alpha + 2) = 0$$
$$\therefore \; \tan\alpha = \tfrac{1}{2} \text{ or } -2$$

But α is in quadrant 1.

$\therefore \; \tan\alpha = \frac{1}{2}$



$\therefore \; \sin\alpha = \frac{1}{\sqrt{5}}$

6 **a**
$$\begin{aligned} &(2\sin\alpha - 1)^2 \\ &= 4\sin^2\alpha - 4\sin\alpha + 1 \end{aligned}$$

b
$$\begin{aligned} &(\cos\alpha - \sin\alpha)^2 \\ &= \cos^2\alpha - 2\sin\alpha\cos\alpha + \sin^2\alpha \\ &= \cos^2\alpha + \sin^2\alpha - 2\sin\alpha\cos\alpha \\ &= 1 - \sin 2\alpha \end{aligned}$$

7 **a** $\dfrac{\cos\theta}{1+\sin\theta}+\dfrac{1+\sin\theta}{\cos\theta}$

$=\dfrac{\cos^2\theta+(1+\sin\theta)^2}{(1+\sin\theta)\cos\theta}$

$=\dfrac{\cos^2\theta+1+2\sin\theta+\sin^2\theta}{(1+\sin\theta)\cos\theta}$

$=\dfrac{2+2\sin\theta}{(1+\sin\theta)\cos\theta}$ $\{\cos^2\theta+\sin^2\theta=1\}$

$=\dfrac{2\cancel{(1+\sin\theta)}}{\cancel{(1+\sin\theta)}\cos\theta}$

$=\dfrac{2}{\cos\theta}$

$=2\sec\theta$

b $\left(1+\dfrac{1}{\cos\theta}\right)\left(\cos\theta-\cos^2\theta\right)$

$=\cancel{\cos\theta}-\cos^2\theta+1-\cancel{\cos\theta}$

$=1-\cos^2\theta$

$=\sin^2\theta$

8 **a** $\arcsin x=\frac{\pi}{3}$

$x=\sin\frac{\pi}{3}$

$=\frac{\sqrt{3}}{2}$

b $\arctan(x-2)=\frac{\pi}{6}$

$x-2=\tan\frac{\pi}{6}$

$x-2=\frac{1}{\sqrt{3}}$

$x=2+\frac{1}{\sqrt{3}}$

9 $\cos 2\theta=1-2\sin^2\theta$

$\therefore\ \cos\left(\frac{\pi}{4}\right)=1-2\sin^2\left(\frac{\pi}{8}\right)$ {letting $\theta=\frac{\pi}{8}$}

$\therefore\ \frac{1}{\sqrt{2}}=1-2\sin^2\left(\frac{\pi}{8}\right)$

$\therefore\ 2\sin^2\left(\frac{\pi}{8}\right)=1-\frac{1}{\sqrt{2}}=\frac{\sqrt{2}-1}{\sqrt{2}}$

$\therefore\ \sin^2\left(\frac{\pi}{8}\right)=\frac{\sqrt{2}-1}{2\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}$

$\therefore\ \sin^2\left(\frac{\pi}{8}\right)=\frac{2-\sqrt{2}}{4}$

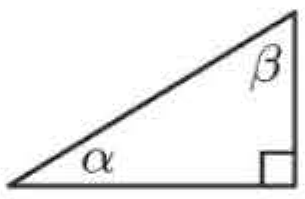
$\therefore\ \sin\left(\frac{\pi}{8}\right)=\pm\dfrac{\sqrt{2-\sqrt{2}}}{2}$

But $\sin\left(\frac{\pi}{8}\right)$ is positive as $\frac{\pi}{8}$ is in quadrant 1.

$\therefore\ \sin\left(\frac{\pi}{8}\right)=\frac{1}{2}\sqrt{2-\sqrt{2}}$

10 $\alpha+\beta=\frac{\pi}{2}$ {angles of a $\triangle$}

$\therefore\ \beta=\frac{\pi}{2}-\alpha$



So, $\sin 2\beta=\sin(\pi-2\alpha)$

$=\sin\pi\cos 2\alpha-\cos\pi\sin 2\alpha$

$=(0)\cos 2\alpha-(-1)\sin 2\alpha$

$=\sin 2\alpha$

11 **a** $(\sin\theta+\cos\theta)^2$

$=\sin^2\theta+2\sin\theta\cos\theta+\cos^2\theta$

$=\sin^2\theta+\cos^2\theta+2\sin\theta\cos\theta$

$=1+\sin 2\theta$

b $\csc(2x)+\cot(2x)$

$=\dfrac{1}{\sin 2x}+\dfrac{\cos 2x}{\sin 2x}$

$=\dfrac{1+\cos 2x}{\sin 2x}$

$=\dfrac{\cancel{1}+2\cos^2 x-\cancel{1}}{2\sin x\cos x}$

$=\dfrac{\cancel{2}\cos x\,\cancel{\cos x}}{\cancel{2}\sin x\,\cancel{\cos x}}$

$=\cot x$

12 P_n is "$\sin^2\theta+\sin^2 2\theta+\sin^2 3\theta+\ldots.+\sin^2(n\theta)=\frac{1}{2}\left[n-\dfrac{\cos[(n+1)\theta]\sin n\theta}{\sin\theta}\right]$", $n\in\mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n=1$, LHS $=\sin^2\theta$, RHS $=\frac{1}{2}\left[1-\dfrac{\cos 2\theta\,\cancel{\sin\theta}}{\cancel{\sin\theta}}\right]$

$=\frac{1}{2}-\frac{1}{2}(1-2\sin^2\theta)$

$=\sin^2\theta$ $\therefore$ P_1 is true.

(2) If P_k is true then

$$\sin^2\theta + \sin^2 2\theta + \sin^2 3\theta + ... + \sin^2(k\theta) = \tfrac{1}{2}\left[k - \frac{\cos(k+1)\theta\sin k\theta}{\sin\theta}\right]$$

$\therefore$ $\sin^2\theta + \sin^2 2\theta + \sin^2 3\theta + ... + \sin^2(k\theta) + \sin^2(k+1)\theta$

$$= \tfrac{1}{2}\left[k - \frac{\cos(k+1)\theta\sin k\theta}{\sin\theta}\right] + \sin^2(k+1)\theta$$

$$= \tfrac{1}{2}\left[k - \frac{\cos(k+1)\theta\sin k\theta}{\sin\theta}\right] + \tfrac{1}{2} - \tfrac{1}{2}\cos 2(k+1)\theta \qquad \{\sin^2\theta = \tfrac{1}{2} - \tfrac{1}{2}\cos 2\theta\}$$

$$= \tfrac{1}{2}(k+1) - \frac{\cos(k+1)\theta\sin k\theta + \cos 2(k+1)\theta\sin\theta}{2\sin\theta}$$

$$= \tfrac{1}{2}(k+1) -$$

$$\frac{\tfrac{1}{2}\sin[k\theta + (k+1)\theta] + \tfrac{1}{2}\sin[k\theta - (k+1)\theta] + \tfrac{1}{2}\sin[\theta + 2(k+1)\theta] + \tfrac{1}{2}\sin[\theta - 2(k+1)\theta]}{2\sin\theta}$$

{products to sums formula}

$$= \tfrac{1}{2}(k+1) - \frac{\tfrac{1}{2}\sin(2k+1)\theta + \tfrac{1}{2}\sin(-\theta) + \tfrac{1}{2}\sin(2k+3)\theta + \tfrac{1}{2}\sin(-2k-1)\theta}{2\sin\theta}$$

$$= \tfrac{1}{2}(k+1) - \frac{\cancel{\tfrac{1}{2}\sin(2k+1)\theta} - \tfrac{1}{2}\sin\theta + \tfrac{1}{2}\sin(2k+3)\theta - \cancel{\tfrac{1}{2}\sin(2k+1)\theta}}{2\sin\theta}$$

$$= \tfrac{1}{2}(k+1) - \frac{\tfrac{1}{2}\sin(2k+3)\theta - \tfrac{1}{2}\sin\theta}{2\sin\theta}$$

$$= \tfrac{1}{2}(k+1) - \frac{\cos\left[\frac{(2k+3)\theta+\theta}{2}\right]\sin\left[\frac{(2k+3)\theta-\theta}{2}\right]}{2\sin\theta} \qquad \{\text{factor formula}\}$$

$$= \tfrac{1}{2}(k+1) - \frac{\cos(k+2)\theta\sin(k+1)\theta}{2\sin\theta}$$

$$= \tfrac{1}{2}\left[(k+1) - \frac{\cos(k+2)\theta\sin(k+1)\theta}{\sin\theta}\right]$$

Thus P_{k+1} is true whenever P_k is true and P_1 is true.

$\therefore$ P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

13 **a** $\cos(\alpha-\beta) - \cos(\alpha+\beta)$

$= [\cancel{\cos\alpha\cos\beta} + \sin\alpha\sin\beta] - [\cancel{\cos\alpha\cos\beta} - \sin\alpha\sin\beta]$ {compound angle formulae}

$= \sin\alpha\sin\beta + \sin\alpha\sin\beta$

$= 2\sin\alpha\sin\beta$

b $2\sin\alpha\sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta)$

$\therefore$ $\sin\alpha\sin\beta = \tfrac{1}{2}(\cos(\alpha-\beta) - \cos(\alpha+\beta))$

c $\sin[(k+1)\theta]\sin\dfrac{\theta}{2}+\sin\dfrac{k\theta}{2}\sin\dfrac{(k+1)\theta}{2}$

$$= \tfrac{1}{2}\left[\cos\left((k+1)\theta-\frac{\theta}{2}\right)-\cos\left((k+1)\theta+\frac{\theta}{2}\right)\right]$$

$$+\tfrac{1}{2}\left[\cos\left(\frac{k\theta}{2}-\frac{(k+1)\theta}{2}\right)-\cos\left(\frac{k\theta}{2}+\frac{(k+1)\theta}{2}\right)\right]$$

$$= \tfrac{1}{2}\left[\cancel{\cos\left(k\theta+\frac{\theta}{2}\right)}-\cos\left((k+1)\theta+\frac{\theta}{2}\right)+\cos\left(-\frac{\theta}{2}\right)-\cancel{\cos\left(k\theta+\frac{\theta}{2}\right)}\right]$$

$$= \tfrac{1}{2}\left[\cos\left(\frac{\theta}{2}\right)-\cos\left((k+1)\theta+\frac{\theta}{2}\right)\right]$$

$$= \tfrac{1}{2}\left[-2\sin\left(\frac{\frac{\theta}{2}+(k+1)\theta+\frac{\theta}{2}}{2}\right)\sin\left(\frac{\frac{\theta}{2}-(k+1)\theta-\frac{\theta}{2}}{2}\right)\right] \quad \{\text{factor formulae}\}$$

$$= -\sin\left(\frac{(k+2)\theta}{2}\right)\sin\left(-\frac{(k+1)\theta}{2}\right)$$

$$= \sin\left(\frac{\theta(k+1)}{2}\right)\sin\left(\frac{\theta(k+2)}{2}\right)$$

d P_n is "$\sin\theta+\sin 2\theta+\sin 3\theta+ .... +\sin(n\theta)=\dfrac{\sin\left[\frac{1}{2}(n+1)\theta\right]\sin\left(\frac{1}{2}n\theta\right)}{\sin\left(\frac{1}{2}\theta\right)}$", $n\in\mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n=1$, LHS $=\sin\theta$, RHS$=\dfrac{\sin\left[\frac{1}{2}(2)\theta\right]\cancel{\sin\left(\frac{1}{2}\theta\right)}}{\cancel{\sin\left(\frac{1}{2}\theta\right)}}=\sin\theta$ $\quad\therefore$ P_1 is true.

(2) If P_k is true, then

$$\sin\theta+\sin 2\theta+\sin 3\theta+ .... +\sin(k\theta)=\frac{\sin\left[\frac{1}{2}(k+1)\theta\right]\sin\left(\frac{1}{2}k\theta\right)}{\sin\left(\frac{1}{2}\theta\right)}$$

$\therefore$ $\sin\theta+\sin 2\theta+\sin 3\theta+ .... +\sin(k\theta)+\sin(k+1)\theta$

$$=\frac{\sin\left[\frac{1}{2}(k+1)\theta\right]\sin\left(\frac{1}{2}k\theta\right)}{\sin\left(\frac{1}{2}\theta\right)}+\sin(k+1)\theta$$

$$=\frac{1}{\sin\left(\frac{1}{2}\theta\right)}\left(\sin\left[\tfrac{1}{2}(k+1)\theta\right]\sin\left(\tfrac{1}{2}k\theta\right)+\sin(k+1)\theta\sin\left(\tfrac{1}{2}\theta\right)\right)$$

$$=\frac{1}{\sin\left(\frac{1}{2}\theta\right)}\left(\sin\frac{(k+1)\theta}{2}\sin\frac{(k+2)\theta}{2}\right) \quad \{\text{using part } \mathbf{c}\}$$

Thus P_{k+1} is true whenever P_k is true and P_1 is true.

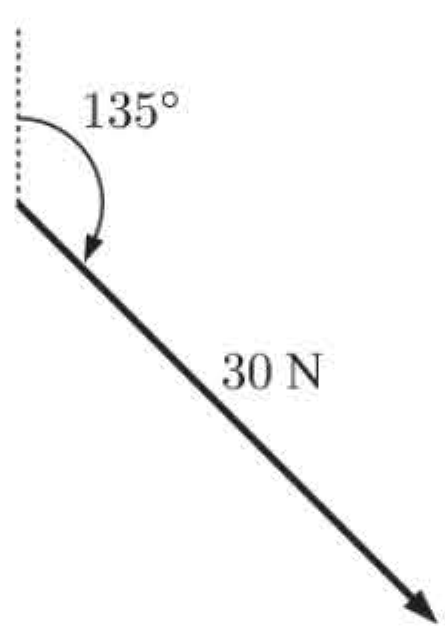
$\therefore$ P_n is true for all $n\in\mathbb{Z}^+$ {Principle of mathematical induction}

Chapter 14

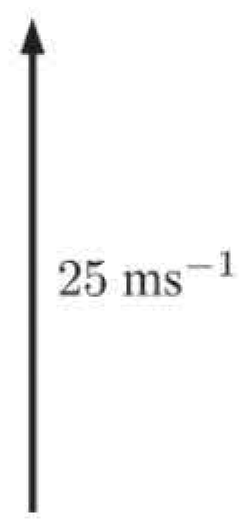
VECTORS

EXERCISE 14A.1

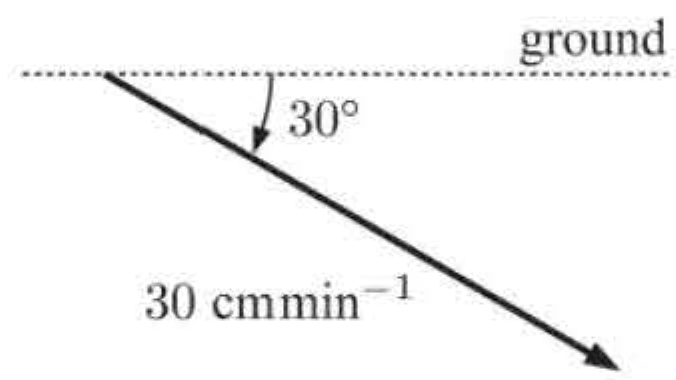
1 **a**



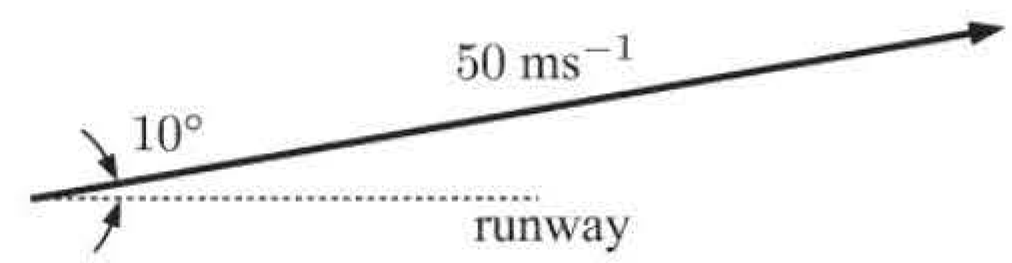
b



c

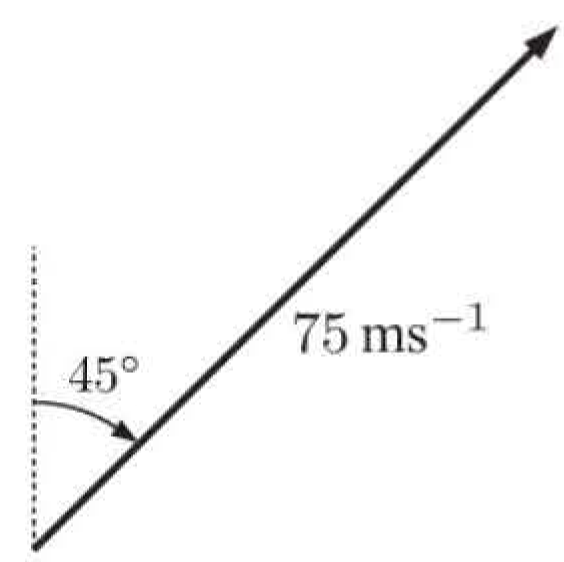


d

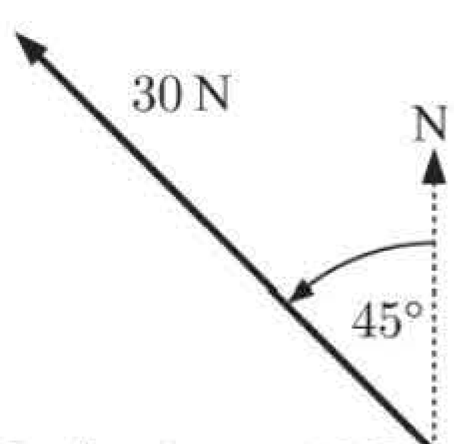


2 **a**

b

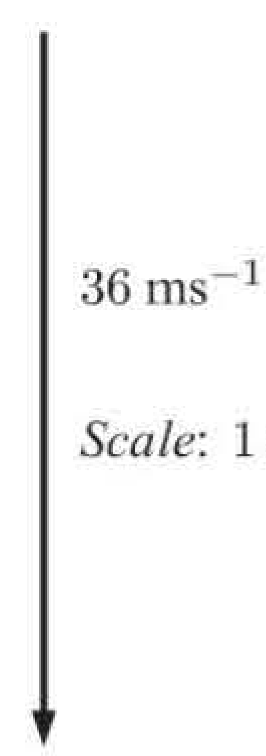


3 **a**



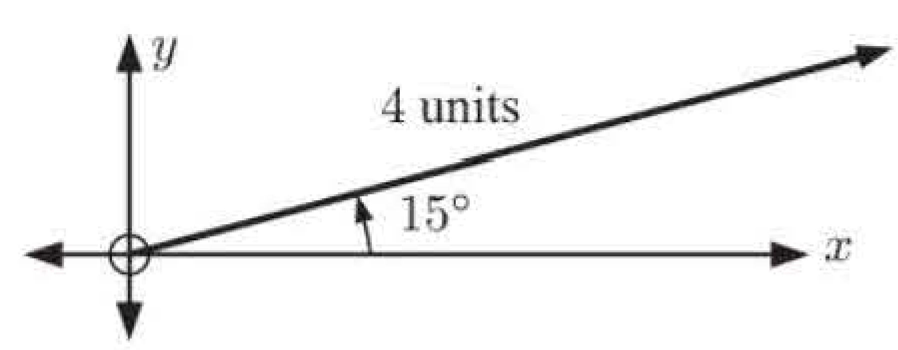
Scale: 1 cm ≡ 10 N

b



Scale: 1 cm ≡ 10 ms^{-1}

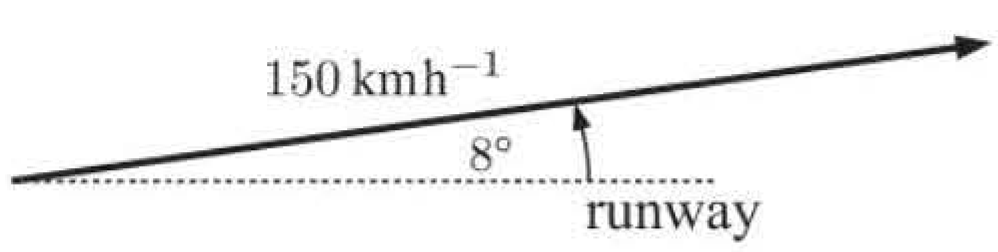
c



Scale: 1 cm ≡ 1 unit

d

Scale: 1 cm ≡ 30 kmh^{-1}



EXERCISE 14A.2

1 **a** If they are equal in magnitude, they have the same length. These are **p**, **q**, **s**, and **t**.

b Those parallel are **p**, **q**, **r**, and **t**.

c Those in the same direction are: **p** and **r**, **q** and **t**.

d To be equal they must have the same direction and be equal in length $\therefore$ $\mathbf{q} = \mathbf{t}$.

e **p** and **q** are negatives (equal length, but opposite direction). Likewise, **p** and **t** are negatives. We write $\mathbf{p} = -\mathbf{q}$ and $\mathbf{p} = -\mathbf{t}$.

2 **a** True, as they have the same length and direction.

b True, as they are sides of an equilateral triangle.

c False, as they do not have the same direction.

d False, as they have opposite directions.

e True, as they have the same length and direction.

f False, as they do not have the same direction.

3 **a** **i** $\overrightarrow{BC}$ is the vector which originates at B and terminates at C.

ii $\overrightarrow{ED} = \overrightarrow{AB}$, as they have the same length and direction.

b **i** $\overrightarrow{FE}$ and $\overrightarrow{BC}$ are negatives of $\overrightarrow{EF}$, as they both have the same length but opposite direction.

ii All sides of the hexagon are equal in length

$\therefore$ the vectors with the same length as $\overrightarrow{ED}$ are

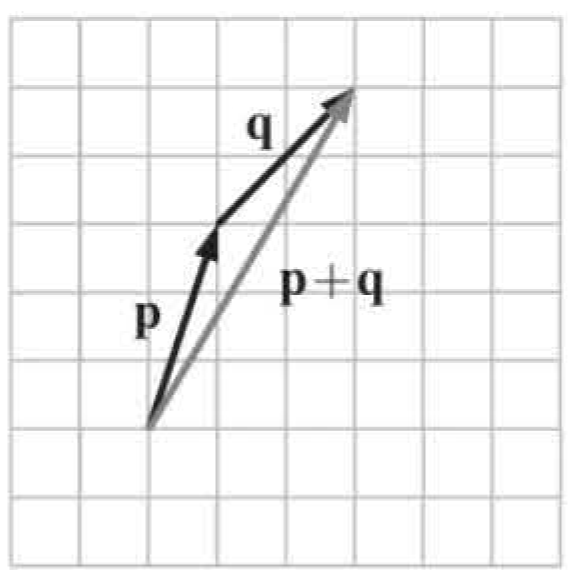
$\overrightarrow{DE}$, $\overrightarrow{EF}$, $\overrightarrow{FE}$, $\overrightarrow{FA}$, $\overrightarrow{AF}$, $\overrightarrow{AB}$, $\overrightarrow{BA}$, $\overrightarrow{BC}$, $\overrightarrow{CB}$, $\overrightarrow{CD}$, and $\overrightarrow{DC}$.

c The vector $\overrightarrow{FC}$ is parallel to $\overrightarrow{AB}$ and twice its length.

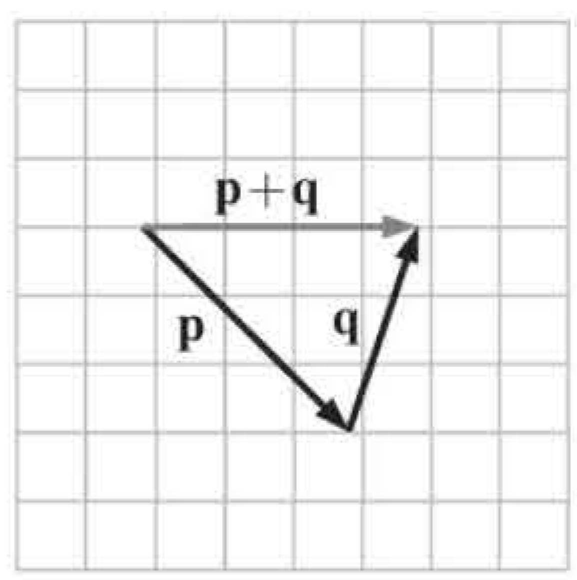
$\overrightarrow{CF}$ is also parallel to $\overrightarrow{AB}$ and twice its length (but in the opposite direction).

EXERCISE 14B.1

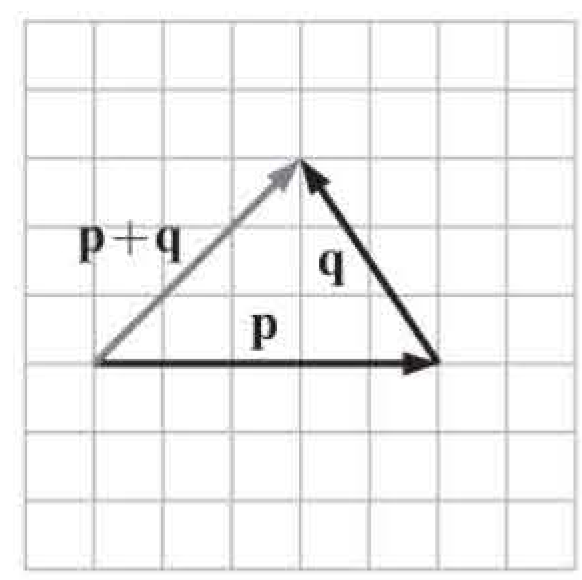
1 **a**



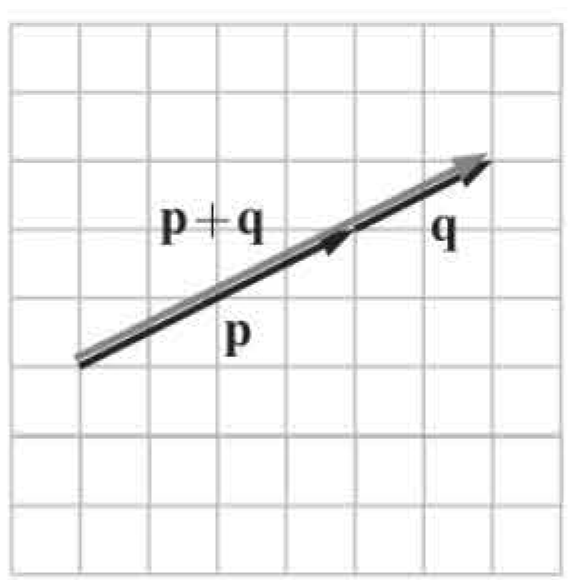
b



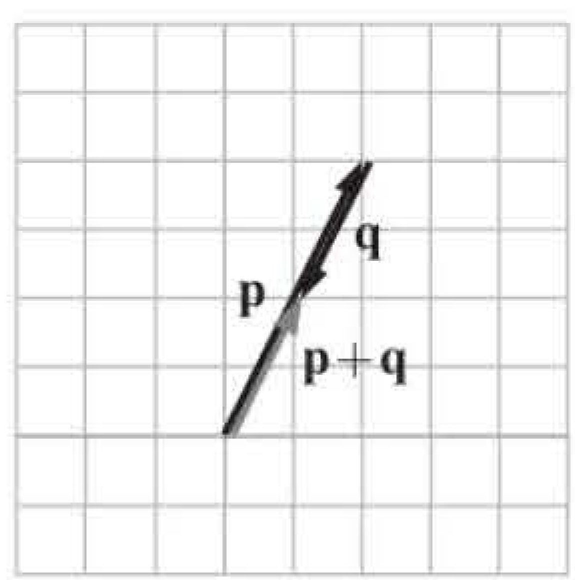
c



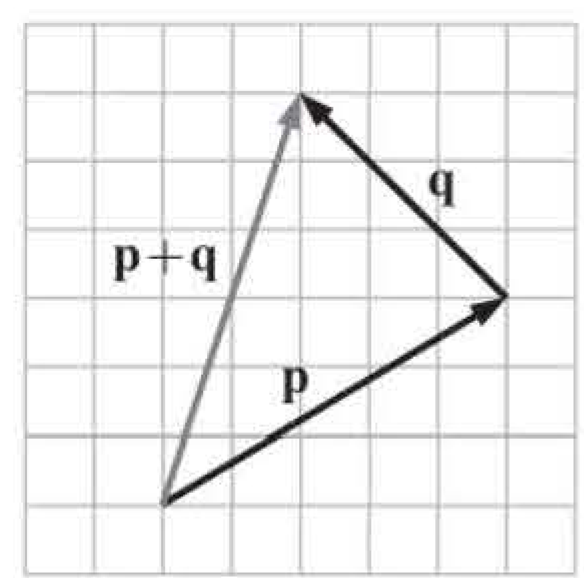
d



e



f



2 **a** $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

b $\overrightarrow{BC} + \overrightarrow{CD} = \overrightarrow{BD}$

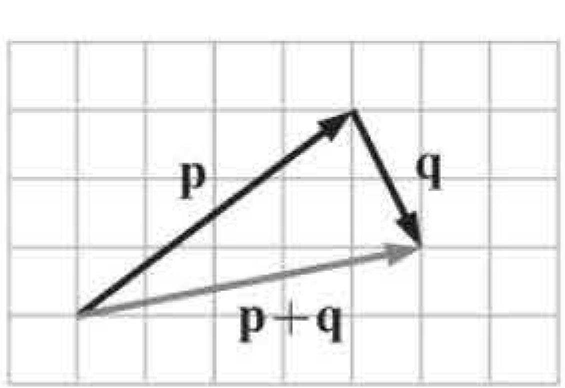
c $\overrightarrow{AB} + \overrightarrow{BA} = \overrightarrow{AA}$
$= \mathbf{0}$

d $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD}$
$= \overrightarrow{AC} + \overrightarrow{CD}$
$= \overrightarrow{AD}$

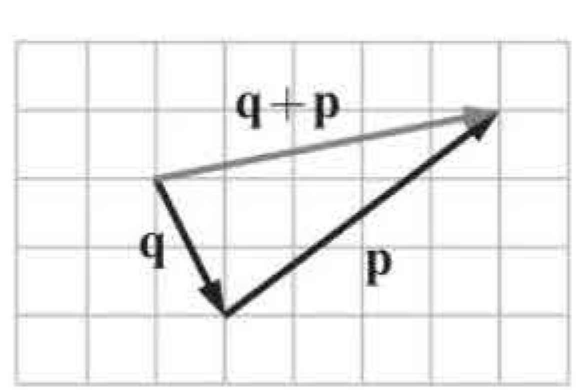
e $\overrightarrow{AC} + \overrightarrow{CB} + \overrightarrow{BD}$
$= \overrightarrow{AB} + \overrightarrow{BD}$
$= \overrightarrow{AD}$

f $\overrightarrow{BC} + \overrightarrow{CA} + \overrightarrow{AB}$
$= \overrightarrow{BA} + \overrightarrow{AB}$
$= \overrightarrow{BB}$
$= \mathbf{0}$

3 **a** **i**



ii



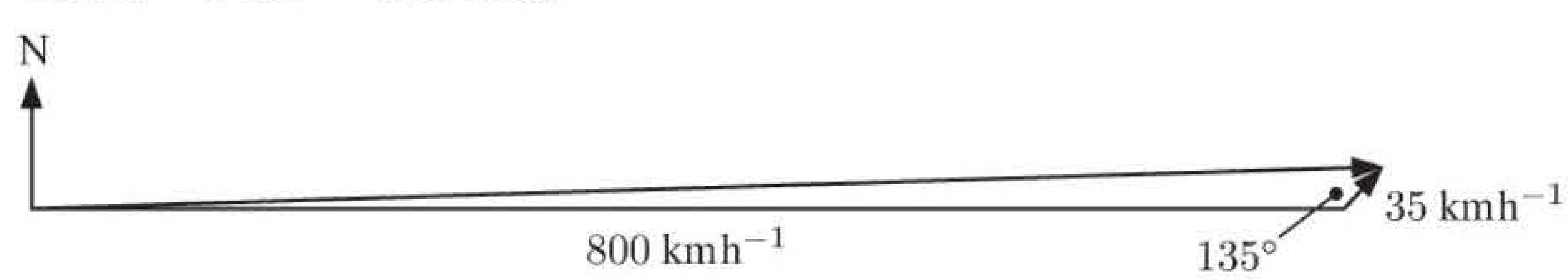
b yes

4 $\overrightarrow{PS} = \overrightarrow{PR} + \overrightarrow{RS}$
$= (\mathbf{a} + \mathbf{b}) + \mathbf{c}$

But $\overrightarrow{PS} = \overrightarrow{PQ} + \overrightarrow{QS}$
$= \mathbf{a} + (\mathbf{b} + \mathbf{c})$

$\therefore$ $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$ {as both are equal to $\overrightarrow{PS}$, associative law}

5 **a** *Scale*: $1\text{ cm} = 100\text{ km h}^{-1}$



b We need to perform vector addition to find the effect of the wind on the aeroplane.

c Measuring the length of the resulting vector, we get 82.5 mm, or 8.25 cm.

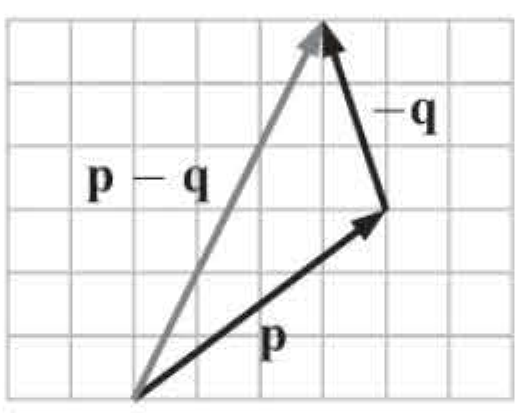
$\therefore$ the resulting speed of the plane is $8.25 \times 100 = 825 \text{ km h}^{-1}$.

Using a protractor to measure the angle between 'true north' and the resulting vector, we get $88°$.

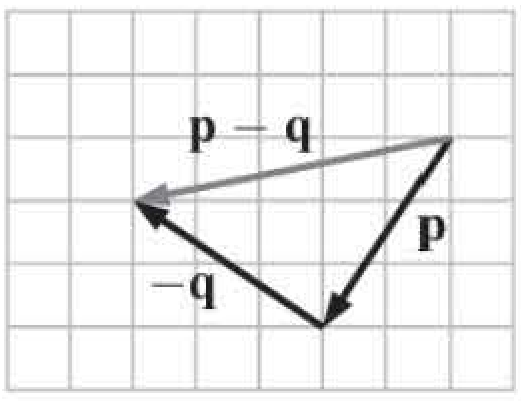
$\therefore$ the direction of the aeroplane is $88°$ east of north.

EXERCISE 14B.2

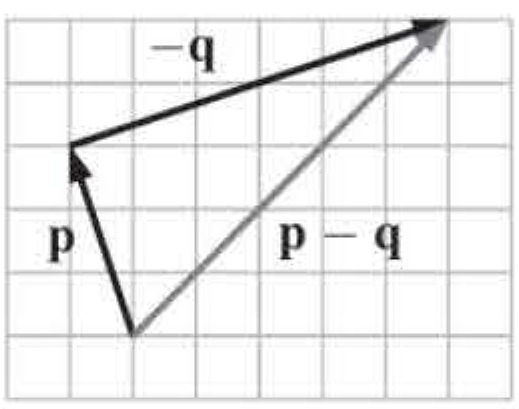
1 **a**



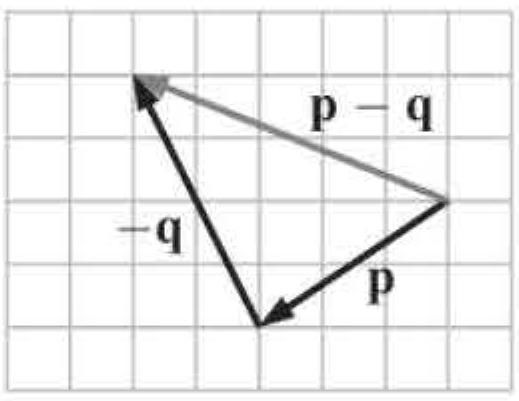
b



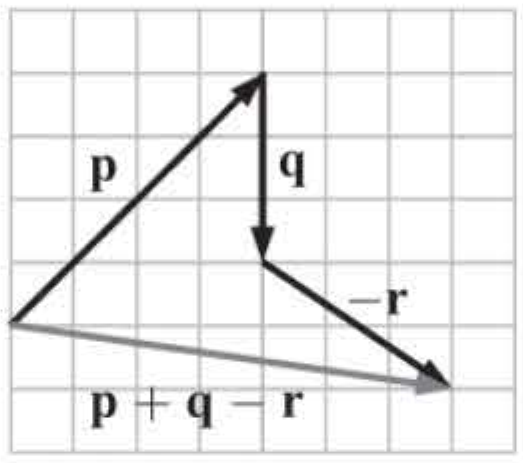
c



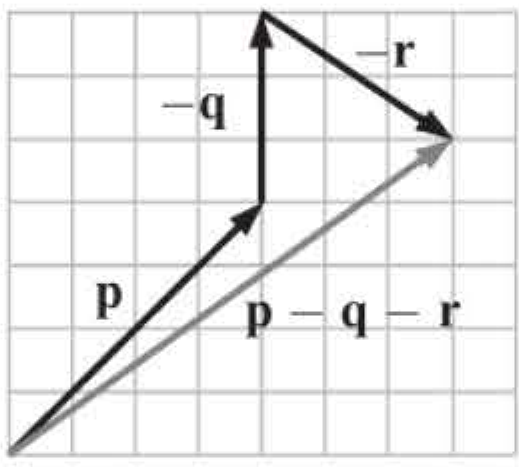
d



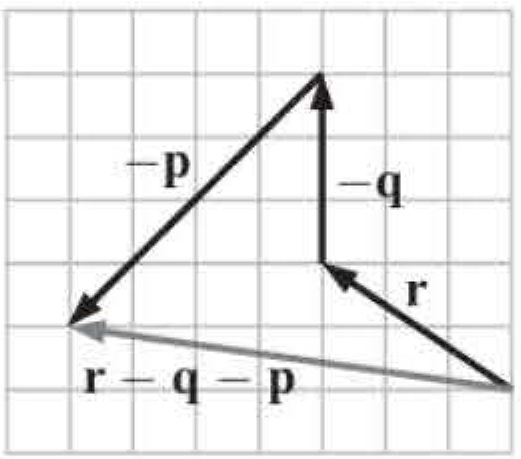
2 **a**



b



c



3 **a** $\overrightarrow{AC} + \overrightarrow{CB} = \overrightarrow{AB}$

b $\overrightarrow{AD} - \overrightarrow{BD} = \overrightarrow{AD} + \overrightarrow{DB}$
$= \overrightarrow{AB}$

c $\overrightarrow{AC} + \overrightarrow{CA} = \overrightarrow{AA}$
$= \mathbf{0}$

d $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD}$
$= \overrightarrow{AC} + \overrightarrow{CD}$
$= \overrightarrow{AD}$

e $\overrightarrow{BA} - \overrightarrow{CA} + \overrightarrow{CB}$
$= \overrightarrow{BA} + \overrightarrow{AC} + \overrightarrow{CB}$
$= \overrightarrow{BC} + \overrightarrow{CB}$
$= \overrightarrow{BB}$
$= \mathbf{0}$

f $\overrightarrow{AB} - \overrightarrow{CB} - \overrightarrow{DC}$
$= \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD}$
$= \overrightarrow{AC} + \overrightarrow{CD}$
$= \overrightarrow{AD}$

EXERCISE 14B.3

1 **a** $\mathbf{t} = \mathbf{r} + \mathbf{s}$ **b** $\mathbf{r} = -\mathbf{s} - \mathbf{t}$ **c** $\mathbf{r} = -\mathbf{p} - \mathbf{q} - \mathbf{s}$

d $\mathbf{r} = \mathbf{q} - \mathbf{p} + \mathbf{s}$ **e** $\mathbf{p} = \mathbf{t} + \mathbf{s} + \mathbf{r} - \mathbf{q}$ **f** $\mathbf{p} = -\mathbf{u} + \mathbf{t} + \mathbf{s} - \mathbf{r} - \mathbf{q}$

2 **a** **i** $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$
$= \mathbf{r} + \mathbf{s}$

ii $\overrightarrow{CA} = \overrightarrow{CB} + \overrightarrow{BA}$
$= -\overrightarrow{BC} - \overrightarrow{AB}$
$= -\mathbf{t} - \mathbf{s}$

iii $\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BC}$
$= \mathbf{r} + \mathbf{s} + \mathbf{t}$

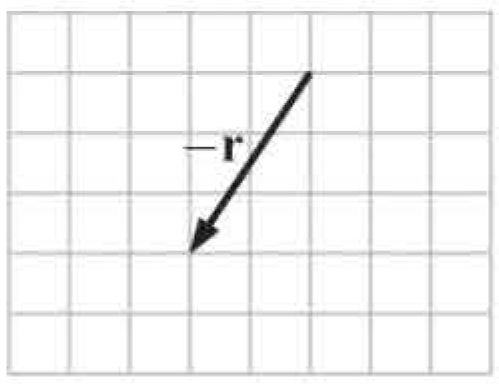
b **i** $\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BD}$
$= \mathbf{p} + \mathbf{q}$

ii $\overrightarrow{BC} = \overrightarrow{BD} + \overrightarrow{DC}$
$= \mathbf{q} + \mathbf{r}$

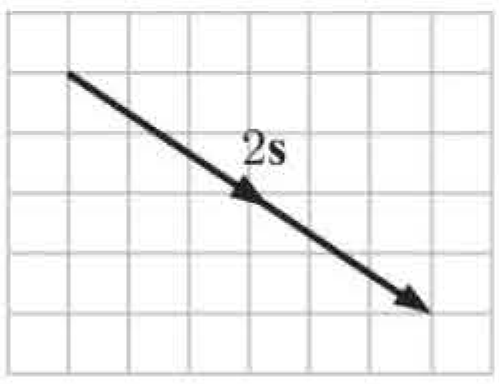
iii $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BD} + \overrightarrow{DC}$
$= \mathbf{p} + \mathbf{q} + \mathbf{r}$

EXERCISE 14B.4

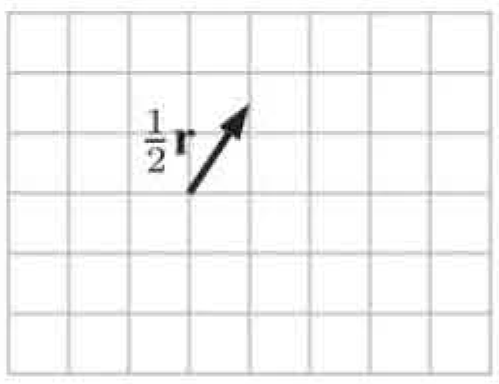
1 **a**



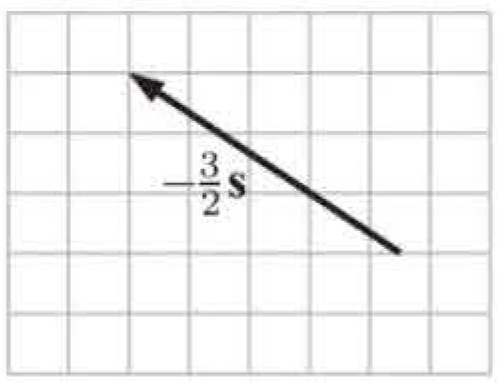
b



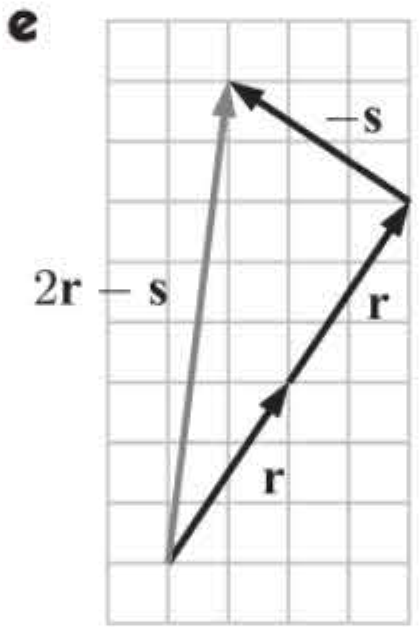
c



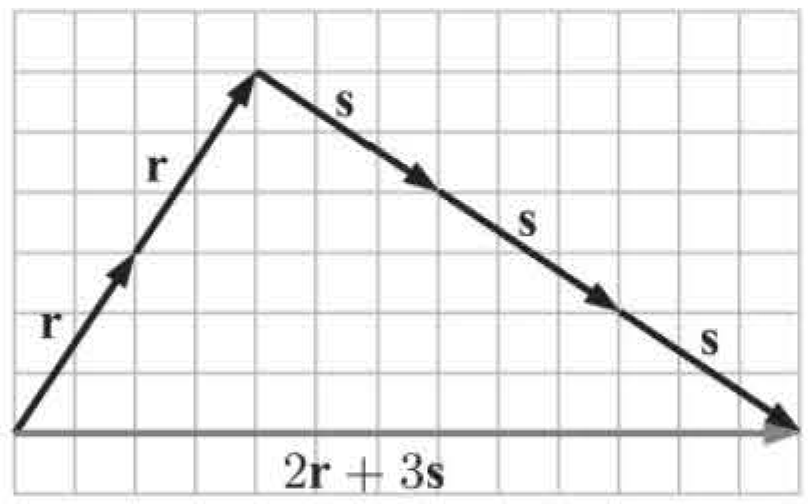
d



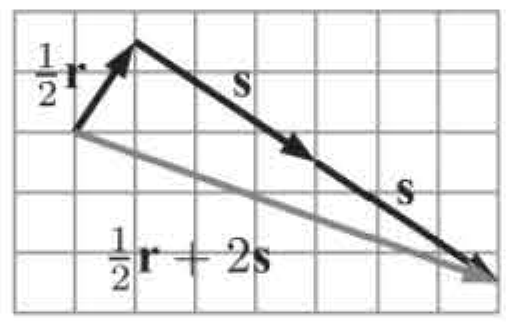
e



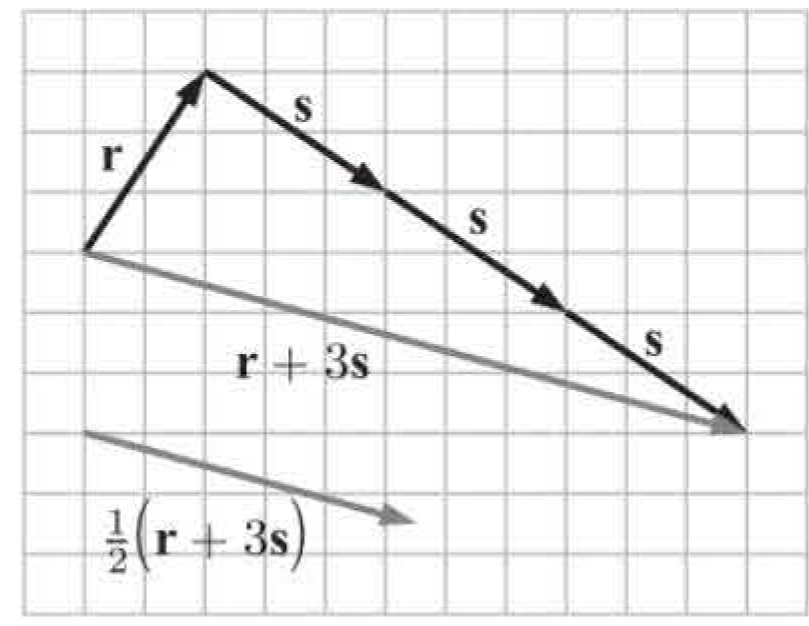
f



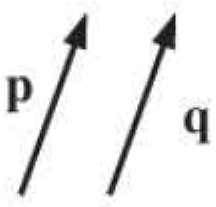
g



h

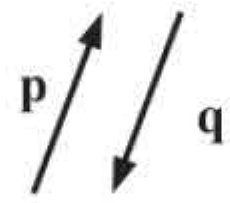


2 **a**



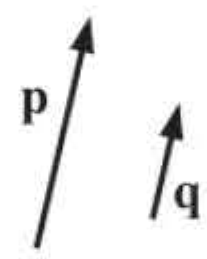
$\mathbf{p}=\mathbf{q}$

b



$\mathbf{p}=-\mathbf{q}$

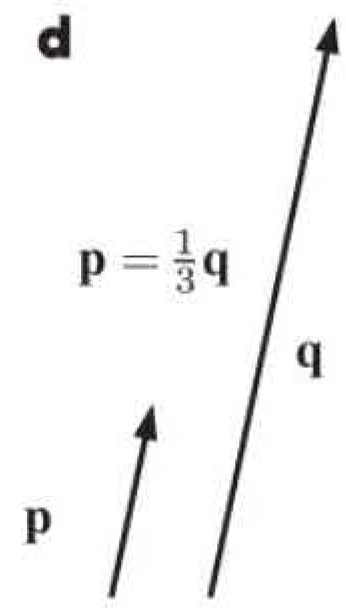
c



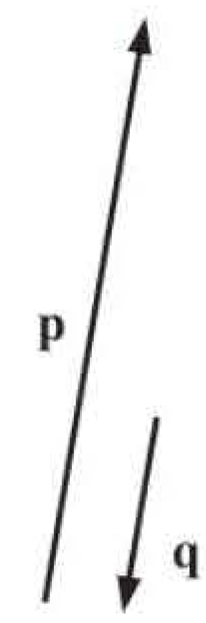
$\mathbf{p}=2\mathbf{q}$

d

$\mathbf{p}=\frac{1}{3}\mathbf{q}$

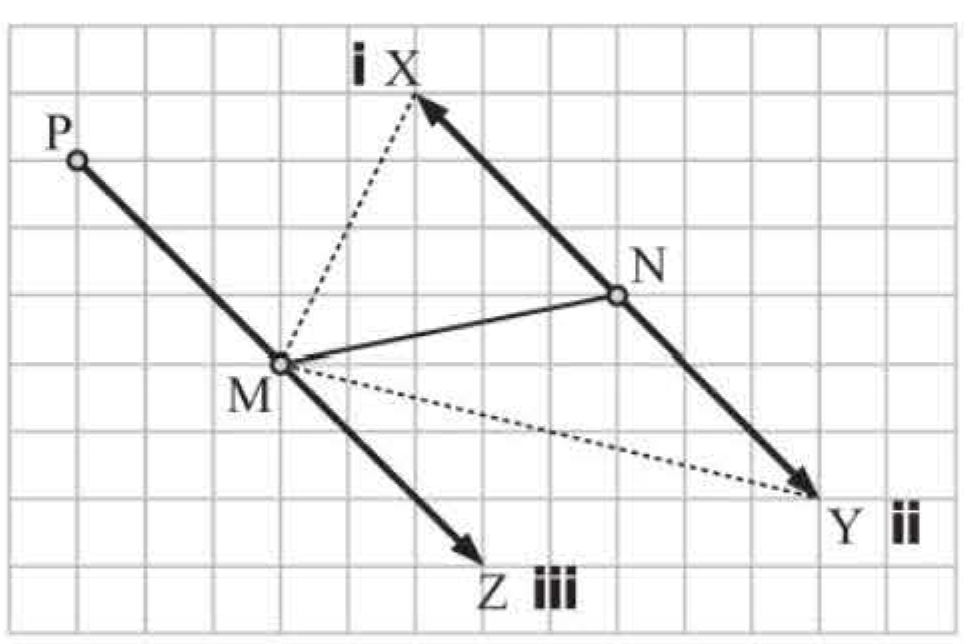


e



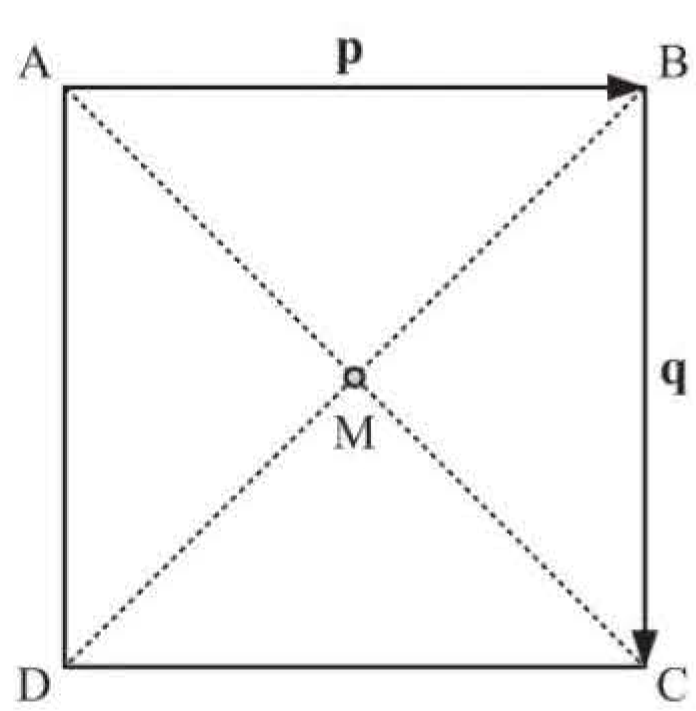
$\mathbf{p}=-3\mathbf{q}$

3 **a**



b a parallelogram

4



a $\overrightarrow{CD}=-\overrightarrow{AB}$
$=-\mathbf{p}$

b $\overrightarrow{AC}=\overrightarrow{AB}+\overrightarrow{BC}$
$=\mathbf{p}+\mathbf{q}$

c $\overrightarrow{AM}=\frac{1}{2}\overrightarrow{AC}$
$=\frac{1}{2}(\mathbf{p}+\mathbf{q})$
{using part **b**}

d $\overrightarrow{BD}=\overrightarrow{BC}+\overrightarrow{CD}$
$=\mathbf{q}+(-\mathbf{p})$ {using part **a**}
$=\mathbf{q}-\mathbf{p}$
and $\overrightarrow{BM}=\frac{1}{2}\overrightarrow{BD}$
$=\frac{1}{2}(\mathbf{q}-\mathbf{p})$

5 **a** $\overrightarrow{PX}=\overrightarrow{QR}$
$=\mathbf{b}$

b $\overrightarrow{PS}=2\overrightarrow{PX}$
$=2\mathbf{b}$
{using part **a**}

c $\overrightarrow{QX}=\overrightarrow{QR}+\overrightarrow{RX}$
$=\mathbf{b}+(-\mathbf{a})$
$=\mathbf{b}-\mathbf{a}$

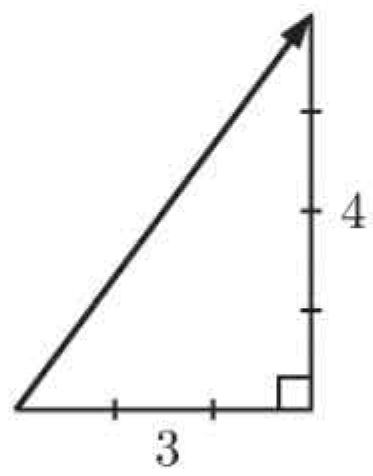
d $\overrightarrow{RS}=\overrightarrow{QX}$
$=\mathbf{b}-\mathbf{a}$
{using part **c**}

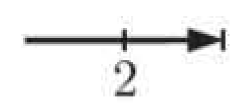
EXERCISE 14C

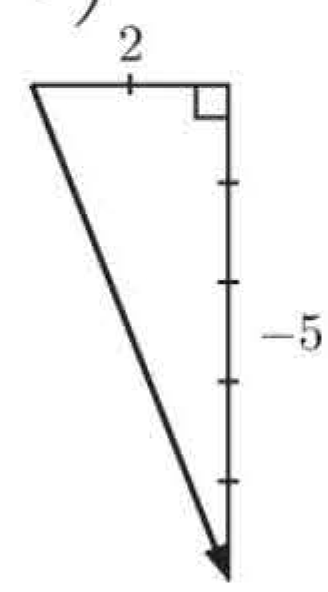
1 **a** $\begin{pmatrix}7\\3\end{pmatrix}$, $7\mathbf{i}+3\mathbf{j}$ **b** $\begin{pmatrix}-6\\0\end{pmatrix}$, $-6\mathbf{i}$ **c** $\begin{pmatrix}2\\-5\end{pmatrix}$, $2\mathbf{i}-5\mathbf{j}$

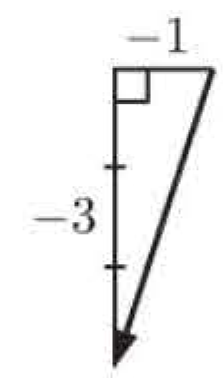
d $\begin{pmatrix}0\\6\end{pmatrix}$, $6\mathbf{j}$ **e** $\begin{pmatrix}-6\\3\end{pmatrix}$, $-6\mathbf{i}+3\mathbf{j}$ **f** $\begin{pmatrix}-5\\-5\end{pmatrix}$, $-5\mathbf{i}-5\mathbf{j}$

2 **a** $\begin{pmatrix} 3 \\ 4 \end{pmatrix} = 3\mathbf{i} + 4\mathbf{j}$ **b** $\begin{pmatrix} 2 \\ 0 \end{pmatrix} = 2\mathbf{i}$ **c** $\begin{pmatrix} 2 \\ -5 \end{pmatrix} = 2\mathbf{i} - 5\mathbf{j}$ **d** $\begin{pmatrix} -1 \\ -3 \end{pmatrix} = -\mathbf{i} - 3\mathbf{j}$





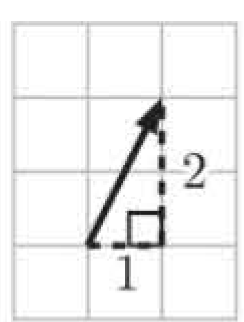


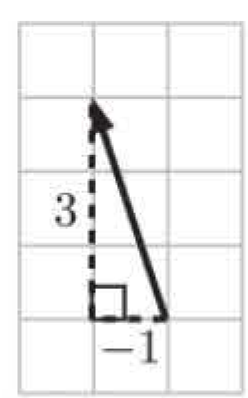


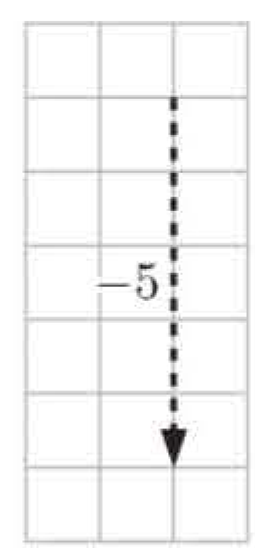
3 **a** $\overrightarrow{BA} = \begin{pmatrix} -4 \\ -1 \end{pmatrix} = -4\mathbf{i} - \mathbf{j}$ **b** $\overrightarrow{BC} = \begin{pmatrix} -1 \\ -5 \end{pmatrix} = -\mathbf{i} - 5\mathbf{j}$ **c** $\overrightarrow{DC} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} = 2\mathbf{i}$

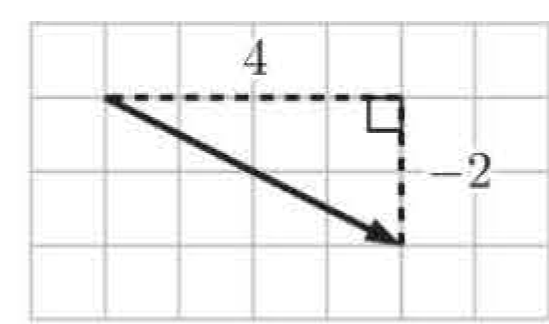
d $\overrightarrow{AC} = \begin{pmatrix} 3 \\ -4 \end{pmatrix} = 3\mathbf{i} - 4\mathbf{j}$ **e** $\overrightarrow{CA} = \begin{pmatrix} -3 \\ 4 \end{pmatrix} = -3\mathbf{i} + 4\mathbf{j}$ **f** $\overrightarrow{DB} = \begin{pmatrix} 3 \\ 5 \end{pmatrix} = 3\mathbf{i} + 5\mathbf{j}$

4 **a** $\mathbf{i} + 2\mathbf{j} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ **b** $-\mathbf{i} + 3\mathbf{j} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$ **c** $-5\mathbf{j} = \begin{pmatrix} 0 \\ -5 \end{pmatrix}$ **d** $4\mathbf{i} - 2\mathbf{j} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$









5 The zero vector in component form is $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

EXERCISE 14D

1 **a** $\left|\begin{pmatrix} 3 \\ 4 \end{pmatrix}\right| = \sqrt{3^2 + 4^2}$
$= \sqrt{9 + 16}$
$= \sqrt{25} = 5$ units

b $\left|\begin{pmatrix} -4 \\ 3 \end{pmatrix}\right| = \sqrt{(-4)^2 + 3^2}$
$= \sqrt{16 + 9}$
$= \sqrt{25} = 5$ units

c $\left|\begin{pmatrix} 2 \\ 0 \end{pmatrix}\right| = \sqrt{2^2 + 0^2}$
$= \sqrt{4}$
$= 2$ units

d $\left|\begin{pmatrix} -2 \\ 2 \end{pmatrix}\right| = \sqrt{(-2)^2 + 2^2}$
$= \sqrt{4 + 4}$
$= \sqrt{8}$ units

e $\left|\begin{pmatrix} 0 \\ -3 \end{pmatrix}\right| = \sqrt{0^2 + (-3)^2}$
$= \sqrt{9}$
$= 3$ units

2 **a** As $\mathbf{i} + \mathbf{j} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$,
$|\mathbf{i} + \mathbf{j}| = \sqrt{1^2 + 1^2}$
$= \sqrt{2}$ units

b As $5\mathbf{i} - 12\mathbf{j} = \begin{pmatrix} 5 \\ -12 \end{pmatrix}$,
$|5\mathbf{i} - 12\mathbf{j}| = \sqrt{5^2 + (-12)^2}$
$= \sqrt{25 + 144}$
$= \sqrt{169} = 13$ units

c As $-\mathbf{i} + 4\mathbf{j} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$,
$|-\mathbf{i} + 4\mathbf{j}| = \sqrt{(-1)^2 + 4^2}$
$= \sqrt{1 + 16}$
$= \sqrt{17}$ units

d As $3\mathbf{i} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$,
$|3\mathbf{i}| = \sqrt{3^2 + 0^2}$
$= \sqrt{9}$
$= 3$ units

e As $k\mathbf{j} = \begin{pmatrix} 0 \\ k \end{pmatrix}$,
$|k\mathbf{j}| = \sqrt{0^2 + k^2}$
$= \sqrt{k^2}$
$= |k|$ units

3 **a** $\sqrt{0^2+(-1)^2}=1$

$\therefore \begin{pmatrix}0\\-1\end{pmatrix}$ is a unit vector.

b $\sqrt{(-\frac{1}{\sqrt{2}})^2+(\frac{1}{\sqrt{2}})^2}=\sqrt{\frac{1}{2}+\frac{1}{2}}=1$

$\therefore \begin{pmatrix}-\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{pmatrix}$ is a unit vector.

c $\sqrt{(\frac{2}{3})^2+(\frac{1}{3})^2}=\sqrt{\frac{4}{9}+\frac{1}{9}}=\frac{\sqrt{5}}{3}$

$\therefore \begin{pmatrix}\frac{2}{3}\\\frac{1}{3}\end{pmatrix}$ is not a unit vector.

d $\sqrt{(-\frac{3}{5})^2+(-\frac{4}{5})^2}=\sqrt{\frac{9}{25}+\frac{16}{25}}=1$

$\therefore \begin{pmatrix}-\frac{3}{5}\\-\frac{4}{5}\end{pmatrix}$ is a unit vector.

e $\sqrt{(\frac{2}{7})^2+(-\frac{5}{7})^2}=\sqrt{\frac{4}{49}+\frac{25}{49}}=\frac{\sqrt{29}}{7}$

$\therefore \begin{pmatrix}\frac{2}{7}\\-\frac{5}{7}\end{pmatrix}$ is not a unit vector.

4 **a** length $=1$

$\therefore \sqrt{0^2+k^2}=1$

$\therefore k^2=1$

$\therefore k=\pm 1$

b length $=1$

$\therefore \sqrt{k^2+0}=1$

$\therefore k^2=1$

$\therefore k=\pm 1$

c length $=1$

$\therefore \sqrt{k^2+1}=1$

$\therefore k^2+1=1$

$\therefore k^2=0$

$\therefore k=0$

d length $=1$

$\therefore \sqrt{k^2+k^2}=1$

$\therefore 2k^2=1$

$\therefore k^2=\frac{1}{2}$

$\therefore k=\pm\sqrt{\frac{1}{2}}=\pm\frac{1}{\sqrt{2}}$

e length $=1$

$\therefore \sqrt{\left(\frac{1}{2}\right)^2+k^2}=1$

$\therefore \frac{1}{4}+k^2=1$

$\therefore k^2=\frac{3}{4}$

$\therefore k=\pm\sqrt{\frac{3}{4}}=\pm\frac{\sqrt{3}}{2}$

5 If $|\mathbf{v}|=\sqrt{73}$ units then $\sqrt{8^2+p^2}=\sqrt{73}$

$\therefore 64+p^2=73$

$\therefore p^2=9 \quad \therefore p=\pm 3$

EXERCISE 14E

1 **a** $\mathbf{a}+\mathbf{b}=\begin{pmatrix}-3\\2\end{pmatrix}+\begin{pmatrix}1\\4\end{pmatrix}=\begin{pmatrix}-2\\6\end{pmatrix}$

b $\mathbf{b}+\mathbf{a}=\begin{pmatrix}1\\4\end{pmatrix}+\begin{pmatrix}-3\\2\end{pmatrix}=\begin{pmatrix}-2\\6\end{pmatrix}$

c $\mathbf{b}+\mathbf{c}=\begin{pmatrix}1\\4\end{pmatrix}+\begin{pmatrix}-2\\-5\end{pmatrix}=\begin{pmatrix}-1\\-1\end{pmatrix}$

d $\mathbf{c}+\mathbf{b}=\begin{pmatrix}-2\\-5\end{pmatrix}+\begin{pmatrix}1\\4\end{pmatrix}=\begin{pmatrix}-1\\-1\end{pmatrix}$

e $\mathbf{a}+\mathbf{c}=\begin{pmatrix}-3\\2\end{pmatrix}+\begin{pmatrix}-2\\-5\end{pmatrix}=\begin{pmatrix}-5\\-3\end{pmatrix}$

f $\mathbf{c}+\mathbf{a}=\begin{pmatrix}-2\\-5\end{pmatrix}+\begin{pmatrix}-3\\2\end{pmatrix}=\begin{pmatrix}-5\\-3\end{pmatrix}$

g $\mathbf{a}+\mathbf{a}=\begin{pmatrix}-3\\2\end{pmatrix}+\begin{pmatrix}-3\\2\end{pmatrix}=\begin{pmatrix}-6\\4\end{pmatrix}$

h $\mathbf{b}+\mathbf{a}+\mathbf{c}=\begin{pmatrix}1\\4\end{pmatrix}+\begin{pmatrix}-3\\2\end{pmatrix}+\begin{pmatrix}-2\\-5\end{pmatrix}$

$=\begin{pmatrix}-2\\6\end{pmatrix}+\begin{pmatrix}-2\\-5\end{pmatrix}=\begin{pmatrix}-4\\1\end{pmatrix}$

2 **a** $\mathbf{p}-\mathbf{q}=\begin{pmatrix}-4\\2\end{pmatrix}-\begin{pmatrix}-1\\-5\end{pmatrix}=\begin{pmatrix}-3\\7\end{pmatrix}$

b $\mathbf{q}-\mathbf{r}=\begin{pmatrix}-1\\-5\end{pmatrix}-\begin{pmatrix}3\\-2\end{pmatrix}=\begin{pmatrix}-4\\-3\end{pmatrix}$

c $\mathbf{p}+\mathbf{q}-\mathbf{r}$

$=\begin{pmatrix}-4\\2\end{pmatrix}+\begin{pmatrix}-1\\-5\end{pmatrix}-\begin{pmatrix}3\\-2\end{pmatrix}=\begin{pmatrix}-8\\-1\end{pmatrix}$

d $\mathbf{p}-\mathbf{q}-\mathbf{r}$

$=\begin{pmatrix}-4\\2\end{pmatrix}-\begin{pmatrix}-1\\-5\end{pmatrix}-\begin{pmatrix}3\\-2\end{pmatrix}=\begin{pmatrix}-6\\9\end{pmatrix}$

e $\mathbf{q}-\mathbf{r}-\mathbf{p}=\begin{pmatrix}-1\\-5\end{pmatrix}-\begin{pmatrix}3\\-2\end{pmatrix}-\begin{pmatrix}-4\\2\end{pmatrix}$
$=\begin{pmatrix}0\\-5\end{pmatrix}$

f $\mathbf{r}+\mathbf{q}-\mathbf{p}=\begin{pmatrix}3\\-2\end{pmatrix}+\begin{pmatrix}-1\\-5\end{pmatrix}-\begin{pmatrix}-4\\2\end{pmatrix}$
$=\begin{pmatrix}6\\-9\end{pmatrix}$

3 **a** $\mathbf{a}+\mathbf{0}=\begin{pmatrix}a_1\\a_2\end{pmatrix}+\begin{pmatrix}0\\0\end{pmatrix}$
$=\begin{pmatrix}a_1+0\\a_2+0\end{pmatrix}=\begin{pmatrix}a_1\\a_2\end{pmatrix}=\mathbf{a}$

b $\mathbf{a}-\mathbf{a}=\begin{pmatrix}a_1\\a_2\end{pmatrix}-\begin{pmatrix}a_1\\a_2\end{pmatrix}$
$=\begin{pmatrix}a_1-a_1\\a_2-a_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}=\mathbf{0}$

4 **a** $-3\mathbf{p}$
$=-3\begin{pmatrix}1\\5\end{pmatrix}$
$=\begin{pmatrix}-3\\-15\end{pmatrix}$

b $\frac{1}{2}\mathbf{q}$
$=\frac{1}{2}\begin{pmatrix}-2\\4\end{pmatrix}$
$=\begin{pmatrix}-1\\2\end{pmatrix}$

c $2\mathbf{p}+\mathbf{q}$
$=2\begin{pmatrix}1\\5\end{pmatrix}+\begin{pmatrix}-2\\4\end{pmatrix}$
$=\begin{pmatrix}2\\10\end{pmatrix}+\begin{pmatrix}-2\\4\end{pmatrix}$
$=\begin{pmatrix}0\\14\end{pmatrix}$

d $\mathbf{p}-2\mathbf{q}$
$=\begin{pmatrix}1\\5\end{pmatrix}-2\begin{pmatrix}-2\\4\end{pmatrix}$
$=\begin{pmatrix}1\\5\end{pmatrix}-\begin{pmatrix}-4\\8\end{pmatrix}$
$=\begin{pmatrix}5\\-3\end{pmatrix}$

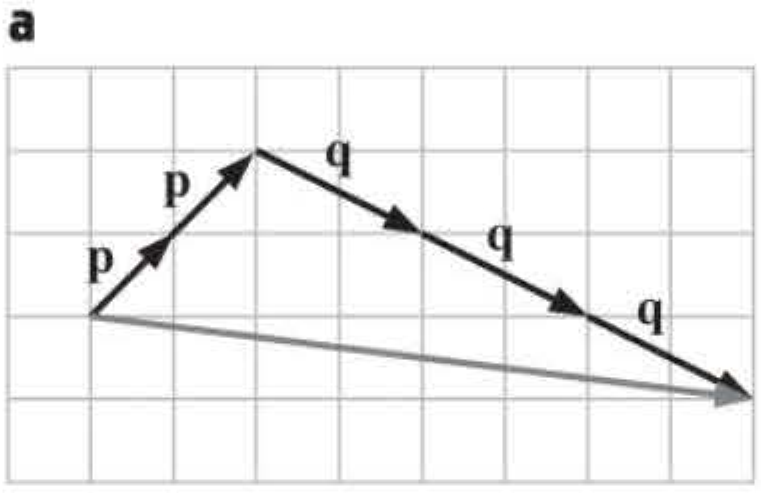
e $\mathbf{p}-\frac{1}{2}\mathbf{r}=\begin{pmatrix}1\\5\end{pmatrix}-\frac{1}{2}\begin{pmatrix}-3\\-1\end{pmatrix}$
$=\begin{pmatrix}1\\5\end{pmatrix}-\begin{pmatrix}-\frac{3}{2}\\-\frac{1}{2}\end{pmatrix}$
$=\begin{pmatrix}\frac{5}{2}\\\frac{11}{2}\end{pmatrix}$

f $2\mathbf{p}+3\mathbf{r}=2\begin{pmatrix}1\\5\end{pmatrix}+3\begin{pmatrix}-3\\-1\end{pmatrix}$
$=\begin{pmatrix}2\\10\end{pmatrix}+\begin{pmatrix}-9\\-3\end{pmatrix}$
$=\begin{pmatrix}-7\\7\end{pmatrix}$

g $2\mathbf{q}-3\mathbf{r}=2\begin{pmatrix}-2\\4\end{pmatrix}-3\begin{pmatrix}-3\\-1\end{pmatrix}$
$=\begin{pmatrix}-4\\8\end{pmatrix}-\begin{pmatrix}-9\\-3\end{pmatrix}$
$=\begin{pmatrix}5\\11\end{pmatrix}$

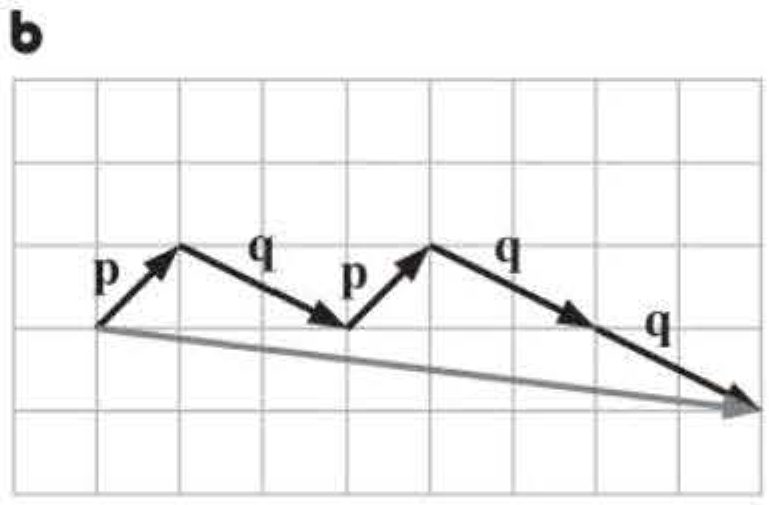
h $2\mathbf{p}-\mathbf{q}+\frac{1}{3}\mathbf{r}$
$=2\begin{pmatrix}1\\5\end{pmatrix}-\begin{pmatrix}-2\\4\end{pmatrix}+\frac{1}{3}\begin{pmatrix}-3\\-1\end{pmatrix}$
$=\begin{pmatrix}2\\10\end{pmatrix}-\begin{pmatrix}-2\\4\end{pmatrix}+\begin{pmatrix}-1\\-\frac{1}{3}\end{pmatrix}$
$=\begin{pmatrix}3\\\frac{17}{3}\end{pmatrix}$

5 **a**



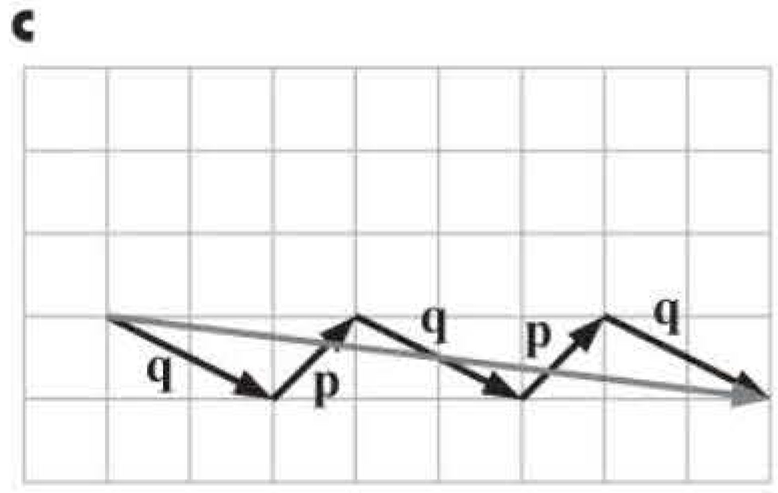
$=\begin{pmatrix}8\\-1\end{pmatrix}$

b



$=\begin{pmatrix}8\\-1\end{pmatrix}$

c



$=\begin{pmatrix}8\\-1\end{pmatrix}$

The vector expressions are equal, as each consists of 2 **p**s and 3 **q**s. Each expression is equal to $2\mathbf{p}+3\mathbf{q}$.

6 **a** $|\,\mathbf{r}\,|=\sqrt{2^2+3^2}$
$=\sqrt{13}$ units

b $|\,\mathbf{s}\,|=\sqrt{(-1)^2+4^2}$
$=\sqrt{17}$ units

c
$$\begin{aligned}&\mathbf{r}+\mathbf{s}\\&=\begin{pmatrix}2\\3\end{pmatrix}+\begin{pmatrix}-1\\4\end{pmatrix}\\&=\begin{pmatrix}1\\7\end{pmatrix}\\\therefore\quad&|\,\mathbf{r}+\mathbf{s}\,|\\&=\sqrt{1^2+7^2}\\&=\sqrt{50}\text{ units}\\&=5\sqrt{2}\text{ units}\end{aligned}$$

d
$$\begin{aligned}&\mathbf{r}-\mathbf{s}\\&=\begin{pmatrix}2\\3\end{pmatrix}-\begin{pmatrix}-1\\4\end{pmatrix}\\&=\begin{pmatrix}3\\-1\end{pmatrix}\\\therefore\quad&|\,\mathbf{r}-\mathbf{s}\,|\\&=\sqrt{3^2+(-1)^2}\\&=\sqrt{10}\text{ units}\end{aligned}$$

e
$$\begin{aligned}&\mathbf{s}-2\mathbf{r}\\&=\begin{pmatrix}-1\\4\end{pmatrix}-\begin{pmatrix}4\\6\end{pmatrix}\\&=\begin{pmatrix}-5\\-2\end{pmatrix}\\\therefore\quad&|\,\mathbf{s}-2\mathbf{r}\,|\\&=\sqrt{(-5)^2+(-2)^2}\\&=\sqrt{29}\text{ units}\end{aligned}$$

7 **a**
$$\begin{aligned}|\,\mathbf{p}\,|&=\sqrt{1^2+3^2}\\&=\sqrt{10}\text{ units}\end{aligned}$$

b
$$\begin{aligned}2\mathbf{p}&=\begin{pmatrix}2\\6\end{pmatrix}\\\therefore\quad|\,2\mathbf{p}\,|&=\sqrt{2^2+6^2}\\&=\sqrt{4+36}\\&=\sqrt{40}\\&=2\sqrt{10}\text{ units}\end{aligned}$$

c
$$\begin{aligned}-2\mathbf{p}&=\begin{pmatrix}-2\\-6\end{pmatrix}\\\therefore\quad|-2\mathbf{p}\,|&=\sqrt{(-2)^2+(-6)^2}\\&=\sqrt{4+36}\\&=\sqrt{40}\\&=2\sqrt{10}\text{ units}\end{aligned}$$

d
$$\begin{aligned}3\mathbf{p}&=\begin{pmatrix}3\\9\end{pmatrix}\\\therefore\quad|\,3\mathbf{p}\,|&=\sqrt{3^2+9^2}\\&=\sqrt{9+81}\\&=\sqrt{90}\\&=3\sqrt{10}\text{ units}\end{aligned}$$

e
$$\begin{aligned}-3\mathbf{p}&=\begin{pmatrix}-3\\-9\end{pmatrix}\\\therefore\quad|-3\mathbf{p}\,|&=\sqrt{(-3)^2+(-9)^2}\\&=\sqrt{9+81}\\&=\sqrt{90}\\&=3\sqrt{10}\text{ units}\end{aligned}$$

f
$$\begin{aligned}|\,\mathbf{q}\,|&=\sqrt{(-2)^2+4^2}\\&=\sqrt{4+16}\\&=\sqrt{20}\\&=2\sqrt{5}\text{ units}\end{aligned}$$

g
$$\begin{aligned}4\mathbf{q}&=\begin{pmatrix}-8\\16\end{pmatrix}\\\therefore\quad|\,4\mathbf{q}\,|&=\sqrt{(-8)^2+16^2}\\&=\sqrt{64+256}\\&=\sqrt{320}\\&=8\sqrt{5}\text{ units}\end{aligned}$$

h
$$\begin{aligned}-4\mathbf{q}&=\begin{pmatrix}8\\-16\end{pmatrix}\\\therefore\quad|-4\mathbf{q}\,|&=\sqrt{8^2+(-16)^2}\\&=\sqrt{64+256}\\&=\sqrt{320}\\&=8\sqrt{5}\text{ units}\end{aligned}$$

i
$$\begin{aligned}\tfrac{1}{2}\mathbf{q}&=\begin{pmatrix}-1\\2\end{pmatrix}\\\therefore\quad\left|\,\tfrac{1}{2}\mathbf{q}\,\right|&=\sqrt{(-1)^2+2^2}\\&=\sqrt{5}\text{ units}\end{aligned}$$

j
$$\begin{aligned}-\tfrac{1}{2}\mathbf{q}&=\begin{pmatrix}1\\-2\end{pmatrix}\\\therefore\quad\left|-\tfrac{1}{2}\mathbf{q}\,\right|&=\sqrt{1^2+(-2)^2}\\&=\sqrt{5}\text{ units}\end{aligned}$$

8
$$\begin{aligned}k\mathbf{x}&=\mathbf{a}\\\therefore\quad k\begin{pmatrix}x_1\\x_2\end{pmatrix}&=\begin{pmatrix}a_1\\a_2\end{pmatrix}\\\therefore\quad kx_1=a_1\quad&\text{and}\quad kx_2=a_2\\\therefore\quad x_1=\tfrac{1}{k}a_1\quad&\text{and}\quad x_2=\tfrac{1}{k}a_2\\\therefore\quad\begin{pmatrix}x_1\\x_2\end{pmatrix}&=\begin{pmatrix}\frac{1}{k}a_1\\\frac{1}{k}a_2\end{pmatrix}=\tfrac{1}{k}\begin{pmatrix}a_1\\a_2\end{pmatrix}\end{aligned}$$

and so $\mathbf{x}=\frac{1}{k}\mathbf{a}$

9
$$\begin{aligned}k\mathbf{v}&=\begin{pmatrix}kv_1\\kv_2\end{pmatrix}\\\therefore\quad|\,k\mathbf{v}\,|&=\sqrt{(kv_1)^2+(kv_2)^2}\\&=\sqrt{k^2{v_1}^2+k^2{v_2}^2}\\&=\sqrt{k^2({v_1}^2+{v_2}^2)}\\&=\sqrt{k^2}\sqrt{{v_1}^2+{v_2}^2}\\&=|k|\,\sqrt{{v_1}^2+{v_2}^2}\\&=|k|\,|\,\mathbf{v}\,|\end{aligned}$$

EXERCISE 14F

1 **a** $\overrightarrow{AB} = \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \end{pmatrix} = \begin{pmatrix} 4-2 \\ 7-3 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

b $\overrightarrow{AB} = \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \end{pmatrix} = \begin{pmatrix} 1-3 \\ 4--1 \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$

c $\overrightarrow{AB} = \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \end{pmatrix} = \begin{pmatrix} 1--2 \\ 4-7 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \end{pmatrix}$

d $\overrightarrow{AB} = \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \end{pmatrix} = \begin{pmatrix} 3-2 \\ 0-5 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \end{pmatrix}$

e $\overrightarrow{AB} = \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \end{pmatrix} = \begin{pmatrix} 6-0 \\ -1-4 \end{pmatrix} = \begin{pmatrix} 6 \\ -5 \end{pmatrix}$

f $\overrightarrow{AB} = \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \end{pmatrix} = \begin{pmatrix} 0--1 \\ 0--3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

2 **a** Let B have coordinates (b_1, b_2).

$\therefore \overrightarrow{AB} = \begin{pmatrix} b_1 - 1 \\ b_2 - 4 \end{pmatrix}$

$\therefore \begin{pmatrix} b_1 - 1 \\ b_2 - 4 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

$\therefore b_1 - 1 = 3$ and $b_2 - 4 = -2$

$\therefore b_1 = 4$ and $b_2 = 2$

$\therefore$ B has coordinates $(4, 2)$.

b Let C have coordinates (c_1, c_2).

$\therefore \overrightarrow{CA} = \begin{pmatrix} 1 - c_1 \\ 4 - c_2 \end{pmatrix}$

$\therefore \begin{pmatrix} 1 - c_1 \\ 4 - c_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

$\therefore 1 - c_1 = -1$ and $4 - c_2 = 2$

$\therefore c_1 = 2$ and $c_2 = 2$

$\therefore$ C has coordinates $(2, 2)$.

3 **a** $\overrightarrow{PC} = \begin{pmatrix} 1-(-1) \\ 2-1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

b Let Q have coordinates (q_1, q_2).

$\therefore \overrightarrow{CQ} = \begin{pmatrix} q_1 - 1 \\ q_2 - 2 \end{pmatrix}$

But $\overrightarrow{CQ} = \overrightarrow{PC}$

$\therefore \begin{pmatrix} q_1 - 1 \\ q_2 - 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\therefore q_1 - 1 = 2$ and $q_2 - 2 = 1$

$\therefore q_1 = 3$ and $q_2 = 3$

$\therefore$ Q has coordinates $(3, 3)$.

4 **a** $\overrightarrow{AB} = \begin{pmatrix} 6-1 \\ 5-4 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$

b $\overrightarrow{CD} = -\overrightarrow{AB} = -\begin{pmatrix} 5 \\ 1 \end{pmatrix} = \begin{pmatrix} -5 \\ -1 \end{pmatrix}$

c Let D have coordinates (d_1, d_2).

$\therefore \overrightarrow{CD} = \begin{pmatrix} d_1 - 4 \\ d_2 - (-1) \end{pmatrix} = \begin{pmatrix} d_1 - 4 \\ d_2 + 1 \end{pmatrix}$

But $\overrightarrow{CD} = \begin{pmatrix} -5 \\ -1 \end{pmatrix}$

$\therefore \begin{pmatrix} d_1 - 4 \\ d_2 + 1 \end{pmatrix} = \begin{pmatrix} -5 \\ -1 \end{pmatrix}$

$\therefore d_1 - 4 = -5$ and $d_2 + 1 = -1$

$\therefore d_1 = -1$ and $d_2 = -2$

$\therefore$ D has coordinates $(-1, -2)$.

5 **a** $\overrightarrow{AB} = \begin{pmatrix} 3-(-1) \\ k-3 \end{pmatrix} = \begin{pmatrix} 4 \\ k-3 \end{pmatrix}$

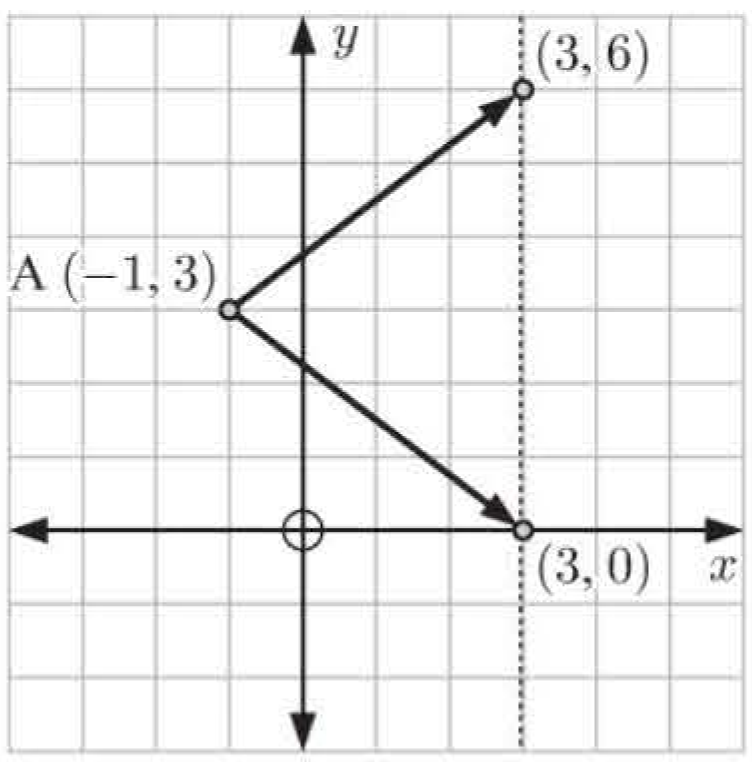
Since A and B are 5 units apart,

$|\overrightarrow{AB}| = 5$ units and $|\overrightarrow{AB}| = \sqrt{4^2 + (k-3)^2}$

$= \sqrt{16 + (k-3)^2}$

b
$$
\begin{aligned}
|\overrightarrow{AB}| &= 5\\
\therefore\ \sqrt{16+(k-3)^2} &= 5\\
\therefore\ 16+k^2-6k+9 &= 25\\
\therefore\ k^2-6k &= 0\\
\therefore\ k(k-6) &= 0\\
\therefore\ k=0 \quad &\text{or} \quad k-6=0\\
\therefore\ k=0 \quad &\text{or} \quad 6
\end{aligned}
$$

c



6 **a** $\overrightarrow{AB} = \begin{pmatrix} 3-1 \\ 5-2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$\overrightarrow{AC} = \begin{pmatrix} 4-1 \\ -1-2 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \end{pmatrix}$

b
$$
\begin{aligned}
&\overrightarrow{BC}\\
&= \overrightarrow{BA} + \overrightarrow{AC}\\
&= -\overrightarrow{AB} + \overrightarrow{AC}
\end{aligned}
$$

c
$$
\begin{aligned}
&\overrightarrow{BC}\\
&= -\overrightarrow{AB} + \overrightarrow{AC}\\
&= -\begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 3 \\ -3 \end{pmatrix}\\
&= \begin{pmatrix} -2 \\ -3 \end{pmatrix} + \begin{pmatrix} 3 \\ -3 \end{pmatrix}\\
&= \begin{pmatrix} -2+3 \\ -3-3 \end{pmatrix}\\
&= \begin{pmatrix} 1 \\ -6 \end{pmatrix}
\end{aligned}
$$

d $\overrightarrow{BC} = \begin{pmatrix} 4-3 \\ -1-5 \end{pmatrix} = \begin{pmatrix} 1 \\ -6 \end{pmatrix}$ ✓

7 **a**
$$
\begin{aligned}
&\overrightarrow{AC}\\
&= \overrightarrow{AB} + \overrightarrow{BC}\\
&= -\overrightarrow{BA} + \overrightarrow{BC}\\
&= -\begin{pmatrix} 2 \\ -3 \end{pmatrix} + \begin{pmatrix} -3 \\ 1 \end{pmatrix}\\
&= \begin{pmatrix} -5 \\ 4 \end{pmatrix}
\end{aligned}
$$

b
$$
\begin{aligned}
&\overrightarrow{CB}\\
&= \overrightarrow{CA} + \overrightarrow{AB}\\
&= \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \begin{pmatrix} -1 \\ 3 \end{pmatrix}\\
&= \begin{pmatrix} 1 \\ 2 \end{pmatrix}
\end{aligned}
$$

c
$$
\begin{aligned}
&\overrightarrow{SP}\\
&= \overrightarrow{SR} + \overrightarrow{RQ} + \overrightarrow{QP}\\
&= -\overrightarrow{RS} + \overrightarrow{RQ} - \overrightarrow{PQ}\\
&= -\begin{pmatrix} -3 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ 4 \end{pmatrix}\\
&= \begin{pmatrix} 6 \\ -5 \end{pmatrix}
\end{aligned}
$$

8 **a** M is $\left(\dfrac{3+-1}{2}, \dfrac{6+2}{2}\right)$

$\therefore$ M is $(1, 4)$

b $\overrightarrow{CA} = \begin{pmatrix} 3--4 \\ 6-1 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$

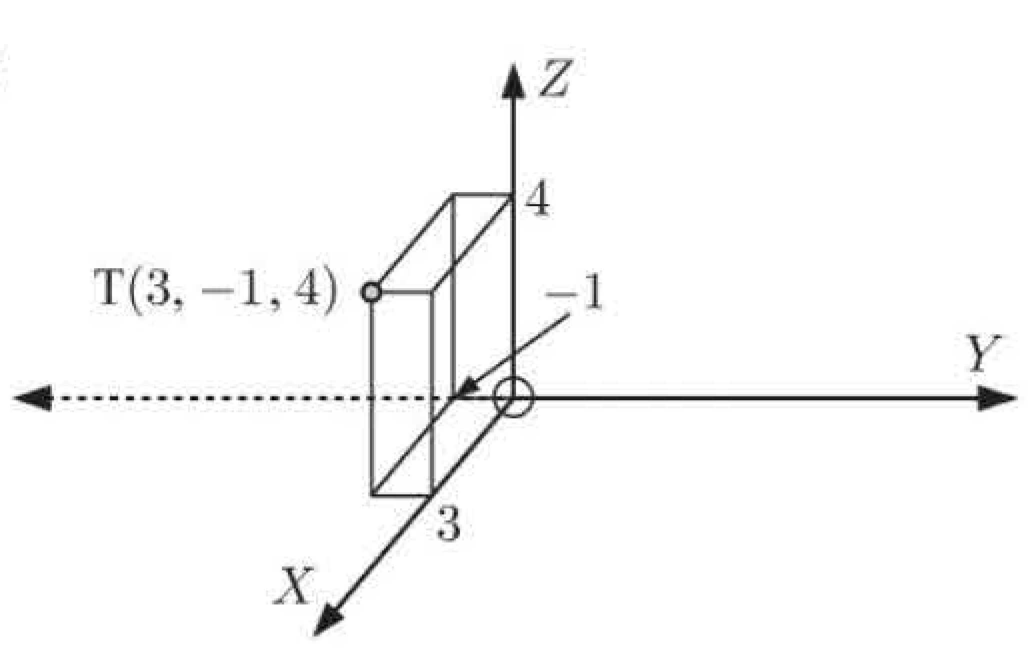
$\overrightarrow{CM} = \begin{pmatrix} 1--4 \\ 4-1 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$

$\overrightarrow{CB} = \begin{pmatrix} -1--4 \\ 2-1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

c
$$
\begin{aligned}
&\tfrac{1}{2}\overrightarrow{CA} + \tfrac{1}{2}\overrightarrow{CB}\\
&= \tfrac{1}{2}\begin{pmatrix} 7 \\ 5 \end{pmatrix} + \tfrac{1}{2}\begin{pmatrix} 3 \\ 1 \end{pmatrix}\\
&= \begin{pmatrix} 5 \\ 3 \end{pmatrix} \text{ which is } \overrightarrow{CM}
\end{aligned}
$$

EXERCISE 14G

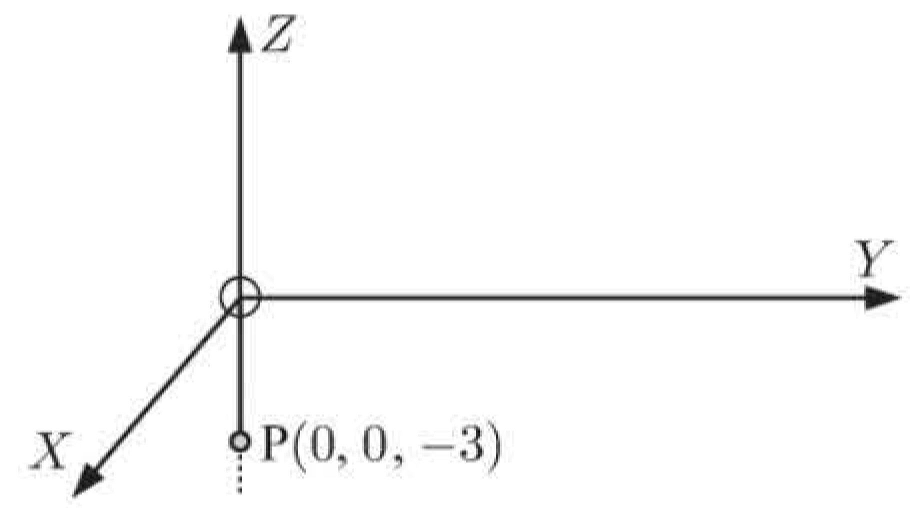
1 **a**



b $\overrightarrow{OT} = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$

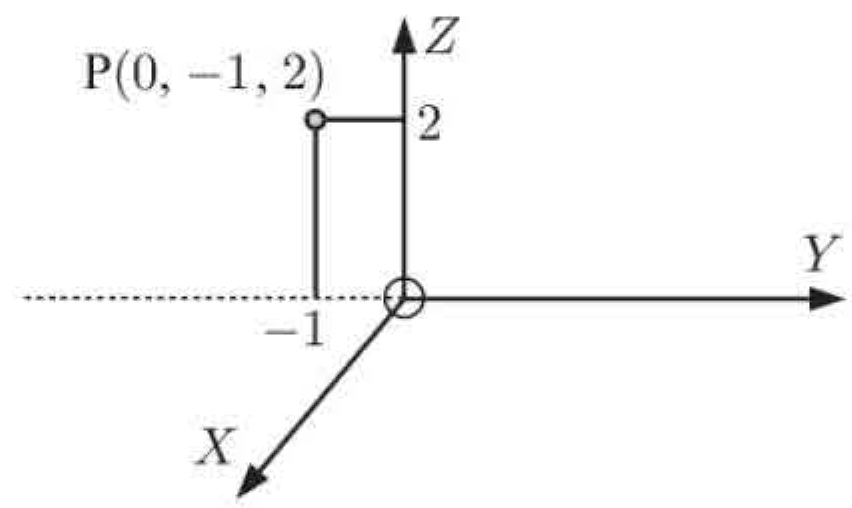
c
$$
\begin{aligned}
OT &= \sqrt{(3-0)^2+(-1-0)^2+(4-0)^2}\\
&= \sqrt{9+1+16}\\
&= \sqrt{26} \text{ units}
\end{aligned}
$$

2 **a**



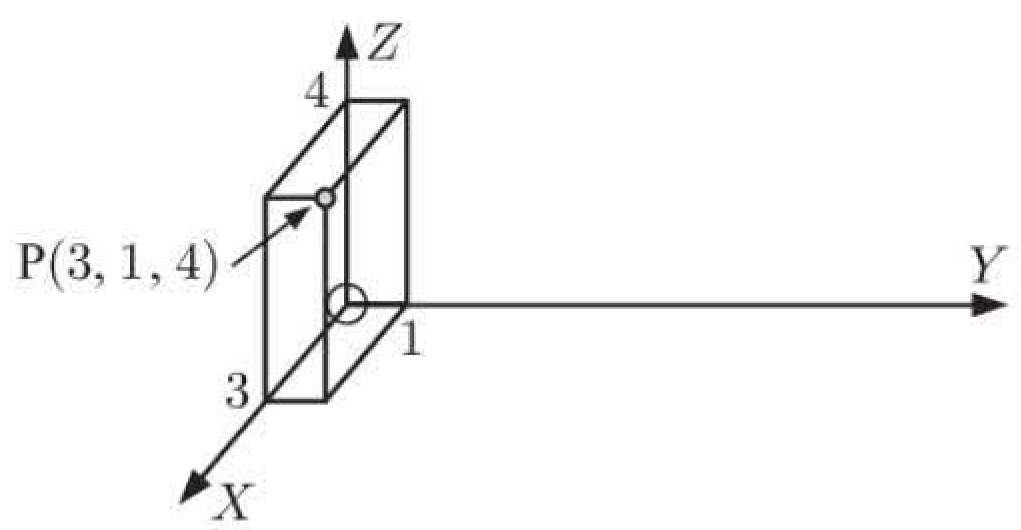
$\text{OP} = \sqrt{0^2 + 0^2 + (-3)^2} = 3$ units

b



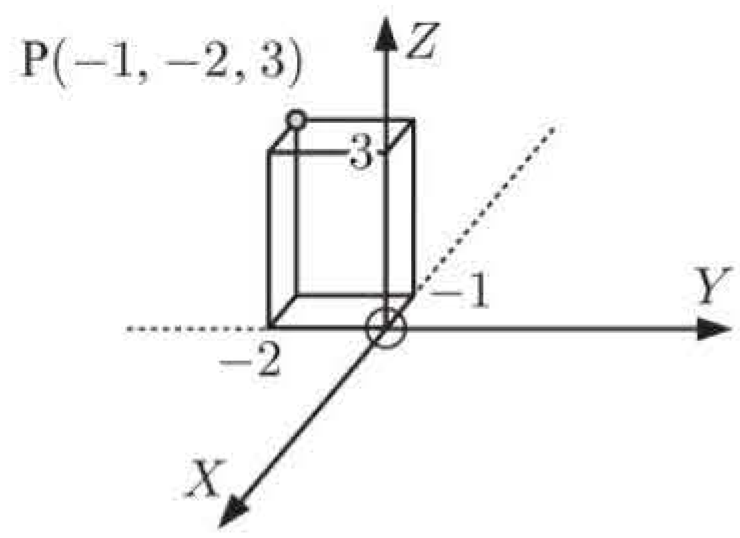
$\text{OP} = \sqrt{0^2 + (-1)^2 + 2^2} = \sqrt{5}$ units

c



$\text{OP} = \sqrt{3^2 + 1^2 + 4^2} = \sqrt{26}$ units

d



$\text{OP} = \sqrt{(-1)^2 + (-2)^2 + 3^2} = \sqrt{14}$ units

3 **a** $\overrightarrow{\text{AB}} = \begin{pmatrix} 1-(-3) \\ 0-1 \\ -1-2 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix}$, $\quad \overrightarrow{\text{BA}} = \begin{pmatrix} -3-1 \\ 1-0 \\ 2-(-1) \end{pmatrix} = \begin{pmatrix} -4 \\ 1 \\ 3 \end{pmatrix}$

b $|\overrightarrow{\text{AB}}| = \sqrt{4^2 + (-1)^2 + (-3)^2} = \sqrt{26}$ units, $\quad |\overrightarrow{\text{BA}}| = \sqrt{(-4)^2 + 1^2 + 3^2} = \sqrt{26}$ units

4 $\overrightarrow{\text{OA}} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$, $\quad \overrightarrow{\text{OB}} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$, $\quad \overrightarrow{\text{AB}} = \begin{pmatrix} -1-3 \\ 1-1 \\ 2-0 \end{pmatrix} = \begin{pmatrix} -4 \\ 0 \\ 2 \end{pmatrix}$

5 **a** The position vector of M relative to N

$$= \overrightarrow{\text{NM}} = \begin{pmatrix} 4-(-1) \\ -2-2 \\ -1-0 \end{pmatrix} = \begin{pmatrix} 5 \\ -4 \\ -1 \end{pmatrix}$$

b The position vector of N relative to M

$$= \overrightarrow{\text{MN}} = \begin{pmatrix} -1-4 \\ 2-(-2) \\ 0-(-1) \end{pmatrix} = \begin{pmatrix} -5 \\ 4 \\ 1 \end{pmatrix}$$

c $\text{MN} = \sqrt{(-5)^2 + 4^2 + 1^2} = \sqrt{25+16+1} = \sqrt{42}$ units

6 **a** The position vector of A relative to O

$$= \overrightarrow{\text{OA}} = \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix}$$

$$\therefore\ \text{OA} = \sqrt{(-1)^2 + 2^2 + 5^2} = \sqrt{1+4+25} = \sqrt{30} \text{ units}$$

b The position vector of B relative to A

$$= \overrightarrow{\text{AB}} = \begin{pmatrix} 2-(-1) \\ 0-2 \\ 3-5 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ -2 \end{pmatrix}$$

$$\therefore\ \text{AB} = \sqrt{3^2 + (-2)^2 + (-2)^2} = \sqrt{9+4+4} = \sqrt{17} \text{ units}$$

c The position vector of C relative to A

$$= \overrightarrow{\text{AC}} = \begin{pmatrix} -3-(-1) \\ 1-2 \\ 0-5 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ -5 \end{pmatrix}$$

$$\therefore\ \text{AC} = \sqrt{(-2)^2 + (-1)^2 + (-5)^2} = \sqrt{4+1+25} = \sqrt{30} \text{ units}$$

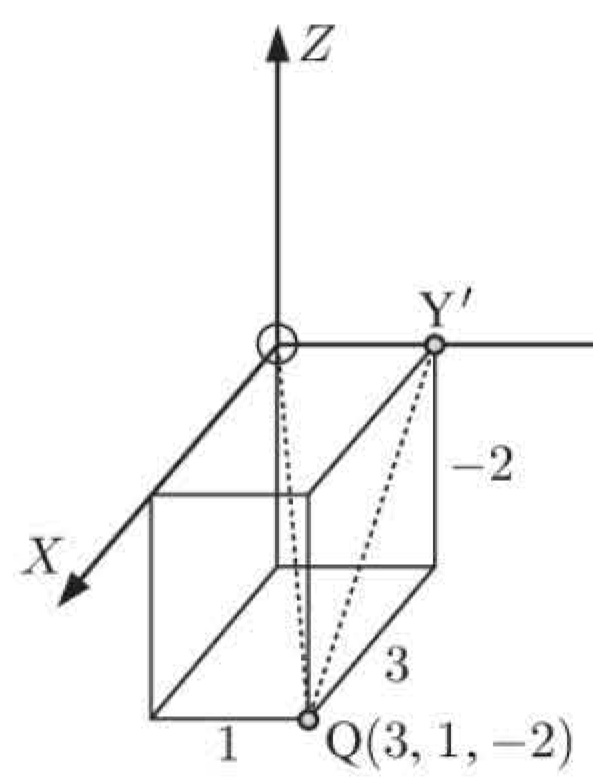
d The position vector of B relative to C

$$= \overrightarrow{\text{CB}} = \begin{pmatrix} 2-(-3) \\ 0-1 \\ 3-0 \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix}$$

$$\therefore\ \text{CB} = \sqrt{5^2 + (-1)^2 + 3^2} = \sqrt{25+1+9} = \sqrt{35} \text{ units}$$

e Triangle ABC has $\text{AC} = \sqrt{30}$ units, $\text{CB} = \sqrt{35}$ units, and $\text{AB} = \sqrt{17}$ units.
All the side lengths are different, and $(\sqrt{17})^2 + (\sqrt{30})^2 \neq (\sqrt{35})^2$.
$\therefore$ triangle ABC is scalene, and not right angled.

7



a The distance from Q to the Y-axis is the distance from Q to $\text{Y}'(0,\ 1,\ 0)$.

$$\therefore\ \text{QY}' = \sqrt{(3-0)^2+(1-1)^2+(-2-0)^2} = \sqrt{9+4} = \sqrt{13} \text{ units}$$

b The distance from Q to the origin is

$$\text{QO} = \sqrt{(3-0)^2+(1-0)^2+(-2-0)^2} = \sqrt{9+1+4} = \sqrt{14} \text{ units}$$

c The distance from Q to the ZOY plane is the distance from Q to $(0,\ 1,\ -2)$, which is 3 units.

8 $\text{P}(0,\ 4,\ 4)$, $\text{Q}(2,\ 6,\ 5)$, $\text{R}(1,\ 4,\ 3)$

$$\text{PQ} = \sqrt{(2-0)^2+(6-4)^2+(5-4)^2} = \sqrt{4+4+1} = 3$$

$$\text{PR} = \sqrt{(1-0)^2+(4-4)^2+(3-4)^2} = \sqrt{1+0+1} = \sqrt{2}$$

$$\text{QR} = \sqrt{(1-2)^2+(4-6)^2+(3-5)^2} = \sqrt{1+4+4} = 3$$

$\therefore$ PQ = QR and so $\triangle$PQR is isosceles.

9 **a** $\text{A}(0,\ 0,\ 3)$, $\text{B}(2,\ 8,\ 1)$, $\text{C}(-9,\ 6,\ 18)$

$$\text{AB} = \sqrt{(2-0)^2+(8-0)^2+(1-3)^2} = \sqrt{4+64+4} = \sqrt{72}$$

$$\text{AC} = \sqrt{(-9-0)^2+(6-0)^2+(18-3)^2} = \sqrt{81+36+225} = \sqrt{342}$$

$$\text{BC} = \sqrt{(-9-2)^2+(6-8)^2+(18-1)^2} = \sqrt{121+4+289} = \sqrt{414}$$

Since $\text{BC}^2 = \text{AB}^2 + \text{AC}^2$, $\triangle$ABC is right angled.

b $\text{A}(1,\ 0,\ -3)$, $\text{B}(2,\ 2,\ 0)$, $\text{C}(4,\ 6,\ 6)$

$$\text{AB} = \sqrt{(2-1)^2+(2-0)^2+(0--3)^2} = \sqrt{1+4+9} = \sqrt{14}$$

$$\text{AC} = \sqrt{(4-1)^2+(6-0)^2+(6--3)^2} = \sqrt{9+36+81} = \sqrt{126} = 3\sqrt{14}$$

$$\text{BC} = \sqrt{(4-2)^2+(6-2)^2+(6-0)^2} = \sqrt{4+16+36} = \sqrt{56} = 2\sqrt{14}$$

Since AB + BC = AC, the points A, B, and C lie on a straight line, so they do not form a triangle.

10 **a** $\overrightarrow{\text{AB}} = \begin{pmatrix} 6-5 \\ 12-6 \\ 9-(-2) \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \\ 11 \end{pmatrix}$

and so $|\,\overrightarrow{\text{AB}}\,| = \sqrt{1^2+6^2+11^2} = \sqrt{1+36+121} = \sqrt{158}$ units

$\overrightarrow{\text{BC}} = \begin{pmatrix} 2-6 \\ 4-12 \\ 2-9 \end{pmatrix} = \begin{pmatrix} -4 \\ -8 \\ -7 \end{pmatrix}$

and so $|\,\overrightarrow{\text{BC}}\,| = \sqrt{(-4)^2+(-8)^2+(-7)^2} = \sqrt{16+64+49} = \sqrt{129}$ units

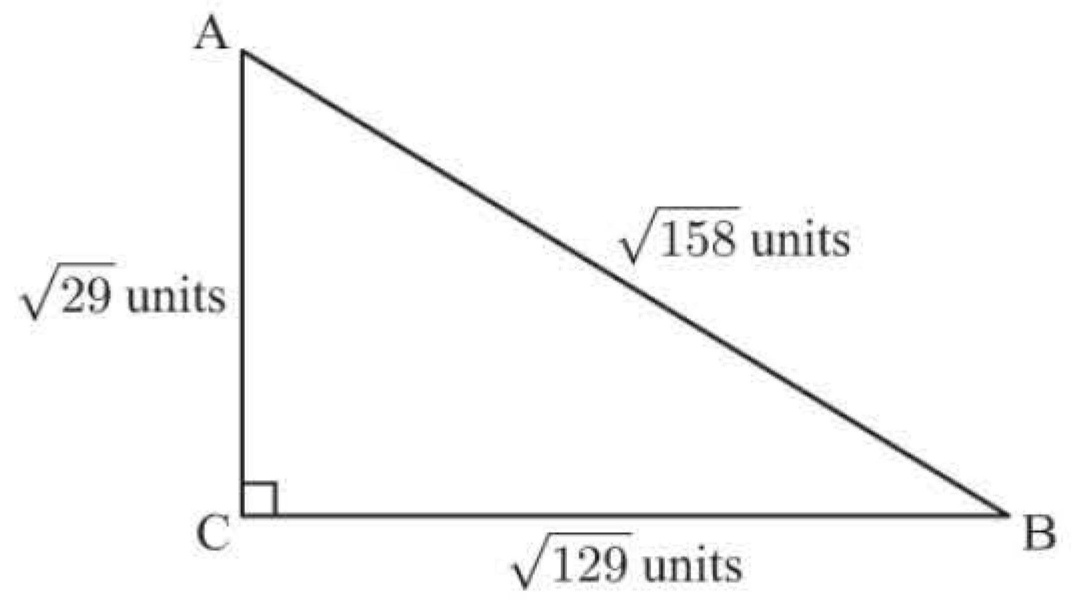
$\overrightarrow{\text{AC}} = \begin{pmatrix} 2-5 \\ 4-6 \\ 2-(-2) \end{pmatrix} = \begin{pmatrix} -3 \\ -2 \\ 4 \end{pmatrix}$

and so $|\,\overrightarrow{\text{AC}}\,| = \sqrt{(-3)^2+(-2)^2+4^2} = \sqrt{9+4+16} = \sqrt{29}$ units

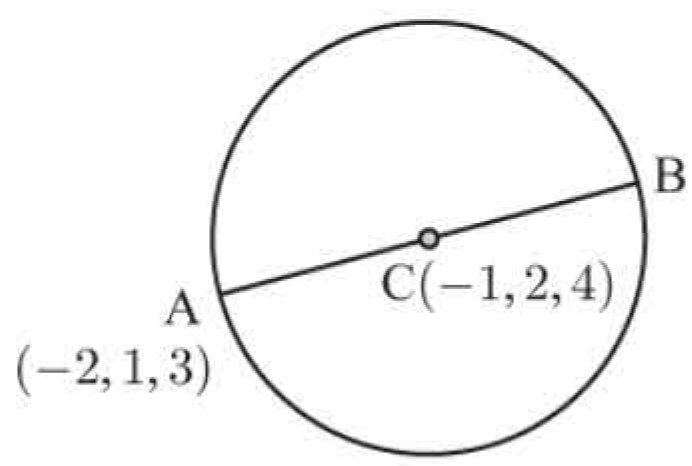
Now, $\left(\sqrt{29}\right)^2 + \left(\sqrt{129}\right)^2 = 29 + 129 = 158 = \left(\sqrt{158}\right)^2$

So, $\text{AC}^2 + \text{BC}^2 = \text{AB}^2$

$\therefore$ triangle ABC is right angled with the right angle at C.

b 

$$\begin{aligned}\text{Area} &= \tfrac{1}{2} \times \text{base} \times \text{height}\\ &= \tfrac{1}{2} \times \sqrt{129} \times \sqrt{29}\\ &\approx 30.6 \text{ units}^2\end{aligned}$$

11 

If B is (a, b, c) then $\dfrac{a-2}{2} = -1, \quad \dfrac{b+1}{2} = 2, \quad \dfrac{c+3}{2} = 4$

$\therefore \quad a = 0, \quad b = 3, \quad c = 5$

$\therefore \quad$ B is $(0, 3, 5)$

$$\begin{aligned} r = \text{AC} &= \sqrt{(-1 - -2)^2 + (2-1)^2 + (4-3)^2}\\ &= \sqrt{1+1+1}\\ &= \sqrt{3} \text{ units}\end{aligned}$$

12 **a** $(0, y, 0)$ for any y

b The distance between $(0, y, 0)$ and B$(-1, -1, 2)$ is $\sqrt{(-1)^2 + (-1-y)^2 + 2^2}$.

$\therefore \quad \sqrt{1 + (y+1)^2 + 4} = \sqrt{14}$

$\therefore \quad (y+1)^2 = 9$

$\therefore \quad y + 1 = \pm 3$

$\therefore \quad y = -1 \pm 3$

$\therefore \quad y = -4$ or 2 $\quad \therefore$ the two points are $(0, -4, 0)$ and $(0, 2, 0)$.

13 **a** $\begin{pmatrix} a-4 \\ b-3 \\ c+2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -4 \end{pmatrix}$

$\therefore \quad \begin{cases} a - 4 = 1 \\ b - 3 = 3 \\ c + 2 = -4 \end{cases}$

$\therefore \quad a = 5, \quad b = 6, \quad c = -6$

b $\begin{pmatrix} a-5 \\ b-2 \\ c+3 \end{pmatrix} = \begin{pmatrix} 3-a \\ 2-b \\ 5-c \end{pmatrix}$

$\therefore \quad \begin{cases} a - 5 = 3 - a \\ b - 2 = 2 - b \\ c + 3 = 5 - c \end{cases}$

$\therefore \quad 2a = 8, \quad 2b = 4, \quad 2c = 2$

$\therefore \quad a = 4, \quad b = 2, \quad c = 1$

14 **a** length $= 1$

$\therefore \quad \sqrt{\tfrac{1}{4} + k^2 + \tfrac{1}{16}} = 1$

$\therefore \quad \sqrt{k^2 + \tfrac{5}{16}} = 1$

$\therefore \quad k^2 = \tfrac{11}{16}$

$\therefore \quad k = \pm\tfrac{\sqrt{11}}{4}$

b length $= 1$

$\therefore \quad \sqrt{k^2 + \tfrac{4}{9} + \tfrac{1}{9}} = 1$

$\therefore \quad \sqrt{k^2 + \tfrac{5}{9}} = 1$

$\therefore \quad k^2 = \tfrac{4}{9}$

$\therefore \quad k = \pm\tfrac{2}{3}$

15 A$(-1, 3, 4)$, B$(2, 5, -1)$, C$(-1, 2, -2)$, D(r, s, t)

a If $\overrightarrow{\text{AC}} = \overrightarrow{\text{BD}}$ then $\begin{pmatrix} -1-(-1) \\ 2-3 \\ -2-4 \end{pmatrix} = \begin{pmatrix} r-2 \\ s-5 \\ t-(-1) \end{pmatrix}$

$\therefore \quad r - 2 = 0, \quad s - 5 = -1,$ and $t + 1 = -6 \quad \therefore \quad r = 2, \quad s = 4,$ and $t = -7$

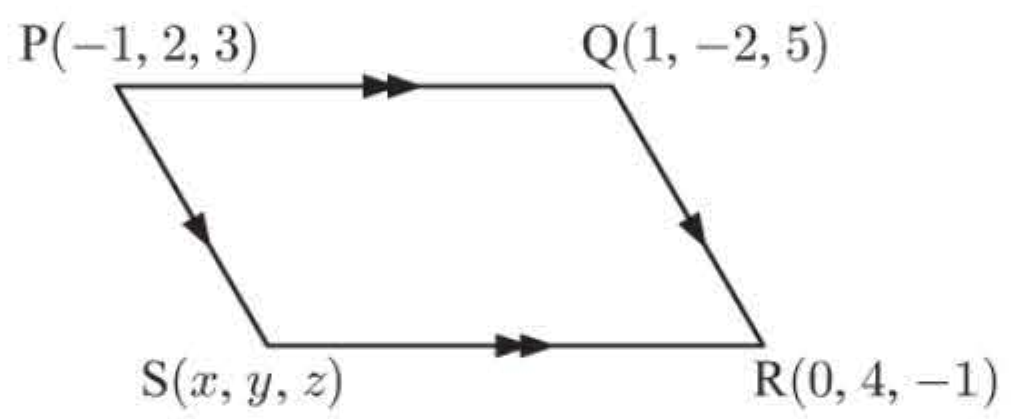
b If $\overrightarrow{\text{AB}} = \overrightarrow{\text{DC}}$ then $\begin{pmatrix} 2-(-1) \\ 5-3 \\ -1-4 \end{pmatrix} = \begin{pmatrix} -1-r \\ 2-s \\ -2-t \end{pmatrix}$

$\therefore \quad -1 - r = 3, \quad 2 - s = 2,$ and $-2 - t = -5 \quad \therefore \quad r = -4, \quad s = 0,$ and $t = 3$

16 **a** $\overrightarrow{AB} = \begin{pmatrix} 3-1 \\ -3-2 \\ 2-3 \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \\ -1 \end{pmatrix}$ and $\overrightarrow{DC} = \begin{pmatrix} 7-5 \\ -4-1 \\ 5-6 \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \\ -1 \end{pmatrix}$.

b ABCD is a parallelogram since its opposite sides are parallel and equal in length.

17 **a** Suppose S is at (x, y, z). $\overrightarrow{PQ} = \overrightarrow{SR}$ {opposite sides are parallel and equal in length}



$$\therefore \begin{pmatrix} 1-(-1) \\ -2-2 \\ 5-3 \end{pmatrix} = \begin{pmatrix} 0-x \\ 4-y \\ -1-z \end{pmatrix}$$

$$\therefore \begin{pmatrix} 2 \\ -4 \\ 2 \end{pmatrix} = \begin{pmatrix} -x \\ 4-y \\ -1-z \end{pmatrix}$$

$$\therefore\ -x = 2 \qquad 4-y = -4 \qquad -1-z = 2$$

$$\therefore\ x = -2 \qquad y = 8 \qquad z = -3$$

$\therefore$ S is at $(-2, 8, -3)$.

b The midpoint of [PR] is $\left(\dfrac{-1+0}{2}, \dfrac{2+4}{2}, \dfrac{3+(-1)}{2}\right)$ which is $(-\frac{1}{2}, 3, 1)$.

The midpoint of [QS] is $\left(\dfrac{1+(-2)}{2}, \dfrac{-2+8}{2}, \dfrac{5+(-3)}{2}\right)$ which is $(-\frac{1}{2}, 3, 1)$.

So, [PR] and [QS] have the same midpoint. ✓

EXERCISE 14H

1 **a**
$$2\mathbf{x} = \mathbf{q}$$
$$\therefore\ \tfrac{1}{2}(2\mathbf{x}) = \tfrac{1}{2}\mathbf{q}$$
$$\therefore\ \mathbf{x} = \tfrac{1}{2}\mathbf{q}$$

b
$$\tfrac{1}{2}\mathbf{x} = \mathbf{n}$$
$$\therefore\ 2(\tfrac{1}{2}\mathbf{x}) = 2\mathbf{n}$$
$$\therefore\ \mathbf{x} = 2\mathbf{n}$$

c
$$-3\mathbf{x} = \mathbf{p}$$
$$\therefore\ 3\mathbf{x} = -\mathbf{p}$$
$$\therefore\ \tfrac{1}{3}(3\mathbf{x}) = -\tfrac{1}{3}\mathbf{p}$$
$$\therefore\ \mathbf{x} = -\tfrac{1}{3}\mathbf{p}$$

d
$$\mathbf{q} + 2\mathbf{x} = \mathbf{r}$$
$$\therefore\ 2\mathbf{x} = \mathbf{r} - \mathbf{q}$$
$$\therefore\ \mathbf{x} = \tfrac{1}{2}(\mathbf{r} - \mathbf{q})$$

e
$$4\mathbf{s} - 5\mathbf{x} = \mathbf{t}$$
$$\therefore\ -5\mathbf{x} = \mathbf{t} - 4\mathbf{s}$$
$$\therefore\ 5\mathbf{x} = 4\mathbf{s} - \mathbf{t}$$
$$\therefore\ \mathbf{x} = \tfrac{1}{5}(4\mathbf{s} - \mathbf{t})$$

f
$$4\mathbf{m} - \tfrac{1}{3}\mathbf{x} = \mathbf{n}$$
$$\therefore\ 4\mathbf{m} - \mathbf{n} = \tfrac{1}{3}\mathbf{x}$$
$$\therefore\ \mathbf{x} = 3(4\mathbf{m} - \mathbf{n})$$

2 **a**
$$2\mathbf{a} + \mathbf{x} = \mathbf{b}$$
$$\therefore\ \mathbf{x} = \mathbf{b} - 2\mathbf{a}$$
$$= \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} - 2\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$$
$$= \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} - \begin{pmatrix} -2 \\ 4 \\ 6 \end{pmatrix}$$
$$\therefore\ \mathbf{x} = \begin{pmatrix} 4 \\ -6 \\ -5 \end{pmatrix}$$

b
$$3\mathbf{x} - \mathbf{a} = 2\mathbf{b}$$
$$\therefore\ 3\mathbf{x} = \mathbf{a} + 2\mathbf{b}$$
$$\therefore\ \mathbf{x} = \tfrac{1}{3}(\mathbf{a} + 2\mathbf{b})$$
$$= \tfrac{1}{3}\left[\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} + 2\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}\right]$$
$$= \tfrac{1}{3}\left[\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ -4 \\ 2 \end{pmatrix}\right]$$
$$\therefore\ \mathbf{x} = \tfrac{1}{3}\begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{2}{3} \\ \frac{5}{3} \end{pmatrix}$$

c
$$2\mathbf{b} - 2\mathbf{x} = -\mathbf{a}$$
$$\therefore\ \mathbf{a} + 2\mathbf{b} = 2\mathbf{x}$$
$$\therefore\ \mathbf{x} = \tfrac{1}{2}(\mathbf{a} + 2\mathbf{b}) = \tfrac{1}{2}\begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix} \qquad \{\text{using } \mathbf{b}\}$$
$$= \begin{pmatrix} \frac{3}{2} \\ -1 \\ \frac{5}{2} \end{pmatrix}$$

3 $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -\overrightarrow{OA} + \overrightarrow{OB} = -\begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix}$

$\therefore\ |\overrightarrow{AB}| = \sqrt{3^2 + 4^2 + (-2)^2} = \sqrt{9+16+4} = \sqrt{29}$ units

4 **a** $\overrightarrow{AB} = \begin{pmatrix} 3-(-1) \\ -2-3 \\ 1-2 \end{pmatrix} = \begin{pmatrix} 4 \\ -5 \\ -1 \end{pmatrix}$
$= 4\mathbf{i} - 5\mathbf{j} - \mathbf{k}$

b $|\overrightarrow{AB}| = \sqrt{4^2 + (-5)^2 + (-1)^2}$
$= \sqrt{16+25+1}$
$= \sqrt{42}$ units

5 **a** $|\mathbf{a}| = \sqrt{1^2 + 0^2 + 3^2}$
$= \sqrt{1+9}$
$= \sqrt{10}$ units

b $|\mathbf{b}| = \sqrt{(-2)^2 + 1^2 + 1^2}$
$= \sqrt{4+1+1}$
$= \sqrt{6}$ units

c $2|\mathbf{a}| = 2\sqrt{10}$ units {using part **a**}

d $2\mathbf{a} = 2\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 6 \end{pmatrix}$

$\therefore\ |2\mathbf{a}| = \sqrt{2^2 + 0^2 + 6^2}$
$= \sqrt{4+36}$
$= \sqrt{40}$
$= \sqrt{4}\sqrt{10}$
$= 2\sqrt{10}$ units

e $-3|\mathbf{b}| = -3\sqrt{6}$ units {using part **b**}

f $-3\mathbf{b} = -3\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \\ -3 \end{pmatrix}$

$\therefore\ |-3\mathbf{b}| = \sqrt{6^2 + (-3)^2 + (-3)^2}$
$= \sqrt{36+9+9}$
$= \sqrt{54}$
$= \sqrt{9}\sqrt{6}$
$= 3\sqrt{6}$ units

g $\mathbf{a} + \mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$
$= \begin{pmatrix} 1-2 \\ 0+1 \\ 3+1 \end{pmatrix}$
$= \begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix}$

$\therefore\ |\mathbf{a} + \mathbf{b}| = \sqrt{(-1)^2 + 1^2 + 4^2}$
$= \sqrt{1+1+16}$
$= \sqrt{18}$
$= \sqrt{9}\sqrt{2}$ units $= 3\sqrt{2}$ units

h $\mathbf{a} - \mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} - \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$
$= \begin{pmatrix} 1-(-2) \\ 0-1 \\ 3-1 \end{pmatrix}$
$= \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$

$\therefore\ |\mathbf{a} - \mathbf{b}| = \sqrt{3^2 + (-1)^2 + 2^2}$
$= \sqrt{9+1+4}$
$= \sqrt{14}$ units

6 $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$
$= (\mathbf{i} - \mathbf{j} + \mathbf{k}) + (-2\mathbf{i} + \mathbf{j} - 3\mathbf{k})$
$= -\mathbf{i} - 2\mathbf{k}$

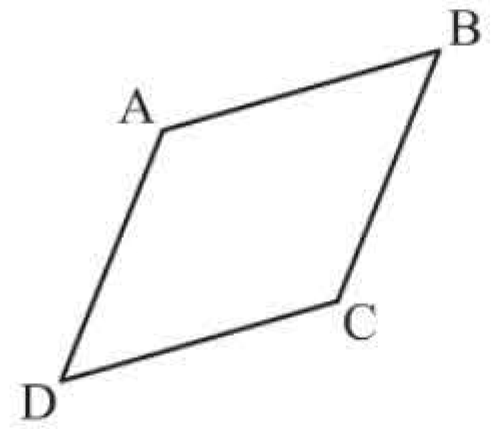
7 A(2, 1, −2), B(0, 3, −4), C(1, −2, 1), D(−2, −3, 2)

$\overrightarrow{AC} = \begin{pmatrix} 1-2 \\ -2-1 \\ 1-(-2) \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ 3 \end{pmatrix}$

$\overrightarrow{BD} = \begin{pmatrix} -2-0 \\ -3-3 \\ 2-(-4) \end{pmatrix} = \begin{pmatrix} -2 \\ -6 \\ 6 \end{pmatrix} = 2\begin{pmatrix} -1 \\ -3 \\ 3 \end{pmatrix} = 2\overrightarrow{AC}$

8 $\overrightarrow{AB} = \begin{pmatrix} 2--1 \\ 3-5 \\ -3-2 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ -5 \end{pmatrix}$ $\therefore$ C is $(2+3,\ 3-2,\ -3-5)$, or $(5,\ 1,\ -8)$,
D is $(5+3,\ 1-2,\ -8-5)$, or $(8,\ -1,\ -13)$,
E is $(8+3,\ -1-2,\ -13-5)$, or $(11,\ -3,\ -18)$.

9



a $\overrightarrow{AB} = \begin{pmatrix} 4-3 \\ 2--1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$\overrightarrow{DC} = \begin{pmatrix} -1--2 \\ 4-1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

Now $\overrightarrow{AB} = \overrightarrow{DC}$
$\therefore$ sides [AB] and [DC] are equal in length and parallel.
This is sufficient to deduce that ABCD is a parallelogram.

b $\overrightarrow{AB} = \begin{pmatrix} -1-5 \\ 2-0 \\ 4-3 \end{pmatrix} = \begin{pmatrix} -6 \\ 2 \\ 1 \end{pmatrix}$

$\overrightarrow{DC} = \begin{pmatrix} 4-10 \\ -3--5 \\ 6-5 \end{pmatrix} = \begin{pmatrix} -6 \\ 2 \\ 1 \end{pmatrix}$

So $\overrightarrow{AB} = \overrightarrow{DC}$
$\therefore$ sides [AB] and [DC] are equal in length and parallel.
This is sufficient to deduce that ABCD is a parallelogram.

c $\overrightarrow{AB} = \begin{pmatrix} 1-2 \\ 4--3 \\ -1-2 \end{pmatrix} = \begin{pmatrix} -1 \\ 7 \\ -3 \end{pmatrix}$

$\overrightarrow{DC} = \begin{pmatrix} -2--1 \\ 6--1 \\ -2-2 \end{pmatrix} = \begin{pmatrix} -1 \\ 7 \\ -4 \end{pmatrix}$

So, $\overrightarrow{AB} \neq \overrightarrow{DC}$
$\therefore$ ABCD cannot be a parallelogram.

10 **a** Let D be $(a,\ b)$.

Now $\overrightarrow{CD} = \overrightarrow{BA}$

$\therefore \begin{pmatrix} a-8 \\ b--2 \end{pmatrix} = \begin{pmatrix} 3-2 \\ 0--1 \end{pmatrix}$

$\therefore \begin{pmatrix} a-8 \\ b+2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\therefore$ $a = 9,\ b = -1$
So, D is $(9,\ -1)$.

b Let R be $(a,\ b,\ c)$.

Now $\overrightarrow{SR} = \overrightarrow{PQ}$

$\therefore \begin{pmatrix} a-4 \\ b-0 \\ c-7 \end{pmatrix} = \begin{pmatrix} -2--1 \\ 5-4 \\ 2-3 \end{pmatrix}$

$\therefore \begin{pmatrix} a-4 \\ b \\ c-7 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$

$\therefore$ $a = 3,\ b = 1,\ c = 6$
So, R is $(3,\ 1,\ 6)$.

c Let X be $(a,\ b,\ c)$.

Now $\overrightarrow{WX} = \overrightarrow{ZY}$

$\therefore \begin{pmatrix} a--1 \\ b-5 \\ c-8 \end{pmatrix} = \begin{pmatrix} 3-0 \\ -2-4 \\ -2-6 \end{pmatrix}$

$\therefore \begin{pmatrix} a+1 \\ b-5 \\ c-8 \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \\ -8 \end{pmatrix}$

$\therefore$ $a = 2,\ b = -1,\ c = 0$
So, X is $(2,\ -1,\ 0)$.

11 **a** $\overrightarrow{BD} = \frac{1}{2}\overrightarrow{OA}$
$= \frac{1}{2}\mathbf{a}$

b $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$
$= -\mathbf{a} + \mathbf{b}$
$= \mathbf{b} - \mathbf{a}$

c $\overrightarrow{BA} = -\overrightarrow{AB}$
$= -(\mathbf{b} - \mathbf{a})$ {using **b**}
$= -\mathbf{b} + \mathbf{a}$ or $\mathbf{a} - \mathbf{b}$

d $\overrightarrow{OD} = \overrightarrow{OB} + \overrightarrow{BD}$
$= \mathbf{b} + \frac{1}{2}\mathbf{a}$ {using **a**}

e $\overrightarrow{AD}$
$= \overrightarrow{AO} + \overrightarrow{OD}$
$= -\mathbf{a} + (\mathbf{b} + \frac{1}{2}\mathbf{a})$ {using **d**}
$= -\frac{1}{2}\mathbf{a} + \mathbf{b}$ or $\mathbf{b} - \frac{1}{2}\mathbf{a}$

f $\overrightarrow{DA} = -\overrightarrow{AD}$
$= \frac{1}{2}\mathbf{a} - \mathbf{b}$ {using **e**}

12 **a** $\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BD} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \\ -1 \end{pmatrix}$

b $\overrightarrow{CB} = \overrightarrow{CA} + \overrightarrow{AB} = -\overrightarrow{AC} + \overrightarrow{AB} = -\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} + \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ -4 \end{pmatrix} + \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ 4 \\ -2 \end{pmatrix}$

c $\overrightarrow{CD} = \overrightarrow{CB} + \overrightarrow{BD} = \begin{pmatrix} -3 \\ 4 \\ -2 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix}$ {using **b**} $= \begin{pmatrix} -3 \\ 6 \\ -5 \end{pmatrix}$

13 **a** $\mathbf{a}+\mathbf{b}=\begin{pmatrix}2\\-1\\1\end{pmatrix}+\begin{pmatrix}1\\2\\-3\end{pmatrix}=\begin{pmatrix}3\\1\\-2\end{pmatrix}$

b $\mathbf{a}-\mathbf{b}=\begin{pmatrix}2\\-1\\1\end{pmatrix}-\begin{pmatrix}1\\2\\-3\end{pmatrix}=\begin{pmatrix}1\\-3\\4\end{pmatrix}$

c $\mathbf{b}+2\mathbf{c}=\begin{pmatrix}1\\2\\-3\end{pmatrix}+2\begin{pmatrix}0\\1\\-3\end{pmatrix}=\begin{pmatrix}1\\2\\-3\end{pmatrix}+\begin{pmatrix}0\\2\\-6\end{pmatrix}=\begin{pmatrix}1\\4\\-9\end{pmatrix}$

d $\mathbf{c}-\frac{1}{2}\mathbf{a}=\begin{pmatrix}0\\1\\-3\end{pmatrix}-\frac{1}{2}\begin{pmatrix}2\\-1\\1\end{pmatrix}=\begin{pmatrix}0\\1\\-3\end{pmatrix}-\begin{pmatrix}1\\-\frac{1}{2}\\\frac{1}{2}\end{pmatrix}=\begin{pmatrix}-1\\\frac{3}{2}\\-\frac{7}{2}\end{pmatrix}$

e $\mathbf{a}-\mathbf{b}-\mathbf{c}=\begin{pmatrix}2\\-1\\1\end{pmatrix}-\begin{pmatrix}1\\2\\-3\end{pmatrix}-\begin{pmatrix}0\\1\\-3\end{pmatrix}=\begin{pmatrix}1\\-4\\7\end{pmatrix}$

f $2\mathbf{b}-\mathbf{c}+\mathbf{a}=2\begin{pmatrix}1\\2\\-3\end{pmatrix}-\begin{pmatrix}0\\1\\-3\end{pmatrix}+\begin{pmatrix}2\\-1\\1\end{pmatrix}=\begin{pmatrix}2\\4\\-6\end{pmatrix}-\begin{pmatrix}0\\1\\-3\end{pmatrix}+\begin{pmatrix}2\\-1\\1\end{pmatrix}=\begin{pmatrix}4\\2\\-2\end{pmatrix}$

14 **a** $|\,\mathbf{a}\,|=\sqrt{(-1)^2+1^2+3^2}=\sqrt{11}$ units

b $|\,\mathbf{b}\,|=\sqrt{1^2+(-3)^2+2^2}=\sqrt{14}$ units

c $|\,\mathbf{b}+\mathbf{c}\,|=\left|\begin{pmatrix}1\\-3\\2\end{pmatrix}+\begin{pmatrix}-2\\2\\4\end{pmatrix}\right|=\left|\begin{pmatrix}-1\\-1\\6\end{pmatrix}\right|$

$=\sqrt{(-1)^2+(-1)^2+6^2}$

$=\sqrt{1+1+36}$

$=\sqrt{38}$ units

d $|\,\mathbf{a}-\mathbf{c}\,|=\left|\begin{pmatrix}-1\\1\\3\end{pmatrix}-\begin{pmatrix}-2\\2\\4\end{pmatrix}\right|=\left|\begin{pmatrix}1\\-1\\-1\end{pmatrix}\right|$

$=\sqrt{1^2+(-1)^2+(-1)^2}$

$=\sqrt{3}$ units

e $|\,\mathbf{a}\,|\,\mathbf{b}=\sqrt{11}\begin{pmatrix}1\\-3\\2\end{pmatrix}$ {using **a**}

$=\begin{pmatrix}\sqrt{11}\\-3\sqrt{11}\\2\sqrt{11}\end{pmatrix}$

f $\dfrac{1}{|\,\mathbf{a}\,|}\mathbf{a}=\dfrac{1}{\sqrt{11}}\begin{pmatrix}-1\\1\\3\end{pmatrix}$ {using **a**}

$=\begin{pmatrix}-\frac{1}{\sqrt{11}}\\\frac{1}{\sqrt{11}}\\\frac{3}{\sqrt{11}}\end{pmatrix}$

15 **a** $2\begin{pmatrix}1\\0\\3a\end{pmatrix}=\begin{pmatrix}b\\c-1\\2\end{pmatrix}$

$\therefore\ \begin{pmatrix}2\\0\\6a\end{pmatrix}=\begin{pmatrix}b\\c-1\\2\end{pmatrix}$

$\therefore\ 2=b,\quad 0=c-1,\quad \text{and}\quad 6a=2$

$\therefore\ a=\frac{1}{3},\quad b=2,\quad \text{and}\quad c=1$

b $a\begin{pmatrix}1\\1\\0\end{pmatrix}+b\begin{pmatrix}2\\0\\-1\end{pmatrix}+c\begin{pmatrix}0\\1\\1\end{pmatrix}=\begin{pmatrix}-1\\3\\3\end{pmatrix}$

$\therefore\ \begin{pmatrix}a\\a\\0\end{pmatrix}+\begin{pmatrix}2b\\0\\-b\end{pmatrix}+\begin{pmatrix}0\\c\\c\end{pmatrix}=\begin{pmatrix}-1\\3\\3\end{pmatrix}$

So, $a+2b=-1$ .... (1)

$a+c=3$

$\therefore\ c=3-a$ .... (2)

$-b+c=3$

$\therefore\ c=b+3$ .... (3)

Substituting (2) into (3), we get

$3-a=b+3$

$\therefore\ -a=b$

Substituting into (1), we get

$a+2(-a)=-1$

$\therefore\ -a=-1$

$\therefore\ a=1$

$\therefore\ b=-1$

and $c=-1+3=2$ {using (3)}

$\therefore\ a=1,\quad b=-1,\quad \text{and}\quad c=2$

c $$a\begin{pmatrix}2\\-3\\1\end{pmatrix}+b\begin{pmatrix}1\\7\\2\end{pmatrix}=\begin{pmatrix}7\\-19\\2\end{pmatrix}$$

$$\therefore \begin{pmatrix}2a\\-3a\\a\end{pmatrix}+\begin{pmatrix}b\\7b\\2b\end{pmatrix}=\begin{pmatrix}7\\-19\\2\end{pmatrix}$$

$$\therefore \begin{pmatrix}2a+b\\-3a+7b\\a+2b\end{pmatrix}=\begin{pmatrix}7\\-19\\2\end{pmatrix}$$

So, $2a+b=7$

$\therefore\ b=7-2a$ .... (1)

$-3a+7b=-19$ .... (2)

$a+2b=2$ .... (3)

Substituting (1) into (3), we get

$a+2(7-2a)=2$

$\therefore\ a+14-4a=2$

$\therefore\ -3a=-12$

$\therefore\ a=4$

and so $b=7-2(4)=-1$

$\therefore\ a=4,\ \ b=-1$

Check: $-3(4)+7(-1)=-12-7=-19$ ✓

EXERCISE 14I

1 Since $\mathbf{a}$ and $\mathbf{b}$ are parallel, then $\mathbf{b}=k\mathbf{a}$. $\therefore \begin{pmatrix}-6\\r\\s\end{pmatrix}=k\begin{pmatrix}2\\-1\\3\end{pmatrix}=\begin{pmatrix}2k\\-k\\3k\end{pmatrix}$

$\therefore\ 2k=-6,\ r=-k,\ s=3k \qquad \therefore\ k=-3,\ r=3,\ s=-9$

2 If $\begin{pmatrix}3\\-1\\2\end{pmatrix}$ and $\begin{pmatrix}a\\2\\b\end{pmatrix}$ are parallel, then $\begin{pmatrix}a\\2\\b\end{pmatrix}=k\begin{pmatrix}3\\-1\\2\end{pmatrix}$.

$\therefore\ a=3k,\ 2=-k,\ b=2k \qquad \therefore\ k=-2,\ a=-6,$ and $b=-4$

3 **a** $\overrightarrow{AB}=3\overrightarrow{CD}$ means that $\overrightarrow{AB}$ is parallel to $\overrightarrow{CD}$ and 3 times its length.

b $\overrightarrow{RS}=-\frac{1}{2}\overrightarrow{KL}$ means that $\overrightarrow{RS}$ is parallel to $\overrightarrow{KL}$, half its length, and in the opposite direction.

c

$\overrightarrow{AB}=2\overrightarrow{BC}$ means that A, B, and C are collinear and the length of $\overrightarrow{AB}$ is twice the length of $\overrightarrow{BC}$.

4 $\overrightarrow{OP}=\begin{pmatrix}3\\2\\-1\end{pmatrix},\ \overrightarrow{OQ}=\begin{pmatrix}1\\4\\-3\end{pmatrix},\ \overrightarrow{OR}=\begin{pmatrix}2\\-1\\2\end{pmatrix},\ \overrightarrow{OS}=\begin{pmatrix}-1\\-2\\3\end{pmatrix}$

a $\overrightarrow{PR}=\overrightarrow{PO}+\overrightarrow{OR}=-\begin{pmatrix}3\\2\\-1\end{pmatrix}+\begin{pmatrix}2\\-1\\2\end{pmatrix}=\begin{pmatrix}-1\\-3\\3\end{pmatrix}$

$\overrightarrow{QS}=\overrightarrow{QO}+\overrightarrow{OS}=-\begin{pmatrix}1\\4\\-3\end{pmatrix}+\begin{pmatrix}-1\\-2\\3\end{pmatrix}=\begin{pmatrix}-2\\-6\\6\end{pmatrix}=2\begin{pmatrix}-1\\-3\\3\end{pmatrix}=2\overrightarrow{PR}$ and so [QS] $\parallel$ [PR].

b Since $\overrightarrow{QS}=2\overrightarrow{PR},\ \ |\overrightarrow{QS}|=2\,|\overrightarrow{PR}|,$ and so [QS] is twice as long as [PR].

5 **a** The vector in the same direction as $\mathbf{a}$ and twice its length is $2\mathbf{a}$.

$$2\mathbf{a} = 2\begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \end{pmatrix}$$

b The vector in the opposite direction to $\mathbf{a}$ and half its length is $-\frac{1}{2}\mathbf{a}$.

$$-\tfrac{1}{2}\mathbf{a} = -\tfrac{1}{2}\begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} -\frac{2}{2} \\ -\frac{4}{2} \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$$

6 **a** $\mathbf{i} + 2\mathbf{j}$ has length $\sqrt{1^2 + 2^2} = \sqrt{5}$ units $\quad \therefore$ unit vector $= \frac{1}{\sqrt{5}}(\mathbf{i} + 2\mathbf{j})$

b $2\mathbf{i} - 3\mathbf{k}$ has length $\sqrt{2^2 + 0^2 + (-3)^2} = \sqrt{4+9} = \sqrt{13}$ units

$\therefore$ unit vector is $\frac{1}{\sqrt{13}}(2\mathbf{i} - 3\mathbf{k})$

c $2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ has length $\sqrt{2^2 + (-2)^2 + 1^2} = \sqrt{9} = 3$ units

$\therefore$ unit vector is $\frac{1}{3}(2\mathbf{i} - 2\mathbf{j} + \mathbf{k})$

7 **a** $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ has length $\sqrt{2^2 + (-1)^2} = \sqrt{5}$ units

$\therefore$ the unit vector in the same direction is $\frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$\therefore$ the vector of length 3 units in the same direction is $\frac{3}{\sqrt{5}}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{6}{\sqrt{5}} \\ -\frac{3}{\sqrt{5}} \end{pmatrix}$

b $\begin{pmatrix} -1 \\ -4 \end{pmatrix}$ has length $\sqrt{(-1)^2 + (-4)^2} = \sqrt{17}$ units

$\therefore$ the unit vector in the opposite direction is $-\frac{1}{\sqrt{17}}\begin{pmatrix} -1 \\ -4 \end{pmatrix} = \frac{1}{\sqrt{17}}\begin{pmatrix} 1 \\ 4 \end{pmatrix}$

$\therefore$ the vector of length 2 units in the opposite direction is $\frac{2}{\sqrt{17}}\begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{17}} \\ \frac{8}{\sqrt{17}} \end{pmatrix}$

8 **a** $\overrightarrow{\text{AB}}$ is a vector in the same direction as $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ with length 4 units.

Now, $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ has length $\sqrt{1^2 + (-1)^2} = \sqrt{2}$ units

$\therefore$ the unit vector in the same direction is $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\therefore$ the vector of length 4 units in the same direction is $\frac{4}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = 2\sqrt{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$= \begin{pmatrix} 2\sqrt{2} \\ -2\sqrt{2} \end{pmatrix}$$

$$\therefore \quad \overrightarrow{\text{AB}} = \begin{pmatrix} 2\sqrt{2} \\ -2\sqrt{2} \end{pmatrix}$$

b $\overrightarrow{\text{OA}} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, $\overrightarrow{\text{AB}} = \begin{pmatrix} 2\sqrt{2} \\ -2\sqrt{2} \end{pmatrix}$

Now $\overrightarrow{\text{OB}} = \overrightarrow{\text{OA}} + \overrightarrow{\text{AB}}$

$$\therefore \quad \overrightarrow{\text{OB}} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} + \begin{pmatrix} 2\sqrt{2} \\ -2\sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} 3 + 2\sqrt{2} \\ 2 - 2\sqrt{2} \end{pmatrix}$$

c If $\overrightarrow{\text{OB}} = \begin{pmatrix} 3 + 2\sqrt{2} \\ 2 - 2\sqrt{2} \end{pmatrix}$, then the coordinates of B are $(3 + 2\sqrt{2},\, 2 - 2\sqrt{2})$.

9 **a** $|\mathbf{a}| = \sqrt{2^2 + (-1)^2 + (-2)^2}$
$= \sqrt{4+1+4}$
$= 3$ units

$\therefore$ the vectors of length 1 unit parallel to $\mathbf{a}$ are $\pm\frac{1}{3}\mathbf{a}$.

$\therefore$ the vectors are $\begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ -\frac{2}{3} \end{pmatrix}$ and $\begin{pmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$.

b $|\mathbf{b}| = \sqrt{(-2)^2 + (-1)^2 + 2^2}$
$= \sqrt{4+1+4}$
$= 3$ units

$\therefore$ the vectors of length 2 units parallel to $\mathbf{b}$ are $\pm\frac{2}{3}\mathbf{b}$.

$\therefore$ the vectors are $\begin{pmatrix} -\frac{4}{3} \\ -\frac{2}{3} \\ \frac{4}{3} \end{pmatrix}$ and $\begin{pmatrix} \frac{4}{3} \\ \frac{2}{3} \\ -\frac{4}{3} \end{pmatrix}$.

10 **a** $\begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$ has length $\sqrt{(-1)^2 + 4^2 + 1^2} = \sqrt{18} = 3\sqrt{2}$ units

$\therefore$ the unit vector in the same direction is $\frac{1}{3\sqrt{2}}\begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$

$\therefore$ the vector of length 6 units in the same direction is $\frac{6}{3\sqrt{2}}\begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} = \sqrt{2}\begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} -\sqrt{2} \\ 4\sqrt{2} \\ \sqrt{2} \end{pmatrix}$

b $\begin{pmatrix} -1 \\ -2 \\ -2 \end{pmatrix}$ has length $\sqrt{(-1)^2 + (-2)^2 + (-2)^2} = \sqrt{9} = 3$ units

$\therefore$ the unit vector in the opposite direction is $-\frac{1}{3}\begin{pmatrix} -1 \\ -2 \\ -2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

$\therefore$ the vector of length 5 units in the opposite direction is $\frac{5}{3}\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{5}{3} \\ \frac{10}{3} \\ \frac{10}{3} \end{pmatrix}$

11 **a** $\overrightarrow{AB} = \begin{pmatrix} 4-(-2) \\ 3-1 \\ 0-4 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ -4 \end{pmatrix} = 2\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$

$\overrightarrow{BC} = \begin{pmatrix} 19-4 \\ 8-3 \\ -10-0 \end{pmatrix} = \begin{pmatrix} 15 \\ 5 \\ -10 \end{pmatrix} = 5\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$

So, $\overrightarrow{AB} = \frac{2}{5}\overrightarrow{BC}$

$\therefore$ A, B, and C are collinear.

b $\overrightarrow{PQ} = \begin{pmatrix} 5-2 \\ -5-1 \\ -2-1 \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \\ -3 \end{pmatrix} = 3\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$

$\overrightarrow{QR} = \begin{pmatrix} -1-5 \\ 7-(-5) \\ 4-(-2) \end{pmatrix} = \begin{pmatrix} -6 \\ 12 \\ 6 \end{pmatrix} = -6\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$

So, $\overrightarrow{PQ} = \frac{3}{-6}\overrightarrow{QR} = -\frac{1}{2}\overrightarrow{QR}$

$\therefore$ P, Q, and R are collinear.

c $\overrightarrow{AB} = \begin{pmatrix} 11-2 \\ -9-(-3) \\ 7-4 \end{pmatrix} = \begin{pmatrix} 9 \\ -6 \\ 3 \end{pmatrix}$

$\overrightarrow{BC} = \begin{pmatrix} -13-11 \\ a-(-9) \\ b-7 \end{pmatrix} = \begin{pmatrix} -24 \\ a+9 \\ b-7 \end{pmatrix}$

A, B, and C are collinear.

$\therefore \overrightarrow{AB} = k\overrightarrow{BC}$

$9 = k \times -24$

$\therefore k = -\frac{3}{8}$

$\therefore \overrightarrow{AB} = -\frac{3}{8}\overrightarrow{BC}$

$\therefore -\frac{8}{3}\overrightarrow{AB} = \overrightarrow{BC}$

So, $-\frac{8}{3} \times -6 = a+9$
$16 = a+9$
$a = 7$

and $-\frac{8}{3} \times 3 = b-7$
$-8 = b-7$
$b = -1$

d $\overrightarrow{\mathrm{KL}} = \begin{pmatrix} 4-1 \\ -3-(-1) \\ 7-0 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 7 \end{pmatrix}$

$\overrightarrow{\mathrm{LM}} = \begin{pmatrix} a-4 \\ 2-(-3) \\ b-7 \end{pmatrix} = \begin{pmatrix} a-4 \\ 5 \\ b-7 \end{pmatrix}$

K, L, and M are collinear.

$\therefore \overrightarrow{\mathrm{KL}} = k\overrightarrow{\mathrm{LM}}$

$-2 = k\,5$

$k = -\frac{2}{5}$

$\therefore \overrightarrow{\mathrm{KL}} = -\frac{2}{5}\overrightarrow{\mathrm{LM}}$

$\therefore -\frac{5}{2}\overrightarrow{\mathrm{KL}} = \overrightarrow{\mathrm{LM}}$

So, $-\frac{5}{2} \times 3 = a - 4$

$a = -\frac{15}{2} + \frac{8}{2}$

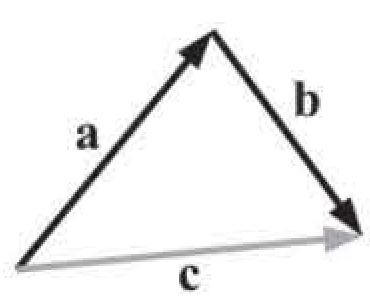
$= -\frac{7}{2}$

and $-\frac{5}{2} \times 7 = b - 7$

$b = -\frac{35}{2} + \frac{14}{2}$

$= -\frac{21}{2}$

12 **Case:** **a** and **b** are not parallel.



Since **a**, **b**, and **c** form a triangle,

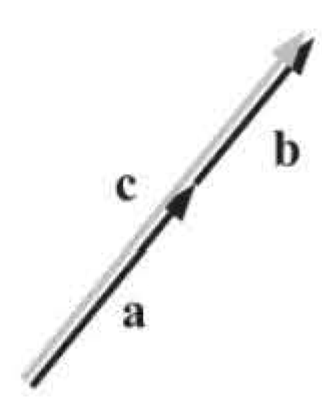
length of **a** + length of **b** > length of **c**

But $\mathbf{c} = \mathbf{a} + \mathbf{b}$, so length of **c** = length of $(\mathbf{a} + \mathbf{b})$

$\therefore$ length of **a** + length of **b** > length of $(\mathbf{a} + \mathbf{b})$

$\therefore |\mathbf{a}| + |\mathbf{b}| > |\mathbf{a} + \mathbf{b}|$

Case: **a** and **b** are parallel.



length of **a** + length of **b** = length of $(\mathbf{a} + \mathbf{b})$

$\therefore |\mathbf{a}| + |\mathbf{b}| = |\mathbf{a} + \mathbf{b}|$

These are the only possible cases, so $\therefore$ for any **a**, **b**, $|\mathbf{a}| + |\mathbf{b}| \geqslant |\mathbf{a} + \mathbf{b}|$

EXERCISE 14J

1 **a** $\mathbf{q} \bullet \mathbf{p}$

$= \begin{pmatrix} -1 \\ 5 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$= -1(3) + 5(2)$

$= -3 + 10$

$= 7$

b $\mathbf{q} \bullet \mathbf{r}$

$= \begin{pmatrix} -1 \\ 5 \end{pmatrix} \bullet \begin{pmatrix} -2 \\ 4 \end{pmatrix}$

$= -1(-2) + 5(4)$

$= 2 + 20$

$= 22$

c $\mathbf{q} \bullet (\mathbf{p} + \mathbf{r})$

$= \begin{pmatrix} -1 \\ 5 \end{pmatrix} \bullet \left[\begin{pmatrix} 3 \\ 2 \end{pmatrix} + \begin{pmatrix} -2 \\ 4 \end{pmatrix}\right]$

$= \begin{pmatrix} -1 \\ 5 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 6 \end{pmatrix}$

$= -1(1) + 5(6)$

$= -1 + 30$

$= 29$

d $3\mathbf{r} \bullet \mathbf{q}$

$= 3\begin{pmatrix} -2 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 5 \end{pmatrix}$

$= \begin{pmatrix} -6 \\ 12 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 5 \end{pmatrix}$

$= -6(-1) + 12(5)$

$= 6 + 60$

$= 66$

e $2\mathbf{p} \bullet 2\mathbf{p}$

$= 2\begin{pmatrix} 3 \\ 2 \end{pmatrix} \bullet 2\begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$= \begin{pmatrix} 6 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} 6 \\ 4 \end{pmatrix}$

$= 6(6) + 4(4)$

$= 36 + 16$

$= 52$

f $\mathbf{i} \bullet \mathbf{p}$

$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$= 1(3) + 0(2)$

$= 3 + 0$

$= 3$

g $\mathbf{q} \bullet \mathbf{j} = \begin{pmatrix} -1 \\ 5 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
$= -1(0) + 5(1)$
$= 0 + 5 = 5$

h $\mathbf{i} \bullet \mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
$= 1(1) + 0(0)$
$= 1 + 0 = 1$

2 **a** $\mathbf{a} \bullet \mathbf{b} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$
$= 2(-1) + 1(1) + 3(1)$
$= -2 + 1 + 3$
$= 2$

b $\mathbf{b} \bullet \mathbf{a} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$
$= (-1)(2) + 1(1) + 1(3)$
$= -2 + 1 + 3$
$= 2$

c $|\mathbf{a}|^2 = \left(\sqrt{2^2 + 1^2 + 3^2}\right)^2$
$= 14$

d $\mathbf{a} \bullet \mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$
$= 2(2) + 1(1) + 3(3)$
$= 14$

e $\mathbf{a} \bullet (\mathbf{b} + \mathbf{c})$
$= \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \bullet \left[\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}\right]$
$= \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$
$= 2(-1) + 1(0) + 3(2) = 4$

f $\mathbf{a} \bullet \mathbf{b} + \mathbf{a} \bullet \mathbf{c}$
$= 2 + \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$ {using **a**}
$= 2 + 2(0) + 1(-1) + 3(1)$
$= 4$

3 **a** $\mathbf{p} \bullet \mathbf{q}$
$= \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$
$= 3(-2) + (-1)(1) + 2(3)$
$= -6 - 1 + 6$
$= -1$

b If the angle between $\mathbf{p}$ and $\mathbf{q}$ is θ, then

$$\cos\theta = \frac{\mathbf{p} \bullet \mathbf{q}}{|\mathbf{p}||\mathbf{q}|} = \frac{-1}{\sqrt{3^2 + (-1)^2 + 2^2}\sqrt{(-2)^2 + 1^2 + 3^2}}$$
$$= \frac{-1}{\sqrt{14}\sqrt{14}}$$
$$\therefore \quad \theta = \cos^{-1}(-\tfrac{1}{14}) \approx 94.1^\circ$$

4 **a** If the angle between $\mathbf{m}$ and $\mathbf{n}$ is θ, then

$$\cos\theta = \frac{\mathbf{m} \bullet \mathbf{n}}{|\mathbf{m}||\mathbf{n}|} = \frac{2(-1) + (-1)(3) + (-1)(2)}{\sqrt{2^2 + (-1)^2 + (-1)^2}\sqrt{(-1)^2 + 3^2 + 2^2}}$$
$$= \frac{-7}{\sqrt{6}\sqrt{14}} = -\frac{7}{\sqrt{84}}$$
$$\therefore \quad \theta = \cos^{-1}\left(-\tfrac{7}{\sqrt{84}}\right) \approx 140^\circ$$

b $\mathbf{m} = 2\mathbf{j} - \mathbf{k} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$ and $\mathbf{n} = \mathbf{i} + 2\mathbf{k} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

If the angle between $\mathbf{m}$ and $\mathbf{n}$ is θ, then

$$\cos\theta = \frac{\mathbf{m} \bullet \mathbf{n}}{|\mathbf{m}||\mathbf{n}|} = \frac{(0)(1) + (2)(0) + (-1)(2)}{\sqrt{0^2 + 2^2 + (-1)^2}\sqrt{1^2 + 0^2 + 2^2}}$$
$$= \frac{-2}{\sqrt{5}\sqrt{5}} = -\frac{2}{5}$$
$$\therefore \quad \theta = \cos^{-1}\left(-\tfrac{2}{5}\right) \approx 114^\circ$$

5 **a** $(\mathbf{i}+\mathbf{j}-\mathbf{k}) \bullet (2\mathbf{j}+\mathbf{k})$
$$= \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$$
$$= 1(0) + 1(2) - 1(1)$$
$$= 1$$

b $\mathbf{i} \bullet \mathbf{i}$
$$= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$
$$= 1(1) + 0(0) + 0(0)$$
$$= 1$$

c $\mathbf{i} \bullet \mathbf{j}$
$$= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$
$$= 1(0) + 0(1) + 0(0)$$
$$= 0$$

6 **a** $\mathbf{p} \bullet \mathbf{q} = |\mathbf{p}||\mathbf{q}|\cos\theta$
$$= 2 \times 5 \times \cos 60°$$
$$= 5$$

b $\mathbf{p} \bullet \mathbf{q} = |\mathbf{p}||\mathbf{q}|\cos\theta$
$$= 6 \times 3 \times \cos 120°$$
$$= -9$$

7 **a** **i** If $\mathbf{v}$ and $\mathbf{w}$ are parallel, then they are either in the same direction (so the angle between them is $0°$) or in opposite directions (so the angle between them is $180°$).

If the angle between them is $0°$, then $\mathbf{v} \bullet \mathbf{w} = |\mathbf{v}||\mathbf{w}|\cos 0°$
$$= 3 \times 4 \times 1$$
$$= 12$$

If the angle between them is $180°$, then $\mathbf{v} \bullet \mathbf{w} = |\mathbf{v}||\mathbf{w}|\cos 180°$
$$= 3 \times 4 \times -1$$
$$= -12$$

ii $\mathbf{v} \bullet \mathbf{w} = |\mathbf{v}||\mathbf{w}|\cos 60°$
$$= 3 \times 4 \times \tfrac{1}{2}$$
$$= 6$$

b **i** $\mathbf{a}$ and $\mathbf{b}$ are not perpendicular as their dot product is not equal to 0.

ii $\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$

If $\mathbf{a}$ and $\mathbf{b}$ are parallel, then $\theta = 0$ or 180
$$\therefore \cos\theta = \pm 1$$
$$\therefore -12 = |\mathbf{a}| \times 1 \times \pm 1$$
$$\therefore |\mathbf{a}| = \pm 12 \quad \text{but} \quad |\mathbf{a}| > 0$$
$$\therefore |\mathbf{a}| = 12 \text{ units}$$

{**Note:** This means that $\cos\theta$ must be -1, so the angle between $\mathbf{a}$ and $\mathbf{b}$ is $180°$
$\therefore$ **a** and **b** are in opposite directions.}

c **i** $\mathbf{c} \bullet \mathbf{d} = |\mathbf{c}||\mathbf{d}|\cos\theta$
$$\therefore 5 = \left|\sqrt{5}\right|\left|\sqrt{5}\right|\cos\theta$$
$$\therefore 5 = 5\cos\theta$$
$$\therefore \cos\theta = 1$$
$$\therefore \theta = 0°$$
So, $\mathbf{c} = \mathbf{d}$

ii $\mathbf{c} \bullet \mathbf{d} = |\mathbf{c}||\mathbf{d}|\cos\theta$
$$\therefore -5 = \left|\sqrt{5}\right|\left|\sqrt{5}\right|\cos\theta$$
$$\therefore -5 = 5\cos\theta$$
$$\therefore \cos\theta = -1$$
$$\therefore \theta = 180°$$
So, $\mathbf{c} = -\mathbf{d}$

8 **a** P has coordinates $(\cos\theta, \sin\theta)$.

b $\overrightarrow{BP} = \overrightarrow{BO} + \overrightarrow{OP}$
$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$
$$= \begin{pmatrix} 1+\cos\theta \\ \sin\theta \end{pmatrix} = \begin{pmatrix} \cos\theta + 1 \\ \sin\theta \end{pmatrix}$$

$\overrightarrow{AP} = \overrightarrow{AO} + \overrightarrow{OP}$
$$= \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$
$$= \begin{pmatrix} -1+\cos\theta \\ \sin\theta \end{pmatrix} = \begin{pmatrix} \cos\theta - 1 \\ \sin\theta \end{pmatrix}$$

c $\overrightarrow{AP} \bullet \overrightarrow{BP} = (\cos\theta + 1)(\cos\theta - 1) + \sin^2\theta$
$$= \cos^2\theta - 1 + \sin^2\theta$$
$$= 1 - 1 \qquad \{\cos^2\theta + \sin^2\theta = 1\}$$
$$= 0$$

d $\overrightarrow{AP} \bullet \overrightarrow{BP} = 0$, which means $\overrightarrow{AP}$ and $\overrightarrow{BP}$ are perpendicular.
Now, triangle APB is in a semi-circle, and the angle at P is $90°$.
So, we have deduced that the angle in a semi-circle is a right angle.

9 $\mathbf{a} \bullet (\mathbf{b}+\mathbf{c})$

$$= \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \bullet \left[\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}\right]$$
$$= \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \bullet \begin{pmatrix} b_1+c_1 \\ b_2+c_2 \\ b_3+c_3 \end{pmatrix}$$
$$= a_1(b_1+c_1) + a_2(b_2+c_2) + a_3(b_3+c_3)$$
$$= a_1b_1 + a_1c_1 + a_2b_2 + a_2c_2 + a_3b_3 + a_3c_3$$
$$= (a_1b_1 + a_2b_2 + a_3b_3) + (a_1c_1 + a_2c_2 + a_3c_3)$$
$$= \mathbf{a} \bullet \mathbf{b} + \mathbf{a} \bullet \mathbf{c}$$

$\therefore\ \mathbf{p} \bullet (\mathbf{c}+\mathbf{d}) = \mathbf{p} \bullet \mathbf{c} + \mathbf{p} \bullet \mathbf{d}$

If we let $\mathbf{p} = \mathbf{a}+\mathbf{b}$,

then
$$\begin{aligned} &(\mathbf{a}+\mathbf{b}) \bullet (\mathbf{c}+\mathbf{d}) \\ &= \mathbf{p} \bullet (\mathbf{c}+\mathbf{d}) \\ &= \mathbf{p} \bullet \mathbf{c} + \mathbf{p} \bullet \mathbf{d} \\ &= (\mathbf{a}+\mathbf{b}) \bullet \mathbf{c} + (\mathbf{a}+\mathbf{b}) \bullet \mathbf{d} \\ &= \mathbf{c} \bullet (\mathbf{a}+\mathbf{b}) + \mathbf{d} \bullet (\mathbf{a}+\mathbf{b}) \\ &= \mathbf{c} \bullet \mathbf{a} + \mathbf{c} \bullet \mathbf{b} + \mathbf{d} \bullet \mathbf{a} + \mathbf{d} \bullet \mathbf{b} \\ &= \mathbf{a} \bullet \mathbf{c} + \mathbf{a} \bullet \mathbf{d} + \mathbf{b} \bullet \mathbf{c} + \mathbf{b} \bullet \mathbf{d} \end{aligned}$$

10 **a** **i** $\begin{pmatrix} 3 \\ t \end{pmatrix} \bullet \begin{pmatrix} -2 \\ 1 \end{pmatrix} = 0$

$\therefore\ -6+t=0$

$\therefore\ t=6$

ii If $\mathbf{p} \parallel \mathbf{q}$ then $\begin{pmatrix} 3 \\ t \end{pmatrix} = k\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

where $k \neq 0$

$\therefore\ 3=-2k$ and $t=k$

$\therefore\ k=-\frac{3}{2}$ and $t=-\frac{3}{2}$

b **i** $\begin{pmatrix} t \\ t+2 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ -4 \end{pmatrix} = 0$

$\therefore\ 3t-4(t+2)=0$

$\therefore\ 3t-4t-8=0$

$\therefore\ -t=8$

$\therefore\ t=-8$

ii If $\mathbf{r} \parallel \mathbf{s}$ then $\begin{pmatrix} t \\ t+2 \end{pmatrix} = k\begin{pmatrix} 3 \\ -4 \end{pmatrix}$

where $k \neq 0$

$\therefore\ t=3k$ and $t+2=-4k$

$\therefore\ t+2=-4\left(\frac{t}{3}\right)$

$\therefore\ 3t+6=-4t$

$\therefore\ 7t=-6$

$\therefore\ t=-\frac{6}{7}$

c **i** $\begin{pmatrix} t \\ t+2 \end{pmatrix} \bullet \begin{pmatrix} 2-3t \\ t \end{pmatrix} = 0$

$\therefore\ 2t-3t^2+t^2+2t=0$

$\therefore\ -2t^2+4t=0$

$\therefore\ t^2-2t=0$

$\therefore\ t(t-2)=0$

$\therefore\ t=0$ or 2

ii If $\mathbf{a} \parallel \mathbf{b}$ then $\begin{pmatrix} t \\ t+2 \end{pmatrix} = k\begin{pmatrix} 2-3t \\ t \end{pmatrix}$

$\therefore\ t=k(2-3t)$ and $t+2=kt$

$\therefore\ \dfrac{t}{2-3t} = \dfrac{t+2}{t}$ {equating ks}

$\therefore\ t^2=(t+2)(2-3t)$

$\therefore\ t^2=2t-3t^2+4-6t$

$\therefore\ 4t^2+4t-4=0$

$\therefore\ t^2+t-1=0$

which has $\Delta = 1^2-4(1)(-1)=5$

$\therefore\ t=\dfrac{-1\pm\sqrt{5}}{2}$

11 **a** $\begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = 1(2)+1(3)+5(-1)=0$

$\therefore\ \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$ are perpendicular.

b $\mathbf{a} \bullet \mathbf{b}$

$= \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

$= 3(-1) + 1(1) + 2(1)$

$= 0$

$\mathbf{b} \bullet \mathbf{c}$

$= \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 5 \\ -4 \end{pmatrix}$

$= (-1)(1) + 1(5) + 1(-4)$

$= 0$

$\mathbf{a} \bullet \mathbf{c}$

$= \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 5 \\ -4 \end{pmatrix}$

$= (3)(1) + 1(5) + 2(-4)$

$= 0$

$\therefore$ **a**, **b**, and **c** are mutually perpendicular.

c **i**

$$\begin{pmatrix} 3 \\ -1 \\ t \end{pmatrix} \bullet \begin{pmatrix} 2t \\ -3 \\ -4 \end{pmatrix} = 0$$

$\therefore\ 3(2t) + (-1)(-3) + t(-4) = 0$

$\therefore\ 6t + 3 - 4t = 0$

$\therefore\ 2t + 3 = 0$

$\therefore\ t = -\frac{3}{2}$

ii

$$\begin{pmatrix} 3 \\ t \\ -2 \end{pmatrix} \bullet \begin{pmatrix} 1-t \\ -3 \\ 4 \end{pmatrix} = 0$$

$\therefore\ 3(1-t) + t(-3) + (-2)4 = 0$

$\therefore\ 3 - 3t - 3t - 8 = 0$

$\therefore\ -6t = 5$

$\therefore\ t = -\frac{5}{6}$

12 **a** We have three points: A$(-2, 1)$, B$(-2, 5)$, C$(3, 1)$.

Then $\overrightarrow{AB} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$, $\overrightarrow{AC} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$, and $\overrightarrow{BC} = \begin{pmatrix} 5 \\ -4 \end{pmatrix}$

Now $\overrightarrow{AB} \bullet \overrightarrow{AC} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} 5 \\ 0 \end{pmatrix} = 0 + 0 = 0$

$\therefore$ $\overrightarrow{AB}$ is perpendicular to $\overrightarrow{AC}$ and so $\triangle$ABC is right angled at A.

b We have three points: A$(4, 7)$, B$(1, 2)$, C$(-1, 6)$

Then $\overrightarrow{AB} = \begin{pmatrix} -3 \\ -5 \end{pmatrix}$, $\overrightarrow{AC} = \begin{pmatrix} -5 \\ -1 \end{pmatrix}$, and $\overrightarrow{BC} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$

Now $\overrightarrow{AB} \bullet \overrightarrow{AC} = \begin{pmatrix} -3 \\ -5 \end{pmatrix} \bullet \begin{pmatrix} -5 \\ -1 \end{pmatrix} = 15 + 5 = 20$

$\overrightarrow{AB} \bullet \overrightarrow{BC} = \begin{pmatrix} -3 \\ -5 \end{pmatrix} \bullet \begin{pmatrix} -2 \\ 4 \end{pmatrix} = 6 + (-20) = -14$

$\overrightarrow{AC} \bullet \overrightarrow{BC} = \begin{pmatrix} -5 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} -2 \\ 4 \end{pmatrix} = 10 + (-4) = 6$

$\therefore$ none of the sides are perpendicular to each other and so $\triangle$ABC is not right angled.

c We have three points: A$(2, -2)$, B$(5, 7)$, C$(-1, -1)$

Then $\overrightarrow{AB} = \begin{pmatrix} 3 \\ 9 \end{pmatrix}$, $\overrightarrow{AC} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$, and $\overrightarrow{BC} = \begin{pmatrix} -6 \\ -8 \end{pmatrix}$

Now $\overrightarrow{AB} \bullet \overrightarrow{AC} = \begin{pmatrix} 3 \\ 9 \end{pmatrix} \bullet \begin{pmatrix} -3 \\ 1 \end{pmatrix} = -9 + 9 = 0$

$\therefore$ $\overrightarrow{AB}$ is perpendicular to $\overrightarrow{AC}$ and so $\triangle$ABC is right angled at A.

d We have three points: A$(10, 1)$, B$(5, 2)$, C$(7, 4)$

Then $\overrightarrow{AB} = \begin{pmatrix} -5 \\ 1 \end{pmatrix}$, $\overrightarrow{AC} = \begin{pmatrix} -3 \\ 3 \end{pmatrix}$, and $\overrightarrow{BC} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

Now $\overrightarrow{AC} \bullet \overrightarrow{BC} = \begin{pmatrix} -3 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 2 \end{pmatrix} = -6 + 6 = 0$

$\therefore$ $\overrightarrow{AC}$ is perpendicular to $\overrightarrow{BC}$ and so $\triangle$ABC is right angled at C.

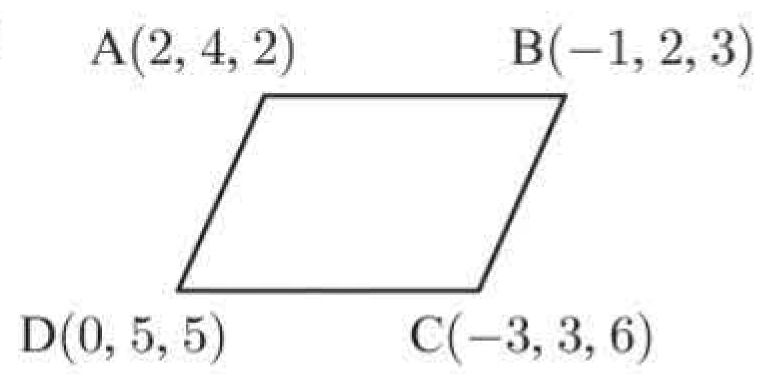
13 We have three points: $A(5, 1, 2)$, $B(6, -1, 0)$, $C(3, 2, 0)$

Then $\overrightarrow{AB} = \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$, $\overrightarrow{AC} = \begin{pmatrix} -2 \\ 1 \\ -2 \end{pmatrix}$, and $\overrightarrow{BC} = \begin{pmatrix} -3 \\ 3 \\ 0 \end{pmatrix}$

Now $\overrightarrow{AB} \bullet \overrightarrow{AC} = \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} -2 \\ 1 \\ -2 \end{pmatrix} = (-2) + (-2) + 4 = 0$

$\therefore$ $\overrightarrow{AB}$ is perpendicular to $\overrightarrow{AC}$ and so $\triangle ABC$ is right angled at A.

14 **a**



$\overrightarrow{AB} = \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix}$, $\overrightarrow{BC} = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$ $\therefore$ $\overrightarrow{AB}$ is parallel to $\overrightarrow{DC}$ and $\overrightarrow{BC}$ is parallel to $\overrightarrow{AD}$.

$\overrightarrow{DC} = \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix}$, $\overrightarrow{AD} = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$ $\therefore$ ABCD is a parallelogram.

b $|\overrightarrow{AB}| = \sqrt{(-3)^2 + (-2)^2 + 1^2} = \sqrt{9+4+1} = \sqrt{14}$ units

and $|\overrightarrow{BC}| = \sqrt{(-2)^2 + 1^2 + 3^2} = \sqrt{4+1+9} = \sqrt{14}$ units

$\therefore$ ABCD is a rhombus.

c $\overrightarrow{AC} \bullet \overrightarrow{BD} = \begin{pmatrix} -5 \\ -1 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = (-5)(1) + (-1)(3) + 4(2) = 0$

$\therefore$ $\overrightarrow{AC}$ is perpendicular to $\overrightarrow{BD}$ which illustrates that the diagonals of a rhombus are perpendicular.

15 **a** $\begin{pmatrix} 5 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} -2 \\ 5 \end{pmatrix} = -10 + 10 = 0$, so $\begin{pmatrix} -2 \\ 5 \end{pmatrix}$ is one such vector.

$\therefore$ required vectors have form $k\begin{pmatrix} -2 \\ 5 \end{pmatrix}$, $k \neq 0$.

b $\begin{pmatrix} -1 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ -1 \end{pmatrix} = -2 + 2 = 0$, so $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ is one such vector.

$\therefore$ required vectors have form $k\begin{pmatrix} 2 \\ -1 \end{pmatrix}$, $k \neq 0$.

c $\begin{pmatrix} 3 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 3 \end{pmatrix} = 3 - 3 = 0$, so $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ is one such vector.

$\therefore$ required vectors have form $k\begin{pmatrix} 1 \\ 3 \end{pmatrix}$, $k \neq 0$.

d $\begin{pmatrix} -4 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 4 \end{pmatrix} = -12 + 12 = 0$, so $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ is one such vector.

$\therefore$ required vectors have form $k\begin{pmatrix} 3 \\ 4 \end{pmatrix}$, $k \neq 0$.

e $\begin{pmatrix} 2 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0 + 0 = 0$, so $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is one such vector.

$\therefore$ required vectors have form $k\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $k \neq 0$.

16 Suppose $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is perpendicular to $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$. $\therefore \begin{pmatrix} a \\ b \\ c \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = 0$

So, to find a vector perpendicular to $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$, we pick two non-zero integer values for a and b, then solve for c.

For example, if $a = 1,\ b = 2$

then $\begin{pmatrix} 1 \\ 2 \\ c \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = 0$

$\therefore\ 1 + 4 - c = 0$

$\therefore\ 5 - c = 0$

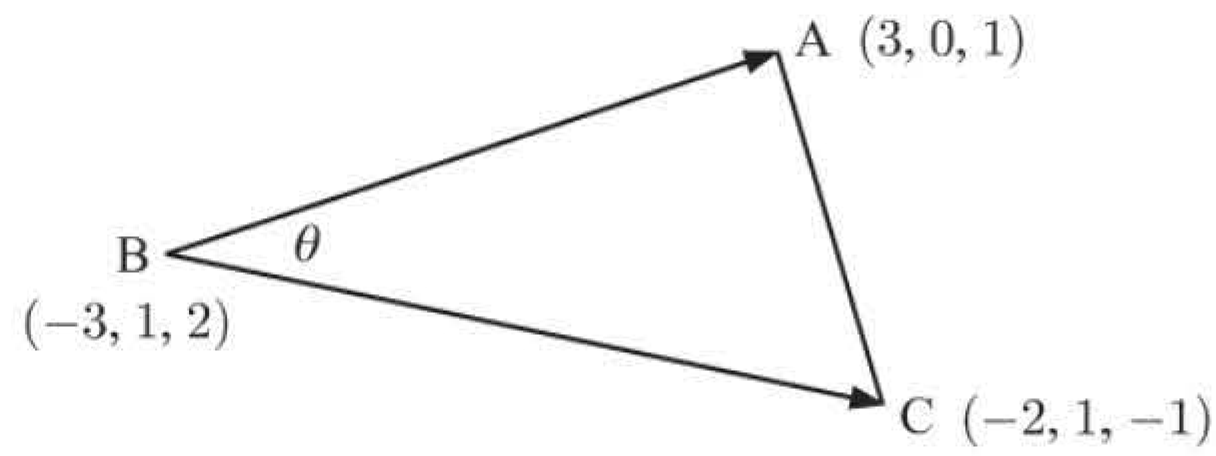
$\therefore\ c = 5$

So, the vector $\begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$ is perpendicular to $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$.

Repeating this process with a different value of a (or b) will give another vector which is perpendicular to $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$.

17 Given $A(3, 0, 1)$, $B(-3, 1, 2)$, and $C(-2, 1, -1)$,

$\overrightarrow{BC} = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}$ and $\overrightarrow{BA} = \begin{pmatrix} 6 \\ -1 \\ -1 \end{pmatrix}$

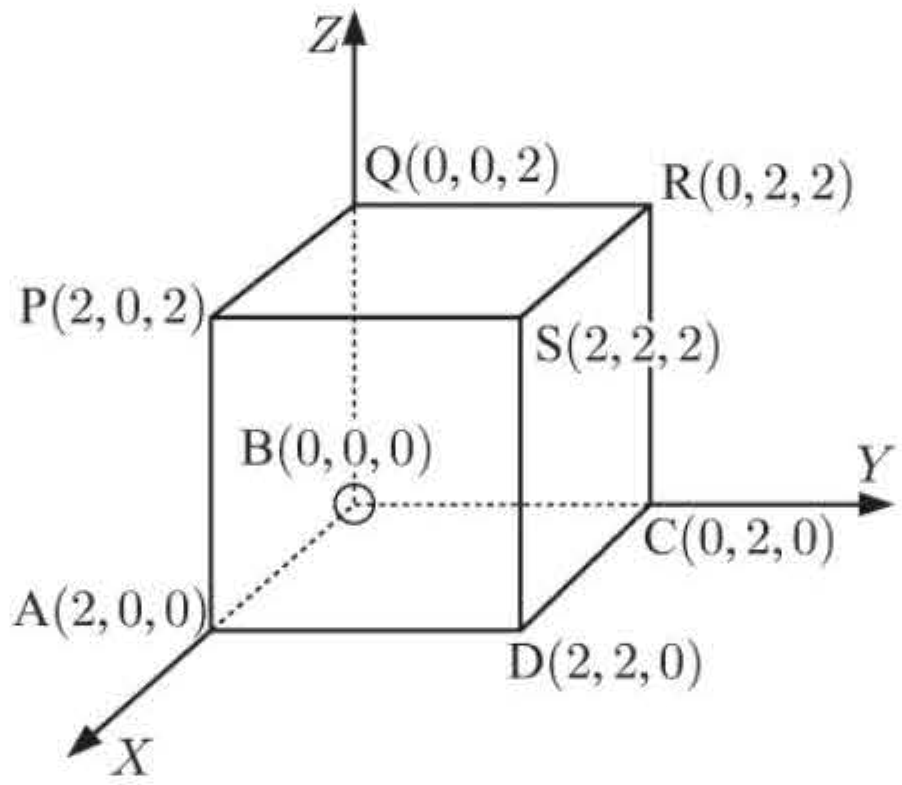


$$\therefore \cos\theta = \frac{\overrightarrow{BC} \bullet \overrightarrow{BA}}{|\overrightarrow{BC}||\overrightarrow{BA}|} = \frac{\begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} \bullet \begin{pmatrix} 6 \\ -1 \\ -1 \end{pmatrix}}{\sqrt{1+9}\sqrt{36+1+1}} = \frac{6+0+3}{\sqrt{10}\sqrt{38}} = \frac{9}{\sqrt{380}}$$

$\therefore\ \theta \approx 62.5°$

If $\overrightarrow{BA}$ and $\overrightarrow{CB}$ are used we would find the exterior angle of the triangle at B, which is $117.5°$.

18



a Suppose the origin is at B.

Now $\overrightarrow{BA} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$ and $\overrightarrow{BS} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$

$\therefore\ \overrightarrow{BA} \bullet \overrightarrow{BS} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} = 4 + 0 + 0 = 4$

$$\therefore \cos A\widehat{B}S = \frac{4}{\sqrt{4+0+0}\sqrt{4+4+4}} = \frac{4}{2 \times 2\sqrt{3}} = \frac{1}{\sqrt{3}}$$

$\therefore\ A\widehat{B}S \approx 54.7°$

b Consider vectors away from B.

$\overrightarrow{BR} = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$ and $\overrightarrow{BP} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$

$\therefore\ \overrightarrow{BR} \bullet \overrightarrow{BP} = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} = 0 + 0 + 4 = 4$

$$\therefore \cos R\widehat{B}P = \frac{4}{\sqrt{0+4+4}\sqrt{4+0+4}} = \frac{4}{\sqrt{8} \times \sqrt{8}} = \tfrac{1}{2} \text{ and so } R\widehat{B}P = 60°$$

c $\overrightarrow{BP} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$ and $\overrightarrow{BS} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$

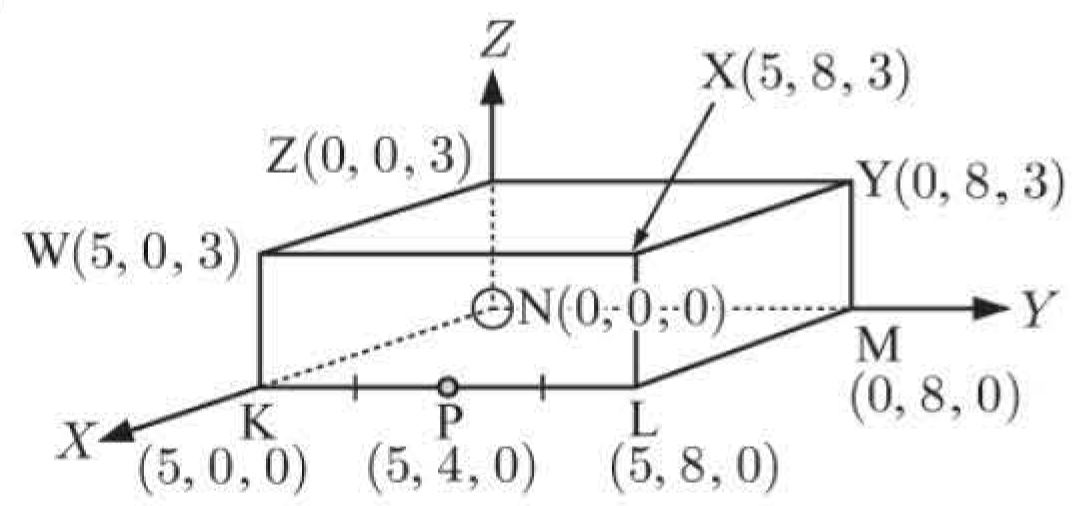
$\therefore\ \overrightarrow{BP} \bullet \overrightarrow{BS} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} = 4 + 0 + 4 = 8$

$$\therefore \cos P\widehat{B}S = \frac{8}{\sqrt{4+4}\sqrt{4+4+4}} = \frac{8}{\sqrt{96}}$$

$\therefore\ P\widehat{B}S \approx 35.3°$

19 Suppose the origin is at N.

a



$$\overrightarrow{NY} = \begin{pmatrix} 0 \\ 8 \\ 3 \end{pmatrix} \quad \text{and} \quad \overrightarrow{NX} = \begin{pmatrix} 5 \\ 8 \\ 3 \end{pmatrix}$$

$$\overrightarrow{NY} \bullet \overrightarrow{NX} = \begin{pmatrix} 0 \\ 8 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 5 \\ 8 \\ 3 \end{pmatrix} = 0 + 64 + 9 = 73$$

$$\therefore \cos Y\widehat{N}X = \frac{73}{\sqrt{64+9}\sqrt{25+64+9}}$$

$$= \frac{73}{\sqrt{73}\sqrt{98}} = \sqrt{\tfrac{73}{98}}$$

$$\therefore Y\widehat{N}X \approx 30.3°$$

b $\overrightarrow{NY} = \begin{pmatrix} 0 \\ 8 \\ 3 \end{pmatrix}$ and $\overrightarrow{NP} = \begin{pmatrix} 5 \\ 4 \\ 0 \end{pmatrix}$

$$\overrightarrow{NY} \bullet \overrightarrow{NP} = \begin{pmatrix} 0 \\ 8 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 5 \\ 4 \\ 0 \end{pmatrix}$$

$$= 0 + 32 + 0$$

$$= 32$$

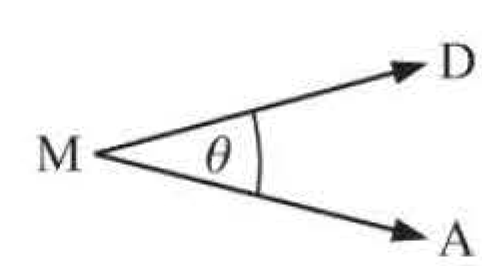
$$\therefore \cos Y\widehat{N}P = \frac{32}{\sqrt{64+9}\sqrt{25+16}}$$

$$= \frac{32}{\sqrt{73}\sqrt{41}}$$

$$\therefore Y\widehat{N}P \approx 54.2°$$

20 **a** M is the midpoint of [BC]. $\therefore$ M is at $\left(\frac{2+1}{2}, \frac{2+3}{2}, \frac{2+1}{2}\right)$, which is $\left(\frac{3}{2}, \frac{5}{2}, \frac{3}{2}\right)$.

b Now $\overrightarrow{MD} = \begin{pmatrix} \frac{3}{2} \\ -\frac{1}{2} \\ -\frac{3}{2} \end{pmatrix}$ and $\overrightarrow{MA} = \begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \\ -\frac{1}{2} \end{pmatrix}$



$$\therefore \cos\theta = \frac{\overrightarrow{MD} \bullet \overrightarrow{MA}}{|\overrightarrow{MD}||\overrightarrow{MA}|} = \frac{\begin{pmatrix} \frac{3}{2} \\ -\frac{1}{2} \\ -\frac{3}{2} \end{pmatrix} \bullet \begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \\ -\frac{1}{2} \end{pmatrix}}{\sqrt{\frac{9}{4}+\frac{1}{4}+\frac{9}{4}}\sqrt{\frac{1}{4}+\frac{9}{4}+\frac{1}{4}}}$$

$$\therefore \cos\theta = \frac{\frac{3}{4}+\frac{3}{4}+\frac{3}{4}}{\sqrt{\frac{19}{4}}\sqrt{\frac{11}{4}}} = \frac{\frac{9}{4}}{\frac{\sqrt{209}}{4}} = \frac{9}{\sqrt{209}} \quad \text{and so} \quad \theta \approx 51.5°$$

21 **a** $\begin{pmatrix} 2 \\ t \\ t-2 \end{pmatrix} \bullet \begin{pmatrix} t \\ 3 \\ t \end{pmatrix} = 0 \quad \therefore \; 2t + 3t + t(t-2) = 0$

$$\therefore 5t + t^2 - 2t = 0$$

$$\therefore t^2 + 3t = 0$$

$$\therefore t(t+3) = 0 \text{ and so } t = 0 \text{ or } t = -3$$

b Given that $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 2 \\ 2 \\ r \end{pmatrix}$, and $\mathbf{c} = \begin{pmatrix} s \\ t \\ 1 \end{pmatrix}$ are mutually perpendicular,

$\mathbf{a} \bullet \mathbf{b} = 0$, $\mathbf{b} \bullet \mathbf{c} = 0$, and $\mathbf{a} \bullet \mathbf{c} = 0$

$$\therefore \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 2 \\ r \end{pmatrix} = 0 \qquad \therefore 2 + 4 + 3r = 0$$

$$\therefore 3r = -6$$

$$\therefore r = -2$$

and $\begin{pmatrix} 2 \\ 2 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} s \\ t \\ 1 \end{pmatrix} = 0 \qquad \therefore 2s + 2t - 2 = 0$

$$\therefore s + t = 1 \quad .... (1)$$

and $\begin{pmatrix}1\\2\\3\end{pmatrix} \bullet \begin{pmatrix}s\\t\\1\end{pmatrix} = 0 \qquad \therefore \quad s + 2t + 3 = 0$

$\therefore \quad s + 2t = -3 \quad .... (2)$

$(2) - (1)$ gives $t = -4$ and so $s = 5$

$\therefore \quad r = -2, \quad s = 5, \quad$ and $\quad t = -4$

22 **a** Let θ be the angle between the vectors $\mathbf{i} = \begin{pmatrix}1\\0\\0\end{pmatrix}$ and $\begin{pmatrix}1\\2\\3\end{pmatrix}$.

Then $\cos\theta = \dfrac{\begin{pmatrix}1\\0\\0\end{pmatrix} \bullet \begin{pmatrix}1\\2\\3\end{pmatrix}}{\left|\begin{pmatrix}1\\0\\0\end{pmatrix}\right|\left|\begin{pmatrix}1\\2\\3\end{pmatrix}\right|} = \dfrac{1}{\sqrt{1}\sqrt{1+4+9}} = \dfrac{1}{\sqrt{14}}$ and so $\theta \approx 74.5°$.

b Let θ be the angle between the vectors $\mathbf{j} = \begin{pmatrix}0\\1\\0\end{pmatrix}$ and $\begin{pmatrix}-1\\1\\3\end{pmatrix}$.

Then $\cos\theta = \dfrac{\begin{pmatrix}0\\1\\0\end{pmatrix} \bullet \begin{pmatrix}-1\\1\\3\end{pmatrix}}{\left|\begin{pmatrix}0\\1\\0\end{pmatrix}\right|\left|\begin{pmatrix}-1\\1\\3\end{pmatrix}\right|} = \dfrac{1}{\sqrt{1}\sqrt{1+1+9}} = \dfrac{1}{\sqrt{11}}$ and so $\theta \approx 72.5°$.

23 For example let $\mathbf{a} = \mathbf{i}$, $\mathbf{b} = \mathbf{j}$, and $\mathbf{c} = \mathbf{k}$

$\mathbf{i} \bullet \mathbf{j} = \mathbf{i} \bullet \mathbf{k} = 0$ and $\mathbf{j} \neq \mathbf{k}$

24 **a** Show $|\mathbf{a}+\mathbf{b}|^2 + |\mathbf{a}-\mathbf{b}|^2 = 2|\mathbf{a}|^2 + 2|\mathbf{b}|^2$ using $|\mathbf{x}|^2 = \mathbf{x} \bullet \mathbf{x}$

$$\begin{aligned}
\text{LHS} &= (\mathbf{a}+\mathbf{b}) \bullet (\mathbf{a}+\mathbf{b}) + (\mathbf{a}-\mathbf{b}) \bullet (\mathbf{a}-\mathbf{b})\\
&= \mathbf{a}\bullet\mathbf{a} + \cancel{\mathbf{a}\bullet\mathbf{b}} + \cancel{\mathbf{b}\bullet\mathbf{a}} + \mathbf{b}\bullet\mathbf{b} + \mathbf{a}\bullet\mathbf{a} - \cancel{\mathbf{a}\bullet\mathbf{b}} - \cancel{\mathbf{b}\bullet\mathbf{a}} + \mathbf{b}\bullet\mathbf{b}\\
&= 2\mathbf{a}\bullet\mathbf{a} + 2\mathbf{b}\bullet\mathbf{b}\\
&= 2|\mathbf{a}|^2 + 2|\mathbf{b}|^2\\
&= \text{RHS}
\end{aligned}$$

b Show $|\mathbf{a}+\mathbf{b}|^2 - |\mathbf{a}-\mathbf{b}|^2 = 4\mathbf{a} \bullet \mathbf{b}$ using $|\mathbf{x}|^2 = \mathbf{x} \bullet \mathbf{x}$

$$\begin{aligned}
\text{LHS} &= (\mathbf{a}+\mathbf{b}) \bullet (\mathbf{a}+\mathbf{b}) - (\mathbf{a}-\mathbf{b}) \bullet (\mathbf{a}-\mathbf{b})\\
&= \mathbf{a}\bullet\mathbf{a} + \mathbf{a}\bullet\mathbf{b} + \mathbf{b}\bullet\mathbf{a} + \mathbf{b}\bullet\mathbf{b} - (\mathbf{a}\bullet\mathbf{a} - \mathbf{a}\bullet\mathbf{b} - \mathbf{b}\bullet\mathbf{a} + \mathbf{b}\bullet\mathbf{b})\\
&= \cancel{\mathbf{a}\bullet\mathbf{a}} + \mathbf{a}\bullet\mathbf{b} + \mathbf{b}\bullet\mathbf{a} + \cancel{\mathbf{b}\bullet\mathbf{b}} - \cancel{\mathbf{a}\bullet\mathbf{a}} + \mathbf{a}\bullet\mathbf{b} + \mathbf{b}\bullet\mathbf{a} - \cancel{\mathbf{b}\bullet\mathbf{b}}\\
&= \mathbf{a}\bullet\mathbf{b} + \mathbf{a}\bullet\mathbf{b} + \mathbf{a}\bullet\mathbf{b} + \mathbf{a}\bullet\mathbf{b}\\
&= 4\mathbf{a}\bullet\mathbf{b}\\
&= \text{RHS}
\end{aligned}$$

25 **a**

If $|\mathbf{a}+\mathbf{b}| = |\mathbf{a}-\mathbf{b}|$

then $|\mathbf{a}+\mathbf{b}|^2 = |\mathbf{a}-\mathbf{b}|^2$

$\therefore \quad (\mathbf{a}+\mathbf{b}) \bullet (\mathbf{a}+\mathbf{b}) = (\mathbf{a}-\mathbf{b}) \bullet (\mathbf{a}-\mathbf{b})$

$\therefore \quad \cancel{\mathbf{a}\bullet\mathbf{a}} + \mathbf{a}\bullet\mathbf{b} + \mathbf{b}\bullet\mathbf{a} + \cancel{\mathbf{b}\bullet\mathbf{b}} = \cancel{\mathbf{a}\bullet\mathbf{a}} - \mathbf{a}\bullet\mathbf{b} - \mathbf{b}\bullet\mathbf{a} + \cancel{\mathbf{b}\bullet\mathbf{b}}$

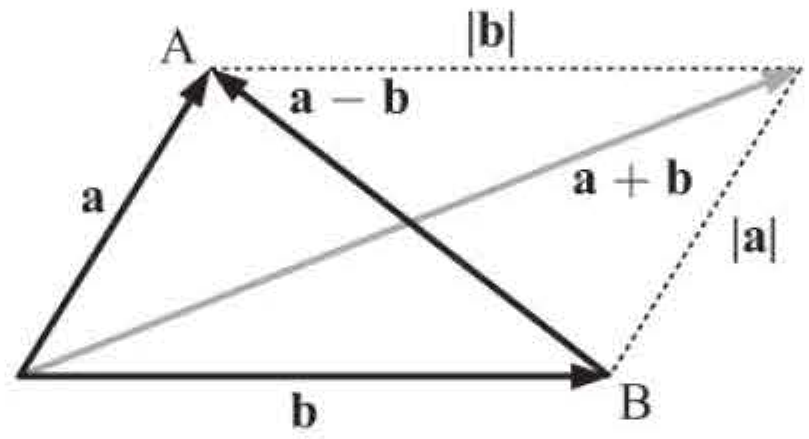
$\therefore \quad \mathbf{a}\bullet\mathbf{b} + \mathbf{a}\bullet\mathbf{b} + \mathbf{a}\bullet\mathbf{b} + \mathbf{a}\bullet\mathbf{b} = 0$

$\therefore \quad 4\mathbf{a}\bullet\mathbf{b} = 0$

$\therefore \quad \mathbf{a}\bullet\mathbf{b} = 0$

$\therefore$ **a** and **b** are perpendicular.

b



As shown $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$ are the diagonals of a parallelogram with side lengths $|\mathbf{a}|$ and $|\mathbf{b}|$.

Now if $|\mathbf{a}+\mathbf{b}| = |\mathbf{a}-\mathbf{b}|$ then the parallelogram must be a square, and **a** is perpendicular to **b**.

26 $(\mathbf{a}+\mathbf{b}) \bullet (\mathbf{a}-\mathbf{b})$
$= \mathbf{a} \bullet \mathbf{a} - \cancel{\mathbf{a} \bullet \mathbf{b}} + \cancel{\mathbf{b} \bullet \mathbf{a}} - \mathbf{b} \bullet \mathbf{b}$
$= |\mathbf{a}|^2 - |\mathbf{b}|^2$
$= 3^2 - 4^2$
$= 9 - 16$
$= -7$

27 The dot product is only defined for two vectors. For $\mathbf{a} \bullet \mathbf{b} \bullet \mathbf{c}$, the result of $\mathbf{a} \bullet \mathbf{b}$ is a scalar, and the dot product of the scalar and $\mathbf{c}$ is meaningless.

EXERCISE 14K.1

1 **a**
$$\begin{aligned}
\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & 1 \\ 1 & 4 & -2 \end{vmatrix} \\
&= \begin{vmatrix} -3 & 1 \\ 4 & -2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & -3 \\ 1 & 4 \end{vmatrix} \mathbf{k} \\
&= ((-3)\times(-2) - 1\times 4)\mathbf{i} - (2\times(-2) - 1\times 1)\mathbf{j} + (2\times 4 - (-3)\times 1)\mathbf{k} \\
&= (6-4)\mathbf{i} - (-4-1)\mathbf{j} + (8+3)\mathbf{k} \\
&= 2\mathbf{i} - (-5)\mathbf{j} + 11\mathbf{k} \\
&= \begin{pmatrix} 2 \\ 5 \\ 11 \end{pmatrix}
\end{aligned}$$

b
$$\begin{aligned}
\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} \times \begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 0 & 2 \\ 3 & -1 & -2 \end{vmatrix} \\
&= \begin{vmatrix} 0 & 2 \\ -1 & -2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -1 & 2 \\ 3 & -2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -1 & 0 \\ 3 & -1 \end{vmatrix} \mathbf{k} \\
&= (0\times(-2) - 2\times(-1))\mathbf{i} - ((-1)\times(-2) - 2\times 3)\mathbf{j} \\
&\qquad + ((-1)\times(-1) - 0\times 3)\mathbf{k} \\
&= (0+2)\mathbf{i} - (2-6)\mathbf{j} + (1-0)\mathbf{k} \\
&= 2\mathbf{i} - (-4)\mathbf{j} + \mathbf{k} \\
&= \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}
\end{aligned}$$

c
$$\begin{aligned}
(\mathbf{i}+\mathbf{j}-2\mathbf{k}) \times (\mathbf{i}-\mathbf{k}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -2 \\ 1 & 0 & -1 \end{vmatrix} \\
&= \begin{vmatrix} 1 & -2 \\ 0 & -1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & -2 \\ 1 & -1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} \mathbf{k} \\
&= (1\times(-1) - (-2)\times 0)\mathbf{i} - (1\times(-1) - (-2)\times 1)\mathbf{j} \\
&\qquad + (1\times 0 - 1\times 1)\mathbf{k} \\
&= (-1-0)\mathbf{i} - (-1+2)\mathbf{j} + (0-1)\mathbf{k} \\
&= -\mathbf{i} - \mathbf{j} - \mathbf{k}
\end{aligned}$$

d
$$\begin{aligned}
(2\mathbf{i}-\mathbf{k}) \times (\mathbf{j}+3\mathbf{k}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -1 \\ 0 & 1 & 3 \end{vmatrix} \\
&= \begin{vmatrix} 0 & -1 \\ 1 & 3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & -1 \\ 0 & 3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} \mathbf{k} \\
&= (0\times 3 - (-1)\times 1)\mathbf{i} - (2\times 3 - (-1)\times 0)\mathbf{j} + (2\times 1 - 0\times 0)\mathbf{k} \\
&= (0+1)\mathbf{i} - (6-0)\mathbf{j} + (2-0)\mathbf{k} \\
&= \mathbf{i} - 6\mathbf{j} + 2\mathbf{k}
\end{aligned}$$

2 **a** $$\begin{pmatrix}1\\2\\3\end{pmatrix}\times\begin{pmatrix}-1\\3\\-1\end{pmatrix}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&2&3\\-1&3&-1\end{vmatrix}$$
$$=\begin{vmatrix}2&3\\3&-1\end{vmatrix}\mathbf{i}-\begin{vmatrix}1&3\\-1&-1\end{vmatrix}\mathbf{j}+\begin{vmatrix}1&2\\-1&3\end{vmatrix}\mathbf{k}$$
$$=(2\times(-1)-3\times3)\mathbf{i}-(1\times(-1)-(-1)\times3)\mathbf{j}+(1\times3-(-1)\times2)\mathbf{k}$$
$$=(-2-9)\mathbf{i}-(-1+3)\mathbf{j}+(3+2)\mathbf{k}$$
$$=-11\mathbf{i}-2\mathbf{j}+5\mathbf{k}$$
$$=\begin{pmatrix}-11\\-2\\5\end{pmatrix}$$

b $$\mathbf{a}\bullet(\mathbf{a}\times\mathbf{b})=\begin{pmatrix}1\\2\\3\end{pmatrix}\bullet\begin{pmatrix}-11\\-2\\5\end{pmatrix}$$
$$=-11-4+15$$
$$=0$$

$$\mathbf{b}\bullet(\mathbf{a}\times\mathbf{b})=\begin{pmatrix}-1\\3\\-1\end{pmatrix}\bullet\begin{pmatrix}-11\\-2\\5\end{pmatrix}$$
$$=11-6-5$$
$$=0$$

$\therefore\ \mathbf{a}\bullet(\mathbf{a}\times\mathbf{b})=0=\mathbf{b}\bullet(\mathbf{a}\times\mathbf{b})$

c $\mathbf{a}\times\mathbf{b}$ is a vector perpendicular to both **a** and **b**.

3 **a** $$\mathbf{i}\times\mathbf{i}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&0&0\\1&0&0\end{vmatrix}$$
$$=\begin{vmatrix}0&0\\0&0\end{vmatrix}\mathbf{i}-\begin{vmatrix}1&0\\1&0\end{vmatrix}\mathbf{j}+\begin{vmatrix}1&0\\1&0\end{vmatrix}\mathbf{k}$$
$$=(0\times0-0\times0)\mathbf{i}-(1\times0-0\times1)\mathbf{j}+(1\times0-0\times1)\mathbf{k}$$
$$=\mathbf{0}$$

$$\mathbf{j}\times\mathbf{j}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\0&1&0\\0&1&0\end{vmatrix}$$
$$=\begin{vmatrix}1&0\\1&0\end{vmatrix}\mathbf{i}-\begin{vmatrix}0&0\\0&0\end{vmatrix}\mathbf{j}+\begin{vmatrix}0&1\\0&1\end{vmatrix}\mathbf{k}$$
$$=(1\times0-0\times1)\mathbf{i}-(0\times0-0\times0)\mathbf{j}+(0\times1-1\times0)\mathbf{k}$$
$$=\mathbf{0}$$

$$\mathbf{k}\times\mathbf{k}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\0&0&1\\0&0&1\end{vmatrix}$$
$$=\begin{vmatrix}0&1\\0&1\end{vmatrix}\mathbf{i}-\begin{vmatrix}0&1\\0&1\end{vmatrix}\mathbf{j}+\begin{vmatrix}0&0\\0&0\end{vmatrix}\mathbf{k}$$
$$=(0\times1-1\times0)\mathbf{i}-(0\times1-1\times0)\mathbf{j}+(0\times0-0\times0)\mathbf{k}$$
$$=\mathbf{0}$$

$\mathbf{a}\times\mathbf{a}=\mathbf{0}$ for all vectors **a**.

b **i** $$\mathbf{i}\times\mathbf{j}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&0&0\\0&1&0\end{vmatrix}$$
$$=\begin{vmatrix}0&0\\1&0\end{vmatrix}\mathbf{i}-\begin{vmatrix}1&0\\0&0\end{vmatrix}\mathbf{j}+\begin{vmatrix}1&0\\0&1\end{vmatrix}\mathbf{k}$$
$$=(0\times0-0\times1)\mathbf{i}-(1\times0-0\times0)\mathbf{j}+(1\times1-0\times0)\mathbf{k}$$
$$=\mathbf{k}$$

$$\mathbf{j}\times\mathbf{i}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\0&1&0\\1&0&0\end{vmatrix}$$
$$=\begin{vmatrix}1&0\\0&0\end{vmatrix}\mathbf{i}-\begin{vmatrix}0&0\\1&0\end{vmatrix}\mathbf{j}+\begin{vmatrix}0&1\\1&0\end{vmatrix}\mathbf{k}$$
$$=(1\times0-0\times0)\mathbf{i}-(0\times0-0\times1)\mathbf{j}+(0\times0-1\times1)\mathbf{k}$$
$$=-\mathbf{k}$$

Notice that we observe $\mathbf{a}\times\mathbf{b}=-\mathbf{b}\times\mathbf{a}$.

ii $\mathbf{j}\times\mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

$$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}\mathbf{k}$$
$$= (1\times 1 - 0\times 0)\mathbf{i} - (0\times 1 - 0\times 0)\mathbf{j} + (0\times 0 - 1\times 0)\mathbf{k}$$
$$= \mathbf{i}$$

$\mathbf{k}\times\mathbf{j} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix}$

$$= \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix}\mathbf{k}$$
$$= (0\times 0 - 1\times 1)\mathbf{i} - (0\times 0 - 1\times 0)\mathbf{j} + (0\times 1 - 0\times 0)\mathbf{k}$$
$$= -\mathbf{i}$$

iii $\mathbf{i}\times\mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

$$= \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}\mathbf{k}$$
$$= (0\times 1 - 0\times 0)\mathbf{i} - (1\times 1 - 0\times 0)\mathbf{j} + (1\times 0 - 0\times 0)\mathbf{k}$$
$$= -\mathbf{j}$$

$\mathbf{k}\times\mathbf{i} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix}$

$$= \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix}\mathbf{k}$$
$$= (0\times 0 - 1\times 0)\mathbf{i} - (0\times 0 - 1\times 1)\mathbf{j} + (0\times 0 - 0\times 1)\mathbf{k}$$
$$= \mathbf{j}$$

$\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$ for all vectors $\mathbf{a}$ and $\mathbf{b}$.

4 **a** $\mathbf{a}\times\mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$

$$= \begin{vmatrix} a_2 & a_3 \\ a_2 & a_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ a_1 & a_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ a_1 & a_2 \end{vmatrix}\mathbf{k}$$
$$= (a_2a_3 - a_2a_3)\mathbf{i} - (a_1a_3 - a_1a_3)\mathbf{j} + (a_1a_2 - a_1a_2)\mathbf{k}$$
$$= \mathbf{0}$$

Hence $\mathbf{a}\times\mathbf{a} = \mathbf{0}$ for all 3-dimensional vectors $\mathbf{a}$.

b $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

$$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k}$$
$$= (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

$$-(\mathbf{b}\times\mathbf{a}) = -\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$$
$$= -\left[\begin{vmatrix} b_2 & b_3 \\ a_2 & a_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} b_1 & b_3 \\ a_1 & a_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} b_1 & b_2 \\ a_1 & a_2 \end{vmatrix}\mathbf{k}\right]$$
$$= -\left[(b_2a_3 - b_3a_2)\mathbf{i} - (b_1a_3 - b_3a_1)\mathbf{j} + (b_1a_2 - b_2a_1)\mathbf{k}\right]$$
$$= (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$
$$= \mathbf{a}\times\mathbf{b}$$

5 **a** $\mathbf{b}\times\mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 1 \\ 0 & 1 & -2 \end{vmatrix}$

$= \begin{vmatrix} -1 & 1 \\ 1 & -2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 1 \\ 0 & -2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix}\mathbf{k}$

$= (2-1)\mathbf{i} - (-4)\mathbf{j} + 2\mathbf{k}$

$= \mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$

$= \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$

b $\mathbf{a}\bullet(\mathbf{b}\times\mathbf{c}) = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}\bullet\begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$

$= 1 + 12 + 4$

$= 17$

6 **a** $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2 \\ 0 & -1 & 1 \end{vmatrix}$

$= \begin{vmatrix} 0 & 2 \\ -1 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}\mathbf{k}$

$= 2\mathbf{i} - \mathbf{j} - \mathbf{k}$

$= \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$

b $\mathbf{a}\times\mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2 \\ 2 & 0 & -1 \end{vmatrix}$

$= \begin{vmatrix} 0 & 2 \\ 0 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix}\mathbf{k}$

$= -(-1-4)\mathbf{j}$

$= 5\mathbf{j}$

$= \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix}$

c $(\mathbf{a}\times\mathbf{b}) + (\mathbf{a}\times\mathbf{c}) = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix}$

$= \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}$

d $(\mathbf{b}+\mathbf{c}) = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$

$= \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$

$\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2 \\ 2 & -1 & 0 \end{vmatrix}$

$= \begin{vmatrix} 0 & 2 \\ -1 & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix}\mathbf{k}$

$= (0-(-2))\mathbf{i} - (0-4)\mathbf{j} + (-1-0)\mathbf{k}$

$= 2\mathbf{i} + 4\mathbf{j} - \mathbf{k}$

$= \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}$

7 $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1+c_1 & b_2+c_2 & b_3+c_3 \end{vmatrix}$

$= \begin{vmatrix} a_2 & a_3 \\ b_2+c_2 & b_3+c_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1+c_1 & b_3+c_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1+c_1 & b_2+c_2 \end{vmatrix}\mathbf{k}$

$= (a_2(b_3+c_3) - a_3(b_2+c_2))\mathbf{i} - (a_1(b_3+c_3) - a_3(b_1+c_1))\mathbf{j}$
$\quad + (a_1(b_2+c_2) - a_2(b_1+c_1))\mathbf{k}$

$= (a_2b_3 + a_2c_3 - a_3b_2 - a_3c_2)\mathbf{i} - (a_1b_3 + a_1c_3 - a_3b_1 - a_3c_1)\mathbf{j}$
$\quad + (a_1b_2 + a_1c_2 - a_2b_1 - a_2c_1)\mathbf{k}$

$\mathbf{a}\times\mathbf{b} + \mathbf{a}\times\mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$$\therefore\ \mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}=\begin{vmatrix}a_2 & a_3\\ b_2 & b_3\end{vmatrix}\mathbf{i}-\begin{vmatrix}a_1 & a_3\\ b_1 & b_3\end{vmatrix}\mathbf{j}+\begin{vmatrix}a_1 & a_2\\ b_1 & b_2\end{vmatrix}\mathbf{k}+\begin{vmatrix}a_2 & a_3\\ c_2 & c_3\end{vmatrix}\mathbf{i}-\begin{vmatrix}a_1 & a_3\\ c_1 & c_3\end{vmatrix}\mathbf{j}+\begin{vmatrix}a_1 & a_2\\ c_1 & c_2\end{vmatrix}\mathbf{k}$$

$$\begin{aligned}&=(a_2b_3-a_3b_2)\mathbf{i}-(a_1b_3-a_3b_1)\mathbf{j}+(a_1b_2-a_2b_1)\mathbf{k}\\ &\quad+(a_2c_3-a_3c_2)\mathbf{i}-(a_1c_3-a_3c_1)\mathbf{j}+(a_1c_2-a_2c_1)\mathbf{k}\\ &=(a_2b_3+a_2c_3-a_3b_2-a_3c_2)\mathbf{i}-(a_1b_3+a_1c_3-a_3b_1-a_3c_1)\mathbf{j}\\ &\quad+(a_1b_2+a_1c_2-a_2b_1-a_2c_1)\mathbf{k}\\ &=\mathbf{a}\times(\mathbf{b}\times\mathbf{c})\end{aligned}$$

8 $$\begin{aligned}(\mathbf{a}+\mathbf{b})\times(\mathbf{c}+\mathbf{d})&=((\mathbf{a}+\mathbf{b})\times\mathbf{c})+((\mathbf{a}+\mathbf{b})\times\mathbf{d}) &&\{\text{using } \mathbf{a}\times(\mathbf{b}+\mathbf{c})=(\mathbf{a}\times\mathbf{b})+(\mathbf{a}\times\mathbf{c})\}\\ &=((\mathbf{a}\times\mathbf{c})+(\mathbf{b}\times\mathbf{c}))+((\mathbf{a}\times\mathbf{d})+(\mathbf{b}\times\mathbf{d}))\\ &=(\mathbf{a}\times\mathbf{c})+(\mathbf{a}\times\mathbf{d})+(\mathbf{b}\times\mathbf{c})+(\mathbf{b}\times\mathbf{d})\end{aligned}$$

9 **a** $$\begin{aligned}\mathbf{a}\times(\mathbf{a}+\mathbf{b})&=(\mathbf{a}\times\mathbf{a})+(\mathbf{a}\times\mathbf{b})\\ &=(\mathbf{a}\times\mathbf{b}) &&\{\text{since } \mathbf{a}\times\mathbf{a}=\mathbf{0}\}\end{aligned}$$

b $$\begin{aligned}(\mathbf{a}+\mathbf{b})\times(\mathbf{a}+\mathbf{b})&=(\mathbf{a}\times\mathbf{a})+(\mathbf{a}\times\mathbf{b})+(\mathbf{b}\times\mathbf{a})+(\mathbf{b}\times\mathbf{b}) &&\{\text{from } \mathbf{8}\}\\ &=(\mathbf{a}\times\mathbf{b})+(\mathbf{b}\times\mathbf{a}) &&\{\text{since } (\mathbf{a}\times\mathbf{a})=\mathbf{0}\}\\ &=(\mathbf{a}\times\mathbf{b})-(\mathbf{a}\times\mathbf{b}) &&\{\text{since } \mathbf{a}\times\mathbf{b}=-\mathbf{b}\times\mathbf{a}\}\\ &=\mathbf{0}\end{aligned}$$

c $$\begin{aligned}(\mathbf{a}+\mathbf{b})\times(\mathbf{a}-\mathbf{b})&=(\mathbf{a}\times\mathbf{a})+(\mathbf{a}\times(-\mathbf{b}))+(\mathbf{b}\times\mathbf{a})+(\mathbf{b}\times(-\mathbf{b}))\\ &=(\mathbf{a}\times(-\mathbf{b}))+(\mathbf{b}\times\mathbf{a})+(\mathbf{b}\times(-\mathbf{b})) &&\{\text{since } (\mathbf{a}\times\mathbf{a})=\mathbf{0}\}\\ &=-(-\mathbf{b}\times\mathbf{a})+(\mathbf{b}\times\mathbf{a})-(-\mathbf{b}\times\mathbf{b}) &&\{\text{since } \mathbf{a}\times\mathbf{b}=-\mathbf{b}\times\mathbf{a}\}\\ &=(\mathbf{b}\times\mathbf{a})+(\mathbf{b}\times\mathbf{a})+(\mathbf{b}\times\mathbf{b})\\ &=2(\mathbf{b}\times\mathbf{a}) &&\{\text{since } (\mathbf{b}\times\mathbf{b})=\mathbf{0}\}\end{aligned}$$

d $$\begin{aligned}2\mathbf{a}\bullet(\mathbf{a}\times\mathbf{b})&=\begin{pmatrix}2a_1\\2a_2\\2a_3\end{pmatrix}\bullet\begin{pmatrix}a_2b_3-a_3b_2\\a_3b_1-a_1b_3\\a_1b_2-a_2b_1\end{pmatrix}\\ &=2a_1(a_2b_3-a_3b_2)+2a_2(a_3b_1-a_1b_3)+2a_3(a_1b_2-a_2b_1)\\ &=\cancel{2a_1a_2b_3}-\cancel{2a_1a_3b_2}+\cancel{2a_2a_3b_1}-\cancel{2a_1a_2b_3}+\cancel{2a_1a_3b_2}-\cancel{2a_2a_3b_1}\\ &=0\end{aligned}$$

or $\mathbf{a}\times\mathbf{b}$ is perpendicular to $2a$. The dot product of two vectors which are perpendicular is 0.

10 **a** $$\begin{aligned}\begin{pmatrix}2\\-1\\3\end{pmatrix}\times\begin{pmatrix}1\\1\\1\end{pmatrix}&=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&-1&3\\1&1&1\end{vmatrix}\\ &=\begin{vmatrix}-1&3\\1&1\end{vmatrix}\mathbf{i}-\begin{vmatrix}2&3\\1&1\end{vmatrix}\mathbf{j}+\begin{vmatrix}2&-1\\1&1\end{vmatrix}\mathbf{k}\\ &=(-1-3)\mathbf{i}-(2-3)\mathbf{j}+(2+1)\mathbf{k}\\ &=-4\mathbf{i}+\mathbf{j}+3\mathbf{k}\end{aligned}$$

$\therefore$ perpendicular vectors have the form $k\begin{pmatrix}-4\\1\\3\end{pmatrix}$, $k\neq 0,\ k\in\mathbb{R}$.

b $$\begin{aligned}\begin{pmatrix}-1\\3\\4\end{pmatrix}\times\begin{pmatrix}5\\0\\2\end{pmatrix}&=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-1&3&4\\5&0&2\end{vmatrix}\\ &=\begin{vmatrix}3&4\\0&2\end{vmatrix}\mathbf{i}-\begin{vmatrix}-1&4\\5&2\end{vmatrix}\mathbf{j}+\begin{vmatrix}-1&3\\5&0\end{vmatrix}\mathbf{k}\\ &=6\mathbf{i}-(-2-20)\mathbf{j}-15\mathbf{k}\\ &=6\mathbf{i}+22\mathbf{j}-15\mathbf{k}\end{aligned}$$

$\therefore$ perpendicular vectors have the form $k\begin{pmatrix}6\\22\\-15\end{pmatrix}$, $k\neq 0,\ k\in\mathbb{R}$.

c $(\mathbf{i}+\mathbf{j}) \times (\mathbf{i}-\mathbf{j}-\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 1 & -1 & -1 \end{vmatrix}$

$$= \begin{vmatrix} 1 & 0 \\ -1 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}\mathbf{k}$$

$$= -\mathbf{i}+\mathbf{j}-2\mathbf{k}$$

$\therefore$ perpendicular vectors have the form $(-\mathbf{i}+\mathbf{j}-2\mathbf{k})n,\ n \neq 0,\ n \in \mathbb{R}$.

d $(\mathbf{i}-\mathbf{j}-\mathbf{k}) \times (2\mathbf{i}+2\mathbf{j}-3\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & -1 \\ 2 & 2 & -3 \end{vmatrix}$

$$= \begin{vmatrix} -1 & -1 \\ 2 & -3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & -1 \\ 2 & -3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & -1 \\ 2 & 2 \end{vmatrix}\mathbf{k}$$

$$= (3+2)\mathbf{i} - (-3+2)\mathbf{j} + (2+2)\mathbf{k}$$

$$= 5\mathbf{i}+\mathbf{j}+4\mathbf{k}$$

$\therefore$ perpendicular vectors have the form $(5\mathbf{i}+\mathbf{j}+4\mathbf{k})n,\ n \neq 0,\ n \in \mathbb{R}$.

11 $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & -1 \\ 1 & -2 & 2 \end{vmatrix}$

$$= \begin{vmatrix} 3 & -1 \\ -2 & 2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix}\mathbf{k}$$

$$= (6-2)\mathbf{i} - (4+1)\mathbf{j} + (-4-3)\mathbf{k}$$

$$= 4\mathbf{i} - 5\mathbf{j} - 7\mathbf{k}$$

Now $|\mathbf{a} \times \mathbf{b}| = \sqrt{4^2 + (-5)^2 + (-7)^2}$

$$= \sqrt{90}$$

$$= 3\sqrt{10}$$

$\therefore$ the two vectors of length 5 units which are perpendicular to both **a** and **b** are $\pm\frac{5}{3\sqrt{10}}\begin{pmatrix} 4 \\ -5 \\ -7 \end{pmatrix}$

$$= \pm\tfrac{5\sqrt{10}}{30}\begin{pmatrix} 4 \\ -5 \\ -7 \end{pmatrix}$$

$$= \pm\tfrac{10}{\sqrt{6}}\begin{pmatrix} 4 \\ -5 \\ -7 \end{pmatrix}$$

12 **a** $A(1, 3, 2),\ \ B(0, 2, -5),\ \ C(3, 1, -4)$

$$\overrightarrow{AB} = \begin{pmatrix} 0-1 \\ 2-3 \\ -5-2 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -7 \end{pmatrix} \qquad \overrightarrow{AC} = \begin{pmatrix} 3-1 \\ 1-3 \\ -4-2 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ -6 \end{pmatrix}$$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -1 & -7 \\ 2 & -2 & -6 \end{vmatrix} = \begin{vmatrix} -1 & -7 \\ -2 & -6 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -1 & -7 \\ 2 & -6 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -1 & -1 \\ 2 & -2 \end{vmatrix}\mathbf{k}$$

$$= (6-14)\mathbf{i} - (6+14)\mathbf{j} + (2+2)\mathbf{k}$$

$$= -8\mathbf{i} - 20\mathbf{j} + 4\mathbf{k}$$

$$= -4(2\mathbf{i} + 5\mathbf{j} - \mathbf{k})$$

$\therefore$ vectors of the form $k\begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix},\ k \neq 0,\ k \in \mathbb{R}$ will be perpendicular to the plane.

b P(2, 0, −1), Q(0, 1, 3), R(1, −1, 1)

$$\overrightarrow{PQ} = \begin{pmatrix} 0-2 \\ 1-0 \\ 3-(-1) \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} \qquad \overrightarrow{PR} = \begin{pmatrix} 1-2 \\ -1-0 \\ 1-(-1) \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 1 & 4 \\ -1 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 4 \\ -1 & 2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -2 & 4 \\ -1 & 2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -2 & 1 \\ -1 & -1 \end{vmatrix}\mathbf{k}$$

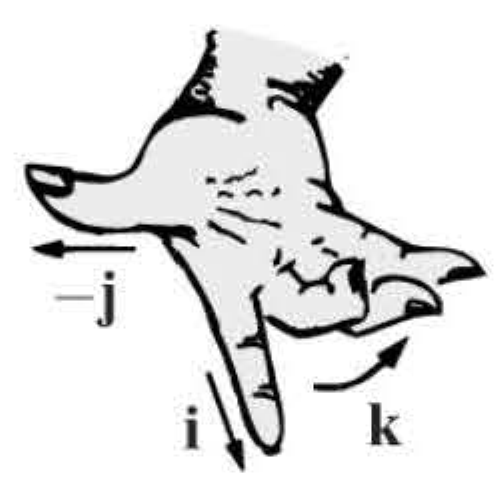
$$= (2+4)\mathbf{i} - (-4+4)\mathbf{j} + (2+1)\mathbf{k}$$
$$= 6\mathbf{i} + 3\mathbf{k}$$
$$= 3(2\mathbf{i} + \mathbf{k})$$

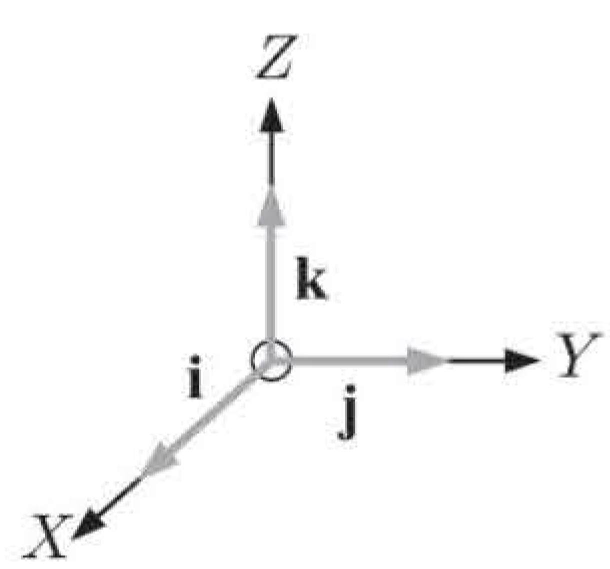
$\therefore$ vectors of the form $k\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$, $k \neq 0$, $k \in \mathbb{R}$ will be perpendicular to the plane.

EXERCISE 14K.2

1 **a** $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$, $\mathbf{k} \times \mathbf{i} = \mathbf{j}$ {from **Exercise 14K.1**, question **3**, part **a**}

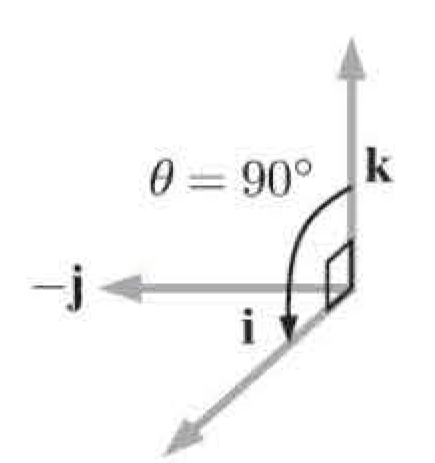
b Yes the **right hand rule** does accurately give the direction.





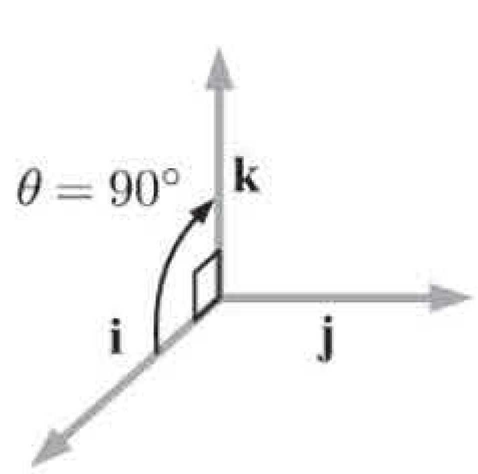
c If $\mathbf{u}$ is the unit vector in the direction $\mathbf{i} \times \mathbf{k}$, then by using the right hand rule, $\mathbf{u} = -\mathbf{j}$.

$$|\mathbf{i}||\mathbf{j}|\sin\theta\mathbf{u} = 1 \times 1 \times \sin 90^\circ \times (-\mathbf{j})$$
$$= -\mathbf{j} \qquad \{\text{since } \sin 90^\circ = 1\}$$



If $\mathbf{u}$ is the unit vector in the direction $\mathbf{k} \times \mathbf{i}$, then by using the right hand rule, $\mathbf{u} = \mathbf{j}$.

$$|\mathbf{i}||\mathbf{j}|\sin\theta\mathbf{u} = 1 \times 1 \times \sin 90^\circ \times \mathbf{j}$$
$$= \mathbf{j} \qquad \{\text{since } \sin 90^\circ = 1\}$$



2 **a** $\mathbf{a} \bullet \mathbf{b} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

$$= 2 + 0 - 3$$
$$= -1$$

$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 3 \\ 1 & 0 & -1 \end{vmatrix}$$
$$= \begin{vmatrix} -1 & 3 \\ 0 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & -1 \\ 1 & 0 \end{vmatrix}\mathbf{k}$$
$$= \mathbf{i} - (-2-3)\mathbf{j} + \mathbf{k}$$
$$= \mathbf{i} + 5\mathbf{j} + \mathbf{k}$$

b $|\mathbf{a}| = \sqrt{2^2+(-1)^2+3^2}$
$= \sqrt{4+1+9}$
$= \sqrt{14}$

$|\mathbf{b}| = \sqrt{1^2+0^2+(-1)^2}$
$= \sqrt{1+0+1}$
$= \sqrt{2}$

$\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta$

$$\cos\theta = \frac{\mathbf{a}\bullet\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|}$$
$$= \frac{-1}{\sqrt{14}\sqrt{2}} \qquad \{\mathbf{a}\bullet\mathbf{b} = -1 \text{ from part } \mathbf{a}\}$$
$$= -\tfrac{1}{\sqrt{28}}$$
$$= -\tfrac{1}{2\sqrt{7}}$$
$$= -\tfrac{\sqrt{7}}{14}$$

c $\sin^2\theta + \cos^2\theta = 1$

$\therefore\ \sin\theta = \pm\sqrt{1-\cos^2\theta}$
$= \pm\sqrt{1-\left(\frac{1}{\sqrt{28}}\right)^2}$
$= \pm\sqrt{\frac{27}{28}}$

But since θ is the angle between two vectors, $0° \leqslant \theta \leqslant 180°$.
$\therefore\ \sin\theta \geqslant \theta$
$\therefore\ \sin\theta = \sqrt{\frac{27}{28}}$

d $|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}|\sin\theta$

$$\sin\theta = \frac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{a}|\,|\mathbf{b}|}$$
$$= \frac{\left|\begin{pmatrix}1\\5\\1\end{pmatrix}\right|}{\sqrt{14}\sqrt{2}} \qquad \{\text{from parts } \mathbf{a} \text{ and } \mathbf{b}\}$$
$$= \frac{\sqrt{1^2+5^2+1^2}}{\sqrt{28}}$$
$$= \frac{\sqrt{1+25+1}}{\sqrt{28}}$$
$$= \frac{\sqrt{27}}{\sqrt{28}}$$

3 $(\Rightarrow)$ $\mathbf{a}\times\mathbf{b} = \mathbf{0}$

$$\therefore\ \begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix} \times \begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix} = \mathbf{0}$$

$\therefore\ a_2b_3 - a_3b_2 = 0$
$a_1b_3 - a_3b_1 = 0$
$a_1b_2 - a_2b_1 = 0$

$\therefore\ \dfrac{a_1b_3}{a_3} = b_1 \qquad \{\text{assuming that } a_3 \neq 0 \text{ since } \mathbf{a} \neq 0\}$

$\dfrac{a_2b_3}{a_3} = b_2$

$\dfrac{a_3b_3}{a_3} = b_3$

$\therefore\ \mathbf{b} = \dfrac{b_3}{a_3}\mathbf{a}$ which implies $\mathbf{b}$ is parallel to $\mathbf{a}$.

$(\Leftarrow)$ If $\mathbf{a} \parallel \mathbf{b}$ then $\mathbf{b} = k\mathbf{a}$, $k \neq 0$, $k \in \mathbb{R}$

$\therefore\ \mathbf{a}\times\mathbf{b} = \mathbf{a}\times k\mathbf{a}$
$= k(\mathbf{a}\times\mathbf{a})$
$= k\times\mathbf{0}$
$= \mathbf{0}$

4 O(0, 0, 0), A(2, 3, -1), B(-1, 1, 2)

a **i** $\overrightarrow{OA} = \begin{pmatrix}2-0\\3-0\\-1-0\end{pmatrix} = \begin{pmatrix}2\\3\\-1\end{pmatrix} \qquad \overrightarrow{OB} = \begin{pmatrix}-1-0\\1-0\\2-0\end{pmatrix} = \begin{pmatrix}-1\\1\\2\end{pmatrix}$

ii $\overrightarrow{OA} \times \overrightarrow{OB} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & -1 \\ -1 & 1 & 2 \end{vmatrix}$

$$= \begin{vmatrix} 3 & -1 \\ 1 & 2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & 3 \\ -1 & 1 \end{vmatrix}\mathbf{k}$$

$$= (6+1)\mathbf{i} - (4-1)\mathbf{j} + (2+3)\mathbf{k}$$

$$= 7\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$$

$$= \begin{pmatrix} 7 \\ -3 \\ 5 \end{pmatrix}$$

iii $|\,\overrightarrow{OA} \times \overrightarrow{OB}\,| = \sqrt{7^2 + (-3)^2 + 5^2}$

$$= \sqrt{49 + 9 + 25}$$

$$= \sqrt{83}$$

b Area of $\triangle OAB = \frac{1}{2}\,|\,\overrightarrow{OA}\,||\,\overrightarrow{OB}\,|\sin\theta$

$$= \tfrac{1}{2}\,|\,\overrightarrow{OA} \times \overrightarrow{OB}\,|$$

$$= \tfrac{\sqrt{83}}{2} \text{ units}^2$$

5 **a** $\mathbf{a} \times \mathbf{c} = \mathbf{b} \times \mathbf{c}$

$\therefore\ \mathbf{0} = \mathbf{b} \times \mathbf{c} - \mathbf{a} \times \mathbf{c}$

$\therefore\ \mathbf{0} = (\mathbf{b} - \mathbf{a}) \times \mathbf{c}$

$\therefore\ \overrightarrow{OC}$ is parallel to $\overrightarrow{AB}$.

b $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$

$\therefore\ \mathbf{b} \times (\mathbf{a} + \mathbf{b} + \mathbf{c}) = \mathbf{0}$

$\therefore\ \mathbf{b} \times \mathbf{a} + \cancel{\mathbf{b} \times \mathbf{b}} + \mathbf{b} \times \mathbf{c} = \mathbf{0}$ {since $\mathbf{b} \times \mathbf{b} = \mathbf{0}$}

$\therefore\ -\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} = \mathbf{0}$ {since $\mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b}$}

$\therefore\ \mathbf{b} \times \mathbf{c} = \mathbf{a} \times \mathbf{b}$

c $\mathbf{b} \times \mathbf{c} = \mathbf{c} \times \mathbf{a}, \quad \mathbf{c} \neq \mathbf{0}$

$\therefore\ \mathbf{b} \times \mathbf{c} - \mathbf{c} \times \mathbf{a} = \mathbf{0}$

$\therefore\ \mathbf{b} \times \mathbf{c} + \mathbf{a} \times \mathbf{c} = \mathbf{0}$ {since $-\mathbf{c} \times \mathbf{a} = \mathbf{a} \times \mathbf{c}$}

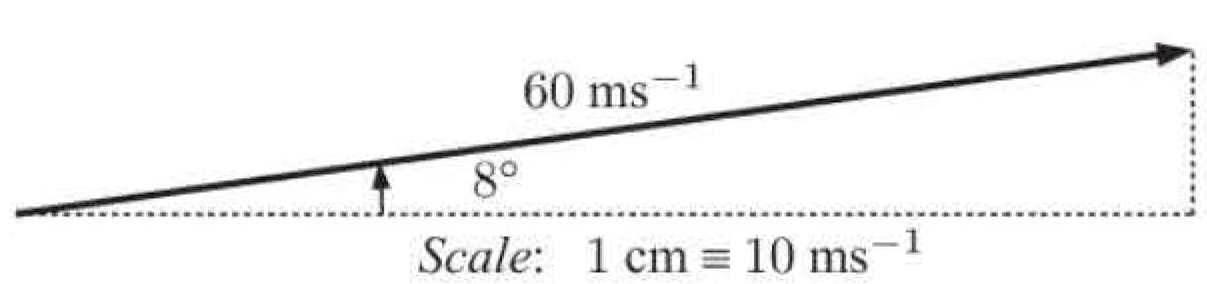
$\therefore\ (\mathbf{b} + \mathbf{a}) \times \mathbf{c} = \mathbf{0}$

$\therefore$ since $\mathbf{c} \neq \mathbf{0}$, $\mathbf{b} + \mathbf{a}$ and $\mathbf{c}$ must be parallel.

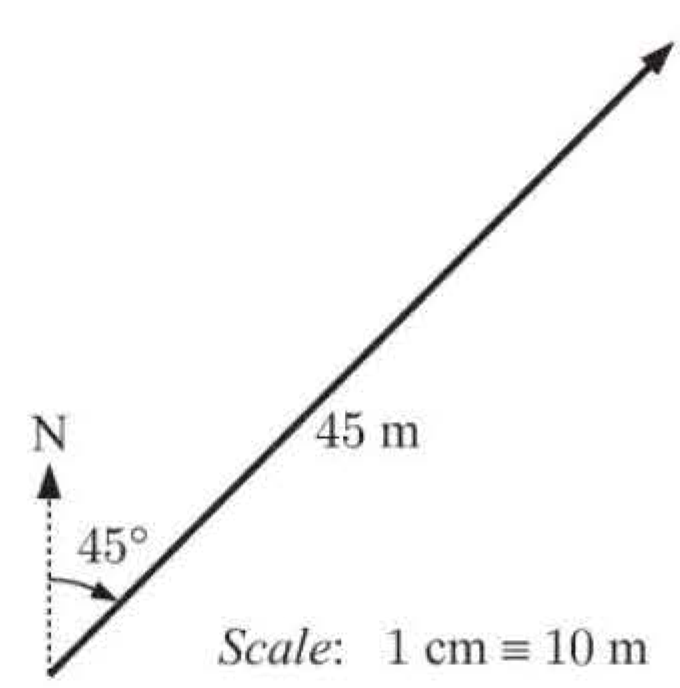
$\therefore\ \mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a} = k\mathbf{c}, \ k \neq 0, \ k \in \mathbb{R}$.

REVIEW SET 14A

1 **a**



b



2 **a** $\overrightarrow{AB} - \overrightarrow{CB} = \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

b $\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{DC} = \overrightarrow{AC} + \overrightarrow{CD} = \overrightarrow{AD}$

3 **a** $\mathbf{q} = \mathbf{p} + \mathbf{r}$

b $\mathbf{l} = \mathbf{k} - \mathbf{j} + \mathbf{n} - \mathbf{m}$

4 $\overrightarrow{SP} = \overrightarrow{SR} + \overrightarrow{RQ} + \overrightarrow{QP}$
$= -\overrightarrow{RS} + \overrightarrow{RQ} - \overrightarrow{PQ}$
$= -\begin{pmatrix} 2 \\ -3 \end{pmatrix} + \begin{pmatrix} -1 \\ 2 \end{pmatrix} - \begin{pmatrix} -4 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$

5 **a** $\overrightarrow{BC} = 2\overrightarrow{OA} = 2\mathbf{p}$

Now $\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OB} + \overrightarrow{BC}$
$= -\mathbf{p} + \mathbf{q} + 2\mathbf{p}$
$= \mathbf{p} + \mathbf{q}$

b $\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM}$
$= \mathbf{p} + \frac{1}{2}\overrightarrow{AC}$
$= \mathbf{p} + \frac{1}{2}(\mathbf{p} + \mathbf{q})$
$= \frac{3}{2}\mathbf{p} + \frac{1}{2}\mathbf{q}$

6 The vectors are parallel, so $\begin{pmatrix} -12 \\ -20 \\ 2 \end{pmatrix} = k\begin{pmatrix} 3 \\ m \\ n \end{pmatrix}$ $\quad \therefore\ 3k = -12,\quad km = -20,\quad kn = 2$
$\therefore\ k = -4,\quad m = 5,\quad n = -\frac{1}{2}$

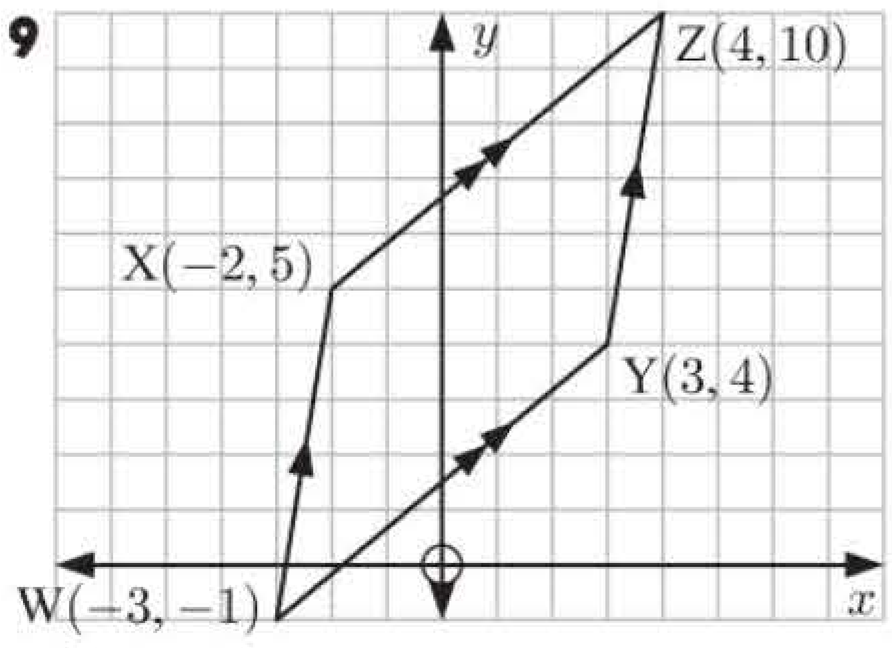
7 $\overrightarrow{CB} = \overrightarrow{CA} + \overrightarrow{AB} = -\overrightarrow{AC} + \overrightarrow{AB} = \begin{pmatrix} 6 \\ -1 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \\ -7 \\ 4 \end{pmatrix} = \begin{pmatrix} 8 \\ -8 \\ 7 \end{pmatrix}$

8 **a** $\mathbf{p} \bullet \mathbf{q} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 5 \end{pmatrix}$
$= -3 + (-10)$
$= -13$

b $\mathbf{p} - \mathbf{r} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} - \begin{pmatrix} -3 \\ 4 \end{pmatrix} = \begin{pmatrix} 6 \\ -6 \end{pmatrix}$

$\therefore\ \mathbf{q} \bullet (\mathbf{p} - \mathbf{r}) = \begin{pmatrix} -1 \\ 5 \end{pmatrix} \bullet \begin{pmatrix} 6 \\ -6 \end{pmatrix} = -6 - 30$
$= -36$

9



$\overrightarrow{WY} = \begin{pmatrix} 3 - -3 \\ 4 - -1 \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \end{pmatrix}$

$\overrightarrow{XZ} = \begin{pmatrix} 4 - -2 \\ 10 - 5 \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \end{pmatrix}$

So, $\overrightarrow{WY} = \overrightarrow{XZ}$

$\therefore$ [WY] is parallel to [XZ] and they are equal in length. This is sufficient to deduce that WYZX is a parallelogram.

10 $\overrightarrow{AB} = \begin{pmatrix} -1 - 2 \\ 4 - 3 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$

$\overrightarrow{AC} = \begin{pmatrix} 3 - 2 \\ k - 3 \end{pmatrix} = \begin{pmatrix} 1 \\ k - 3 \end{pmatrix}$

Now $\overrightarrow{AB} \bullet \overrightarrow{AC} = 0$ $\{$as $\widehat{BAC} = 90°\}$

$\therefore\ \begin{pmatrix} -3 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ k - 3 \end{pmatrix} = 0$
$\therefore\ -3 + k - 3 = 0$
$\therefore\ k = 6$

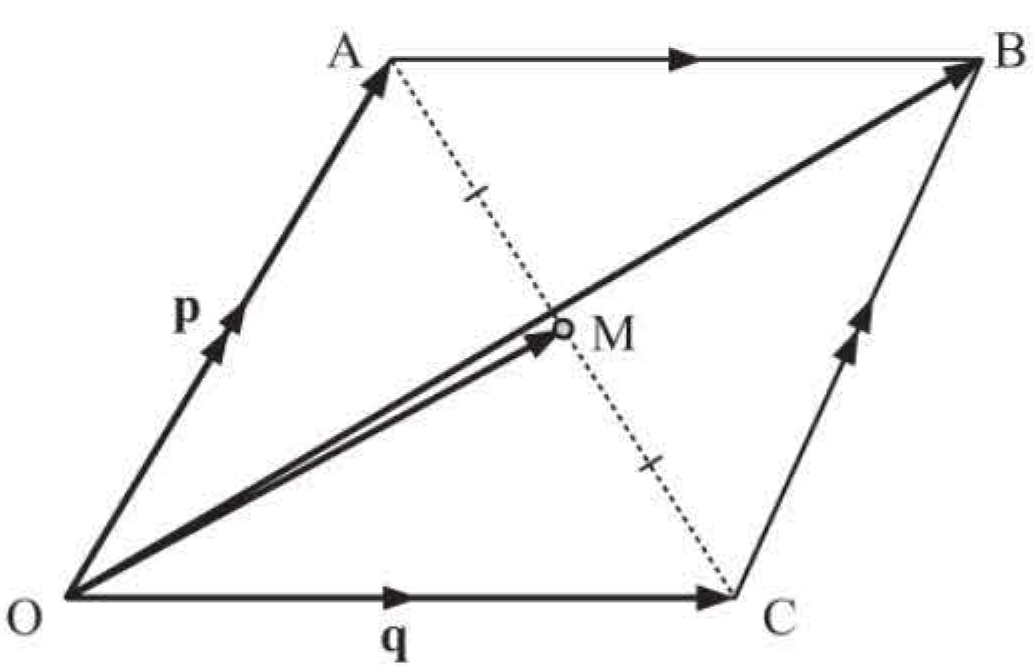
11 **a** $\mathbf{a} \bullet \mathbf{b}$ is a scalar, so $\mathbf{a} \bullet \mathbf{b} \bullet \mathbf{c}$ is a scalar dotted with a vector, which is meaningless.

b $\mathbf{b} \times \mathbf{c}$ must be done first otherwise we have the cross product of a scalar with a vector, which is meaningless.

12 One vector perpendicular to $\begin{pmatrix} -4 \\ 5 \end{pmatrix}$ is $\begin{pmatrix} 5 \\ 4 \end{pmatrix}$ as the dot product $= -20 + 20 = 0$

$\therefore$ all vectors have the form $t\begin{pmatrix} 5 \\ 4 \end{pmatrix}$, $t \neq 0$.

13



a **i** $\overrightarrow{OB}$
$= \overrightarrow{OA} + \overrightarrow{AB}$
$= \overrightarrow{OA} + \overrightarrow{OC}$
$= \mathbf{p} + \mathbf{q}$

ii $\overrightarrow{OM}$
$= \overrightarrow{OA} + \overrightarrow{AM}$
$= \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AC}$
$= \mathbf{p} + \frac{1}{2}(\overrightarrow{AO} + \overrightarrow{OC})$
$= \mathbf{p} + \frac{1}{2}(-\mathbf{p} + \mathbf{q})$
$= \mathbf{p} - \frac{1}{2}\mathbf{p} + \frac{1}{2}\mathbf{q}$
$= \frac{1}{2}\mathbf{p} + \frac{1}{2}\mathbf{q}$

b We notice that $\overrightarrow{OM} = \frac{1}{2}\overrightarrow{OB}$

$\therefore$ [OM] $\parallel$ [OB] and OM $= \frac{1}{2}$OB

So, O, M, and B are collinear (as O is common) and hence M is the midpoint of [OB].

14 **a**
$$1^2 = \left(\tfrac{4}{7}\right)^2 + \left(\frac{1}{k}\right)^2$$
$$= \tfrac{16}{49} + \frac{1}{k^2}$$
$$\therefore\ \frac{1}{k^2} = 1 - \tfrac{16}{49}$$
$$= \tfrac{33}{49}$$
$$\therefore\ k^2 = \tfrac{49}{33}$$
$$\therefore\ k = \pm\sqrt{\tfrac{49}{33}}$$
$$= \pm\tfrac{7}{\sqrt{33}}$$

b
$$1^2 = k^2 + k^2$$
$$\therefore\ 1 = 2k^2$$
$$\therefore\ k^2 = \tfrac{1}{2}$$
$$\therefore\ k = \pm\sqrt{\tfrac{1}{2}}$$
$$= \pm\tfrac{1}{\sqrt{2}}$$

15 **a**
$$\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta$$
$$= 2 \times 4 \times \cos 120^\circ$$
$$= 8(-\tfrac{1}{2})$$
$$= -4$$

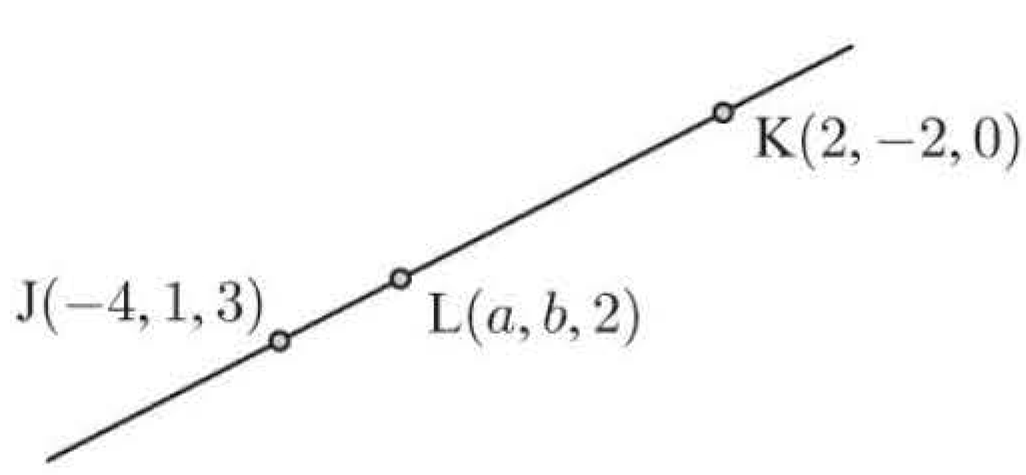
b
$$\mathbf{b} \bullet \mathbf{c} = |\mathbf{b}|\,|\mathbf{c}|\cos\theta$$
$$= 4 \times 5 \times \cos 60^\circ$$
$$= 20(\tfrac{1}{2})$$
$$= 10$$

c
$$\mathbf{a} \bullet \mathbf{c} = |\mathbf{a}|\,|\mathbf{c}|\cos\theta$$
$$= 2 \times 5 \times \cos 180^\circ$$
$$= 10(-1)$$
$$= -10$$

16
$$\overrightarrow{JK} = \begin{pmatrix} 2-(-4) \\ -2-1 \\ 0-3 \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \\ -3 \end{pmatrix}$$
$$\overrightarrow{JL} = \begin{pmatrix} a-(-4) \\ b-1 \\ 2-3 \end{pmatrix} = \begin{pmatrix} a+4 \\ b-1 \\ -1 \end{pmatrix}$$

If J, K, and L are collinear then $\overrightarrow{JK} \parallel \overrightarrow{JL}$

$$\therefore\ \begin{pmatrix} 6 \\ -3 \\ -3 \end{pmatrix} = k\begin{pmatrix} a+4 \\ b-1 \\ -1 \end{pmatrix} \quad \text{for some } k \neq 0,\ k \in \mathbb{R}$$



$$\therefore\ -3 = k(-1)$$
$$\therefore\ k = 3$$

$\therefore\ -3 = k(b-1)$ and $6 = k(a+4)$

$\therefore\ -3 = 3(b-1)$ and $6 = 3(a+4)$

$\therefore\ -1 = b-1$ and $2 = a+4$

$\therefore\ b = 0$ and $a = -2$

17 $|\mathbf{u}\times\mathbf{v}| = \sqrt{1^2+(-3)^2+(-4)^2}$
$= \sqrt{1+9+16}$
$= \sqrt{26}$

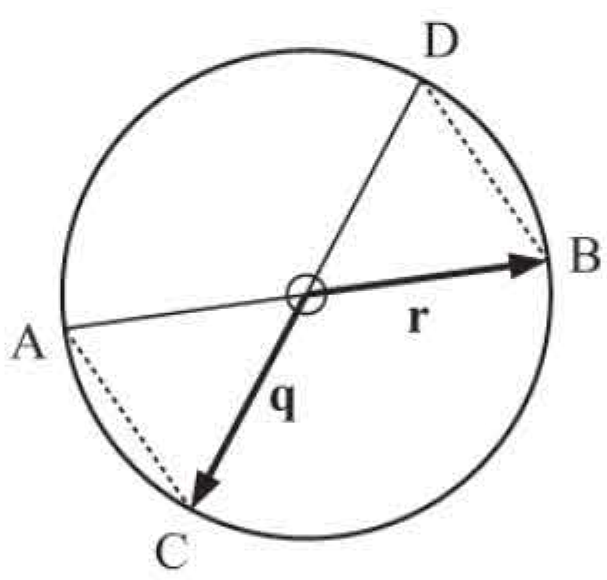
$\sin\theta = \dfrac{|\mathbf{u}\times\mathbf{v}|}{|\mathbf{u}|\,|\mathbf{v}|}$
$= \dfrac{\sqrt{26}}{3\times 5}$
$= \frac{\sqrt{26}}{15}$

$\sin^2\theta + \cos^2\theta = 1$
$\cos\theta = \pm\sqrt{1-\sin^2\theta}$
$= \pm\sqrt{1-\frac{26}{225}}$
$= \pm\sqrt{\frac{199}{225}}$
$= \pm\frac{\sqrt{199}}{15}$

$\mathbf{u}\bullet\mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\cos\theta$
$= 3\times 5\times\left(\pm\frac{\sqrt{199}}{15}\right)$
$= \pm\sqrt{199}$

So, if θ is acute, $\mathbf{u}\bullet\mathbf{v} = \sqrt{199}$
and if θ is obtuse, $\mathbf{u}\bullet\mathbf{v} = -\sqrt{199}$

18



a **i** $\overrightarrow{DB}$
$= \overrightarrow{DO} + \overrightarrow{OB}$
$= \overrightarrow{OC} + \overrightarrow{OB}$
$= \mathbf{q} + \mathbf{r}$

ii $\overrightarrow{AC}$
$= \overrightarrow{AO} + \overrightarrow{OC}$
$= \overrightarrow{OB} + \overrightarrow{OC}$
$= \mathbf{r} + \mathbf{q}$

b We see that $\overrightarrow{DB} = \overrightarrow{AC}$
$\therefore$ [DB] is parallel to [AC] and equal in length.

19 **a**
$$\begin{pmatrix}2-t\\3\\t\end{pmatrix}\bullet\begin{pmatrix}t\\4\\t+1\end{pmatrix} = 0$$
$\therefore\ (2-t)t + 12 + t(t+1) = 0$
$\therefore\ 2t - \cancel{t^2} + 12 + \cancel{t^2} + t = 0$
$\therefore\ 3t + 12 = 0$
$\therefore\ t = -4$

b $\overrightarrow{KL} = \begin{pmatrix}-3-4\\4-3\\2-(-1)\end{pmatrix} = \begin{pmatrix}-7\\1\\3\end{pmatrix}$, $\overrightarrow{LM} = \begin{pmatrix}2-(-3)\\1-4\\-2-2\end{pmatrix} = \begin{pmatrix}5\\-3\\-4\end{pmatrix}$,

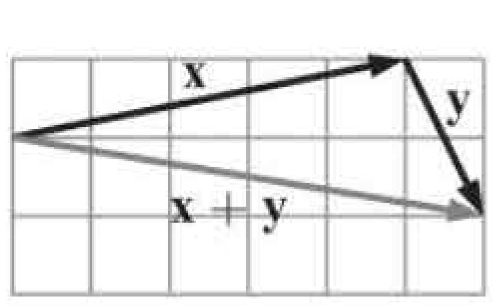
$\overrightarrow{MK} = \begin{pmatrix}2-4\\1-3\\-2-(-1)\end{pmatrix} = \begin{pmatrix}-2\\-2\\-1\end{pmatrix}$

Now $\overrightarrow{LM}\bullet\overrightarrow{MK} = \begin{pmatrix}5\\-3\\-4\end{pmatrix}\bullet\begin{pmatrix}-2\\-2\\-1\end{pmatrix} = 5\times(-2)+(-3)\times(-2)+(-4)\times(-1)$
$= -10+6+4$
$= 0$

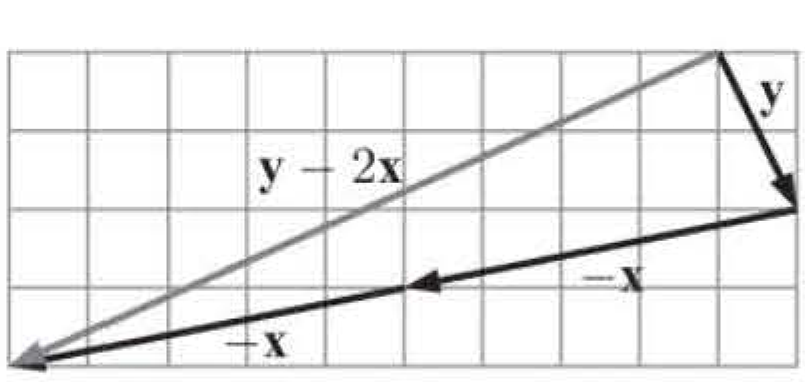
($\overrightarrow{KL}\bullet\overrightarrow{LM} = -50$, and $\overrightarrow{MK}\bullet\overrightarrow{KL} = 9$)
$\therefore$ [LM] and [MK] are perpendicular.
$\therefore$ $\triangle$KLM is right angled at M.

REVIEW SET 14B

1 **a**



b



2 $\overrightarrow{AB} = \begin{pmatrix} 4-(-2) \\ 0-(-1) \\ -1-3 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \\ -4 \end{pmatrix}$, $\quad \overrightarrow{AC} = \begin{pmatrix} -2-(-2) \\ 1-(-1) \\ -4-3 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ -7 \end{pmatrix}$,

$\overrightarrow{BC} = \begin{pmatrix} -2-4 \\ 1-0 \\ -4-(-1) \end{pmatrix} = \begin{pmatrix} -6 \\ 1 \\ -3 \end{pmatrix}$

$\therefore\ AB = \sqrt{6^2+1^2+(-4)^2} = \sqrt{53}$ units $\qquad AC = \sqrt{0^2+2^2+(-7)^2} = \sqrt{53}$ units $\qquad BC = \sqrt{(-6)^2+1^2+(-3)^2} = \sqrt{46}$ units

$\therefore\ AB = AC$, so ABC is an isosceles triangle.

3 **a** $|\mathbf{s}| = \sqrt{(-3)^2+2^2} = \sqrt{13}$ units

b $\mathbf{r}+\mathbf{s} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} + \begin{pmatrix} -3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$\therefore\ |\mathbf{r}+\mathbf{s}| = \sqrt{1^2+3^2} = \sqrt{10}$ units

c $2\mathbf{s}-\mathbf{r} = 2\begin{pmatrix} -3 \\ 2 \end{pmatrix} - \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} -10 \\ 3 \end{pmatrix}$

$\therefore\ |2\mathbf{s}-\mathbf{r}| = \sqrt{(-10)^2+3^2} = \sqrt{109}$ units

4 $r\begin{pmatrix} -2 \\ 1 \end{pmatrix} + s\begin{pmatrix} 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 13 \\ -24 \end{pmatrix}$

$\therefore\ \begin{pmatrix} -2r+3s \\ r-4s \end{pmatrix} = \begin{pmatrix} 13 \\ -24 \end{pmatrix}$

$\therefore\ -2r+3s = 13$

$r-4s = -24 \quad \text{.... (1)}$

$\therefore\ -2r+3s = 13$

$2r-8s = -48 \quad \{2\times(1)\}$

adding $\ -5s = -35$

$\therefore\ s = 7$

Now using (1), $r-4(7) = -24$

$\therefore\ r = -24+28$

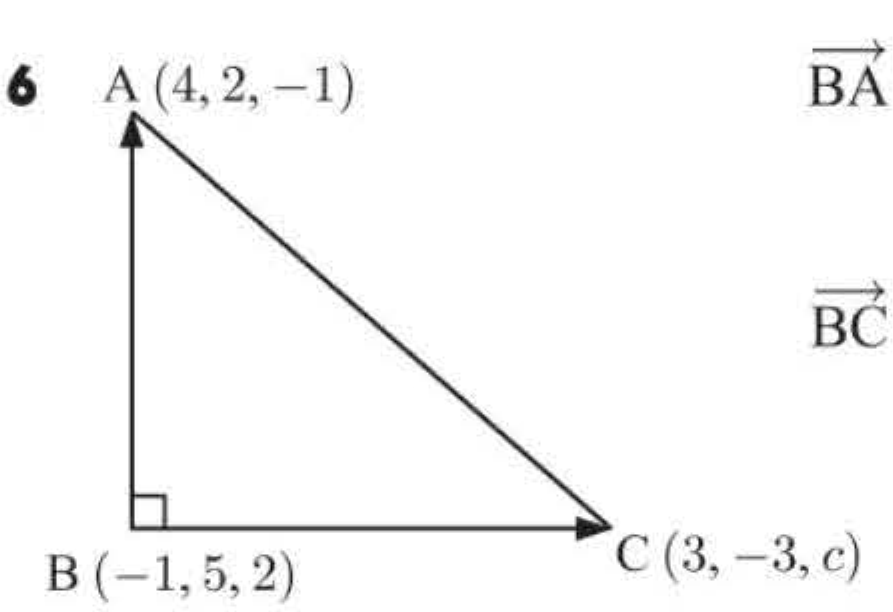
$\therefore\ r = 4$ and $s = 7$

5 **a** $\overrightarrow{PQ} = \begin{pmatrix} -4-2 \\ 4-3 \\ 2--1 \end{pmatrix} = \begin{pmatrix} -6 \\ 1 \\ 3 \end{pmatrix}$

b $PQ = |\overrightarrow{PQ}| = \sqrt{36+1+9} = \sqrt{46}$ units

c The midpoint is at $\left(\frac{2+(-4)}{2}, \frac{3+4}{2}, \frac{-1+2}{2}\right)$ which is $(-1, \frac{7}{2}, \frac{1}{2})$ or $(-1, 3\frac{1}{2}, \frac{1}{2})$.

6



$\overrightarrow{BA} = \begin{pmatrix} 4--1 \\ 2-5 \\ -1-2 \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \\ -3 \end{pmatrix}$

$\overrightarrow{BC} = \begin{pmatrix} 3--1 \\ -3-5 \\ c-2 \end{pmatrix} = \begin{pmatrix} 4 \\ -8 \\ c-2 \end{pmatrix}$

But $\overrightarrow{BA} \bullet \overrightarrow{BC} = 0$

$\therefore\ 20+24-3(c-2) = 0$

$\therefore\ 44 = 3(c-2)$

$\therefore\ 3c-6 = 44$

$\therefore\ 3c = 50$

$\therefore\ c = \frac{50}{3}$

7 $\mathbf{a}-3\mathbf{x} = \mathbf{b}$

$\therefore\ \mathbf{a}-\mathbf{b} = 3\mathbf{x}$

$\therefore\ \mathbf{x} = \frac{1}{3}(\mathbf{a}-\mathbf{b}) = \frac{1}{3}\begin{pmatrix} 2-(-1) \\ -3-2 \\ 1-3 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 3 \\ -5 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{5}{3} \\ -\frac{2}{3} \end{pmatrix}$

8 If the angle is θ then using $\mathbf{a} \bullet \mathbf{b} = |\,\mathbf{a}\,|\,|\,\mathbf{b}\,| \cos\theta$,

$$\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} = \sqrt{9+1+4}\sqrt{4+25+1}\cos\theta$$

$$\therefore\ 6+5-2 = \sqrt{14}\sqrt{30}\cos\theta$$

$$\therefore\ \frac{9}{\sqrt{14 \times 30}} = \cos\theta$$

$$\therefore\ \theta \approx 64.0^\circ$$

9 Let $\mathrm{Q}(0,\,0,\,z)$ be a point on the Z-axis.

$$\overrightarrow{\mathrm{PQ}} = \begin{pmatrix} 0-(-4) \\ 0-2 \\ z-5 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ z-5 \end{pmatrix}$$

$$\mathrm{PQ} = \sqrt{4^2+(-2)^2+(z-5)^2} = 6$$

$$\therefore\ 16+4+(z-5)^2 = 36$$

$$\therefore\ (z-5)^2 = 16$$

$$\therefore\ z-5 = \pm 4$$

$$\therefore\ z = 1 \text{ or } 9$$

$\therefore$ Q is $(0,\,0,\,1)$ or $(0,\,0,\,9)$.

10 Since they are perpendicular,

$$\begin{pmatrix} 3 \\ 3-2t \end{pmatrix} \bullet \begin{pmatrix} t^2+t \\ -2 \end{pmatrix} = 0$$

$$\therefore\ 3(t^2+t) - 2(3-2t) = 0$$

$$\therefore\ 3t^2+3t-6+4t = 0$$

$$\therefore\ 3t^2+7t-6 = 0$$

$$\therefore\ (3t-2)(t+3) = 0$$

$$\therefore\ t = \tfrac{2}{3} \text{ or } -3$$

11
$$\overrightarrow{\mathrm{PQ}} = \begin{pmatrix} 4-(-6) \\ 6-8 \\ 8-2 \end{pmatrix} = \begin{pmatrix} 10 \\ -2 \\ 6 \end{pmatrix} = 2\begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix}$$

$$\overrightarrow{\mathrm{QR}} = \begin{pmatrix} 19-4 \\ 3-6 \\ 17-8 \end{pmatrix} = \begin{pmatrix} 15 \\ -3 \\ 9 \end{pmatrix} = 3\begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix}$$

So, $\overrightarrow{\mathrm{PQ}} = \frac{2}{3}\overrightarrow{\mathrm{QR}}$

$\therefore$ $\overrightarrow{\mathrm{QR}}$ is a scalar multiple of $\overrightarrow{\mathrm{PQ}}$.

$\therefore$ P, Q, and R are collinear.

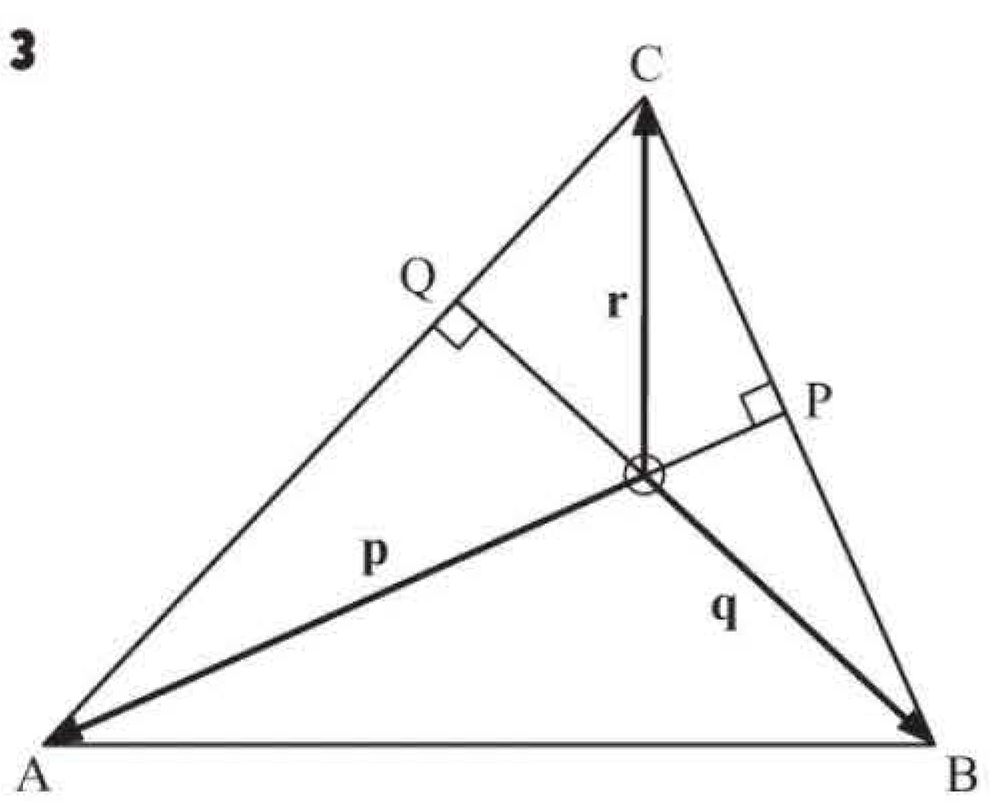
12 **a**
$$\begin{aligned} &\mathbf{u} \bullet \mathbf{v} \\ &= \begin{pmatrix} -4 \\ 2 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 3 \\ -2 \end{pmatrix} \\ &= -4(-1) + 2(3) + 1(-2) \\ &= 4+6-2 \\ &= 8 \end{aligned}$$

b If θ is the angle between $\mathbf{u}$ and $\mathbf{v}$ then

$$\begin{aligned} \cos\theta &= \frac{\mathbf{u} \bullet \mathbf{v}}{|\,\mathbf{u}\,|\,|\,\mathbf{v}\,|} \\ &= \frac{8}{\sqrt{(-4)^2+2^2+1^2}\sqrt{(-1)^2+3^2+(-2)^2}} \\ &= \frac{8}{\sqrt{21}\sqrt{14}} \end{aligned}$$

$$\therefore\ \theta \approx 62.2^\circ$$

13



a
$$\begin{aligned} \overrightarrow{\mathrm{AC}} &= \overrightarrow{\mathrm{AO}} + \overrightarrow{\mathrm{OC}} \\ &= -\mathbf{p} + \mathbf{r} \\ &= \mathbf{r} - \mathbf{p} \end{aligned} \qquad \begin{aligned} \overrightarrow{\mathrm{BC}} &= \overrightarrow{\mathrm{BO}} + \overrightarrow{\mathrm{OC}} \\ &= -\mathbf{q} + \mathbf{r} \\ &= \mathbf{r} - \mathbf{q} \end{aligned}$$

b [AP] $\perp$ [BC] and [BQ] $\perp$ [AC]

$$\begin{aligned} &\therefore\ \mathbf{p} \perp \mathbf{r} - \mathbf{q} \\ &\therefore\ \mathbf{p} \bullet (\mathbf{r} - \mathbf{q}) = 0 \\ &\therefore\ \mathbf{p} \bullet \mathbf{r} - \mathbf{p} \bullet \mathbf{q} = 0 \\ &\therefore\ \mathbf{p} \bullet \mathbf{r} = \mathbf{p} \bullet \mathbf{q} \end{aligned} \qquad \begin{aligned} &\therefore\ \mathbf{q} \perp (\mathbf{r} - \mathbf{p}) \\ &\therefore\ \mathbf{q} \bullet (\mathbf{r} - \mathbf{p}) = 0 \\ &\therefore\ \mathbf{q} \bullet \mathbf{r} - \mathbf{q} \bullet \mathbf{p} = 0 \\ &\therefore\ \mathbf{q} \bullet \mathbf{r} = \mathbf{p} \bullet \mathbf{q} \end{aligned}$$

$$\therefore\ \mathbf{q} \bullet \mathbf{r} = \mathbf{p} \bullet \mathbf{q} = \mathbf{p} \bullet \mathbf{r}$$

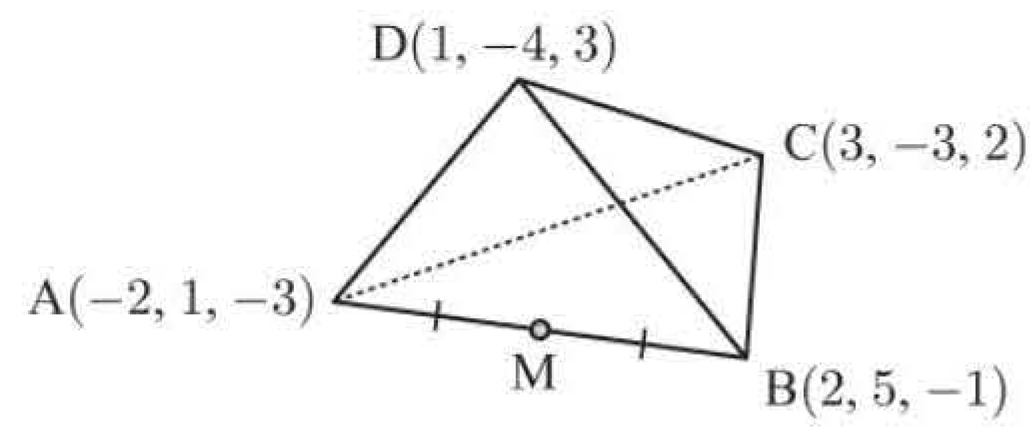
c
$$\begin{aligned} \mathbf{r} \bullet \overrightarrow{\mathrm{AB}} &= \mathbf{r} \bullet (-\mathbf{p} + \mathbf{q}) \\ &= -\mathbf{r} \bullet \mathbf{p} + \mathbf{r} \bullet \mathbf{q} \\ &= -\mathbf{p} \bullet \mathbf{q} + \mathbf{p} \bullet \mathbf{q} \qquad \{\text{from } \mathbf{b}\} \\ &= 0 \qquad \text{and so } \mathbf{r} \perp \overrightarrow{\mathrm{AB}} \qquad \therefore\ [\mathrm{OC}] \perp [\mathrm{AB}] \end{aligned}$$

14 $|\,3\mathbf{i} - 2\mathbf{j} + \mathbf{k}\,| = \sqrt{3^2 + (-2)^2 + 1^2} = \sqrt{14}$

$\therefore$ a unit vector in the direction $3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ is $\frac{1}{\sqrt{14}}(3\mathbf{i} - 2\mathbf{j} + \mathbf{k})$

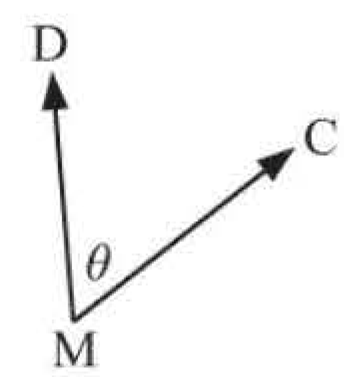
$\therefore$ two vectors 4 units long and parallel to $3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ are $\pm\frac{4}{\sqrt{14}}(3\mathbf{i} - 2\mathbf{j} + \mathbf{k})$.

15 M is $\left(\frac{-2+2}{2}, \frac{1+5}{2}, \frac{-3-1}{2}\right)$ or $(0, 3, -2)$.



$$\therefore\ \overrightarrow{MD} = \begin{pmatrix} 1-0 \\ -4-3 \\ 3-(-2) \end{pmatrix} = \begin{pmatrix} 1 \\ -7 \\ 5 \end{pmatrix},$$

$$\overrightarrow{MC} = \begin{pmatrix} 3-0 \\ -3-3 \\ 2-(-2) \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \\ 4 \end{pmatrix}$$



$$\therefore\ \overrightarrow{MD} \bullet \overrightarrow{MC} = |\overrightarrow{MD}||\overrightarrow{MC}|\cos\theta$$

$$\therefore\ 3+42+20 = \sqrt{1+49+25}\sqrt{9+36+16}\cos\theta$$

$$\therefore\ 65 = \sqrt{75}\sqrt{61}\cos\theta$$

$$\therefore\ \theta \approx 16.1^\circ$$

16 $$\begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & -2 \\ 1 & 1 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} -1 & -2 \\ 1 & 2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & -2 \\ 1 & 2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix}\mathbf{k}$$

$$= (-2+2)\mathbf{i} - (4+2)\mathbf{j} + (2+1)\mathbf{k}$$

$$= -6\mathbf{j} + 3\mathbf{k}$$

$\therefore$ perpendicular vectors have the form $k\begin{pmatrix} 0 \\ -6 \\ 3 \end{pmatrix}$, $k \neq 0$, $k \in \mathbb{R}$.

17 **a** $$\sqrt{k^2 + (\tfrac{1}{\sqrt{2}})^2 + (-k)^2} = 1$$

$$\therefore\ k^2 + \tfrac{1}{2} + k^2 = 1$$

$$\therefore\ 2k^2 = \tfrac{1}{2}$$

$$\therefore\ k^2 = \tfrac{1}{4}$$

$$\therefore\ k = \pm\tfrac{1}{2}$$

b $\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$ has length $\sqrt{3^2+2^2+(-1)^2} = \sqrt{14}$ units

$\therefore$ a unit vector in the opposite direction is $-\frac{1}{\sqrt{14}}\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$

$\therefore$ a vector of length 5 units in the opposite direction is $-\frac{5}{\sqrt{14}}\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$.

18 Using $\theta = \cos^{-1}\left(\dfrac{\mathbf{u} \bullet \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}\right)$,

$$\mathbf{u} \bullet \mathbf{v} = \begin{pmatrix} 2 \\ -4 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} = -2-4+9 = 3$$

$$|\mathbf{u}| = \sqrt{2^2+(-4)^2+3^2} = \sqrt{4+16+9} = \sqrt{29}$$

$$|\mathbf{v}| = \sqrt{(-1)^2+1^2+3^2} = \sqrt{1+1+9} = \sqrt{11}$$

$$\therefore\ \theta = \cos^{-1}\left(\frac{3}{\sqrt{29}\sqrt{11}}\right) \approx 80.3^\circ$$

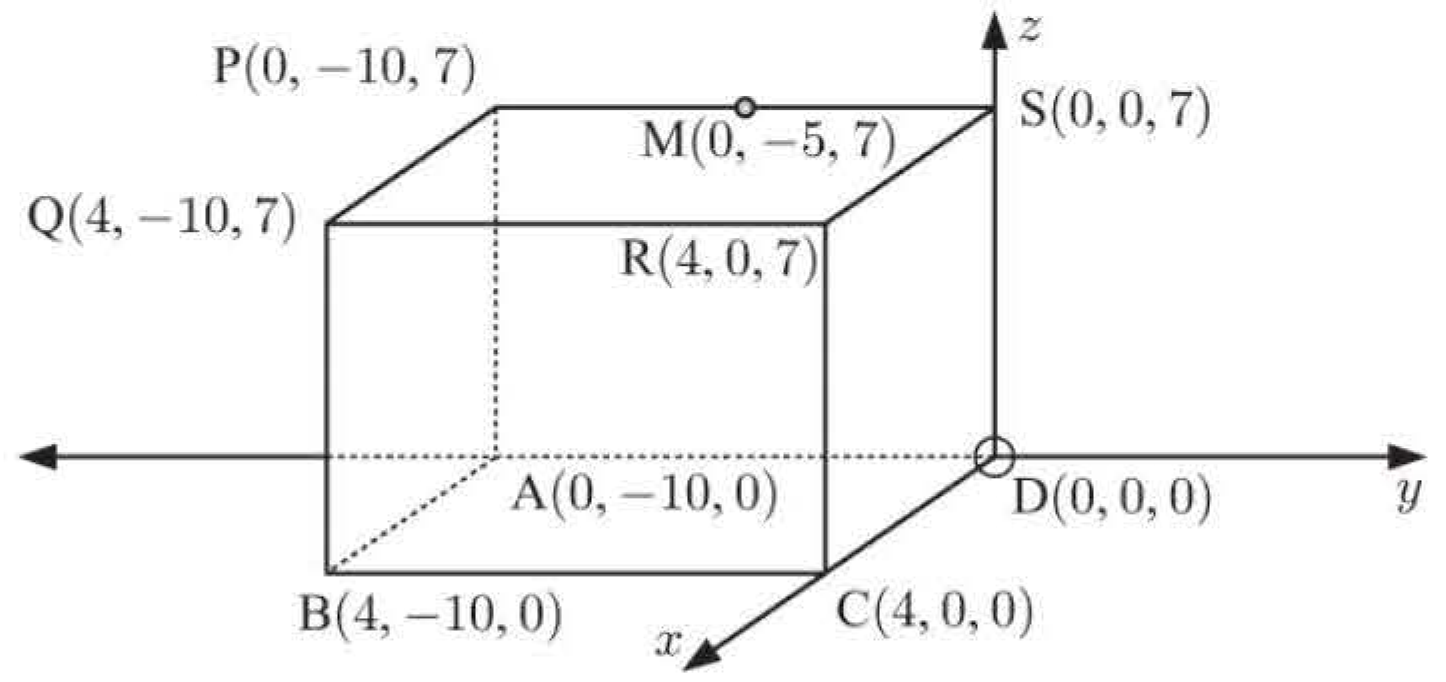
or using $\theta = \sin^{-1}\left(\dfrac{|\mathbf{u} \times \mathbf{v}|}{|\mathbf{u}||\mathbf{v}|}\right)$,

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -4 & 3 \\ -1 & 1 & 3 \end{vmatrix} = -15\mathbf{i} - 9\mathbf{j} - 2\mathbf{k}$$

$$|\mathbf{u} \times \mathbf{v}| = \sqrt{(-15)^2+(-9)^2+(-2)^2} = \sqrt{310}$$

$$\therefore\ \theta = \sin^{-1}\left(\frac{\sqrt{310}}{\sqrt{29}\sqrt{11}}\right) \approx 80.3^\circ$$

19 Let D be the origin.



$\overrightarrow{DM} = \begin{pmatrix} 0 \\ -5 \\ 7 \end{pmatrix}$, $\overrightarrow{DQ} = \begin{pmatrix} 4 \\ -10 \\ 7 \end{pmatrix}$

$$\begin{aligned} DM \bullet DQ &= \begin{pmatrix} 0 \\ -5 \\ 7 \end{pmatrix} \bullet \begin{pmatrix} 4 \\ -10 \\ 7 \end{pmatrix} \\ &= 0 + 50 + 49 \\ &= 99 \end{aligned}$$

$$\begin{aligned} |\,DM\,| &= \sqrt{0^2 + (-5)^2 + 7^2} \\ &= \sqrt{25 + 49} \\ &= \sqrt{74} \end{aligned} \qquad \begin{aligned} |\,DQ\,| &= \sqrt{4^2 + (-10)^2 + 7^2} \\ &= \sqrt{16 + 100 + 49} \\ &= \sqrt{165} \end{aligned}$$

$$\begin{aligned} D\widehat{Q}M &= \cos^{-1}\left(\frac{DM \bullet DQ}{|\,DM\,||\,DQ\,|}\right) \\ &= \cos^{-1}\left(\frac{99}{\sqrt{74}\sqrt{165}}\right) \\ &\approx 26.4^\circ \end{aligned}$$

or $$\begin{aligned} \overrightarrow{DM} \times \overrightarrow{DQ} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -5 & 7 \\ 4 & -10 & 7 \end{vmatrix} \\ &= 35\mathbf{i} + 28\mathbf{j} + 20\mathbf{k} \end{aligned}$$

$$\begin{aligned} |\,\overrightarrow{DM} \times \overrightarrow{DQ}\,| &= \sqrt{35^2 + 28^2 + 20^2} \\ &= \sqrt{2409} \end{aligned}$$

$$\begin{aligned} D\widehat{Q}M &= \sin^{-1}\left(\frac{|\,\overrightarrow{DM} \times \overrightarrow{DQ}\,|}{|\,DM\,||\,DQ\,|}\right) \\ &= \sin^{-1}\left(\frac{\sqrt{2409}}{\sqrt{74}\sqrt{165}}\right) \\ &\approx 26.4^\circ \end{aligned}$$

REVIEW SET 14C

1 **a** $\overrightarrow{PR} + \overrightarrow{RQ} = \overrightarrow{PQ}$

b $\begin{aligned} \overrightarrow{PS} + \overrightarrow{SQ} + \overrightarrow{QR} &= \overrightarrow{PQ} + \overrightarrow{QR} \\ &= \overrightarrow{PR} \end{aligned}$

2 **a** $\mathbf{m} - \mathbf{n} + \mathbf{p} = \begin{pmatrix} 6 \\ -3 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} + \begin{pmatrix} -1 \\ 3 \\ 6 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 11 \end{pmatrix}$

b $2\mathbf{n} - 3\mathbf{p} = 2\begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} - 3\begin{pmatrix} -1 \\ 3 \\ 6 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ -8 \end{pmatrix} - \begin{pmatrix} -3 \\ 9 \\ 18 \end{pmatrix} = \begin{pmatrix} 7 \\ -3 \\ -26 \end{pmatrix}$

c $\mathbf{m} + \mathbf{p} = \begin{pmatrix} 6 \\ -3 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 3 \\ 6 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 7 \end{pmatrix}$ $\therefore$ $\begin{aligned} |\,\mathbf{m} + \mathbf{p}\,| &= \sqrt{25 + 0 + 49} \\ &= \sqrt{74} \text{ units} \end{aligned}$

3 **a** If $\overrightarrow{AB} = \frac{1}{2}\overrightarrow{CD}$ then [AB] $\parallel$ [CD] and [AB] is half the length of [CD].

b If $\overrightarrow{AB} = 2\overrightarrow{AC}$ then [AB] $\parallel$ [AC] and AB = 2AC

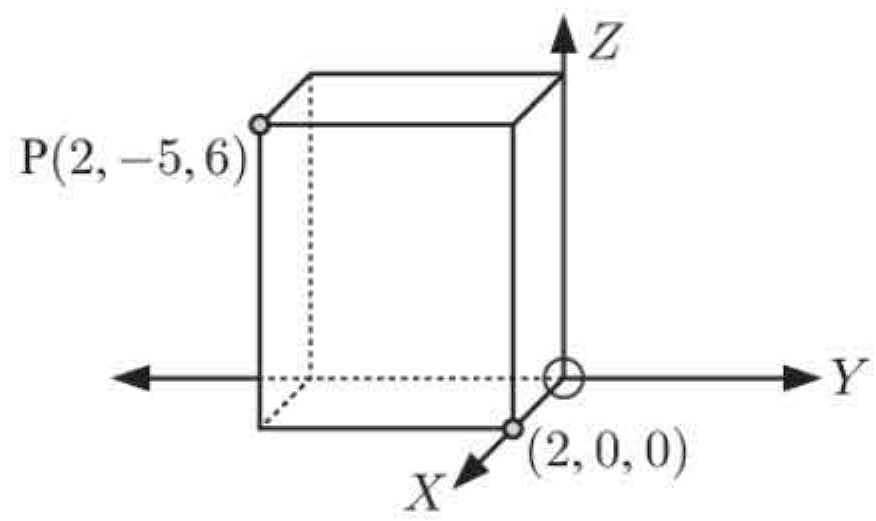
$\therefore$ A, B, and C are collinear and AB = 2AC.

So, C is the midpoint of [AB].

4 **a** $\overrightarrow{PQ} = \begin{pmatrix} -1-2 \\ 7--5 \\ 9-6 \end{pmatrix} = \begin{pmatrix} -3 \\ 12 \\ 3 \end{pmatrix}$

b $PQ = \sqrt{(-3)^2+12^2+3^2}$
$= \sqrt{9+144+9}$
$= \sqrt{162}$ units

c



$\therefore$ the *shortest* distance from P to the x-axis
$= \sqrt{(2-2)^2+(0--5)^2+(0-6)^2}$
$= \sqrt{0+25+36}$
$= \sqrt{61}$ units

5 **a** $\overrightarrow{OQ} = \overrightarrow{OR} + \overrightarrow{RQ} = \mathbf{r} + \mathbf{q}$

b $\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OR} + \overrightarrow{RQ} = -\mathbf{p} + \mathbf{r} + \mathbf{q}$

c $\overrightarrow{ON} = \overrightarrow{OR} + \overrightarrow{RN} = \mathbf{r} + \frac{1}{2}\mathbf{q}$

d $\overrightarrow{MN} = \overrightarrow{MQ} + \overrightarrow{QN}$
$= \frac{1}{2}\overrightarrow{PQ} + \frac{1}{2}\overrightarrow{QR}$
$= \frac{1}{2}(-\mathbf{p}+\mathbf{r}+\mathbf{q}) + \frac{1}{2}(-\mathbf{q})$ {from part **b**}
$= -\frac{1}{2}\mathbf{p} + \frac{1}{2}\mathbf{r} + \frac{1}{2}\mathbf{q} - \frac{1}{2}\mathbf{q}$
$= -\frac{1}{2}\mathbf{p} + \frac{1}{2}\mathbf{r}$

6 **a** $\mathbf{p} - 3\mathbf{x} = \mathbf{0}$
$\therefore\ 3\mathbf{x} = \mathbf{p}$
$\therefore\ \mathbf{x} = \frac{1}{3}\mathbf{p}$
$= \frac{1}{3}\begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$
$\therefore\ \mathbf{x} = \begin{pmatrix} -1 \\ \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$

b $2\mathbf{q} - \mathbf{x} = \mathbf{r}$
$\therefore\ \mathbf{r} + \mathbf{x} = 2\mathbf{q}$
$\therefore\ \mathbf{x} = 2\mathbf{q} - \mathbf{r}$
$= 2\begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}$
$\therefore\ \mathbf{x} = \begin{pmatrix} 4 \\ -8 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 4-3 \\ -8-2 \\ 2-0 \end{pmatrix} = \begin{pmatrix} 1 \\ -10 \\ 2 \end{pmatrix}$

7 Since $\mathbf{v}$ is parallel to $\mathbf{w}$, the angle θ between $\mathbf{v}$ and $\mathbf{w}$ is either $0°$ or $180°$.

Now, $\mathbf{v} \bullet \mathbf{w} = |\mathbf{v}|\,|\mathbf{w}|\cos\theta$
$= 3 \times 2 \times \cos 0°$ or $3 \times 2 \times \cos 180°$
$= 6(1)$ or $6(-1)$
$= \pm 6$

8 Vectors parallel to $\mathbf{i} + r\mathbf{j} + 2\mathbf{k}$ have form $k\begin{pmatrix} 1 \\ r \\ 2 \end{pmatrix}$, $k \neq 0$, $k \in \mathbb{R}$. If these vectors are perpendicular to $2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ then $k\begin{pmatrix} 1 \\ r \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} = 0$

$k(2+2r-2) = 0$
$2kr = 0$
but $k \neq 0 \quad \therefore\ r = 0$

length of $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \sqrt{1^2+0^2+2^2} = \sqrt{1+4}$
$= \sqrt{5}$

$\therefore$ the unit vector is $\frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ or $\begin{pmatrix} \frac{1}{\sqrt{5}} \\ 0 \\ \frac{2}{\sqrt{5}} \end{pmatrix}$

or $\frac{1}{\sqrt{5}}\mathbf{i} + \frac{2}{\sqrt{5}}\mathbf{k}$.

9 As the vectors are perpendicular,

$\begin{pmatrix} -4 \\ t+2 \\ t \end{pmatrix} \bullet \begin{pmatrix} t \\ 1+t \\ -3 \end{pmatrix} = 0$
$\therefore\ -4t + (t+2)(1+t) - 3t = 0$
$\therefore\ -4t + t + t^2 + 2 + 2t - 3t = 0$
$\therefore\ t^2 - 4t + 2 = 0$
$\therefore\ t = \dfrac{4 \pm \sqrt{16-4(1)(2)}}{2}$
$\therefore\ t = \dfrac{4 \pm \sqrt{8}}{2} = 2 \pm \sqrt{2}$

10
$$\overrightarrow{MK} = \begin{pmatrix} 3-4 \\ 1-1 \\ 4-3 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \qquad \overrightarrow{LK} = \begin{pmatrix} 3--2 \\ 1-1 \\ 4-3 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix}$$

$$\overrightarrow{ML} = \begin{pmatrix} -2-4 \\ 1-1 \\ 3-3 \end{pmatrix} = \begin{pmatrix} -6 \\ 0 \\ 0 \end{pmatrix} \qquad \overrightarrow{LM} = \begin{pmatrix} 4--2 \\ 1-1 \\ 3-3 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}$$

$$\overrightarrow{MK} \bullet \overrightarrow{ML} = |\overrightarrow{MK}||\overrightarrow{ML}| \cos\widehat{M}$$
$$\therefore\ 6+0+0 = \sqrt{1+0+1}\sqrt{36+0+0}\cos\widehat{M}$$
$$\therefore\ 6 = \sqrt{2}\times 6\cos\widehat{M}$$
$$\therefore\ \cos\widehat{M} = \tfrac{1}{\sqrt{2}}$$
$$\therefore\ \widehat{M} = 45^\circ$$
$$\text{and}\quad \widehat{K} \approx 180^\circ - 45^\circ - 11.3^\circ \approx 123.7^\circ$$

$$\therefore\ \overrightarrow{LK} \bullet \overrightarrow{LM} = |\overrightarrow{LK}||\overrightarrow{LM}| \cos\widehat{L}$$
$$\therefore\ 30+0+0 = \sqrt{25+0+1}\sqrt{36+0+0}\cos\widehat{L}$$
$$\therefore\ 30 = \sqrt{26}\times 6\cos\widehat{L}$$
$$\therefore\ \tfrac{5}{\sqrt{26}} = \cos\widehat{L}$$
$$\therefore\ \widehat{L} \approx 11.3^\circ$$

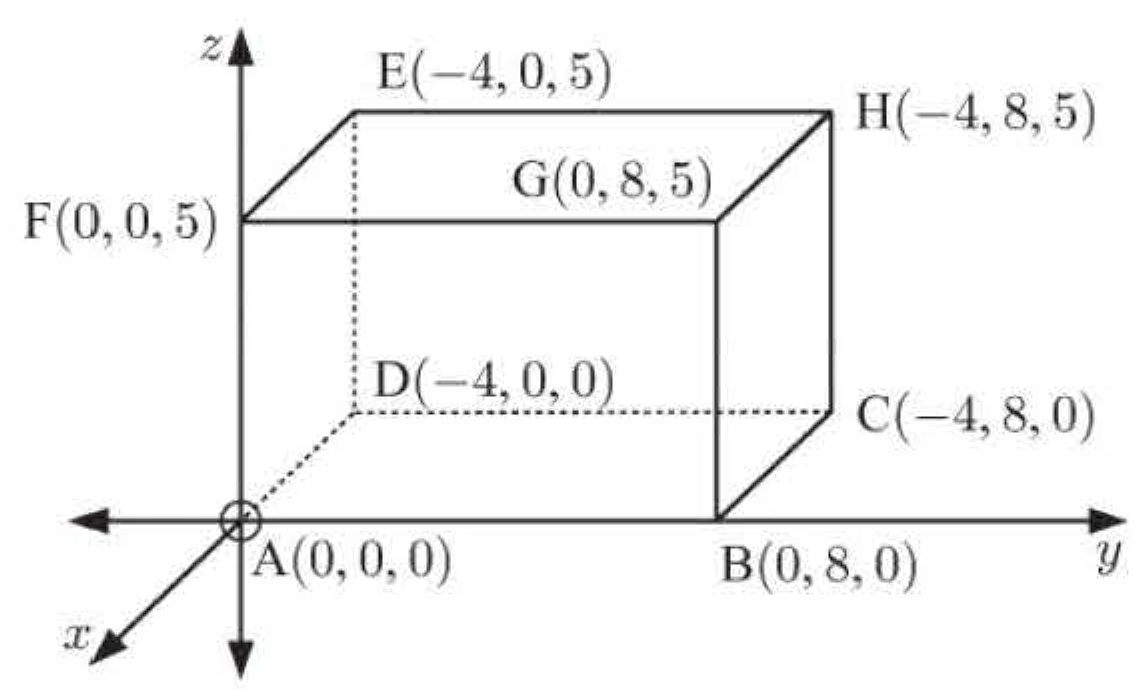
11 **a** $\begin{pmatrix} \frac{5}{13} \\ k \end{pmatrix}$ is a unit vector if

$$\sqrt{\left(\tfrac{5}{13}\right)^2 + k^2} = 1$$
$$\therefore\ \tfrac{25}{169} + k^2 = 1$$
$$\therefore\ k^2 = \tfrac{144}{169}$$
$$\therefore\ k = \pm\tfrac{12}{13}$$

b $\begin{pmatrix} k \\ k \\ k \end{pmatrix}$ is a unit vector if

$$\sqrt{k^2+k^2+k^2} = 1$$
$$\therefore\ 3k^2 = 1$$
$$\therefore\ k^2 = \tfrac{1}{3}$$
$$\therefore\ k = \pm\tfrac{1}{\sqrt{3}}$$

12 Let A be the origin.



$$\overrightarrow{AG} = \begin{pmatrix} 0 \\ 8 \\ 5 \end{pmatrix}, \qquad \overrightarrow{AC} = \begin{pmatrix} -4 \\ 8 \\ 0 \end{pmatrix}$$

$$\begin{aligned} AG \bullet AC &= \begin{pmatrix} 0 \\ 8 \\ 5 \end{pmatrix} \bullet \begin{pmatrix} -4 \\ 8 \\ 0 \end{pmatrix} \\ &= 0\times(-4) + 8\times 8 + 5\times 0 \\ &= 64 \end{aligned}$$

$$\begin{aligned} |AG| &= \sqrt{0^2+8^2+5^2} \\ &= \sqrt{64+25} \\ &= \sqrt{89} \end{aligned} \qquad \begin{aligned} |AC| &= \sqrt{(-4)^2+8^2+0^2} \\ &= \sqrt{16+64} \\ &= \sqrt{80} \end{aligned}$$

$$\begin{aligned} G\widehat{A}C &= \cos^{-1}\left(\frac{AG \bullet AC}{|AG||AC|}\right) \\ &= \cos^{-1}\left(\frac{64}{\sqrt{89}\sqrt{80}}\right) \\ &\approx 40.7^\circ \end{aligned}$$

or
$$\begin{aligned} \overrightarrow{AG}\times\overrightarrow{AC} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 8 & 5 \\ -4 & 8 & 0 \end{vmatrix} \\ &= 40\mathbf{i} + 20\mathbf{j} + 32\mathbf{k} \end{aligned} \qquad \begin{aligned} |\overrightarrow{AG}\times\overrightarrow{AC}| &= \sqrt{40^2+20^2+32^2} \\ &= \sqrt{1600+400+1024} \\ &= \sqrt{3024} \end{aligned}$$

$$\begin{aligned} G\widehat{A}C &= \sin^{-1}\left(\frac{|\overrightarrow{AG}\times\overrightarrow{AC}|}{|AG||AC|}\right) \\ &= \sin^{-1}\left(\frac{\sqrt{3024}}{\sqrt{89}\sqrt{80}}\right) \\ &\approx 40.7^\circ \end{aligned}$$

13
$$\begin{aligned}\text{LHS} &= \mathbf{p} \bullet (\mathbf{q} - \mathbf{r})\\ &= \begin{pmatrix} 3 \\ -2 \end{pmatrix} \bullet \left[\begin{pmatrix} -2 \\ 5 \end{pmatrix} - \begin{pmatrix} 1 \\ -3 \end{pmatrix}\right]\\ &= \begin{pmatrix} 3 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} -3 \\ 8 \end{pmatrix}\\ &= -9 - 16\\ &= -25\end{aligned}$$

$$\begin{aligned}\text{RHS} &= \mathbf{p} \bullet \mathbf{q} - \mathbf{p} \bullet \mathbf{r}\\ &= \begin{pmatrix} 3 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} -2 \\ 5 \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -3 \end{pmatrix}\\ &= (-6 - 10) - (3 + 6)\\ &= -16 - 9\\ &= -25\end{aligned}$$

$\therefore$ LHS = RHS ✓

14 **a**
$$\begin{aligned}\overrightarrow{\text{PQ}} &= \begin{pmatrix} 4 - (-1) \\ 0 - 2 \\ -1 - 3 \end{pmatrix}\\ &= \begin{pmatrix} 5 \\ -2 \\ -4 \end{pmatrix}\end{aligned}$$

b Let X be the point $(1, 0, 0)$ on the x-axis.

$\overrightarrow{\text{OX}} = (1, 0, 0)$,
$$\begin{aligned}\theta &= \cos^{-1}\left(\frac{\overrightarrow{\text{PQ}} \bullet \overrightarrow{\text{OX}}}{|\overrightarrow{\text{PQ}}||\overrightarrow{\text{OX}}|}\right)\\ &= \cos^{-1}\left(\frac{5 + 0 + 0}{\sqrt{25 + 4 + 16}\sqrt{1}}\right)\\ &= \cos^{-1}\left(\frac{5}{\sqrt{45}}\right)\\ &\approx 41.8^\circ\end{aligned}$$

15
$$\overrightarrow{\text{MP}} \bullet \overrightarrow{\text{PT}} = 0$$

Now, $\begin{pmatrix} 5 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 5 \end{pmatrix} = 5 + (-5) = 0$

$\therefore$ $\overrightarrow{\text{PT}}$ has the form $k\begin{pmatrix} 1 \\ 5 \end{pmatrix}$, $k \neq 0$, $k \in \mathbb{R}$.

Also, $|\overrightarrow{\text{MP}}| = |\overrightarrow{\text{PT}}|$

$$\begin{aligned}\therefore\ \sqrt{5^2 + (-1)^2} &= \sqrt{k^2 + (5k)^2}\\ \therefore\ \sqrt{26} &= \sqrt{26k^2}\\ &= |k|\sqrt{26}\\ \therefore\ k &= \pm 1\end{aligned}$$

So $\overrightarrow{\text{PT}} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$ or $\begin{pmatrix} -1 \\ -5 \end{pmatrix}$

Finally, $\overrightarrow{\text{OT}} = \overrightarrow{\text{OM}} + \overrightarrow{\text{MP}} + \overrightarrow{\text{PT}}$

$$= \begin{pmatrix} -2 \\ 4 \end{pmatrix} + \begin{pmatrix} 5 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

or $\begin{pmatrix} -2 \\ 4 \end{pmatrix} + \begin{pmatrix} 5 \\ -1 \end{pmatrix} + \begin{pmatrix} -1 \\ -5 \end{pmatrix}$

$\therefore$ $\overrightarrow{\text{OT}} = \begin{pmatrix} 4 \\ 8 \end{pmatrix}$ or $\begin{pmatrix} 2 \\ -2 \end{pmatrix}$

16
$$\begin{aligned}\mathbf{p} \bullet \mathbf{q} &= \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ -4 \\ 2 \end{pmatrix}\\ &= 2(-1) + (-1)(-4) + 4 \times 2\\ &= -2 + 4 + 8\\ &= 10\end{aligned}$$

$$\begin{aligned}|\mathbf{p}| &= \sqrt{2^2 + (-1)^2 + 4^2}\\ &= \sqrt{4 + 1 + 16}\\ &= \sqrt{21}\end{aligned}$$

$$\begin{aligned}|\mathbf{q}| &= \sqrt{(-1)^2 + (-4)^2 + 2^2}\\ &= \sqrt{1 + 16 + 4}\\ &= \sqrt{21}\end{aligned}$$

If θ is the angle between $\mathbf{p}$ and $\mathbf{q}$ then
$$\begin{aligned}\theta &= \cos^{-1}\left(\frac{\mathbf{p} \bullet \mathbf{q}}{|\mathbf{p}||\mathbf{q}|}\right)\\ &= \cos^{-1}\left(\frac{10}{\sqrt{21}\sqrt{21}}\right)\\ &\approx 61.6^\circ\end{aligned}$$

17 $\mathbf{u} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$

$$\cos\theta = \frac{\mathbf{u} \bullet \mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \frac{\begin{pmatrix} 2 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 3 \end{pmatrix}}{\sqrt{2^2+1^2}\sqrt{3^2}}$$

$$= \frac{3}{\sqrt{5}\sqrt{9}} = \frac{1}{\sqrt{5}}$$

Now $\sin^2\theta + \cos^2\theta = 1$

$\therefore \sin^2\theta + \frac{1}{5} = 1$

$\therefore \sin^2\theta = \frac{4}{5}$

$\therefore \sin\theta = \pm\frac{2}{\sqrt{5}}$

But θ is acute, so $\sin\theta = \frac{2}{\sqrt{5}}$.

or $\sin\theta = \dfrac{|\mathbf{u} \times \mathbf{v}|}{|\mathbf{u}||\mathbf{v}|}$ $\quad |\mathbf{u} \times \mathbf{v}| = (2 \times 3 - 1 \times 0)$
$= 6 - 0$
$= 6$

$\therefore \sin\theta = \dfrac{6}{\sqrt{5}\sqrt{9}} = \frac{6}{3\sqrt{5}} = \frac{2}{\sqrt{5}}$

18
$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 0 & 3 \\ 2 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 3 \\ -1 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -1 & 3 \\ 2 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -1 & 0 \\ 2 & -1 \end{vmatrix}\mathbf{k}$$

$$= (0 \times 1 - (-1) \times 3)\mathbf{i} - ((-1) \times 1 - 2 \times 3)\mathbf{j} + ((-1) \times (-1) - 2 \times 0)\mathbf{k}$$

$$= 3\mathbf{i} + 7\mathbf{j} + \mathbf{k}$$

$= \begin{pmatrix} 3 \\ 7 \\ 1 \end{pmatrix}$ is a perpendicular vector

$$\sqrt{3^2+7^2+1^2} = \sqrt{9+49+1}$$
$$= \sqrt{59}$$

$\therefore \frac{1}{\sqrt{59}}\begin{pmatrix} 3 \\ 7 \\ 1 \end{pmatrix}$ is a perpendicular unit vector.

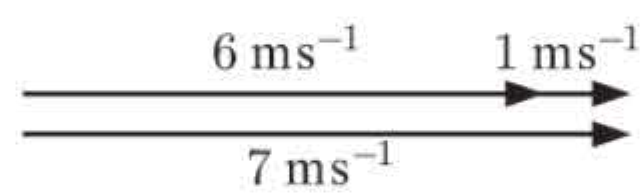
$\therefore \pm\frac{3}{\sqrt{59}}\begin{pmatrix} 3 \\ 7 \\ 1 \end{pmatrix}$ are the two 3 unit length perpendicular vectors.

Chapter 15

VECTOR APPLICATIONS

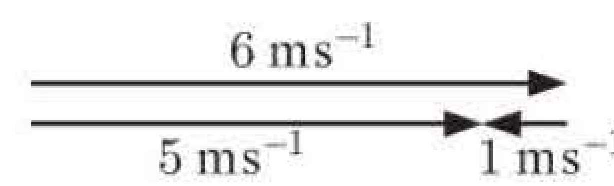
EXERCISE 15A

1 **a**



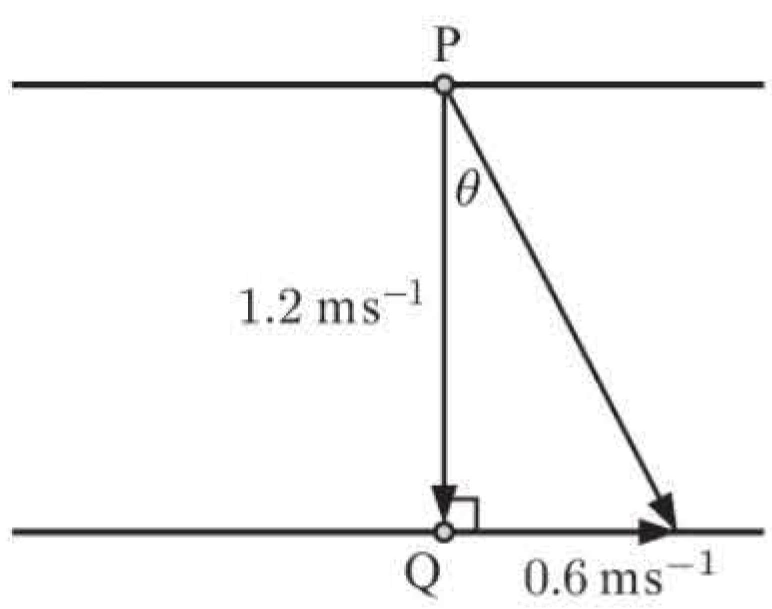
If the athlete is assisted by a wind of $1\ \text{m s}^{-1}$ his speed will be $7\ \text{m s}^{-1}$.

b



If the athlete runs into a head wind of $1\ \text{m s}^{-1}$ his speed will be $5\ \text{m s}^{-1}$.

2 **a**



$$\begin{aligned}(\text{actual speed})^2 &= (\text{swimming speed})^2 + (\text{current})^2\\ &= 1.2^2 + 0.6^2\\ &= 1.8\end{aligned}$$

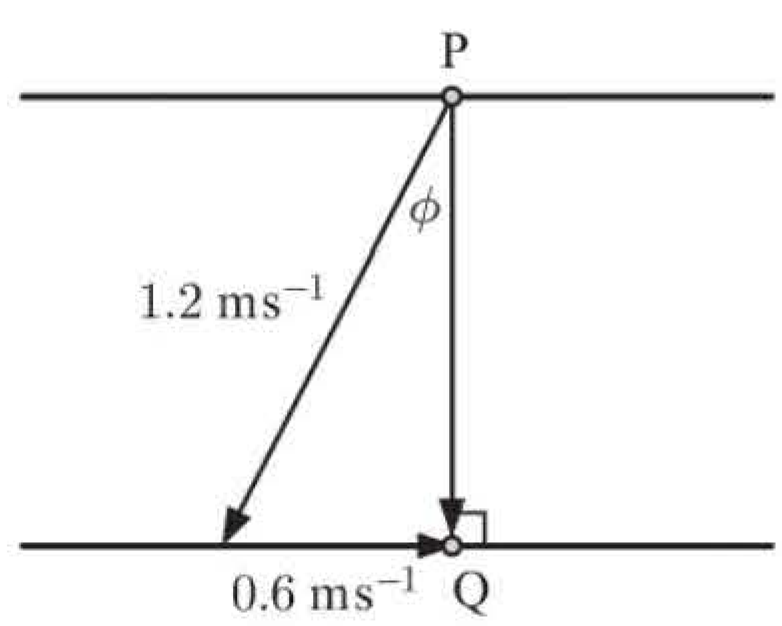
$\therefore$ actual speed $= \sqrt{1.8} \approx 1.34\ \text{m s}^{-1}$

$\tan\theta = \dfrac{0.6}{1.2}$

$\therefore \theta \approx 26.6°$

$\therefore$ Mary's actual velocity is approximately $1.34\ \text{m s}^{-1}$ in the direction $26.6°$ to the left of her intended line.

b **i**



Mary needs to aim to the right of Q so the current will correct her direction.

$\sin\phi = \dfrac{0.6}{1.2}$

$\therefore \phi = 30°$

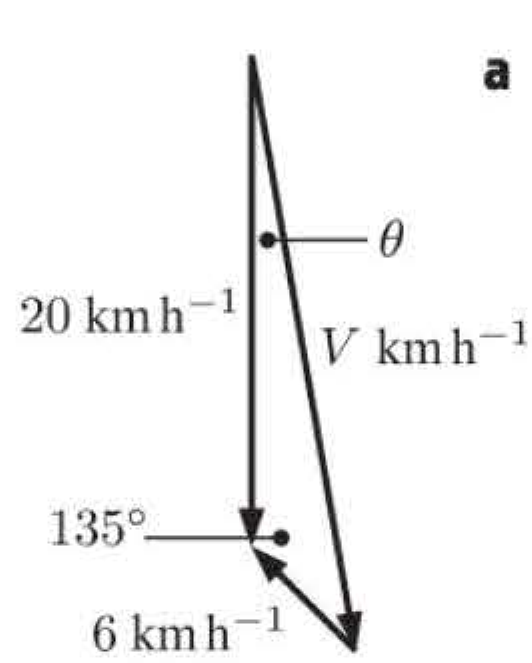
$\therefore$ Mary should aim to swim $30°$ to the right of Q.

ii $(\text{swimming speed})^2 = (\text{actual speed})^2 + (\text{current})^2$

$$\begin{aligned}\therefore (\text{actual speed})^2 &= 1.2^2 - 0.6^2\\ &= 1.08\end{aligned}$$

$\therefore$ Mary's actual speed $= \sqrt{1.08} \approx 1.04\ \text{m s}^{-1}$

3



a Using the cosine rule,

$V^2 = 20^2 + 6^2 - 2 \times 20 \times 6 \times \cos 135°$

$\therefore V \approx 24.6$

$\therefore$ the equivalent speed in still water is $24.6\ \text{km h}^{-1}$.

b Using the sine rule,

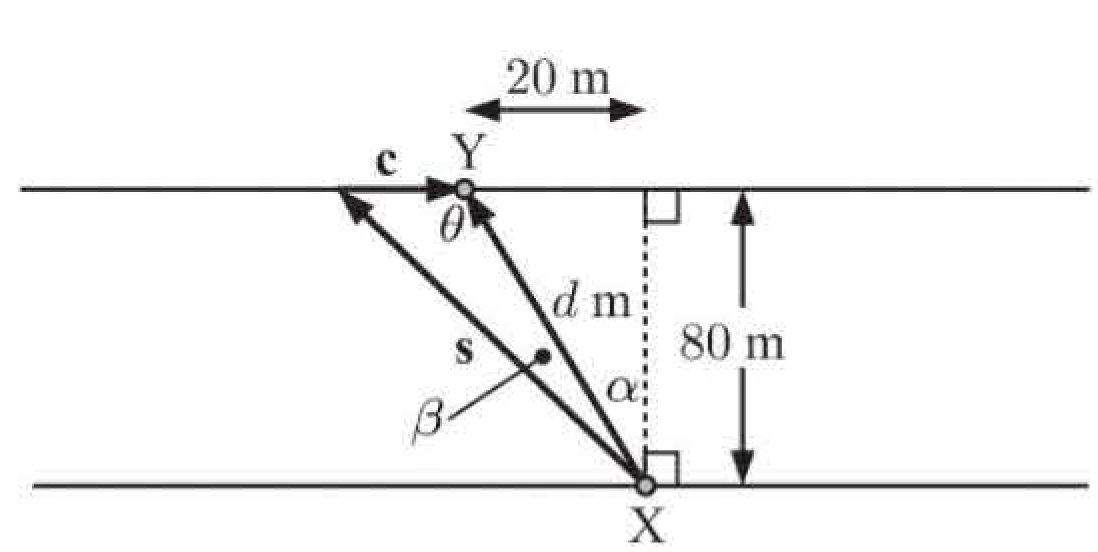
$\dfrac{\sin\theta}{6} \approx \dfrac{\sin 135°}{24.61}$

$\therefore \theta \approx \sin^{-1}\left(\dfrac{6 \times \sin 135°}{24.61}\right)$

$\therefore \theta \approx 9.93°$

$\therefore$ the boat should head $9.93°$ east of south.

4



a $d^2 = 80^2 + 20^2$ {Pythagoras}

$\therefore d = \sqrt{80^2 + 20^2}$ $\{d > 0\}$

$\therefore d \approx 82.5$

$\therefore$ the distance from X to Y is about 82.5 m.

b $\alpha = \tan^{-1}(\frac{20}{80}) \approx 14.04°$

$\therefore\ \theta \approx 90° + 14.04°$ {exterior angle of $\triangle$}

$\therefore\ \theta \approx 104.04°$

In t seconds, Stephanie can swim $1.8t$ metres, and the current will move $0.3t$ metres.

$\therefore\ |\mathbf{s}| = 1.8t$ and $|\mathbf{c}| = 0.3t$

Using the sine rule,

$$\frac{\sin\beta}{0.3t} = \frac{\sin\theta}{1.8t}$$

$$\beta \approx \sin^{-1}\left(\frac{0.3 \times \sin 104.04°}{1.8}\right)$$

$\therefore\ \beta \approx 9.31°$

$\therefore\ \alpha + \beta \approx 23.3°$

$\therefore$ Stephanie should head $23.3°$ to the left of the perpendicular across the river.

c $\tan(\alpha+\beta) = \dfrac{20 + 0.3t}{80}$

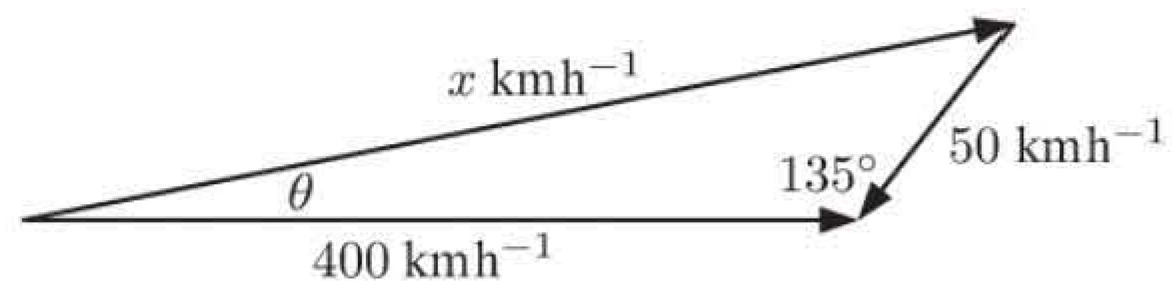
$\therefore\ 20 + 0.3t \approx 80\tan(23.34°)$

$\therefore\ t \approx \dfrac{80\tan(23.34°) - 20}{0.3}$

$\therefore\ t \approx 48.4$

$\therefore$ Stephanie will take 48.4 seconds to cross the river.

5



a Using the cosine rule,

$$x^2 = 50^2 + 400^2 - 2 \times 50 \times 400\cos 135°$$

$\therefore\ x \approx 436.79$

The aeroplane should fly so that its speed in still air would be 437 km h^{-1}.

The wind slows the aeroplane down to 400 km h^{-1}.

b Using the sine rule,

$$\frac{\sin\theta}{50} \approx \frac{\sin 135°}{436.79}$$

$\therefore\ \theta \approx 4.64°$

The aeroplane should head $4.64°$ north of east.

EXERCISE 15B

1 **a** Given A(2, 1, 1), B(4, 3, 0), and C(1, 3, −2), $\overrightarrow{AB} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix}$.

$$\therefore\ \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & -1 \\ -1 & 2 & -3 \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 2 & -3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & -1 \\ -1 & -3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & 2 \\ -1 & 2 \end{vmatrix}\mathbf{k}$$

$$= -4\mathbf{i} + 7\mathbf{j} + 6\mathbf{k}$$

$\therefore$ area $= \frac{1}{2}|-4\mathbf{i} + 7\mathbf{j} + 6\mathbf{k}|$ {area $= \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$}

$= \frac{1}{2}\sqrt{(-4)^2 + 7^2 + 6^2}$

$= \frac{1}{2}\sqrt{101}$ units^2

b Given A(0, 0, 0), B(−1, 2, 3), and C(1, 2, 6), $\overrightarrow{AB} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}$.

$$\therefore\ \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & 3 \\ 1 & 2 & 6 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 2 & 6 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -1 & 3 \\ 1 & 6 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -1 & 2 \\ 1 & 2 \end{vmatrix}\mathbf{k}$$

$$= 6\mathbf{i} + 9\mathbf{j} - 4\mathbf{k}$$

$\therefore$ area $= \frac{1}{2}|6\mathbf{i} + 9\mathbf{j} - 4\mathbf{k}|$ {area $= \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$}

$= \frac{1}{2}\sqrt{6^2 + 9^2 + (-4)^2}$

$= \frac{1}{2}\sqrt{133}$ units^2

c Given A(1, 3, 2), B(2, −1, 0), and C(1, 10, 6), $\overrightarrow{AB} = \begin{pmatrix} 1 \\ -4 \\ -2 \end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix} 0 \\ 7 \\ 4 \end{pmatrix}$.

$$\therefore\ \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -4 & -2 \\ 0 & 7 & 4 \end{vmatrix} = \begin{vmatrix} -4 & -2 \\ 7 & 4 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & -2 \\ 0 & 4 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & -4 \\ 0 & 7 \end{vmatrix}\mathbf{k}$$

$$= -2\mathbf{i} - 4\mathbf{j} + 7\mathbf{k}$$

$$\therefore\ \text{area} = \tfrac{1}{2}\,|-2\mathbf{i} - 4\mathbf{j} + 7\mathbf{k}| = \tfrac{1}{2}\sqrt{(-2)^2 + (-4)^2 + 7^2} = \tfrac{1}{2}\sqrt{69}\ \text{units}^2$$

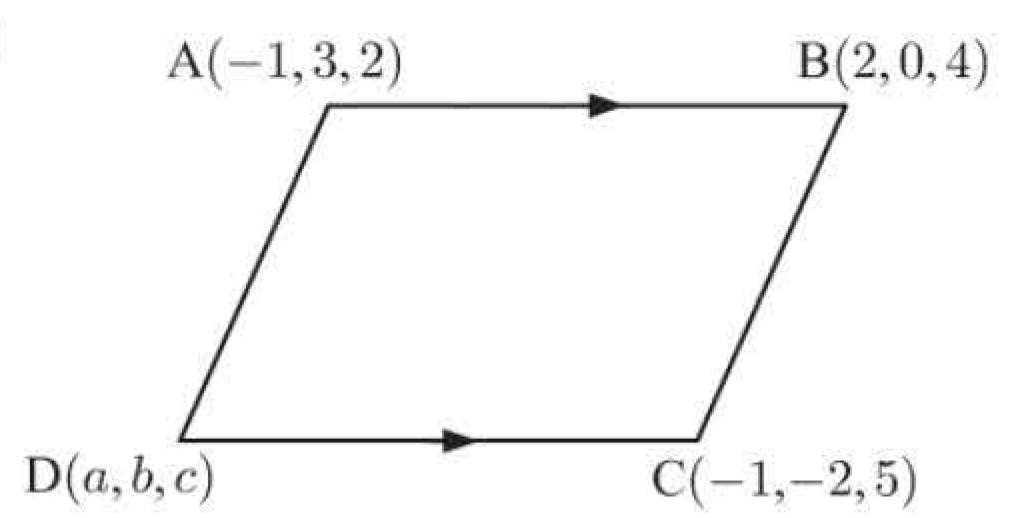
2 Given A(−1, 2, 2), B(2, −1, 4), and C(0, 1, 0), $\overrightarrow{AB} = \begin{pmatrix} 3 \\ -3 \\ 2 \end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$.

$$\therefore\ \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -3 & 2 \\ 1 & -1 & -2 \end{vmatrix} = \begin{vmatrix} -3 & 2 \\ -1 & -2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & 2 \\ 1 & -2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & -3 \\ 1 & -1 \end{vmatrix}\mathbf{k}$$

$$= 8\mathbf{i} + 8\mathbf{j}$$

$$\therefore\ \text{area of parallelogram} = |\,8\mathbf{i} + 8\mathbf{j}\,| = \sqrt{8^2 + 8^2} = 8\sqrt{2}\ \text{units}^2$$

3 **a**



Suppose D is at (a, b, c).

Since $\overrightarrow{AB} = \overrightarrow{DC}$,

$$\begin{pmatrix} 3 \\ -3 \\ 2 \end{pmatrix} = \begin{pmatrix} -1-a \\ -2-b \\ 5-c \end{pmatrix}$$

$\therefore\ -1-a = 3,\quad -2-b = -3,\quad \text{and}\quad 5-c = 2$

$\therefore\ a = -4,\quad b = 1,\quad \text{and}\quad c = 3$

$\therefore$ D is at (−4, 1, 3).

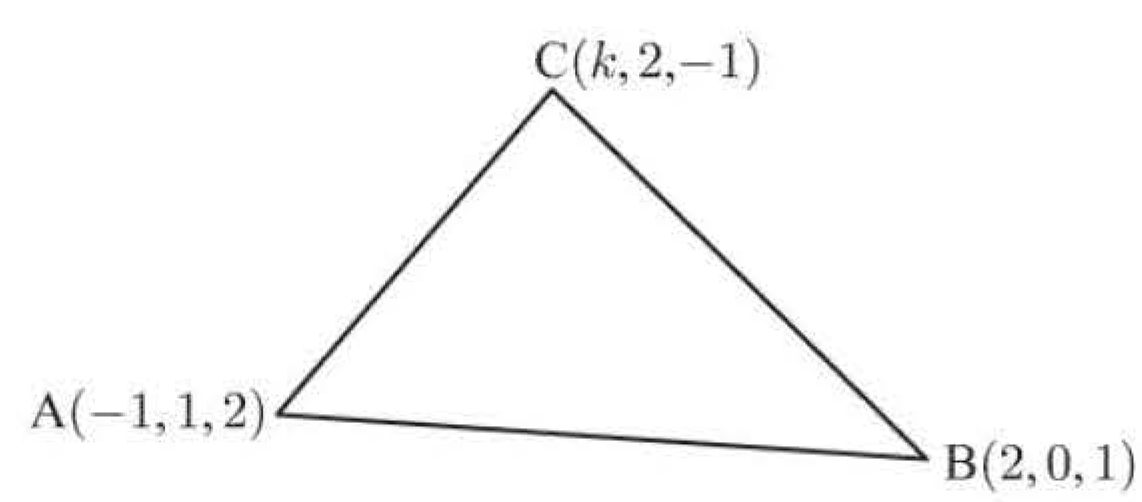
b $\overrightarrow{BC} = \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix}$ and $\overrightarrow{BA} = \begin{pmatrix} -3 \\ 3 \\ -2 \end{pmatrix}$

$$\therefore\ \overrightarrow{BC} \times \overrightarrow{BA} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & -2 & 1 \\ -3 & 3 & -2 \end{vmatrix} = \begin{vmatrix} -2 & 1 \\ 3 & -2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -3 & 1 \\ -3 & -2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -3 & -2 \\ -3 & 3 \end{vmatrix}\mathbf{k}$$

$$= \mathbf{i} - 9\mathbf{j} - 15\mathbf{k}$$

$$\therefore\ \text{area} = |\mathbf{i} - 9\mathbf{j} - 15\mathbf{k}| = \sqrt{1^2 + (-9)^2 + (-15)^2} = \sqrt{307}\ \text{units}^2$$

4 Now $\overrightarrow{AB} = \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix} k+1 \\ 1 \\ -3 \end{pmatrix}$



Area of $\triangle$ABC $= \frac{1}{2}\,|\,\overrightarrow{AC} \times \overrightarrow{AB}\,|$

$$\therefore\ \sqrt{88} = \tfrac{1}{2}\left|\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ k+1 & 1 & -3 \\ 3 & -1 & -1 \end{vmatrix}\right| = \tfrac{1}{2}\left|\begin{vmatrix} 1 & -3 \\ -1 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} k+1 & -3 \\ 3 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} k+1 & 1 \\ 3 & -1 \end{vmatrix}\mathbf{k}\right|$$

$$\therefore\ \sqrt{352} = |(-1-3)\mathbf{i} - (-(k+1) - -9)\mathbf{j} + (-(k+1) - 3)\mathbf{k}|$$

$$\therefore\ \sqrt{352} = |-4\mathbf{i} + (k-8)\mathbf{j} + (-k-4)\mathbf{k}|$$

$$\therefore\ \sqrt{352} = \sqrt{16 + (k-8)^2 + (-k-4)^2}$$

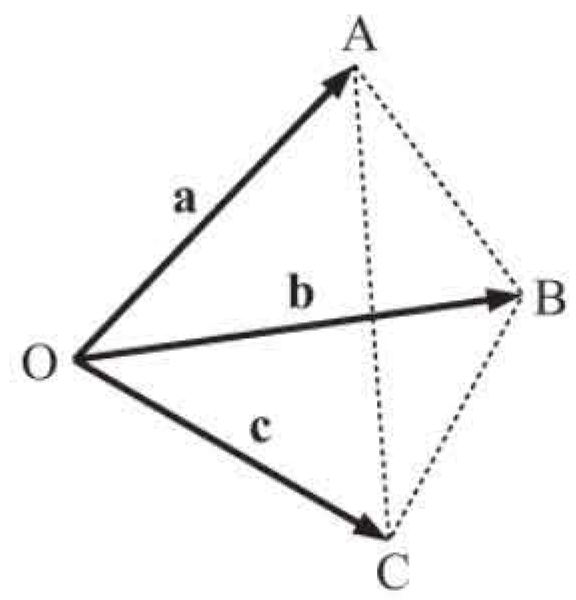
$$\therefore\ 352 = 16 + k^2 - 16k + 64 + k^2 + 8k + 16$$

$$\therefore\ 2k^2 - 8k - 256 = 0$$

$$\therefore\ k^2 - 4k - 128 = 0$$

$$\therefore\ k = \frac{4 \pm \sqrt{16 + 4(1)(128)}}{2} = 2 \pm \sqrt{132} = 2 \pm 2\sqrt{33}$$

5



Total surface area S of the tetrahedron is the sum of the areas of the 4 triangular faces.

Now $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -\mathbf{a} + \mathbf{b} = \mathbf{b} - \mathbf{a}$

and $\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC} = -\mathbf{a} + \mathbf{c} = \mathbf{c} - \mathbf{a}$

$$\therefore\ S = \tfrac{1}{2}|\mathbf{a}\times\mathbf{b}| + \tfrac{1}{2}|\mathbf{a}\times\mathbf{c}| + \tfrac{1}{2}|\mathbf{b}\times\mathbf{c}| + \tfrac{1}{2}|(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})|$$
$$= \tfrac{1}{2}\{|\mathbf{a}\times\mathbf{b}| + |\mathbf{a}\times\mathbf{c}| + |\mathbf{b}\times\mathbf{c}| + |(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})|\}$$

EXERCISE 15C

1 **a** **i** $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 4 \end{pmatrix},\ \ \lambda \in \mathbb{R}$

ii $x = 3 + \lambda,\ \ y = -4 + 4\lambda,\ \ \lambda \in \mathbb{R}$

iii $\lambda = x - 3 = \dfrac{y+4}{4}$

$\therefore\ 4x - 12 = y + 4$

$\therefore\ 4x - y = 16$

b **i** If the line has direction vector $\mathbf{b}$ perpendicular to $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$, then

$\mathbf{b} \bullet \begin{pmatrix} 5 \\ 2 \end{pmatrix} = 0$

$\therefore\ \mathbf{b} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$ is a reasonable choice

$\therefore\ \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} + \lambda\begin{pmatrix} -2 \\ 5 \end{pmatrix},\ \ \lambda \in \mathbb{R}$

ii $x = 5 - 2\lambda,\ \ y = 2 + 5\lambda,\ \ \lambda \in \mathbb{R}$

iii $\lambda = \dfrac{x-5}{-2} = \dfrac{y-2}{5}$

$\therefore\ 5x - 25 = -2y + 4$

$\therefore\ 5x + 2y = 29$

c **i** $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -6 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} 3 \\ 7 \end{pmatrix},\ \ \lambda \in \mathbb{R}$

ii $x = -6 + 3\lambda,\ \ y = 7\lambda,\ \ \lambda \in \mathbb{R}$

iii $\lambda = \dfrac{x+6}{3} = \dfrac{y}{7}$

$\therefore\ 7x + 42 = 3y$

$\therefore\ 7x - 3y = -42$

d **i** Take $(-1,\ 11)$ as our fixed point,

so $\mathbf{a} = \begin{pmatrix} -1 \\ 11 \end{pmatrix}$.

The direction vector $\mathbf{b} = \begin{pmatrix} -3-(-1) \\ 12-11 \end{pmatrix}$

$= \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$\therefore\ \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 11 \end{pmatrix} + \lambda\begin{pmatrix} -2 \\ 1 \end{pmatrix},\ \ \lambda \in \mathbb{R}$

ii $x = -1 - 2\lambda,\ \ y = 11 + \lambda,\ \ \lambda \in \mathbb{R}$

iii $\lambda = \dfrac{x+1}{-2} = y - 11$

$\therefore\ x + 1 = -2y + 22$

$\therefore\ x + 2y = 21$

2 **a** $x = -1 + 2\lambda,\ \ y = 4 - \lambda,\ \ \lambda \in \mathbb{R}$

b When $\lambda = 0,\ x = -1 + 2(0) = -1$ and $y = 4 - 0 = 4$ $\therefore$ the point is $(-1,\ 4)$.

When $\lambda = 1,\ x = -1 + 2(1) = 1$ and $y = 4 - 1 = 3$ $\therefore$ the point is $(1,\ 3)$.

When $\lambda = 3,\ x = -1 + 2(3) = 5$ and $y = 4 - 3 = 1$ $\therefore$ the point is $(5,\ 1)$.

When $\lambda = -1,\ x = -1 + 2(-1) = -3$ and $y = 4 - (-1) = 5$ $\therefore$ the point is $(-3,\ 5)$.

When $\lambda = -4,\ x = -1 + 2(-4) = -9$ and $y = 4 - (-4) = 8$ $\therefore$ the point is $(-9,\ 8)$.

3 **a** If $\lambda + 2 = 3$ and $1 - 3\lambda = -2$,

then $\lambda = 1$ and $-3\lambda = -3$

$\therefore\ \lambda = 1$

Since $\lambda = 1$ in each case,
$(3,\ -2)$ lies on the line.

b If $(k,\ 4)$ lies on $x = 1 - 2\lambda,\ \ y = 1 + \lambda$, then

$k = 1 - 2\lambda$ and $4 = 1 + \lambda$

$\therefore\ \lambda = 3$

and $k = 1 - 6$

$\therefore\ k = -5$

4 **a** When $\lambda = 1$, $\mathbf{r} = \begin{pmatrix} 1 \\ 5 \end{pmatrix} + 1 \times \begin{pmatrix} -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1-1 \\ 5+3 \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \end{pmatrix}$

$\therefore$ the point is $(0, 8)$.

b $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$ is a non-zero scalar multiple of $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$. It is parallel and in the opposite direction, so it could also be used to describe the direction of the line.

c The line passes through point $(0, 8)$ and has direction vector $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$.

$\therefore$ $\mathbf{r} = \begin{pmatrix} 0 \\ 8 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -3 \end{pmatrix}$, $\mu \in \mathbb{R}$ is an alternative vector equation for the line.

5 **a** **i** $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -7 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$, $\lambda \in \mathbb{R}$

ii $x = 1 + 2\lambda$, $y = 3 + \lambda$, $z = -7 + 3\lambda$, $\lambda \in \mathbb{R}$ **iii** $\lambda = \dfrac{x-1}{2} = y - 3 = \dfrac{z+7}{3}$

b **i** $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$, $\lambda \in \mathbb{R}$

ii $x = \lambda$, $y = 1 + \lambda$, $z = 2 - 2\lambda$, $\lambda \in \mathbb{R}$ **iii** $\lambda = x = y - 1 = \dfrac{-z+2}{2}$

c **i** Since the line is parallel to the X-axis, it has direction vector $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$\therefore$ $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\lambda \in \mathbb{R}$

ii $x = -2 + \lambda$, $y = 2$, $z = 1$, $\lambda \in \mathbb{R}$ **iii** $y = 2$, $z = 1$

d **i** $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$, $\lambda \in \mathbb{R}$

ii $x = 2\lambda$, $y = 2 - \lambda$, $z = -1 + 3\lambda$, $\lambda \in \mathbb{R}$ **iii** $\lambda = \dfrac{x}{2} = -y + 2 = \dfrac{z+1}{3}$

e **i** Since the line is perpendicular to the XOY plane, it has direction vector $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\therefore$ $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $\lambda \in \mathbb{R}$

ii $x = 3$, $y = 2$, $z = -1 + \lambda$, $\lambda \in \mathbb{R}$ **iii** $x = 3$, $y = 2$

6 **a** $\overrightarrow{AB} = \begin{pmatrix} -1-1 \\ 3-2 \\ 2-1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$ $\therefore$ $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$, $\lambda \in \mathbb{R}$

b $\overrightarrow{CD} = \begin{pmatrix} 3-0 \\ 1-1 \\ -1-3 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ -4 \end{pmatrix}$ $\therefore$ $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 0 \\ -4 \end{pmatrix}$, $\lambda \in \mathbb{R}$

c $\overrightarrow{EF} = \begin{pmatrix} 1-1 \\ -1-2 \\ 5-5 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \\ 0 \end{pmatrix}$ $\therefore$ $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ -3 \\ 0 \end{pmatrix}$, $\lambda \in \mathbb{R}$

d $\overrightarrow{GH} = \begin{pmatrix} 5-0 \\ -1-1 \\ 3--1 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 4 \end{pmatrix}$ $\therefore$ $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ -2 \\ 4 \end{pmatrix}$, $\lambda \in \mathbb{R}$

7 **a** $\begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix}$ **b** $\begin{pmatrix} -1 \\ 1 \\ -3 \end{pmatrix}$

c $\lambda = \dfrac{x-2}{3} = \dfrac{y+1}{2} = z - 1$

$\therefore\ x - 2 = 3\lambda$ and $y + 1 = 2\lambda$ and $z - 1 = \lambda$

$\therefore\ x = 2 + 3\lambda$ $\quad y = -1 + 2\lambda$ $\quad z = 1 + \lambda$

$\therefore$ direction vector is $\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$

d $\mu = \dfrac{1-x}{2} = \dfrac{y}{4} = \dfrac{z-3}{3}$

$\therefore\ 2\mu = 1 - x$ and $y = 4\mu$ and $z - 3 = 3\mu$

$\therefore\ x = 1 - 2\mu$ $\quad y = 4\mu$ $\quad z = 3 + 3\mu$

$\therefore$ direction vector is $\begin{pmatrix} -2 \\ 4 \\ 3 \end{pmatrix}$

8 Given $x = 1 - \lambda,\ \ y = 3 + \lambda,\ \ z = 3 - 2\lambda$:

a The line meets the XOY plane when $z = 0$ $\quad \therefore\ 3 - 2\lambda = 0$

$\therefore\ \lambda = \frac{3}{2}$

Then $x = 1 - \frac{3}{2} = -\frac{1}{2}$ and $y = 3 + \frac{3}{2} = \frac{9}{2}$, so the point is $\left(-\frac{1}{2}, \frac{9}{2}, 0\right)$.

b The line meets the YOZ plane when $x = 0$ $\quad \therefore\ 1 - \lambda = 0$

$\therefore\ \lambda = 1$

Then $y = 3 + 1 = 4$ and $z = 3 - 2 = 1$, so the point is $(0, 4, 1)$.

c The line meets the XOZ plane when $y = 0$ $\quad \therefore\ 3 + \lambda = 0$

$\therefore\ \lambda = -3$

Then $x = 1 - (-3) = 4$ and $z = 3 - 2(-3) = 9$, so the point is $(4, 0, 9)$.

9 **a** When $\lambda = 0$, $x = x_0$, $y = y_0$, and $z = z_0$

$\therefore\ (x_0, y_0, z_0)$

b $\begin{pmatrix} l \\ m \\ n \end{pmatrix}$

c $\lambda = \dfrac{x - x_0}{l} = \dfrac{y - y_0}{m} = \dfrac{z - z_0}{n},\quad l, m, n \neq 0$

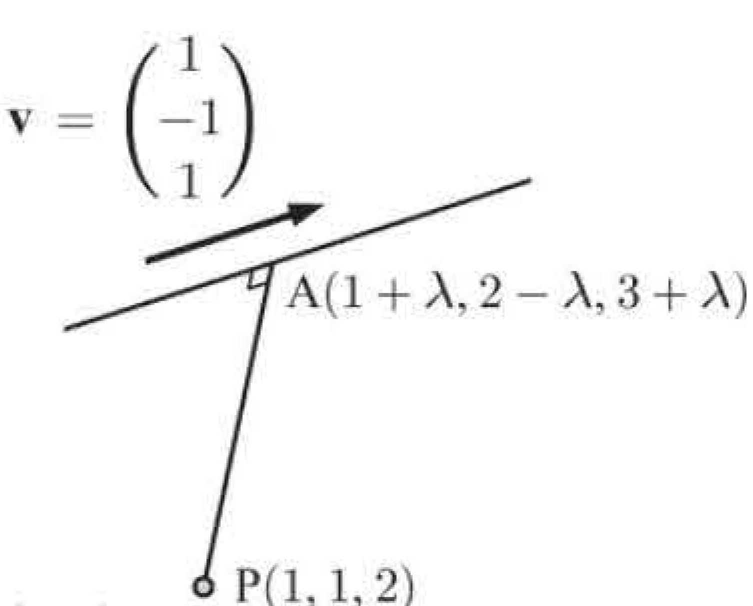
10 Given a line with equations $x = 2 - \lambda$, $y = 3 + 2\lambda$, and $z = 1 + \lambda$,
the distance to the point $(1, 0, -2)$ is $\sqrt{(2 - \lambda - 1)^2 + (3 + 2\lambda - 0)^2 + (1 + \lambda + 2)^2}$.

But this distance $= 5\sqrt{3}$ units

$$\therefore\ \sqrt{(1-\lambda)^2 + (3+2\lambda)^2 + (\lambda+3)^2} = 5\sqrt{3}$$
$$\therefore\ (1-\lambda)^2 + (3+2\lambda)^2 + (\lambda+3)^2 = 75$$
$$\therefore\ 1 - 2\lambda + \lambda^2 + 9 + 12\lambda + 4\lambda^2 + \lambda^2 + 6\lambda + 9 = 75$$
$$\therefore\ 6\lambda^2 + 16\lambda - 56 = 0$$
$$\therefore\ 3\lambda^2 + 8\lambda - 28 = 0$$
$$\therefore\ (3\lambda + 14)(\lambda - 2) = 0$$
$$\therefore\ \lambda = -\tfrac{14}{3} \text{ or } \lambda = 2$$

When $\lambda = 2$, the point is $(0, 7, 3)$, and when $\lambda = -\frac{14}{3}$, the point is $\left(\frac{20}{3}, -\frac{19}{3}, -\frac{11}{3}\right)$.

11 **a** Let $A(1+\lambda, 2-\lambda, 3+\lambda)$ be a point on the line such that $\overrightarrow{PA}$ is perpendicular to the line.



Then $\overrightarrow{PA} = \begin{pmatrix} 1+\lambda-1 \\ 2-\lambda-1 \\ 3+\lambda-2 \end{pmatrix} = \begin{pmatrix} \lambda \\ 1-\lambda \\ 1+\lambda \end{pmatrix}$

and $PA = \sqrt{\lambda^2 + (1-\lambda)^2 + (1+\lambda)^2}$

$= \sqrt{\lambda^2 + (1-2\lambda+\lambda^2) + (1+2\lambda+\lambda^2)}$

$= \sqrt{3\lambda^2+2}$ units

[PA] is perpendicular to the line when $PA^2 = 3\lambda^2 + 2$ is minimised,

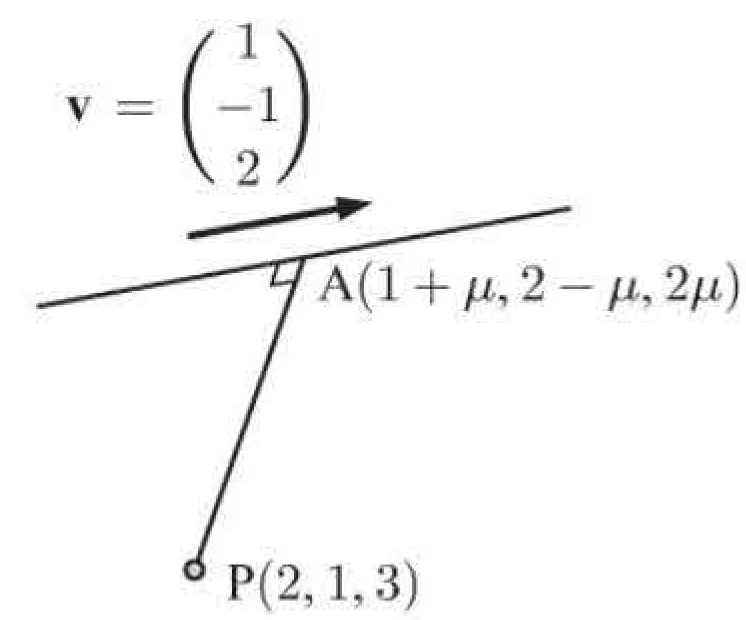
which occurs when $\lambda = -\dfrac{b}{2a} = -\dfrac{0}{6} = 0$

$\therefore$ A is at $(1+0, 2-0, 3+0)$

$\therefore$ the foot of the perpendicular is $(1, 2, 3)$.

b Let A be a point on the line such that $\overrightarrow{PA}$ is perpendicular to the line.

$\therefore$ A is at $(1+\mu, 2-\mu, 2\mu)$ for some μ.



Now $\overrightarrow{PA} = \begin{pmatrix} 1+\mu-2 \\ 2-\mu-1 \\ 2\mu-3 \end{pmatrix} = \begin{pmatrix} \mu-1 \\ 1-\mu \\ 2\mu-3 \end{pmatrix}$

and $PA = \sqrt{(\mu-1)^2 + (1-\mu)^2 + (2\mu-3)^2}$

$= \sqrt{\mu^2 - 2\mu + 1 + 1 - 2\mu + \mu^2 + 4\mu^2 - 12\mu + 9}$

$= \sqrt{6\mu^2 - 16\mu + 11}$ units

[PA] is perpendicular to the line when $PA^2 = 6\mu^2 - 16\mu + 11$ is minimised,

which occurs when $\mu = -\dfrac{b}{2a} = -\dfrac{-16}{12} = \frac{4}{3}$

$\therefore$ A is at $\left(1+\frac{4}{3}, 2-\frac{4}{3}, 2(\frac{4}{3})\right)$

$\therefore$ the foot of the perpendicular is $\left(\frac{7}{3}, \frac{2}{3}, \frac{8}{3}\right)$.

EXERCISE 15D

1 L_1 has direction vector $\begin{pmatrix} 12 \\ 5 \end{pmatrix}$ and L_2 has direction vector $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$.

If θ is the angle between them, $\cos\theta = \dfrac{\left|\begin{pmatrix} 12 \\ 5 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ -4 \end{pmatrix}\right|}{\sqrt{144+25}\sqrt{9+16}}$

$= \dfrac{|36 + (-20)|}{13 \times 5}$

$= \frac{16}{65}$

$\therefore \theta = \cos^{-1}\left(\frac{16}{65}\right)$

$\therefore \theta \approx 75.7°$

2 Line 1 has direction vector $\begin{pmatrix} 5 \\ -2 \end{pmatrix}$ and line 2 has direction vector $\begin{pmatrix} 4 \\ 10 \end{pmatrix}$

and $\begin{pmatrix} 5 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} 4 \\ 10 \end{pmatrix} = 20 + (-20) = 0$

$\therefore$ the lines are perpendicular.

3 L_1 has direction vector $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$ and L_2 has direction vector $\begin{pmatrix} 5 \\ 4 \end{pmatrix}$.

If θ is the angle between them,
$$\cos\theta = \frac{\left|\begin{pmatrix} 4 \\ -3 \end{pmatrix} \bullet \begin{pmatrix} 5 \\ 4 \end{pmatrix}\right|}{\sqrt{16+9}\sqrt{25+16}} = \frac{|20+(-12)|}{\sqrt{25\times 41}} = \frac{8}{\sqrt{25\times 41}}$$
$$\therefore\ \theta = \cos^{-1}\left(\frac{8}{\sqrt{25\times 41}}\right)$$
$$\therefore\ \theta \approx 75.5^\circ$$

$\therefore$ the required angle measures 75.5°.

4 **a** L_1: $s = \dfrac{x-8}{3} = \dfrac{9-y}{16} = \dfrac{z-10}{7}$

$3s = x-8$ and $16s = 9-y$ and $7s = z-10$

$\therefore\ x = 8+3s \qquad y = 9-16s \qquad z = 10+7s$

$\therefore$ line 1 has direction vector $\begin{pmatrix} 3 \\ -16 \\ 7 \end{pmatrix}$ and line 2 has direction vector $\begin{pmatrix} 3 \\ 8 \\ -5 \end{pmatrix}$.

If θ is the angle between them,
$$\cos\theta = \frac{\left|\begin{pmatrix} 3 \\ -16 \\ 7 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 8 \\ -5 \end{pmatrix}\right|}{\sqrt{9+256+49}\sqrt{9+64+25}} = \frac{|9-128-35|}{\sqrt{314}\sqrt{98}} = \frac{154}{\sqrt{314\times 98}}$$

$\therefore\ \theta \approx 28.6^\circ$

b Since $L_1 \perp L_3$,
$$\begin{pmatrix} 3 \\ -16 \\ 7 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ -3 \\ x \end{pmatrix} = 0$$
$$\therefore\ 48 + 7x = 0$$
$$\therefore\ x = -\tfrac{48}{7}$$

5 **a** $x - y = 3$ has gradient $+\frac{1}{1}$ and so has direction vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

$3x + 2y = 11$ has gradient $-\frac{3}{2}$ and so has direction vector $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$.

$$\therefore\ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ -3 \end{pmatrix} = \sqrt{1+1}\sqrt{4+9}\cos\theta$$
$$\therefore\ 2-3 = \sqrt{2}\sqrt{13}\cos\theta$$
$$\therefore\ \tfrac{-1}{\sqrt{26}} = \cos\theta$$
$$\therefore\ \theta \approx 101.3^\circ$$

$\therefore$ the angle is $180^\circ - 101.3^\circ \approx 78.7^\circ$

b $y = x + 2$ has gradient $1 = \frac{1}{1}$ and so has direction vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

$y = 1 - 3x$ has gradient $-3 = \frac{-3}{1}$ and so has direction vector $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$.

$$\therefore\ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -3 \end{pmatrix} = \sqrt{1+1}\sqrt{1+9}\cos\theta$$
$$\therefore\ 1-3 = \sqrt{2}\sqrt{10}\cos\theta$$
$$\therefore\ \tfrac{-2}{\sqrt{20}} = \cos\theta$$
$$\therefore\ \theta \approx 116.6^\circ$$

$\therefore$ the angle is $180^\circ - 116.6^\circ \approx 63.4^\circ$

c $y+x=7$ has gradient $-1=\frac{-1}{1}$ and so has direction vector $\begin{pmatrix}1\\-1\end{pmatrix}$.

$x-3y+2=0$ has gradient $\frac{1}{3}$ and so has direction vector $\begin{pmatrix}3\\1\end{pmatrix}$.

$\therefore \begin{pmatrix}1\\-1\end{pmatrix}\bullet\begin{pmatrix}3\\1\end{pmatrix}=\sqrt{1+1}\sqrt{9+1}\cos\theta$

$\therefore 3-1=\sqrt{2}\sqrt{10}\cos\theta$

$\therefore \frac{2}{\sqrt{20}}=\cos\theta$

$\therefore \theta\approx 63.4°$

$\therefore$ the angle is $63.4°$.

d $y=2-x$ has gradient $-1=\frac{-1}{1}$ and so has direction vector $\begin{pmatrix}1\\-1\end{pmatrix}$.

$x-2y=7$ has gradient $\frac{1}{2}$ and so has direction vector $\begin{pmatrix}2\\1\end{pmatrix}$.

$\therefore \begin{pmatrix}1\\-1\end{pmatrix}\bullet\begin{pmatrix}2\\1\end{pmatrix}=\sqrt{1+1}\sqrt{4+1}\cos\theta$

$\therefore 2-1=\sqrt{2}\sqrt{5}\cos\theta$

$\therefore \frac{1}{\sqrt{10}}=\cos\theta$

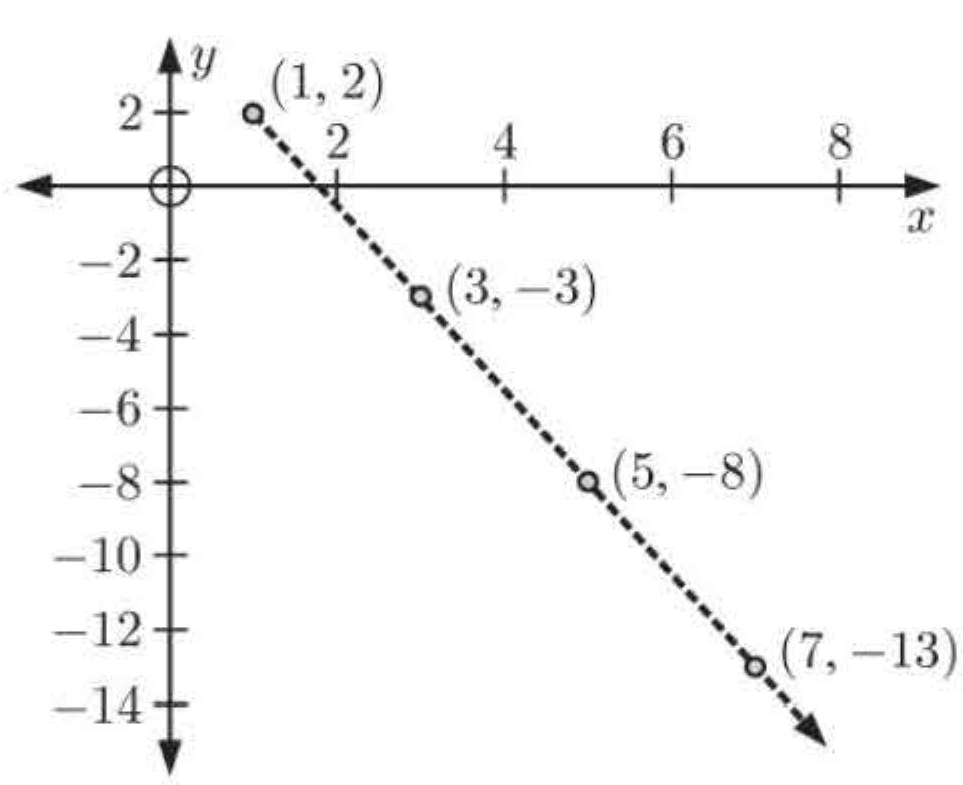
$\therefore \theta\approx 71.6°$

$\therefore$ the angle is $71.6°$.

EXERCISE 15E

1 **a** $x(0)=1$ and $y(0)=2$,
$\therefore$ the initial position is $(1, 2)$

b



c The velocity vector is $\begin{pmatrix}2\\-5\end{pmatrix}$.

d In 1 second, the x-step is 2 and y-step is -5, which is a distance of $\sqrt{2^2+(-5)^2}=\sqrt{29}$ cm
$\therefore$ the speed is $\sqrt{29}$ cm s^{-1}.

2 **a** $\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}2\\3\end{pmatrix}+t\begin{pmatrix}4\\-5\end{pmatrix}$, $t\geqslant 0$

b 90 minutes = 1.5 hours
When $t=1.5$, $x=2+4(1.5)=8$
and $y=3-5(1.5)=-4.5$
$\therefore$ the boat is at $(8, -4.5)$ after 90 minutes.

c When the boat reaches the point $(5, -0.75)$,
$2+4t=5$ and $3-5t=-0.75$
$\therefore 4t=3$ $\quad -5t=-3.75$
$\therefore t=0.75$ $\quad t=0.75$
It will take 0.75 hours = 45 minutes for the boat to reach point $(5, -0.75)$.

3 **a** $\mathbf{r}=\mathbf{a}+t\mathbf{b}$

$\therefore \begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-3\\-2\end{pmatrix}+t\begin{pmatrix}2\\4\end{pmatrix}$, $t\geqslant 0$

$\therefore \begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-3+2t\\-2+4t\end{pmatrix}$

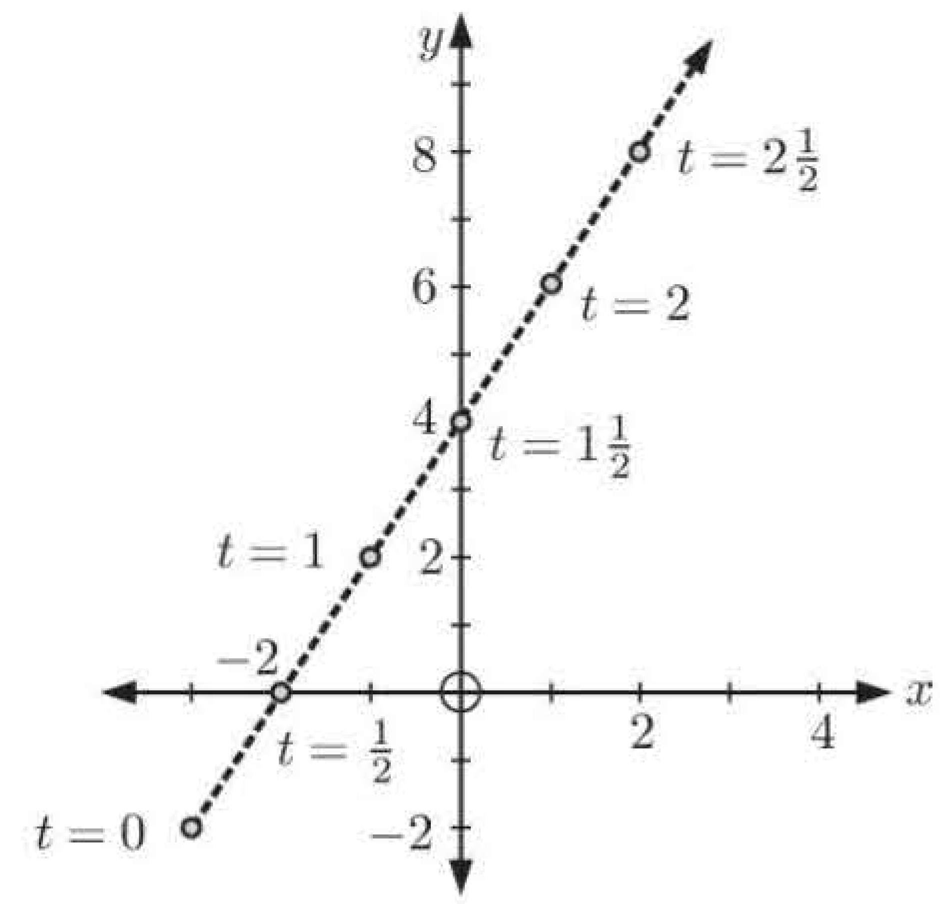
b At $t=2.5$, $-3+2t=-3+5=2$
and $-2+4t=-2+10=8$

So, the position vector is $\begin{pmatrix}2\\8\end{pmatrix}$.

c **i** When the car is due north, $x=0$
$\therefore -3+2t=0$
$\therefore t=1.5$ seconds

ii When the car is due west, $y=0$
$\therefore -2+4t=0$
$\therefore t=0.5$ seconds

d



4 **a** **i** When $t=0$, $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$

$\therefore$ the object is at $(-4, 3)$.

ii The velocity vector is $\begin{pmatrix} 12 \\ 5 \end{pmatrix}$.

iii The speed is $\sqrt{12^2+5^2} = 13\ \text{m s}^{-1}$

b **i** When $t=0$, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$

$\therefore$ the object is at $(3, 0, 4)$.

ii The velocity vector is $\begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$.

iii The speed is $\sqrt{2^2+(-1)^2+(-2)^2} = 3\ \text{m s}^{-1}$

5 **a** $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$ has length $\sqrt{4^2+(-3)^2} = 5$

$\therefore$ $30\begin{pmatrix} 4 \\ -3 \end{pmatrix}$ has length 150

$\therefore$ the velocity vector is $\begin{pmatrix} 120 \\ -90 \end{pmatrix}$.

b $2\mathbf{i}+\mathbf{j} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ has length $\sqrt{2^2+1^2} = \sqrt{5}$

$\therefore$ $10\sqrt{5}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ has length 50

$\therefore$ the velocity vector is $\begin{pmatrix} 20\sqrt{5} \\ 10\sqrt{5} \end{pmatrix}$.

6 $-2\mathbf{i}+5\mathbf{j}-14\mathbf{k}$ has length $\sqrt{(-2)^2+5^2+(-14)^2} = \sqrt{4+25+196} = \sqrt{225} = 15$

$\therefore$ $6\begin{pmatrix} -2 \\ 5 \\ -14 \end{pmatrix}$ has length 90, so the velocity vector is $\begin{pmatrix} -12 \\ 30 \\ -84 \end{pmatrix}$.

7 Yacht A: $\begin{pmatrix} x_A \\ y_A \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix} + t\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ Yacht B: $\begin{pmatrix} x_B \\ y_B \end{pmatrix} = \begin{pmatrix} 1 \\ -8 \end{pmatrix} + t\begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $t \geqslant 0$

a When $t=0$, $\begin{pmatrix} x_A \\ y_A \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$ $\therefore$ A is at $(4, 5)$

and $\begin{pmatrix} x_B \\ y_B \end{pmatrix} = \begin{pmatrix} 1 \\ -8 \end{pmatrix}$ $\therefore$ B is at $(1, -8)$.

b For A, the velocity vector is $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$, and for B it is $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

c Speed of A $= \sqrt{1^2+(-2)^2} = \sqrt{5}\ \text{km h}^{-1}$. Speed of B $= \sqrt{2^2+1^2} = \sqrt{5}\ \text{km h}^{-1}$.

d A has direction vector $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ and B has direction vector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Since $\begin{pmatrix} 1 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2-2 = 0$, the paths of the yachts are at right angles to each other.

8 **a** P's torpedo has position $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} -5 \\ 4 \end{pmatrix} + t\begin{pmatrix} 3 \\ -1 \end{pmatrix}$ and at $t=0$, the time is 1:34 pm

$\therefore$ $x_1(t) = -5+3t$, $y_1(t) = 4-t$.

b Speed $= \sqrt{3^2+(-1)^2} = \sqrt{10}\ \text{km min}^{-1}$

c Q fires its torpedo after a minutes.

$\therefore$ at time t, its torpedo has travelled for $(t-a)$ minutes.

$\therefore$ $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 15 \\ 7 \end{pmatrix} + (t-a)\begin{pmatrix} -4 \\ -3 \end{pmatrix}$, $t \geqslant a$

$\therefore$ $x_2(t) = 15-4(t-a)$ and $y_2(t) = 7-3(t-a)$

$\therefore$ position is $Q(15-4(t-a), 7-3(t-a))$.

d They meet when $x_1(t) = x_2(t)$ and $y_1(t) = y_2(t)$

$\therefore\ -5 + 3t = 15 - 4(t-a)$ and $4 - t = 7 - 3(t-a)$

$\therefore\ 7t - 4a = 20$ .... (1) and $2t - 3a = 3$ .... (2)

Solving simultaneously,

$$\begin{array}{rl} 21t - 12a = 60 & \{3 \times (1)\} \\ -8t + 12a = -12 & \{-4 \times (2)\} \\ \hline \text{adding}\quad 13t \qquad = 48 & \end{array}$$

$\therefore\ t = \frac{48}{13}$ and $7\left(\frac{48}{13}\right) - 4a = 20$

$\therefore\ t \approx 3.6923$ $\qquad \therefore\ 5.8462 = 4a$

$\therefore\ t \approx 3$ min 41.54 sec $\qquad \therefore\ a \approx 1.4615 \approx 1$ min 27.7 sec

So, as $a \approx 1.4615$, Q fired at 1:35:28 pm, and the explosion occurred at 1:37:42 pm.

9 **a** $\overrightarrow{AB} = \begin{pmatrix} 3-6 \\ 10-9 \\ 2.5-3 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ -0.5 \end{pmatrix}$

b $|\overrightarrow{AB}| = \sqrt{(-3)^2 + 1^2 + (-0.5)^2}$
$= \sqrt{10.25}$ km

The helicopter travels $\sqrt{10.25}$ km in 10 minutes.

$\therefore$ the helicopter's speed is $6 \times \sqrt{10.25} \approx 19.2\ \text{km h}^{-1}$.

c $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 9 \\ 3 \end{pmatrix} + t\begin{pmatrix} -3 \\ 1 \\ -0.5 \end{pmatrix},\quad t \in \mathbb{R}$

d If $z = 0,\ 3 + (-0.5)t = 0$

$\therefore\ t = 6$

$t = 1$ represents 10 minutes, so $t = 6$ represents 60 minutes.

$\therefore$ the helicopter lands on the helipad after 1 hour.

EXERCISE 15F

1 **a** Let N be the point on the line closest to P.
N has coordinates $(2+t, 3+2t)$ for some $t \in \mathbb{R}$.

$\overrightarrow{PN}$ is $\begin{pmatrix} 2+t-3 \\ 3+2t-2 \end{pmatrix} = \begin{pmatrix} t-1 \\ 2t+1 \end{pmatrix}$.

Now $\overrightarrow{PN} \bullet \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 0$, as $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is the direction vector of the line.

$\therefore\ \begin{pmatrix} t-1 \\ 2t+1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 0$

$\therefore\ (t-1) + 2(2t+1) = 0$

$\therefore\ t - 1 + 4t + 2 = 0$

$\therefore\ 5t = -1$

$\therefore\ t = -\frac{1}{5}$

Thus $\overrightarrow{PN} = \begin{pmatrix} -\frac{1}{5} - 1 \\ -\frac{2}{5} + 1 \end{pmatrix} = \begin{pmatrix} -\frac{6}{5} \\ \frac{3}{5} \end{pmatrix}$

$= \frac{3}{5}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

and $|\overrightarrow{PN}| = \frac{3}{5}\left|\begin{pmatrix} -2 \\ 1 \end{pmatrix}\right| = \frac{3}{5}\sqrt{(-2)^2 + 1^2}$

$= \frac{3}{5}\sqrt{5}$

So the shortest distance from P to the line is $\frac{3}{5}\sqrt{5}$ units.

b Let N be the point on the line closest to Q.
N has coordinates $(t, 1-t)$ for some $t \in \mathbb{R}$.

$\overrightarrow{QN}$ is $\begin{pmatrix} t-(-1) \\ 1-t-1 \end{pmatrix} = \begin{pmatrix} t+1 \\ -t \end{pmatrix}$.

Now $\overrightarrow{QN} \bullet \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0$, as $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is the direction vector of the line.

$\therefore\ \begin{pmatrix} t+1 \\ -t \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0$

$\therefore\ (t+1) + (-1)(-t) = 0$

$\therefore\ t + 1 + t = 0$

$\therefore\ 2t = -1$

$\therefore\ t = -\frac{1}{2}$

Thus $\overrightarrow{QN} = \begin{pmatrix} -\frac{1}{2} + 1 \\ -\left(-\frac{1}{2}\right) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$

$= \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

and $|\overrightarrow{QN}| = \frac{1}{2}\left|\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right| = \frac{1}{2}\sqrt{1^2 + 1^2}$

$= \frac{1}{2}\sqrt{2}$

So the shortest distance from Q to the line is $\frac{1}{2}\sqrt{2}$ units.

c Let N be the point on the line closest to R.
N has coordinates $(2+s, 3-s)$ for some $s \in \mathbb{R}$.

$\overrightarrow{RN}$ is $\begin{pmatrix} 2+s-(-3) \\ 3-s-(-1) \end{pmatrix} = \begin{pmatrix} s+5 \\ 4-s \end{pmatrix}$.

Now $\overrightarrow{RN} \bullet \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0$, as $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is the direction vector of the line.

$$\therefore \begin{pmatrix} s+5 \\ 4-s \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0$$

$$\therefore (s+5)+(-1)(4-s) = 0$$
$$\therefore s+5-4+s = 0$$
$$\therefore 2s = -1$$
$$\therefore s = -\tfrac{1}{2}$$

Thus $\overrightarrow{RN} = \begin{pmatrix} -\frac{1}{2}+5 \\ 4-(-\frac{1}{2}) \end{pmatrix} = \begin{pmatrix} \frac{9}{2} \\ \frac{9}{2} \end{pmatrix}$

$$= \tfrac{9}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and $|\overrightarrow{RN}| = \frac{9}{2}\left|\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right| = \frac{9}{2}\sqrt{1^2+1^2}$

$$= \tfrac{9}{2}\sqrt{2}$$

So the shortest distance from R to the line is $\frac{9}{2}\sqrt{2}$ units.

d Let N be the point on the line closest to S.
N has coordinates $(2+3t, 5-7t)$ for some $t \in \mathbb{R}$.

$\overrightarrow{SN}$ is $\begin{pmatrix} 2+3t-5 \\ 5-7t-(-2) \end{pmatrix} = \begin{pmatrix} 3t-3 \\ 7-7t \end{pmatrix}$.

Now $\overrightarrow{SN} \bullet \begin{pmatrix} 3 \\ -7 \end{pmatrix} = 0$, as $\begin{pmatrix} 3 \\ -7 \end{pmatrix}$ is the direction vector of the line.

$$\therefore \begin{pmatrix} 3t-3 \\ 7-7t \end{pmatrix} \bullet \begin{pmatrix} 3 \\ -7 \end{pmatrix} = 0$$

$$\therefore 3(3t-3)-7(7-7t) = 0$$
$$\therefore 9t-9-49+49t = 0$$
$$\therefore 58t = -58$$
$$\therefore t = 1$$

Thus $\overrightarrow{SN} = \begin{pmatrix} 3(1)-3 \\ 7-7(1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\therefore |\overrightarrow{SN}| = 0$

So S actually lies on the line, and the shortest distance is 0 units.

2 **a** $6\mathbf{i} - 6\mathbf{j}$

b The length of $\begin{pmatrix} -3 \\ 4 \end{pmatrix} = \sqrt{(-3)^2+4^2} = \sqrt{25} = 5$

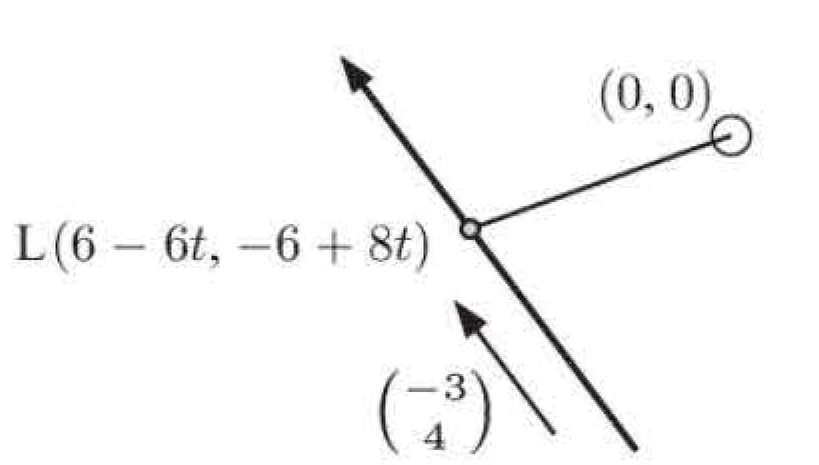
As the speed is 10 km h^{-1}, the liner has velocity vector $2\begin{pmatrix} -3 \\ 4 \end{pmatrix} = \begin{pmatrix} -6 \\ 8 \end{pmatrix}$.

$\therefore$ the liner has position $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 \\ -6 \end{pmatrix} + t\begin{pmatrix} -6 \\ 8 \end{pmatrix} = \begin{pmatrix} 6-6t \\ -6+8t \end{pmatrix}$, $t \geqslant 0$, t in hours.

c The liner is due east when $y = 0$

$$\therefore -6+8t = 0$$
$$\therefore \text{at } t = \tfrac{3}{4} \text{ hours}$$

d The liner L is nearest the fishing boat O when $\overrightarrow{OL} \perp \binom{-3}{4}$



$$\therefore \overrightarrow{OL} \bullet \begin{pmatrix} -3 \\ 4 \end{pmatrix} = 0$$

$$\therefore \begin{pmatrix} 6-6t \\ -6+8t \end{pmatrix} \bullet \begin{pmatrix} -3 \\ 4 \end{pmatrix} = 0$$

$$\therefore (-18+18t)+(-24+32t) = 0$$
$$\therefore 50t = 42$$
$$\therefore t = 0.84 \text{ hours} = 50.4 \text{ minutes}$$

and when $t = 0.84$, $\overrightarrow{OL} = \begin{pmatrix} 6-6(0.84) \\ -6+8(0.84) \end{pmatrix} = \begin{pmatrix} 0.96 \\ 0.72 \end{pmatrix}$

$\therefore$ the liner is closest to the fishing boat after 0.84 hours or 50.4 minutes, when it is at $(0.96, 0.72)$.

3 **a** $|\mathbf{b}| = \sqrt{(-3)^2+(-1)^2} = \sqrt{10}$

As the speed is $40\sqrt{10}$ km h^{-1}, the velocity vector is $40\begin{pmatrix}-3\\-1\end{pmatrix} = \begin{pmatrix}-120\\-40\end{pmatrix}$.

b $\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}200\\100\end{pmatrix} + t\begin{pmatrix}-120\\-40\end{pmatrix}$, $t \geqslant 0$ $\{t = 0$ at 12:00 noon$\}$

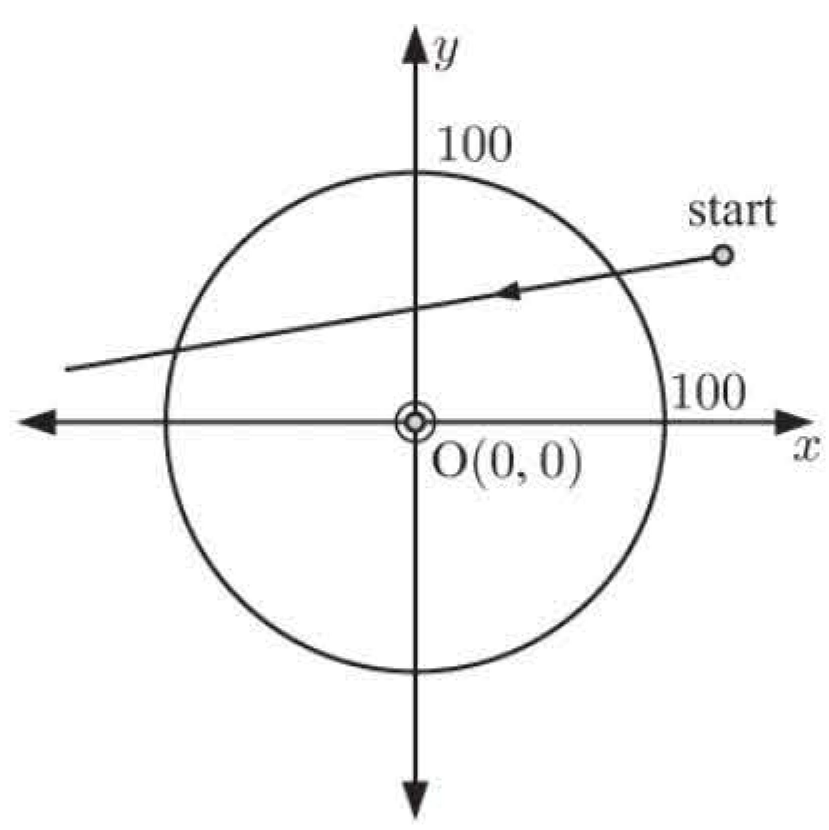
c At 1:00 pm, $t = 1$ and $\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}200-120\\100-40\end{pmatrix} = \begin{pmatrix}80\\60\end{pmatrix}$

$\therefore$ the aircraft is at (80, 60).

d The distance from O(0, 0) to $P_1(80, 60)$ is $\left|\begin{pmatrix}80\\60\end{pmatrix}\right| = \sqrt{80^2+60^2} = 100$ km,

which is when it becomes visible to radar. {within 100 km of O(0, 0)}

e



A general point on the path is $P(200-120t, 100-40t)$.

Now $\overrightarrow{OP} = \begin{pmatrix}200-120t\\100-40t\end{pmatrix}$,

and for the closest point $\overrightarrow{OP} \bullet \begin{pmatrix}-3\\-1\end{pmatrix} = 0$

$$\therefore\ -3(200-120t) - 1(100-40t) = 0$$
$$\therefore\ -700 + 400t = 0$$
$$\therefore\ t = \tfrac{7}{4} = 1\tfrac{3}{4} \text{ hours}$$

$\therefore$ the time when the aircraft is closest is 1:45 pm, and

at this time $\overrightarrow{OP} = \begin{pmatrix}200-120(\frac{7}{4})\\100-40(\frac{7}{4})\end{pmatrix} = \begin{pmatrix}-10\\30\end{pmatrix}$

$$\therefore\ d_{\min} = \sqrt{(-10)^2+30^2} \approx 31.6 \text{ km}$$

f It disappears from radar when $|\overrightarrow{OP}| = 100$ and $t > 1\frac{3}{4}$

$$\therefore\ \sqrt{(200-120t)^2+(100-40t)^2} = 100$$
$$\therefore\ 40\,000 - 48\,000t + 14\,400t^2 + 10\,000 - 8000t + 1600t^2 = 10\,000$$
$$\therefore\ 16\,000t^2 - 56\,000t + 40\,000 = 0$$
$$\therefore\ 16t^2 - 56t + 40 = 0 \qquad \{\div 1000\}$$
$$\therefore\ 2t^2 - 7t + 5 = 0 \qquad \{\div 8\}$$
$$\therefore\ (2t-5)(t-1) = 0$$
$$\therefore\ t = \tfrac{5}{2} \qquad \{\text{as } t > 1\tfrac{3}{4}\}$$

So, the aircraft disappears from the radar screen $2\frac{1}{2}$ hours after noon, or at 2:30 pm.

4 **a** At A, $y = 0$

$\therefore\ 2x = 36$

$\therefore\ x = 18$

At B, $x = 0$

$\therefore\ 3y = 36$

$\therefore\ y = 12$

So A is (18, 0) and B is (0, 12).

b $2x + 3y = 36$

$\therefore\ 3y = 36 - 2x$

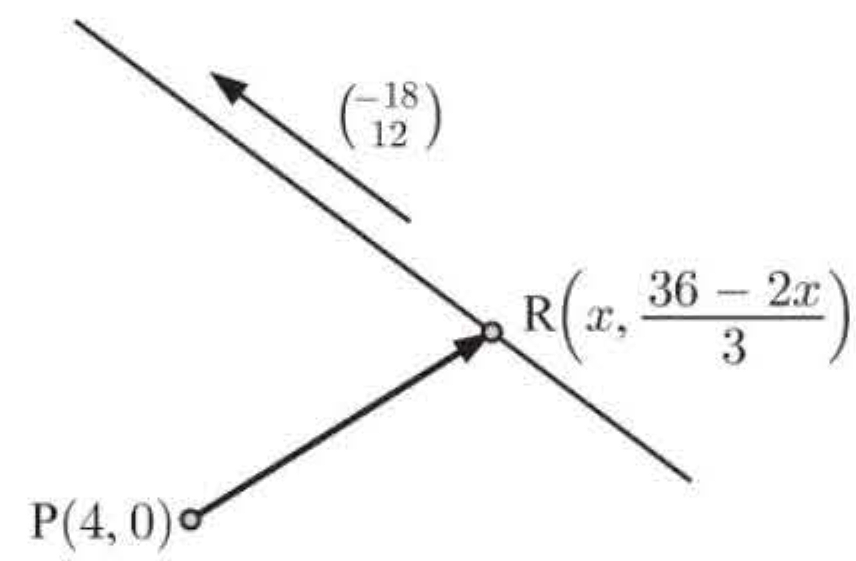
$\therefore\ y = \dfrac{36-2x}{3}$

$\therefore$ any point R on the railway track can be written $R\left(x, \dfrac{36-2x}{3}\right)$.

c $\overrightarrow{PR} = \begin{pmatrix}x-4\\\frac{36-2x}{3}-0\end{pmatrix} = \begin{pmatrix}x-4\\\frac{36-2x}{3}\end{pmatrix}$,

$\overrightarrow{AB} = \begin{pmatrix}0-18\\12-0\end{pmatrix} = \begin{pmatrix}-18\\12\end{pmatrix}$

d The point closest to the railway track is R such that $\overrightarrow{PR} \perp \overrightarrow{AB}$.



$$\therefore \overrightarrow{PR} \bullet \overrightarrow{AB} = 0$$

$$\therefore \begin{pmatrix} x-4 \\ \frac{36-2x}{3} \end{pmatrix} \bullet \begin{pmatrix} -18 \\ 12 \end{pmatrix} = 0$$

$$\therefore -18(x-4) + 4(36-2x) = 0$$

$$\therefore -18x + 72 + 144 - 8x = 0$$

$$\therefore 26x = 216$$

$$\therefore x = \tfrac{108}{13}$$

Now when $x = \frac{108}{13}$, $\dfrac{36-2x}{3} = 12 - \frac{2}{3}x = 12 - \frac{2}{3}\left(\frac{108}{13}\right) = \frac{84}{13}$. So R is $\left(\frac{108}{13}, \frac{84}{13}\right)$.

$$|\overrightarrow{PR}| = \sqrt{\left(\tfrac{108}{13} - 4\right)^2 + \left(\tfrac{84}{13} - 0\right)^2} = \sqrt{\tfrac{784}{13}} \approx 7.77 \text{ km}$$

The closest point on the track to the camp is $\left(\frac{108}{13}, \frac{84}{13}\right)$, a distance of 7.77 km.

5 For A, $x_A(t) = 3 - t$, $y_A(t) = 2t - 4$ For B, $x_B(t) = 4 - 3t$, $y_B(t) = 3 - 2t$

a When $t = 0$, $x_A(0) = 3$, $y_A(0) = -4$ and $x_B(0) = 4$, $y_B(0) = 3$

$\therefore$ A is at $(3, -4)$. $\therefore$ B is at $(4, 3)$.

b The velocity vector of A is $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ and the velocity vector of B is $\begin{pmatrix} -3 \\ -2 \end{pmatrix}$.

c If the angle is θ, $\begin{pmatrix} -1 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} -3 \\ -2 \end{pmatrix} = \sqrt{1+4}\sqrt{9+4}\cos\theta$

$$\therefore 3 - 4 = \sqrt{5}\sqrt{13}\cos\theta$$

$$\therefore \tfrac{-1}{\sqrt{65}} = \cos\theta \quad \text{and so} \quad \theta \approx 97.1°$$

$\therefore$ the acute angle between the paths is $\approx 82.9°$.

d If D is the distance between them, then

$$\begin{aligned} D &= \sqrt{[(4-3t) - (3-t)]^2 + [(3-2t) - (2t-4)]^2} \\ &= \sqrt{[1-2t]^2 + [7-4t]^2} \\ &= \sqrt{1 - 4t + 4t^2 + 49 - 56t + 16t^2} \\ &= \sqrt{20t^2 - 60t + 50} \end{aligned}$$

Under the square root we have a quadratic in t, so D is a minimum when

$$t = -\frac{b}{2a} = \tfrac{60}{40} = 1\tfrac{1}{2}$$

$\therefore$ $t = 1.5$ hours

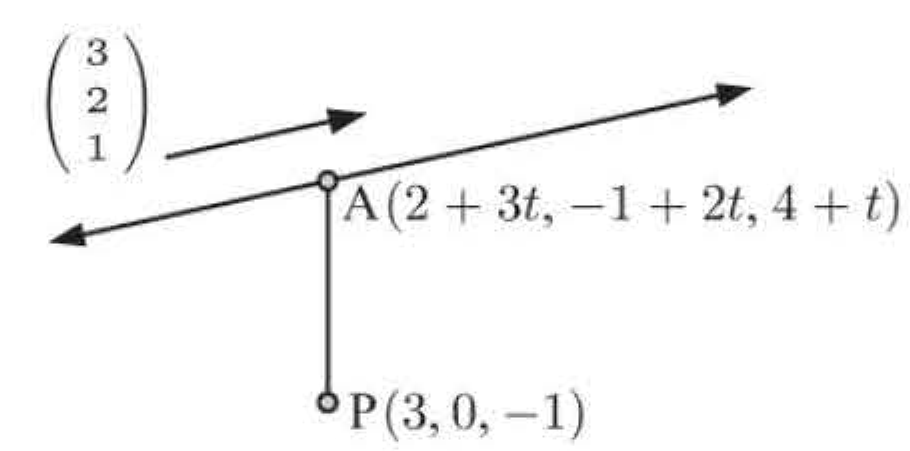
$\therefore$ the boats are closest after $1\frac{1}{2}$ hours.

6 **a** The direction vector of the line is $\mathbf{b} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$.

Let the point $(3, 0, -1)$ be P, and $A(2+3t, -1+2t, 4+t)$ be any point on the line.

$$\therefore \overrightarrow{PA} = \begin{pmatrix} 2+3t-3 \\ -1+2t-0 \\ 4+t-(-1) \end{pmatrix} = \begin{pmatrix} -1+3t \\ -1+2t \\ 5+t \end{pmatrix}$$

Now $\overrightarrow{PA}$ and $\mathbf{b}$ are perpendicular, so $\overrightarrow{PA} \bullet \mathbf{b} = 0$.



$$\therefore \begin{pmatrix} -1+3t \\ -1+2t \\ 5+t \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = 0$$

$$\therefore -3 + 9t - 2 + 4t + 5 + t = 0$$

$$\therefore 14t = 0$$

$$\therefore t = 0$$

Substituting $t = 0$ into the parametric equations, we obtain the foot of the perpendicular $(2, -1, 4)$.

b When $t = 0$, $\overrightarrow{PA} = \begin{pmatrix} -1 \\ -1 \\ 5 \end{pmatrix}$, so $PA = \sqrt{1+1+25}$
$= \sqrt{27}$ units

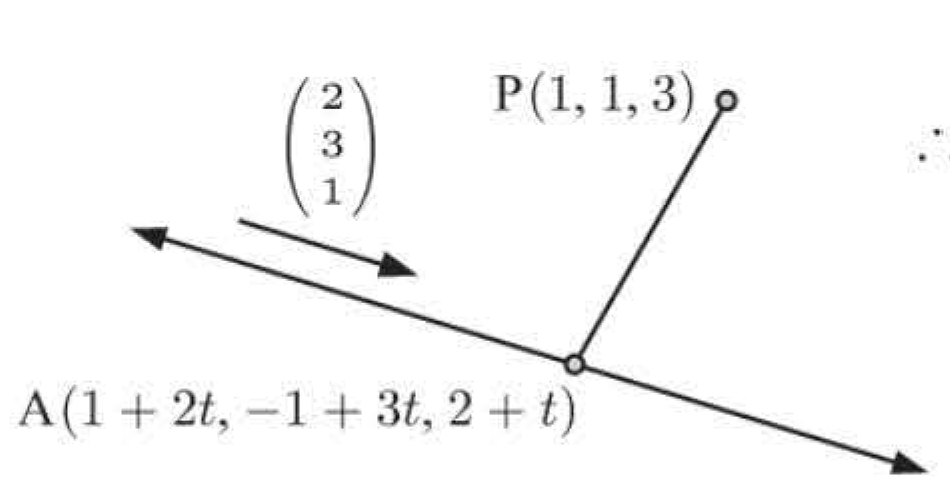
$\therefore$ the shortest distance from the point to the line is $\sqrt{27}$ units.

7 **a** The line has direction vector $\mathbf{b} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$.

Let the point $(1, 1, 3)$ be P and $A(1+2t, -1+3t, 2+t)$ be any point on the line.

$\therefore \overrightarrow{PA} = \begin{pmatrix} 1+2t-1 \\ -1+3t-1 \\ 2+t-3 \end{pmatrix} = \begin{pmatrix} 2t \\ -2+3t \\ -1+t \end{pmatrix}$.

Now $\overrightarrow{PA}$ and $\mathbf{b}$ are perpendicular, so $\overrightarrow{PA} \bullet \mathbf{b} = 0$



$\therefore \begin{pmatrix} 2t \\ -2+3t \\ -1+t \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = 0$

$\therefore 4t - 6 + 9t - 1 + t = 0$

$\therefore 14t = 7$

$\therefore t = \frac{1}{2}$

Substituting $t = \frac{1}{2}$ into the parametric equations, we obtain the foot of the perpendicular $(2, \frac{1}{2}, \frac{5}{2})$.

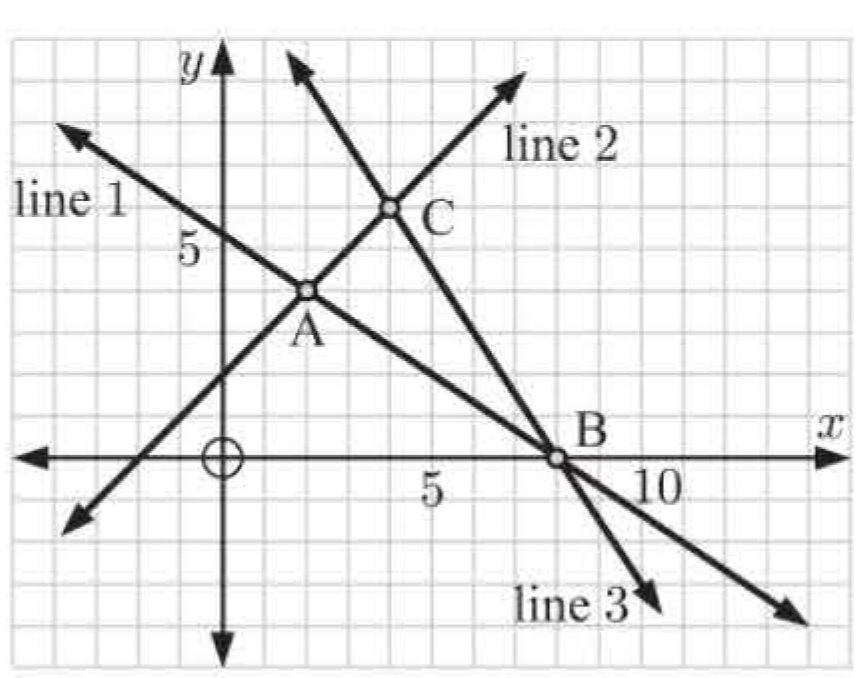
b When $t = \frac{1}{2}$, $\overrightarrow{PA} = \begin{pmatrix} 1 \\ -2+\frac{3}{2} \\ -1+\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$

$\therefore PA = \sqrt{1 + \frac{1}{4} + \frac{1}{4}} = \sqrt{\frac{3}{2}}$ units

$\therefore$ the shortest distance from the point to the line is $\sqrt{\frac{3}{2}}$ units.

EXERCISE 15G

1 **a**



b A is $(2, 4)$, B is $(8, 0)$, C is $(4, 6)$

c $BC = \sqrt{(4-8)^2 + (6-0)^2} = \sqrt{16+36}$
$= \sqrt{52}$ units

$AB = \sqrt{(8-2)^2 + (0-4)^2} = \sqrt{36+16}$
$= \sqrt{52}$ units

$\therefore BC = AB$ and so $\triangle ABC$ is isosceles.

d Line 1 and Line 2 meet at A.

$\therefore \begin{pmatrix} -1 \\ 6 \end{pmatrix} + r\begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} + s\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\therefore \begin{pmatrix} 3r - s \\ -2r - s \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$

$\therefore 3r - s = 1$

and $2r + s = 4$

Adding, $5r = 5$ $\therefore r = 1$

$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 6 \end{pmatrix} + \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ ✓

Line 2 and Line 3 meet at C.

$\therefore \begin{pmatrix} 0 \\ 2 \end{pmatrix} + s\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 10 \\ -3 \end{pmatrix} + t\begin{pmatrix} -2 \\ 3 \end{pmatrix}$

$\therefore \begin{pmatrix} s + 2t \\ s - 3t \end{pmatrix} = \begin{pmatrix} 10 \\ -5 \end{pmatrix}$

$\therefore s + 2t = 10$

$-s + 3t = 5$

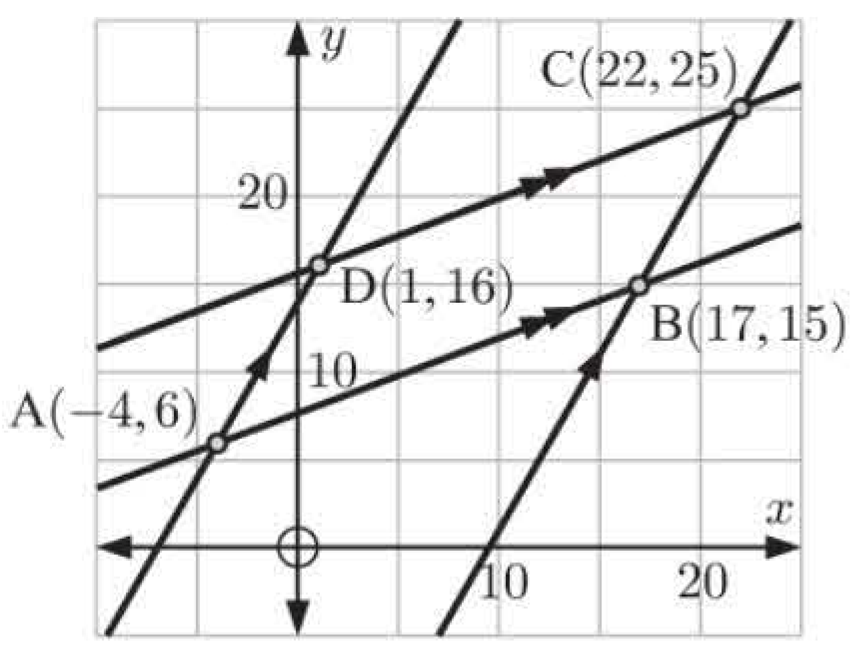
Adding, $5t = 15$ $\therefore t = 3$

$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 10 \\ -3 \end{pmatrix} + 3\begin{pmatrix} -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$ ✓

Line 1 and Line 3 meet at B.
$$\therefore \begin{pmatrix}-1\\6\end{pmatrix}+r\begin{pmatrix}3\\-2\end{pmatrix}=\begin{pmatrix}10\\-3\end{pmatrix}+t\begin{pmatrix}-2\\3\end{pmatrix}$$
$$\therefore \begin{pmatrix}3r+2t\\-2r-3t\end{pmatrix}=\begin{pmatrix}11\\-9\end{pmatrix}$$
$$\therefore\ 3r+2t=11 \quad .... (1)$$
$$-2r-3t=-9 \quad .... (2)$$
$$\therefore\ 9r+6t=33 \quad \{3\times(1)\}$$
$$-4r-6t=-18 \quad \{2\times(2)\}$$
$$\text{Adding, } 5r=15$$
$$\therefore\ r=3$$

So, $\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-1\\6\end{pmatrix}+3\begin{pmatrix}3\\-2\end{pmatrix}=\begin{pmatrix}8\\0\end{pmatrix}$ ✓

2 **a**



b A(−4, 6), B(17, 15), C(22, 25), D(1, 16)

c Lines (AB) and (AD) meet at A.
$$\therefore \begin{pmatrix}-4\\6\end{pmatrix}+r\begin{pmatrix}7\\3\end{pmatrix}=\begin{pmatrix}-4\\6\end{pmatrix}+s\begin{pmatrix}1\\2\end{pmatrix}$$
$$\therefore \begin{pmatrix}7r-s\\3r-2s\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$
$$\therefore\ 7r-s=0 \quad .... (1)$$
$$\text{and } 3r-2s=0$$
$$-14r+2s=0 \quad \{-2\times(1)\}$$
$$\text{Adding, } -11r=0$$
$$\therefore\ r=0$$
$$\therefore \begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-4\\6\end{pmatrix} \ ✓$$

Lines (AB) and (CB) meet at B.
$$\therefore \begin{pmatrix}-4\\6\end{pmatrix}+r\begin{pmatrix}7\\3\end{pmatrix}=\begin{pmatrix}22\\25\end{pmatrix}+u\begin{pmatrix}-1\\-2\end{pmatrix}$$
$$\therefore \begin{pmatrix}7r+u\\3r+2u\end{pmatrix}=\begin{pmatrix}26\\19\end{pmatrix}$$
$$\therefore\ 7r+u=26 \quad .... (1)$$
$$\text{and } 3r+2u=19$$
$$-14r-2u=-52 \quad \{-2\times(1)\}$$
$$\text{Adding, } -11r=-33$$
$$\therefore\ r=3$$
$$\therefore \begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-4\\6\end{pmatrix}+3\begin{pmatrix}7\\3\end{pmatrix}=\begin{pmatrix}17\\15\end{pmatrix} \ ✓$$

Lines (CD) and (CB) meet at C.
$$\therefore \begin{pmatrix}22\\25\end{pmatrix}+t\begin{pmatrix}-7\\-3\end{pmatrix}=\begin{pmatrix}22\\25\end{pmatrix}+u\begin{pmatrix}-1\\-2\end{pmatrix}$$
$$\therefore \begin{pmatrix}-7t+u\\-3t+2u\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$
$$\therefore\ -7t+u=0 \quad .... (1)$$
$$\text{and } -3t+2u=0$$
$$14t-2u=0 \quad \{-2\times(1)\}$$
$$\text{Adding, } 11t=0$$
$$\therefore\ t=0$$
$$\therefore \begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}22\\25\end{pmatrix} \ ✓$$

Lines (AD) and (CD) meet at D.
$$\therefore \begin{pmatrix}-4\\6\end{pmatrix}+s\begin{pmatrix}1\\2\end{pmatrix}=\begin{pmatrix}22\\25\end{pmatrix}+t\begin{pmatrix}-7\\-3\end{pmatrix}$$
$$\therefore \begin{pmatrix}s+7t\\2s+3t\end{pmatrix}=\begin{pmatrix}26\\19\end{pmatrix}$$
$$\therefore\ s+7t=26 \quad .... (1)$$
$$\text{and } 2s+3t=19$$
$$-2s-14t=-52 \quad \{-2\times(1)\}$$
$$\text{Adding, } -11t=-33$$
$$\therefore\ t=3$$
$$\therefore \begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}22\\25\end{pmatrix}+3\begin{pmatrix}-7\\-3\end{pmatrix}=\begin{pmatrix}1\\16\end{pmatrix} \ ✓$$

3 **a** Lines (AB) and (AC) meet at A.

$$\therefore \begin{pmatrix}0\\2\end{pmatrix}+r\begin{pmatrix}2\\1\end{pmatrix}=\begin{pmatrix}0\\5\end{pmatrix}+t\begin{pmatrix}1\\-1\end{pmatrix}$$

$$\therefore \begin{pmatrix}2r-t\\r+t\end{pmatrix}=\begin{pmatrix}0\\3\end{pmatrix}$$

$$\therefore\ 2r-t=0$$
$$r+t=3$$
$$\text{Adding, } 3r=3$$
$$\therefore\ r=1$$

$$\therefore \begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\2\end{pmatrix}+\begin{pmatrix}2\\1\end{pmatrix}=\begin{pmatrix}2\\3\end{pmatrix}$$

$\therefore$ A is (2, 3)

Lines (AB) and (BC) meet at B.

$$\therefore \begin{pmatrix}0\\2\end{pmatrix}+r\begin{pmatrix}2\\1\end{pmatrix}=\begin{pmatrix}8\\6\end{pmatrix}+s\begin{pmatrix}-1\\-2\end{pmatrix}$$

$$\therefore \begin{pmatrix}2r+s\\r+2s\end{pmatrix}=\begin{pmatrix}8\\4\end{pmatrix}$$

$$\therefore\ -4r-2s=-16$$
$$r+2s=4$$
$$\text{Adding, } -3r=-12$$
$$\therefore\ r=4$$

$$\therefore \begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\2\end{pmatrix}+4\begin{pmatrix}2\\1\end{pmatrix}=\begin{pmatrix}8\\6\end{pmatrix}$$

$\therefore$ B is (8, 6)

Lines (BC) and (AC) meet at C.

$$\therefore \begin{pmatrix}8\\6\end{pmatrix}+s\begin{pmatrix}-1\\-2\end{pmatrix}=\begin{pmatrix}0\\5\end{pmatrix}+t\begin{pmatrix}1\\-1\end{pmatrix}$$

$$\therefore \begin{pmatrix}-s-t\\-2s+t\end{pmatrix}=\begin{pmatrix}-8\\-1\end{pmatrix}$$

$$\therefore\ -s-t=-8$$
$$-2s+t=-1$$
$$\text{Adding, } -3s=-9$$
$$\therefore\ s=3$$

$$\therefore \begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}8\\6\end{pmatrix}+3\begin{pmatrix}-1\\-2\end{pmatrix}=\begin{pmatrix}5\\0\end{pmatrix}$$

$\therefore$ C is (5, 0)

b A(2, 3), B(8, 6), C(5, 0)

$$\text{AB}=\sqrt{(8-2)^2+(6-3)^2}=\sqrt{36+9}=\sqrt{45}$$

$$\text{BC}=\sqrt{(5-8)^2+(0-6)^2}=\sqrt{9+36}=\sqrt{45}$$

$$\text{AC}=\sqrt{(5-2)^2+(0-3)^2}=\sqrt{9+9}=\sqrt{18}$$

The two equal sides are [AB] and [BC] and they have length $\sqrt{45}$ units. [AC] has length $\sqrt{18}$ units.

4 **a** Lines (QP) and (PR) meet at P.

$$\therefore \begin{pmatrix}3\\-1\end{pmatrix}+r\begin{pmatrix}14\\10\end{pmatrix}=\begin{pmatrix}0\\18\end{pmatrix}+t\begin{pmatrix}5\\-7\end{pmatrix}$$

$$\therefore \begin{pmatrix}14r-5t\\10r+7t\end{pmatrix}=\begin{pmatrix}-3\\19\end{pmatrix}$$

$$\therefore\ 14r-5t=-3 \quad .... (1)$$
$$10r+7t=19 \quad .... (2)$$

$$\therefore\ 98r-35t=-21 \quad \{7\times(1)\}$$
$$50r+35t=95 \quad \{5\times(2)\}$$
$$\text{Adding, } 148r=74$$
$$\therefore\ r=\tfrac{1}{2}$$

$$\therefore \begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}3\\-1\end{pmatrix}+\tfrac{1}{2}\begin{pmatrix}14\\10\end{pmatrix}=\begin{pmatrix}10\\4\end{pmatrix}$$

$\therefore$ P is (10, 4)

Lines (QR) and (PR) meet at R.

$$\therefore \begin{pmatrix}3\\-1\end{pmatrix}+s\begin{pmatrix}17\\-9\end{pmatrix}=\begin{pmatrix}0\\18\end{pmatrix}+t\begin{pmatrix}5\\-7\end{pmatrix}$$

$$\therefore \begin{pmatrix}17s-5t\\-9s+7t\end{pmatrix}=\begin{pmatrix}-3\\19\end{pmatrix}$$

$$\therefore\ 17s-5t=-3 \quad .... (1)$$
$$-9s+7t=19 \quad .... (2)$$

$$\therefore\ 119s-35t=-21 \quad \{7\times(1)\}$$
$$-45s+35t=95 \quad \{5\times(2)\}$$
$$\text{Adding, } 74s=74$$
$$\therefore\ s=1$$

$$\therefore \begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}3\\-1\end{pmatrix}+\begin{pmatrix}17\\-9\end{pmatrix}=\begin{pmatrix}20\\-10\end{pmatrix}$$

$\therefore$ R is (20, −10)

Lines (QP) and (PR) meet at Q.

$$\begin{pmatrix} 3 \\ -1 \end{pmatrix} + r\begin{pmatrix} 14 \\ 10 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} + s\begin{pmatrix} 17 \\ -9 \end{pmatrix}$$

$$\therefore \quad r\begin{pmatrix} 14 \\ 10 \end{pmatrix} = s\begin{pmatrix} 17 \\ -9 \end{pmatrix}$$

$$\therefore \quad r = s = 0$$

$$\text{So,} \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

$\therefore$ Q is $(3, -1)$

b $\overrightarrow{PQ} = \begin{pmatrix} 3-10 \\ -1-4 \end{pmatrix} = \begin{pmatrix} -7 \\ -5 \end{pmatrix}$

$\overrightarrow{PR} = \begin{pmatrix} 20-10 \\ -10-4 \end{pmatrix} = \begin{pmatrix} 10 \\ -14 \end{pmatrix}$

and $\overrightarrow{PQ} \bullet \overrightarrow{PR} = -70 + 70 = 0$

c $[PQ] \perp [PR] \quad \therefore \quad Q\widehat{P}R = 90°$

d
$$\begin{aligned} \text{Area} &= \tfrac{1}{2} \mid \overrightarrow{PQ} \mid \mid \overrightarrow{PR} \mid \\ &= \tfrac{1}{2}\sqrt{49+25}\sqrt{100+196} \\ &= 74 \text{ units}^2 \end{aligned}$$

5 **a** Lines (AB) and (AD) meet at A.

$$\therefore \quad \begin{pmatrix} 2 \\ 5 \end{pmatrix} + r\begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} + u\begin{pmatrix} -3 \\ 12 \end{pmatrix}$$

$$\therefore \quad \begin{pmatrix} 4r+3u \\ r-12u \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$$

$$\begin{aligned} \therefore \quad 4r + 3u &= 1 \\ r - 12u &= -4 \quad \text{.... (1)} \end{aligned}$$

$$\begin{aligned} \therefore \quad 4r + 3u &= 1 \\ -4r + 48u &= 16 \quad \{-4 \times (1)\} \\ \hline \text{Adding,} \quad 51u &= 17 \\ \therefore \quad u &= \tfrac{1}{3} \end{aligned}$$

$$\therefore \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \tfrac{1}{3}\begin{pmatrix} -3 \\ 12 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

$\therefore$ A is $(2, 5)$

Lines (AB) and (BC) meet at B.

$$\therefore \quad \begin{pmatrix} 2 \\ 5 \end{pmatrix} + r\begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 18 \\ 9 \end{pmatrix} + s\begin{pmatrix} -8 \\ 32 \end{pmatrix}$$

$$\therefore \quad \begin{pmatrix} 4r+8s \\ r-32s \end{pmatrix} = \begin{pmatrix} 16 \\ 4 \end{pmatrix}$$

$$\begin{aligned} \therefore \quad 4r + 8s &= 16 \quad \text{.... (1)} \\ r - 32s &= 4 \quad \text{.... (2)} \end{aligned}$$

$$\begin{aligned} \therefore \quad r + 2s &= 4 \quad \{(1) \div 4\} \\ -r + 32s &= -4 \quad \{-1 \times (2)\} \\ \hline \text{Adding,} \quad 34s &= 0 \\ \therefore \quad s &= 0 \end{aligned}$$

$$\therefore \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 18 \\ 9 \end{pmatrix}$$

$\therefore$ B is $(18, 9)$

Lines (BC) and (CD) meet at C.

$$\therefore \quad \begin{pmatrix} 18 \\ 9 \end{pmatrix} + s\begin{pmatrix} -8 \\ 32 \end{pmatrix} = \begin{pmatrix} 14 \\ 25 \end{pmatrix} + t\begin{pmatrix} -8 \\ -2 \end{pmatrix}$$

$$\therefore \quad \begin{pmatrix} -8s+8t \\ 32s+2t \end{pmatrix} = \begin{pmatrix} -4 \\ 16 \end{pmatrix}$$

$$\begin{aligned} \therefore \quad -8s + 8t &= -4 \quad \text{.... (1)} \\ 32s + 2t &= 16 \end{aligned}$$

$$\begin{aligned} \therefore \quad 2s - 2t &= 1 \quad \{(1) \div -4\} \\ 32s + 2t &= 16 \\ \hline \text{Adding,} \quad 34s \quad &= 17 \\ \therefore \quad s &= \tfrac{1}{2} \end{aligned}$$

$$\therefore \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 18 \\ 9 \end{pmatrix} + \tfrac{1}{2}\begin{pmatrix} -8 \\ 32 \end{pmatrix} = \begin{pmatrix} 14 \\ 25 \end{pmatrix}$$

$\therefore$ C is $(14, 25)$

Lines (CD) and (AD) meet at D.

$$\therefore \quad \begin{pmatrix} 14 \\ 25 \end{pmatrix} + t\begin{pmatrix} -8 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} + u\begin{pmatrix} -3 \\ 12 \end{pmatrix}$$

$$\therefore \quad \begin{pmatrix} -8t+3u \\ -2t-12u \end{pmatrix} = \begin{pmatrix} -11 \\ -24 \end{pmatrix}$$

$$\begin{aligned} \therefore \quad -8t + 3u &= -11 \quad \text{.... (1)} \\ -2t - 12u &= -24 \quad \text{.... (2)} \end{aligned}$$

$$\begin{aligned} \therefore \quad 16t - 6u &= 22 \quad \{(-2) \times (1)\} \\ t + 6u &= 12 \quad \{(2) \div -2\} \\ \hline \text{Adding,} \quad 17t \quad &= 34 \\ \therefore \quad t &= 2 \end{aligned}$$

$$\therefore \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 14 \\ 25 \end{pmatrix} + 2\begin{pmatrix} -8 \\ -2 \end{pmatrix} = \begin{pmatrix} -2 \\ 21 \end{pmatrix}$$

$\therefore$ D is $(-2, 21)$

b $\overrightarrow{AC} = \begin{pmatrix} 14-2 \\ 25-5 \end{pmatrix} = \begin{pmatrix} 12 \\ 20 \end{pmatrix}$

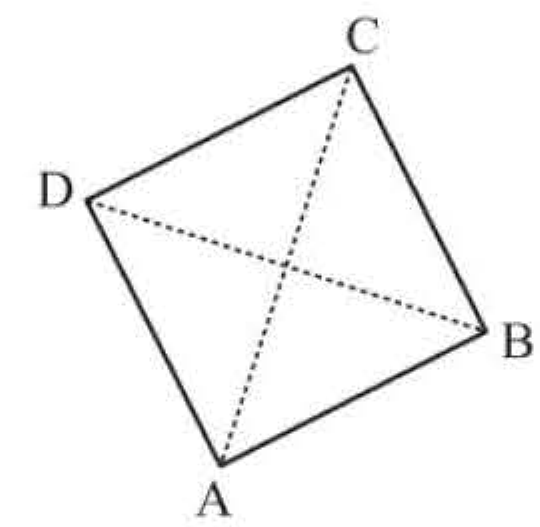
$\overrightarrow{DB} = \begin{pmatrix} 18--2 \\ 9-21 \end{pmatrix} = \begin{pmatrix} 20 \\ -12 \end{pmatrix}$

i $|\overrightarrow{AC}| = \sqrt{12^2+20^2} = \sqrt{544}$ units

ii $|\overrightarrow{DB}| = \sqrt{20^2+(-12)^2} = \sqrt{544}$ units

iii $\overrightarrow{AC} \bullet \overrightarrow{DB} = 240 - 240 = 0$

c The diagonals are perpendicular and equal in length, and as their midpoints are the same (at (8, 15)), ABCD is a square.



EXERCISE 15H.1

1 **a** In augmented matrix form, the system is:

$$\left[\begin{array}{cc|c} 1 & -2 & 8 \\ 4 & 1 & 5 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -2 & 8 \\ 0 & 9 & -27 \end{array}\right] \quad R_2 \to R_2 - 4R_1 \qquad \begin{array}{rrr} 4 & 1 & 5 \\ -4 & 8 & -32 \\ \hline 0 & 9 & -27 \end{array}$$

From R_2, $9y = -27$
$\therefore\ y = -3$

Now $x - 2y = 8$
$\therefore\ x - 2(-3) = 8$
$\therefore\ x = 2$

So, the solution is $x = 2,\ y = -3$.

b In augmented matrix form, the system is:

$$\left[\begin{array}{cc|c} 4 & 5 & 21 \\ 5 & -3 & -20 \end{array}\right] \sim \left[\begin{array}{cc|c} 4 & 5 & 21 \\ 0 & -37 & -185 \end{array}\right] \quad R_2 \to 4R_2 - 5R_1 \qquad \begin{array}{rrr} 20 & -12 & -80 \\ -20 & -25 & -105 \\ \hline 0 & -37 & -185 \end{array}$$

From R_2, $-37y = -185$
$\therefore\ y = 5$

Now $4x + 5y = 21$
$\therefore\ 4x + 25 = 21$
$\therefore\ 4x = -4$
$\therefore\ x = -1$

So, the solution is $x = -1,\ y = 5$.

c In augmented matrix form, the system is:

$$\left[\begin{array}{cc|c} 3 & 1 & -10 \\ 2 & 5 & -24 \end{array}\right] \sim \left[\begin{array}{cc|c} 3 & 1 & -10 \\ 0 & 13 & -52 \end{array}\right] \quad R_2 \to 3R_2 - 2R_1 \qquad \begin{array}{rrr} 6 & 15 & -72 \\ -6 & -2 & 20 \\ \hline 0 & 13 & -52 \end{array}$$

From R_2, $13y = -52$
$\therefore\ y = -4$

Now $3x + y = -10$
$\therefore\ 3x + (-4) = -10$
$\therefore\ 3x = -6$
$\therefore\ x = -2$

So, the solution is $x = -2,\ y = -4$.

2 **a** One equation is not a multiple of the other and their gradients are not the same, so the lines are intersecting.

b $x + y = 7$ can be written as $3x + 3y = 21$ and the other line is $3x + 3y = 1$,
$\therefore$ the lines are parallel.

c The lines intersect at $(2\frac{1}{2}, 2)$.

d $x - 2y = 4$ can be written as $2x - 4y = 8$, so the lines are coincident.

e The lines are intersecting.

f $3x - 4y = 5$ can be written as $-3x + 4y = -5$ and the other line is $-3x + 4y = 2$,
$\therefore$ the lines are parallel.

3 **a** $x + 2y = 3$ can be written as $2x + 4y = 6$, $\therefore$ the equations represent coincident lines.
So, there are an infinite number of solutions (all the points on the line).

b As the second equation is an exact multiple of the first, it will give the same solutions as the first so it can be ignored.

c **i** If $x = t$, $t + 2y = 3$

$$\therefore\ 2y = 3 - t$$

$$\therefore\ y = \frac{3-t}{2} \quad \therefore \text{ the solutions are } x = t,\ y = \frac{3-t}{2},\ t \in \mathbb{R}.$$

ii If $y = s$, $x + 2s = 3$

$$\therefore\ x = 3 - 2s \quad \therefore \text{ the solutions are } x = 3 - 2s,\ y = s,\ s \in \mathbb{R}.$$

4 **a** In augmented matrix form, the system is:

$$\left[\begin{array}{cc|c} 2 & 3 & 5 \\ 2 & 3 & 11 \end{array}\right] \sim \left[\begin{array}{cc|c} 2 & 3 & 5 \\ 0 & 0 & 6 \end{array}\right] \quad R_2 \to R_2 - R_1$$

$$\begin{array}{ccc} 2 & 3 & 11 \\ -2 & -3 & -5 \\ \hline 0 & 0 & 6 \end{array}$$

b R_2 shows $0x + 0y = 6$ $\therefore$ there are no solutions

c the lines are parallel

5 **a** In augmented matrix form, the system is:

$$\left[\begin{array}{cc|c} 2 & 3 & 5 \\ 4 & 6 & 10 \end{array}\right] \sim \left[\begin{array}{cc|c} 2 & 3 & 5 \\ 0 & 0 & 0 \end{array}\right] \quad R_2 \to R_2 - 2R_1$$

$$\begin{array}{ccc} 4 & 6 & 10 \\ -4 & -6 & -10 \\ \hline 0 & 0 & 0 \end{array}$$

b R_2 shows $0x + 0y = 0$, which is true for all x and y.

All solutions come from $2x + 3y = 5$. Let $x = t$, $y = \dfrac{5-2t}{3}$ for all values of t

$\therefore$ there are infinitely many solutions of the form $x = t$, $y = \dfrac{5-2t}{3}$, $t \in \mathbb{R}$.

The lines are coincident.

6 **a** In augmented matrix form, the system is:

$$\left[\begin{array}{cc|c} 3 & -1 & 2 \\ 6 & -2 & 4 \end{array}\right] \sim \left[\begin{array}{cc|c} 3 & -1 & 2 \\ 0 & 0 & 0 \end{array}\right] \quad R_2 \to R_2 - 2R_1$$

$$\begin{array}{ccc} 6 & -2 & 4 \\ -6 & 2 & -4 \\ \hline 0 & 0 & 0 \end{array}$$

R_2 shows $0x + 0y = 0$, which is true for all x and y.

So, there are infinitely many solutions {the lines are coincident}.

Substitute $x = t$ in the first equation

$$3x - y = 2$$
$$\therefore\ 3t - y = 2$$
$$y = 3t - 2$$

So, the solutions have form $x = t$, $y = 3t - 2$, $t \in \mathbb{R}$.

b $3x - y = 2$ .... (1)

$6x - 2y = k$ .... (2)

If $k = 4$ then $6x - 2y = 4$, which is an exact multiple ($\times 2$) of equation (1), $\therefore$ the lines are coincident and there are an infinite number of solutions of the form $x = t$, $y = 3t - 2$, $t \in \mathbb{R}$.

If $k \neq 4$ then the equations represent parallel lines $\therefore$ there are no solutions.

7 **a** In augmented matrix form, the system is:

$$\left[\begin{array}{cc|c} 3 & -1 & 8 \\ 6 & -2 & k \end{array}\right] \sim \left[\begin{array}{cc|c} 3 & -1 & 8 \\ 0 & 0 & k-16 \end{array}\right] \quad R_2 \to R_2 - 2R_1$$

$$\begin{array}{ccc} 6 & -2 & k \\ -6 & 2 & -16 \\ \hline 0 & 0 & k-16 \end{array}$$

b **i** If $k = 16$ there are infinitely many solutions.

ii Substitute $x = t$ in $3x - y = 8$, then $3t - y = 8$ $\therefore$ $y = 3t - 8$.

The solutions are $x = t$, $y = 3t - 8$, $t \in \mathbb{R}$.

c **i** The system has no solutions when $k - 16 \neq 0$, $\therefore$ $k \neq 16$.

ii The lines are parallel but not coincident.

8 **a** In augmented matrix form, the system is:

$$\left[\begin{array}{cc|c} 4 & 8 & 1 \\ 2 & -a & 11 \end{array}\right] \sim \left[\begin{array}{cc|c} 4 & 8 & 1 \\ 0 & -2a-8 & 21 \end{array}\right] \quad R_2 \to 2R_2 - R_1$$

$$\begin{array}{ccc} 4 & -2a & 22 \\ -4 & -8 & -1 \\ \hline 0 & -2a-8 & 21 \end{array}$$

$$\sim \left[\begin{array}{cc|c} 4 & 8 & 1 \\ 0 & 2a+8 & -21 \end{array}\right] \quad R_2 \to -R_2$$

b A unique solution exists provided $2a+8 \neq 0 \quad \therefore \quad a \neq -4$.

c From R_2, $(2a+8)y = -21$

$$\therefore \quad y = \frac{-21}{2a+8}$$

and $4x + 8y = 1$

$$\therefore \quad 4x + 8\left(\frac{-21}{2a+8}\right) = 1$$

$$\therefore \quad 4x(2a+8) - 168 = 2a+8$$
$$\therefore \quad 2x(2a+8) - 84 = a+4$$
$$\therefore \quad 2x(2a+8) = a+88$$
$$\therefore \quad x = \frac{a+88}{4a+16}$$

The solution is $x = \dfrac{a+88}{4a+16}$, $y = \dfrac{-21}{2a+8}$, $a \neq -4$.

d When $a = -4$ there are no solutions as the lines are parallel.

9 **a** In augmented matrix form, the system is:

$$\left[\begin{array}{cc|c} m & 2 & 6 \\ 2 & m & 6 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} m & 2 & 6 \\ 0 & m^2-4 & 6m-12 \end{array}\right] \quad R_2 \to mR_2 - 2R_1$$

$$\begin{array}{ccc} 2m & m^2 & 6m \\ -2m & -4 & -12 \\ \hline 0 & m^2-4 & 6m-12 \end{array}$$

A unique solution exists provided $m^2 - 4 \neq 0$.

So, there is a unique solution for all m except $m = \pm 2$.

b In R_2, $(m^2-4)y = 6m-12$

$$\therefore \quad y = \frac{6(m-2)}{(m-2)(m+2)}$$

$$\therefore \quad y = \frac{6}{m+2} \quad \text{provided} \quad m \neq \pm 2$$

Substituting in $mx + 2y = 6$

gives $mx + 2\left(\dfrac{6}{m+2}\right) = 6$

$$\therefore \quad m(m+2)x + 12 = 6(m+2)$$
$$\therefore \quad m(m+2)x = 6m + 12 - 12$$
$$\therefore \quad m(m+2)x = 6m$$
$$\therefore \quad x = \frac{6}{m+2}$$

So, the unique solution is $x = \dfrac{6}{m+2}$, $y = \dfrac{6}{m+2}$ when $m \neq \pm 2$.

c When $m = 2$, the equations are $2x+2y=6$ and $2x+2y=6$, $\therefore$ the lines are coincident. So, there are an infinite number of solutions of the form $x = t$, $y = \dfrac{6-2t}{2} = 3-t$ for all $t \in \mathbb{R}$.

When $m = -2$, the equations are $-2x+2y=6$ and $2x-2y=6$
or $-2x+2y=-6$

$\therefore$ the lines are parallel and there are no solutions.

EXERCISE 15H.2

1 **a** Line 1 has direction vector $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ and line 2 has direction vector $\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$.

As one vector is not a scalar multiple of the other, the lines are not parallel.

Now $\quad 1+2t = -2+3s \qquad 2-t = 3-s \qquad 3+t = 1+2s$

$\therefore \quad 2t - 3s = -3$ .... (1) $\qquad \therefore \quad -t+s = 1$ .... (2) $\qquad \therefore \quad t - 2s = -2$ .... (3)

Solving (2) and (3) simultaneously:
$$\begin{aligned} -t+s&=1\\ t-2s&=-2\\ \hline -s&=-1 \quad \therefore\ s=1 \text{ and } t=0\end{aligned}$$

and in (1), $\text{LHS} = 2t-3s = 2(0)-3(1) = -3$ ✓

$\therefore$ $s=1$, $t=0$ satisfies all three equations

$\therefore$ the two lines meet at $(1, 2, 3)$ $\{$using $t=0$ or $s=1\}$

The acute angle between the lines has $\cos\theta = \dfrac{|6+1+2|}{\sqrt{4+1+1}\,\sqrt{9+1+4}} = \dfrac{9}{\sqrt{84}}$

and so $\theta \approx 10.9°$

b Line 1 has direction vector $\begin{pmatrix} 2\\ -12\\ 12\end{pmatrix}$ and line 2 has direction vector $\begin{pmatrix} 4\\ 3\\ -1\end{pmatrix}$.

As one vector is not a scalar multiple of the other, the lines are not parallel.

Now $-1+2\lambda = 4\mu-3$ $\quad$ $2-12\lambda = 3\mu+2$ $\quad$ $4+12\lambda = -\mu-1$

$\therefore$ $2\lambda-4\mu=-2$ $\quad$ $-12\lambda-3\mu=0$ $\quad$ $12\lambda+\mu=-5$ .... (3)

$\therefore$ $\lambda-2\mu=-1$ .... (1) $\quad$ $\mu=-4\lambda$ .... (2)

Solving (1) and (2) simultaneously: $\lambda-2(-4\lambda)=-1$

$\therefore$ $9\lambda=-1$

$\therefore$ $\lambda=-\frac{1}{9}$ and so $\mu=\frac{4}{9}$

In (3), $12\lambda+\mu = 12\left(-\frac{1}{9}\right)+\frac{4}{9} = -\frac{12}{9}+\frac{4}{9} = -\frac{8}{9}$, which is not -5.

Since the system is inconsistent, the lines do not intersect, so the lines are skew.

The acute angle between the lines has $\cos\theta = \dfrac{|8-36-12|}{\sqrt{292}\,\sqrt{26}} = \dfrac{40}{\sqrt{7592}}$ and so $\theta \approx 62.7°$.

c Line 1 has direction vector $\begin{pmatrix} 6\\ 8\\ 2\end{pmatrix}$ and line 2 has direction vector $\begin{pmatrix} 3\\ 4\\ 1\end{pmatrix}$.

As $\begin{pmatrix} 6\\ 8\\ 2\end{pmatrix} = 2\begin{pmatrix} 3\\ 4\\ 1\end{pmatrix}$ the two lines are parallel. Hence, $\theta = 0°$.

To see if the lines are coincident, try to find a shared point.

The point on line 1 where $t=1$ is $(6, 11, 1)$.

The unique point on line 2 with z-coordinate 1 is the point where $1+s=1$ $\therefore$ $s=0$.

This point is $(2, 0, 1)$. Since $(6, 11, 1) \neq (2, 0, 1)$ the lines are not coincident.

d In line 1 let $x = 2-y = z+2 = t$, so $x=t$, $y=2-t$, and $z=t-2$, $t\in\mathbb{R}$.

Line 1 has direction vector $\begin{pmatrix} 1\\ -1\\ 1\end{pmatrix}$ and line 2 has direction vector $\begin{pmatrix} 3\\ -2\\ 2\end{pmatrix}$.

As one vector is not a scalar multiple of the other, the lines are not parallel.

Now $t=1+3s$ .... (1) $\quad$ $2-t=-2-2s$ $\quad$ $-2+t=2s+\frac{1}{2}$

$-t+2s=-4$ .... (2) $\quad$ $t-2s=2\frac{1}{2}$ .... (3)

Solving (1) and (2) simultaneously: $-(1+3s)+2s=-4$

$\therefore$ $-1-3s+2s=-4$

$\therefore$ $-s=-3$

$\therefore$ $s=3$ and so $t=1+3(3)=10$

Substituting in (3), $t-2s = 10-2(3) = 4 \neq 2\frac{1}{2}$

Since the system is inconsistent, the lines do not meet. $\therefore$ they are skew.

The acute angle between the lines has $\cos\theta = \dfrac{|3+2+2|}{\sqrt{1+1+1}\sqrt{9+4+4}} = \dfrac{7}{\sqrt{3}\sqrt{17}}$

$\therefore$ $\theta \approx 11.4°$

e Line 1 has direction vector $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ and line 2 has direction vector $\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$.

As one vector is not a scalar multiple of the other, the lines are not parallel.

$$\begin{array}{lll} 1+\lambda = 2+3\mu & 2-\lambda = 3-2\mu & 3+2\lambda = \mu - 5 \\ \lambda - 3\mu = 1 \ \text{.... (1)} & -\lambda + 2\mu = 1 \ \text{.... (2)} & 2\lambda - \mu = -8 \ \text{.... (3)} \end{array}$$

Solving (1) and (2) simultaneously: $\lambda - 3\mu = 1$

$$\begin{array}{r} -\lambda + 2\mu = 1 \\ \hline \text{Adding,} \quad -\mu = 2 \end{array}$$

$\therefore \ \mu = -2$ and $\lambda - 3(-2) = 1 \quad \therefore \ \lambda = -5$

Checking in (3), $2\lambda - \mu = 2(-5) - (-2) = -10 + 2 = -8$ ✓

Since $\mu = -2, \ \lambda = -5$ satisfies all three equations, the lines meet.

They meet at $x = 1 + (-5), \ y = 2 - (-5), \ z = 3 + 2(-5)$, or at $(-4, 7, -7)$.

The acute angle between the lines has $\cos\theta = \dfrac{|3+2+2|}{\sqrt{1+1+4}\sqrt{9+4+1}} = \dfrac{7}{\sqrt{84}}$

and so $\theta \approx 40.2°$

f Line 1 has direction vector $\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$ and line 2 has direction vector $\begin{pmatrix} 4 \\ -2 \\ 0 \end{pmatrix}$.

Now $\begin{pmatrix} 4 \\ -2 \\ 0 \end{pmatrix} = -2\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$, so the lines are parallel and hence $\theta = 0°$.

All points on line 1 have z-coordinate 5 and all points on line 2 have z-coordinate 3.

$\therefore$ the lines are not coincident.

g Line 1 has direction vector $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ and line 2 has direction vector $\begin{pmatrix} -4 \\ 2 \\ -6 \end{pmatrix}$.

As $\begin{pmatrix} -4 \\ 2 \\ -6 \end{pmatrix} = -2\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$, the two lines are parallel. Hence $\theta = 0°$.

The point on line 1 where $\lambda = 1$ is $(3, -1, 4)$.

The unique point on line 2 with x-coordinate 3 is the point where $3 - 4\mu = 3 \quad \therefore \ \mu = 0$.

This point is $(3, -1, 4)$.

Lines 1 and 2 are parallel and share the point $(3, -1, 4)$. $\therefore$ they are coincident and $\theta = 0°$.

2 Line 1 is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$ with direction vector $\mathbf{a} = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$.

Line 2 is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \mu\begin{pmatrix} -2 \\ 4 \\ 2 \end{pmatrix}$ with direction vector $\mathbf{b} = \begin{pmatrix} -2 \\ 4 \\ 2 \end{pmatrix}$.

Line 3 is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ with direction vector $\mathbf{c} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$.

Line 1 and line 2:

Since $\begin{pmatrix} -2 \\ 4 \\ 2 \end{pmatrix} = -2\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$, line 1 and line 2 are parallel.

When $\lambda = 0$, the point on line 1 is $(3, -1, 2)$.

For line 2, $y = 4\mu$, so the unique point on line 2 with y-coordinate -1 is the point where $\mu = -\frac{1}{4}$.

This point is $(\frac{3}{2}, -1, -\frac{3}{2})$.

Since $(\frac{3}{2}, -1, -\frac{3}{2}) \neq (3, -1, 2)$, line 1 and line 2 are not coincident.

Line 1 and line 3:

Equating x, y, and z values in lines 1 and 3 gives

$$\underbrace{\begin{array}{ll} 3+\lambda = t & -1-2\lambda = 1+2t \\ \therefore\ t = 3+\lambda & \therefore\ 2t = -2-2\lambda \\ & \therefore\ t = -1-\lambda \end{array}} \qquad \begin{array}{l} 2-\lambda = 1+t \\ \therefore\ \lambda + t = 1 \ \ .... \ (1) \\ \ \end{array}$$

Solving these we get $3+\lambda = -1-\lambda$

$$\therefore\ 2\lambda = -4$$

$$\therefore\ \lambda = -2 \quad \text{and so} \quad t = 3-2 \quad \therefore\ t = 1$$

Checking in (1): $\lambda + t = -2+1 = -1 \neq 1$

So, there is no simultaneous solution to all 3 equations.

$\therefore$ the lines do not intersect.

$\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} \neq k \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ for any $k \in \mathbb{R}$.

$\therefore$ lines 1 and 3 are not parallel.

Since they do not intersect and are not parallel, they are skew.

If θ is the acute angle between **a** and **c**, then $\cos\theta = \dfrac{|\mathbf{a} \bullet \mathbf{c}|}{|\mathbf{a}||\mathbf{c}|} = \dfrac{|1-4-1|}{\sqrt{1+4+1}\sqrt{1+4+1}} = \dfrac{4}{\sqrt{6}\sqrt{6}} = \frac{2}{3}$

$$\therefore\ \theta \approx 48.2^\circ$$

So, line 1 and line 3 are skew with an angle of about 48.2° between them.

Line 2 and line 3:

Equating x, y, and z values in lines 2 and 3 gives

$$\underbrace{\begin{array}{ll} 1-2\mu = t & 4\mu = 1+2t \\ \therefore\ t = 1-2\mu & \therefore\ 2t = -1+4\mu \\ \therefore\ 2t = 2-4\mu & \end{array}} \qquad \begin{array}{l} -1+2\mu = 1+t \\ \therefore\ 2\mu - t = 2 \ \ .... \ (2) \\ \ \end{array}$$

Solving these we get $2-4\mu = -1+4\mu$

$$\therefore\ 8\mu = 3$$

$$\therefore\ \mu = \tfrac{3}{8} \quad \text{and so} \quad t = 1-2(\tfrac{3}{8}) \quad \therefore\ t = \tfrac{1}{4}$$

Checking in (2): $2\mu - t = 2(\frac{3}{8}) - \frac{1}{4} = \frac{6}{8} - \frac{2}{8} = \frac{4}{8} = \frac{1}{2} \neq 2$

So, there is no simultaneous solution to all 3 equations.

$\therefore$ the lines do not intersect.

$\begin{pmatrix} -2 \\ 4 \\ 2 \end{pmatrix} \neq k \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ for any $k \in \mathbb{R}$.

$\therefore$ lines 2 and 3 are not parallel.

Since they do not intersect and are not parallel, they are skew.

If ϕ is the acute angle between **b** and **c** then

$$\cos\phi = \frac{|\mathbf{b} \bullet \mathbf{c}|}{|\mathbf{b}||\mathbf{c}|} = \frac{|-2+8+2|}{\sqrt{4+16+4}\sqrt{1+4+1}} = \frac{8}{\sqrt{144}} = \frac{2}{3}$$

$$\therefore\ \phi \approx 48.2^\circ$$

So, line 2 and line 3 are skew with an angle of about 48.2° between them.

EXERCISE 15I

1 **a** $\mathbf{n} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$, and the point $(-1, 2, 4)$ lies on the plane.

$\therefore$ the equation is $2x - y + 3z = 2(-1) - 2 + 3(4)$ which is $2x - y + 3z = 8$.

b $\overrightarrow{AB} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$ is a vector normal to the plane, and $(2, 3, 1)$ lies on the plane.

$\therefore$ the equation is $3x + 4y + z = 3(2) + 4(3) + 1$

$\therefore\ 3x + 4y + z = 19$

c The line $x = 1 + t,\ \ y = 2 - t,\ \ z = 3 + 2t$ has direction vector $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$.

Also, letting $t = 0$, the point $(1, 2, 3)$ lies on the plane and we call this point B.

$\therefore\ \overrightarrow{AB} = \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix}$ and so a vector normal to the plane is $\overrightarrow{AB} \times \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

$$\therefore\ \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 0 & 2 \\ 1 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 0 & 2 \\ -1 & 2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -2 & 2 \\ 1 & 2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -2 & 0 \\ 1 & -1 \end{vmatrix}\mathbf{k}$$

$= 2\mathbf{i} + 6\mathbf{j} + 2\mathbf{k}$ or $2(\mathbf{i} + 3\mathbf{j} + \mathbf{k})$

$\therefore$ since A(3, 2, 1) lies on the plane, it has equation $x + 3y + z = 3 + 3(2) + 1$

or $x + 3y + z = 10$

2 **a** $2x + 3y - z = 8$ has $\mathbf{n} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$

b $3x - y + 0z = 11$ has $\mathbf{n} = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$

c $0x + 0y + z = 2$ has $\mathbf{n} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

d $1x + 0y + 0z = 0$ has $\mathbf{n} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

3 **a** The y-axis is perpendicular to the XOZ plane $\therefore$ a normal vector is $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$\therefore$ since the origin lies on the plane, it has equation $y = 0$.

b Since the plane is perpendicular to the Z-axis, it has normal vector $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\therefore$ since $(2, -1, 4)$ lies on the plane, it has equation $z = 4$.

4 **a** **i** $\overrightarrow{AB} = \begin{pmatrix} 1 \\ 1 \\ -4 \end{pmatrix}$, $\overrightarrow{AC} = \begin{pmatrix} -1 \\ 0 \\ -2 \end{pmatrix}$, so $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -4 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 0 \\ -2 \end{pmatrix}$

ii If $\mathbf{n}$ is the normal vector, then

$$\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -4 \\ -1 & 0 & -2 \end{vmatrix} = \begin{vmatrix} 1 & -4 \\ 0 & -2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & -4 \\ -1 & -2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 1 \\ -1 & 0 \end{vmatrix}\mathbf{k} = -2\mathbf{i} + 6\mathbf{j} + \mathbf{k}$$

$\therefore$ since A(0, 2, 6) lies on the plane, it has equation $-2x + 6y + z = -2(0) + 6(2) + 6$

$\therefore\ -2x + 6y + z = 18$

b **i** $\overrightarrow{AB} = \begin{pmatrix} -3 \\ 3 \\ -2 \end{pmatrix}$, $\overrightarrow{AC} = \begin{pmatrix} -3 \\ -1 \\ -1 \end{pmatrix}$, so $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 3 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} -3 \\ -1 \\ -1 \end{pmatrix}$

ii If $\mathbf{n}$ is the normal vector, then

$$\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 3 & -2 \\ -3 & -1 & -1 \end{vmatrix} = \begin{vmatrix} 3 & -2 \\ -1 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -3 & -2 \\ -3 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -3 & 3 \\ -3 & -1 \end{vmatrix}\mathbf{k}$$

$= -5\mathbf{i} + 3\mathbf{j} + 12\mathbf{k}$

$\therefore$ since C(0, 0, 1) lies on the plane, it has equation $-5x + 3y + 12z = 12$.

c **i** $\overrightarrow{AB} = \begin{pmatrix} -2 \\ -1 \\ -1 \end{pmatrix}$, $\overrightarrow{AC} = \begin{pmatrix} 2 \\ -3 \\ -3 \end{pmatrix}$, so $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ -1 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -3 \\ -3 \end{pmatrix}$

ii If $\mathbf{n}$ is the normal vector, then

$$\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & -1 & -1 \\ 2 & -3 & -3 \end{vmatrix} = \begin{vmatrix} -1 & -1 \\ -3 & -3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -2 & -1 \\ 2 & -3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -2 & -1 \\ 2 & -3 \end{vmatrix} \mathbf{k}$$

$$= -8\mathbf{j} + 8\mathbf{k} \quad \text{or} \quad -8(\mathbf{j} - \mathbf{k})$$

$\therefore$ since $A(2, 0, 3)$ lies on the plane, it has equation $y - z = -3$.

5 **a** The normal to $x - 3y + 4z = 8$ is $\begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix}$, and this is the direction vector of the line.

$\therefore$ since the line passes through $(1, -2, 0)$, it has equation

$$x = 1 + \lambda, \quad y = -2 - 3\lambda, \quad z = 4\lambda, \quad \lambda \in \mathbb{R}.$$

b The normal to $x - y - 2z = 11$ is $\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$.

$\therefore$ since the line passes through $(3, 4, -1)$, it has equation

$$x = 3 + \lambda, \quad y = 4 - \lambda, \quad z = -1 - 2\lambda, \quad \lambda \in \mathbb{R}.$$

6 The line has direction vector $\overrightarrow{AB} = \begin{pmatrix} -1 \\ 3 \\ -3 \end{pmatrix}$.

$\therefore$ since the line passes through $A(2, -1, 3)$, it has parametric equations

$$x = 2 - t, \quad y = -1 + 3t, \quad z = 3 - 3t, \quad t \in \mathbb{R}.$$

This line meets $x + 2y - z = 5$ when $(2 - t) + 2(-1 + 3t) - (3 - 3t) = 5$

$$\therefore \; 2 - t - 2 + 6t - 3 + 3t = 5$$

$\therefore$ $8t = 8$ $\therefore$ $t = 1$, and so they meet at $(1, 2, 0)$.

7 **a** The direction vector of the line is $\overrightarrow{PQ} = \begin{pmatrix} 1 \\ 2 \\ -5 \end{pmatrix}$.

$\therefore$ since it passes through $P(1, -2, 4)$, it has parametric equations

$$x = 1 + t, \quad y = -2 + 2t, \quad z = 4 - 5t, \quad t \in \mathbb{R}.$$

b **i** The line meets the YOZ plane when $x = 0$, or when $t = -1$.
This corresponds to the point $(0, -4, 9)$.

ii The line meets $y + z = 2$ when $-2 + 2t + 4 - 5t = 2$ $\therefore$ $-3t = 0$ $\therefore$ $t = 0$
This corresponds to the point $(1, -2, 4)$.

iii The line meets $\dfrac{x-3}{2} = \dfrac{y+2}{3} = \dfrac{z-30}{-1}$

when $\dfrac{1+t-3}{2} = \dfrac{-2+2t+2}{3} = \dfrac{4-5t-30}{-1}$

$$\therefore \; \frac{t-2}{2} = \frac{2t}{3} = 5t + 26$$

$$\therefore \; 3t - 6 = 4t = 30t + 156$$

$$\therefore \; 3t - 6 = 4t \quad \text{and} \quad 4t = 30t + 156$$

$$\therefore \; t = -6 \quad \text{and} \quad -26t = 156$$

$$\therefore \; t = -6 \quad \text{is a common solution}$$

$\therefore$ the lines meet at the point corresponding to $t = -6$, which is $(-5, -14, 34)$.

8 **a** The plane $2x+y-2z=-11$ has $\mathbf{n}=\begin{pmatrix}2\\1\\-2\end{pmatrix}$.

$\therefore$ the parametric equations of (AN) are $x=1+2t,\ y=0+t,\ z=2-2t,\ t\in\mathbb{R}$.

This line meets the plane when $2(1+2t)+t-2(2-2t)=-11$

$$\therefore\ 2+4t+t-4+4t=-11$$
$$\therefore\ 9t=-9$$
$$\therefore\ t=-1$$

Thus N is $(-1,-1,4)$ and $\therefore\ \overrightarrow{\text{AN}}=\begin{pmatrix}-2\\-1\\2\end{pmatrix}$ and $\text{AN}=\sqrt{(-2)^2+(-1)^2+2^2}$
$=\sqrt{9}=3$ units

b The plane $x-y+3z=-10$ has $\mathbf{n}=\begin{pmatrix}1\\-1\\3\end{pmatrix}$

$\therefore$ the parametric equations of (AN) are $x=2+t,\ y=-1-t,\ z=3+3t,\ t\in\mathbb{R}$.

This line meets the plane when $(2+t)-(-1-t)+3(3+3t)=-10$

$$\therefore\ 2+t+1+t+9+9t=-10$$
$$\therefore\ 11t=-22$$
$$\therefore\ t=-2$$

$\therefore$ N is $(0,1,-3)$

$\therefore\ \overrightarrow{\text{AN}}=\begin{pmatrix}-2\\2\\-6\end{pmatrix}$ and $\text{AN}=\sqrt{(-2)^2+2^2+(-6)^2}=\sqrt{44}=2\sqrt{11}$ units.

c The plane $4x-y-2z=8$ has $\mathbf{n}=\begin{pmatrix}4\\-1\\-2\end{pmatrix}$.

$\therefore$ the parametric equations of (AN) are

$$x=1+4t,\quad y=-4-t,\quad z=-3-2t,\quad t\in\mathbb{R}.$$

This line meets the plane when $4(1+4t)-(-4-t)-2(-3-2t)=8$

$$\therefore\ 4+16t+4+t+6+4t=8$$
$$\therefore\ 21t=-6$$
$$\therefore\ t=-\tfrac{2}{7}$$

$\therefore$ N is $\left(-\frac{1}{7},-\frac{26}{7},-\frac{17}{7}\right)$, $\therefore\ \overrightarrow{\text{AN}}=\begin{pmatrix}-\frac{8}{7}\\\frac{2}{7}\\\frac{4}{7}\end{pmatrix}$ and so $\text{AN}=\sqrt{\left(-\frac{8}{7}\right)^2+\left(\frac{2}{7}\right)^2+\left(\frac{4}{7}\right)^2}$
$=\sqrt{\frac{84}{49}}=2\sqrt{\frac{3}{7}}$ units

9 The mirror image lies on the normal line to the plane through the object point.

Now $x+2y+z=1$ has $\mathbf{n}=\begin{pmatrix}1\\2\\1\end{pmatrix}$

$\therefore$ the normal at A has parametric equations $x=3+t,\ y=1+2t,\ z=2+t,\ t\in\mathbb{R}$.

This line meets the plane when $(3+t)+2(1+2t)+2+t=1$

$$\therefore\ 3+t+2+4t+2+t=1$$
$$\therefore\ 6t=-6$$
$$\therefore\ t=-1$$

$\therefore$ N is $(2,-1,1)$

If A′ is the mirror image of A, then $\overrightarrow{\text{AN}}=\overrightarrow{\text{NA}'}$

$\therefore$ letting A′ have coordinates (a,b,c), $\begin{pmatrix}-1\\-2\\-1\end{pmatrix}=\begin{pmatrix}a-2\\b+1\\c-1\end{pmatrix}$

$\therefore\ a-2=-1,\ b+1=-2,\ c-1=-1$ and so A′ is at $(1,-3,0)$.

10 The plane $x+4y-z=-2$ has normal $\mathbf{n}=\begin{pmatrix}1\\4\\-1\end{pmatrix}$ which passes through $(3, 4, -1)$.

$\therefore$ the normal has parametric equations $x=3+t,\ y=4+4t,\ z=-1-t,\ t\in\mathbb{R}$ and will meet any of the coordinate axes if any two of the values of x, y, and z are zero at the same time.

$\therefore$ since $x=0$ when $t=-3$ and $y=z=0$ when $t=-1$, the normal meets the X-axis when $t=-1$, at the point $(2, 0, 0)$.

11 $\overrightarrow{\mathrm{AB}}=\begin{pmatrix}-1\\-3\\-1\end{pmatrix}$

a The normal $\mathbf{n}$ is perpendicular to both the X-axis and $\overrightarrow{\mathrm{AB}}$.

Since the X-axis has direction vector $\begin{pmatrix}1\\0\\0\end{pmatrix}$,

$$\mathbf{n}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&0&0\\-1&-3&-1\end{vmatrix}$$

$$=\begin{vmatrix}0&0\\-3&-1\end{vmatrix}\mathbf{i}-\begin{vmatrix}1&0\\-1&-1\end{vmatrix}\mathbf{j}+\begin{vmatrix}1&0\\-1&-3\end{vmatrix}\mathbf{k}$$

$$=\mathbf{j}-3\mathbf{k}$$

Since $\mathrm{A}(1, 2, 3)$ is in the plane, the plane has equation $y-3z=1(2)-3(3)$

or $y-3z=-7$

b The normal $\mathbf{n}$ is perpendicular to both the Y-axis and $\overrightarrow{\mathrm{AB}}$.

Since the Y-axis has direction vector $\begin{pmatrix}0\\1\\0\end{pmatrix}$,

$$\mathbf{n}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\0&1&0\\-1&-3&-1\end{vmatrix}$$

$$=\begin{vmatrix}1&0\\-3&-1\end{vmatrix}\mathbf{i}-\begin{vmatrix}0&0\\-1&-1\end{vmatrix}\mathbf{j}+\begin{vmatrix}0&1\\-1&-3\end{vmatrix}\mathbf{k}$$

$$=-\mathbf{i}+\mathbf{k}$$

Since $\mathrm{A}(1, 2, 3)$ is in the plane, the plane has equation $-x+z=-1(1)+1(3)=2$

$\therefore\ x-z=-2$

c The normal $\mathbf{n}$ is perpendicular to both the Z-axis and $\overrightarrow{\mathrm{AB}}$.

Since the Z-axis has direction vector $\begin{pmatrix}0\\0\\1\end{pmatrix}$,

$$\mathbf{n}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\0&0&1\\-1&-3&-1\end{vmatrix}$$

$$=\begin{vmatrix}0&1\\-3&-1\end{vmatrix}\mathbf{i}-\begin{vmatrix}0&1\\-1&-1\end{vmatrix}\mathbf{j}+\begin{vmatrix}0&0\\-1&-3\end{vmatrix}\mathbf{k}$$

$$=3\mathbf{i}-\mathbf{j}$$

Since $\mathrm{A}(1, 2, 3)$ is in the plane, the plane has equation $3x-y=3(1)-1(2)$

or $3x-y=1$

12 Now $x-1=\dfrac{y-2}{2}=z+3$ has direction vector $\begin{pmatrix}1\\2\\1\end{pmatrix}$

and $x+1=y-3=2z+5$ has direction vector $\begin{pmatrix}1\\1\\\frac{1}{2}\end{pmatrix}$ $\left\{\text{since } 2z+5=\dfrac{z+\frac{5}{2}}{\frac{1}{2}}\right\}$

$\therefore$ a vector perpendicular to both lines is:

$$\begin{pmatrix}1\\2\\1\end{pmatrix}\times\begin{pmatrix}1\\1\\\frac{1}{2}\end{pmatrix}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&2&1\\1&1&\frac{1}{2}\end{vmatrix}=\begin{vmatrix}2&1\\1&\frac{1}{2}\end{vmatrix}\mathbf{i}-\begin{vmatrix}1&1\\1&\frac{1}{2}\end{vmatrix}\mathbf{j}+\begin{vmatrix}1&2\\1&1\end{vmatrix}\mathbf{k}$$

$$=0\mathbf{i}+\tfrac{1}{2}\mathbf{j}-\mathbf{k}$$

$\therefore$ $\begin{pmatrix}0\\1\\-2\end{pmatrix}$ is perpendicular to both lines.

A plane with normal $\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$ has equation $y - 2z = c$ for some c.

Now for line 1, $\dfrac{y-2}{2} = z + 3$ and for line 2, $y - 3 = 2z + 5$

$$\therefore\ y - 2 = 2z + 6 \qquad \therefore\ y - 2z = 8 \text{ also.}$$

$$\therefore\ y - 2z = 8$$

$\therefore$ $y - 2z = 8$ is a plane containing both lines, so the lines are coplanar.

13 **a** Since $A(1, 2, k)$ lies on $x + 2y - 2z = 8$,

$$1 + 2(2) - 2k = 8$$
$$\therefore\ 1 + 4 - 2k = 8$$
$$\therefore\ -2k = 3$$
$$\therefore\ k = -\tfrac{3}{2}$$

b Since $x + 2y - 2z = 8$, the plane has normal vector $\mathbf{n} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$.

$\therefore$ the normal from A has parametric equations

$$x = 1 + t,\quad y = 2 + 2t,\quad z = -\tfrac{3}{2} - 2t,\quad t \in \mathbb{R}.$$

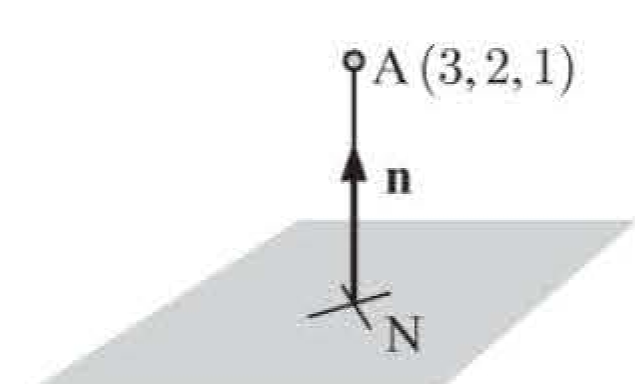
$\therefore$ points of the normal that are 6 units from A have

$$\sqrt{(1 + t - 1)^2 + (2 + 2t - 2)^2 + (-\tfrac{3}{2} - 2t + \tfrac{3}{2})^2} = 6$$
$$\therefore\ \sqrt{t^2 + 4t^2 + 4t^2} = 6$$
$$\therefore\ 9t^2 = 36$$
$$\therefore\ t^2 = 4$$
$$\therefore\ t = \pm 2$$

$\therefore$ B is $(3, 6, -\frac{11}{2})$ or $(-1, -2, \frac{5}{2})$

14 **a** The normal from $A(3, 2, 1)$ to the plane has direction vector

$$\mathbf{n} = (2\mathbf{i} + \mathbf{j} + \mathbf{k}) \times (4\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ 4 & 2 & -2 \end{vmatrix}$$



$$= \begin{vmatrix} 1 & 1 \\ 2 & -2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 1 \\ 4 & -2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix}\mathbf{k}$$
$$= -4\mathbf{i} + 8\mathbf{j}$$

$\therefore$ if N is the foot of the normal from A, (AN)

has equation $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + t\begin{pmatrix} -4 \\ 8 \\ 0 \end{pmatrix}$

So, N has coordinates of the form $(3 - 4t, 2 + 8t, 1)$

But N lies on the plane $\therefore$ $\begin{pmatrix} 3 - 4t \\ 2 + 8t \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + \mu\begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix}$

$$\therefore\ \begin{cases} 3 - 4t = 3 + 2\lambda + 4\mu \\ 2 + 8t = 1 + \lambda + 2\mu \\ 1 = 2 + \lambda - 2\mu \end{cases} \text{ and so } \begin{cases} 2\lambda + 4\mu + 4t = 0 \\ \lambda + 2\mu - 8t = 1 \\ \lambda - 2\mu \qquad = -1 \end{cases}$$

Solving simultaneously using technology gives $\lambda = -0.4$, $\mu = 0.3$, $t = -0.1$

$\therefore$ N is $(3 - 4(-0.1), 2 + 8(-0.1), 1)$ or $(3.4, 1.2, 1)$

$\therefore$ $\overrightarrow{AN} = \begin{pmatrix} 3.4 - 3 \\ 1.2 - 2 \\ 1 - 1 \end{pmatrix} = \begin{pmatrix} 0.4 \\ -0.8 \\ 0 \end{pmatrix}$ and $|\overrightarrow{AN}| = \sqrt{(0.4)^2 + (0.8)^2 + 0^2} = \frac{2}{\sqrt{5}}$ units

b The normal from $A(1, 0, -2)$ to the plane has direction vector

$$\mathbf{n} = (3\mathbf{i} - \mathbf{j} + 2\mathbf{k}) \times (-\mathbf{i} + \mathbf{j} - \mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 2 \\ -1 & 1 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & 2 \\ -1 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & -1 \\ -1 & 1 \end{vmatrix}\mathbf{k}$$

$$= -\mathbf{i} + \mathbf{j} + 2\mathbf{k}$$

$\therefore$ (AN) has equation $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$

So, N has coordinates of the form $(1-t,\ t,\ -2+2t)$

But N lies on the plane $\therefore$ $\begin{pmatrix} 1-t \\ t \\ -2+2t \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + \mu\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$

$\therefore$ $\begin{cases} 1 + 3\lambda - \mu = 1 - t \\ -1 - \lambda + \mu = t \\ 1 + 2\lambda - \mu = -2 + 2t \end{cases}$ and so $\begin{cases} 3\lambda - \mu + t = 0 \\ -\lambda + \mu - t = 1 \\ 2\lambda - \mu - 2t = -3 \end{cases}$

Solving simultaneously using technology gives $\lambda = \frac{1}{2}$, $\mu = 2\frac{1}{3}$, $t = \frac{5}{6}$

$\therefore$ N is $\left(1 - \frac{5}{6},\ \frac{5}{6},\ -2 + \frac{5}{3}\right)$ or $\left(\frac{1}{6},\ \frac{5}{6},\ -\frac{1}{3}\right)$

$\therefore$ $\overrightarrow{\text{AN}} = \begin{pmatrix} \frac{1}{6} - 1 \\ \frac{5}{6} - 0 \\ -\frac{1}{3} - -2 \end{pmatrix} = \begin{pmatrix} -\frac{5}{6} \\ \frac{5}{6} \\ \frac{5}{3} \end{pmatrix}$ and $|\overrightarrow{\text{AN}}| = \sqrt{\frac{25}{36} + \frac{25}{36} + \frac{25}{9}} = \frac{5\sqrt{6}}{6}$

$= \frac{5}{\sqrt{6}}$ units

15 $\mathbf{b}$ and $\mathbf{c}$ are non-parallel vectors in the plane. The normal vector to the plane is $\mathbf{n} = \mathbf{b} \times \mathbf{c}$.
We want to show that $\mathbf{n} \perp (\lambda\mathbf{b} + \mu\mathbf{c})$ for all $\lambda, \mu \in \mathbb{R}$ except when $\lambda = \mu = 0$.
So, we want to show that $\mathbf{n} \bullet (\lambda\mathbf{b} + \mu\mathbf{c}) = 0$ $\{\mathbf{v} \bullet \mathbf{w} = 0 \Leftrightarrow \mathbf{v}$ and $\mathbf{w}$ are perpendicular$\}$

$\mathbf{n} \bullet (\lambda\mathbf{b} + \mu\mathbf{c}) = (\mathbf{b} \times \mathbf{c}) \bullet (\lambda\mathbf{b} + \mu\mathbf{c})$ $\{\mathbf{n} = \mathbf{b} \times \mathbf{c}\}$

$= (\mathbf{b} \times \mathbf{c}) \bullet \lambda\mathbf{b} + (\mathbf{b} \times \mathbf{c}) \bullet \mu\mathbf{c}$ $\{\mathbf{v} \bullet (\mathbf{w} + \mathbf{x}) = \mathbf{v} \bullet \mathbf{w} + \mathbf{v} \bullet \mathbf{x}\}$

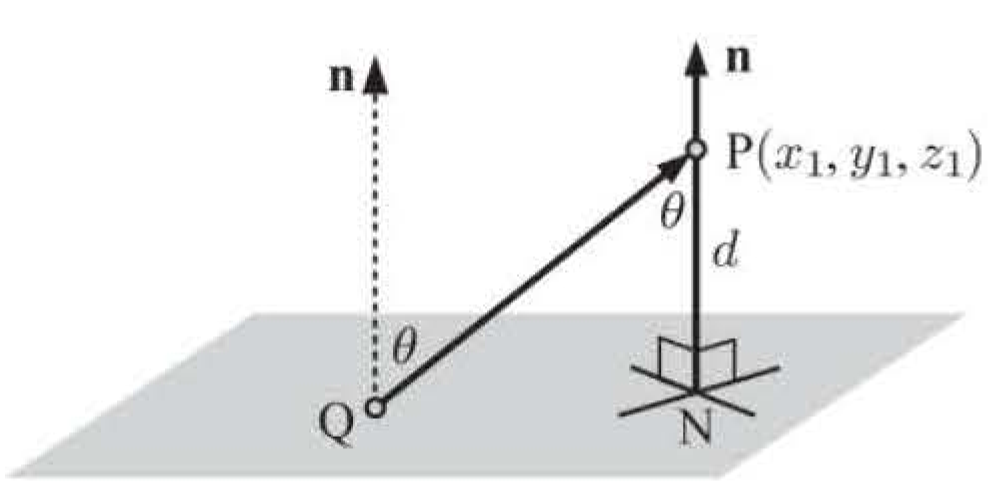
$= 0 + 0$ $\{(\mathbf{b} \times \mathbf{c}) \perp \mathbf{b}$ $\therefore$ $(\mathbf{b} \times \mathbf{c}) \perp$ to any non-zero scalar multiple of $\mathbf{b}$. Similarly for $\mathbf{c}$.$\}$

$= 0$ for all $\lambda, \mu \in \mathbb{R}$ except when $\lambda = \mu = 0$

16 **a** If N is the point on the plane such that (NP) is a normal to it, then $\triangle$NPQ is right angled at N.
Draw a line parallel to $\mathbf{n}$ through Q.

Now θ is the angle between vectors $\mathbf{n}$ and $\overrightarrow{\text{QP}}$.

$\therefore$ $\cos\theta = \dfrac{|\overrightarrow{\text{QP}} \bullet \mathbf{n}|}{|\overrightarrow{\text{QP}}|\,|\mathbf{n}|}$ .... (1)



But $\text{Q}\widehat{\text{P}}\text{N} = \theta$ {alternate angles}

$\therefore$ $\cos\theta = \dfrac{d}{|\overrightarrow{\text{QP}}|}$ .... (2)

Equating (1) and (2), $\dfrac{d}{|\overrightarrow{\text{QP}}|} = \dfrac{|\overrightarrow{\text{QP}} \bullet \mathbf{n}|}{|\overrightarrow{\text{QP}}|\,|\mathbf{n}|}$

$\therefore$ $d = \dfrac{|\overrightarrow{\text{QP}} \bullet \mathbf{n}|}{|\mathbf{n}|}$

b Since Q is any point on the plane, it has coordinates (x, y, z) such that $Ax + By + Cz + D = 0$.

The normal vector to the plane is $\mathbf{n} = \begin{pmatrix} A \\ B \\ C \end{pmatrix}$.

$$\therefore \text{ using } \mathbf{a}, \quad d = \frac{|\overrightarrow{\mathrm{QP}} \bullet \mathbf{n}|}{|\mathbf{n}|} = \frac{\left|\begin{pmatrix} x_1 - x \\ y_1 - y \\ z_1 - z \end{pmatrix} \bullet \begin{pmatrix} A \\ B \\ C \end{pmatrix}\right|}{\sqrt{A^2 + B^2 + C^2}}$$

$$= \frac{|Ax_1 - Ax + By_1 - By + Cz_1 - Cz|}{\sqrt{A^2 + B^2 + C^2}}$$

$$= \frac{|Ax_1 + By_1 + Cz_1 - (Ax + By + Cz)|}{\sqrt{A^2 + B^2 + C^2}}$$

$$= \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

c **8 a** *check*: Given A(1, 0, 2) and the plane $2x + y - 2z + 11 = 0$,

$$d = \frac{|2x_1 + y_1 - 2z_1 + 11|}{\sqrt{2^2 + 1^2 + (-2)^2}} = \frac{|2(1) + 1(0) - 2(2) + 11|}{\sqrt{9}} = \tfrac{9}{3} = 3 \text{ units}$$

8 b *check*: Given A(2, −1, 3) and the plane $x - y + 3z = -10$,

$$d = \frac{|x_1 - y_1 + 3z_1 + 10|}{\sqrt{1^2 + (-1)^2 + 3^2}}$$

$$= \frac{|2 - (-1) + 3(3) + 10|}{\sqrt{11}}$$

$$= \tfrac{22}{\sqrt{11}} = 2\sqrt{11} \text{ units}$$

8 c *check*: Given A(1, −4, −3) and the plane $4x - y - 2z = 8$,

$$d = \frac{|4x_1 - y_1 - 2z_1 - 8|}{\sqrt{4^2 + (-1)^2 + (-2)^2}}$$

$$= \frac{|4 - (-4) - 2(-3) - 8|}{\sqrt{21}}$$

$$= \tfrac{6}{\sqrt{21}} \text{ units or } 2\sqrt{\tfrac{3}{7}} \text{ units}$$

d Using the formula derived in **b**,

i $d = \dfrac{|x_1 + 2y_1 - z_1 - 10|}{\sqrt{1^2 + 2^2 + (-1)^2}} = \dfrac{|0 + 2(0) - 0 - 10|}{\sqrt{6}} = \dfrac{10}{\sqrt{6}}$ units

ii $d = \dfrac{|x_1 + y_1 - z_1 - 2|}{\sqrt{1^2 + 1^2 + (-1)^2}} = \dfrac{|1 + (-3) - 2 - 2|}{\sqrt{3}} = \dfrac{|-6|}{\sqrt{3}} = \dfrac{6}{\sqrt{3}}$ units or $2\sqrt{3}$ units

17 **a** First choose a point on the first plane $x + y + 2z = 4$, for example, (0, 0, 2).

Using the formula obtained in **16 b** to calculate the distance from this point to the second plane,

$$d = \frac{|2x_1 + 2y_1 + 4z_1 + 11|}{\sqrt{2^2 + 2^2 + 4^2}} = \frac{|2(0) + 2(0) + 4(2) + 11|}{\sqrt{24}} = \frac{19}{\sqrt{24}} \text{ units.}$$

b Choose a point on the plane $ax + by + cz + d_1 = 0$, for example, $\left(0, 0, -\dfrac{d_1}{c}\right)$.

Using the formula obtained in **16 b** to calculate the distance from this point to the second plane,

$$d = \frac{|ax_1 + by_1 + cz_1 + d_2|}{\sqrt{a^2 + b^2 + c^2}} = \frac{\left|a(0) + b(0) + c\left(-\dfrac{d_1}{c}\right) + d_2\right|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|d_2 - d_1|}{\sqrt{a^2 + b^2 + c^2}} \text{ units}$$

18 The line $x = 2 + t, \ y = -1 + 2t, \ z = -3t$ has direction vector $\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$,

and $\begin{pmatrix} 11 \\ -4 \\ 1 \end{pmatrix}$ is a vector normal to the plane $11x - 4y + z = 0$.

But $\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} \bullet \begin{pmatrix} 11 \\ -4 \\ 1 \end{pmatrix} = 11 - 8 - 3 = 0$

$\therefore$ these vectors are perpendicular and so the line is parallel to the plane.

Choose any point on the line, say $t = 0$, which corresponds to the point $(2, -1, 0)$.

Then the distance $d = \dfrac{|11x_1 - 4y_1 + z_1|}{\sqrt{11^2 + (-4)^2 + 1^2}} = \dfrac{|11(2) - 4(-1) + 0|}{\sqrt{138}} = \dfrac{26}{\sqrt{138}}$ units

19 Since the planes are parallel to $2x - y + 2z = 5$, they have equation $2x - y + 2z = a$ for some a.

Choose any point on $2x - y + 2z = 5$, for example, $(0, -5, 0)$.

Then the distance from this point to the plane $2x - y + 2z = a$ is

$$d = \frac{|2x_1 - y_1 + 2z_1 - a|}{\sqrt{2^2 + (-1)^2 + 2^2}}$$

$$\therefore\ 2 = \frac{|2(0) - (-5) + 2(0) - a|}{3}$$

$$\therefore\ 6 = |5 - a|$$

$$\therefore\ 5 - a = \pm 6$$

$$\therefore\ a = 5 \pm 6$$

$$\therefore\ a = -1 \text{ or } a = 11$$

$\therefore$ the planes are $2x - y + 2z = -1$
and $2x - y + 2z = 11$

EXERCISE 15J

1 **a** $\mathbf{n} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ and $\mathbf{d} = \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix}$

$\therefore\ \sin\phi = \dfrac{|\mathbf{n} \bullet \mathbf{d}|}{|\mathbf{n}||\mathbf{d}|} = \dfrac{|4 - 3 + 1|}{\sqrt{3}\sqrt{26}} = \dfrac{2}{\sqrt{78}}$

and so $\phi \approx 13.1^\circ$

b $\mathbf{n} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ and $\mathbf{d} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$

$\therefore\ \sin\phi = \dfrac{|\mathbf{n} \bullet \mathbf{d}|}{|\mathbf{n}||\mathbf{d}|} = \dfrac{|2 - 3 + 1|}{\sqrt{6}\sqrt{11}} = 0$

and so $\phi \approx 0^\circ$

So, the line and plane are parallel.

c $\mathbf{n} = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}$ So, if $x - 4 = 3 - y = 2(z + 1) = t$

or equivalently $x = 4 + t,\ y = 3 - t,\ z = -1 + \frac{1}{2}t$ then $\mathbf{d} = \begin{pmatrix} 1 \\ -1 \\ \frac{1}{2} \end{pmatrix}$.

$\therefore\ \sin\phi = \dfrac{|\mathbf{n} \bullet \mathbf{d}|}{|\mathbf{n}||\mathbf{d}|} = \dfrac{\left|3 + (-4) + (-\frac{1}{2})\right|}{\sqrt{26}\sqrt{\frac{9}{4}}} = \dfrac{\left|-\frac{3}{2}\right|}{\frac{3}{2}\sqrt{26}} = \dfrac{1}{\sqrt{26}}$ and so $\phi \approx 11.3^\circ$

d The plane has normal vector

$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -4 & -1 \\ 1 & 1 & -2 \end{vmatrix} = \begin{vmatrix} -4 & -1 \\ 1 & -2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & -1 \\ 1 & -2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & -4 \\ 1 & 1 \end{vmatrix}\mathbf{k} = 9\mathbf{i} + 5\mathbf{j} + 7\mathbf{k}$$

and the line has direction vector $\mathbf{d} = \mathbf{i} - \mathbf{j} + \mathbf{k}$

$$\therefore\ \sin\phi = \frac{|\mathbf{n} \bullet \mathbf{d}|}{|\mathbf{n}||\mathbf{d}|} = \frac{|9 - 5 + 7|}{\sqrt{81 + 25 + 49}\sqrt{1 + 1 + 1}} = \frac{11}{\sqrt{155}\sqrt{3}}$$

$$\therefore\ \phi \approx 30.7^\circ$$

2 **a** $\mathbf{n}_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ and $\mathbf{n}_2 = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$

$\therefore\ \cos\theta = \dfrac{|\mathbf{n}_1 \bullet \mathbf{n}_2|}{|\mathbf{n}_1||\mathbf{n}_2|} = \dfrac{|2 - 3 + 2|}{\sqrt{6}\sqrt{14}} = \frac{1}{\sqrt{84}}$

$\therefore\ \theta \approx 83.7^\circ$

b $\mathbf{n}_1 = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$ and $\mathbf{n}_2 = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$

$\therefore\ \cos\theta = \dfrac{|\mathbf{n}_1 \bullet \mathbf{n}_2|}{|\mathbf{n}_1||\mathbf{n}_2|} = \dfrac{|3 - 1 - 3|}{\sqrt{11}\sqrt{11}} = \frac{1}{11}$

$\therefore\ \theta \approx 84.8^\circ$

c $\mathbf{n}_1 = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$ and $\mathbf{n}_2 = \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}$ $\therefore\ \cos\theta = \dfrac{|\mathbf{n}_1 \bullet \mathbf{n}_2|}{|\mathbf{n}_1|\,|\mathbf{n}_2|} = \dfrac{|6-4-1|}{\sqrt{11}\sqrt{21}} = \dfrac{1}{\sqrt{231}}$

$\therefore\ \theta \approx 86.2°$

d $\mathbf{n}_1 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & -1 \\ 2 & -4 & 3 \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ -4 & 3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -1 & -1 \\ 2 & 3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -1 & 1 \\ 2 & -4 \end{vmatrix}\mathbf{k}$

$= -\mathbf{i} + \mathbf{j} + 2\mathbf{k}$

$\mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & -1 & -1 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} -1 & -1 \\ 1 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -2 & -1 \\ 1 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -2 & -1 \\ 1 & 1 \end{vmatrix}\mathbf{k}$

$= \mathbf{j} - \mathbf{k}$

$\therefore\ \cos\theta = \dfrac{|\mathbf{n}_1 \bullet \mathbf{n}_2|}{|\mathbf{n}_1|\,|\mathbf{n}_2|} = \dfrac{|-1(0)+1(1)+2(-1)|}{\sqrt{1+1+4}\sqrt{0+1+1}} = \dfrac{|1-2|}{\sqrt{6}\sqrt{2}} = \dfrac{1}{\sqrt{12}}$

$\therefore\ \theta \approx 73.2°$

e $\mathbf{n}_1 = \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix}$ and $\mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 0 \\ 2 & 1 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 0 \\ 1 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & 0 \\ 2 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & -1 \\ 2 & 1 \end{vmatrix}\mathbf{k}$

$= -\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$

$\therefore\ \cos\theta = \dfrac{|\mathbf{n}_1 \bullet \mathbf{n}_2|}{|\mathbf{n}_1|\,|\mathbf{n}_2|} = \dfrac{|(3)(-1)+(-4)(-3)+(1)(5)|}{\sqrt{9+16+1}\sqrt{1+9+25}} = \dfrac{14}{\sqrt{26\times 35}}$

$\therefore\ \theta \approx 62.3°$

EXERCISE 15K

1 **a** In augmented matrix form, the system is:

$$\left[\begin{array}{ccc|c} 1 & -2 & 5 & 1 \\ 2 & -4 & 8 & 2 \\ -3 & 6 & 7 & -3 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 5 & 1 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 22 & 0 \end{array}\right] \begin{array}{l} \\ R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + 3R_1 \end{array}$$

$$\begin{array}{rrrr} 2 & -4 & 8 & 2 \\ -2 & 4 & -10 & -2 \\ \hline 0 & 0 & -2 & 0 \end{array} \qquad \begin{array}{rrrr} -3 & 6 & 7 & -3 \\ 3 & -6 & 15 & 3 \\ \hline 0 & 0 & 22 & 0 \end{array}$$

Rows 2 and 3 show $-2z = 0$ and $22z = 0$, so $z = 0$.

Row 1 becomes $x - 2y + 5(0) = 1$

let $y = t$, then $x - 2t = 1$

$\therefore\ x = 1 + 2t$

$\therefore$ there are infinitely many solutions of the form $x = 1 + 2t,\ y = t,\ z = 0,\ t \in \mathbb{R}$.

b In augmented matrix form, the system is:

$$\left[\begin{array}{ccc|c} 1 & 4 & 11 & 7 \\ 1 & 6 & 17 & 9 \\ 1 & 4 & 8 & 4 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 4 & 11 & 7 \\ 0 & 2 & 6 & 2 \\ 0 & 0 & -3 & -3 \end{array}\right] \begin{array}{l} \\ R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$\begin{array}{rrrr} 1 & 6 & 17 & 9 \\ -1 & -4 & -11 & -7 \\ \hline 0 & 2 & 6 & 2 \end{array} \qquad \begin{array}{rrrr} 1 & 4 & 8 & 4 \\ -1 & -4 & -11 & -7 \\ \hline 0 & 0 & -3 & -3 \end{array}$$

The last row gives $-3z = -3$ $\therefore\ z = 1$

$\therefore$ in row 2, $2y + 6z = 2$

$\therefore\ 2y + 6 = 2$

$\therefore\ y = -2$

and in row 1, $x + 4y + 11z = 7$

$\therefore\ x + 4(-2) + 11(1) = 7$

$\therefore\ x + 3 = 7$

$\therefore\ x = 4$

Thus we have a unique solution $x = 4,\ y = -2,\ z = 1$.

c In augmented matrix form, the system is:

$$\left[\begin{array}{ccc|c} 2 & -1 & 3 & 17 \\ 2 & -2 & -5 & 4 \\ 3 & 2 & 2 & 10 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 2 & -1 & 3 & 17 \\ 0 & -1 & -8 & -13 \\ 0 & 7 & -5 & -31 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to 2R_3 - 3R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 2 & -1 & 3 & 17 \\ 0 & -1 & -8 & -13 \\ 0 & 0 & -61 & -122 \end{array}\right] \quad R_3 \to R_3 + 7R_2$$

$$\begin{array}{rrrr} 2 & -2 & -5 & 4 \\ -2 & 1 & -3 & -17 \\ \hline 0 & -1 & -8 & -13 \end{array}$$

$$\begin{array}{rrrr} 6 & 4 & 4 & 20 \\ -6 & 3 & -9 & -51 \\ \hline 0 & 7 & -5 & -31 \end{array}$$

$$\begin{array}{rrrr} 0 & 7 & -5 & -31 \\ 0 & -7 & -56 & -91 \\ \hline 0 & 0 & -61 & -122 \end{array}$$

The last row gives $-61z = -122$ $\therefore$ $z = 2$

$\therefore$ in row 2, $-y - 8z = -13$
$\therefore$ $-y - 16 = -13$
$\therefore$ $y = -3$

and in row 1, $2x - y + 3z = 17$
$\therefore$ $2x + 3 + 6 = 17$
$\therefore$ $2x = 8$
$\therefore$ $x = 4$

Thus we have a unique solution $x = 4$, $y = -3$, $z = 2$.

d In augmented matrix form, the system is:

$$\left[\begin{array}{ccc|c} 2 & 3 & 4 & 1 \\ 5 & 6 & 7 & 2 \\ 8 & 9 & 10 & 4 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 2 & 3 & 4 & 1 \\ 0 & -3 & -6 & -1 \\ 0 & -3 & -6 & 0 \end{array}\right] \quad \begin{array}{l} R_2 \to 2R_2 - 5R_1 \\ R_3 \to R_3 - 4R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 2 & 3 & 4 & 1 \\ 0 & -3 & -6 & -1 \\ 0 & 0 & 0 & 1 \end{array}\right] \quad R_3 \to R_3 - R_2$$

$$\begin{array}{rrrr} 10 & 12 & 14 & 4 \\ -10 & -15 & -20 & -5 \\ \hline 0 & -3 & -6 & -1 \end{array}$$

$$\begin{array}{rrrr} 8 & 9 & 10 & 4 \\ -8 & -12 & -16 & -4 \\ \hline 0 & -3 & -6 & 0 \end{array}$$

In row 3, $0z = 1$

$\therefore$ the system is inconsistent and has no real solutions.

2 **a** Either (1) no solutions or (2) an infinite number of solutions.

b **i** They are parallel if $a_1 = ka_2$, $b_1 = kb_2$, $c_1 = kc_2$ and $d_1 \neq kd_2$ for some k.

ii They are coincident if $a_1 = ka_2$, $b_1 = kb_2$, $c_1 = kc_2$ and $d_1 = kd_2$ for some k.

c **i** $$\left[\begin{array}{ccc|c} 1 & -3 & 2 & 8 \\ 3 & -9 & 2 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -3 & 2 & 8 \\ 0 & 0 & -4 & -20 \end{array}\right] \quad R_2 \to R_2 - 3R_1$$

$\therefore$ $-4z = -20$ and $x - 3y + 2z = 8$ or $x = 3y - 2z + 8$

$\therefore$ $z = 5$ and if we let $y = t$, then $x = 3t - 2(5) + 8 = -2 + 3t$

$\therefore$ the planes meet in the line $x = -2 + 3t$, $y = t$, $z = 5$, $t \in \mathbb{R}$

ii $$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 1 & -1 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 0 & -3 & 1 & 1 \end{array}\right] \quad R_2 \to 2R_2 - R_1$$

$\therefore$ $-3y + z = 1$ and $2x + y + z = 5$

$\therefore$ if we let $y = t$, then $z = 1 + 3y = 1 + 3t$ and
$$\begin{aligned} 2x &= 5 - y - z \\ &= 5 - t - (1 + 3t) \\ &= 4 - 4t \\ \therefore\ x &= 2 - 2t \end{aligned}$$

$\therefore$ the planes meet in the line $x = 2 - 2t$, $y = t$, $z = 1 + 3t$, $t \in \mathbb{R}$

iii $$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 6 \\ 3 & 6 & -9 & 18 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & -3 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_2 \to R_2 - 3R_1$$

$\therefore$ there are infinitely many solutions, as the planes are coincident.

Let $y = s$ and $z = t$ in $x + 2y - 3z = 6$, $s, t \in \mathbb{R}$
$\therefore\ x = 3t - 2s + 6$
$\therefore\ x = 3t - 2s + 6,\ y = s,\ z = t,\quad s, t \in \mathbb{R}$ is the general solution of the plane.

3 **a**
$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 6 \\ 2 & 4 & k & 12 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 6 \\ 0 & 0 & k+2 & 0 \end{array}\right] \quad R_2 \to R_2 - 2R_1$$

If $k = -2$, the two planes are coincident.
$\therefore$ infinitely many solutions.
If $k \neq -2$, the two planes meet in a line.
$\therefore$ infinitely many solutions.

b
$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & 8 \\ 2 & -2 & 6 & k \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 3 & 8 \\ 0 & 0 & 0 & k-16 \end{array}\right] \quad R_2 \to R_2 - 2R_1$$

If $k = 16$, the planes are coincident.
$\therefore$ infinitely many solutions.
If $k \neq 16$, the planes are parallel but not coincident.
$\therefore$ no solutions exist.

4
(1) $P_1 = P_2 = P_3$: infinitely many solutions where x, y, and z are in terms of two parameters, s and t say. The solution is a plane.

(2) $P_1 = P_2$ are coincident and cut by P_3: infinitely many solutions where x, y, and z are in terms of one parameter, t say. The solution is a line.

(3) $P_1 = P_2$ with P_3 parallel but not coincident: no solutions exist.

(4) P_1 and P_2 are parallel but not coincident, and P_3 cuts both planes: no solutions exist.

(5) P_1, P_2, and P_3 are all parallel but not coincident: no solutions exist.

(6) P_1, P_2, and P_3 meet in a unique point (a, b, c), so that $x = a$, $y = b$, $z = c$.

(7) P_1, P_2, and P_3 meet in a common line: infinitely many solutions where x, y, and z are in terms of one parameter, t say.

(8) P_1, P_2, and P_3 are such that the line of intersection between any two is parallel to the third plane: no solutions exist.

5 **a** The system has augmented matrix
$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & -5 \\ 1 & -1 & 2 & 11 \\ 4 & 1 & -5 & -18 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & -5 \\ 0 & -2 & 3 & 16 \\ 0 & -3 & -1 & 2 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 4R_1 \end{array}$$
$$\sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & -5 \\ 0 & -2 & 3 & 16 \\ 0 & 0 & -11 & -44 \end{array}\right] \quad R_3 \to 2R_3 - 3R_2$$
$\therefore$ the planes meet at the unique point $(1, -2, 4)$

Now $-11z = -44 \quad \therefore\ z = 4$
and $-2y + 3z = 16$
$\therefore\ -2y + 12 = 16$
$\therefore\ -2y = 4$
$\therefore\ y = -2$
and $x + y - z = -5$
$\therefore\ x = -5 - (-2) + 4$
$\therefore\ x = 1$

b The system has augmented matrix
$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 2 & 1 & -1 & 8 \\ 5 & -2 & 5 & 11 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 3 & -5 & 6 \\ 0 & 3 & -5 & 6 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 5R_1 \end{array}$$
$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 3 & -5 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_3 \to R_3 - R_2$$
$\therefore$ the three planes meet in a common line
$$x = \frac{9-t}{3}, \quad y = \frac{5t+6}{3}, \quad z = t, \quad t \in \mathbb{R}$$

Let $z = t$
As $3y - 5z = 6$
$3y = 5t + 6$
$\therefore\ y = \dfrac{5t+6}{3}$
But $x - y + 2z = 1$
$\therefore\ x = 1 + \dfrac{5t+6}{3} - 2t$
$\therefore\ x = \dfrac{3 + 5t + 6 - 6t}{3}$
$\therefore\ x = \dfrac{9-t}{3}$

c The system has augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 8 \\ 2 & -1 & -1 & 5 \\ 3 & -4 & -1 & 2 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 8 \\ 0 & -5 & 1 & -11 \\ 0 & -10 & 2 & -22 \end{array}\right] \begin{array}{l} \\ R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 8 \\ 0 & -5 & 1 & -11 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ \\ R_3 \to R_3 - 2R_2 \end{array}$$

Let $y = t$

Now $-5y + z = -11$

$\therefore\ z = -11 + 5t$

Also $x + 2y - z = 8$

$\therefore\ x = 8 - 2y + z$

$\therefore\ x = 8 - 2t - 11 + 5t$

$\therefore\ x = -3 + 3t$

$\therefore$ the three planes meet in a common line $x = 3t - 3,\quad y = t,\quad z = 5t - 11,\quad t \in \mathbb{R}$

d The system has augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 8 \\ 2 & -2 & 2 & 11 \\ 1 & 3 & -1 & -2 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 8 \\ 0 & 0 & 0 & -5 \\ 0 & 4 & -2 & -10 \end{array}\right] \begin{array}{l} \\ R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

The first two planes are parallel and are cut by the third plane.

$\therefore$ the equations are inconsistent and there are no solutions.

e The system has augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 1 & -1 & 1 & 4 \\ 3 & 3 & -6 & 3 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 0 & -2 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$$

There are two coincident planes cut by a third plane.

$\therefore$ infinitely many solutions in a line:

$$x = \frac{t+5}{2},\quad y = \frac{3t-3}{2},\quad z = t,\quad t \in \mathbb{R}$$

Let $z = t$

Now $-2y + 3z = 3$

$\therefore\ 2y = 3z - 3$

$\therefore\ y = \dfrac{3t-3}{2}$

and as $x + y - 2z = 1$

$\therefore\ x = 1 - y + 2z$

$\therefore\ x = 1 - \dfrac{3t-3}{2} + 2t$

$\therefore\ x = \dfrac{2 - 3t + 3 + 4t}{2}$

$\therefore\ x = \dfrac{t+5}{2}$

f The system has augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 5 \\ 1 & 1 & 1 & 1 \\ 5 & -1 & 2 & 17 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & -1 & 5 \\ 0 & 2 & 2 & -4 \\ 0 & 4 & 7 & -8 \end{array}\right] \begin{array}{l} \\ R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 5R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & -1 & 5 \\ 0 & 2 & 2 & -4 \\ 0 & 0 & 3 & 0 \end{array}\right] \begin{array}{l} \\ \\ R_3 \to R_3 - 2R_2 \end{array}$$

Now $3z = 0$

$\therefore\ z = 0$

As $2y + 2z = -4$

$\therefore\ 2y = -4$

$\therefore\ y = -2$

and as $x - y - z = 5$

$\therefore\ x = 5 + (-2) + 0$

$\therefore\ x = 3$

$\therefore$ the planes meet at the unique point $(3, -2, 0)$.

6 The system has augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & 1 \\ 2 & -3 & -1 & 3 \\ 3 & -5 & -5 & k \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 3 & -1 \\ 0 & -1 & -7 & 1 \\ 0 & -2 & -14 & k-3 \end{array}\right] \begin{array}{l} \\ R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 3 & -1 \\ 0 & -1 & -7 & 1 \\ 0 & 0 & 0 & k-5 \end{array}\right] \begin{array}{l} \\ \\ R_3 \to R_3 - 2R_2 \end{array}$$

(1) If $k = 5$, the planes meet in a line {as we have a row of zeros}

$$\begin{aligned}
&\text{Let}\quad z = t\\
\text{Now}\quad &-y - 7z = 1\\
&\therefore\ y = -1 - 7t\\
\text{and}\quad &x - y + 3z = 1\\
&\therefore\ x = 1 + y - 3z\\
&\therefore\ x = 1 - 1 - 7t - 3t\\
&\therefore\ x = -10t
\end{aligned}$$

$\therefore\ x = -10t,\ y = -1 - 7t,\ z = t,\ t \in \mathbb{R}$

(2) If $k \neq 5$ there are no solutions.

Since no two planes are parallel, the line of intersection of any two planes is parallel to the third plane.

7 **a** In augmented matrix form, the system is:

$$\left[\begin{array}{ccc|c} 1 & 3 & 3 & a-1 \\ 2 & -1 & 1 & 7 \\ 3 & -5 & a & 16 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & 3 & a-1 \\ 0 & -7 & -5 & 9-2a \\ 0 & -14 & a-9 & 19-3a \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & 3 & a-1 \\ 0 & -7 & -5 & 9-2a \\ 0 & 0 & a+1 & a+1 \end{array}\right] \quad R_3 \to R_3 - 2R_2$$

$$\begin{array}{cccc} 2 & -1 & 1 & 7 \\ -2 & -6 & -6 & -2a+2 \\ \hline 0 & -7 & -5 & 9-2a \end{array}$$

$$\begin{array}{cccc} 3 & -5 & a & 16 \\ -3 & -9 & -9 & -3a+3 \\ \hline 0 & -14 & a-9 & 19-3a \end{array}$$

$$\begin{array}{cccc} 0 & -14 & a-9 & 19-3a \\ 0 & 14 & 10 & 4a-18 \\ \hline 0 & 0 & a+1 & a+1 \end{array}$$

b If $a = -1$, row 3 is a row of zeros, so there are infinitely many solutions, as we have 2 equations in 3 unknowns.

Let $z = t$ in row 2, then $-7y - 5t = 9 - 2(-1)$

$$\therefore\ -7y - 5t = 11$$

$$\therefore\ y = \frac{-5t - 11}{7}$$

and substituting in row 1 gives $x + 3\left(\dfrac{-5t - 11}{7}\right) + 3t = (-1) - 1$

$$\therefore\ x - \frac{15t}{7} - \frac{33}{7} + 3t = -2 \text{ and so, } x = \frac{19}{7} - \frac{6t}{7}$$

$\therefore$ there are infinitely many solutions of form $x = \dfrac{19 - 6t}{7},\ y = \dfrac{-5t - 11}{7},\ z = t,\ t \in \mathbb{R}$.

In this case we have three planes which meet in a line.

c If $a \neq -1$, then $(a+1)z = a + 1 \quad \therefore\ z = 1$

From row 2, $-7y - 5(1) = 9 - 2a$

$$\therefore\ -7y = -2a + 14$$

$$y = \frac{2a}{7} - 2$$

and substituting in row 1 gives $x + 3\left(\dfrac{2a}{7} - 2\right) + 3(1) = a - 1$

$$\therefore\ x + \frac{6a}{7} - 6 + 3 = a - 1$$

$$\therefore\ x = \frac{a}{7} + 2$$

$\therefore$ the unique solution is $x = \frac{1}{7}a + 2,\ \ y = \frac{2}{7}a - 2,\ \ z = 1$.

In this case we have three planes which meet at a point.

8 In augmented matrix form, the system is:

$$\left[\begin{array}{ccc|c} 1 & 2 & m & -1 \\ m & -2 & 1 & 1 \\ 2 & 1 & -1 & 3 \end{array}\right]$$

$$\begin{array}{cccc} m & -2 & 1 & 1 \\ -m & -2m & -m^2 & m \\ \hline 0 & -2(m+1) & 1-m^2 & 1+m \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & m & -1 \\ 0 & -2(m+1) & 1-m^2 & 1+m \\ 0 & -3 & -1-2m & 5 \end{array}\right] \begin{array}{l} R_2 \to R_2 - mR_1 \\ R_3 \to R_3 - 2R_1 \end{array}$$

$$\begin{array}{cccc} 2 & 1 & -1 & 3 \\ -2 & -4 & -2m & 2 \\ \hline 0 & -3 & -1-2m & 5 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & m & -1 \\ 0 & -2(m+1) & 1-m^2 & 1+m \\ 0 & 0 & (m+1)(m+5) & -7(m+1) \end{array}\right] \quad R_3 \to -2(m+1)R_3 + 3R_2$$

$$\begin{array}{cccc} 0 & 6(m+1) & 2(m+1)(1+2m) & -10(m+1) \\ 0 & -6(m+1) & 3(1-m^2) & 3(1+m) \\ \hline 0 & 0 & 2(2m^2+3m+1)+3-3m^2 & -7(m+1) \\ & & = m^2+6m+5 & \\ & & = (m+1)(m+5) & \end{array}$$

a If $m=-5$, row 3 becomes $0x+0y+0z=28$

$\therefore$ the system is inconsistent and there are no solutions.

In this case we have three planes with no common point of intersection. No two planes are coincident or parallel. So, the line of intersection of any two planes is parallel to the third.

b If $m=-1$, row 3 is a row of zeros, so we have 2 equations in 3 unknowns.

$\therefore$ there are infinitely many solutions.

When $m=-1$ the equations become

$$\begin{cases} x+2y-z=-1 \\ -x-2y+z=1 \\ 2x+y-z=3 \end{cases}$$

equations 1 and 2 are equivalent. In this case two planes are coincident and the third meets in a line.

c **i** If $m \neq -5$ or -1, there is a unique solution. The planes meet at a point.

ii If $m \neq -1$ and $m \neq -5$, then row 3 becomes

$(m+1)(m+5)z = -7(m+1)$

$\therefore \; z = \dfrac{-7}{m+5}$ and substituting in row 2 gives

$$-2(m+1)y + (1-m^2)\left(\frac{-7}{m+5}\right) = 1+m$$

$$\therefore \; -2(m+1)(m+5)y - 7(1-m^2) = (1+m)(m+5)$$

$$\therefore \; -2(m+1)(m+5)y = m^2+6m+5+7-7m^2$$

$$\therefore \; -2(m+1)(m+5)y = -6m^2+6m+12$$

$$\therefore \; -2(m+1)(m+5)y = -6(m^2-m-2)$$

$$\therefore \; (m+1)(m+5)y = 3(m+1)(m-2)$$

$$\therefore \; y = \frac{3(m-2)}{m+5}$$

and substituting in row 1 gives $\quad x + \dfrac{6(m-2)}{m+5} + \dfrac{-7m}{m+5} = -1$

$$\therefore \; x(m+5) + 6(m-2) - 7m = -(m+5)$$

$$\therefore \; x(m+5) + 6m - 12 - 7m = -m-5$$

$$\therefore \; x(m+5) = 7$$

$$\therefore \; x = \frac{7}{m+5}$$

$\therefore$ the system has a unique solution for all m except $m=-5$ and $m=-1$, and the solution is $x = \dfrac{7}{m+5}$, $y = \dfrac{3(m-2)}{m+5}$, $z = \dfrac{-7}{m+5}$.

9 P_1 meets P_2 where

$$\begin{pmatrix}2\\-1\\0\end{pmatrix}+\lambda\begin{pmatrix}3\\0\\1\end{pmatrix}+\mu\begin{pmatrix}1\\1\\-1\end{pmatrix}=\begin{pmatrix}3\\-1\\3\end{pmatrix}+r\begin{pmatrix}2\\0\\-1\end{pmatrix}+s\begin{pmatrix}1\\1\\0\end{pmatrix}$$

$$\therefore \begin{cases}2+3\lambda+\mu=3+2r+s\\-1+\mu=-1+s\\\lambda-\mu=3-r\end{cases} \quad \text{which gives} \quad \begin{cases}3\lambda+\mu=2r+s+1\\\mu=s\\\lambda-\mu=3-r\end{cases}$$

If $\mu=a$ say, then $s=a$, $3\lambda+a=2r+a+1$, and $\lambda-a=3-r$

$$\therefore\ r=3-\lambda+a$$

$$\therefore\ 3\lambda+a=6-2\lambda+2a+a+1$$

$$\therefore\ 5\lambda=2a+7$$

$$\therefore\ \lambda=\frac{2a+7}{5} \quad \text{and} \quad r=3+a-\frac{2a+7}{5}$$

$$\therefore\ r=\frac{3a+8}{5}$$

$\therefore$ if $\mu=a$, $\lambda=\dfrac{2a+7}{5}$, $r=\dfrac{3a+8}{5}$, $s=a$ .... (1)

P_2 meets P_3 where

$$\begin{pmatrix}3\\-1\\3\end{pmatrix}+r\begin{pmatrix}2\\0\\-1\end{pmatrix}+s\begin{pmatrix}1\\1\\0\end{pmatrix}=\begin{pmatrix}2\\-1\\2\end{pmatrix}+t\begin{pmatrix}1\\-1\\0\end{pmatrix}-u\begin{pmatrix}0\\-1\\2\end{pmatrix}$$

$$\therefore \begin{cases}3+2r+s=2+t\\-1+s=-1-t+u\\3-r=2-2u\end{cases} \quad \text{which gives} \quad \begin{cases}2r+s+1=t\\s=u-t\\2u-r=-1\end{cases}$$

So, if $u=b$ say, then $r=2b+1$ and $4b+2+b-t+1=t$

$$\therefore\ 5b+3=2t \quad \text{and so} \quad t=\frac{5b+3}{2}$$

$$\text{and} \quad s=u-t=b-\frac{5b+3}{2}=\frac{-3b-3}{2}$$

So, if $u=b$, $r=2b+1$, $t=\dfrac{5b+3}{2}$, $s=\dfrac{-3b-3}{2}$ .... (2)

From (1) and (2), $\dfrac{3a+8}{5}=2b+1$ and $a=\dfrac{-3b-3}{2}$

$$\therefore\ 3a+8=10b+5 \quad \text{and} \quad 2a=-3b-3$$

$$\therefore \begin{cases}3a-10b=-3\\2a+3b=-3\end{cases} \quad \text{which has solutions} \quad a=-\tfrac{39}{29},\ b=-\tfrac{3}{29}$$

In (2), $u=-\frac{3}{29}$, $t=\dfrac{5\left(-\frac{3}{29}\right)+3}{2}=1\frac{7}{29}$ or $\frac{36}{29}$

$$\therefore\ \mathbf{r}_3=\begin{pmatrix}2\\-1\\2\end{pmatrix}+\tfrac{36}{29}\begin{pmatrix}1\\-1\\0\end{pmatrix}+\tfrac{3}{29}\begin{pmatrix}0\\-1\\2\end{pmatrix}=\begin{pmatrix}3\frac{7}{29}\\-2\frac{10}{29}\\2\frac{6}{29}\end{pmatrix}=\begin{pmatrix}\frac{94}{29}\\-\frac{68}{29}\\\frac{64}{29}\end{pmatrix}$$

$\therefore$ all 3 planes meet at $\left(\frac{94}{29}, -\frac{68}{29}, \frac{64}{29}\right)$.

REVIEW SET 15A

1 **a** The vector equation is

$$\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-6\\3\end{pmatrix}+t\begin{pmatrix}4\\-3\end{pmatrix}, \quad t\in\mathbb{R}$$

b The parametric equations are
$x=-6+4t$, $y=3-3t$, $t\in\mathbb{R}$

c $\dfrac{x+6}{4} = \dfrac{y-3}{-3} = t$

$\therefore \quad -3x - 18 = 4y - 12$

So, the Cartesian equation is $3x + 4y = -6$.

2 $(-3, m)$ lies on the line, so $\begin{pmatrix} -3 \\ m \end{pmatrix} = \begin{pmatrix} 18 \\ -2 \end{pmatrix} + \begin{pmatrix} -7t \\ 4t \end{pmatrix}$

$\therefore \quad -3 = 18 - 7t \quad \text{and} \quad m = -2 + 4t$

$\therefore \quad 7t = 21$

$\therefore \quad t = 3 \quad \text{and so} \quad m = -2 + 4(3) = 10$

3 **a** When $t = 1$, $\mathbf{r} = \begin{pmatrix} 3 \\ -3 \end{pmatrix} + 1\begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 3+2 \\ -3+5 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$

$\therefore$ the point is $(5, 2)$.

b $\begin{pmatrix} 4 \\ 10 \end{pmatrix}$ is a non-zero scalar multiple of $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$, so it could also be used to describe the direction of the line.

c The line passes through point $(5, 2)$ and has direction vector $\begin{pmatrix} 4 \\ 10 \end{pmatrix}$.

$\therefore \quad \mathbf{r} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} + s\begin{pmatrix} 4 \\ 10 \end{pmatrix}$, $s \in \mathbb{R}$ is an alternative vector equation for the line.

4 $\text{P}(2, 0, 1)$, $\text{Q}(3, 4, -2)$, $\text{R}(-1, 3, 2)$

a $\overrightarrow{\text{PQ}} = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}$

Since $\overrightarrow{\text{PQ}} = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}$ and P is at $(2, 0, 1)$, the line has parametric equations

$x = 2 + t, \quad y = 0 + 4t, \quad z = 1 - 3t$

$\therefore \quad x = 2 + t, \quad y = 4t, \quad z = 1 - 3t, \quad t \in \mathbb{R}$

b $\overrightarrow{\text{QR}} = \begin{pmatrix} -4 \\ -1 \\ 4 \end{pmatrix}$

$|\,\overrightarrow{\text{PQ}}\,| = \sqrt{1^2 + 4^2 + (-3)^2} = \sqrt{26}$

$|\,\overrightarrow{\text{QR}}\,| = \sqrt{(-4)^2 + (-1)^2 + 4^2} = \sqrt{33}$

$$\begin{aligned}\overrightarrow{\text{PQ}} \bullet \overrightarrow{\text{QR}} &= \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix} \bullet \begin{pmatrix} -4 \\ -1 \\ 4 \end{pmatrix} \\ &= 1 \times (-4) + 4 \times (-1) + (-3) \times 4 \\ &= -20\end{aligned}$$

If $\theta = \text{P}\widehat{\text{Q}}\text{R}$, then $\cos\theta = \dfrac{|\,\overrightarrow{\text{PQ}} \bullet \overrightarrow{\text{QR}}\,|}{|\,\overrightarrow{\text{PQ}}\,|\,|\,\overrightarrow{\text{QR}}\,|} = \dfrac{|-20|}{\sqrt{26}\sqrt{33}}$

$$= \frac{20}{\sqrt{26}\sqrt{33}}$$

5 **a** Lines (AB) and (AC) meet at A.

$$\therefore \begin{pmatrix}4\\-1\end{pmatrix}+t\begin{pmatrix}1\\3\end{pmatrix}=\begin{pmatrix}-1\\0\end{pmatrix}+u\begin{pmatrix}3\\1\end{pmatrix}$$

$$\therefore \begin{pmatrix}4+t\\-1+3t\end{pmatrix}=\begin{pmatrix}-1+3u\\u\end{pmatrix}$$

$$\therefore\ t-3u=-5 \quad .... (1)$$
$$3t-u=1$$
$$\therefore\ -3t+9u=15 \quad \{-3\times(1)\}$$
$$3t-u=1$$
$$\text{Adding,}\quad 8u=16$$
$$\therefore\ u=2$$

$$\text{So,}\quad \begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-1\\0\end{pmatrix}+2\begin{pmatrix}3\\1\end{pmatrix}=\begin{pmatrix}5\\2\end{pmatrix}$$

$\therefore$ A is (5, 2).

Lines (AB) and (BC) meet at B.

$$\therefore \begin{pmatrix}4\\-1\end{pmatrix}+t\begin{pmatrix}1\\3\end{pmatrix}=\begin{pmatrix}7\\4\end{pmatrix}+s\begin{pmatrix}1\\-1\end{pmatrix}$$

$$\therefore \begin{pmatrix}4+t\\-1+3t\end{pmatrix}=\begin{pmatrix}7+s\\4-s\end{pmatrix}$$

$$\therefore\ t-s=3$$
$$3t+s=5$$
$$\text{Adding,}\quad 4t=8$$
$$\therefore\ t=2$$

$$\text{So,}\quad \begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}4\\-1\end{pmatrix}+2\begin{pmatrix}1\\3\end{pmatrix}=\begin{pmatrix}6\\5\end{pmatrix}$$

$\therefore$ B is (6, 5).

Lines (BC) and (AC) meet at C.

$$\therefore \begin{pmatrix}7\\4\end{pmatrix}+s\begin{pmatrix}1\\-1\end{pmatrix}=\begin{pmatrix}-1\\0\end{pmatrix}+u\begin{pmatrix}3\\1\end{pmatrix}$$

$$\therefore \begin{pmatrix}7+s\\4-s\end{pmatrix}=\begin{pmatrix}-1+3u\\u\end{pmatrix}$$

$$\therefore\ s-3u=-8$$
$$-s-u=-4$$
$$\text{Adding,}\quad -4u=-12$$
$$\therefore\ u=3$$

$$\text{So,}\quad \begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-1\\0\end{pmatrix}+3\begin{pmatrix}3\\1\end{pmatrix}=\begin{pmatrix}8\\3\end{pmatrix}$$

$\therefore$ C is (8, 3).

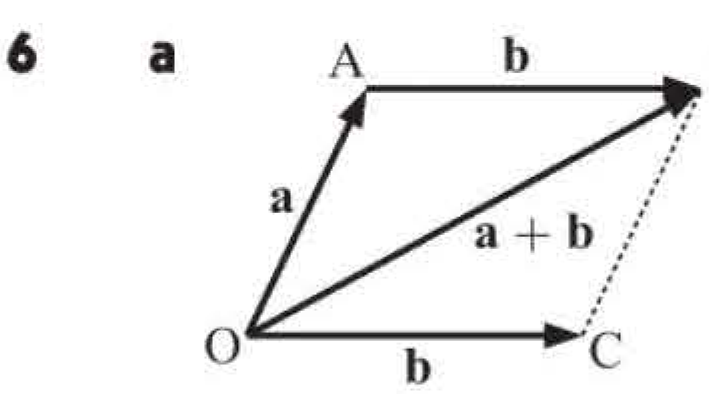
b $\overrightarrow{AB}=\begin{pmatrix}1\\3\end{pmatrix}$, so $|\overrightarrow{AB}|=\sqrt{1+9}=\sqrt{10}$ units

$\overrightarrow{BC}=\begin{pmatrix}2\\-2\end{pmatrix}$, so $|\overrightarrow{BC}|=\sqrt{4+4}=\sqrt{8}$ units

$\overrightarrow{AC}=\begin{pmatrix}3\\1\end{pmatrix}$, so $|\overrightarrow{AC}|=\sqrt{9+1}=\sqrt{10}$ units

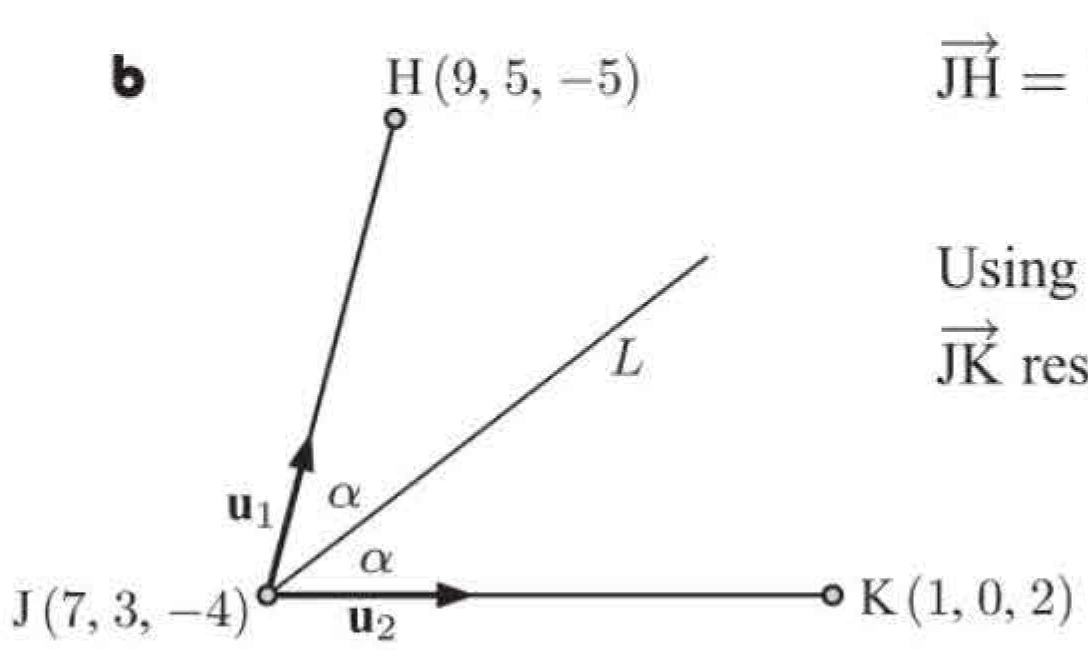
c Triangle ABC is isosceles.

6 **a**



As **a** and **b** are unit vectors, OABC is a rhombus.
But the angles of a rhombus are bisected by its diagonals, so **a** + **b** bisects the angle between vector **a** and vector **b**.

b



$$\overrightarrow{JH}=\begin{pmatrix}9-7\\5-3\\-5--4\end{pmatrix}=\begin{pmatrix}2\\2\\-1\end{pmatrix},\quad \overrightarrow{JK}=\begin{pmatrix}1-7\\0-3\\2--4\end{pmatrix}=\begin{pmatrix}-6\\-3\\6\end{pmatrix}$$

Using **a**, we write unit vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ in the direction of $\overrightarrow{JH}$ and $\overrightarrow{JK}$ respectively.

$\therefore \quad \mathbf{u}_1 = \dfrac{1}{\sqrt{4+4+1}}\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \\ -\frac{1}{3} \end{pmatrix}$ and $\mathbf{u}_2 = \dfrac{1}{\sqrt{36+9+36}}\begin{pmatrix} -6 \\ -3 \\ 6 \end{pmatrix} = \begin{pmatrix} -\frac{2}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$

and $\mathbf{u}_1 + \mathbf{u}_2 = \begin{pmatrix} 0 \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$, which bisects $\widehat{\text{HJK}}$, by **a**.

$\therefore$ the equation of the line L is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \\ -4 \end{pmatrix} + \lambda\begin{pmatrix} 0 \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}, \quad \lambda \in \mathbb{R}$

c $\overrightarrow{\text{HK}} = \begin{pmatrix} 1-9 \\ 0-5 \\ 2--5 \end{pmatrix} = \begin{pmatrix} -8 \\ -5 \\ 7 \end{pmatrix}$ so (HK) has equation $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + s\begin{pmatrix} -8 \\ -5 \\ 7 \end{pmatrix}, \quad s \in \mathbb{R}.$

This line meets L where

$7 = 1 - 8s, \qquad 3 + \dfrac{t}{3} = -5s, \qquad$ and $\qquad -4 + \dfrac{t}{3} = 2 + 7s \quad .... \ (*)$

$\therefore \quad 8s = -6$

$\therefore \quad s = -\frac{3}{4}$ and so $3 + \dfrac{t}{3} = \frac{15}{4}$

$\therefore \quad \dfrac{t}{3} = \frac{3}{4}$

$\therefore \quad t = \frac{9}{4}$

In $(*)$, $\quad \text{LHS} = -4 + \dfrac{t}{3} = -4 + \frac{3}{4} = -\frac{13}{4} \qquad \text{RHS} = 2 + 7s = 2 + 7\left(-\frac{3}{4}\right) = \frac{8}{4} - \frac{21}{4} = -\frac{13}{4}$ ✓

$\therefore \quad s = -\frac{3}{4}, \ t = \frac{9}{4}$ satisfy all 3 equations.

So, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} - \frac{3}{4}\begin{pmatrix} -8 \\ -5 \\ 7 \end{pmatrix} = \begin{pmatrix} 1+6 \\ 0+\frac{15}{4} \\ 2-\frac{21}{4} \end{pmatrix} = \begin{pmatrix} 7 \\ 3\frac{3}{4} \\ -3\frac{1}{4} \end{pmatrix}$

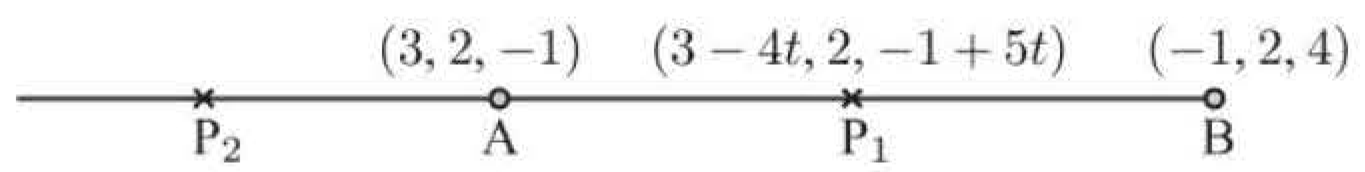
$\therefore \quad L$ meets (HK) at $(7, 3\frac{3}{4}, -3\frac{1}{4})$.

7 **a** $\overrightarrow{\text{AB}} = \begin{pmatrix} -4 \\ 0 \\ 5 \end{pmatrix} \quad \therefore$ the line is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} + t\begin{pmatrix} -4 \\ 0 \\ 5 \end{pmatrix}, \quad t \in \mathbb{R}$

b The equation of the plane is

$-4x + 0y + 5z = -4(-1) + 5(4)$

$\therefore \quad -4x + 5z = 24$



c The distance from a point on the line to A is $d = \sqrt{(-4t)^2 + 0^2 + (5t)^2} = \sqrt{41t^2}$

$\therefore$ since $d = 2\sqrt{41}$ units, $\sqrt{41t^2} = 2\sqrt{41}$

$\therefore \quad t^2 = 4$

$\therefore \quad t = \pm 2 \qquad \therefore$ the points are $(-5, 2, 9)$ and $(11, 2, -11)$.

8 Given $\text{C}(-3, 2, -1)$ and $\text{D}(0, 1, -4)$, $\overrightarrow{\text{CD}} = \begin{pmatrix} 3 \\ -1 \\ -3 \end{pmatrix}$

$\therefore$ the line passing through C and D has parametric equations

$x = -3 + 3t, \quad y = 2 - t, \quad z = -1 - 3t$

The line meets $2x - y + z = 3$ when
$$\begin{aligned} 2(-3+3t) - (2-t) + (-1-3t) &= 3 \\ \therefore\ -6 + 6t - 2 + t - 1 - 3t &= 3 \\ \therefore\ 4t &= 12 \\ \therefore\ t &= 3 \end{aligned}$$

$\therefore$ they meet at $(6,\ -1,\ -10)$

9 $|\mathbf{a}| = 3$, $|\mathbf{b}| = \sqrt{7}$ and $\mathbf{a} \times \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$

a
$$\begin{aligned} |\mathbf{a} \times \mathbf{b}| &= |\mathbf{a}|\,|\mathbf{b}| \sin\theta \\ \sqrt{1+4+9} &= 3 \times \sqrt{7} \times \sin\theta \\ \sin\theta &= \tfrac{\sqrt{14}}{3\sqrt{7}} = \tfrac{\sqrt{2}}{3} \end{aligned}$$

But $\cos^2\theta = 1 - \sin^2\theta$

$\therefore\ \cos\theta = \pm\frac{\sqrt{7}}{3}$

Hence
$$\begin{aligned} \mathbf{a} \bullet \mathbf{b} &= |\mathbf{a}|\,|\mathbf{b}| \cos\theta \\ &= 3 \times \sqrt{7} \times (\pm\tfrac{\sqrt{7}}{3}) \\ &= \pm 7 \end{aligned}$$

b Area
$$\begin{aligned} \triangle\text{OAB} &= \tfrac{1}{2}\,|\mathbf{a} \times \mathbf{b}| \\ &= \tfrac{1}{2}\sqrt{14} \text{ units}^2 \end{aligned}$$

10 **a** The distance of $\text{X}(-1,\ 1,\ 3)$ from $x - 2y - 2z = 8$

is $d = \dfrac{|x_1 - 2y_1 - 2z_1 - 8|}{\sqrt{1^2 + (-2)^2 + (-2)^2}} = \dfrac{|-1-2-6-8|}{3} = \dfrac{|-17|}{3} = \frac{17}{3}$ units

b Since $2 - x = y - 3 = -\frac{1}{2}z$, $\dfrac{x-2}{-1} = \dfrac{y-3}{1} = \dfrac{z}{-2}$

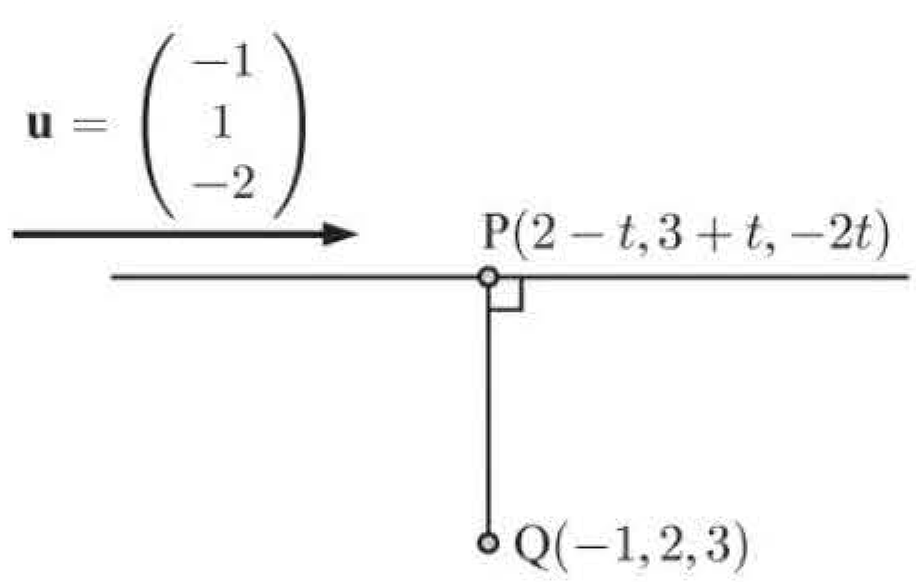
$\therefore$ the line has direction vector $\mathbf{u} = \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}$, and passes through $(2,\ 3,\ 0)$

$\therefore$ if P is a point on the line with coordinates $(2-t,\ 3+t,\ -2t)$, then

$$\overrightarrow{\text{QP}} = \begin{pmatrix} 2-t--1 \\ 3+t-2 \\ -2t-3 \end{pmatrix} = \begin{pmatrix} 3-t \\ 1+t \\ -2t-3 \end{pmatrix}$$

If P is chosen such that $\overrightarrow{\text{QP}}$ is perpendicular to the line, then $\mathbf{u} \bullet \overrightarrow{\text{QP}} = 0$

$$\begin{aligned} \therefore\ \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} 3-t \\ 1+t \\ -2t-3 \end{pmatrix} &= 0 \\ \therefore\ -(3-t) + 1(1+t) - 2(-2t-3) &= 0 \\ \therefore\ -3 + t + 1 + t + 4t + 6 &= 0 \\ \therefore\ 6t &= -4 \\ \therefore\ t &= -\tfrac{2}{3} \end{aligned}$$



$\therefore$ P is at $(2 + \frac{2}{3},\ 3 - \frac{2}{3},\ 2(\frac{2}{3}))$, so the foot of the perpendicular is at $(\frac{8}{3},\ \frac{7}{3},\ \frac{4}{3})$.

11 **a** $\overrightarrow{\text{LM}} = \begin{pmatrix} -2 \\ 2 \\ -2 \end{pmatrix} = -2\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

$\therefore$ since L lies on the line, it has parametric equations

$$x = 1 + t,\quad y = 0 - t,\quad z = 1 + t,\quad t \in \mathbb{R}$$

The line meets $x - 2y - 3z = 14$ if

$$\begin{aligned} (1+t) - 2(-t) - 3(1+t) &= 14 \\ \therefore\ 1 + t + 2t - 3 - 3t &= 14 \\ \therefore\ -2 &= 14 \quad \text{which is absurd} \end{aligned}$$

$\therefore$ the line and plane do not meet, but rather are parallel.

b The distance $d = \dfrac{|x_1 - 2y_1 - 3z_1 - 14|}{\sqrt{1+4+9}} = \dfrac{|1 - 2(0) - 3(1) - 14|}{\sqrt{14}} = \frac{16}{\sqrt{14}}$ units

12 **a** L_1 meets $2x + y - z = 2$

where $2(3t-4) + (t+2) - (2t-1) = 2$

$\therefore\ 6t - 8 + t + 2 - 2t + 1 = 2$

$\therefore\ 5t = 7$

$\therefore\ t = \frac{7}{5}$

$\therefore$ the lines meet at

$\left(3(\frac{7}{5}) - 4,\ \frac{7}{5} + 2,\ 2(\frac{7}{5}) - 1\right)$

which is $(\frac{1}{5}, \frac{17}{5}, \frac{9}{5})$

b L_1 meets L_2

where $3t - 4 = \dfrac{t+2-5}{2} = \dfrac{-(2t-1)-1}{2}$

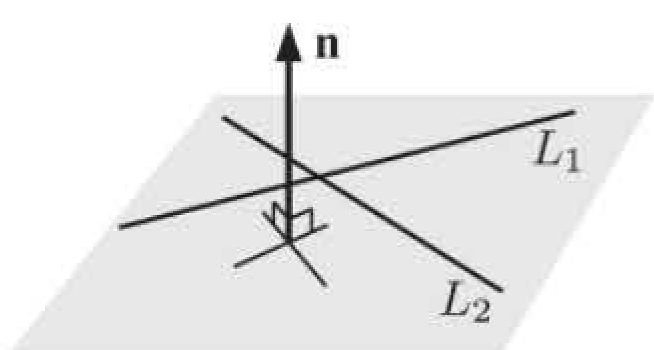
$\therefore\ 6t - 8 = t - 3 = -2t$

$\therefore\ 5t = 5$ and $3t = 3$

$\therefore\ t = 1$

So, L_1 and L_2 meet at $(-1, 3, 1)$.

c



$$\mathbf{n} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & 2 \\ 1 & 2 & -2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2 \\ 2 & -2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & 2 \\ 1 & -2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix}\mathbf{k} = -6\mathbf{i} + 8\mathbf{j} + 5\mathbf{k}$$

$\therefore$ equation is $-6x + 8y + 5z = -6(-1) + 8(3) + 5(1)$

$\therefore\ -6x + 8y + 5z = 35$

$\therefore\ 6x - 8y - 5z = -35$

13 $x - 1 = \dfrac{y+2}{2} = \dfrac{z-3}{4}$ has direction vector $\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$ and $6x + 7y - 5z = 8$ has $\mathbf{n} = \begin{pmatrix} 6 \\ 7 \\ -5 \end{pmatrix}$.

Now $\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} 6 \\ 7 \\ -5 \end{pmatrix} = 6 + 14 - 20 = 0$

$\therefore$ since these two vectors are perpendicular, the line is parallel to the plane.

Choose any point on the line, for example, $(1, -2, 3)$.

Then the distance from the line to the plane is $d = \dfrac{|6x_1 + 7y_1 - 5z_1 - 8|}{\sqrt{6^2 + 7^2 + (-5)^2}}$

$$= \frac{|6(1) + 7(-2) - 5(3) - 8|}{\sqrt{110}}$$

$$= \tfrac{31}{\sqrt{110}} \text{ units}$$

14 If A is $(3, -1, -2)$ and B$(5, 3, -4)$ then $\overrightarrow{AB} = \begin{pmatrix} 5-3 \\ 3--1 \\ -4--2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

$\therefore$ the line has equation $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$, $t \in \mathbb{R}$

and it meets $x^2 + y^2 + z^2 = 26$ where

$(3+t)^2 + (-1+2t)^2 + (-2-t)^2 = 26$

$\therefore\ 9 + 6t + t^2 + 1 - 4t + 4t^2 + 4 + 4t + t^2 - 26 = 0$

$\therefore\ 6t^2 + 6t - 12 = 0$

$\therefore\ t^2 + t - 2 = 0$

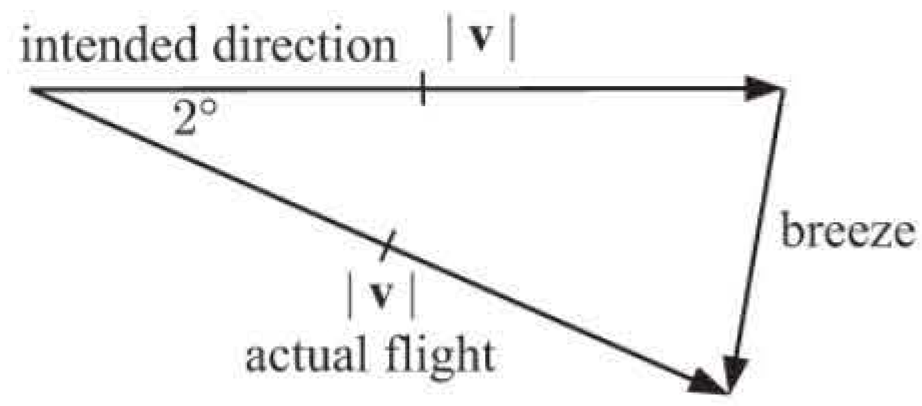
$\therefore\ (t+2)(t-1) = 0$

$\therefore\ t = -2$ or 1

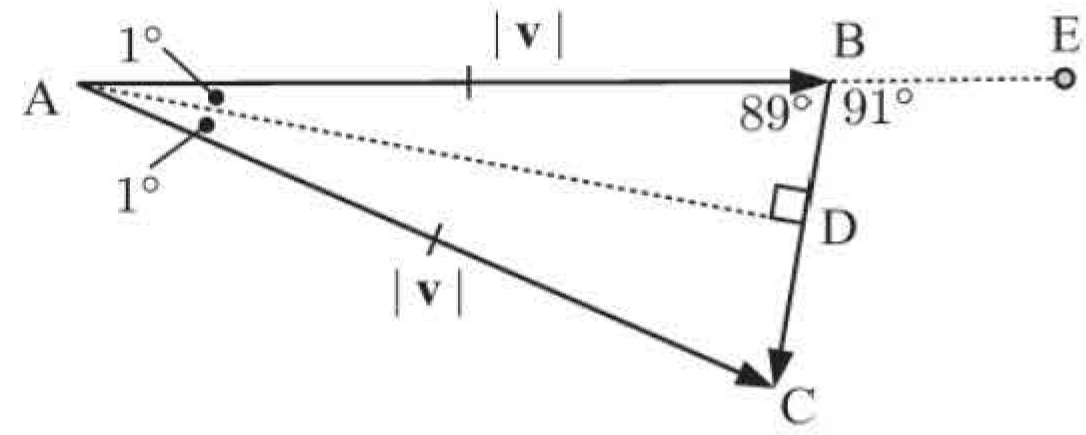
$\therefore\ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix} - 2\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ or $\begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

$\therefore$ the line meets the sphere at $(1, -5, 0)$ and $(4, 1, -3)$.

15 **a**



b **i**



Triangle ABC is isosceles.

$\therefore$ line (AD) that meets the base at the midpoint, D, bisects the angle at A and is perpendicular to the base.

The triangle ABD is right angled at D and angle DAB $= 1°$.

$\therefore$ angle ABD $= 180 - 90 - 1 = 89°$.

If line (AB) is extended to E, then angle DBE $= 180 - 89 = 91°$.

Line (AB) is the arrow's intended path and line (BC) is the breeze, so the breeze is $91°$ to the intended direction of the arrow.

ii The speed of the breeze is the length of (BC) $= 2\times \mid \overrightarrow{BD} \mid$.

$$\sin 1° = \frac{|\overrightarrow{BD}|}{|\mathbf{v}|}$$

$$\therefore\ |\overrightarrow{BD}| = |\mathbf{v}|\sin 1°$$

$$\therefore\ |\overrightarrow{BC}| = 2|\mathbf{v}|\sin 1°$$

So the speed of the breeze is $2|\mathbf{v}|\sin 1°$.

16 **a** X is $\left(\frac{4+10}{2}, \frac{4+2}{2}, \frac{-2+0}{2}\right)$ or $(7, 3, -1)$

If D is (a, b, c), $\overrightarrow{AD} = \begin{pmatrix} a-1 \\ b-3 \\ c--4 \end{pmatrix} = \begin{pmatrix} a-1 \\ b-3 \\ c+4 \end{pmatrix}$ and $\overrightarrow{BC} = \begin{pmatrix} 6 \\ -2 \\ 2 \end{pmatrix}$

Since $\overrightarrow{AD} = \overrightarrow{BC}$, $a - 1 = 6$, $b - 3 = -2$, $c + 4 = 2$

$\therefore\ a = 7$, $b = 1$, $c = -2$ $\therefore$ D is $(7, 1, -2)$

b $\overrightarrow{OY} = \overrightarrow{OA} + \overrightarrow{AY} = \begin{pmatrix} 1 \\ 3 \\ -4 \end{pmatrix} + \frac{2}{3}\overrightarrow{AX} = \begin{pmatrix} 1 \\ 3 \\ -4 \end{pmatrix} + \frac{2}{3}\begin{pmatrix} 7-1 \\ 3-3 \\ -1--4 \end{pmatrix}$

$\therefore\ \overrightarrow{OY} = \begin{pmatrix} 1 \\ 3 \\ -4 \end{pmatrix} + \frac{2}{3}\begin{pmatrix} 6 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ -2 \end{pmatrix}$ and so Y is $(5, 3, -2)$

c $\overrightarrow{BY} = \begin{pmatrix} 5-4 \\ 3-4 \\ -2--2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and $\overrightarrow{BD} = \begin{pmatrix} 7-4 \\ 1-4 \\ -2--2 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix}$

$\therefore\ \overrightarrow{BD} = 3\overrightarrow{BY}$ $\therefore\ \overrightarrow{BD} \parallel \overrightarrow{BY}$ and so B, D, and Y are collinear

17 The system has augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 2 & 1 & -1 & -1 \\ 7 & 2 & k & -k \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 3 & -3 & -11 \\ 0 & 9 & k-7 & -k-35 \end{array}\right] \begin{array}{l} \\ R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 7R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 3 & -3 & -11 \\ 0 & 0 & k+2 & -k-2 \end{array}\right] \begin{array}{l} \\ \\ R_3 \to R_3 - 3R_2 \end{array}$$

Thus $(k+2)z = -(k+2)$

If $k \neq -2$ then $z = -1$, and as $3y - 3z = -11$,

then $3y = -14$

$\therefore\ y = -\frac{14}{3}$

and $x - y + z = 5$,

so $x = 5 - \frac{14}{3} + 1 = \frac{4}{3}$

$\therefore$ we have three planes that meet at the unique point $(\frac{4}{3}, -\frac{14}{3}, -1)$.

If $k = -2$, then the 3 planes meet in a common line and hence there are an infinite number of solutions.

In this case, let $z = t$, $t \in \mathbb{R}$.

Now $3y - 3z = -11$,

$\therefore\ 3y = -11 + 3t$

$\therefore\ y = -\frac{11}{3} + t$

and as $x - y + z = 5$

$\therefore\ x = 5 + y - z$

$\therefore\ x = 5 - \frac{11}{3} + t - t$

$\therefore\ x = \frac{4}{3}$

$\therefore\ x = \frac{4}{3},\quad y = -\frac{11}{3} + t,\quad z = t,\quad t \in \mathbb{R}$

REVIEW SET 15B

1 The vector equation is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 4 \end{pmatrix}$, $\lambda \in \mathbb{R}$

2 **a** **i** The yacht is initially at $(-6, 10)$, so its initial position vector is $\begin{pmatrix} -6 \\ 10 \end{pmatrix}$ or $-6\mathbf{i} + 10\mathbf{j}$.

ii $-\mathbf{i} - 3\mathbf{j}$ has length $\sqrt{(-1)^2 + (-3)^2} = \sqrt{10}$

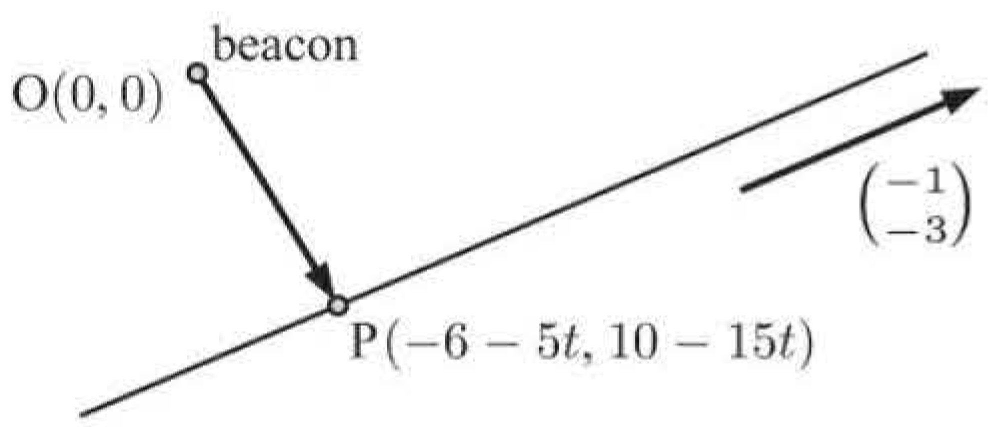
$\therefore$ $5(-\mathbf{i} - 3\mathbf{j})$ has length $5\sqrt{10}$

$\therefore$ the velocity vector is $-5\mathbf{i} - 15\mathbf{j}$

iii $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -6 \\ 10 \end{pmatrix} + t\begin{pmatrix} -5 \\ -15 \end{pmatrix}$ $\therefore$ the position vector is $-6\mathbf{i} + 10\mathbf{j} + t(-5\mathbf{i} - 15\mathbf{j})$

$= (-6 - 5t)\mathbf{i} + (10 - 15t)\mathbf{j},\quad t \geqslant 0$

b Let P be the point on the yacht's path when it is closest to the beacon.



Then $\overrightarrow{OP} = \begin{pmatrix} -6 - 5t \\ 10 - 15t \end{pmatrix}$ and $\overrightarrow{OP} \bullet \begin{pmatrix} -1 \\ -3 \end{pmatrix} = 0$

$\therefore\ -1(-6 - 5t) - 3(10 - 15t) = 0$

$\therefore\ 6 + 5t - 30 + 45t = 0$

$\therefore\ 50t = 24$

$\therefore\ t = 0.48$ h

(or 28.8 min)

c When $t = 0.48$, $\overrightarrow{OP} = \begin{pmatrix} -6 - 5(0.48) \\ 10 - 15(0.48) \end{pmatrix} = \begin{pmatrix} -8.4 \\ 2.8 \end{pmatrix}$

and $|\overrightarrow{OP}| = \sqrt{(-8.4)^2 + (2.8)^2} \approx 8.85$ km

As the closest distance is 8.85 km and the radius is 8 km, the yacht will miss the reef.

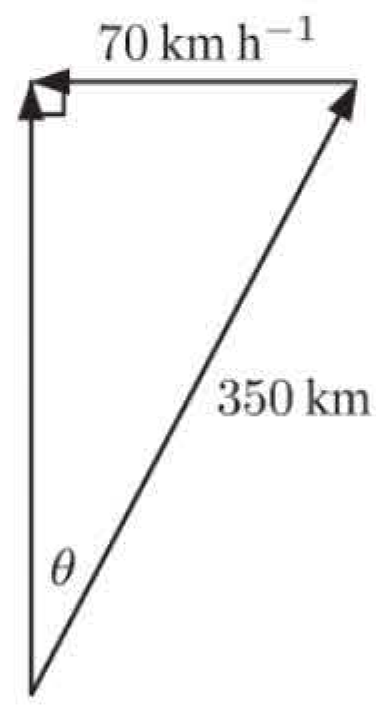
3 **a** **i** $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix} + t\begin{pmatrix} 4 \\ -1 \end{pmatrix}$, $t \in \mathbb{R}$ **ii** $x = 2 + 4t$, $y = -3 - t$, $t \in \mathbb{R}$

b **i** The line has direction vector $\begin{pmatrix} 5 - (-1) \\ -2 - 6 \\ 0 - 3 \end{pmatrix} = \begin{pmatrix} 6 \\ -8 \\ -3 \end{pmatrix}$

$\therefore\ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 6 \\ 3 \end{pmatrix} + t\begin{pmatrix} 6 \\ -8 \\ -3 \end{pmatrix}$, $t \in \mathbb{R}$

ii $x = -1 + 6t$, $y = 6 - 8t$, $z = 3 - 3t$, $t \in \mathbb{R}$

4



a $\sin\theta = \dfrac{70}{350}$

$\therefore \ \theta \approx 11.5°$

So the pilot should face the plane $11.5°$ east of north.

b $x^2 + 70^2 = 350^2$

$\therefore \ x^2 = 350^2 - 70^2$

$\therefore \ x \approx 343 \text{ km h}^{-1}$

So the speed of the plane will be 343 km h^{-1}.

5 L_1 has direction vector $\mathbf{b}_1 = \begin{pmatrix} 5-0 \\ -2-3 \end{pmatrix} = \begin{pmatrix} 5 \\ -5 \end{pmatrix}$

L_2 has direction vector $\mathbf{b}_2 = \begin{pmatrix} -6-(-2) \\ 7-4 \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$

If the angle between the lines is θ,

$$\cos\theta = \frac{|\,\mathbf{b}_1 \bullet \mathbf{b}_2\,|}{|\,\mathbf{b}_1\,|\,|\,\mathbf{b}_2\,|} = \frac{|-20-15|}{\sqrt{25+25}\sqrt{16+9}} = \frac{35}{\sqrt{50}\times 5}$$

$\therefore \ \theta \approx 8.13°$

$\therefore$ the angle between L_1 and L_2 is about $8.13°$.

6 **a** $\begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} + t\begin{pmatrix} 1 \\ -3 \end{pmatrix}$ where $t \geqslant 0$. When $t = 0$, the time is 2:17 pm.

$\therefore \ x_1(t) = 2 + t, \quad y_1(t) = 4 - 3t, \quad t \geqslant 0$

b After time t has passed, submarine Y18's torpedo has been moving for time $(t-2)$.

$\therefore \ x_2(t) = 11 - (t-2), \qquad y_2(t) = 3 + a(t-2)$

$\therefore \ x_2(t) = 13 - t \qquad y_2(t) = (3-2a) + at, \quad t \geqslant 2$

c They meet where $2 + t = 13 - t$ and $4 - 3t = (3-2a) + at$

$\therefore \ 2t = 11$

$\therefore \ t = \frac{11}{2}$ $\quad \therefore$ the time would be 2:17 pm plus $5\frac{1}{2}$ min, or 2:22:30 pm

d When $t = \frac{11}{2}$,

$4 - 3\left(\frac{11}{2}\right) = (3-2a) + a\left(\frac{11}{2}\right)$

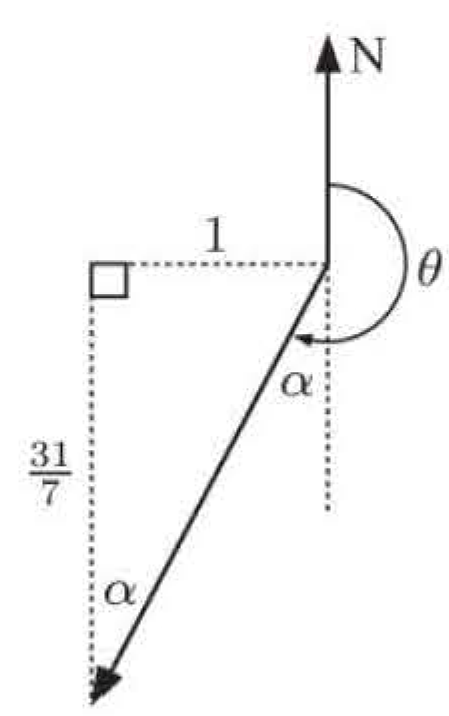
$\therefore \ -\frac{25}{2} = 3 + \dfrac{7a}{2}$

$\therefore \ -25 = 6 + 7a$

$\therefore \ 7a = -31$

$\therefore \ a = -\frac{31}{7}$

Y18's torpedo has velocity vector $\begin{pmatrix} -1 \\ -\frac{31}{7} \end{pmatrix}$



with speed $= \sqrt{(-1)^2 + \left(-\frac{31}{7}\right)^2}$

≈ 4.54 units per minute

$\tan\alpha = \dfrac{1}{\frac{31}{7}} = \frac{7}{31}$

$\therefore \ \alpha = \tan^{-1}\left(\frac{7}{31}\right) \approx 12.7°$

So, the torpedo has speed 4.54 units per minute and direction $12.7°$ west of south.

7 **a** $\mathbf{n} = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$ and $\mathbf{l} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

$\therefore \ \sin\phi = \dfrac{|2 + (-2) + 2|}{\sqrt{9}\sqrt{6}}$

$= \dfrac{2}{3\sqrt{6}}$

$\therefore \ \phi \approx 15.8°$

b The planes have normals

$\mathbf{n}_1 = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$ and $\mathbf{n}_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$

$\therefore \ \cos\theta = \dfrac{|2 + (-1) + (-4)|}{\sqrt{9}\sqrt{6}}$

$= \dfrac{3}{3\sqrt{6}}$ or $\frac{1}{\sqrt{6}}$

$\therefore \ \theta \approx 65.9°$

8 **a** $\dfrac{x-8}{3}=\dfrac{y+9}{-16}=\dfrac{z-10}{7}$ has direction vector $\begin{pmatrix}3\\-16\\7\end{pmatrix}$

$x=15+3t,\quad y=29+8t,\quad z=5-5t$ has direction vector $\begin{pmatrix}3\\8\\-5\end{pmatrix}$

$\therefore$ since the direction vectors are not scalar multiples of each other, the lines are not parallel.

If they intersect then $\dfrac{15+3t-8}{3}=\dfrac{29+8t+9}{-16}=\dfrac{5-5t-10}{7}$

$\therefore\ t+\frac{7}{3}=-\frac{1}{2}t-\frac{38}{16}=-\frac{5}{7}t-\frac{5}{7}$

Now $t+\frac{7}{3}=-\frac{1}{2}t-\frac{38}{16}$ requires $\frac{3}{2}t=-\frac{19}{8}-\frac{7}{3}=-\frac{113}{24}$ $\therefore\ t=-\frac{113}{36}$

and $t+\frac{7}{3}=-\frac{5}{7}t-\frac{5}{7}$ requires $\frac{12}{7}t=-\frac{5}{7}-\frac{7}{3}=-\frac{64}{21}$ $\therefore\ t=-\frac{16}{9}$

Hence the lines do not intersect, and since they are not parallel, they are skew.

b If θ is the acute angle between the two lines, and $\mathbf{v}_1$ and $\mathbf{v}_2$ are their direction vectors,

then $\cos\theta=\dfrac{|\mathbf{v}_1\bullet\mathbf{v}_2|}{|\mathbf{v}_1|\,|\mathbf{v}_2|}=\dfrac{\left|\begin{pmatrix}3\\-16\\7\end{pmatrix}\bullet\begin{pmatrix}3\\8\\-5\end{pmatrix}\right|}{\sqrt{3^2+(-16)^2+7^2}\sqrt{3^2+8^2+(-5)^2}}$

$\therefore\ \cos\theta=\dfrac{|9-128-35|}{\sqrt{314}\sqrt{98}}=\dfrac{154}{\sqrt{30\,772}}$

and so $\theta\approx 28.6^\circ$

c $\mathbf{v}_1\times\mathbf{v}_2=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\3&-16&7\\3&8&-5\end{vmatrix}=\begin{vmatrix}-16&7\\8&-5\end{vmatrix}\mathbf{i}-\begin{vmatrix}3&7\\3&-5\end{vmatrix}\mathbf{j}+\begin{vmatrix}3&-16\\3&8\end{vmatrix}\mathbf{k}$

$=24\mathbf{i}+36\mathbf{j}+72\mathbf{k}$ or $12(2\mathbf{i}+3\mathbf{j}+6\mathbf{k})$

$\therefore\ \begin{pmatrix}2\\3\\6\end{pmatrix}$ is the normal of a plane containing L_1 and parallel to L_2.

$\therefore$ the plane with normal $\begin{pmatrix}2\\3\\6\end{pmatrix}$ and containing L_2 also contains L_3.

$\therefore$ the plane has equation $2x+3y+6z=2(15)+3(29)+6(5)$

$\therefore\ 2x+3y+6z=147$

d The shortest distance between L_1 and L_2 is the shortest distance between the plane containing L_2 and L_3 and a point on L_1.

From **Exercise 15I** question **16 b**,

$$d=\frac{|Ax_1+By_1+Cz_1+D|}{\sqrt{A^2+B^2+C^2}}=\frac{|2(8)+3(-9)+6(10)-147|}{\sqrt{2^2+3^2+6^2}}$$

$$=\frac{98}{\sqrt{49}}=14\text{ units}$$

9 **a** Given A$(-1, 2, 3)$, B$(1, 0, -1)$, and C$(0, -1, 5)$,

$\overrightarrow{AB}=\begin{pmatrix}2\\-2\\-4\end{pmatrix}$ and $\overrightarrow{AC}=\begin{pmatrix}1\\-3\\2\end{pmatrix}$

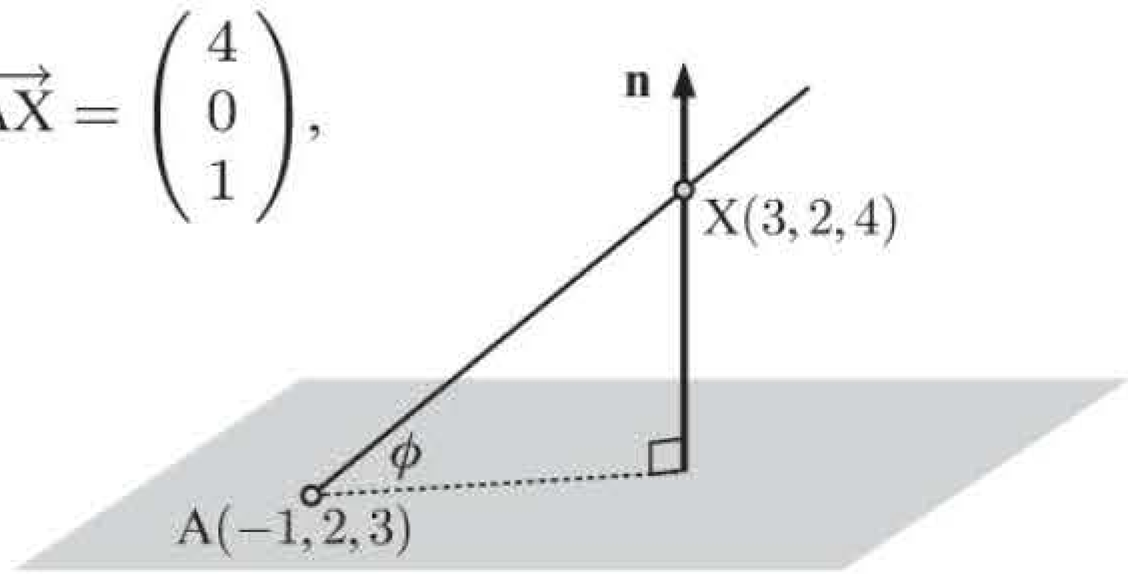
$\therefore$ a normal to the plane is $\mathbf{n}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&-2&-4\\1&-3&2\end{vmatrix}=\begin{vmatrix}-2&-4\\-3&2\end{vmatrix}\mathbf{i}-\begin{vmatrix}2&-4\\1&2\end{vmatrix}\mathbf{j}+\begin{vmatrix}2&-2\\1&-3\end{vmatrix}\mathbf{k}$

$=-16\mathbf{i}-8\mathbf{j}-4\mathbf{k}$ or $-4(4\mathbf{i}+2\mathbf{j}+\mathbf{k})$

$\therefore$ the plane has equation $4x+2y+z=4(1)+2(0)+1(-1)$

$\therefore\ 4x+2y+z=3$

b Given the plane has normal $\mathbf{n} = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$ and $\overrightarrow{AX} = \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}$,



$$\begin{aligned} \sin\phi &= \frac{|\,\mathbf{n} \bullet \overrightarrow{AX}\,|}{|\,\mathbf{n}\,|\,|\overrightarrow{AX}|} \\ &= \frac{|4\times 4 + 2\times 0 + 1\times 1|}{\sqrt{21}\sqrt{17}} \\ &= \frac{17}{\sqrt{21}\sqrt{17}} \quad \text{and so} \quad \phi \approx 64.1^\circ \end{aligned}$$

10 **a** All vectors normal to $x - y + z = 6$ have the form $t\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} t \\ -t \\ t \end{pmatrix}$, $t \in \mathbb{R}$

$\therefore$ if the vector has length 3 units,

$$\begin{aligned} \sqrt{t^2+t^2+t^2} &= 3 \\ \therefore\ 3t^2 &= 9 \\ \therefore\ t^2 &= 3 \\ \therefore\ t &= \pm\sqrt{3} \end{aligned}$$

$\therefore$ the vectors are $\begin{pmatrix} \sqrt{3} \\ -\sqrt{3} \\ \sqrt{3} \end{pmatrix}$ and $\begin{pmatrix} -\sqrt{3} \\ \sqrt{3} \\ -\sqrt{3} \end{pmatrix}$.

b Any vector parallel to $\mathbf{i} + r\mathbf{j} + 3\mathbf{k}$ has the form $t\begin{pmatrix} 1 \\ r \\ 3 \end{pmatrix} = \begin{pmatrix} t \\ rt \\ 3t \end{pmatrix}$, $t \in \mathbb{R}$.

This is perpendicular to $\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$ if $\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} t \\ rt \\ 3t \end{pmatrix} = 0$

$$\begin{aligned} \therefore\ 2t - rt + 6t &= 0 \\ \therefore\ 8t - rt &= 0 \\ \therefore\ t(8-r) &= 0 \\ \therefore\ t = 0 \ \text{or}\ r &= 8 \end{aligned}$$

But if $t = 0$, the vector has zero length. $\therefore\ r = 8$ and so a vector is $\begin{pmatrix} 1 \\ 8 \\ 3 \end{pmatrix}$.

$\therefore$ the unit vectors are $\mathbf{u} = \frac{1}{\sqrt{74}}\mathbf{i} + \frac{8}{\sqrt{74}}\mathbf{j} + \frac{3}{\sqrt{74}}\mathbf{k}$ or $-\frac{1}{\sqrt{74}}\mathbf{i} - \frac{8}{\sqrt{74}}\mathbf{j} - \frac{3}{\sqrt{74}}\mathbf{k}$

c The distance from the plane to A is $d = \dfrac{|2x_1 - y_1 + 2z_1 - k|}{\sqrt{9}}$

$$\begin{aligned} \therefore\ \frac{|2(-1) - (2) + 2(3) - k|}{3} &= 3 \\ \therefore\ |2 - k| &= 9 \\ \therefore\ 2 - k = 9 \ \text{or}\ k - 2 &= 9 \\ \therefore\ k = -7 \ \text{or}\ &11 \end{aligned}$$

11 $4x - 5y = 11$ has gradient $\frac{4}{5}$ $\quad\therefore$ it has direction vector $\begin{pmatrix} 5 \\ 4 \end{pmatrix}$.

$2x + 3y = 7$ has gradient $-\frac{2}{3}$ $\quad\therefore$ it has direction vector $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$.

If the angle is θ, $\begin{pmatrix} 5 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \sqrt{5^2+4^2}\sqrt{3^2+(-2)^2}\cos\theta$

$$\begin{aligned} \therefore\ 15 - 8 &= \sqrt{41}\sqrt{13}\cos\theta \\ \therefore\ \frac{7}{\sqrt{41\times 13}} &= \cos\theta \\ \therefore\ \theta &\approx 72.35^\circ \end{aligned}$$

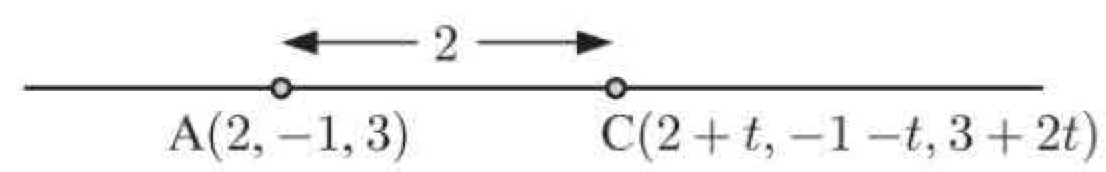
$\therefore$ the angle is 72.35° (or 107.65°)

12 **a** $\overrightarrow{AB} = \begin{pmatrix} -2 \\ 2 \\ -4 \end{pmatrix} = -2\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

$\therefore$ since A lies on the line, it has equations $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \quad \lambda \in \mathbb{R}$

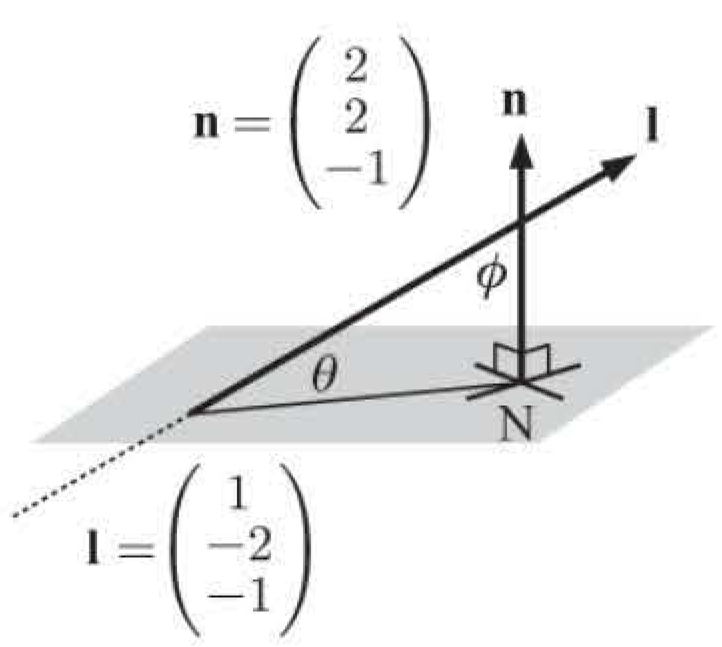
b If C lies on (AB) and is 2 units from A, then C corresponds to λ such that

$$\sqrt{(\lambda)^2 + (-\lambda)^2 + (2\lambda)^2} = 2$$
$$\therefore \sqrt{6\lambda^2} = 2$$
$$\therefore 6\lambda^2 = 4$$
$$\therefore \lambda^2 = \tfrac{4}{6}$$
$$\therefore \lambda = \pm\tfrac{2}{\sqrt{6}}$$



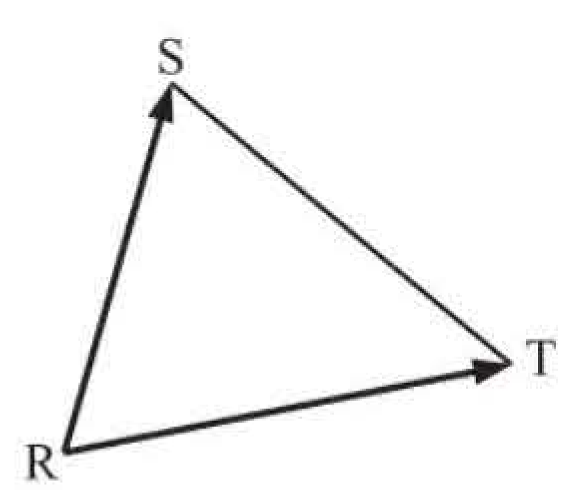
$\therefore$ C is $\left(2 + \frac{2}{\sqrt{6}}, -1 - \frac{2}{\sqrt{6}}, 3 + \frac{4}{\sqrt{6}}\right)$ or $\left(2 - \frac{2}{\sqrt{6}}, -1 + \frac{2}{\sqrt{6}}, 3 - \frac{4}{\sqrt{6}}\right)$

13



$$\mathbf{n} \bullet \mathbf{l} = |\mathbf{n}|\,|\mathbf{l}|\cos\phi$$
$$\therefore \cos\phi = \frac{\mathbf{n} \bullet \mathbf{l}}{|\mathbf{n}|\,|\mathbf{l}|}$$
$$\therefore \sin\theta = \frac{|\mathbf{n} \bullet \mathbf{l}|}{|\mathbf{n}|\,|\mathbf{l}|} \quad \text{as} \quad \cos\phi = \cos\left(\tfrac{\pi}{2} - \theta\right) = \sin\theta$$
$$= \frac{|2 - 4 + 1|}{\sqrt{4+4+1}\sqrt{1+4+1}} = \tfrac{1}{\sqrt{54}}$$
$$\therefore \theta \approx 7.82^\circ$$

14



$$\overrightarrow{RS} = \overrightarrow{RO} + \overrightarrow{OS} = \overrightarrow{OS} - \overrightarrow{OR}$$
$$\therefore \overrightarrow{RS} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k} - 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$$
$$= 3\mathbf{j} + 3\mathbf{k}$$

Likewise $\overrightarrow{RT} = \overrightarrow{OT} - \overrightarrow{OR} = \mathbf{i} + 2\mathbf{j} - \mathbf{k} - 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$
$$= -\mathbf{i} + 4\mathbf{j}$$

Now area $= \frac{1}{2}\,|\overrightarrow{RS} \times \overrightarrow{RT}|$

$$= \tfrac{1}{2}\left\|\begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 3 & 3 \\ -1 & 4 & 0 \end{matrix}\right\|$$
$$= \tfrac{1}{2}\left|\begin{vmatrix} 3 & 3 \\ 4 & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 0 & 3 \\ -1 & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 0 & 3 \\ -1 & 4 \end{vmatrix}\mathbf{k}\right|$$
$$= \tfrac{1}{2}\,|-12\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}|$$
$$= \tfrac{1}{2}\sqrt{144 + 9 + 9}$$
$$= \tfrac{1}{2}\sqrt{162}$$
$$= \tfrac{1}{2}\,9\sqrt{2}$$
$$= \tfrac{9\sqrt{2}}{2} \text{ units}^2$$

15 **a** Line 1 has direction vector $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ and line 2 has direction vector $\begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix}$.

As one vector is not a scalar multiple of the other, the lines are not parallel.

Now, $2 + t = -8 + 4s$ .... (1) $\quad -1 + 2t = s$ .... (2) $\quad 3 - t = 7 - 2s$ .... (3)

Substituting (2) into (1), $2+t=-8+4(-1+2t)$

$\therefore\ 2+t=-8-4+8t$

$\therefore\ 7t=14$

$\therefore\ t=2$

$\therefore\ s=-1+2(2)=3$

In (3), LHS $=3-2=1$ RHS $=7-2(3)=1$ ✓

$\therefore\ s=3,\ t=2$ satisfies all three equations.

$\therefore$ the lines meet at $(4,\,3,\,1)$ {substituting $t=2$ into line 1}

The angle θ between the lines has $\cos\theta=\dfrac{\left|\begin{pmatrix}1\\2\\-1\end{pmatrix}\bullet\begin{pmatrix}4\\1\\-2\end{pmatrix}\right|}{\sqrt{1+4+1}\sqrt{16+1+4}}=\dfrac{|4+2+2|}{\sqrt{6}\sqrt{21}}=\dfrac{8}{3\sqrt{14}}$

$\therefore\ \theta\approx 44.5^\circ$

b Line 1 has direction vector $\begin{pmatrix}1\\-2\\3\end{pmatrix}$ and line 2 has direction vector $\begin{pmatrix}-1\\3\\1\end{pmatrix}$.

As one vector is not a scalar multiple of the other, the lines are not parallel.

Now, $3+t=2-s$ $\quad 5-2t=1+3s$ $\quad -1+3t=4+s$

$\therefore\ t+s=-1$ .... (1) $\quad 2t+3s=4$ .... (2) $\quad 3t-s=5$ .... (3)

Solving (1) and (3) simultaneously:

$$\begin{array}{r}t+s=-1\\ 3t-s=5\\ \hline 4t\quad=4\end{array}$$

Adding, $\therefore\ t=1 \quad \therefore\ s=-2$

In (2), LHS $=2(1)+3(-2)=-4$ ✗

$\therefore$ the system of equations is inconsistent and so the lines are skew.

The angle θ between them has $\cos\theta=\dfrac{\left|\begin{pmatrix}1\\-2\\3\end{pmatrix}\bullet\begin{pmatrix}-1\\3\\1\end{pmatrix}\right|}{\sqrt{1+4+9}\sqrt{1+9+1}}=\dfrac{|-1-6+3|}{\sqrt{14}\sqrt{11}}=\dfrac{4}{\sqrt{154}}$

$\therefore\ \theta\approx 71.2^\circ$

16 **a** $\mathbf{p}\times\mathbf{q}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&-1&2\\2&3&-1\end{vmatrix}=\begin{vmatrix}-1&2\\3&-1\end{vmatrix}\mathbf{i}-\begin{vmatrix}1&2\\2&-1\end{vmatrix}\mathbf{j}+\begin{vmatrix}1&-1\\2&3\end{vmatrix}\mathbf{k}$

$=-5\mathbf{i}+5\mathbf{j}+5\mathbf{k}$

$=5\begin{pmatrix}-1\\1\\1\end{pmatrix}$

b l has direction vector $\begin{pmatrix}2\\1\\m\end{pmatrix}$

$\therefore$ $\mathbf{p}\times\mathbf{q}$ is perpendicular to l if $\begin{pmatrix}-1\\1\\1\end{pmatrix}\bullet\begin{pmatrix}2\\1\\m\end{pmatrix}=0$

$\therefore\ -2+1+m=0$

$\therefore\ m=1$

c P has normal vector $\mathbf{n}=\begin{pmatrix}-1\\1\\1\end{pmatrix}$, and $(1,\,-2,\,3)$ lies on the plane {letting $\lambda=0$}

$\therefore$ P has equation $-x+y+z=-1+(-2)+3$

$\therefore\ x-y-z=0$

d A lies on the plane if $4 - t - 2 = 0$
$$\therefore\ t = 2$$

e $\overrightarrow{AB} = \begin{pmatrix} 2 \\ -5 \\ 3 \end{pmatrix}$ and P has normal vector $\mathbf{n} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

$\therefore$ if θ is the angle between (AB) and P then
$$\sin\theta = \frac{\left|\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ -5 \\ 3 \end{pmatrix}\right|}{\sqrt{(-1)^2 + 1^2 + 1^2}\,\sqrt{2^2 + (-5)^2 + 3^2}}$$
$$= \frac{|-2-5+3|}{\sqrt{3}\sqrt{38}} = \frac{4}{\sqrt{114}}$$

17 **a** We substitute L_1 into the LHS of the plane's equation
$$2(-2t+2) + (t) + (3t+1) = -4t + 4 + t + 3t + 1 = 5 \quad \checkmark$$
$\therefore$ the plane contains the line.

b If $x + ky + z = 3$ contains L_1 then
$$(-2t+2) + k(t) + 3t + 1 = 3$$
$$\therefore\ t[-2 + k + 3] + 2 + 1 = 3$$
$$\therefore\ t[k+1] = 0$$
$$\therefore\ k = -1 \text{ as } t \in \mathbb{R}$$

c From **a** and **b**, both $2x + y + z = 5$ and $x - y + z = 3$ contain L_1.
So, substituting L_1 into plane 3 gives
$$-2(-2t+2) + pt + 2(3t+1) = q \quad \text{for all } t \in \mathbb{R}$$
$$\therefore\ 4t - 4 + pt + 6t + 2 = q \quad \text{for all } t \in \mathbb{R}$$
$$\therefore\ (10+p)t - 2 = q \quad \text{for all } t$$
This equation has infinitely many solutions for t
when $10 + p = 0$ and $-2 = q$ {equating coefficients}
$\therefore\ p = -10$ and $q = -2$

18 **a** In augmented matrix form, the system is:
$$\left[\begin{array}{ccc|c} 1 & -3 & 2 & -5 \\ 3 & 1 & (2-k) & 10 \\ -2 & 6 & k & 5 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|c} 1 & -3 & 2 & -5 \\ 0 & 10 & -4-k & 25 \\ 0 & 0 & k+4 & -5 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 + 2R_1 \end{array}$$

$$\begin{array}{rrrr} 3 & 1 & 2-k & 10 \\ -3 & 9 & -6 & 15 \\ \hline 0 & 10 & -4-k & 25 \end{array}$$
$$\begin{array}{rrrr} -2 & 6 & k & 5 \\ 2 & -6 & 4 & -10 \\ \hline 0 & 0 & k+4 & -5 \end{array}$$

b If $k = -4$, row 3 becomes $0x + 0y + 0z = -5$
$\therefore$ the system is inconsistent and there are no solutions.
When $k = -4$ the system becomes
$$\begin{cases} x - 3y + 2z = -5 \\ 3x + y + 6z = 10 \\ -2x + 6y - 4z = 5 \end{cases}$$
Planes 1 and 3 are parallel.
In this case two planes are parallel and they are intersected by the third plane.

c **i** There is a unique solution when $k \neq -4$.

ii If $k \neq -4$, then row 3 becomes $(k+4)z = -5$
$$\therefore\ z = \frac{-5}{k+4}$$
and substituting in row 2 gives $10y + -(4+k)\left(\dfrac{-5}{k+4}\right) = 25$
$$\therefore\ 10y + 5 = 25$$
$$\therefore\ 10y = 20$$
$$\therefore\ y = 2$$

and substituting in row 1 gives $x - 3(2) + 2\left(\frac{-5}{k+4}\right) = -5$

$$\therefore\ x = 1 + \frac{10}{k+4}$$

$\therefore$ the unique solution is $x = 1 + \frac{10}{k+4}$, $y = 2$, $z = \frac{-5}{k+4}$.

In this case we have three planes which meet at a point.

iii When $k = 1$, $x = 1 + \frac{10}{1+4} = 3$, $y = 2$, and $z = \frac{-5}{1+4} = -1$

So, when $k = 1$ the unique solution is $(3,\ 2,\ -1)$.

REVIEW SET 15C

1 The direction vector is $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$ which has length $\sqrt{3^2 + (-1)^2} = \sqrt{10}$ units

$\therefore\ 2\sqrt{10}\begin{pmatrix} 3 \\ -1 \end{pmatrix}$ has length 20. So, the velocity vector is $\begin{pmatrix} 6\sqrt{10} \\ -2\sqrt{10} \end{pmatrix}$ or $2\sqrt{10}(3\mathbf{i} - \mathbf{j})$.

2 **a** $x(0) = -4$ and $y(0) = 3$, so the initial position is $(-4,\ 3)$.

b $x(4) = -4 + 8(4) = 28$ and $y(4) = 3 + 6(4) = 27$, so at $t = 4$ the position is $(28,\ 27)$.

c The velocity vector is $\begin{pmatrix} 8 \\ 6 \end{pmatrix}$.

d The speed is $\sqrt{8^2 + 6^2} = 10\ \text{m s}^{-1}$.

3 **a** (KL) has direction vector $\begin{pmatrix} 5 \\ -2 \end{pmatrix}$ and (MN) has direction vector $\begin{pmatrix} -5 \\ 2 \end{pmatrix}$.

Now $\begin{pmatrix} 5 \\ -2 \end{pmatrix} = -\begin{pmatrix} -5 \\ 2 \end{pmatrix}$, so (KL) $\parallel$ (MN).

b $\overrightarrow{\text{KL}} = a\begin{pmatrix} 5 \\ -2 \end{pmatrix}$, $\overrightarrow{\text{NK}} = b\begin{pmatrix} 4 \\ 10 \end{pmatrix}$, $\overrightarrow{\text{MN}} = c\begin{pmatrix} -5 \\ 2 \end{pmatrix}$ {for some constants a, b, c}

$\therefore\ \overrightarrow{\text{KL}} \bullet \overrightarrow{\text{NK}} = ab(20 - 20) = 0$ and $\overrightarrow{\text{NK}} \bullet \overrightarrow{\text{MN}} = bc(-20 + 20) = 0$

$\therefore$ (NK) is perpendicular to both (KL) and (MN).

c (KL) and (NK) meet at K.

$$\therefore\ \begin{pmatrix} 2 \\ 19 \end{pmatrix} + p\begin{pmatrix} 5 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \end{pmatrix} + r\begin{pmatrix} 4 \\ 10 \end{pmatrix}$$

$$\therefore\ \begin{pmatrix} 5p - 4r \\ -2p - 10r \end{pmatrix} = \begin{pmatrix} 1 \\ -12 \end{pmatrix}$$

$$\begin{aligned} \therefore\ 5p - 4r &= 1 \quad \text{.... (1)} \\ 2p + 10r &= 12 \quad \text{.... (2)} \\ \therefore\ 25p - 20r &= 5 \quad \{5 \times (1)\} \\ 4p + 20r &= 24 \quad \{2 \times (2)\} \\ \text{Adding,}\quad 29p \qquad &= 29 \\ \therefore\ p &= 1 \end{aligned}$$

and $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 19 \end{pmatrix} + \begin{pmatrix} 5 \\ -2 \end{pmatrix} = \begin{pmatrix} 7 \\ 17 \end{pmatrix}$

$\therefore$ K is $(7,\ 17)$.

(KL) and (ML) meet at L.

$$\therefore\ \begin{pmatrix} 2 \\ 19 \end{pmatrix} + p\begin{pmatrix} 5 \\ -2 \end{pmatrix} = \begin{pmatrix} 33 \\ -5 \end{pmatrix} + q\begin{pmatrix} -11 \\ 16 \end{pmatrix}$$

$$\therefore\ \begin{pmatrix} 5p + 11q \\ -2p - 16q \end{pmatrix} = \begin{pmatrix} 31 \\ -24 \end{pmatrix}$$

$$\begin{aligned} \therefore\ 5p + 11q &= 31 \quad \text{.... (1)} \\ -2p - 16q &= -24 \quad \text{.... (2)} \\ \therefore\ 10p + 22q &= 62 \quad \{2 \times (1)\} \\ -10p - 80q &= -120 \quad \{5 \times (2)\} \\ \text{Adding,}\quad -58q &= -58 \\ \therefore\ q &= 1 \end{aligned}$$

and $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 33 \\ -5 \end{pmatrix} + \begin{pmatrix} -11 \\ 16 \end{pmatrix} = \begin{pmatrix} 22 \\ 11 \end{pmatrix}$

$\therefore$ L is $(22,\ 11)$.

(ML) and (MN) meet at M.

$$\therefore \begin{pmatrix}33\\-5\end{pmatrix} + q\begin{pmatrix}-11\\16\end{pmatrix} = \begin{pmatrix}43\\-9\end{pmatrix} + s\begin{pmatrix}-5\\2\end{pmatrix}$$

$$\therefore \begin{pmatrix}-11q+5s\\16q-2s\end{pmatrix} = \begin{pmatrix}10\\-4\end{pmatrix}$$

$$\begin{aligned}\therefore\ -11q+5s &= 10 \quad .... (1)\\ 16q-2s &= -4 \quad .... (2)\\ \therefore\ -22q+10s &= 20 \quad \{2\times(1)\}\\ 80q-10s &= -20 \quad \{5\times(2)\}\\ \text{Adding,}\quad 58q \quad &= 0\end{aligned}$$

$$\therefore\ q = 0 \quad \text{and so} \quad \begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}33\\-5\end{pmatrix}$$

$\therefore$ M is (33, -5).

(NK) and (MN) meet at N.

$$\therefore \begin{pmatrix}3\\7\end{pmatrix} + r\begin{pmatrix}4\\10\end{pmatrix} = \begin{pmatrix}43\\-9\end{pmatrix} + s\begin{pmatrix}-5\\2\end{pmatrix}$$

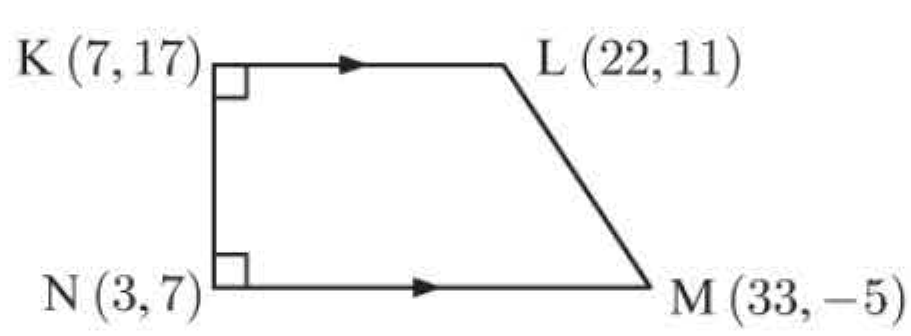
$$\therefore \begin{pmatrix}4r+5s\\10r-2s\end{pmatrix} = \begin{pmatrix}40\\-16\end{pmatrix}$$

$$\begin{aligned}\therefore\ 4r+5s &= 40 \quad .... (1)\\ 10r-2s &= -16 \quad .... (2)\\ \therefore\ 8r+10s &= 80 \quad \{2\times(1)\}\\ 50r-10s &= -80 \quad \{5\times(2)\}\\ \text{Adding,}\quad 58r \quad &= 0\end{aligned}$$

$$\therefore\ r = 0 \quad \text{and so} \quad \begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}3\\7\end{pmatrix}$$

$\therefore$ N is (3, 7).

d



$$\begin{aligned}\text{KL} &= \sqrt{(22-7)^2+(11-17)^2}\\ &= \sqrt{225+36}\\ &= \sqrt{261}\ \text{units}\end{aligned}$$

$$\begin{aligned}\text{KN} &= \sqrt{(7-3)^2+(17-7)^2}\\ &= \sqrt{16+100}\\ &= \sqrt{116}\ \text{units}\end{aligned}$$

$$\begin{aligned}\text{NM} &= \sqrt{(33-3)^2+(-5-7)^2}\\ &= \sqrt{900+144}\\ &= \sqrt{1044}\ \text{units}\end{aligned}$$

$$\begin{aligned}\therefore\ \text{area} &= \left(\frac{\sqrt{261}+\sqrt{1044}}{2}\right)\times\sqrt{116}\\ &= 261\ \text{units}^2\end{aligned}$$

4 L_1 has direction vector $\mathbf{b}_1 = \begin{pmatrix}-4\\3\end{pmatrix}$, L_2 has direction vector $\mathbf{b}_2 = \begin{pmatrix}5\\-12\end{pmatrix}$

If θ is the angle between them,

$$\cos\theta = \frac{|\,\mathbf{b}_1 \bullet \mathbf{b}_2\,|}{|\,\mathbf{b}_1\,|\,|\,\mathbf{b}_2\,|} = \frac{|-20-36|}{\sqrt{16+9}\sqrt{25+144}} = \frac{56}{5\times 13}$$

$$\therefore\ \theta \approx 30.5^\circ$$

$\therefore$ the angle between L_1 and L_2 is about 30.5°.

5 **a**

$$\overrightarrow{\text{AB}} = \begin{pmatrix}0-3\\2--1\\-2-1\end{pmatrix} = \begin{pmatrix}-3\\3\\-3\end{pmatrix}$$

$$\therefore\ |\,\overrightarrow{\text{AB}}\,| = \sqrt{9+9+9} = \sqrt{27}\ \text{units}$$

b

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0\\2\\-2\end{pmatrix} + t\begin{pmatrix}-3\\3\\-3\end{pmatrix}, \quad t\in\mathbb{R}$$

$$\therefore \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0\\2\\-2\end{pmatrix} + \lambda\begin{pmatrix}-1\\1\\-1\end{pmatrix}, \quad \text{where}\ \lambda = 3t$$

$$\therefore\ \mathbf{r} = 2\mathbf{j} - 2\mathbf{k} + \lambda(-\mathbf{i}+\mathbf{j}-\mathbf{k}), \quad \text{where}\ \lambda\in\mathbb{R}$$

A lies on the line $\mathbf{r}$ when $\lambda = -3$ and B lies on $\mathbf{r}$ when $\lambda = 0$.

$\therefore$ the line between A and B is the same line as $\mathbf{r}$, so it can be described by $\mathbf{r}$.

c The line with equation $t(\mathbf{i}+\mathbf{j}+\mathbf{k})$ has direction vector $\begin{pmatrix}1\\1\\1\end{pmatrix}$.

$$\therefore\ \overrightarrow{AB}\bullet\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}-3\\3\\-3\end{pmatrix}\bullet\begin{pmatrix}1\\1\\1\end{pmatrix}$$
$$=-3+3-3$$
$$=-3$$

$$\left|\begin{pmatrix}1\\1\\1\end{pmatrix}\right|=\sqrt{1^2+1^2+1^2}=\sqrt{3}$$

$$\therefore\ \cos\theta=\frac{|-3|}{3\sqrt{3}\times\sqrt{3}}=\frac{3}{9}=\frac{1}{3}$$

$$\therefore\ \theta\approx 70.5^\circ$$

$\therefore$ the angle between the two lines is about 70.5°.

6 **a** Road A has direction vector $\begin{pmatrix}15--9\\-16-2\end{pmatrix}=\begin{pmatrix}24\\-18\end{pmatrix}=6\begin{pmatrix}4\\-3\end{pmatrix}$.

So, Road A has equation $\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-9\\2\end{pmatrix}+t\begin{pmatrix}4\\-3\end{pmatrix}$, $t\in\mathbb{R}$.

Road B has direction vector $\begin{pmatrix}21-6\\18--18\end{pmatrix}=\begin{pmatrix}15\\36\end{pmatrix}=3\begin{pmatrix}5\\12\end{pmatrix}$

So, Road B has equation $\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}6\\-18\end{pmatrix}+s\begin{pmatrix}5\\12\end{pmatrix}$, $s\in\mathbb{R}$.

b Let $A(-9+4t,\ 2-3t)$ be any point on Road A.

$$\therefore\ \overrightarrow{HA}=\begin{pmatrix}-9+4t-4\\2-3t-11\end{pmatrix}=\begin{pmatrix}-13+4t\\-9-3t\end{pmatrix}$$

The closest point to $H(4,\ 11)$ on Road A is such that $\overrightarrow{HA}\perp\begin{pmatrix}4\\-3\end{pmatrix}$.

$$\therefore\ \begin{pmatrix}-13+4t\\-9-3t\end{pmatrix}\bullet\begin{pmatrix}4\\-3\end{pmatrix}=0$$
$$\therefore\ -52+16t+27+9t=0$$
$$\therefore\ 25t=25$$
$$\therefore\ t=1$$

So A is $(-9+4,\ 2-3)$ or $(-5,\ -1)$.

$$\therefore\ \overrightarrow{HA}=\begin{pmatrix}-13+4\\-9-3\end{pmatrix}=\begin{pmatrix}-9\\-12\end{pmatrix}$$

$$\therefore\ |\overrightarrow{HA}|=\sqrt{81+144}=\sqrt{225}=15\text{ km}$$

Now, let $B(6+5s,\ -18+12s)$ be any point on Road B.

$$\therefore\ \overrightarrow{HB}=\begin{pmatrix}6+5s-4\\-18+12s-11\end{pmatrix}=\begin{pmatrix}2+5s\\-29+12s\end{pmatrix}$$

The closest point to $H(4,\ 11)$ on Road B is such that $\overrightarrow{HB}\perp\begin{pmatrix}5\\12\end{pmatrix}$.

$$\therefore\ \begin{pmatrix}2+5s\\-29+12s\end{pmatrix}\bullet\begin{pmatrix}5\\12\end{pmatrix}=0$$
$$\therefore\ 10+25s-348+144s=0$$
$$\therefore\ 169s=338$$
$$\therefore\ s=2$$

So B is $(6+5(2), -18+12(2))$ or $(16, 6)$.

$$\therefore \ \overrightarrow{HB} = \begin{pmatrix} 2+5(2) \\ -29+12(2) \end{pmatrix} = \begin{pmatrix} 12 \\ -5 \end{pmatrix}$$

$$\therefore \ |\overrightarrow{HB}| = \sqrt{144+25} = \sqrt{169} = 13 \text{ km}$$

The hiker should head toward Road B, a distance of 13 km.

7 **a** $\overrightarrow{AB} = \begin{pmatrix} 2-4 \\ 1-2 \\ 5--1 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ 6 \end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix} 9-4 \\ 4-2 \\ 1--1 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ 2 \end{pmatrix}$

$$\therefore \ \overrightarrow{AB} \bullet \overrightarrow{AC} = (-2)(5)+(-1)(2)+(6)(2) = -10-2+12 = 0$$

$$\therefore \ \overrightarrow{AB} \perp \overrightarrow{AC}$$

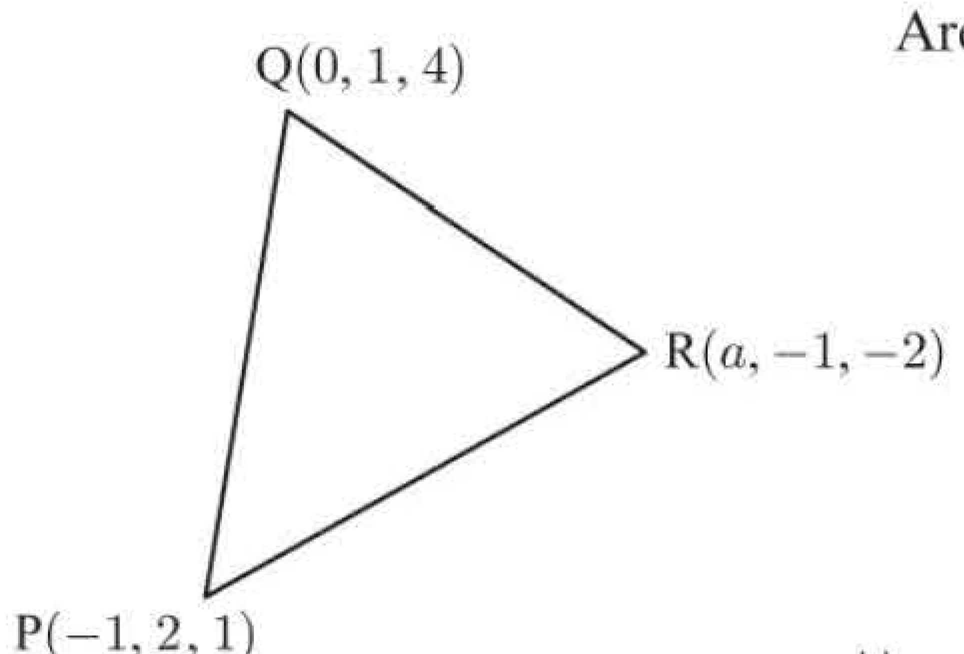
b **i** The equation is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix} + t\begin{pmatrix} -2 \\ -1 \\ 6 \end{pmatrix}, \quad t \in \mathbb{R}$

or $x = 4-2t, \quad y = 2-t, \quad z = -1+6t, \quad t \in \mathbb{R}$

ii The equation is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix} + s\begin{pmatrix} 5 \\ 2 \\ 2 \end{pmatrix}, \quad s \in \mathbb{R}$

or $x = 4+5s, \quad y = 2+2s, \quad z = -1+2s, \quad s \in \mathbb{R}$

8



$$\text{Area of } \triangle = \tfrac{1}{2}\,|\overrightarrow{PQ} \times \overrightarrow{PR}| = \tfrac{1}{2}\left|\begin{pmatrix} 0--1 \\ 1-2 \\ 4-1 \end{pmatrix} \times \begin{pmatrix} a--1 \\ -1-2 \\ -2-1 \end{pmatrix}\right|$$

$$= \tfrac{1}{2}\left|\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} \times \begin{pmatrix} a+1 \\ -3 \\ -3 \end{pmatrix}\right|$$

$$\therefore \ \tfrac{1}{2}\left\|\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 3 \\ a+1 & -3 & -3 \end{vmatrix}\right\| = \sqrt{118}$$

$$\therefore \ \left|\begin{vmatrix} -1 & 3 \\ -3 & -3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 3 \\ a+1 & -3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & -1 \\ a+1 & -3 \end{vmatrix}\mathbf{k}\right| = 2\sqrt{118}$$

$$|12\mathbf{i} - (-3-3a-3)\mathbf{j} + (-3+a+1)\mathbf{k}| = 2\sqrt{118}$$

$$\therefore \ \sqrt{144+(3a+6)^2+(a-2)^2} = 2\sqrt{118}$$

$$\therefore \ 144+9a^2+36a+36+a^2-4a+4 = 472$$

$$\therefore \ 10a^2+32a-288 = 0$$

$$\therefore \ 5a^2+16a-144 = 0$$

$$\therefore \ (5a+36)(a-4) = 0$$

$$\therefore \ a = -\tfrac{36}{5} \text{ or } 4$$

9 Given: A$(-1, 2, 3)$, B$(2, 0, -1)$, and C$(-3, 2, -4)$

a $\overrightarrow{AB} = \begin{pmatrix} 3 \\ -2 \\ -4 \end{pmatrix} \quad \overrightarrow{AC} = \begin{pmatrix} -2 \\ 0 \\ -7 \end{pmatrix} \quad \therefore$ a normal vector to the plane is

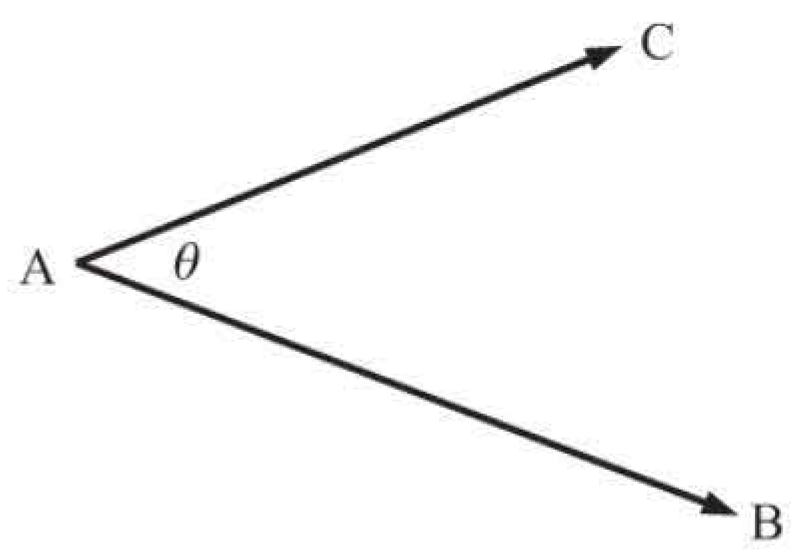
$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 0 & -7 \\ 3 & -2 & -4 \end{vmatrix} = \begin{vmatrix} 0 & -7 \\ -2 & -4 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -2 & -7 \\ 3 & -4 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -2 & 0 \\ 3 & -2 \end{vmatrix}\mathbf{k}$$

$$= -14\mathbf{i} - 29\mathbf{j} + 4\mathbf{k}$$

$\therefore$ since B lies on the plane, it has equation $14x+29y-4z = 14(2)+29(0)-4(-1)$

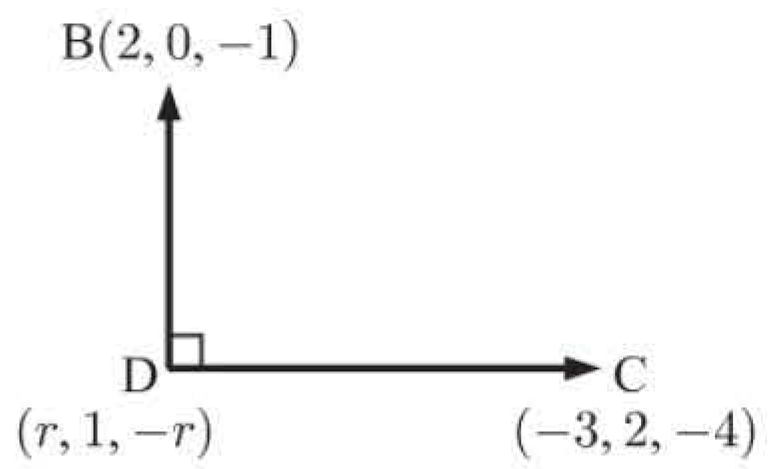
$$\therefore \ 14x+29y-4z = 32$$

b



$$\cos\theta = \frac{|\overrightarrow{AB} \bullet \overrightarrow{AC}|}{|\overrightarrow{AB}||\overrightarrow{AC}|}$$
$$= \frac{|3 \times -2 + -2 \times 0 + -4 \times -7|}{\sqrt{9+4+16}\sqrt{4+0+49}}$$
$$= \frac{22}{\sqrt{29 \times 53}} \quad \text{and so} \quad \theta \approx 55.9^\circ$$

c



If D is at $(r,\ 1,\ -r)$ then $\overrightarrow{DB} = \begin{pmatrix} 2-r \\ -1 \\ -1+r \end{pmatrix}$

and $\overrightarrow{DC} = \begin{pmatrix} -3-r \\ 1 \\ -4+r \end{pmatrix}$

Now $\mathrm{B}\widehat{\mathrm{D}}\mathrm{C}$ is a right angle, so $\overrightarrow{DB} \bullet \overrightarrow{DC} = 0$

$$\therefore\ (2-r)(-3-r) + (-1) + (-1+r)(-4+r) = 0$$
$$\therefore\ -6 - 2r + 3r + r^2 - 1 + 4 - r - 4r + r^2 = 0$$
$$\therefore\ 2r^2 - 4r - 3 = 0$$
$$\therefore\ r = \frac{4 \pm \sqrt{16+24}}{4}$$
$$\therefore\ r = \frac{2 \pm \sqrt{10}}{2}$$

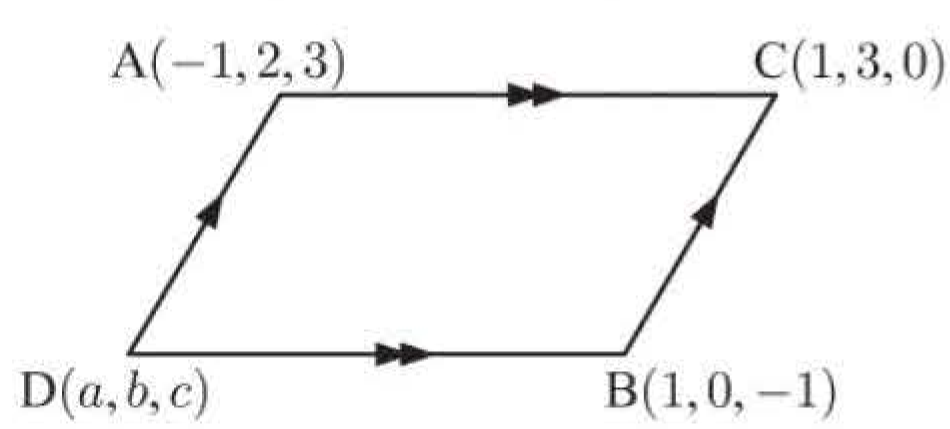
10 **a** Given $A(-1,\ 2,\ 3)$, $B(1,\ 0,\ -1)$, and $C(1,\ 3,\ 0)$,

$$\overrightarrow{AB} = \begin{pmatrix} 2 \\ -2 \\ -4 \end{pmatrix} = 2\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \quad \text{and} \quad \overrightarrow{AC} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$$

$\therefore$ a normal to the plane containing A, B, and C is

$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & -2 \\ 2 & 1 & -3 \end{vmatrix} = \begin{vmatrix} -1 & -2 \\ 1 & -3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & -2 \\ 2 & -3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix}\mathbf{k} = 5\mathbf{i} - \mathbf{j} + 3\mathbf{k} \quad \text{or} \quad \begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix}$$

b



Suppose D has coordinates $(a,\ b,\ c)$

$\therefore$ since $\overrightarrow{AD} = \overrightarrow{CB}$, $\begin{pmatrix} a+1 \\ b-2 \\ c-3 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \\ -1 \end{pmatrix}$

$\therefore\ a = -1,\quad b = -1 \quad \text{and} \quad c = 2$

$\therefore$ D is at $(-1,\ -1,\ 2)$

c From **a**, $\overrightarrow{AC} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$ and from **b**, $\overrightarrow{AD} = \begin{pmatrix} 0 \\ -3 \\ -1 \end{pmatrix}$

$$\overrightarrow{AC} \times \overrightarrow{AD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -3 \\ 0 & -3 & -1 \end{vmatrix} = \begin{vmatrix} 1 & -3 \\ -3 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & -3 \\ 0 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & 1 \\ 0 & -3 \end{vmatrix}\mathbf{k}$$
$$= -10\mathbf{i} + 2\mathbf{j} - 6\mathbf{k}$$

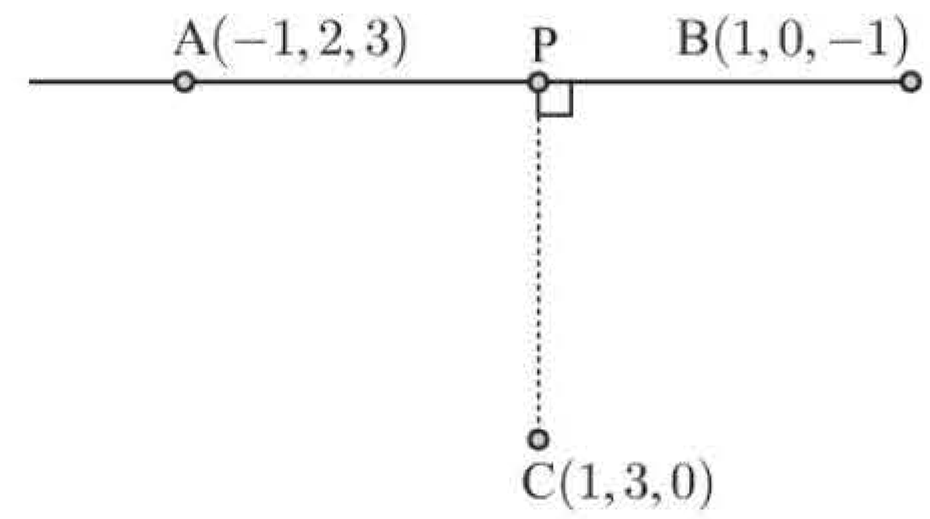
$\therefore$ area of parallelogram $= |-10\mathbf{i} + 2\mathbf{j} - 6\mathbf{k}|$

$$= \sqrt{100+4+36} = \sqrt{140} \approx 11.8 \text{ units}^2$$

d From **a**, $\overrightarrow{AB}$ has direction vector $\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$

$\therefore$ the line through A and B has parametric equations
$x = 1 + t,\quad y = 0 - t,\quad z = -1 - 2t,\quad t \in \mathbb{R}.$

If $P(1+t,\ -t,\ -1-2t)$ is the foot of the perpendicular,



then $\overrightarrow{CP} = \begin{pmatrix} t \\ -t-3 \\ -1-2t \end{pmatrix}$ and $\overrightarrow{CP} \bullet \overrightarrow{AB} = 0$

$$\therefore \begin{pmatrix} t \\ -t-3 \\ -1-2t \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} = 0$$
$$\therefore t + t + 3 + 2 + 4t = 0$$
$$\therefore 6t = -5$$
$$\therefore t = -\tfrac{5}{6}$$

$\therefore$ P is $(1-\frac{5}{6},\ \frac{5}{6},\ -1+\frac{10}{6})$ or $(\frac{1}{6},\ \frac{5}{6},\ \frac{2}{3})$.

11 $P(2,\ 0,\ 1)$, $Q(3,\ 4,\ -2)$, $R(-1,\ 3,\ 2)$

a $\overrightarrow{PQ} = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}$

$|\overrightarrow{PQ}| = \sqrt{1+16+9} = \sqrt{26}$ units

and $\overrightarrow{QR} = \begin{pmatrix} -4 \\ -1 \\ 4 \end{pmatrix}$

b Since $\overrightarrow{PQ} = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}$ and P is at $(2,\ 0,\ 1)$,

the line has equation
$x = 2 + \lambda,\quad y = 0 + 4\lambda,\quad z = 1 - 3\lambda$
$\therefore\ x = 2 + \lambda,\quad y = 4\lambda,\quad z = 1 - 3\lambda,\quad \lambda \in \mathbb{R}$

c The vector equation of the plane is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} -4 \\ -1 \\ 4 \end{pmatrix}$, $\lambda,\ \mu \in \mathbb{R}$.

12 **a** Given $A(-1,\ 3,\ 2)$ and the plane $2x - y + 2z = 8$,

the distance from A to the plane is $d = \dfrac{|2x_1 - y_1 + 2z_1 - 8|}{\sqrt{2^2 + (-1)^2 + 2^2}} = \dfrac{|2(-1) - 3 + 2(2) - 8|}{3}$

$$= \frac{|-9|}{3} = 3 \text{ units}$$

b The point on the plane nearest A is the foot of the normal to the plane that passes through A.

Since the normal has direction vector $\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$ and passes through $(-1,\ 3,\ 2)$,

it has equation $x = -1 + 2t,\ y = 3 - t,\ z = 2 + 2t,\ t \in \mathbb{R}$

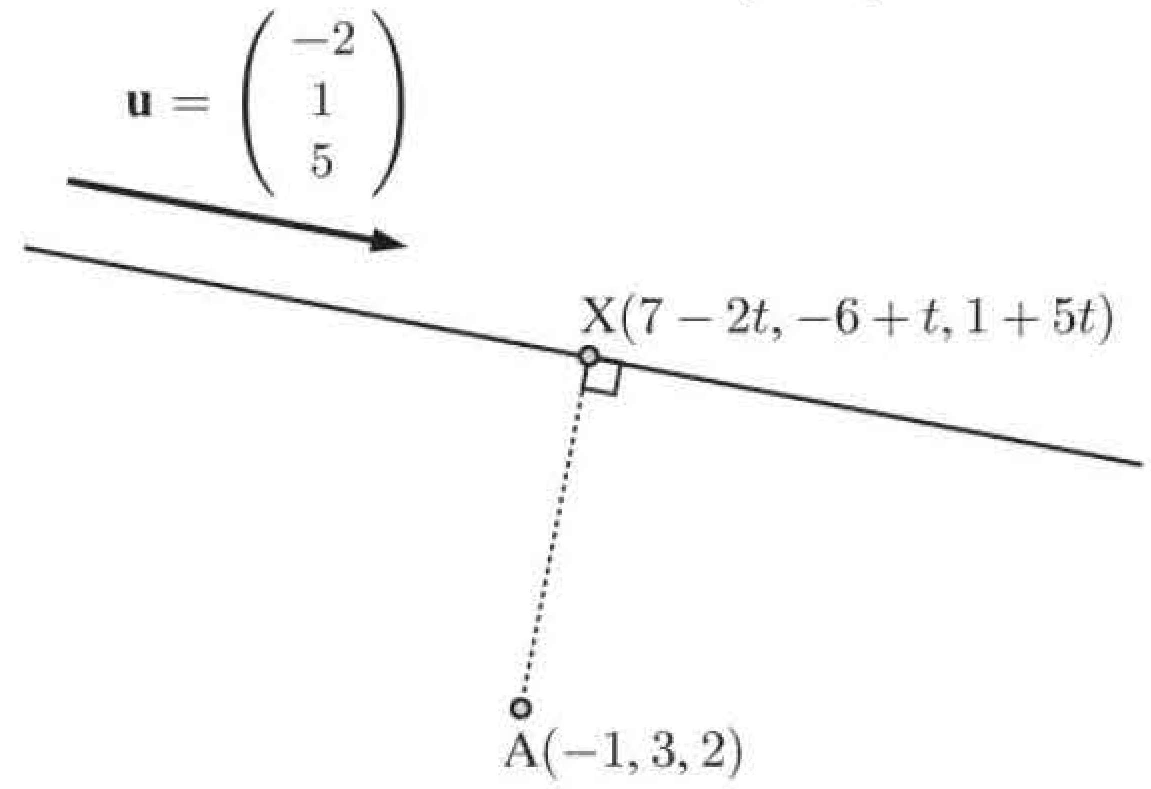
and meets the plane when $2(-1+2t) - (3-t) + 2(2+2t) = 8$

$$\therefore -2 + 4t - 3 + t + 4 + 4t = 8$$
$$\therefore 9t = 9$$
$$\therefore t = 1 \quad \therefore \text{ the point is } (1,\ 2,\ 4).$$

c Suppose X is the foot of the perpendicular from A to the line, so X has coordinates $(7-2t,\ -6+t,\ 1+5t)$ for some $t \in \mathbb{R}$. Then the shortest distance from A to the line is AX.

Now $\overrightarrow{AX} = \begin{pmatrix} 8-2t \\ t-9 \\ -1+5t \end{pmatrix}$ and since the line has direction vector $\mathbf{u} = \begin{pmatrix} -2 \\ 1 \\ 5 \end{pmatrix}$,

$$\mathbf{u} \bullet \overrightarrow{AX} = 0$$
$$\begin{pmatrix} -2 \\ 1 \\ 5 \end{pmatrix} \bullet \begin{pmatrix} 8-2t \\ t-9 \\ -1+5t \end{pmatrix} = 0$$
$$-16 + 4t + t - 9 - 5 + 25t = 0$$
$$\therefore 30t = 30$$
$$\therefore t = 1$$



$$\therefore |AX| = \sqrt{6^2 + (-8)^2 + 4^2}$$
$$= \sqrt{36 + 64 + 16} = \sqrt{116} \text{ units}$$

13 Given A$(-1, 0, 2)$, B$(0, -1, 1)$, and C$(1, 2, -1)$

a $\overrightarrow{AB} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix}$

$\therefore \quad \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & -1 \\ 2 & 2 & -3 \end{vmatrix} = \begin{vmatrix} -1 & -1 \\ 2 & -3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & -1 \\ 2 & -3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & -1 \\ 2 & 2 \end{vmatrix}\mathbf{k} = \begin{pmatrix} 5 \\ 1 \\ 4 \end{pmatrix}$

$\therefore$ since A lies on the plane it has equation $5x + y + 4z = 5(-1) + 0 + 4(2)$

$\therefore \quad 5x + y + 4z = 3$

b Since the normal has direction $\begin{pmatrix} 5 \\ 1 \\ 4 \end{pmatrix}$ and passes through $(0, 0, 0)$, it has equation

$x = 0 + 5t, \quad y = 0 + t, \quad z = 0 + 4t$

$\therefore \quad x = 5t, \quad y = t, \quad z = 4t, \quad t \in \mathbb{R}$

c The line meets the plane when $5(5t) + t + 4(4t) = 3$

$\therefore \quad 25t + t + 16t = 3$

$\therefore \quad 42t = 3$

$\therefore \quad t = \frac{1}{14}$

So, the line meets the plane at $(\frac{5}{14}, \frac{1}{14}, \frac{2}{7})$.

14 **a** $\dfrac{x-3}{2} = \dfrac{y-4}{1} = \dfrac{z+1}{-2}$ has direction vector $\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$

while $x = -1 + 3t, \quad y = 2 + 2t, \quad z = 3 - t$ has direction vector $\mathbf{v}_2 = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$

$\therefore$ the lines are not parallel.

If the lines intersect, then $\dfrac{-1+3t-3}{2} = \dfrac{2+2t-4}{1} = \dfrac{3-t+1}{-2}$

$\therefore \quad \frac{3}{2}t - 2 = 2t - 2 = \frac{t}{2} - 2$

Now $t = 0$ satisfies this relation, so the lines intersect at $(-1, 2, 3)$.

b If θ is the acute angle between the lines, then

$\cos\theta = \dfrac{|\mathbf{v}_1 \bullet \mathbf{v}_2|}{|\mathbf{v}_1|\,|\mathbf{v}_2|} = \dfrac{|2 \times 3 + 1 \times 2 + -2 \times -1|}{\sqrt{9}\sqrt{14}} = \dfrac{|6+2+2|}{3\sqrt{14}} = \dfrac{10}{3\sqrt{14}}$

$\therefore \quad \theta \approx 27.0°$

15 **a** The lines meet when

$\dfrac{(15+3\lambda)-8}{3} = \dfrac{(29+8\lambda)+9}{-16} = \dfrac{(5-5\lambda)-10}{7}$ for some $\lambda \in \mathbb{R}$

$\therefore \quad \dfrac{3\lambda+7}{3} = \dfrac{8\lambda+38}{-16} = \dfrac{-5\lambda-5}{7}$

$\therefore \quad -48\lambda - 112 = 24\lambda + 114$ and $56\lambda + 266 = 80\lambda + 80$

$\therefore \quad -72\lambda = 226$ and $186 = 24\lambda$

$\therefore \quad \lambda = -\frac{113}{36}$ and $\lambda = \frac{31}{4}$

$\therefore$ no value of λ satisfies both equations.

$\therefore$ the lines do not meet.

Their direction vectors are $\begin{pmatrix} 3 \\ -16 \\ 7 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 8 \\ -5 \end{pmatrix}$ $\therefore$ they are not parallel.

$\therefore$ line 1 and line 2 are skew.

b Line 3 is parallel to line 1 and so has direction vector $\begin{pmatrix} 3 \\ -16 \\ 7 \end{pmatrix}$.

$\therefore$ the plane containing lines 2 and 3 has normal vector

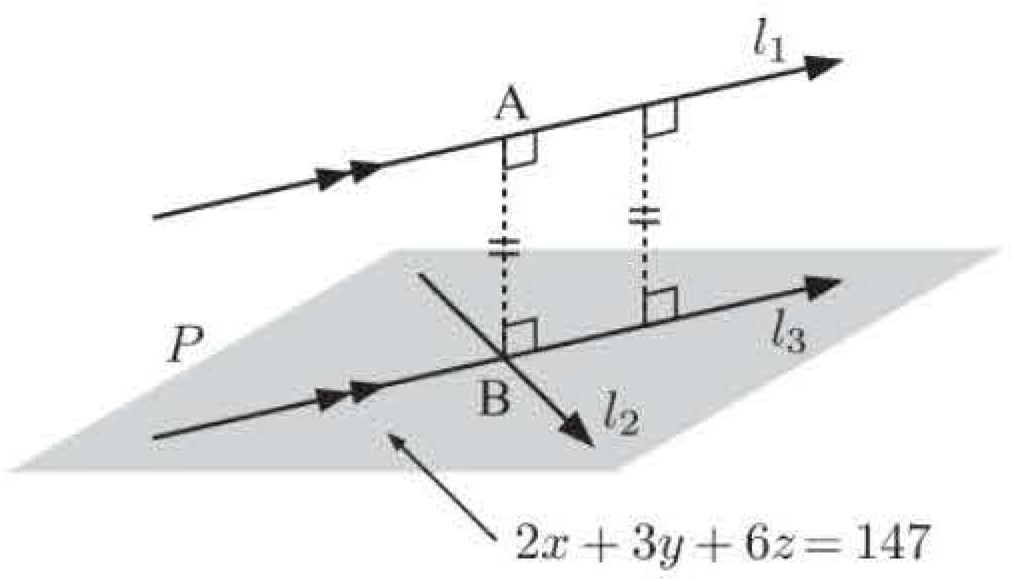
$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 8 & -5 \\ 3 & -16 & 7 \end{vmatrix} = \begin{vmatrix} 8 & -5 \\ -16 & 7 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & -5 \\ 3 & 7 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & 8 \\ 3 & -16 \end{vmatrix}\mathbf{k}$$
$$= -24\mathbf{i} - 36\mathbf{j} - 72\mathbf{k}$$
$$= -12(2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k})$$

$\therefore$ the equation of the plane is $2x + 3y + 6z = 2(15) + 3(29) + 6(5)$

$\therefore$ $2x + 3y + 6z = 147$

c Since line 1 is parallel to line 3, line 1 is parallel to the plane containing lines 2 and 3.

$\therefore$ to find the shortest distance between lines 1 and 2, we choose a point on line 1 then find the shortest distance between this point and the plane.



$(8, -9, 10)$ is a point on line 1.

$$\therefore \text{ distance } d = \frac{|2(8) + 3(-9) + 6(10) - 147|}{\sqrt{2^2 + 3^2 + 6^2}}$$
$$= \frac{|-98|}{\sqrt{49}}$$
$$= \tfrac{98}{7}$$
$$= 14 \text{ units}$$

d Let the common perpendicular meet lines 1 and 2 at A and B respectively.

A is $(8 + 3\mu, -9 - 16\mu, 10 + 7\mu)$ for some $\mu \in \mathbb{R}$

B is $(15 + 3\lambda, 29 + 8\lambda, 5 - 5\lambda)$ for some $\lambda \in \mathbb{R}$

The common perpendicular has direction vector $\begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}$

$\therefore$ we need to find values for μ and λ so that

$$\overrightarrow{AB} = k\begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix} \quad \text{for some scalar } k$$

$$\therefore \begin{pmatrix} 15 + 3\lambda - (8 + 3\mu) \\ 29 + 8\lambda - (-9 - 16\mu) \\ 5 - 5\lambda - (10 + 7\mu) \end{pmatrix} = k\begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}$$

$$\therefore \begin{pmatrix} 7 + 3\lambda - 3\mu \\ 38 + 8\lambda + 16\mu \\ -5 - 5\lambda - 7\mu \end{pmatrix} = \begin{pmatrix} 2k \\ 3k \\ 6k \end{pmatrix}$$

$$\therefore \begin{cases} 3\lambda - 3\mu - 2k = -7 \\ 8\lambda + 16\mu - 3k = -38 \\ -5\lambda - 7\mu - 6k = 5 \end{cases}$$

Solving simultaneously using technology gives $\lambda = -2$, $\mu = -1$, $k = 2$.

$\therefore$ A is $(8 + 3(-1), -9 - 16(-1), 10 + 7(-1))$

and B is $(15 + 3(-2), 29 + 8(-2), 5 - 5(-2))$

$\therefore$ the common perpendicular meets lines 1 and 2 at $(5, 7, 3)$ and $(9, 13, 15)$.

16 **a** The lines meet where

$$\begin{pmatrix}3\\-2\\-2\end{pmatrix}+s\begin{pmatrix}-1\\1\\2\end{pmatrix}=\begin{pmatrix}3\\0\\-1\end{pmatrix}+t\begin{pmatrix}-1\\-1\\1\end{pmatrix}$$

$$\therefore \begin{pmatrix}-s\\s\\2s\end{pmatrix}=\begin{pmatrix}-t\\-t+2\\t+1\end{pmatrix}$$

$$\therefore \begin{cases}-s=-t \;\Rightarrow\; s=t & \text{.... (1)}\\ s=-t+2 & \text{.... (2)}\\ 2s=t+1 & \text{.... (3)}\end{cases}$$

Substituting (1) into (2) gives

$t=-t+2$

$\therefore\ 2t=2$

$\therefore\ t=1$ and $s=1$

Checking with (3): $2(1)=1+1$ ✓

$$\therefore \begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}3\\-2\\-2\end{pmatrix}+1\begin{pmatrix}-1\\1\\2\end{pmatrix}=\begin{pmatrix}2\\-1\\0\end{pmatrix}$$

$\therefore$ A is $(2,\ -1,\ 0)$

b $B(0,\ -3,\ 2)$ lies on L_2 if

$$\begin{pmatrix}3\\0\\-1\end{pmatrix}+t\begin{pmatrix}-1\\-1\\1\end{pmatrix}=\begin{pmatrix}0\\-3\\2\end{pmatrix}$$

for some $t\in\mathbb{R}$

$$\therefore \begin{cases}3-t=0\\ -t=-3\\ -1+t=2\end{cases}$$

$t=3$ satisfies all these equations

$\therefore$ $B(0,\ -3,\ 2)$ lies on L_2

c $\overrightarrow{BC}=\begin{pmatrix}3\\1\\-4\end{pmatrix}$

$\therefore$ the equation of the line (BC) is

$$\mathbf{r}=\begin{pmatrix}0\\-3\\2\end{pmatrix}+u\begin{pmatrix}3\\1\\-4\end{pmatrix},\quad u\in\mathbb{R}$$

d $\overrightarrow{AB}=\begin{pmatrix}-2\\-2\\2\end{pmatrix}$, $\overrightarrow{AC}=\begin{pmatrix}1\\-1\\-2\end{pmatrix}$

$\therefore$ the plane containing A, B, and C has normal vector

$$\mathbf{n}=\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-2&-2&2\\1&-1&-2\end{vmatrix}=\begin{vmatrix}-2&2\\-1&-2\end{vmatrix}\mathbf{i}-\begin{vmatrix}-2&2\\1&-2\end{vmatrix}\mathbf{j}+\begin{vmatrix}-2&-2\\1&-1\end{vmatrix}\mathbf{k}$$

$$=6\mathbf{i}-2\mathbf{j}+4\mathbf{k}$$

$\therefore$ the equation is $6x-2y+4z=6(2)-2(-1)+4(0)=14$ {using A}

or $3x-y+2z=7$

e Area of triangle ABC

$=\frac{1}{2}\,|\overrightarrow{AB}\times\overrightarrow{AC}|$

$=\frac{1}{2}\,|6\mathbf{i}-2\mathbf{j}+4\mathbf{k}|$

$=\frac{1}{2}\sqrt{6^2+(-2)^2+4^2}$

$=\frac{1}{2}\sqrt{56}$

$=\sqrt{14}$ units^2

f The normal to the plane has direction vector $\begin{pmatrix}3\\-1\\2\end{pmatrix}$.

$\therefore$ the normal to the plane passing through $C(3,\ -2,\ -2)$ is

$$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}3\\-2\\-2\end{pmatrix}+\lambda\begin{pmatrix}3\\-1\\2\end{pmatrix},\quad \lambda\in\mathbb{R}$$

$D(9,\ -4,\ 2)$ lies on this line if

$$\begin{pmatrix}3\\-2\\-2\end{pmatrix}+\lambda\begin{pmatrix}3\\-1\\2\end{pmatrix}=\begin{pmatrix}9\\-4\\2\end{pmatrix}\quad \text{for some } \lambda\in\mathbb{R}$$

$$\therefore \begin{cases}3+3\lambda=9\\ -2-\lambda=-4\\ -2+2\lambda=2\end{cases}$$

$\lambda=2$ satisfies all three equations

$\therefore$ D lies on this line.

17 **a** $\left[\begin{array}{ccc|c} 1 & 3 & 2 & 5 \\ 2 & 1 & 9 & 20 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 2 & 5 \\ 0 & -5 & 5 & 10 \end{array}\right] \quad R_2 \to R_2 - 2R_1$

$\sim \left[\begin{array}{ccc|c} 1 & 3 & 2 & 5 \\ 0 & -1 & 1 & 2 \end{array}\right] \quad R_2 \to \frac{1}{5}R_2$

$\therefore\ -y+z=2$ and $x+3y+2z=5$

$\therefore$ if we let $y=s$, then $z=2+s$ and $x = 5-3y-2z$

$= 5-3s-2(2+s)$

$= 5-3s-4-2s$

$\therefore\ x = 1-5s$

$\therefore$ planes A and B intersect in the line L_1: $x=1-5s,\ y=s,\ z=2+s,\ s\in\mathbb{R}$

b $\left[\begin{array}{ccc|c} 1 & -1 & 6 & 8 \\ 2 & 1 & 9 & 20 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 6 & 8 \\ 0 & 3 & -3 & 4 \end{array}\right] \quad R_2 \to R_2 - 2R_1$

$\therefore\ 3y-3z=4$ and $x-y+6z=8$

$\therefore$ if we let $y=t$, then $z=-\frac{4}{3}+y=-\frac{4}{3}+t$ and $x=8+y-6z$

$=8+t-6(-\frac{4}{3}+t)$

$=8+t+8-6t$

$\therefore\ x=16-5t$

$\therefore$ planes B and C intersect in the line L_2: $x=16-5t,\ y=t,\ z=-\frac{4}{3}+t,\ t\in\mathbb{R}$

c $\left[\begin{array}{ccc|c} 1 & 3 & 2 & 5 \\ 1 & -1 & 6 & 8 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 2 & 5 \\ 0 & -4 & 4 & 3 \end{array}\right] \quad R_2 \to R_2 - R_1$

$\therefore\ -4y+4z=3$ and $x+3y+2z=5$

$\therefore$ if we let $y=u$, then $z=\frac{3}{4}+y=\frac{3}{4}+u$ and $x=5-3y-2z$

$=5-3u-2(\frac{3}{4}+u)$

$=5-3u-\frac{3}{2}-2u$

$\therefore\ x=\frac{7}{2}-5u$

$\therefore$ planes A and C intersect in the line L_3: $x=\frac{7}{2}-5u,\ y=u,\ z=\frac{3}{4}+u,\ u\in\mathbb{R}$

d L_1 is $\begin{pmatrix}1\\0\\2\end{pmatrix}+s\begin{pmatrix}-5\\1\\1\end{pmatrix}$ with direction vector $\mathbf{a}=\begin{pmatrix}-5\\1\\1\end{pmatrix}$.

L_2 is $\begin{pmatrix}16\\0\\-\frac{4}{3}\end{pmatrix}+t\begin{pmatrix}-5\\1\\1\end{pmatrix}$ with direction vector $\mathbf{b}=\begin{pmatrix}-5\\1\\1\end{pmatrix}$.

L_3 is $\begin{pmatrix}\frac{7}{2}\\0\\\frac{3}{4}\end{pmatrix}+u\begin{pmatrix}-5\\1\\1\end{pmatrix}$ with direction vector $\mathbf{c}=\begin{pmatrix}-5\\1\\1\end{pmatrix}$.

Since $\mathbf{a}=\mathbf{b}=\mathbf{c}$, L_1, L_2, and L_3 are a parallel.

When $s=0$, the point on L_1 is $(1, 0, 2)$.

For L_2, $y=t$, so the unique point on L_2 with y-coordinate 0 is the point where $t=0$. This point is $(16, 0, -\frac{4}{3})$.

For L_3, $y=u$, so the unique point on L_3 with y-coordinate 0 is the point where $u=0$. This point is $(\frac{7}{2}, 0, \frac{3}{4})$.

$(1, 0, 2)\neq(16, 0, -\frac{4}{3})$, $(16, 0, -\frac{4}{3})\neq(\frac{7}{2}, 0, \frac{3}{4})$, and $(1, 0, 2)\neq(\frac{7}{2}, 0, \frac{3}{4})$.

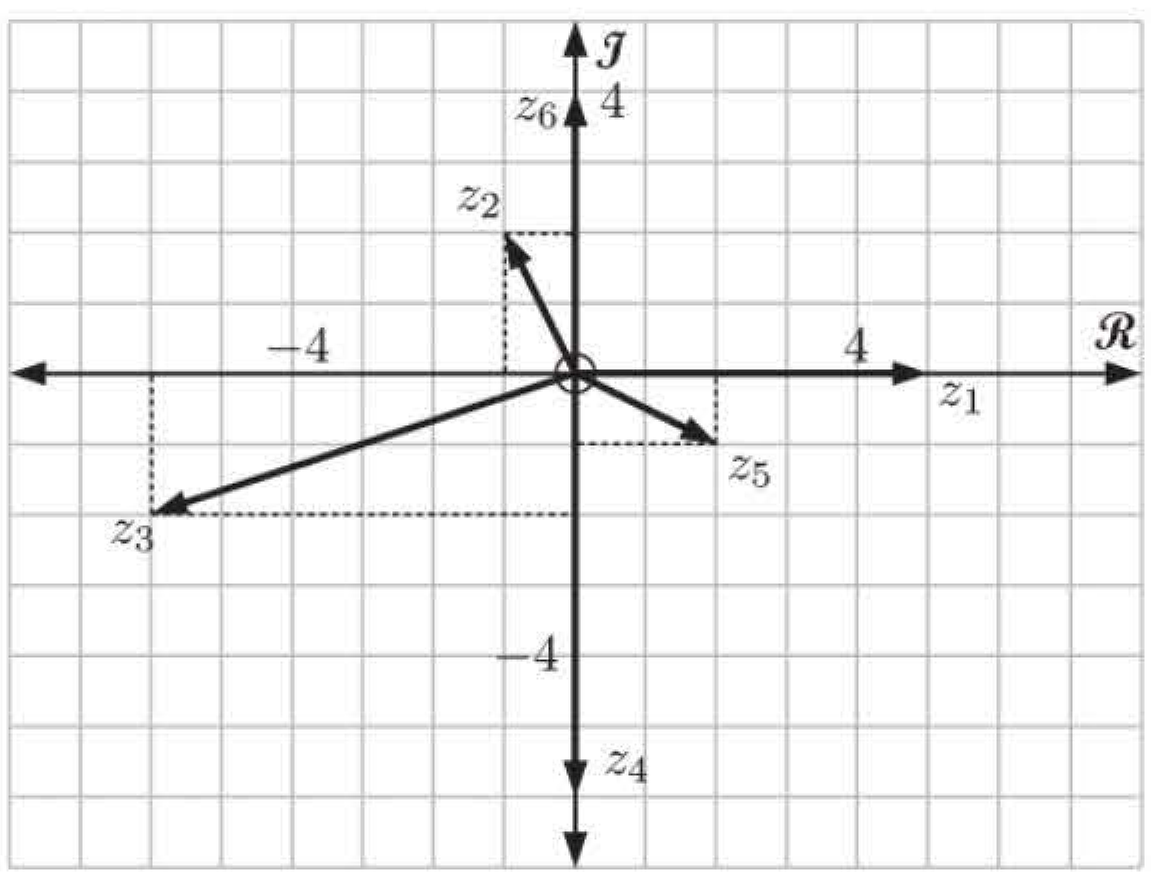
$\therefore$ none of the lines are coincident.

e The three planes have no common point of intersection. The line of intersection of any two planes is parallel to the third plane.

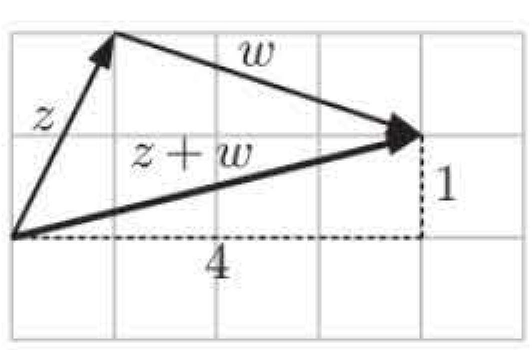
Chapter 16
COMPLEX NUMBERS

EXERCISE 16A

1

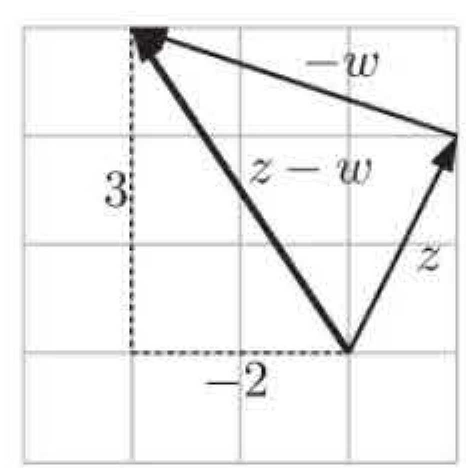


2 **a**



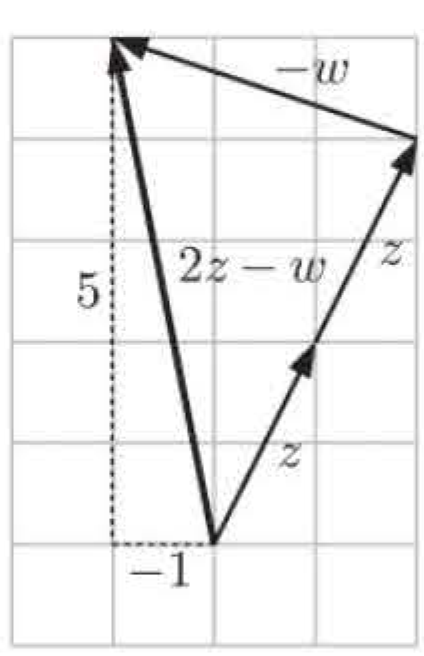
$$z + w$$
$$= (1+2i) + (3-i)$$
$$= 4 + i$$

b



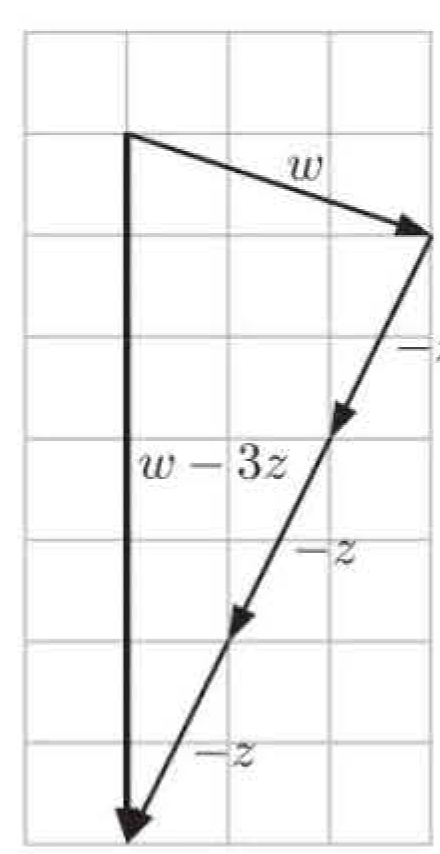
$$z - w$$
$$= (1+2i) - (3-i)$$
$$= 1 + 2i - 3 + i$$
$$= -2 + 3i$$

c



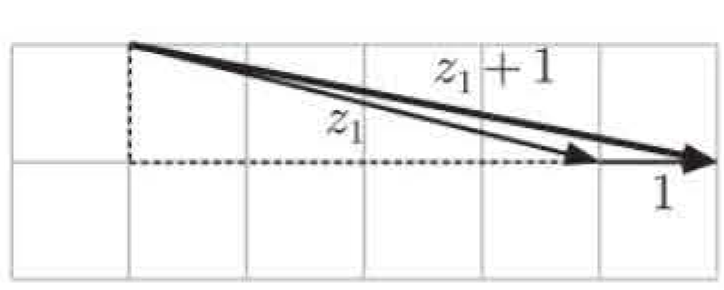
$$2z - w$$
$$= 2(1+2i) - (3-i)$$
$$= 2 + 4i - 3 + i$$
$$= -1 + 5i$$

d



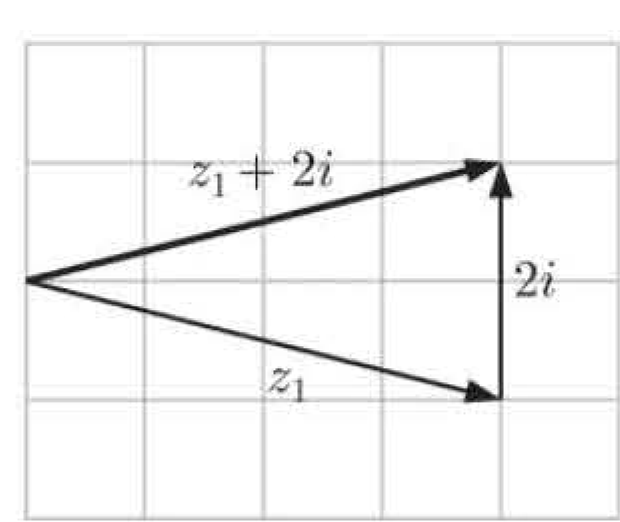
$$w - 3z$$
$$= (3-i) - 3(1+2i)$$
$$= 3 - i - 3 - 6i$$
$$= -7i$$

3 **a**



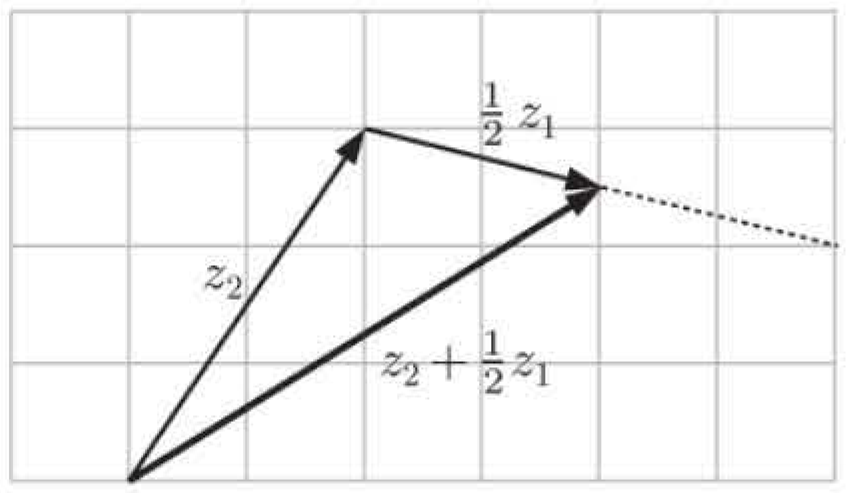
$$z_1 + 1$$
$$= 4 - i + 1$$
$$= 5 - i$$

b



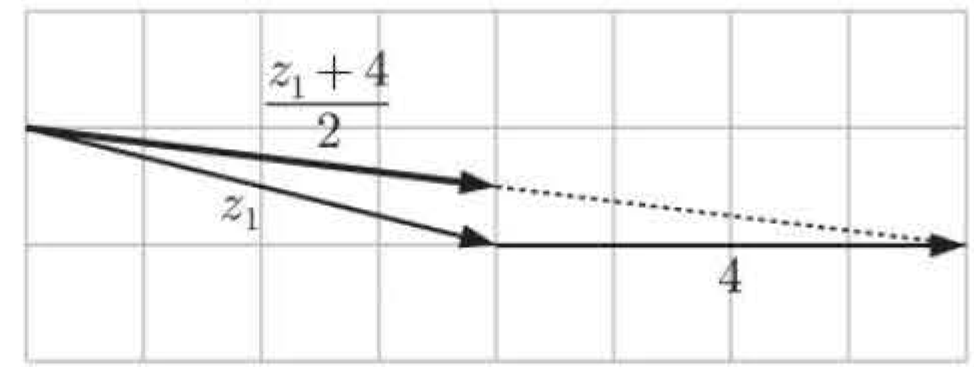
$$z_1 + 2i$$
$$= 4 - i + 2i$$
$$= 4 + i$$

c



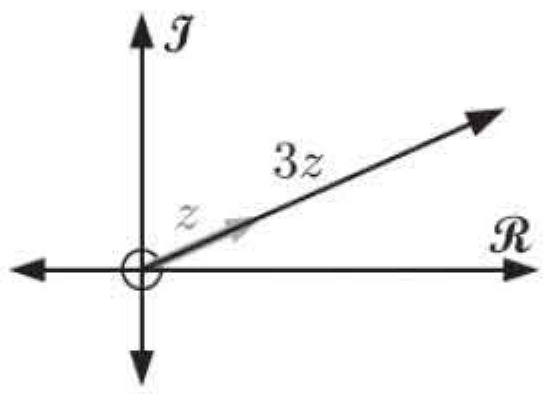
$$\begin{aligned} z_2 + \tfrac{1}{2}z_1 &= (2+3i) + \tfrac{1}{2}(4-i) \\ &= 2 + 3i + 2 - \tfrac{1}{2}i \\ &= 4 + \tfrac{5}{2}i \end{aligned}$$

d



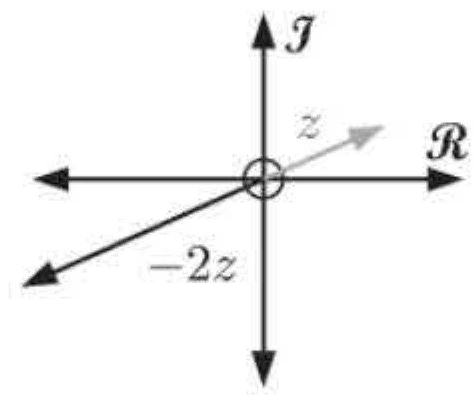
$$\begin{aligned} \frac{z_1+4}{2} &= \frac{4-i+4}{2} \\ &= \frac{8-i}{2} \\ &= 4 - \tfrac{1}{2}i \end{aligned}$$

4 **a**



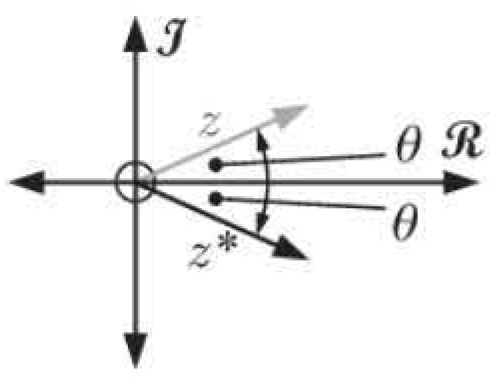
$3z$ is parallel to z and 3 times its length.

b



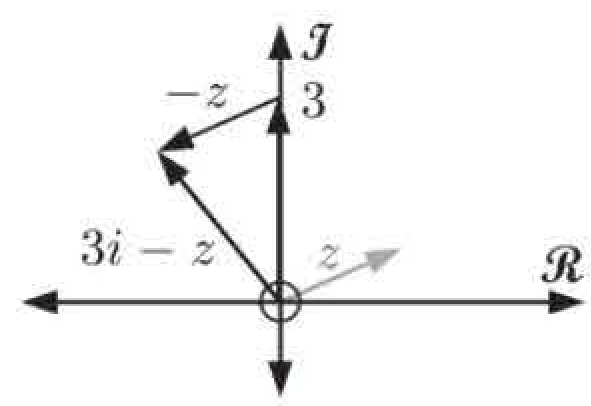
$-2z$ is parallel to z, in the opposite direction and twice its length.

c



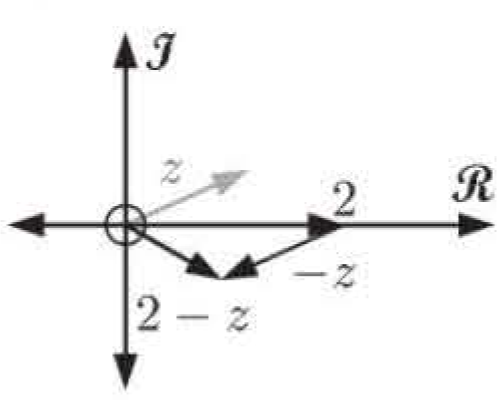
Reflect z in a horizontal line through the start of z.

d



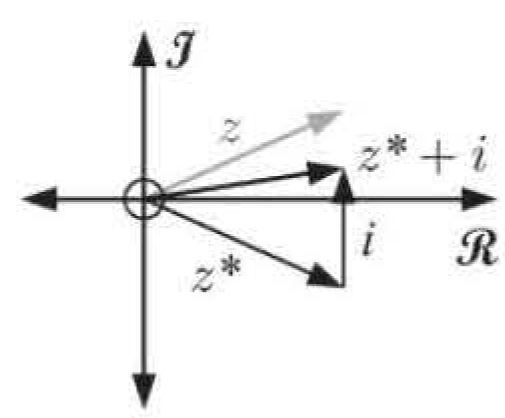
Add $-z$ to $3i$.

e



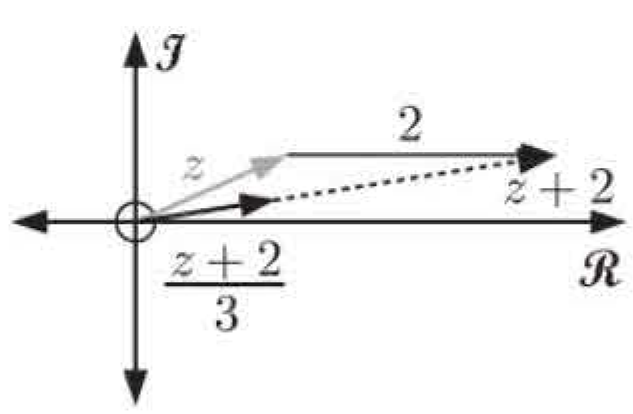
Add $-z$ to 2.

f



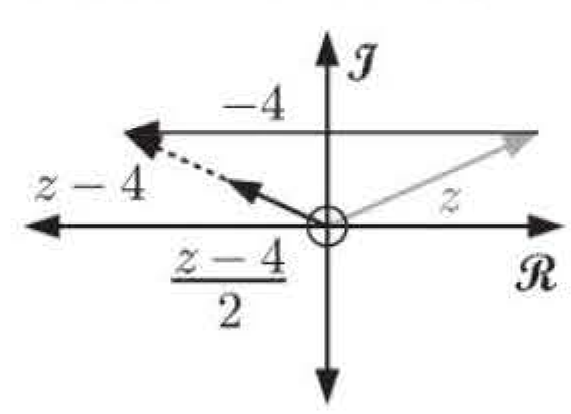
Reflect z in a horizontal line through the start of z and then add i.

g



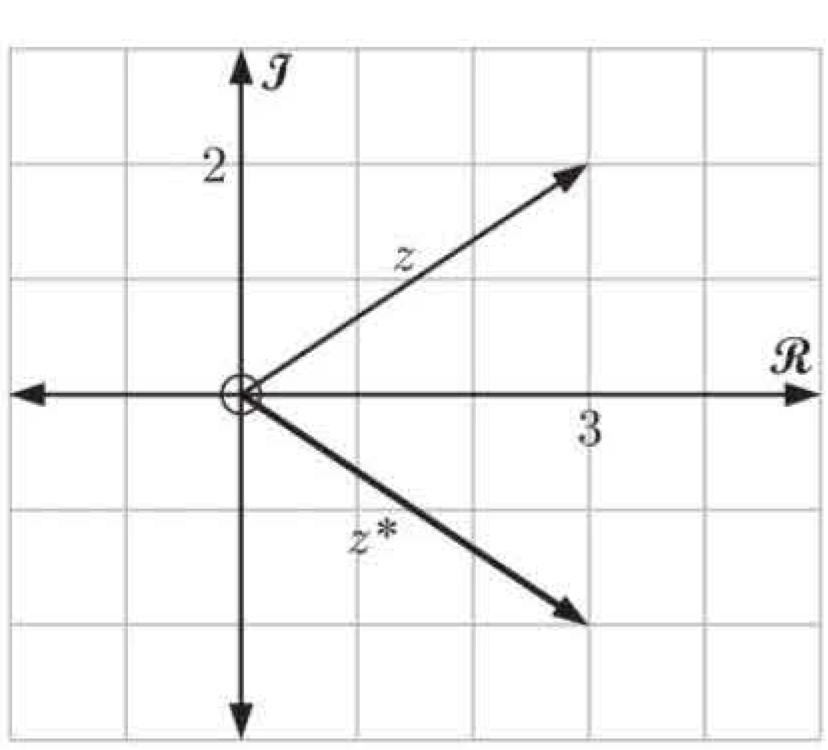
Add z and 2 and find the vector $\frac{1}{3}$ of the length of the result.

h

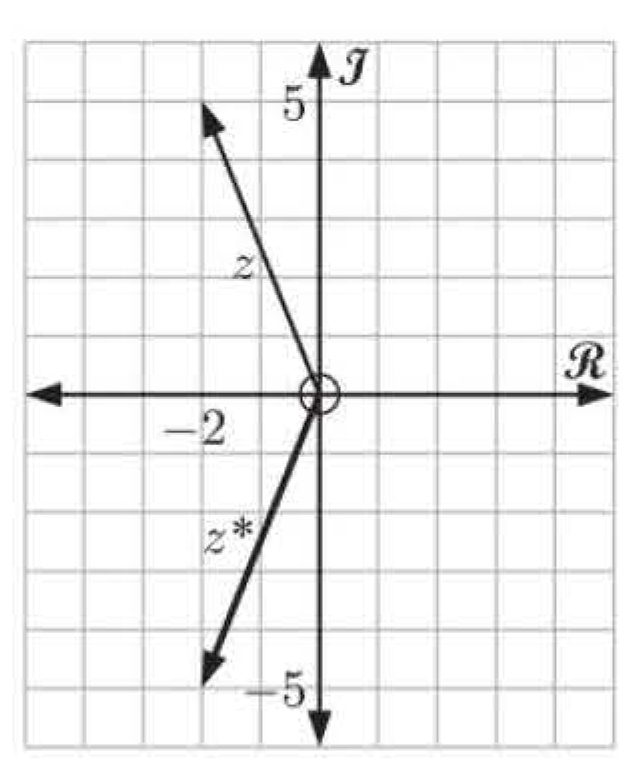


Add z and -4 and find the vector $\frac{1}{2}$ of the length of the result.

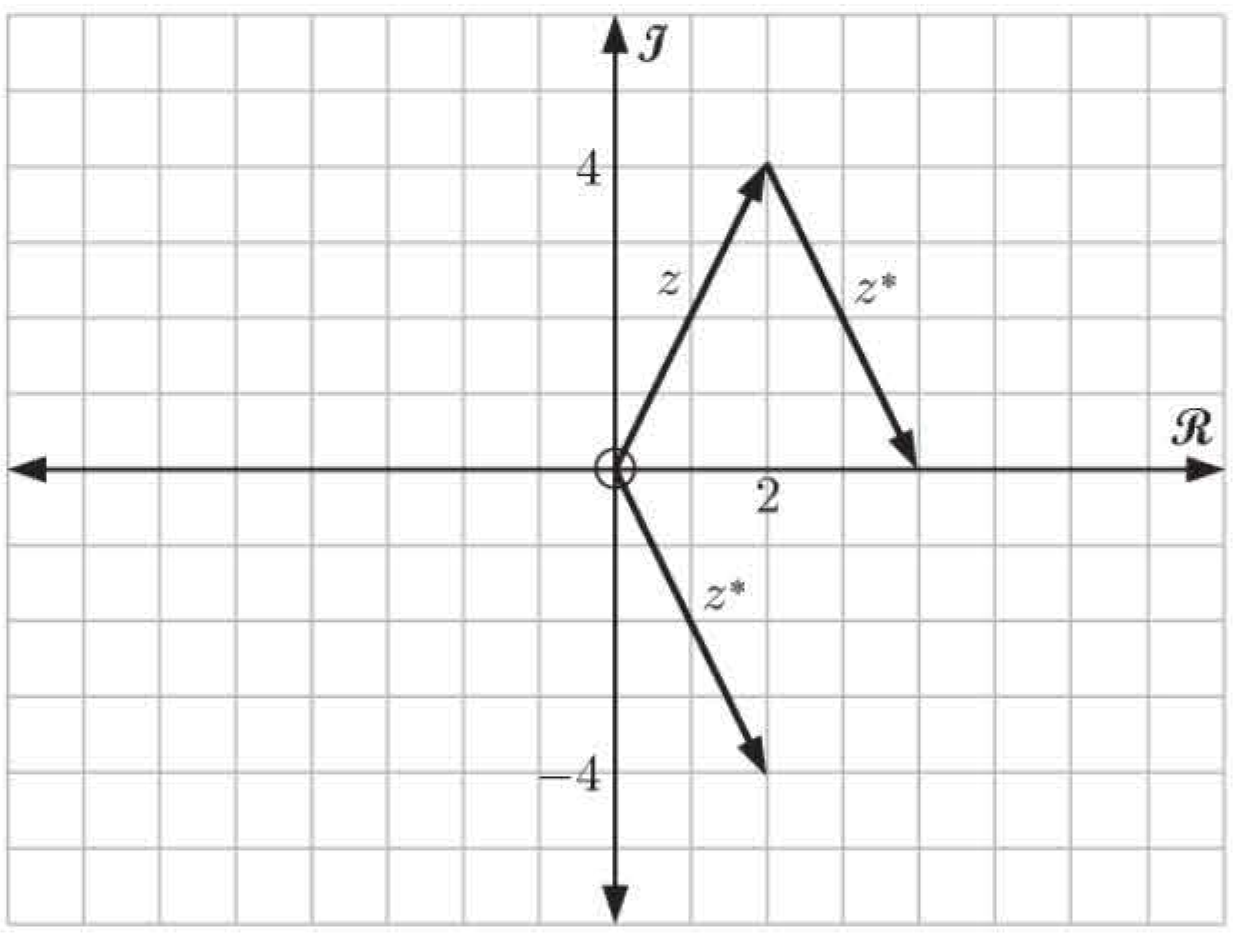
5 **a**



b



6



Suppose $z = a + bi$, where $a, b \in \mathbb{R}$

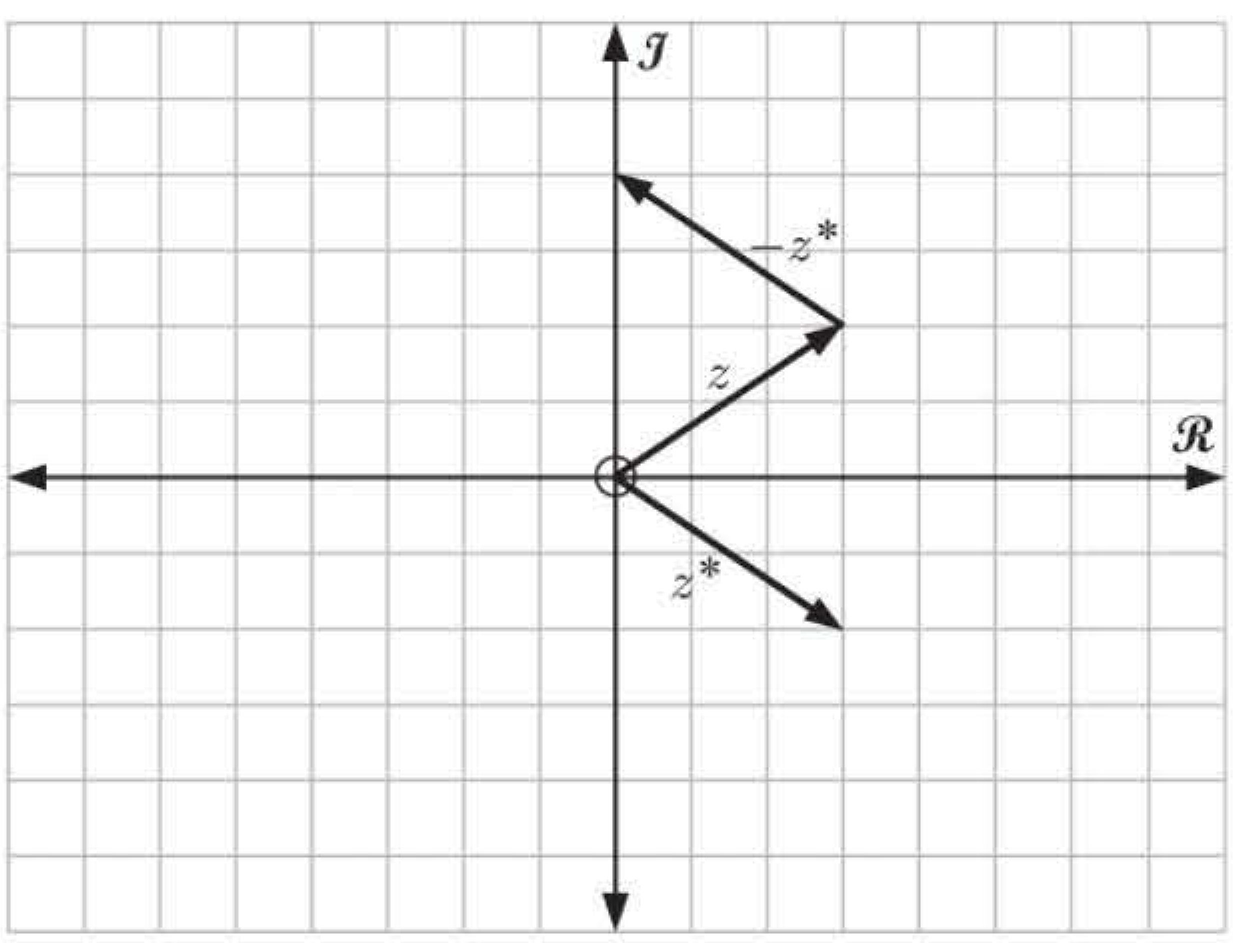
$\therefore\ z^* = a - bi$

and $z + z^* = a + \cancel{bi} + a - \cancel{bi}$

$= 2a$, which is real (since a is real)

$\therefore\ z + z^*$ is always real for any complex number z.

7



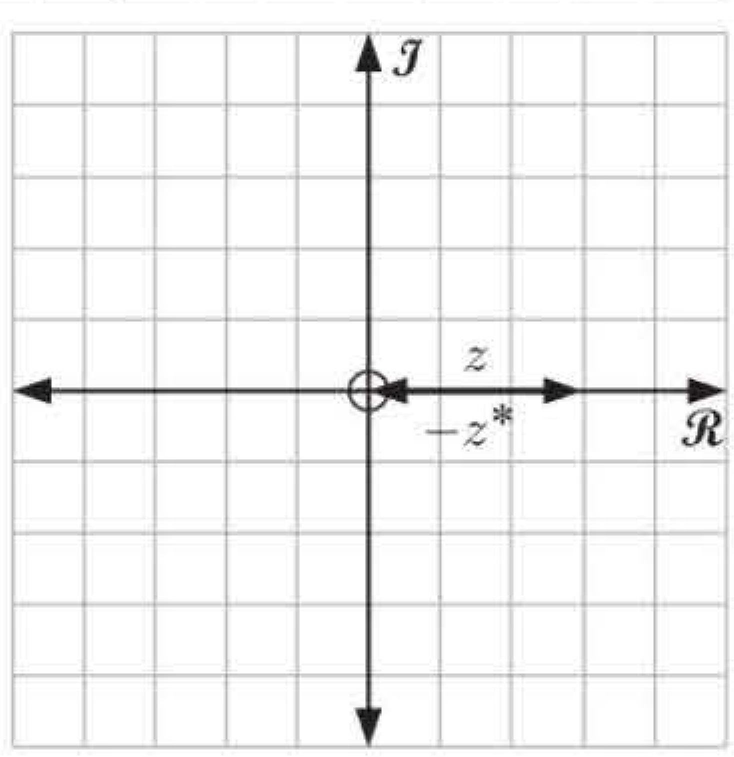
Suppose $z = a + bi$, where $a, b \in \mathbb{R}$

$\therefore\ z^* = a - bi$

and $z - z^* = (a + bi) - (a - bi)$

$= a + bi - a + bi$

$= 2bi$

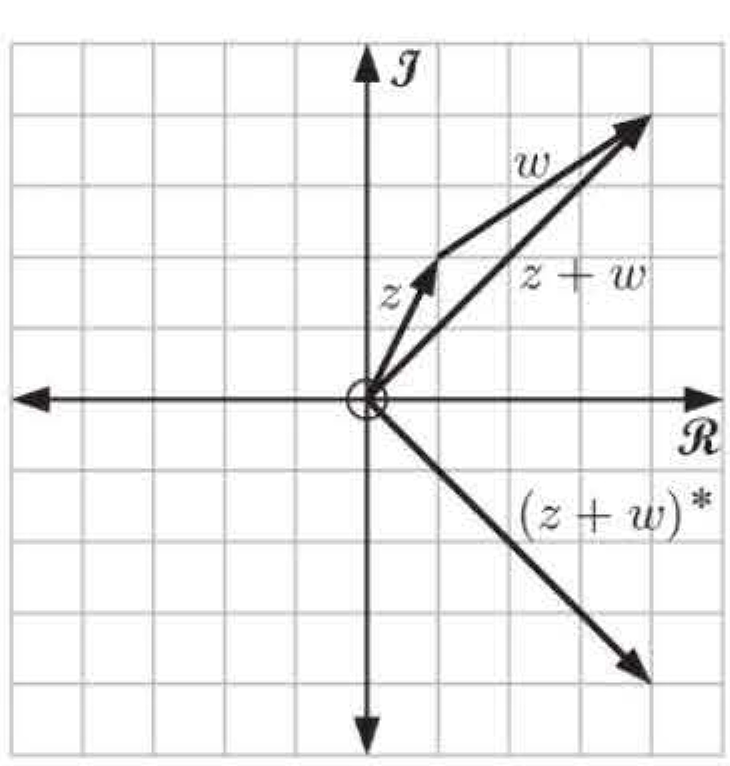


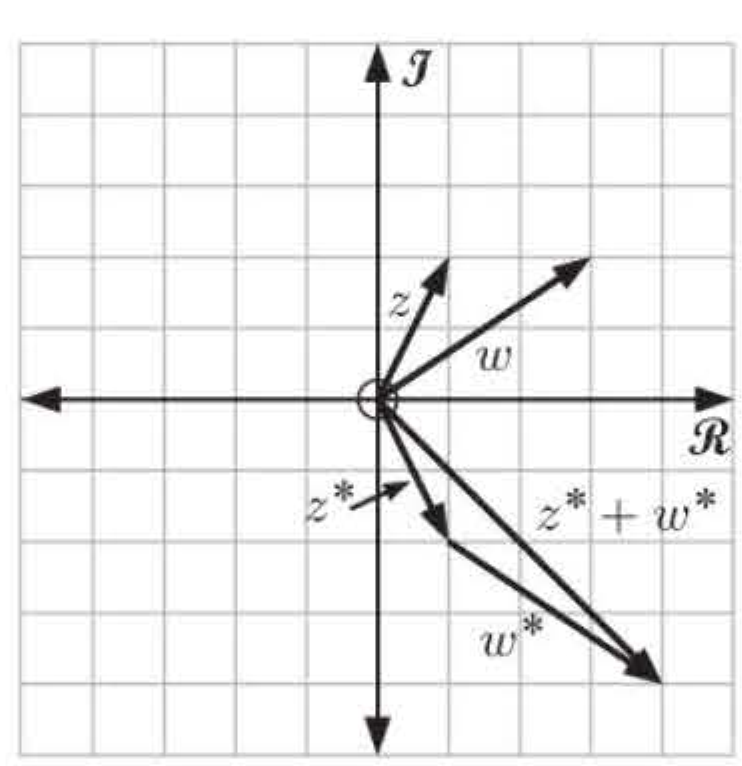
Since b is real, $z - z^*$ is purely imaginary for $b \neq 0$.

If $b = 0$ then $z - z^* = 0$.

$\therefore\ z - z^*$ is purely imaginary, unless z is real, then $z - z^* = 0$.

8 **a**



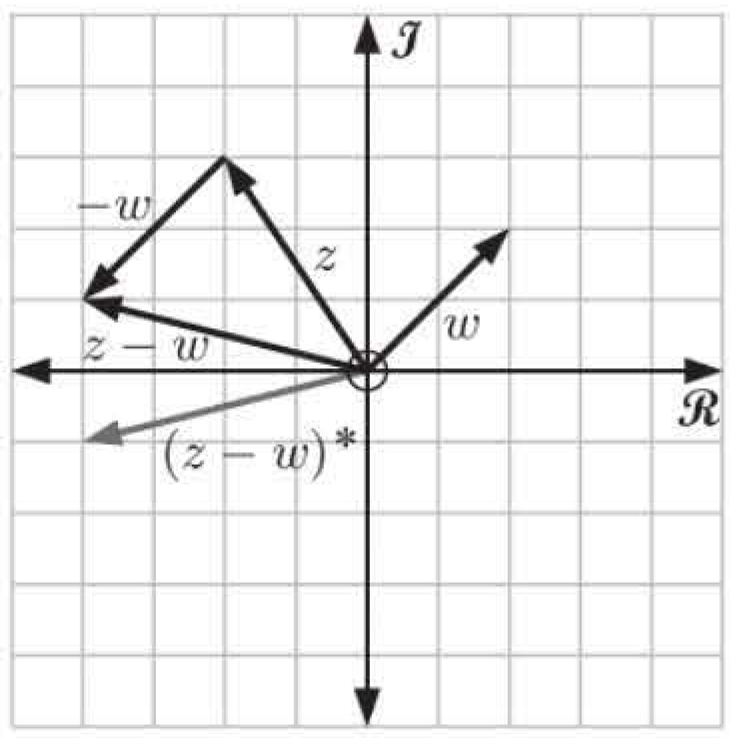


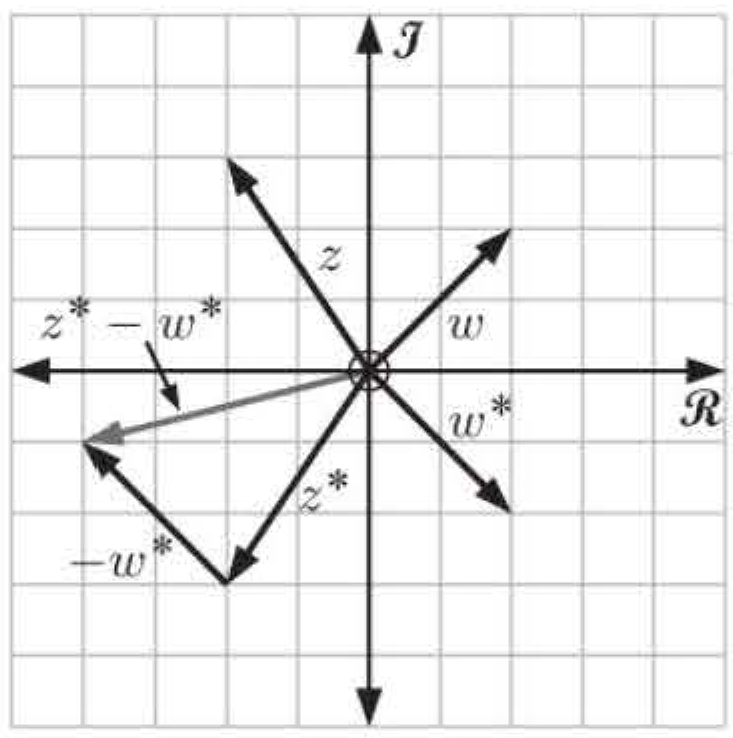
Let $z = a + bi$ and $w = c + di$

$$\begin{aligned}
\therefore\ (z+w)^* &= (a + bi + c + di)^* \\
&= ((a+c) + (b+d)i)^* \\
&= (a+c) - (b+d)i \\
&= a + c - bi - di \\
&= (a - bi) + (c - di) \\
&= z^* + w^*
\end{aligned}$$

$\therefore\ (z+w)^* = z^* + w^*$ for all complex z, w

b





Let $z = a + bi$ and $w = c + di$

$$\begin{aligned} \therefore\ (z-w)^* &= ((a+bi)-(c+di))^* \\ &= (a+bi-c-di)^* \\ &= ((a-c)+(b-d)i)^* \\ &= (a-c)-(b-d)i \\ &= a-c-bi+di \\ &= (a-bi)-(c-di) \\ &= z^* - w^* \end{aligned}$$

$\therefore\ (z-w)^* = z^* - w^*$ for all complex z, w

EXERCISE 16B.1

1 **a** $|3-4i|$
$= \sqrt{3^2+(-4)^2}$
$= \sqrt{9+16}$
$= \sqrt{25}$
$= 5$

b $|5+12i|$
$= \sqrt{5^2+12^2}$
$= \sqrt{25+144}$
$= \sqrt{169}$
$= 13$

c $|-8+2i|$
$= \sqrt{(-8)^2+2^2}$
$= \sqrt{64+4}$
$= \sqrt{68}$
$= 2\sqrt{17}$

d $|3i|$
$= \sqrt{0^2+3^2}$
$= \sqrt{9}$
$= 3$

e $|-4|$
$= \sqrt{(-4)^2+0^2}$
$= \sqrt{16}$
$= 4$

2 **a** $|z|$
$= |2+i|$
$= \sqrt{2^2+1^2}$
$= \sqrt{5}$

b $|z^*|$
$= |2-i|$
$= \sqrt{2^2+(-1)^2}$
$= \sqrt{5}$

c $|z^*|^2$
$= (\sqrt{5})^2$ {from **b**}
$= 5$

d zz^*
$= (2+i)(2-i)$
$= 4 - \cancel{2i} + \cancel{2i} - i^2$
$= 4+1$
$= 5$

e $|zw|$
$= |(2+i)(-1+3i)|$
$= \left|-2+6i-i+3i^2\right|$
$= |-5+5i|$
$= \sqrt{(-5)^2+5^2}$
$= \sqrt{50}$ or $5\sqrt{2}$

f $|z|\,|w|$
$= |2+i|\,|-1+3i|$
$= \sqrt{2^2+1^2}\sqrt{(-1)^2+3^2}$
$= \sqrt{5} \times \sqrt{10}$
$= \sqrt{50}$ or $5\sqrt{2}$

g
$$\begin{aligned}&\left|\frac{z}{w}\right|\\ &=\left|\frac{2+i}{-1+3i}\right|\\ &=\left|\frac{(2+i)}{(-1+3i)}\times\frac{(-1-3i)}{(-1-3i)}\right|\\ &=\left|\frac{-2-6i-i-3i^2}{(-1)^2-(3i)^2}\right|\\ &=\left|\frac{-2+3-7i}{10}\right|\\ &=\left|\tfrac{1}{10}-\tfrac{7}{10}i\right|\\ &=\sqrt{\left(\tfrac{1}{10}\right)^2+\left(\tfrac{-7}{10}\right)^2}\\ &=\sqrt{\tfrac{1+49}{100}}\\ &=\sqrt{\tfrac{50}{100}}\\ &=\tfrac{1}{\sqrt{2}}\end{aligned}$$

h
$$\begin{aligned}&\frac{|z|}{|w|}\\ &=\frac{|2+i|}{|-1+3i|}\\ &=\frac{\sqrt{2^2+1^2}}{\sqrt{(-1)^2+3^2}}\\ &=\tfrac{\sqrt{5}}{\sqrt{10}}\\ &=\sqrt{\tfrac{5}{10}}\\ &=\tfrac{1}{\sqrt{2}}\end{aligned}$$

i
$$\begin{aligned}z^2&=(2+i)^2\\ &=4+4i+i^2\\ &=3+4i\\ \therefore\ \left|z^2\right|&=\sqrt{3^2+4^2}\\ &=\sqrt{25}\\ &=5\end{aligned}$$

j
$$\begin{aligned}&|z|^2\\ &=(\sqrt{5})^2 \quad \{\text{from } \mathbf{a}\}\\ &=5\end{aligned}$$

k
$$\begin{aligned}z^3&=z^2\times z\\ &=(3+4i)(2+i) \qquad \{\text{from } \mathbf{i}\}\\ &=6+3i+8i+4i^2\\ &=2+11i\\ \therefore\ \left|z^3\right|&=\sqrt{2^2+11^2}\\ &=\sqrt{4+121}\\ &=\sqrt{125}\\ &=5\sqrt{5}\end{aligned}$$

l
$$\begin{aligned}&|z|^3\\ &=(\sqrt{5})^3 \quad \{\text{from } \mathbf{a}\}\\ &=\sqrt{125}\\ &=5\sqrt{5}\end{aligned}$$

3 **a** Let $z=a+bi$ where $a, b\in\mathbb{R}$
$$\begin{aligned}\therefore\ |z^*|&=|a-bi|\\ &=\sqrt{a^2+(-b)^2}\\ &=\sqrt{a^2+b^2}\\ &=|a+bi|\\ &=|z| \quad \text{as required}\end{aligned}$$

b Let $z=a+bi$ where $a, b\in\mathbb{R}$
$$\begin{aligned}\therefore\ zz^*&=(a+bi)(a-bi)\\ &=a^2-\cancel{abi}+\cancel{abi}-b^2i^2\\ &=a^2+b^2\\ &=\left(\sqrt{a^2+b^2}\right)^2\\ &=|z|^2 \quad \text{as required}\end{aligned}$$

4 **a**
$$\begin{aligned}z&=\cos\theta+i\sin\theta\\ \therefore\ |z|&=\sqrt{\cos^2\theta+\sin^2\theta}\\ &=\sqrt{1}\\ &=1\end{aligned}$$

b
$$\begin{aligned}z&=r(\cos\theta+i\sin\theta), \quad r\in\mathbb{R}\\ &=r\cos\theta+ri\sin\theta\\ \therefore\ |z|&=\sqrt{r^2\cos^2\theta+r^2\sin^2\theta}\\ &=\sqrt{r^2(\cos^2\theta+\sin^2\theta)}\\ &=\sqrt{r^2}\\ &=|r|\end{aligned}$$

5
$$\left|\frac{z}{w}\right|\times|w|=\left|\frac{z}{w}\times w\right| \qquad \{\text{using } |z_1|\,|z_2|=|z_1z_2|\,\}$$
$$\therefore\ \left|\frac{z}{w}\right|\times|w|=|z|$$
$$\therefore\ \left|\frac{z}{w}\right|=\frac{|z|}{|w|} \quad \text{provided } w\neq 0 \quad \{\text{dividing both sides by } |w|\,\}$$

6 **a** **i** $|z_1z_2z_3| = |(z_1z_2)z_3|$
$= |z_1z_2|\,|z_3|$ {as $|zw| = |z|\,|w|$ }
$= |z_1|\,|z_2|\,|z_3|$ { $|zw| = |z|\,|w|$ again}

ii Now extending this result by the same argument,
$|z_1z_2z_3z_4| = |(z_1z_2z_3)z_4|$
$= |z_1z_2z_3|\,|z_4|$
$= |z_1|\,|z_2|\,|z_3|\,|z_4|$

b The generalisation of **a** is: $|z_1z_2z_3....z_n| = |z_1|\,|z_2|\,|z_3|\,....\,|z_n|$

c P_n is "$|z_1z_2....z_n| = |z_1|\,|z_2|\,....\,|z_n|$" for $n \in \mathbb{Z}^+$.

Proof: (By the principle of mathematical induction)

(1) If $n = 1$ then LHS $= |z_1|$, RHS $= |z_1|$ $\therefore$ P_1 is true.

(2) If P_k is true then $|z_1z_2....z_k| = |z_1|\,|z_2|\,....\,|z_k|$

$|z_1z_2....z_kz_{k+1}| = |(z_1z_2....z_k)z_{k+1}|$
$= |z_1z_2....z_k|\,|z_{k+1}|$ { $|z|\,|w| = |zw|$ }
$= |z_1|\,|z_2|\,....\,|z_k|\,|z_{k+1}|$ $\{P_k\}$

$\therefore$ P_{k+1} is true whenever P_k is true, and P_1 is true.
$\therefore$ P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

d $|z_1z_2....z_n| = |z_1|\,|z_2|\,....\,|z_n|$
Letting $z_1 = z_2 = ... = z_n = z$ we have $|zz....z| = |z|\,|z|\,....\,|z|$
$\therefore$ $|z^n| = |z|^n$

e If $z = 1 - i\sqrt{3}$ then $|z| = \sqrt{1^2 + (-\sqrt{3})^2}$
$= \sqrt{4} = 2$

$\therefore$ $\left|z^{20}\right| = |z|^{20}$
$= 2^{20} = 1\,048\,576$

7 **a** $|2z|$
$= |2|\,|z|$
$= 2 \times 3$
$= 6$

b $|-3z|$
$= |-3|\,|z|$
$= 3 \times 3$
$= 9$

c $|(1+2i)z|$
$= |1+2i| \times |z|$
$= \sqrt{1+4} \times 3$
$= 3\sqrt{5}$

d $|iz|$
$= |i|\,|z|$
$= 1 \times 3$
$= 3$

e $\left|\dfrac{1}{z}\right|$
$= \dfrac{|1|}{|z|}$
$= \frac{1}{3}$

f $\left|\dfrac{2i}{z^2}\right|$
$= \dfrac{|2i|}{|z^2|}$
$= \dfrac{|2i|}{|z|^2}$
$= \dfrac{2}{3^2} = \frac{2}{9}$

8 **a** $w = \dfrac{z+1}{z-1}$. Let $z = a + bi$ where $a, b \in \mathbb{R}$.

$$\therefore\ w = \frac{a+bi+1}{a+bi-1}$$
$$= \frac{(a+1)+bi}{(a-1)+bi}$$
$$= \frac{(a+1)+bi}{(a-1)+bi} \times \frac{(a-1)-bi}{(a-1)-bi}$$
$$= \frac{(a+1)(a-1) - b(a+1)i + b(a-1)i - b^2i^2}{(a-1)^2 - (bi)^2}$$
$$= \frac{a^2 - 1 - \cancel{abi} - bi + \cancel{abi} - bi + b^2}{(a-1)^2 + b^2}$$
$$= \left(\frac{a^2+b^2-1}{(a-1)^2+b^2}\right) + \left(\frac{-2b}{(a-1)^2+b^2}\right)i$$

b $\mathcal{R}e\,(w) = \dfrac{a^2+b^2-1}{(a-1)^2+b^2} = \dfrac{a^2+b^2-1}{a^2-2a+1+b^2} = \dfrac{a^2+b^2-1}{a^2+b^2-2a+1}$

Since $|z| = 1$ $\sqrt{a^2+b^2} = 1$ $\therefore$ $a^2+b^2 = 1$

$\therefore$ $\mathcal{R}e\,(w) = \dfrac{1-1}{1-2a+1} = 0$ provided $a \neq 1$

If $a = 1$, then $\mathcal{R}e\,(w)$ is undefined.

9 **a**

$$\begin{aligned}
|z+9| &= 3\,|z+1| \\
\therefore\ |z+9|^2 &= 9\,|z+1|^2 \\
\therefore\ (z+9)(z+9)^* &= 9(z+1)(z+1)^* && \{zz^* = |z|^2\} \\
\therefore\ (z+9)(z^*+9) &= 9(z+1)(z^*+1) && \{(z\pm w)^* = z^* \pm w^*\} \\
\therefore\ zz^* + \cancel{9z} + \cancel{9z^*} + 81 &= 9zz^* + \cancel{9z} + \cancel{9z^*} + 9 \\
\therefore\ 72 &= 8zz^* \\
\therefore\ zz^* &= 9 \\
\therefore\ |z|^2 &= 9 \\
\therefore\ |z| &= 3 && \{|z| > 0\}
\end{aligned}$$

b

$$\begin{aligned}
\left|\frac{z+4}{z+1}\right| &= 2 \\
\therefore\ \frac{|z+4|}{|z+1|} &= 2 && \left\{\left|\frac{z}{w}\right| = \frac{|z|}{|w|}\right\} \\
\therefore\ |z+4| &= 2\,|z+1| \\
\therefore\ |z+4|^2 &= 4\,|z+1|^2 \\
\therefore\ (z+4)(z+4)^* &= 4(z+1)(z+1)^* && \{zz^* = |z|^2\} \\
\therefore\ (z+4)(z^*+4) &= 4(z+1)(z^*+1) && \{(z\pm w)^* = z^* \pm w^*\} \\
\therefore\ zz^* + \cancel{4z} + \cancel{4z^*} + 16 &= 4zz^* + \cancel{4z} + \cancel{4z^*} + 4 \\
\therefore\ 12 &= 3zz^* \\
\therefore\ zz^* &= 4 \\
\therefore\ |z|^2 &= 4 \\
\therefore\ |z| &= 2 && \{|z| > 0\}
\end{aligned}$$

10

$$\begin{aligned}
|z+w| &= |z-w| \\
\therefore\ |z+w|^2 &= |z-w|^2 \\
\therefore\ (z+w)(z+w)^* &= (z-w)(z-w)^* && \{zz^* = |z|^2\} \\
\therefore\ (z+w)(z^*+w^*) &= (z-w)(z^*-w^*) && \{(z\pm w)^* = z^* \pm w^*\} \\
\therefore\ \cancel{zz^*} + zw^* + wz^* + \cancel{ww^*} &= \cancel{zz^*} - zw^* - wz^* + \cancel{ww^*} \\
\therefore\ zw^* + wz^* &= -zw^* - wz^* \\
\therefore\ 2zw^* &= -2wz^* \\
\therefore\ \frac{z}{z^*} &= -\frac{w}{w^*}
\end{aligned}$$

EXERCISE 16B.2

1 **a** A(3, 6), B(−1, 2), $z = 3+6i$, $w = -1+2i$

i
$$\begin{aligned}
z - w &= (3+6i) - (-1+2i) \\
&= 4+4i \\
|z-w| &= \sqrt{4^2+4^2} \\
&= \sqrt{32} \\
&= 4\sqrt{2}
\end{aligned}$$
$\therefore$ AB $= 4\sqrt{2}$ units

ii
$$\begin{aligned}
\frac{z+w}{2} &= \frac{(3+6i)+(-1+2i)}{2} \\
&= \frac{2+8i}{2} \\
&= 1+4i
\end{aligned}$$
and so M is at $(1,\ 4)$

b A$(-4, 7)$, B$(1, -3)$, $z = -4 + 7i$, $w = 1 - 3i$

i $$\begin{aligned} z - w &= (-4+7i) - (1-3i) \\ &= -5 + 10i \\ |z - w| &= \sqrt{(-5)^2 + 10^2} \\ &= \sqrt{125} \\ &= 5\sqrt{5} \end{aligned}$$
$\therefore$ AB $= 5\sqrt{5}$ units

ii $$\begin{aligned} \frac{z+w}{2} &= \frac{(-4+7i)+(1-3i)}{2} \\ &= \frac{-3+4i}{2} \\ &= -\tfrac{3}{2} + 2i \end{aligned}$$
and so M is at $\left(-\frac{3}{2}, 2\right)$

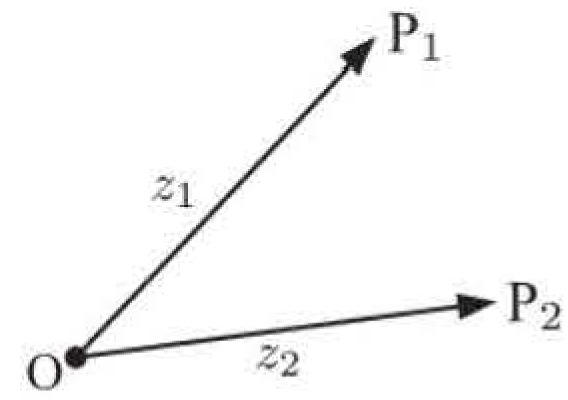
2 **a** **i** $\overrightarrow{\text{OQ}} = z + w$ **ii** $\overrightarrow{\text{PR}} = w - z$

b In $\triangle$OPQ, $|z + w|$ represents the length of OQ, $|z| =$ length of OP, and $|w|$ the length of PQ. Now if w, z are not parallel, we will form the $\triangle$OPQ and this means OQ $<$ OP $+$ PQ.
$\therefore$ $|z + w| < |z| + |w|$
If w and z are parallel then we form a straight line and OQ $=$ OP $+$ PQ
$\therefore$ $|z + w| = |z| + |w|$
Consequently $|z + w| \leqslant |z| + |w|$

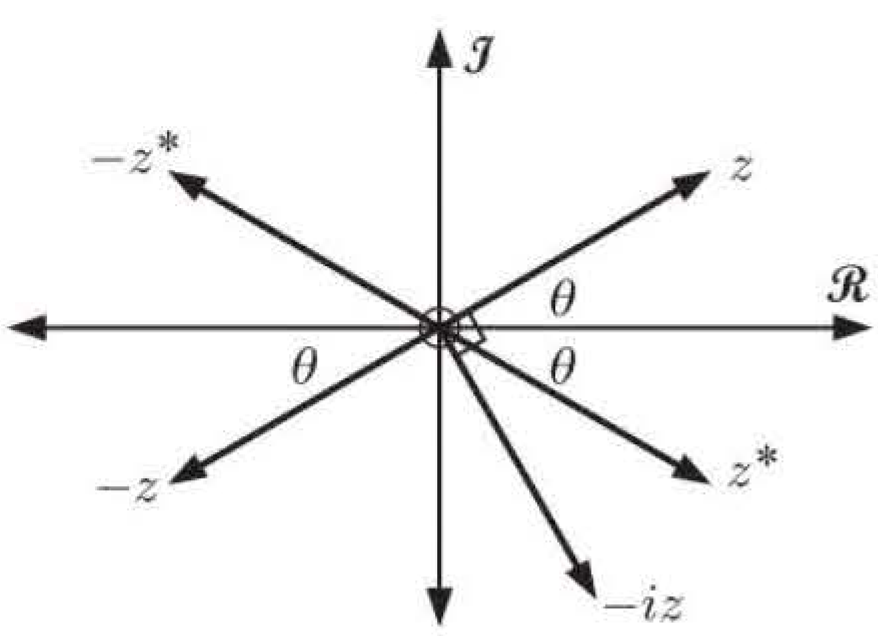
c In $\triangle$OPR, the length of RP is represented by $|z - w|$. If w and z are not parallel, we form a triangle and RP $+$ OP $>$ OR. $\therefore$ $|z - w| + |z| > |w|$
$\therefore$ $|z - w| > |w| - |z|$
Equality will occur when w is parallel to z, in which case a straight line OPR is formed and
$|z - w| = |w| - |z|$
Consequently $|z - w| \geqslant |w| - |z|$

3 $z_1 \equiv \overrightarrow{\text{OP}_1}$ and $z_2 \equiv \overrightarrow{\text{OP}_2}$

$$\begin{aligned} \therefore\ z_1 - z_2 &= \overrightarrow{\text{OP}_1} - \overrightarrow{\text{OP}_2} \\ &= \overrightarrow{\text{OP}_1} + \overrightarrow{\text{P}_2\text{O}} \quad \{\text{since } \overrightarrow{\text{P}_2\text{O}} = -\overrightarrow{\text{OP}_2}\} \\ &= \overrightarrow{\text{P}_2\text{O}} + \overrightarrow{\text{OP}_1} \\ &= \overrightarrow{\text{P}_2\text{P}_1} \end{aligned}$$



4 **a** $z \mapsto z^*$. Reflection in the real axis.
b $z \mapsto -z$. Rotation of π about O.
c $z \mapsto -z^*$. Reflection in the imaginary axis.
d $z \mapsto -iz$. Clockwise rotation of $\frac{\pi}{2}$ about O.



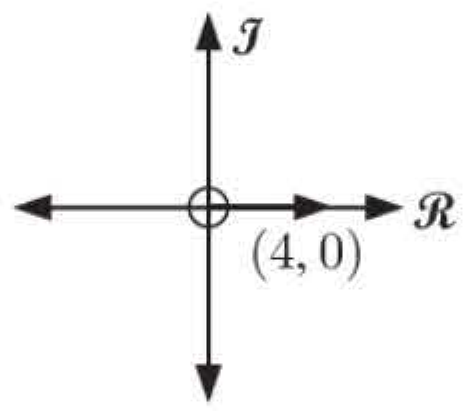
5 $\dfrac{50}{z^*} - \dfrac{10}{z} = 2 + 9i$ where $z = a + bi$, $a, b \in \mathbb{R}$

$\therefore$ $50z - 10z^* = (2+9i)(|z|^2)$ {multiply both sides by $zz^* = |z|^2$ }
$\therefore$ $50(a+bi) - 10(a-bi) = (2+9i)(40)$ { $|z| = 2\sqrt{10}$ $\therefore$ $|z|^2 = 40$}
$\therefore$ $50a + 50bi - 10a + 10bi = 80 + 360i$
$\therefore$ $40a + 60bi = 80 + 360i$

Equating real and imaginary parts, $40a = 80$ and $60b = 360$
$\therefore$ $a = 2$ and $b = 6$
$\therefore$ $z = 2 + 6i$

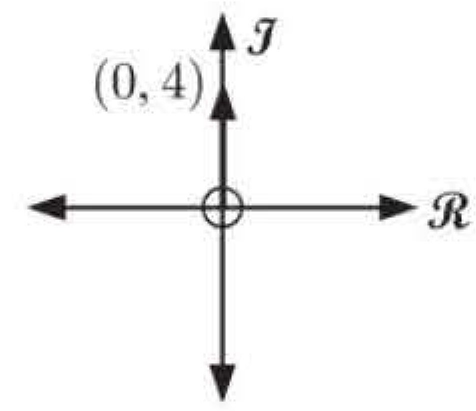
EXERCISE 16C.1

1 **a** $z = 4$



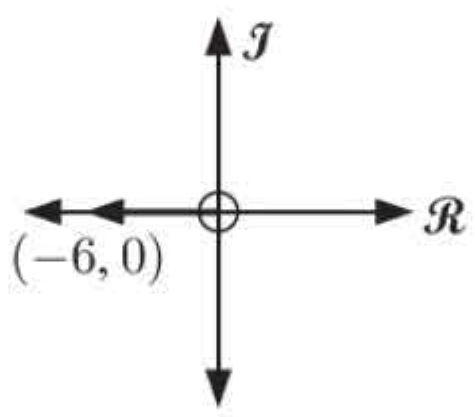
$\arg z = 0$
$|z| = 4$
$\therefore \; z = 4 \text{ cis } 0$

b $z = 4i$



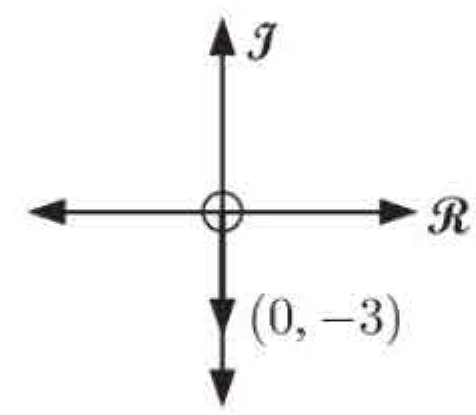
$\arg z = \frac{\pi}{2}$
$|z| = 4$
$\therefore \; z = 4 \text{ cis} \left(\frac{\pi}{2}\right)$

c $z = -6$



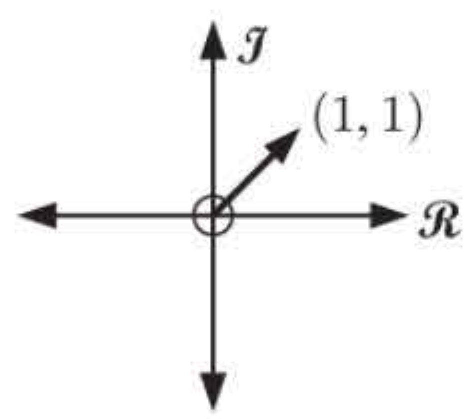
$\arg z = \pi$
$|z| = 6$
$\therefore \; z = 6 \text{ cis } \pi$

d $z = -3i$



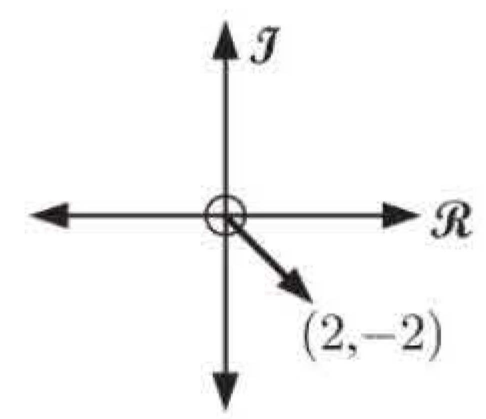
$\arg z = -\frac{\pi}{2}$
$|z| = 3$
$\therefore \; z = 3 \text{ cis} \left(-\frac{\pi}{2}\right)$

e $z = 1 + i$



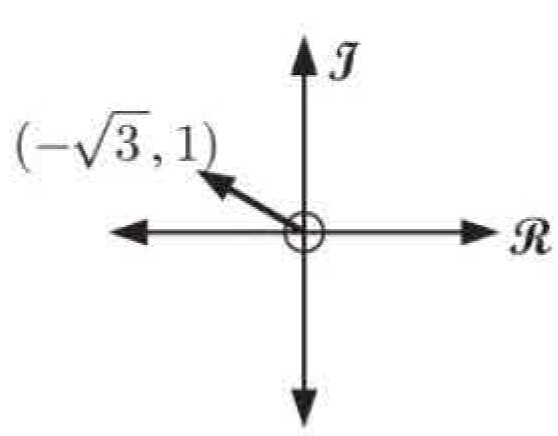
$\arg z = \frac{\pi}{4}$
$$|z| = \sqrt{1^2 + 1^2} = \sqrt{2}$$
$\therefore \; z = \sqrt{2} \text{ cis} \left(\frac{\pi}{4}\right)$

f $z = 2 - 2i$



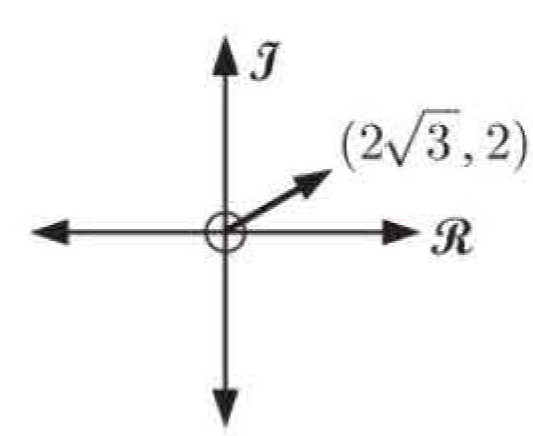
$\arg z = -\frac{\pi}{4}$
$$|z| = \sqrt{2^2 + 2^2} = 2\sqrt{2}$$
$\therefore \; z = 2\sqrt{2} \text{ cis} \left(-\frac{\pi}{4}\right)$

g $z = -\sqrt{3} + i$, $\arg z = \frac{5\pi}{6}$



$$|z| = \sqrt{(-\sqrt{3})^2 + 1^2} = \sqrt{4} = 2$$
$\therefore \; z = 2 \text{ cis} \left(\frac{5\pi}{6}\right)$

h $z = 2\sqrt{3} + 2i$ $\quad \arg z = \frac{\pi}{6}$

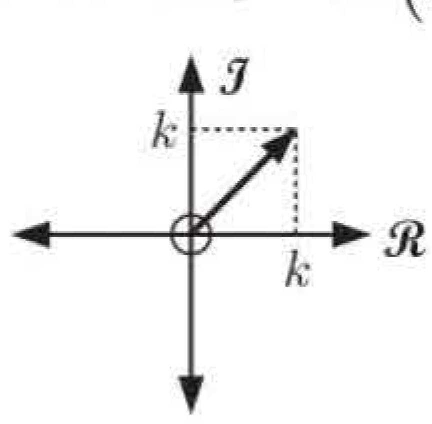


$$|z| = \sqrt{12 + 4} = \sqrt{16} = 4$$
$\therefore \; z = 4 \text{ cis} \left(\frac{\pi}{6}\right)$

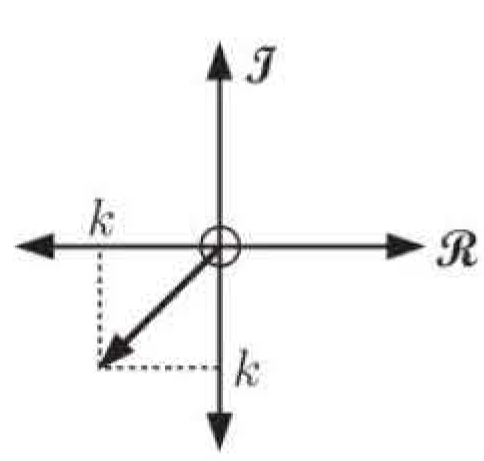
2 $z = 0 = 0 + 0i$ cannot be written in polar form. The vector representing $\overrightarrow{\text{OP}}$ has length zero, and an argument is not defined (no angle can be formed with the positive x-axis).

3 If $k = 0$ it is not possible.

If $k > 0$, $\quad |z| = \sqrt{k^2 + k^2} = k\sqrt{2}$
$\arg z = \frac{\pi}{4}$
$\therefore \; z = k\sqrt{2} \text{ cis} \left(\frac{\pi}{4}\right)$



If $k < 0$, $\quad |z| = \sqrt{k^2 + k^2} = |k|\sqrt{2}$
Since $k < 0$
$|z| = -k\sqrt{2}$
$\arg z = -\frac{3\pi}{4}$
$\therefore \; z = -k\sqrt{2} \text{ cis} \left(-\frac{3\pi}{4}\right)$



4 **a**
$$\begin{aligned} & 2 \text{ cis} \left(\tfrac{\pi}{2}\right) \\ &= 2(\cos\left(\tfrac{\pi}{2}\right) + i \sin\left(\tfrac{\pi}{2}\right)) \\ &= 2(0 + i) \\ &= 2i \end{aligned}$$

b
$$\begin{aligned} & 8 \text{ cis} \left(\tfrac{\pi}{4}\right) \\ &= 8\left(\cos\left(\tfrac{\pi}{4}\right) + i \sin\left(\tfrac{\pi}{4}\right)\right) \\ &= 8\left(\tfrac{1}{\sqrt{2}} + \tfrac{1}{\sqrt{2}} i\right) \\ &= 4\sqrt{2} + 4\sqrt{2} i \end{aligned}$$

c $4 \text{ cis}\left(\frac{\pi}{6}\right)$
$= 4\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right)$
$= 4\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$
$= 2\sqrt{3} + 2i$

d $\sqrt{2} \text{ cis}\left(-\frac{\pi}{4}\right)$
$= \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$
$= \sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)$
$= 1 - i$

e $\sqrt{3} \text{ cis}\left(\frac{2\pi}{3}\right)$
$= \sqrt{3}\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)$
$= \sqrt{3}\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$
$= -\frac{\sqrt{3}}{2} + \frac{3}{2}i$

f $5 \text{ cis } \pi$
$= 5(\cos\pi + i\sin\pi)$
$= 5(-1)$
$= -5$

5 **a** $\text{cis } 0$
$= \cos 0 + i\sin 0$
$= 1$

b $|\text{cis } \theta|$
$= |\cos\theta + i\sin\theta|$
$= \sqrt{\cos^2\theta + \sin^2\theta}$
$= \sqrt{1} = 1$

c $\text{cis } \alpha \times \text{cis } \beta = (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)$
$= \cos\alpha\cos\beta + i\cos\alpha\sin\beta + i\sin\alpha\cos\beta + i^2\sin\alpha\sin\beta$
$= [\cos\alpha\cos\beta - \sin\alpha\sin\beta] + i[\sin\alpha\cos\beta + \sin\beta\cos\alpha]$
$= \cos(\alpha+\beta) + i\sin(\alpha+\beta)$
$= \text{cis}(\alpha+\beta)$

EXERCISE 16C.2

1 **a** $\text{cis } \theta \text{ cis } 2\theta$
$= \text{cis}(\theta + 2\theta)$
$= \text{cis } 3\theta$

b $\frac{\text{cis } 3\theta}{\text{cis } \theta}$
$= \text{cis}(3\theta - \theta)$
$= \text{cis } 2\theta$

c $[\text{cis } \theta]^3$
$= (\text{cis } \theta)(\text{cis } \theta)(\text{cis } \theta)$
$= (\text{cis } 2\theta)(\text{cis } \theta)$
$= \text{cis } 3\theta$

d $\text{cis}\left(\frac{\pi}{18}\right) \times \text{cis}\left(\frac{\pi}{9}\right)$
$= \text{cis}\left(\frac{\pi}{18} + \frac{\pi}{9}\right)$
$= \text{cis}\left(\frac{3\pi}{18}\right)$
$= \text{cis}\left(\frac{\pi}{6}\right)$
$= \cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)$
$= \frac{\sqrt{3}}{2} + \frac{1}{2}i$

e $2 \text{ cis}\left(\frac{\pi}{12}\right) \text{cis}\left(\frac{\pi}{6}\right)$
$= 2 \text{ cis}\left(\frac{\pi}{12} + \frac{\pi}{6}\right)$
$= 2 \text{ cis}\left(\frac{3\pi}{12}\right)$
$= 2 \text{ cis}\left(\frac{\pi}{4}\right)$
$= 2(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right))$
$= 2\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)$
$= \sqrt{2} + i\sqrt{2}$

f $2 \text{ cis}\left(\frac{2\pi}{5}\right) \times 4 \text{ cis}\left(\frac{8\pi}{5}\right)$
$= 8 \text{ cis}\left(\frac{2\pi}{5} + \frac{8\pi}{5}\right)$
$= 8 \text{ cis}\left(\frac{10\pi}{5}\right)$
$= 8 \text{ cis } 2\pi$
$= 8(\cos 2\pi + i\sin 2\pi)$
$= 8(1)$
$= 8$

g $\frac{4 \text{ cis}\left(\frac{\pi}{12}\right)}{2 \text{ cis}\left(\frac{7\pi}{12}\right)}$
$= 2 \text{ cis}\left(\frac{\pi}{12} - \frac{7\pi}{12}\right)$
$= 2 \text{ cis}\left(-\frac{6\pi}{12}\right)$
$= 2 \text{ cis}\left(-\frac{\pi}{2}\right)$
$= 2(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right))$
$= 2(-i)$
$= -2i$

h $\frac{\sqrt{32} \text{ cis}\left(\frac{\pi}{8}\right)}{\sqrt{2} \text{ cis}\left(-\frac{7\pi}{8}\right)}$
$= \frac{\sqrt{32}}{\sqrt{2}} \text{ cis}\left(\frac{\pi}{8} - \left(-\frac{7\pi}{8}\right)\right)$
$= \sqrt{16} \text{ cis}\left(\frac{8\pi}{8}\right)$
$= 4 \text{ cis } \pi$
$= 4(\cos\pi + i\sin\pi)$
$= 4(-1)$
$= -4$

i $\left[\sqrt{2} \text{ cis}\left(\frac{\pi}{8}\right)\right]^4$
$= \sqrt{2} \text{ cis}\left(\frac{\pi}{8}\right) \times \sqrt{2} \text{ cis}\left(\frac{\pi}{8}\right)$
$\times \sqrt{2} \text{ cis}\left(\frac{\pi}{8}\right) \times \sqrt{2} \text{ cis}\left(\frac{\pi}{8}\right)$
$= (\sqrt{2})^4 \text{ cis}\left(\frac{\pi}{8} + \frac{\pi}{8} + \frac{\pi}{8} + \frac{\pi}{8}\right)$
$= 4 \text{ cis}\left(\frac{\pi}{2}\right)$
$= 4(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right))$
$= 4i$

2 **a** $\begin{aligned} &\text{cis } 17\pi \\ &= \text{cis}\,(\pi + 8(2\pi)) \\ &= \text{cis}\,\pi \\ &= -1 \end{aligned}$

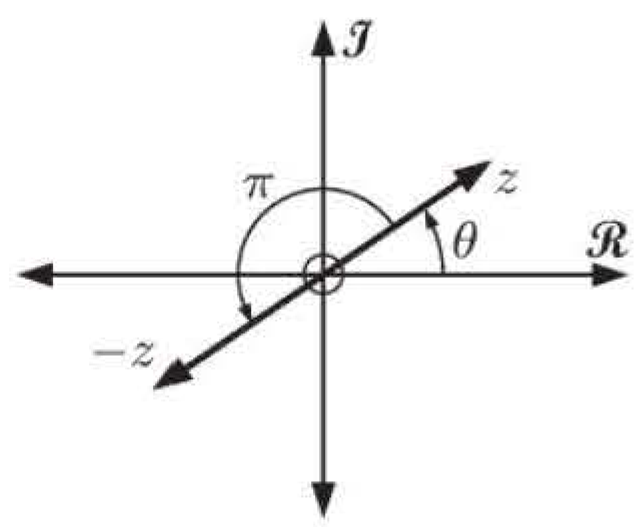
b $\begin{aligned} &\text{cis}\,(-37\pi) \\ &= \text{cis}\,(\pi - 19(2\pi)) \\ &= \text{cis}\,\pi \\ &= -1 \end{aligned}$

c $\begin{aligned} &\text{cis}\left(\tfrac{91\pi}{3}\right) \\ &= \text{cis}\left(\tfrac{\pi}{3} + 15(2\pi)\right) \\ &= \text{cis}\left(\tfrac{\pi}{3}\right) \\ &= \tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i \end{aligned}$

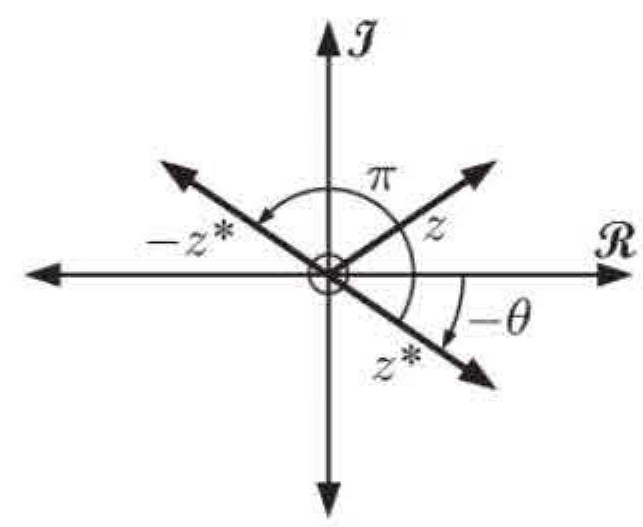
3 **a** $\begin{aligned} z &= 2\text{ cis }\theta \\ |z| &= 2 \\ \arg z &= \theta \end{aligned}$

b $z^* = 2\text{ cis}\,(-\theta)$

c $-z = 2\text{ cis}\,(\theta + \pi)$



d $-z^* = 2\text{ cis}\,(\pi - \theta)$



4 **a** $i = 1\text{ cis}\left(\frac{\pi}{2}\right) = \text{cis}\left(\frac{\pi}{2}\right)$

b $\begin{aligned} iz &= \text{cis}\left(\tfrac{\pi}{2}\right) \times r\text{ cis }\theta \\ &= r\text{ cis}\left(\theta + \tfrac{\pi}{2}\right) \end{aligned}$

c $z \mapsto -iz \qquad \begin{aligned} -iz &= \text{cis}\left(-\tfrac{\pi}{2}\right) \times r\text{ cis }\theta \\ &= r\text{ cis}\left(\theta - \tfrac{\pi}{2}\right) \end{aligned}$

So, a clockwise rotation of $\frac{\pi}{2}$ about O maps z onto $-iz$.

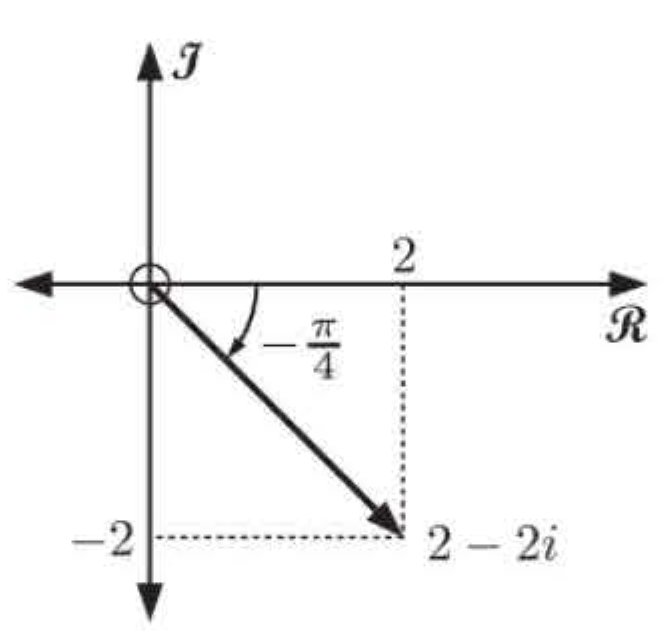
5 **a** **i** $\begin{aligned} &\cos\theta - i\sin\theta \\ &= \cos(-\theta) + i\sin(-\theta) \\ &= \text{cis}(-\theta) \end{aligned}$

ii $\begin{aligned} &\sin\theta - i\cos\theta \\ &= -i\cos\theta - i^2\sin\theta \\ &= -i(\cos\theta + i\sin\theta) \\ &= \text{cis}\left(-\tfrac{\pi}{2}\right)\text{ cis }\theta \\ &= \text{cis}\left(\theta - \tfrac{\pi}{2}\right) \end{aligned}$

b If $z = r\text{ cis }\theta$ then $z^* = r\text{ cis}(-\theta)$ in polar form.

EXERCISE 16C.3

1 **a**

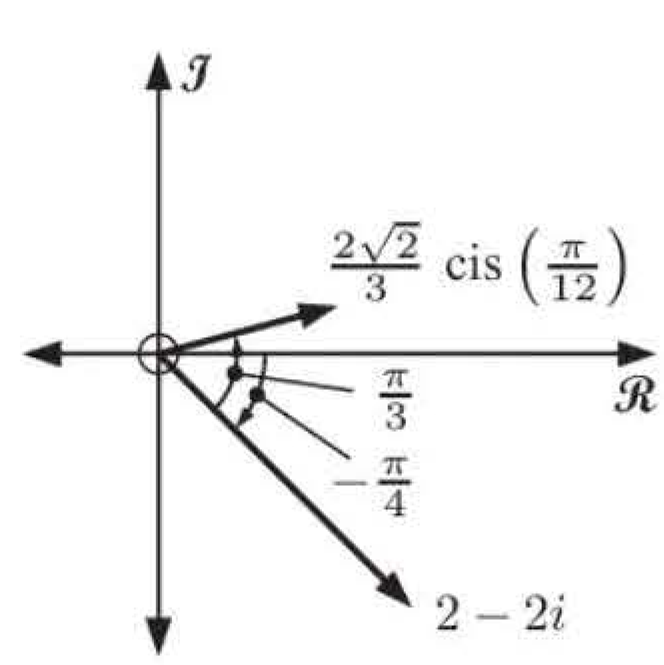


$\begin{aligned} z &= 2 - 2i \\ \therefore\ |z| &= \sqrt{2^2 + (-2)^2} \\ &= 2\sqrt{2} \end{aligned}$

$\begin{aligned} \therefore\ z &= 2\sqrt{2}\left(\tfrac{2}{2\sqrt{2}} - \tfrac{2}{2\sqrt{2}}i\right) \\ &= 2\sqrt{2}\left(\tfrac{1}{\sqrt{2}} - \tfrac{1}{\sqrt{2}}i\right) \\ &= 2\sqrt{2}\left(\cos\left(-\tfrac{\pi}{4}\right) + i\sin\left(-\tfrac{\pi}{4}\right)\right) \\ &= 2\sqrt{2}\text{ cis}\left(-\tfrac{\pi}{4}\right) \end{aligned}$

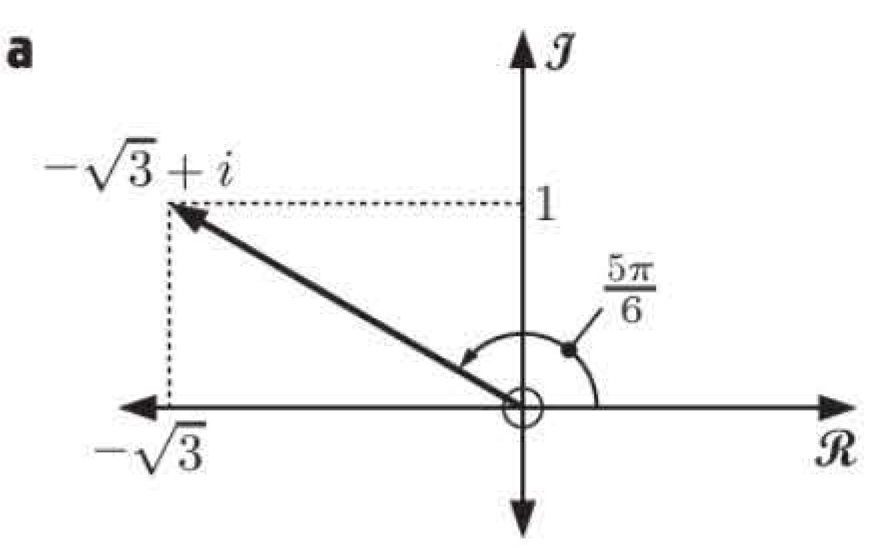
b $\begin{aligned} z \times \tfrac{1}{3}\text{ cis}\left(\tfrac{\pi}{3}\right) &= 2\sqrt{2}\text{ cis}\left(-\tfrac{\pi}{4}\right) \times \tfrac{1}{3}\text{ cis}\left(\tfrac{\pi}{3}\right) \\ &= \tfrac{2\sqrt{2}}{3}\text{ cis}\left(\tfrac{\pi}{3} - \tfrac{\pi}{4}\right) \\ &= \tfrac{2\sqrt{2}}{3}\text{ cis}\left(\tfrac{4\pi - 3\pi}{12}\right) \\ &= \tfrac{2\sqrt{2}}{3}\text{ cis}\left(\tfrac{\pi}{12}\right) \end{aligned}$

c



d When z was multiplied by $\frac{1}{3}\text{ cis}\left(\frac{\pi}{3}\right)$, its modulus was multiplied by $\frac{1}{3}$, and it was rotated anti-clockwise through $\frac{\pi}{3}$ about the origin.

2 **a**

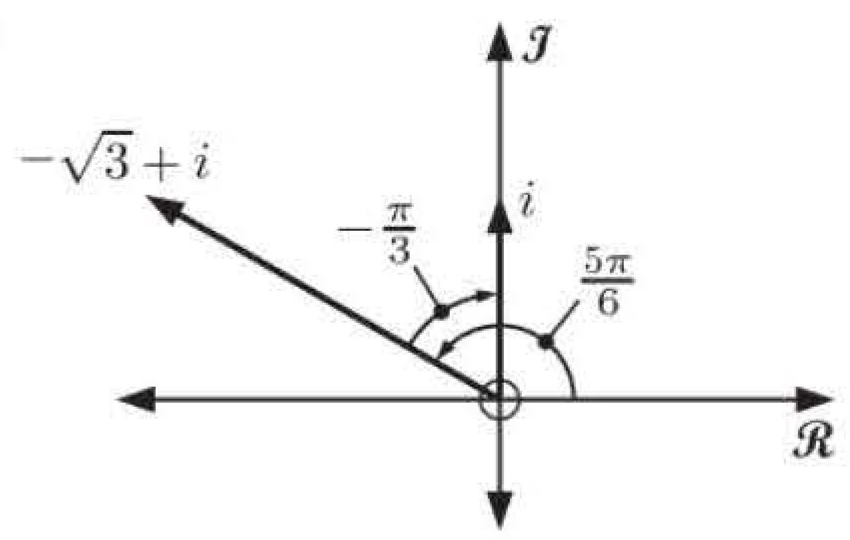


$$\begin{aligned} z &= -\sqrt{3}+i \\ \therefore\ |z| &= \sqrt{(-\sqrt{3})^2+1^2} \\ &= \sqrt{3+1} \\ &= 2 \end{aligned}$$

$$\begin{aligned} \therefore\ z &= 2\left(-\tfrac{\sqrt{3}}{2}+\tfrac{1}{2}i\right) \\ &= 2\left(\cos\left(\tfrac{5\pi}{6}\right)+i\sin\left(\tfrac{5\pi}{6}\right)\right) \\ &= 2\ \text{cis}\left(\tfrac{5\pi}{6}\right) \end{aligned}$$

b $$\begin{aligned} z \times \tfrac{1}{2}\ \text{cis}\left(-\tfrac{\pi}{3}\right) &= 2\ \text{cis}\left(\tfrac{5\pi}{6}\right) \times \tfrac{1}{2}\ \text{cis}\left(-\tfrac{\pi}{3}\right) \\ &= \text{cis}\left(\tfrac{5\pi}{6}-\tfrac{\pi}{3}\right) \\ &= \text{cis}\left(\tfrac{3\pi}{6}\right) \\ &= \text{cis}\left(\tfrac{\pi}{2}\right) \\ &= i \end{aligned}$$

c



d When z was multiplied by $\frac{1}{2}$ cis $\left(-\frac{\pi}{3}\right)$, its modulus was halved, and it was rotated clockwise through $\frac{\pi}{3}$ about the origin.

3 **a** Consider
$$\begin{aligned} \cos\left(\tfrac{\pi}{12}\right)+i\sin\left(\tfrac{\pi}{12}\right) &= \text{cis}\left(\tfrac{\pi}{12}\right) \\ &= \text{cis}\left(\tfrac{4\pi}{12}-\tfrac{3\pi}{12}\right) \\ &= \text{cis}\left(\tfrac{\pi}{3}-\tfrac{\pi}{4}\right) \\ &= \text{cis}\left(\tfrac{\pi}{3}\right) \times \text{cis}\left(-\tfrac{\pi}{4}\right) \\ &= \left(\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}i\right)\left(\tfrac{1}{\sqrt{2}}-\tfrac{1}{\sqrt{2}}i\right) \\ &= \left(\tfrac{1}{2\sqrt{2}}+\tfrac{\sqrt{3}}{2\sqrt{2}}\right)+i\left(\tfrac{\sqrt{3}}{2\sqrt{2}}-\tfrac{1}{2\sqrt{2}}\right) \end{aligned}$$

Equating real parts: $\quad \cos\left(\frac{\pi}{12}\right) = \dfrac{1+\sqrt{3}}{2\sqrt{2}} = \dfrac{\sqrt{2}+\sqrt{6}}{4}$

Equating imaginary parts: $\quad \sin\left(\frac{\pi}{12}\right) = \dfrac{\sqrt{3}-1}{2\sqrt{2}} = \dfrac{\sqrt{6}-\sqrt{2}}{4}$

b Consider
$$\begin{aligned} \cos\left(\tfrac{11\pi}{12}\right)+i\sin\left(\tfrac{11\pi}{12}\right) &= \text{cis}\left(\tfrac{11\pi}{12}\right) \\ &= \text{cis}\left(\tfrac{3\pi}{12}+\tfrac{8\pi}{12}\right) \\ &= \text{cis}\left(\tfrac{\pi}{4}+\tfrac{2\pi}{3}\right) \\ &= \text{cis}\left(\tfrac{\pi}{4}\right) \times \text{cis}\left(\tfrac{2\pi}{3}\right) \\ &= \left(\tfrac{1}{\sqrt{2}}+\tfrac{1}{\sqrt{2}}i\right)\left(-\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}i\right) \\ &= \left(-\tfrac{1}{2\sqrt{2}}-\tfrac{\sqrt{3}}{2\sqrt{2}}\right)+i\left(\tfrac{\sqrt{3}}{2\sqrt{2}}-\tfrac{1}{2\sqrt{2}}\right) \end{aligned}$$

Equating real parts: $\quad \cos\left(\frac{11\pi}{12}\right) = \dfrac{-1-\sqrt{3}}{2\sqrt{2}} = \dfrac{-\sqrt{2}-\sqrt{6}}{4}$

Equating imaginary parts: $\quad \sin\left(\frac{11\pi}{12}\right) = \dfrac{\sqrt{3}-1}{2\sqrt{2}} = \dfrac{\sqrt{6}-\sqrt{2}}{4}$

4 P_n is "$\arg(z^n) = n\arg(z)$", $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n = 1$ then $\text{LHS} = \arg(z^1) = \arg(z)$, $\text{RHS} = 1 \times \arg(z) = \arg(z)$ $\quad\therefore\ P_1$ is true.

(2) If P_k is true then:
$$\begin{aligned}\arg(z^k) &= k\arg(z)\\ \therefore\ \arg(z^{k+1}) &= \arg(z^k z^1)\\ &= \arg(z^k) + \arg(z) \quad \{\text{since } \arg(zw) = \arg(z) + \arg(w)\}\\ &= k\arg(z) + \arg(z) \quad \{P_k\}\\ &= (k+1)\arg(z)\end{aligned}$$

$\therefore$ P_{k+1} is true whenever P_k is true, and P_1 is true.

$\therefore$ P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

5 Let $z = R \text{ cis } \theta$ and $w = r \text{ cis } \phi$, $w \neq 0$

$$\frac{z}{w} = \frac{R \text{ cis } \theta}{r \text{ cis } \phi} = \frac{R}{r} \text{ cis } (\theta - \phi)$$

$$\therefore\ \left|\frac{z}{w}\right| = \frac{R}{r} = \frac{|z|}{|w|}$$

and $\arg\left(\dfrac{z}{w}\right) = \theta - \phi = \arg z - \arg w$ if $w \neq 0$

6 **a**
$$\begin{aligned}z &= 3 \text{ cis } \theta\\ \therefore\ -z &= -1 \times 3 \text{ cis } \theta\\ &= \text{cis } (\pi) \times 3 \text{ cis } \theta\\ &= 3 \text{ cis } (\theta + \pi)\end{aligned}$$

$\therefore$ $|-z| = 3$, but θ is acute, so $\theta + \pi \notin\ ]-\pi, \pi]$

$$\begin{aligned}\therefore\ \arg(-z) &= (\theta + \pi) - 2\pi\\ &= \theta - \pi\end{aligned}$$

b $z^* = 3 \text{ cis } (-\theta)$

$\therefore$ $|z^*| = 3$ and $\arg z^* = -\theta$

c
$$\begin{aligned}iz &= \text{cis}\left(\tfrac{\pi}{2}\right) \times 3 \text{ cis } \theta\\ &= 3 \text{ cis}\left(\tfrac{\pi}{2} + \theta\right)\end{aligned}$$

$\therefore$ $|iz| = 3$ and $\arg(iz) = \theta + \frac{\pi}{2}$

d
$$\begin{aligned}(1+i)z &= \sqrt{2} \text{ cis}\left(\tfrac{\pi}{4}\right) \times 3 \text{ cis } \theta\\ &= 3\sqrt{2} \text{ cis}\left(\theta + \tfrac{\pi}{4}\right)\end{aligned}$$

$\therefore$ $|(1+i)\,z| = 3\sqrt{2}$ and $\arg[(1+i)\,z] = \theta + \frac{\pi}{4}$

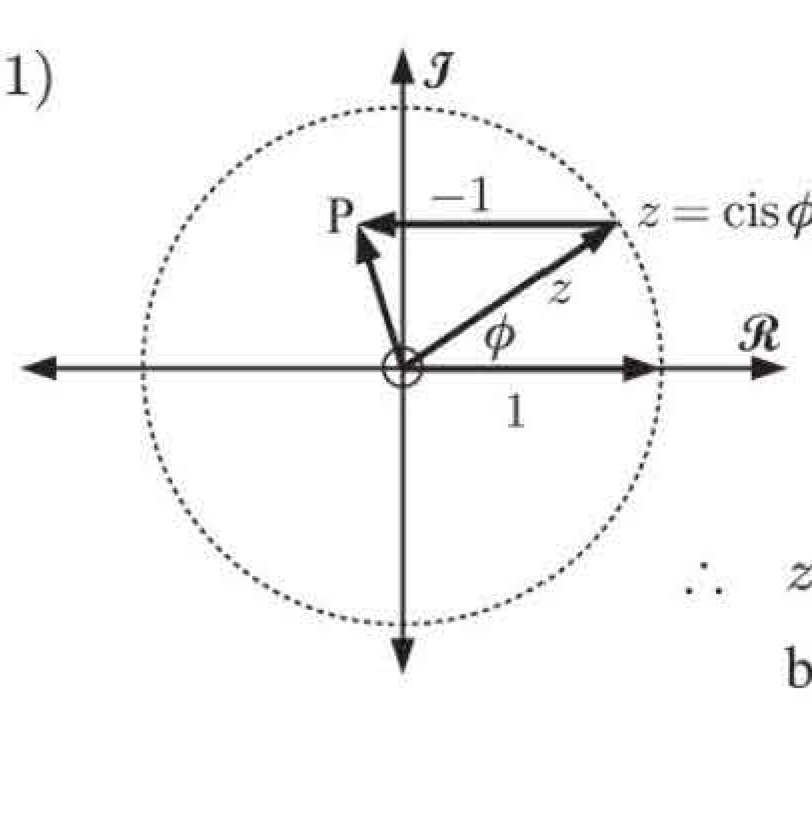
e
$$\begin{aligned}\frac{z}{i} &= \frac{3 \text{ cis } \theta}{\text{cis}\left(\frac{\pi}{2}\right)}\\ &= 3 \text{ cis}\left(\theta - \tfrac{\pi}{2}\right)\end{aligned}$$

$\therefore$ $\left|\dfrac{z}{i}\right| = 3$ and $\arg\left(\dfrac{z}{i}\right) = \theta - \frac{\pi}{2}$

f
$$\begin{aligned}\frac{z}{1-i} &= \frac{3 \text{ cis } \theta}{\sqrt{2} \text{ cis}\left(-\frac{\pi}{4}\right)}\\ &= \tfrac{3}{\sqrt{2}} \text{ cis}\left(\theta - \left(-\tfrac{\pi}{4}\right)\right)\\ &= \tfrac{3}{\sqrt{2}} \text{ cis}\left(\theta + \tfrac{\pi}{4}\right)\end{aligned}$$

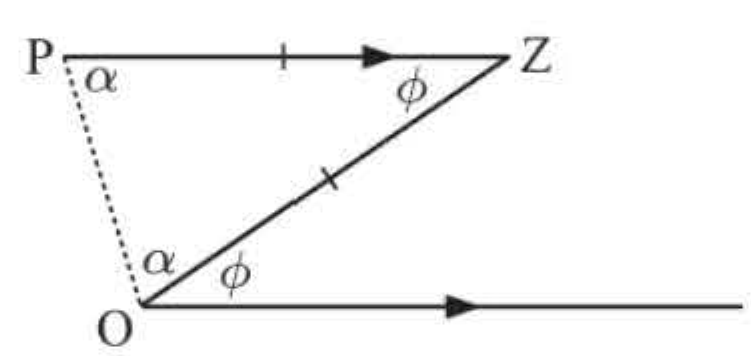
$\therefore$ $\left|\dfrac{z}{1-i}\right| = \frac{3}{\sqrt{2}}$ and $\arg\left(\dfrac{z}{1-i}\right) = \theta + \frac{\pi}{4}$

7 **a** $z - 1 = z + (-1)$



$\therefore$ $z - 1$ is represented by the vector $\overrightarrow{\text{OP}}$

Considering the $\triangle$OZP



$\widehat{PZO} = \phi$ {alternate angles}

$OZ = ZP = 1$

$\therefore$ $\triangle$OPZ is isosceles

$\therefore$ $\widehat{POZ} = \widehat{OPZ} = \alpha$

$\therefore\ 2\alpha + \phi = \pi$ and so $\alpha = \dfrac{\pi - \phi}{2}$

$$\therefore\ \arg(z-1) = \frac{\pi-\phi}{2} + \phi$$
$$= \tfrac{\pi}{2} - \tfrac{\phi}{2} + \phi$$
$$= \tfrac{\pi}{2} + \tfrac{\phi}{2} \quad .... (*)$$

Using the cosine rule in $\triangle$OZP:

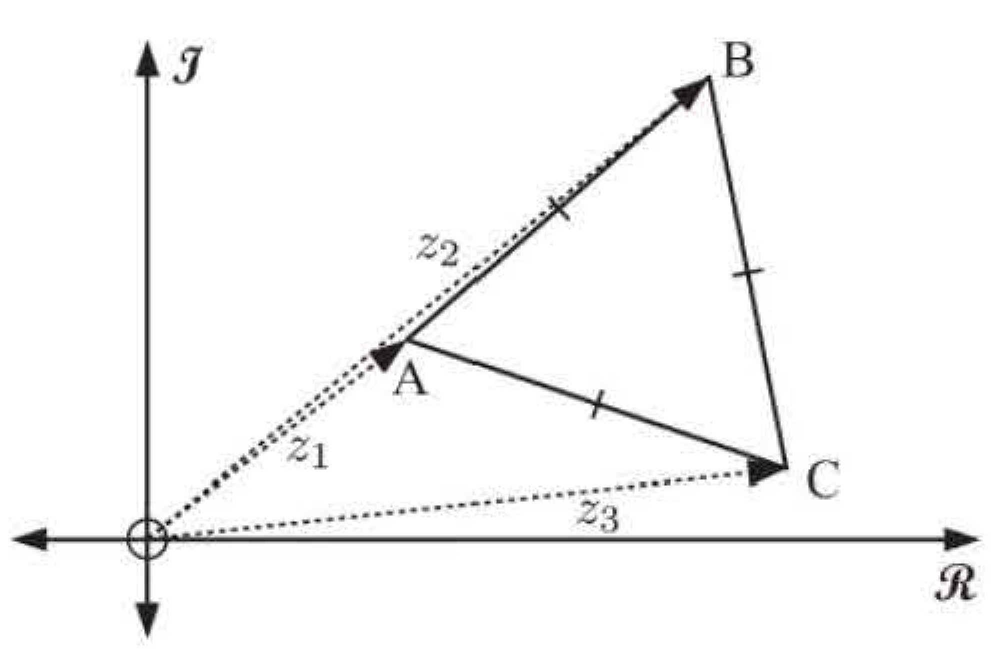
$$OP^2 = 1^2 + 1^2 - 2(1)(1)\cos\phi$$
$$\therefore\ OP^2 = 2 - 2\cos\phi$$
$$\therefore\ OP^2 = 2 - 2\left(1 - 2\sin^2\left(\tfrac{\phi}{2}\right)\right)$$
$$\therefore\ OP^2 = 2 - 2 + 4\sin^2\left(\tfrac{\phi}{2}\right)$$
$$\therefore\ OP^2 = 4\sin^2\left(\tfrac{\phi}{2}\right)$$
$$\therefore\ |z-1| = 2\sin\left(\tfrac{\phi}{2}\right) \quad .... (**)$$

b $z - 1 = 2\sin\left(\frac{\phi}{2}\right)\text{cis}\left(\frac{\pi}{2} + \frac{\phi}{2}\right)$ {using $(*)$ and $(**)$}

c $(z-1)^* = 2\sin\left(\frac{\phi}{2}\right)\text{cis}\left(-\frac{\pi}{2} - \frac{\phi}{2}\right)$

8 **a** Now $z_2 - z_1 = \overrightarrow{AB}$

$z_3 - z_2 = \overrightarrow{BC}$



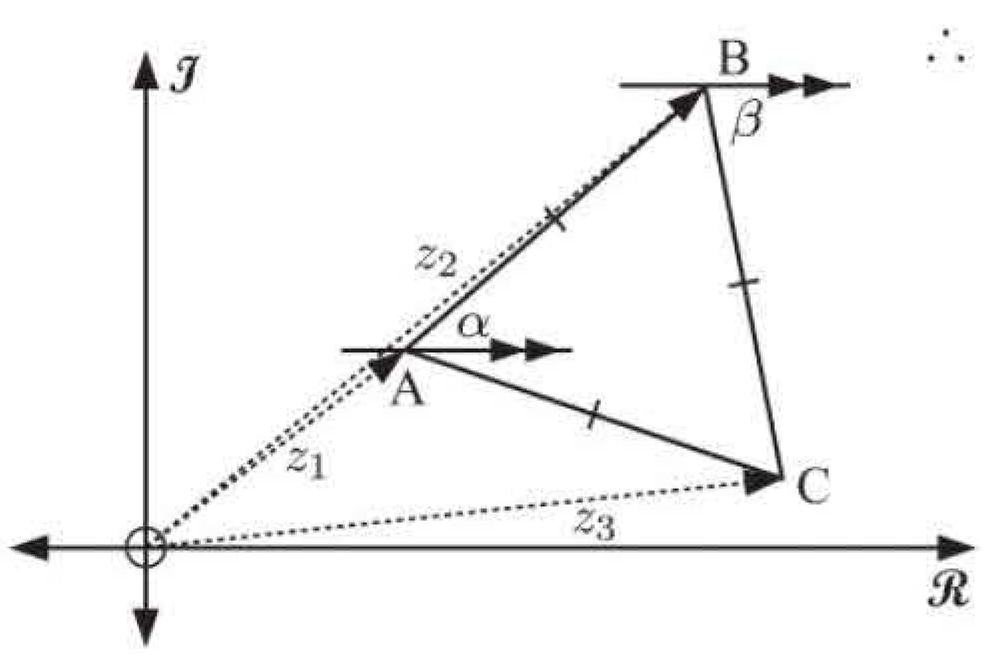
b $$\left|\frac{z_2 - z_1}{z_3 - z_2}\right| = \frac{|z_2 - z_1|}{|z_3 - z_2|} = \frac{|\overrightarrow{AB}|}{|\overrightarrow{BC}|}$$

But $\triangle$ABC is equilateral

$\therefore\ |\overrightarrow{AB}| = |\overrightarrow{BC}|$

$$\therefore\ \left|\frac{z_2 - z_1}{z_3 - z_2}\right| = 1$$

c



$z_2 - z_1 = \overrightarrow{AB}$ and $z_3 - z_2 = \overrightarrow{BC}$ {from **a**}

$\therefore$ let $\arg(z_2 - z_1) = \alpha$

and $\arg(z_3 - z_2) = -\beta$ as shown

$$\therefore\ \arg\left(\frac{z_2 - z_1}{z_3 - z_2}\right) = \arg(z_2 - z_1) - \arg(z_3 - z_2)$$
$$= \alpha - (-\beta)$$
$$= \alpha + \beta$$

But $\widehat{ABC} = \frac{\pi}{3}$ since the triangle is equilateral

$\therefore\ \alpha + \beta + \frac{\pi}{3} = \pi$ {co-interior angles}

$\therefore\ \alpha + \beta = \frac{2\pi}{3}$

$$\therefore\ \arg\left(\frac{z_2 - z_1}{z_3 - z_2}\right) = \tfrac{2\pi}{3}$$

d From **b** and **c**, $\dfrac{z_2 - z_1}{z_3 - z_2} = 1\ \text{cis}\left(\frac{2\pi}{3}\right)$

$$\therefore\ \left(\frac{z_2 - z_1}{z_3 - z_2}\right)^3 = \left(\text{cis}\left(\tfrac{2\pi}{3}\right)\right)^3$$
$$= \text{cis}\left(\tfrac{2\pi}{3} + \tfrac{2\pi}{3} + \tfrac{2\pi}{3}\right)$$
$$= \text{cis}\ 2\pi$$
$$= 1$$

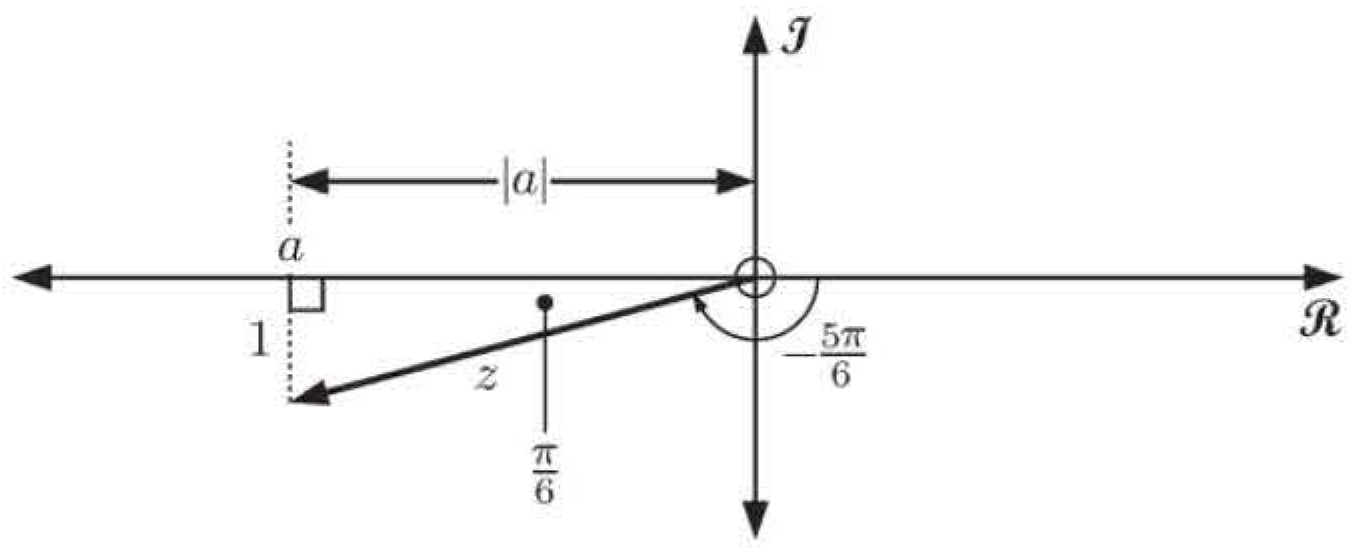
9 $\tan\frac{\pi}{6} = \dfrac{1}{|a|}$

$\therefore\ |a| = \dfrac{1}{\tan\frac{\pi}{6}}$

$\therefore\ a = \pm\sqrt{3}$

but from the graph, a is negative

$\therefore\ a = -\sqrt{3}$



EXERCISE 16C.4

1 Using technology

a $\sqrt{3}\operatorname{cis}(2.5187)$
$= \sqrt{3}(\cos(2.5187) + i\sin(2.5187))$
$\approx -1.41 + 1.01i$

b $\sqrt{11}\operatorname{cis}\left(-\frac{3\pi}{8}\right)$
$= \sqrt{11}\left(\cos\left(-\frac{3\pi}{8}\right) + i\sin\left(-\frac{3\pi}{8}\right)\right)$
$\approx 1.27 - 3.06i$

c $2.836\,49\operatorname{cis}(-2.684\,32) = 2.836\,49(\cos(-2.684\,32) + i\sin(-2.684\,32))$
$\approx -2.55 - 1.25i$

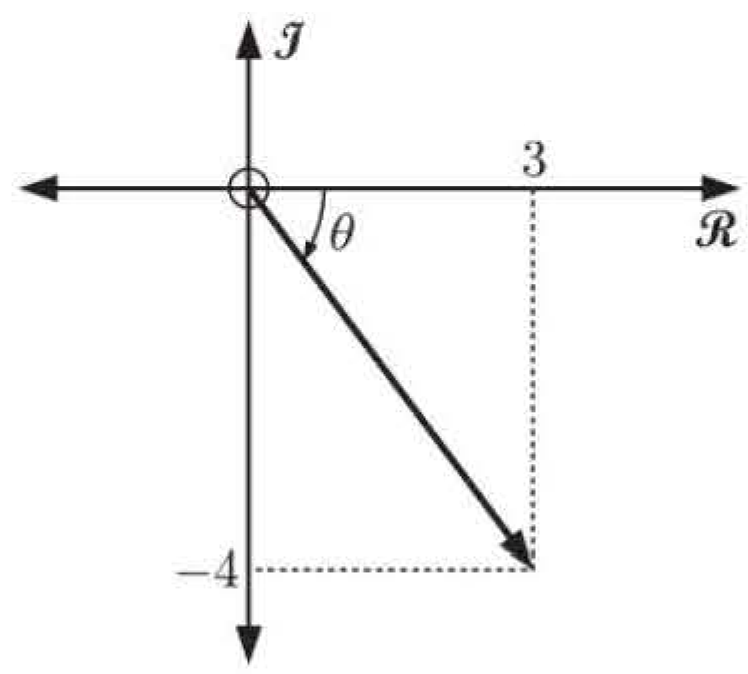
2 Using technology

a $3 - 4i$ has $r = \sqrt{3^2 + (-4)^2}$
$= \sqrt{9+16} = \sqrt{25} = 5$

$\therefore\ \cos\theta = \frac{3}{5}$ and $\sin\theta = -\frac{4}{5}$

$\therefore\ \theta = \arcsin(-\frac{4}{5}) \approx -0.927\,295\,218$

$\therefore\ 3 - 4i \approx 5\operatorname{cis}(-0.927)$

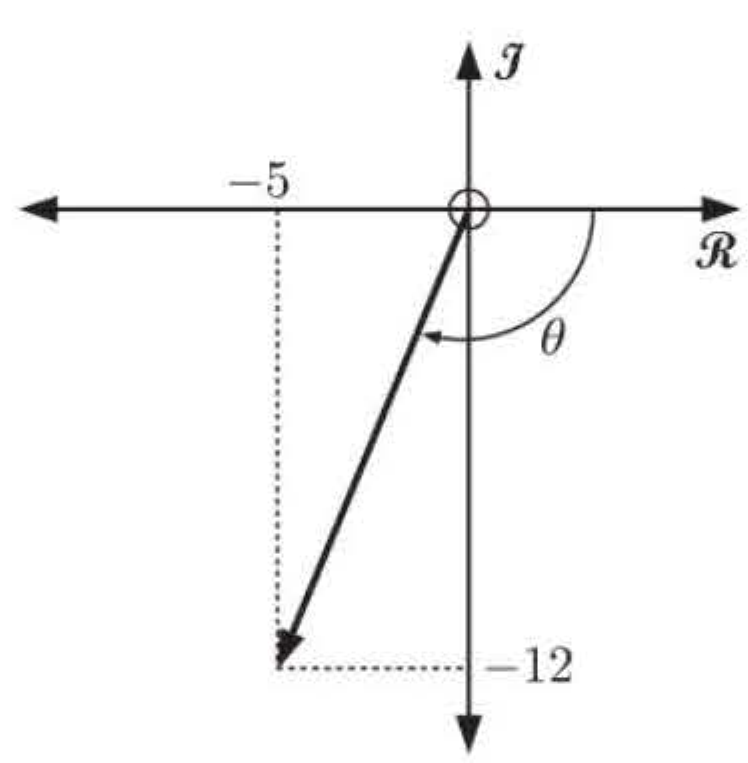


b $-5 - 12i$ has $r = \sqrt{(-5)^2 + (-12)^2}$
$= \sqrt{25 + 144} = \sqrt{169} = 13$

$\therefore\ \cos\theta = -\frac{5}{13}$ and $\sin\theta = -\frac{12}{13}$

$\therefore\ \theta = -\arccos(-\frac{5}{13})$ {quadrant 4}
$\approx -1.965\,587\,446$

$\therefore\ -5 - 12i \approx 13\operatorname{cis}(-1.97)$



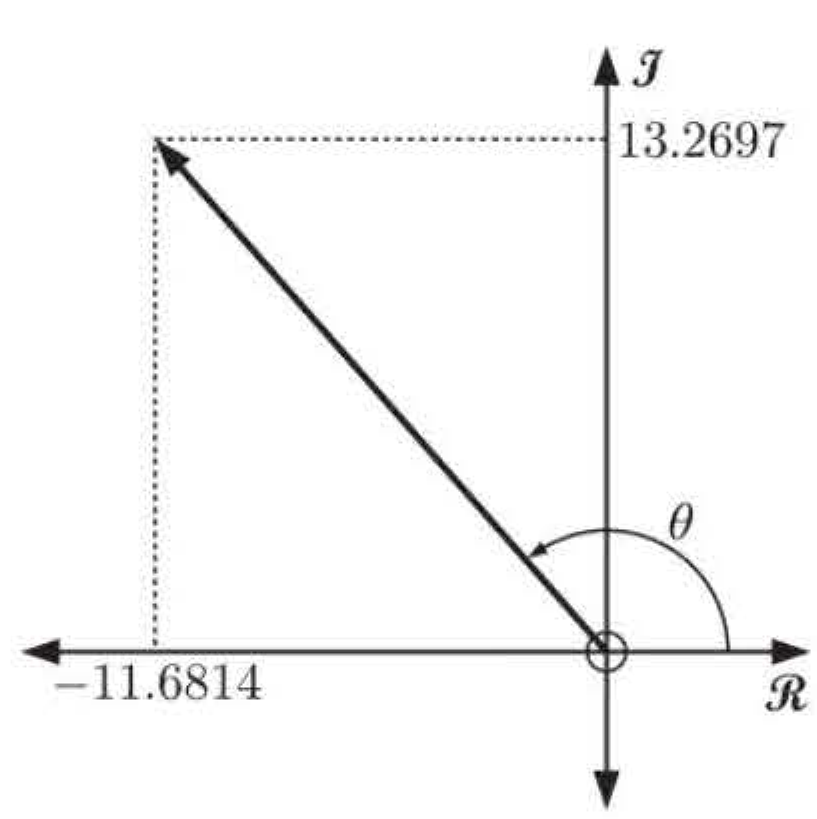
c $-11.6814 + 13.2697i$

has $r = \sqrt{(-11.6814)^2 + 13.2697^2}$
$\approx \sqrt{136.455\,106 + 176.084\,938\,1}$
$\approx \sqrt{312.540\,044\,1}$
$\approx 17.678\,802\,11$

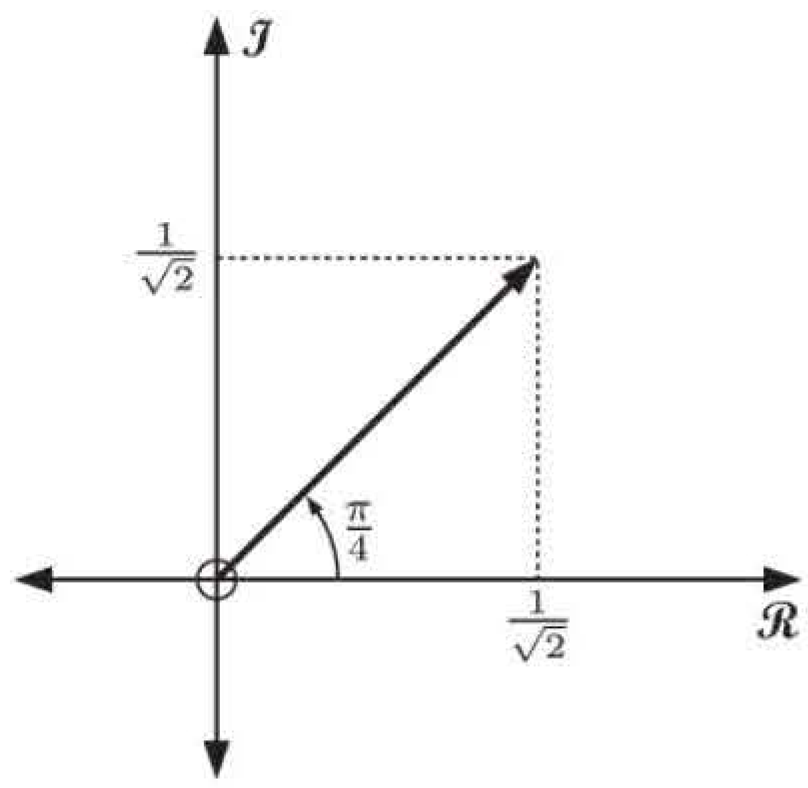
$\therefore\ \cos\theta = -\dfrac{11.6814}{17.678\,802\,11}$ and $\sin\theta = \dfrac{13.2697}{17.678\,802\,11}$

$\therefore\ \theta = \pi - \arcsin\left(\dfrac{13.2697}{17.678\,802\,11}\right)$ {quadrant 2}
$\approx 2.292\,623\,752$

$\therefore\ -11.6814 + 13.2697i \approx 17.7\operatorname{cis}(2.29)$

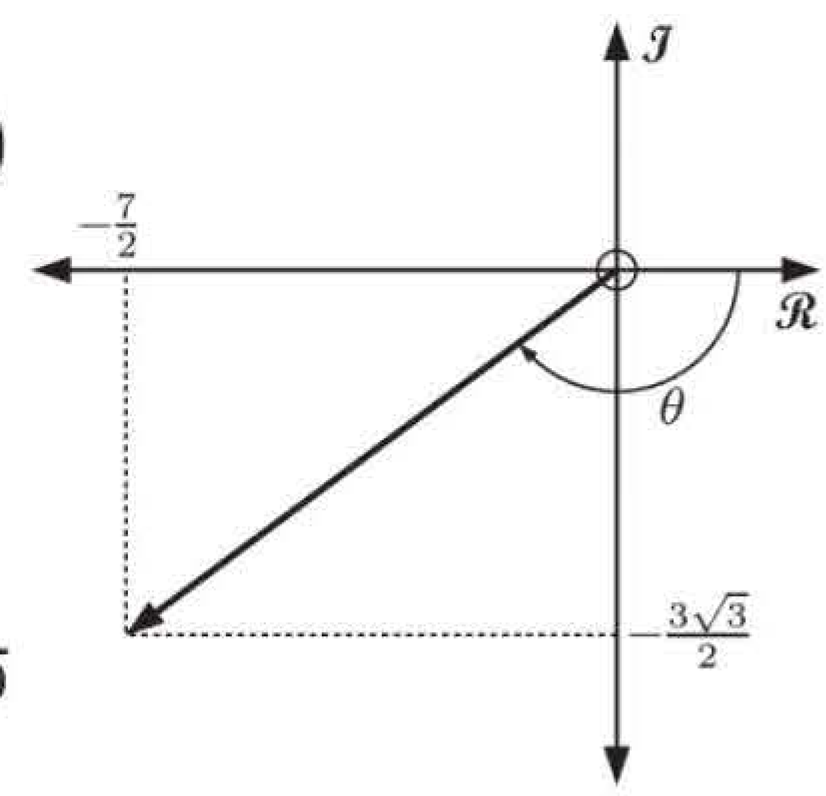


3 **a**
$$\begin{aligned}
&3 \text{ cis}\left(\tfrac{\pi}{4}\right) + \text{cis}\left(-\tfrac{3\pi}{4}\right)\\
&= 3\left(\cos\left(\tfrac{\pi}{4}\right) + i\sin\left(\tfrac{\pi}{4}\right)\right) + \left(\cos\left(-\tfrac{3\pi}{4}\right) + i\sin\left(-\tfrac{3\pi}{4}\right)\right)\\
&= 3\left(\tfrac{1}{\sqrt{2}} + \tfrac{1}{\sqrt{2}}i\right) + \left(-\tfrac{1}{\sqrt{2}} - \tfrac{1}{\sqrt{2}}i\right)\\
&= \tfrac{3}{\sqrt{2}} + \tfrac{3}{\sqrt{2}}i - \tfrac{1}{\sqrt{2}} - \tfrac{1}{\sqrt{2}}i\\
&= \tfrac{2}{\sqrt{2}} + \tfrac{2}{\sqrt{2}}i\\
&= 2\left(\tfrac{1}{\sqrt{2}} + \tfrac{1}{\sqrt{2}}i\right)\\
&= 2 \text{ cis}\left(\tfrac{\pi}{4}\right)
\end{aligned}$$



b
$$\begin{aligned}
&2 \text{ cis}\left(\tfrac{2\pi}{3}\right) + 5 \text{ cis}\left(-\tfrac{2\pi}{3}\right)\\
&= 2\left(\cos\left(\tfrac{2\pi}{3}\right) + i\sin\left(\tfrac{2\pi}{3}\right)\right) + 5\left(\cos\left(-\tfrac{2\pi}{3}\right) + i\sin\left(-\tfrac{2\pi}{3}\right)\right)\\
&= 2\left(-\tfrac{1}{2} + i\tfrac{\sqrt{3}}{2}\right) + 5\left(-\tfrac{1}{2} - i\tfrac{\sqrt{3}}{2}\right)\\
&= -1 + \sqrt{3}i - \tfrac{5}{2} - \tfrac{5\sqrt{3}}{2}i\\
&= -\tfrac{7}{2} - \tfrac{3\sqrt{3}}{2}i
\end{aligned}$$

Now $r = \sqrt{\left(-\frac{7}{2}\right)^2 + \left(-\frac{3\sqrt{3}}{2}\right)^2} = \sqrt{\frac{49+27}{4}} = \sqrt{\frac{76}{4}} = \sqrt{19}$



$\therefore\ \cos\theta = -\frac{7}{2\sqrt{19}}$ and $\sin\theta = -\frac{3\sqrt{3}}{2\sqrt{19}}$

$\therefore\ \theta = -\arccos\left(-\frac{7}{2\sqrt{19}}\right)$ {quadrant 4}

$\approx -2.503\,032\,957$

$\therefore\ 2 \text{ cis}\left(\frac{2\pi}{3}\right) + 5 \text{ cis}\left(-\frac{2\pi}{3}\right) \approx \sqrt{19} \text{ cis}(-2.50)$

4 **a** Sum of roots
$$\begin{aligned}
&= 2 \text{ cis}\left(\tfrac{2\pi}{3}\right) + 2 \text{ cis}\left(\tfrac{4\pi}{3}\right)\\
&= 2\left(\cos\left(\tfrac{2\pi}{3}\right) + i\sin\left(\tfrac{2\pi}{3}\right)\right) + 2\left(\cos\left(\tfrac{4\pi}{3}\right) + i\sin\left(\tfrac{4\pi}{3}\right)\right)\\
&= 2\left(-\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i\right) + 2\left(-\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i\right)\\
&= -1 + \sqrt{3}i - 1 - \sqrt{3}i\\
&= -2
\end{aligned}$$

Product of roots
$$\begin{aligned}
&= 2 \text{ cis}\left(\tfrac{2\pi}{3}\right) \times 2 \text{ cis}\left(\tfrac{4\pi}{3}\right)\\
&= 4 \text{ cis}\left(\tfrac{2\pi}{3} + \tfrac{4\pi}{3}\right)\\
&= 4 \text{ cis}(2\pi)\\
&= 4(1 + 0i)\\
&= 4
\end{aligned}$$

$\therefore$ the equations are
$a(x^2 - (-2)x + 4) = 0$
$\therefore\ a(x^2 + 2x + 4) = 0,\ a \neq 0,\ a \in \mathbb{R}.$

b Sum of roots
$$\begin{aligned}
&= \sqrt{2} \text{ cis}\left(\tfrac{\pi}{4}\right) + \sqrt{2} \text{ cis}\left(-\tfrac{\pi}{4}\right)\\
&= \sqrt{2}\left(\cos\left(\tfrac{\pi}{4}\right) + i\sin\left(\tfrac{\pi}{4}\right)\right) + \sqrt{2}\left(\cos\left(-\tfrac{\pi}{4}\right) + i\sin\left(-\tfrac{\pi}{4}\right)\right)\\
&= \sqrt{2}\left(\tfrac{1}{\sqrt{2}} + \tfrac{1}{\sqrt{2}}i\right) + \sqrt{2}\left(\tfrac{1}{\sqrt{2}} - \tfrac{1}{\sqrt{2}}i\right)\\
&= 1 + i + 1 - i\\
&= 2
\end{aligned}$$

Product of roots

$= \sqrt{2}\text{ cis}\left(\frac{\pi}{4}\right) \times \sqrt{2}\text{ cis}\left(-\frac{\pi}{4}\right)$

$= 2\text{ cis}\left(\frac{\pi}{4} - \frac{\pi}{4}\right)$

$= 2\text{ cis}(0)$

$= 2(1 + 0i)$

$= 2$

$\therefore$ the equations are $a(x^2 - 2x + 2) = 0,\ a \neq 0,\ a \in \mathbb{R}$.

EXERCISE 16D

1 **a** $e^{i\pi} = \cos\pi + i\sin\pi$
$= -1 + i(0)$
$= -1$

b $e^{i\frac{\pi}{3}} = \cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)$
$= \frac{1}{2} + \frac{\sqrt{3}}{2}i$

c $e^{-i\frac{\pi}{2}} = \cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)$
$= 0 + i(-1)$
$= -i$

2 **a** $\text{cis }\theta\text{ cis }\phi = e^{i\theta}e^{i\phi}$
$= e^{i(\theta+\phi)}$
$= \text{cis}(\theta+\phi)$

b $\dfrac{\text{cis }\theta}{\text{cis }\phi} = \dfrac{e^{i\theta}}{e^{i\phi}}$
$= e^{i(\theta-\phi)}$
$= \text{cis}(\theta-\phi)$

3 **a** $\sqrt{z} = (\text{cis }\theta)^{\frac{1}{2}}$
$= (e^{i\theta})^{\frac{1}{2}}$
$= e^{\frac{i\theta}{2}}$
$= \text{cis}\left(\frac{\theta}{2}\right)$
$\therefore\ \arg(\sqrt{z}) = \frac{\theta}{2}$

b $iz = \text{cis}\left(\frac{\pi}{2}\right)\text{ cis }\theta$
$= \text{cis}\left(\frac{\pi}{2} + \theta\right)$
$\therefore\ \arg(iz) = \frac{\pi}{2} + \theta$

c $-iz^2 = \text{cis}\left(-\frac{\pi}{2}\right)\text{cis}^2\theta$
$= \text{cis}\left(-\frac{\pi}{2}\right)\text{ cis }\theta\text{ cis }\theta$
$= \text{cis}\left(-\frac{\pi}{2} + \theta + \theta\right)$
$= \text{cis}\left(2\theta - \frac{\pi}{2}\right)$
$\therefore\ \arg(-iz^2) = 2\theta - \frac{\pi}{2}$

d $\dfrac{i}{z} = \dfrac{\text{cis}\left(\frac{\pi}{2}\right)}{\text{cis }\theta}$
$= \text{cis}\left(\frac{\pi}{2} - \theta\right)$
$\therefore\ \arg\left(\dfrac{i}{z}\right) = \frac{\pi}{2} - \theta$

4 **a** $e^i = \cos(1) + i\sin(1) \quad \{\theta = 1\}$
$\approx 0.540 + 0.841i$

b $3^i = (e^{\ln 3})^i$
$= e^{i\ln 3}$
$= \cos(\ln 3) + i\sin(\ln 3)$
$\approx 0.455 + 0.891i$

c $i = 0 + 1i$
$= \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)$
$= e^{i\frac{\pi}{2}}$

$\therefore\ i^i = (e^{i\frac{\pi}{2}})^i$
$= e^{i^2\frac{\pi}{2}}$
$= e^{-\frac{\pi}{2}}$

d $i^i = e^{-\frac{\pi}{2}} \quad$ {from **c**}
$\therefore\ (i^i)^i = (e^{-\frac{\pi}{2}})^i$
$= e^{-i\frac{\pi}{2}}$

$\therefore\ \left((i^i)^i\right)^2 = (e^{-i\frac{\pi}{2}})^2$
$= e^{-i\pi}$
$= \cos(-\pi) + i\sin(-\pi)$
$= -1 + i(0)$
$= -1$

EXERCISE 16E

1 $$\begin{aligned}(|z|\text{ cis }\theta)^n &= (|z|\,e^{i\theta})^n \quad \{\text{Euler form}\}\\ &= |z|^n\,(e^{i\theta})^n\\ &= |z|^n\,e^{in\theta}\\ &= |z|^n\text{ cis }n\theta \quad \{\text{polar form}\}\end{aligned}$$

2 **a** $$\begin{aligned}&\left(\sqrt{2}\text{ cis}\left(\tfrac{\pi}{5}\right)\right)^{10}\\ &= (\sqrt{2})^{10}\text{ cis}\left(\tfrac{10\pi}{5}\right)\\ &= 2^5\text{ cis }2\pi\\ &= 2^5\\ &= 32\end{aligned}$$

b $$\begin{aligned}&\left(\text{cis}\left(\tfrac{\pi}{12}\right)\right)^{36}\\ &= \text{cis}\left(\tfrac{36\pi}{12}\right)\\ &= \text{cis }3\pi\\ &= -1\end{aligned}$$

c $$\begin{aligned}&\left(\sqrt{2}\text{ cis}\left(\tfrac{\pi}{8}\right)\right)^{12}\\ &= \left(\sqrt{2}\right)^{12}\text{ cis}\left(\tfrac{12\pi}{8}\right)\\ &= 2^6\text{ cis}\left(\tfrac{3\pi}{2}\right)\\ &= 64(-i)\\ &= -64i\end{aligned}$$

d $$\begin{aligned}&\sqrt{5\text{ cis}\left(\tfrac{\pi}{7}\right)}\\ &= \left(5\text{ cis}\left(\tfrac{\pi}{7}\right)\right)^{\frac{1}{2}}\\ &= \sqrt{5}\text{ cis}\left(\tfrac{1}{2}\times\tfrac{\pi}{7}\right)\\ &= \sqrt{5}\text{ cis}\left(\tfrac{\pi}{14}\right)\\ (\text{or } &\approx 2.180 + 0.498i)\end{aligned}$$

e $$\begin{aligned}&\sqrt[3]{8\text{ cis}\left(\tfrac{\pi}{2}\right)}\\ &= \left(8\text{ cis}\left(\tfrac{\pi}{2}\right)\right)^{\frac{1}{3}}\\ &= 2\text{ cis}\left(\tfrac{\pi}{6}\right)\\ &= 2\left(\tfrac{\sqrt{3}}{2}+\tfrac{1}{2}i\right)\\ &= \sqrt{3}+i\end{aligned}$$

f $$\begin{aligned}&\left(8\text{ cis}\left(\tfrac{\pi}{5}\right)\right)^{\frac{5}{3}}\\ &= 8^{\frac{5}{3}}\text{ cis}\left(\tfrac{5}{3}\times\tfrac{\pi}{5}\right)\\ &= 2^5\text{ cis}\left(\tfrac{\pi}{3}\right)\\ &= 32\left(\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}i\right)\\ &= 16+16\sqrt{3}i\end{aligned}$$

3 **a** $$\begin{aligned}&(1+i)^{15}\\ &= \left(\sqrt{2}\left(\tfrac{1}{\sqrt{2}}+\tfrac{1}{\sqrt{2}}i\right)\right)^{15}\\ &= \left(\sqrt{2}\text{ cis}\left(\tfrac{\pi}{4}\right)\right)^{15}\\ &= \sqrt{2}^{14}\sqrt{2}\text{ cis}\left(\tfrac{15\pi}{4}\right)\\ &= 2^7\sqrt{2}\text{ cis}\left(\tfrac{7\pi}{4}\right)\\ &= 128\sqrt{2}\left(\tfrac{1}{\sqrt{2}}-\tfrac{1}{\sqrt{2}}i\right)\\ &= 128-128i\end{aligned}$$

b $$\begin{aligned}&(1-i\sqrt{3})^{11}\\ &= \left(2\left(\tfrac{1}{2}-\tfrac{\sqrt{3}}{2}i\right)\right)^{11}\\ &= \left(2\text{ cis}\left(-\tfrac{\pi}{3}\right)\right)^{11}\\ &= 2^{11}\text{ cis}\left(-\tfrac{11\pi}{3}\right)\\ &= 2048\text{ cis}\left(\tfrac{\pi}{3}\right)\\ &= 2048\left(\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}i\right)\\ &= 1024+1024\sqrt{3}i\end{aligned}$$

c $$\begin{aligned}&(\sqrt{2}-i\sqrt{2})^{-19}\\ &= \left(2\left(\tfrac{1}{\sqrt{2}}-\tfrac{1}{\sqrt{2}}i\right)\right)^{-19}\\ &= \left(2\text{ cis}\left(-\tfrac{\pi}{4}\right)\right)^{-19}\\ &= 2^{-19}\text{ cis}\left(\tfrac{19\pi}{4}\right)\\ &= 2^{-19}\text{ cis}\left(\tfrac{3\pi}{4}\right)\\ &= 2^{-19}\left(-\tfrac{1}{\sqrt{2}}+\tfrac{1}{\sqrt{2}}i\right)\\ &= \frac{1}{524\,288}\left(-\tfrac{1}{\sqrt{2}}+\tfrac{1}{\sqrt{2}}i\right)\end{aligned}$$

d $$\begin{aligned}&(-1+i)^{-11}\\ &= \left(\sqrt{2}\left(-\tfrac{1}{\sqrt{2}}+\tfrac{1}{\sqrt{2}}i\right)\right)^{-11}\\ &= \left(\sqrt{2}\text{ cis}\left(\tfrac{3\pi}{4}\right)\right)^{-11}\\ &= (\sqrt{2})^{-11}\text{ cis}\left(-\tfrac{33\pi}{4}\right)\\ &= (\sqrt{2})^{-11}\text{ cis}\left(-\tfrac{\pi}{4}\right)\\ &= (\sqrt{2})^{-11}\left(\tfrac{1}{\sqrt{2}}-\tfrac{1}{\sqrt{2}}i\right)\\ &= (\sqrt{2})^{-12}(1-i)\\ &= \tfrac{1}{64}(1-i)\end{aligned}$$

e $(\sqrt{3}-i)^{\frac{1}{2}}$
$= \left(2\left(\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)\right)^{\frac{1}{2}}$
$= \left(2 \text{ cis}\left(-\frac{\pi}{6}\right)\right)^{\frac{1}{2}}$
$= 2^{\frac{1}{2}} \text{ cis}\left(-\frac{\pi}{12}\right)$
$= \sqrt{2} \text{ cis}\left(-\frac{\pi}{12}\right)$
$= \sqrt{2}\cos\left(-\frac{\pi}{12}\right) + i\sqrt{2}\sin\left(-\frac{\pi}{12}\right)$

f $(2+2i\sqrt{3})^{-\frac{5}{2}}$
$= \left(4\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\right)^{-\frac{5}{2}}$
$= \left(4 \text{ cis}\left(\frac{\pi}{3}\right)\right)^{-\frac{5}{2}}$
$= 2^{-\frac{5}{2}\times 2} \text{ cis}\left(-\frac{5}{2}\times\frac{\pi}{3}\right)$
$= 2^{-5} \text{ cis}\left(-\frac{5\pi}{6}\right)$
$= \frac{1}{32}\left(-\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)$
$= -\frac{1}{64}\left(\sqrt{3}+i\right)$

4 **a** $z = |z| \text{ cis } \theta$
$\sqrt{z} = (|z| \text{ cis } \theta)^{\frac{1}{2}}$
$\sqrt{z} = |z|^{\frac{1}{2}} \text{ cis}\left(\frac{\theta}{2}\right)$ {De Moivre}

b $-\frac{\pi}{2} < \phi \leqslant \frac{\pi}{2}$

c True: $\cos\phi \geqslant 0$ for all $-\frac{\pi}{2} < \phi \leqslant \frac{\pi}{2}$

5 $\text{cis}(-\theta) = \cos(-\theta) + i\sin(-\theta)$
$= \cos\theta - i\sin\theta$

$\therefore (\cos\theta - i\sin\theta)^{-3} = [\text{cis}(-\theta)]^{-3}$
$= \text{cis } 3\theta$ {De Moivre}

6 **a** $z = 1+i = \sqrt{2} \text{ cis}\left(\frac{\pi}{4}\right)$
$\therefore z^n = (\sqrt{2})^n \text{ cis}\left(\frac{n\pi}{4}\right)$ {De Moivre}

b **i** If z^n is real then $\sin\left(\frac{n\pi}{4}\right) = 0$
$\therefore \frac{n\pi}{4} = 0 + k\pi$
$\therefore n = 4k$ where k is any integer

ii If z^n is purely imaginary then $\cos\left(\frac{n\pi}{4}\right) = 0$
$\therefore \frac{n\pi}{4} = \frac{\pi}{2} + k\pi$
$\therefore n = 2+4k$ where k is any integer

7 **a** $z = 2 \text{ cis } \theta$
$\therefore z^3 = 2^3 \text{ cis } 3\theta$
$\therefore \left|z^3\right| = 8$
and $\arg z^3 = 3\theta$

b $z = 2 \text{ cis } \theta$
$\therefore iz^2 = i(2 \text{ cis } \theta)^2$
$= \text{cis}\left(\frac{\pi}{2}\right)(4 \text{ cis } 2\theta)$
$= 4 \text{ cis}\left(\frac{\pi}{2}+2\theta\right)$
$\therefore \left|iz^2\right| = 4$
and $\arg(iz^2) = \frac{\pi}{2} + 2\theta$

c $z = 2 \text{ cis } \theta$
$\therefore \dfrac{1}{z} = (2 \text{ cis } \theta)^{-1}$
$= \frac{1}{2} \text{ cis}(-\theta)$
$\therefore \left|\dfrac{1}{z}\right| = \frac{1}{2}$
and $\arg\left(\dfrac{1}{z}\right) = -\theta$

d $z = 2 \text{ cis } \theta$
$\therefore -\dfrac{i}{z^2} = -i \times z^{-2}$
$= \text{cis}\left(-\frac{\pi}{2}\right) \times (2 \text{ cis } \theta)^{-2}$
$= 2^{-2} \text{ cis}\left(-\frac{\pi}{2}\right) \text{ cis}(-2\theta)$
$= \frac{1}{4} \text{ cis}\left(-\frac{\pi}{2} - 2\theta\right)$
$\therefore \left|-\dfrac{i}{z^2}\right| = \frac{1}{4}$ and $\arg\left(-\dfrac{i}{z^2}\right) = -\frac{\pi}{2} - 2\theta$

8 If $z = \text{cis}\,\theta$, then
$$\begin{aligned}\frac{z^2-1}{z^2+1} &= \frac{(\text{cis}\,\theta)^2-1}{(\text{cis}\,\theta)^2+1}\\ &= \frac{\text{cis}\,2\theta-1}{\text{cis}\,2\theta+1} \quad \{\text{De Moivre}\}\\ &= \frac{\cos 2\theta + i\sin 2\theta - 1}{\cos 2\theta + i\sin 2\theta + 1}\\ &= \frac{(\cancel{1}-2\sin^2\theta)+i\sin 2\theta - \cancel{1}}{(2\cos^2\theta-\cancel{1})+i\sin 2\theta+\cancel{1}}\\ &= \frac{-2\sin^2\theta + 2i\cos\theta\sin\theta}{2\cos^2\theta+2i\cos\theta\sin\theta}\\ &= \frac{2\sin\theta(i\cos\theta + i^2\sin\theta)}{2\cos\theta(\cos\theta+i\sin\theta)}\\ &= \frac{i\sin\theta\ \text{cis}\,\theta}{\cos\theta\ \text{cis}\,\theta}\\ &= i\tan\theta\end{aligned}$$

9 **a**
$$\begin{aligned}\cos 3\theta + i\sin 3\theta &= \text{cis}\,3\theta\\ &= (\text{cis}\,\theta)^3 \quad \{\text{De Moivre's theorem}\}\\ &= (\cos\theta + i\sin\theta)^3\\ &= \cos^3\theta + 3\cos^2\theta(i\sin\theta) + 3\cos\theta(i\sin\theta)^2 + (i\sin\theta)^3\\ &= \left[\cos^3\theta - 3\cos\theta\sin^2\theta\right] + i\left[3\cos^2\theta\sin\theta - \sin^3\theta\right]\end{aligned}$$

i
$$\begin{aligned}\cos 3\theta &= \cos^3\theta - 3\cos\theta\sin^2\theta\\ &\quad \{\text{equating real parts}\}\\ &= \cos^3\theta - 3\cos\theta(1-\cos^2\theta)\\ &= \cos^3\theta - 3\cos\theta + 3\cos^3\theta\\ &= 4\cos^3\theta - 3\cos\theta\end{aligned}$$

ii
$$\begin{aligned}\sin 3\theta &= 3\cos^2\theta\sin\theta - \sin^3\theta\\ &\quad \{\text{equating imaginary parts}\}\\ &= 3(1-\sin^2\theta)\sin\theta - \sin^3\theta\\ &= 3\sin\theta - 3\sin^3\theta - \sin^3\theta\\ &= 3\sin\theta - 4\sin^3\theta\end{aligned}$$

b
$$\begin{aligned}\tan 3\theta = \frac{\sin 3\theta}{\cos 3\theta} &= \frac{3\sin\theta - 4\sin^3\theta}{4\cos^3\theta - 3\cos\theta} \quad \{\text{using } \mathbf{a}\}\\ &= \frac{\sin\theta(3-4\sin^2\theta)}{\cos\theta(4\cos^2\theta-3)}\\ &= \frac{\sin\theta\left[3(\cos^2\theta+\sin^2\theta)-4\sin^2\theta\right]}{\cos\theta\left[4\cos^2\theta-3(\cos^2\theta+\sin^2\theta)\right]} \quad \{\cos^2\theta+\sin^2\theta=1\}\\ &= \frac{\sin\theta(3\cos^2\theta-\sin^2\theta)}{\cos\theta(\cos^2\theta-3\sin^2\theta)}\\ &= \frac{3\sin\theta\cos^2\theta-\sin^3\theta}{\cos^3\theta-3\sin^2\theta\cos\theta}\\ &= \frac{3\dfrac{\sin\theta}{\cos\theta}-\dfrac{\sin^3\theta}{\cos^3\theta}}{1-3\dfrac{\sin^2\theta}{\cos^2\theta}} \quad \{\div \text{ top and bottom by } \cos^3\theta\}\\ &= \frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}\end{aligned}$$

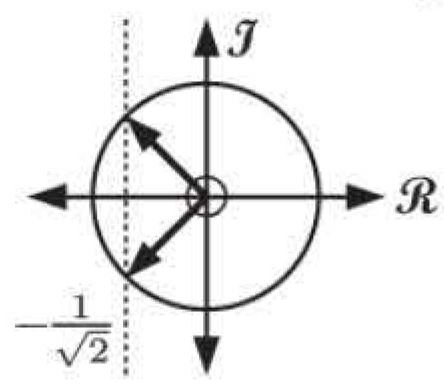
c **i**

$$4x^3 - 3x = -\tfrac{1}{\sqrt{2}}$$

Let $x = \cos\theta$

$$\therefore\ 4\cos^3\theta - 3\cos\theta = -\tfrac{1}{\sqrt{2}}$$

$$\therefore\ \cos 3\theta = -\tfrac{1}{\sqrt{2}}$$



$$3\theta = \tfrac{3\pi}{4}, \tfrac{5\pi}{4}, \tfrac{11\pi}{4}, \tfrac{13\pi}{4}, \tfrac{19\pi}{4}, \tfrac{21\pi}{4}$$

$\{0 \leqslant 3\theta \leqslant 6\pi\}$

$$\therefore\ \theta = \tfrac{\pi}{4}, \tfrac{5\pi}{12}, \tfrac{11\pi}{12}, \tfrac{13\pi}{12}, \tfrac{19\pi}{12}, \tfrac{7\pi}{4}$$

$\{0 \leqslant \theta \leqslant 2\pi\}$

$\therefore\ x = \cos\theta = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ $\quad \{= \cos\frac{7\pi}{4}\}$

or $\cos\frac{5\pi}{12}$ $\quad \{= \cos\frac{19\pi}{12}\}$

or $\cos\frac{11\pi}{12}$ $\quad \{= \cos\frac{13\pi}{12}\}$

ii $x^3 - 3\sqrt{3}x^2 - 3x + \sqrt{3} = 0$

$$\therefore\ \sqrt{3} - 3\sqrt{3}x^2 = 3x - x^3$$

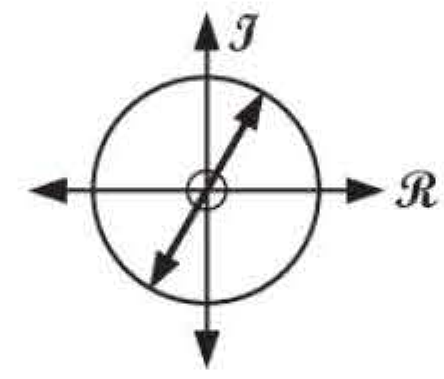
$$\therefore\ \sqrt{3}(1 - 3x^2) = 3x - x^3$$

$$\therefore\ \sqrt{3} = \frac{3x - x^3}{1 - 3x^2}$$

Let $x = \tan\theta$

$$\therefore\ \sqrt{3} = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

$$\therefore\ \tan 3\theta = \sqrt{3}$$



$$3\theta = \tfrac{\pi}{3}, \tfrac{4\pi}{3}, \tfrac{7\pi}{3}, \tfrac{10\pi}{3}, \tfrac{13\pi}{3}, \tfrac{16\pi}{3}$$

$\{0 \leqslant 3\theta \leqslant 6\pi\}$

$$\therefore\ \theta = \tfrac{\pi}{9}, \tfrac{4\pi}{9}, \tfrac{7\pi}{9}, \tfrac{10\pi}{9}, \tfrac{13\pi}{9}, \tfrac{16\pi}{9}$$

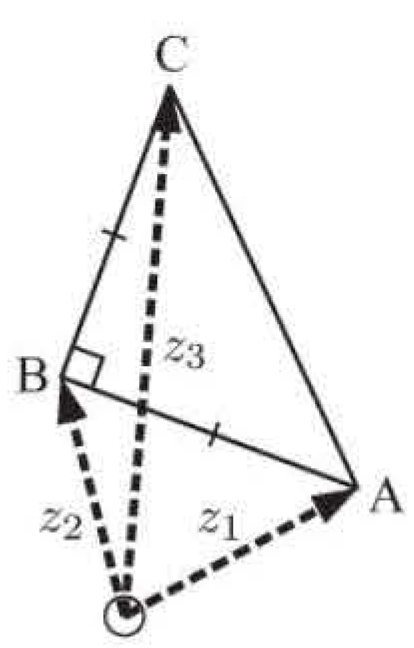
$\{0 \leqslant \theta \leqslant 2\pi\}$

$\therefore\ x = \tan\theta = \tan\frac{\pi}{9}$ $\quad \{= \tan\frac{10\pi}{9}\}$

or $\tan\frac{4\pi}{9}$ $\quad \{= \tan\frac{13\pi}{9}\}$

or $\tan\frac{7\pi}{9}$ $\quad \{= \tan\frac{16\pi}{9}\}$

10 **a**



$\overrightarrow{BC} = \overrightarrow{BO} + \overrightarrow{OC} = -z_2 + z_3 = z_3 - z_2$

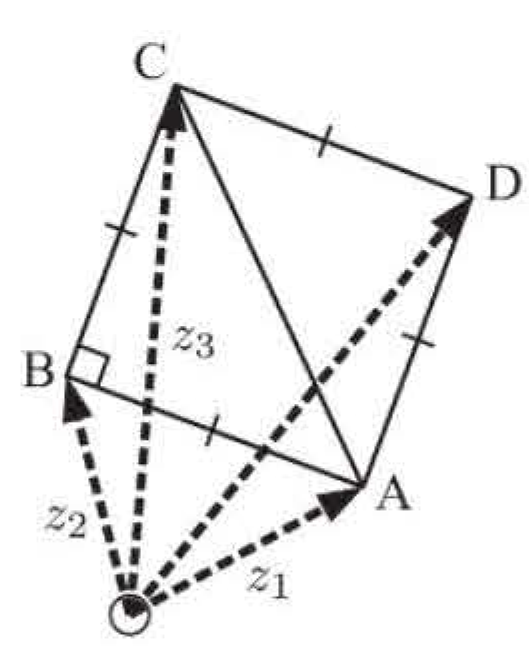
$\overrightarrow{BA} = \overrightarrow{BO} + \overrightarrow{OA} = -z_2 + z_1 = z_1 - z_2$

Suppose $\overrightarrow{BA}$ has length r and argument θ.

$\therefore\ \overrightarrow{BA} = r \text{ cis } \theta$ and $\overrightarrow{BC} = r \text{ cis } (\theta + \frac{\pi}{2})$

$$\begin{aligned}\therefore\ -(z_3 - z_2)^2 &= -1 \times (\overrightarrow{BC})^2 \\ &= \text{cis } \pi \times r^2 \text{ cis } (2\theta + \pi) \\ &= r^2 \text{ cis } (2\theta + 2\pi) \\ &= r^2 \text{ cis } 2\theta \\ &= (r \text{ cis } \theta)^2 \\ &= (z_1 - z_2)^2\end{aligned}$$

b



$\overrightarrow{CD} = \overrightarrow{BA} = z_1 - z_2$

Now $\overrightarrow{OD} = \overrightarrow{OC} + \overrightarrow{CD}$

$\therefore\ \overrightarrow{OD} = z_3 + z_1 - z_2$

$\therefore\ z_3 + z_1 - z_2$ represents D

11
$$\begin{aligned}\cos 4\theta + i\sin 4\theta &= \text{cis } 4\theta \\ &= (\text{cis } \theta)^4 \qquad \{\text{De Moivre's theorem}\} \\ &= (\cos\theta + i\sin\theta)^4 \\ &= \cos^4\theta + 4\cos^3\theta(i\sin\theta) + 6\cos^2\theta(i\sin\theta)^2 + 4\cos\theta(i\sin\theta)^3 + (i\sin\theta)^4 \\ &= \cos^4\theta + [4\cos^3\theta\sin\theta]i - 6\cos^2\theta\sin^2\theta - [4\cos\theta\sin^3\theta]i + \sin^4\theta \\ &= [\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta] + [4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta]i\end{aligned}$$

a Equating real parts gives
$$\begin{aligned}\cos 4\theta &= \cos^4\theta - 6\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2\\ &= \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta\\ &= 8\cos^4\theta - 8\cos^2\theta + 1\end{aligned}$$

b Equating imaginary parts gives $\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$

12 **a** **i** If $z = \text{cis}\,\theta$, then
$$\begin{aligned}z^n + \frac{1}{z^n} &= z^n + z^{-n}\\ &= (\text{cis}\,\theta)^n + (\text{cis}\,\theta)^{-n}\\ &= \text{cis}\,(n\theta) + \text{cis}\,(-n\theta) \qquad \{\text{De Moivre}\}\\ &= (\cos n\theta + i\sin n\theta) + (\cos(-n\theta) + i\sin(-n\theta))\\ &= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta\\ &= 2\cos n\theta\end{aligned}$$

ii In **a i** if we let $n = 1$ we get $z + \dfrac{1}{z} = 2\cos\theta$.

iii
$$\begin{aligned}\left(z+\frac{1}{z}\right)^3 &= z^3 + 3z^2\left(\frac{1}{z}\right) + 3z\left(\frac{1}{z}\right)^2 + \left(\frac{1}{z}\right)^3\\ &= z^3 + 3z + \frac{3}{z} + \frac{1}{z^3}\end{aligned}$$

iv From **a iii**, $\left(z+\dfrac{1}{z}\right)^3 = \left(z^3 + \dfrac{1}{z^3}\right) + 3\left(z + \dfrac{1}{z}\right)$

Using **a i** and **a ii**, $(2\cos\theta)^3 = 2\cos 3\theta + 3(2\cos\theta)$

$\therefore\ 8\cos^3\theta = 2\cos 3\theta + 6\cos\theta$

$\therefore\ \cos^3\theta = \frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta$

v If we let $\theta = \frac{13\pi}{12}$ in **a iv** we get
$$\begin{aligned}\cos^3\left(\tfrac{13\pi}{12}\right) &= \tfrac{1}{4}\cos\left(\tfrac{39\pi}{12}\right) + \tfrac{3}{4}\cos\left(\tfrac{13\pi}{12}\right)\\ &= \tfrac{1}{4}\cos\left(\tfrac{13\pi}{4}\right) + \tfrac{3}{4}\cos\left(\tfrac{3\pi}{4} + \tfrac{\pi}{3}\right)\\ &= \tfrac{1}{4}\cos\left(\tfrac{5\pi}{4}\right) + \tfrac{3}{4}\left[\cos\left(\tfrac{3\pi}{4}\right)\cos\left(\tfrac{\pi}{3}\right) - \sin\left(\tfrac{3\pi}{4}\right)\sin\left(\tfrac{\pi}{3}\right)\right]\\ &= \tfrac{1}{4}\left(-\tfrac{1}{\sqrt{2}}\right) + \tfrac{3}{4}\left[\left(-\tfrac{1}{\sqrt{2}}\right)\left(\tfrac{1}{2}\right) - \left(\tfrac{1}{\sqrt{2}}\right)\left(\tfrac{\sqrt{3}}{2}\right)\right]\\ &= -\tfrac{1}{4\sqrt{2}} - \tfrac{3}{8\sqrt{2}} - \tfrac{3\sqrt{3}}{8\sqrt{2}}\\ &= \tfrac{-2-3-3\sqrt{3}}{8\sqrt{2}} \times \tfrac{\sqrt{2}}{\sqrt{2}}\\ &= \tfrac{-5\sqrt{2}-3\sqrt{6}}{16}\end{aligned}$$

b If $z = \text{cis}\,\theta$, then
$$\begin{aligned}z^n - \frac{1}{z^n} &= z^n - z^{-n}\\ &= (\text{cis}\,\theta)^n - (\text{cis}\,\theta)^{-n}\\ &= \text{cis}\,n\theta - \text{cis}\,(-n\theta) \qquad \{\text{De Moivre}\}\\ &= \cos n\theta + i\sin n\theta - [\cos(-n\theta) + i\sin(-n\theta)]\\ &= \cos n\theta + i\sin n\theta - \cos(-n\theta) - i\sin(-n\theta)\\ &= \cos n\theta + i\sin n\theta - \cos n\theta + i\sin n\theta\\ &= 2i\sin n\theta \quad .... \ (*)\end{aligned}$$

If we let $n = 1$, $z - \dfrac{1}{z} = 2i\sin\theta$

$$\begin{aligned}\therefore\ [2i\sin\theta]^3 &= \left(z - \frac{1}{z}\right)^3\\ \therefore\ 8i^3\sin^3\theta &= z^3 + 3z^2\left(-\frac{1}{z}\right) + 3z\left(-\frac{1}{z}\right)^2 + \left(-\frac{1}{z}\right)^3\\ &= z^3 - \frac{1}{z^3} - 3\left(z - \frac{1}{z}\right)\end{aligned}$$

$\therefore\ 8i^3\sin^3\theta = 2i\sin 3\theta - 3\times 2i\sin\theta$ {using $(*)$}

$\therefore\ -8i\sin^3\theta = 2i\sin 3\theta - 6i\sin\theta$

$\therefore\ \sin^3\theta = -\frac{1}{4}\sin 3\theta + \frac{3}{4}\sin\theta$

$\therefore\ \sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$

c $\sin^3\theta\cos^3\theta = (\frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta)(\frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta)$ {using **12 a iv** and **b**}

$= \frac{3}{16}\sin\theta\cos 3\theta + \frac{9}{16}\sin\theta\cos\theta - \frac{1}{16}\sin 3\theta\cos 3\theta - \frac{3}{16}\sin 3\theta\cos\theta$

$= \frac{3}{16}(\sin\theta\cos 3\theta - \sin 3\theta\cos\theta) + \frac{9}{32}(2\sin\theta\cos\theta) - \frac{1}{32}(2\sin 3\theta\cos 3\theta)$

$= \frac{3}{16}(\sin(\theta - 3\theta)) + \frac{9}{32}\sin 2\theta - \frac{1}{32}\sin 6\theta$

$= -\frac{3}{16}\sin 2\theta + \frac{9}{32}\sin 2\theta - \frac{1}{32}\sin 6\theta$

$= \frac{3}{32}\sin 2\theta - \frac{1}{32}\sin 6\theta$

$= \frac{1}{32}(3\sin 2\theta - \sin 6\theta)$

EXERCISE 16F.1

1 **a** The cube roots of 1 are solutions to $z^3 = 1$, or $z^3 - 1 = 0$

Now $z = 1$ is a solution, so $z - 1$ is a factor.

$$\begin{array}{c|cccc} 1 & 1 & 0 & 0 & -1 \\ & 0 & 1 & 1 & 1 \\ \hline & 1 & 1 & 1 & 0 \end{array}$$

$\therefore\ (z-1)(z^2+z+1) = 0$

$\therefore\ z = 1$ or $\dfrac{-1\pm\sqrt{1-4}}{2}$

$\therefore\ z = 1$ or $-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

b $z^3 = 1$

$\therefore\ z^3 = 1\text{ cis}\,(0 + k2\pi)$

$\therefore\ z = (1\text{ cis}\,(k2\pi))^{\frac{1}{3}}$

$\therefore\ z = 1\text{ cis}\left(\frac{k2\pi}{3}\right)$ {De Moivre}

$\therefore\ z = \text{cis}\,0,\ \text{cis}\left(\frac{2\pi}{3}\right)$ or $\text{cis}\left(\frac{4\pi}{3}\right)$ {when $k = 0, 1, 2$}

$\therefore\ z = 1,\ -\frac{1}{2} + i\frac{\sqrt{3}}{2},\ -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

2 **a** $z^3 = -8i$

$\therefore\ z^3 = 8\text{ cis}\left(-\frac{\pi}{2} + k2\pi\right)$

$\therefore\ z = \left(8\text{ cis}\left(-\frac{\pi}{2} + k2\pi\right)\right)^{\frac{1}{3}}$

$\therefore\ z = 8^{\frac{1}{3}}\text{ cis}\left(-\frac{\pi}{6} + \frac{k2\pi}{3}\right)$ {De Moivre}

$\therefore\ z = 2\text{ cis}\left(-\frac{\pi}{6}\right),\ 2\text{ cis}\left(\frac{3\pi}{6}\right)$ or $2\text{ cis}\left(\frac{7\pi}{6}\right)$ {when $k = 0, 1, 2$}

$\therefore\ z = 2\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right),\ 2\text{ cis}\,\frac{\pi}{2}$ or $2\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$

$\therefore\ z = \sqrt{3} - i,\ 2i,\ -\sqrt{3} - i$

b $z^3 = -27i$ $\therefore\ z^3 = 27\text{ cis}\left(-\frac{\pi}{2} + k2\pi\right)$

$\therefore\ z = \left(27\text{ cis}\left(-\frac{\pi}{2} + k2\pi\right)\right)^{\frac{1}{3}}$

$\therefore\ z = 27^{\frac{1}{3}}\text{ cis}\left(-\frac{\pi}{6} + \frac{k4\pi}{6}\right)$ {De Moivre}

$\therefore\ z = 3\text{ cis}\left(-\frac{\pi}{6}\right),\ 3\text{ cis}\left(\frac{3\pi}{6}\right)$ or $3\text{ cis}\left(\frac{7\pi}{6}\right)$ {when $k = 0, 1, 2$}

$\therefore\ z = 3\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right),\ 3\text{ cis}\,\frac{\pi}{2}$ or $3\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$

$\therefore\ z = \frac{3\sqrt{3}}{2} - \frac{3}{2}i,\ 3i,\ \frac{-3\sqrt{3}}{2} - \frac{3}{2}i$

3 $z^3 = -1$

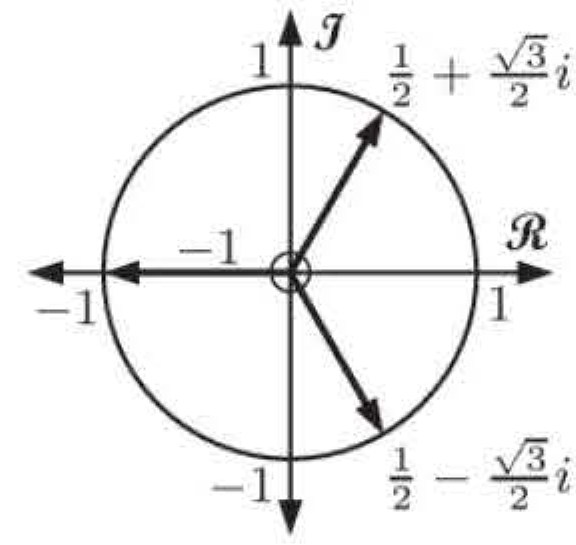
$\therefore\ z^3 = 1 \text{ cis}\,(\pi + k2\pi)$

$\therefore\ z = (\text{cis}\,(\pi + k2\pi))^{\frac{1}{3}}$

$\therefore\ z = \text{cis}\left(\frac{\pi}{3} + \frac{k2\pi}{3}\right)$ {De Moivre}

$\therefore\ z = \text{cis}\left(\frac{\pi}{3}\right),\ \text{cis}\,\pi,\ \text{cis}\left(\frac{5\pi}{3}\right)$ {when $k = 0,\ 1,\ 2$}

$\therefore\ z = \frac{1}{2} + i\frac{\sqrt{3}}{2},\ -1,\ \frac{1}{2} - i\frac{\sqrt{3}}{2}$



4 **a** $z^4 = 16 \quad \therefore\ z^4 = 16 \text{ cis}\,(0 + k2\pi)$

$\therefore\ z = (16 \text{ cis}\,(k2\pi))^{\frac{1}{4}}$

$\therefore\ z = 16^{\frac{1}{4}} \text{ cis}\left(\frac{k\pi}{2}\right)$ {De Moivre}

$\therefore\ z = 2 \text{ cis } 0,\ 2 \text{ cis}\left(\frac{\pi}{2}\right),\ 2 \text{ cis}\,\pi,\ 2 \text{ cis}\left(\frac{3\pi}{2}\right)$ {when $k = 0,\ 1,\ 2$, or 3}

$\therefore\ z = \pm 2$ or $\pm 2i$

b $z^4 = -16 \quad \therefore\ z^4 = 16 \text{ cis}\,(\pi + k2\pi)$

$\therefore\ z = (16 \text{ cis}\,(\pi + k2\pi))^{\frac{1}{4}}$

$\therefore\ z = 16^{\frac{1}{4}} \text{ cis}\left(\frac{\pi}{4} + \frac{k2\pi}{4}\right)$ {De Moivre}

$\therefore\ z = 2 \text{ cis}\left(\frac{\pi}{4}\right),\ 2 \text{ cis}\left(\frac{3\pi}{4}\right),\ 2 \text{ cis}\left(\frac{5\pi}{4}\right),\ 2 \text{ cis}\left(\frac{7\pi}{4}\right)$ {when $k = 0,\ 1,\ 2$, or 3}

$\therefore\ z = 2\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right),\ 2\left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right),\ 2\left(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right),\ 2\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)$

$\therefore\ z = \sqrt{2} \pm i\sqrt{2},\ -\sqrt{2} \pm i\sqrt{2}$

5 $z^4 = -i$

$\therefore\ z^4 = \text{cis}\left(-\frac{\pi}{2} + k2\pi\right)$

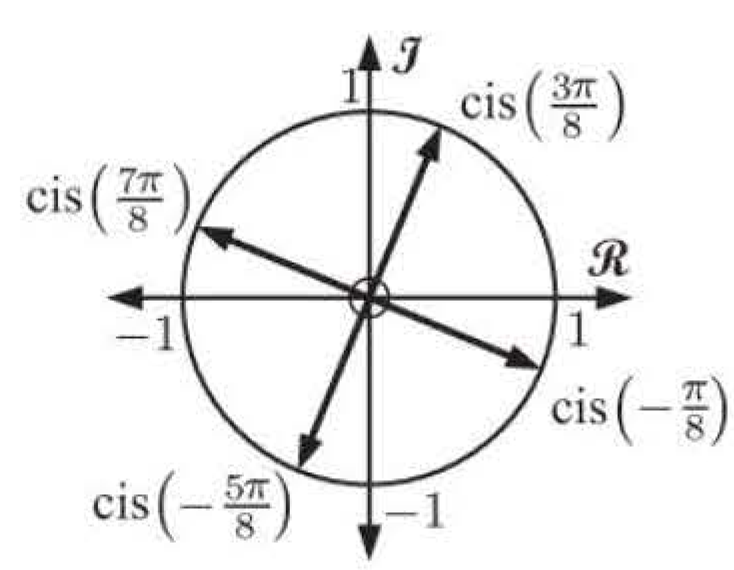
$\therefore\ z = \left(\text{cis}\left(-\frac{\pi}{2} + k2\pi\right)\right)^{\frac{1}{4}}$

$\therefore\ z = \text{cis}\left(-\frac{\pi}{8} + \frac{k\pi}{2}\right)$ {De Moivre}

$\therefore\ z = \text{cis}\left(-\frac{\pi}{8} + \frac{k4\pi}{8}\right)$

$\therefore\ z = \text{cis}\left(-\frac{5\pi}{8}\right),\ \text{cis}\left(-\frac{\pi}{8}\right),\ \text{cis}\left(\frac{3\pi}{8}\right),\ \text{cis}\left(\frac{7\pi}{8}\right)$

{when $k = -1,\ 0,\ 1,\ 2$}



6 **a** $z^3 = 2 + 2i$

$\therefore\ z^3 = 2\sqrt{2} \text{ cis}\left(\frac{\pi}{4} + k2\pi\right)$

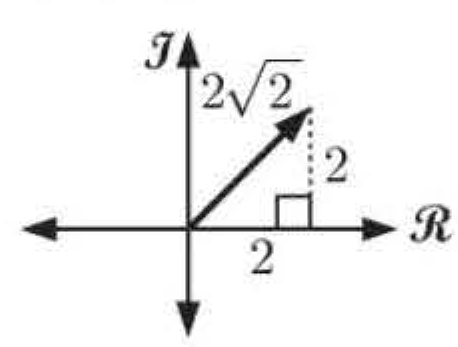
$\therefore\ z = \left[2\sqrt{2} \text{ cis}\left(\frac{\pi}{4} + k2\pi\right)\right]^{\frac{1}{3}}$

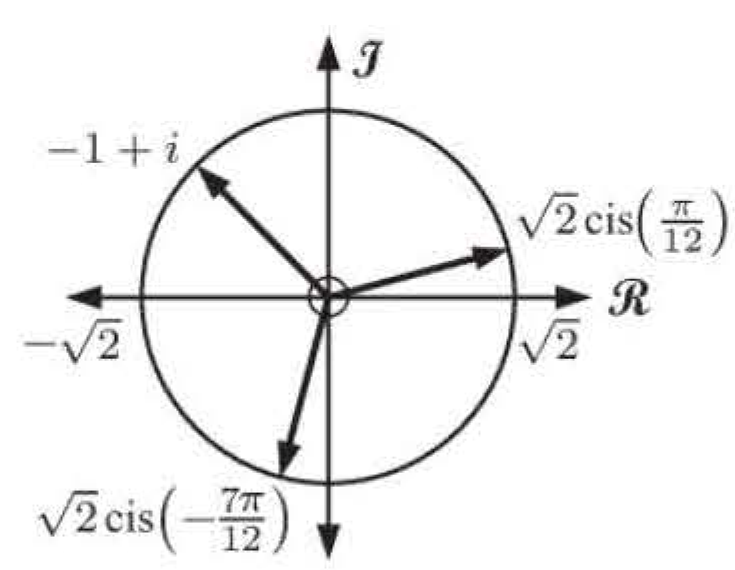
$\therefore\ z = \left(2\sqrt{2}\right)^{\frac{1}{3}} \text{ cis}\left(\frac{\pi}{12} + \frac{k2\pi}{3}\right)$ {De Moivre}

$\therefore\ z = \sqrt{2} \text{ cis}\left(\frac{\pi}{12} + \frac{k8\pi}{12}\right)$

$\therefore\ z = \sqrt{2} \text{ cis}\left(-\frac{7\pi}{12}\right),\ \sqrt{2} \text{ cis}\left(\frac{\pi}{12}\right),\ \sqrt{2} \text{ cis}\left(\frac{3\pi}{4}\right) = -1 + i$

{when $k = -1,\ 0,\ 1$}





b $z^3 = -2 + 2i$

$\therefore\ z^3 = 2\sqrt{2} \text{ cis}\left(\frac{3\pi}{4} + k2\pi\right)$

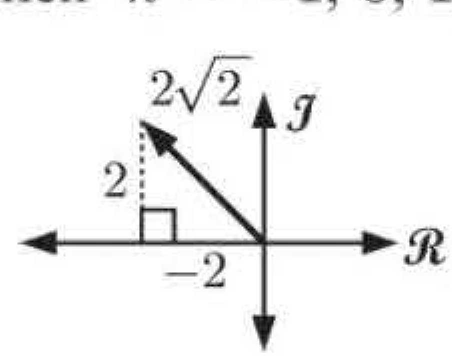
$\therefore\ z = \left[2\sqrt{2} \text{ cis}\left(\frac{3\pi}{4} + k2\pi\right)\right]^{\frac{1}{3}}$

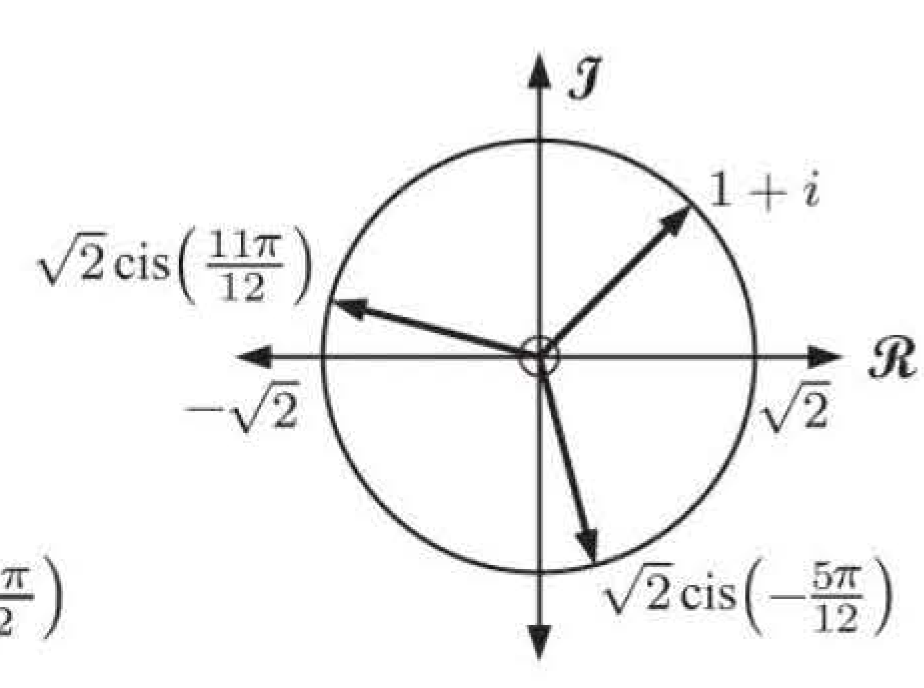
$\therefore\ z = \left(2\sqrt{2}\right)^{\frac{1}{3}} \text{ cis}\left(\frac{\pi}{4} + \frac{k2\pi}{3}\right)$ {De Moivre}

$\therefore\ z = \sqrt{2} \text{ cis}\left(\frac{3\pi}{12} + \frac{k8\pi}{12}\right)$

$\therefore\ z = \sqrt{2} \text{ cis}\left(-\frac{5\pi}{12}\right),\ \sqrt{2} \text{ cis}\left(\frac{\pi}{4}\right) = 1 + i,\ \sqrt{2} \text{ cis}\left(\frac{11\pi}{12}\right)$

{when $k = -1,\ 0,\ 1$}





c $z^2 = \frac{1}{2} + \frac{\sqrt{3}}{2}i$

$\therefore\ z^2 = \text{cis}\left(\frac{\pi}{3} + k2\pi\right)$

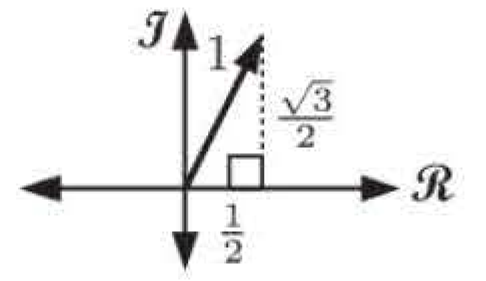
$\therefore\ z = \left[\text{cis}\left(\frac{\pi}{3} + k2\pi\right)\right]^{\frac{1}{2}}$

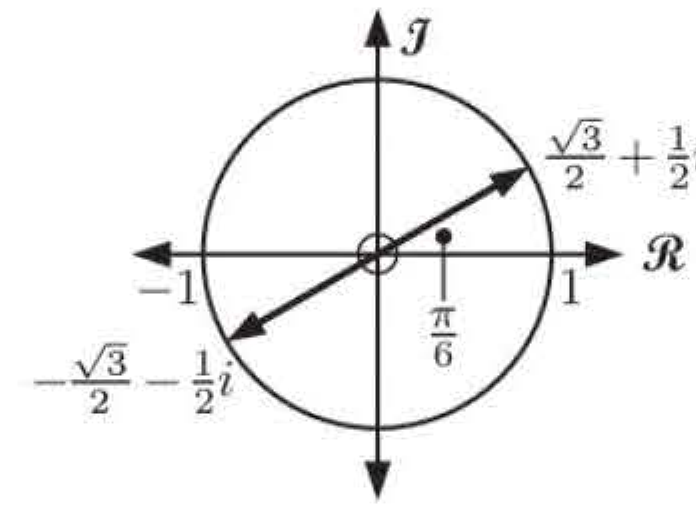
$\therefore\ z = \text{cis}\left(\frac{\pi}{6} + k\pi\right)$ {De Moivre}

$\therefore\ z = \text{cis}\left(\frac{\pi}{6} + \frac{k6\pi}{6}\right)$

$\therefore\ z = \text{cis}\left(-\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2} - \frac{1}{2}i,\quad \text{cis}\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} + \frac{1}{2}i$

{when $k = -1, 0$}





d $z^4 = \sqrt{3} + i$

$\therefore\ z^4 = 2\ \text{cis}\left(\frac{\pi}{6} + k2\pi\right)$

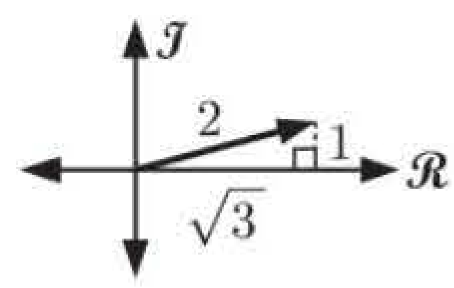
$\therefore\ z = \left[2\ \text{cis}\left(\frac{\pi}{6} + k2\pi\right)\right]^{\frac{1}{4}}$

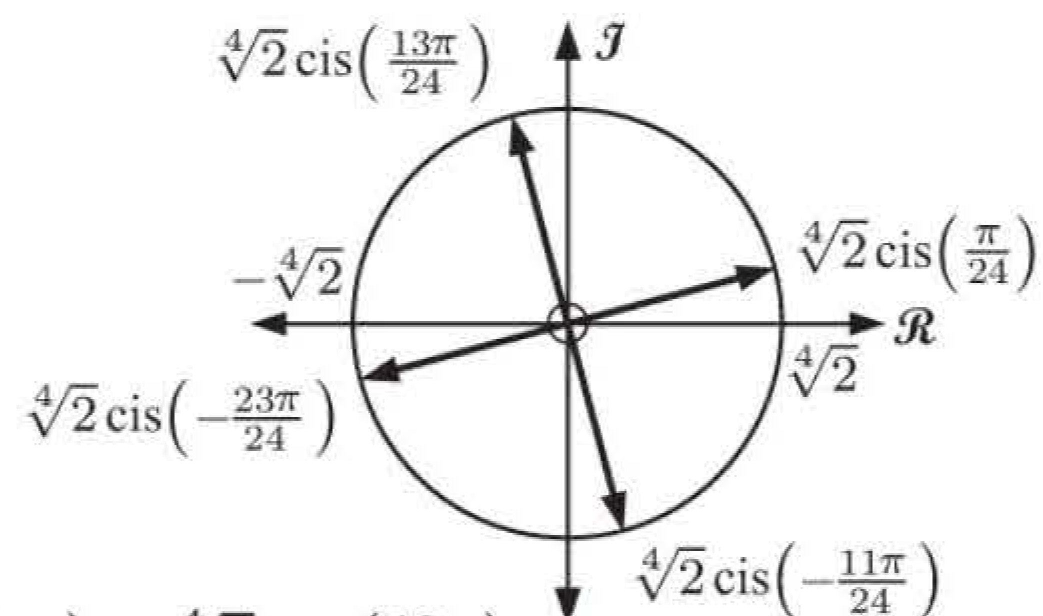
$\therefore\ z = 2^{\frac{1}{4}}\ \text{cis}\left(\frac{\pi}{24} + \frac{k\pi}{2}\right)$ {De Moivre}

$\therefore\ z = \sqrt[4]{2}\ \text{cis}\left(\frac{\pi}{24} + \frac{k12\pi}{24}\right)$

$\therefore\ z = \sqrt[4]{2}\ \text{cis}\left(-\frac{23\pi}{24}\right),\ \sqrt[4]{2}\ \text{cis}\left(-\frac{11\pi}{24}\right),\ \sqrt[4]{2}\ \text{cis}\left(\frac{\pi}{24}\right),\ \sqrt[4]{2}\ \text{cis}\left(\frac{13\pi}{24}\right)$

{when $k = -2, -1, 0, 1$}





e $z^5 = -4 - 4i$

$\therefore\ z^5 = 4\sqrt{2}\ \text{cis}\left(-\frac{3\pi}{4} + k2\pi\right)$

$\therefore\ z = \left[4\sqrt{2}\ \text{cis}\left(-\frac{3\pi}{4} + k2\pi\right)\right]^{\frac{1}{5}}$

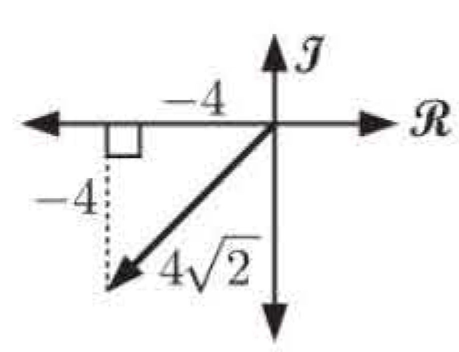
$\therefore\ z = \left(4\sqrt{2}\right)^{\frac{1}{5}}\ \text{cis}\left(-\frac{3\pi}{20} + \frac{k2\pi}{5}\right)$ {De Moivre}

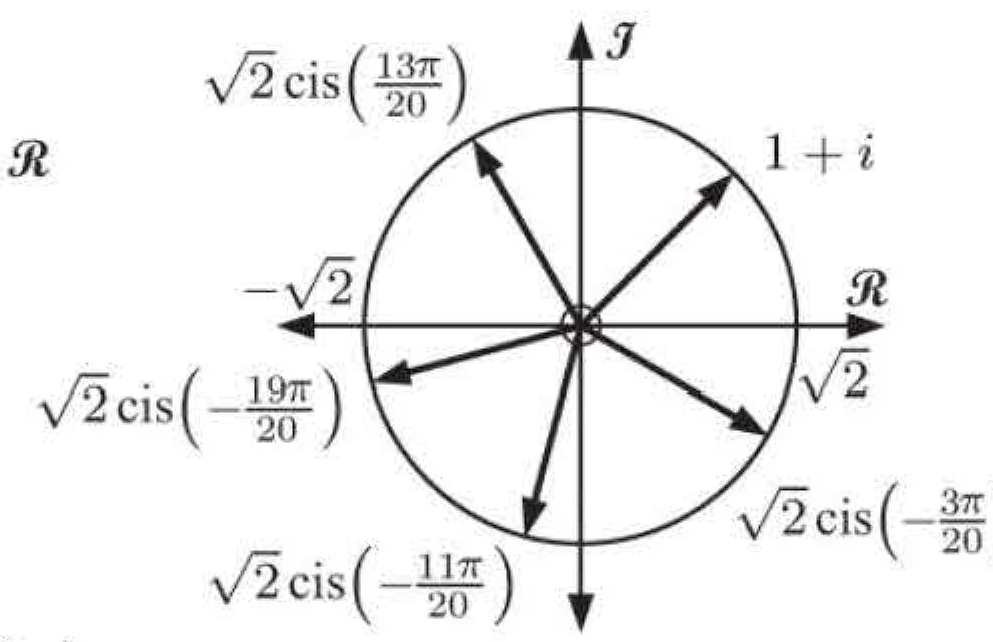
$\therefore\ z = \sqrt{2}\ \text{cis}\left(-\frac{3\pi}{20} + \frac{k8\pi}{20}\right)$

$\therefore\ z = \sqrt{2}\ \text{cis}\left(-\frac{19\pi}{20}\right),\ \sqrt{2}\ \text{cis}\left(-\frac{11\pi}{20}\right),\ \sqrt{2}\ \text{cis}\left(-\frac{3\pi}{20}\right),$

$\sqrt{2}\ \text{cis}\left(\frac{\pi}{4}\right) = 1 + i,\ \sqrt{2}\ \text{cis}\left(\frac{13\pi}{20}\right)$

{when $k = -2, -1, 0, 1, 2$}





f $z^3 = -2\sqrt{3} - 2i$

$\therefore\ z^3 = 4\ \text{cis}\left(-\frac{5\pi}{6} + k2\pi\right)$

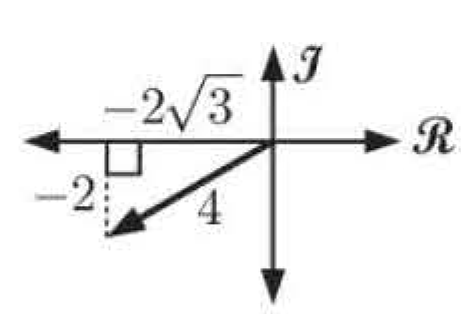
$\therefore\ z = \left[4\ \text{cis}\left(-\frac{5\pi}{6} + k2\pi\right)\right]^{\frac{1}{3}}$

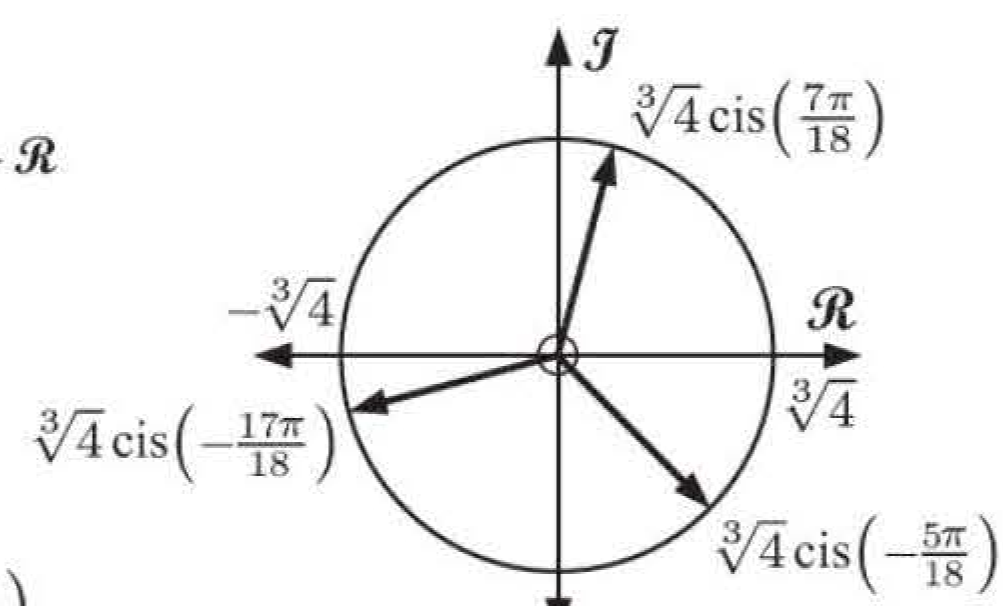
$\therefore\ z = 4^{\frac{1}{3}}\ \text{cis}\left(-\frac{5\pi}{18} + \frac{k2\pi}{3}\right)$ {De Moivre}

$\therefore\ z = \sqrt[3]{4}\ \text{cis}\left(-\frac{5\pi}{18} + \frac{k12\pi}{18}\right)$

$\therefore\ z = \sqrt[3]{4}\ \text{cis}\left(-\frac{17\pi}{18}\right),\ \sqrt[3]{4}\ \text{cis}\left(-\frac{5\pi}{18}\right),\ \sqrt[3]{4}\ \text{cis}\left(\frac{7\pi}{18}\right)$

{when $k = -1, 0, 1$}





7 **a** $z^4 + 1 = 0$

$\therefore\ z^4 = -1$

$\therefore\ z^4 = \text{cis}\left(\pi + k2\pi\right)$

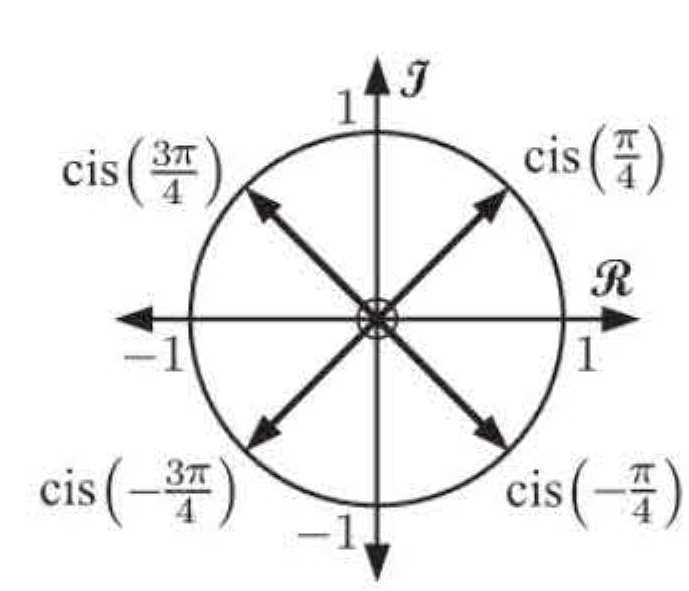
$\therefore\ z = \left[\text{cis}\left(\pi + k2\pi\right)\right]^{\frac{1}{4}}$

$\therefore\ z = \text{cis}\left(\frac{\pi}{4} + \frac{k2\pi}{4}\right)$ {De Moivre}

$\therefore\ z = \text{cis}\left(-\frac{3\pi}{4}\right),\ \text{cis}\left(-\frac{\pi}{4}\right),\ \text{cis}\left(\frac{\pi}{4}\right),\ \text{cis}\left(\frac{3\pi}{4}\right)$

{when $k = -2, -1, 0,$ or 1}

$\therefore\ z = \frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}}i,\ -\frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}}i$



b For the pair of roots $\frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}}i$,

$$\text{sum} = \frac{1}{\sqrt{2}} + \cancel{\frac{1}{\sqrt{2}}i} + \frac{1}{\sqrt{2}} - \cancel{\frac{1}{\sqrt{2}}i} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

$$\text{product} = \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) = \frac{1}{2} + \frac{1}{2} = 1$$

and so we have a quadratic factor of $z^2 - \sqrt{2}z + 1$

For the pair of roots $-\frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}}i$,

$$\text{sum} = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i = -\frac{2}{\sqrt{2}} = -\sqrt{2}$$

$$\text{product} = \left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) = \frac{1}{2} + \frac{1}{2} = 1$$

and so we have a quadratic factor of $z^2 + \sqrt{2}z + 1$

$\therefore \quad z^4 + 1 = \left(z^2 - \sqrt{2}z + 1\right)\left(z^2 + \sqrt{2}z + 1\right)$

8 **a**

$$\begin{aligned} z &= \frac{\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)^2}{\left(\cos\frac{\pi}{10} - i\sin\frac{\pi}{10}\right)^5\left(\cos\frac{\pi}{30} + i\sin\frac{\pi}{30}\right)^{25}} \\ &= \frac{\text{cis}\left(-\frac{\pi}{6}\right)^2}{\left(\cos\left(-\frac{\pi}{10}\right) + i\sin\left(-\frac{\pi}{10}\right)\right)^5\left(\cos\frac{\pi}{30} + i\sin\frac{\pi}{30}\right)^{25}} \\ &= \frac{\text{cis}\left(-\frac{\pi}{3}\right)}{\left(\text{cis}\left(-\frac{\pi}{10}\right)\right)^5\left(\text{cis}\left(\frac{\pi}{30}\right)\right)^{25}} \\ &= \frac{\text{cis}\left(-\frac{\pi}{3}\right)}{\text{cis}\left(-\frac{\pi}{2}\right)\text{cis}\left(\frac{5\pi}{6}\right)} \\ &= \frac{\text{cis}\left(-\frac{\pi}{3}\right)}{\text{cis}\left(\frac{\pi}{3}\right)} \\ &= \text{cis}\left(-\frac{2\pi}{3}\right) \end{aligned}$$

$\therefore \quad |z| = 1, \quad \arg z = -\frac{2\pi}{3}$

b

$$\begin{aligned} z^3 &= \left[\text{cis}\left(-\frac{2\pi}{3}\right)\right]^3 \\ &= \text{cis}\,(-2\pi) \quad \{\text{De Moivre}\} \\ &= 1 \end{aligned}$$

$\therefore \quad z$ is a cube root of 1.

c

$$\begin{aligned} (1-2z)(2z^2-1) &= 2z^2 - 1 - 4z^3 + 2z \\ &= 2z^2 - 1 - 4(1) + 2z \quad \{z^3 = 1\} \\ &= 2z^2 + 2z - 5 \end{aligned}$$

$$\begin{aligned} \text{Now} \quad z^3 = 1 \quad \therefore \quad z^2 &= z^{-1}, \quad z \neq 0 \\ &= \left[\text{cis}\left(-\frac{2\pi}{3}\right)\right]^{-1} \\ &= \text{cis}\left(\frac{2\pi}{3}\right) \quad \{\text{De Moivre}\} \\ &= z^* \end{aligned}$$

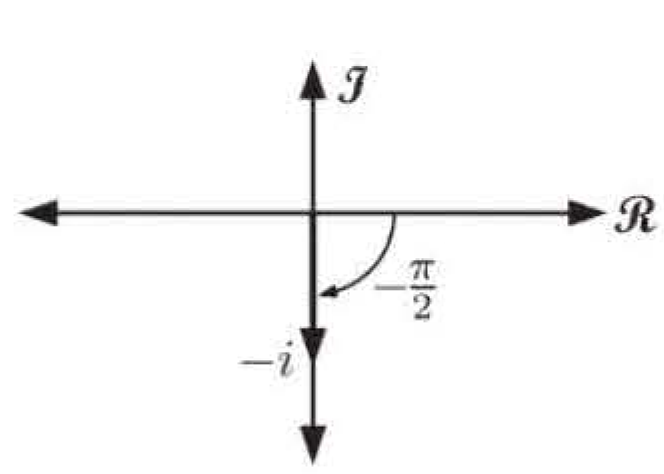
$$\begin{aligned} \therefore \quad 2z^2 + 2z - 5 &= 2z^* + 2z - 5 \\ &= 2(z + z^*) - 5 \end{aligned}$$

which is real as $z + z^*$ is always real.

9 **a** $-16i = 16 \text{ cis}\left(-\frac{\pi}{2}\right)$

b

$$\begin{aligned} z^4 &= -16i \\ \therefore \quad z^4 &= 16 \text{ cis}\left(-\frac{\pi}{2} + k2\pi\right) \\ \therefore \quad z &= \left[16 \text{ cis}\left(-\frac{\pi}{2} + k2\pi\right)\right]^{\frac{1}{4}} \\ \therefore \quad z &= 16^{\frac{1}{4}} \text{ cis}\left(-\frac{\pi}{8} + \frac{k\pi}{2}\right) \quad \{\text{De Moivre}\} \\ \therefore \quad z &= 2 \text{ cis}\left(-\frac{\pi}{8} + \frac{k4\pi}{8}\right) \end{aligned}$$



$\therefore \quad z = 2 \text{ cis}\left(-\frac{5\pi}{8}\right), \ 2 \text{ cis}\left(-\frac{\pi}{8}\right), \ 2 \text{ cis}\left(\frac{3\pi}{8}\right), \ 2 \text{ cis}\left(\frac{7\pi}{8}\right) \quad \{\text{when } k = -1, 0, 1, 2\}$

i The 4th root in the second quadrant is $z = 2\text{ cis}\left(\frac{7\pi}{8}\right)$ $\quad\{\frac{\pi}{2} < \frac{7\pi}{8} < \pi\}$

ii In Cartesian form, $z = 2\left[\cos\left(\frac{7\pi}{8}\right) + i\sin\left(\frac{7\pi}{8}\right)\right]$

$$= 2\cos\left(\frac{7\pi}{8}\right) + \left[2\sin\left(\frac{7\pi}{8}\right)\right]i$$

EXERCISE 16F.2

1 **a** **i** $(z+3)^3 = 1$

$\therefore\ z+3 = 1,\ w,$ or w^2 where $w = \text{cis}\left(\frac{2\pi}{3}\right)$

$\therefore\ z+3 = w^n$ where $n = 0, 1, 2$

$\therefore\ z = w^n - 3$ where $n = 0, 1, 2$ and $w = \text{cis}\left(\frac{2\pi}{3}\right)$

ii $(z-1)^3 = 8$

$\therefore\ \left[\dfrac{z-1}{2}\right]^3 = 1$

$\therefore\ \dfrac{z-1}{2} = 1,\ w,$ or w^2 where $w = \text{cis}\left(\frac{2\pi}{3}\right)$

$\therefore\ \dfrac{z-1}{2} = w^n$ where $n = 0, 1, 2$

$\therefore\ z = 2w^n + 1$ where $n = 0, 1, 2$ and $w = \text{cis}\left(\frac{2\pi}{3}\right)$

iii $(2z-1)^3 = -1$

$\therefore\ (2z-1)^3 = (-1)^3$

$\therefore\ \left(\dfrac{2z-1}{-1}\right)^3 = 1$

$\therefore\ (1-2z)^3 = 1$

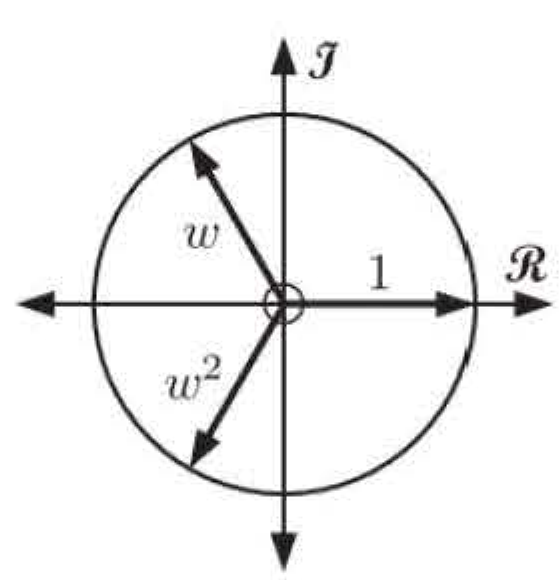
$\therefore\ 1-2z = 1,\ w,\ w^2$ where $w = \text{cis}\left(\frac{2\pi}{3}\right)$

$\therefore\ 1-2z = w^n$ where $n = 0, 1, 2$

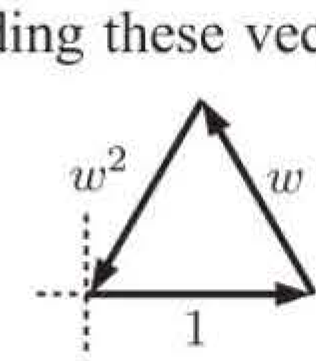
$\therefore\ 2z = 1 - w^n$

$\therefore\ z = \dfrac{1-w^n}{2}$ where $n = 0, 1, 2$ and $w = \text{cis}\left(\frac{2\pi}{3}\right)$

b The following represents the cube roots of unity:



Adding these vectorially



the resultant vector is $\mathbf{0}$

$\therefore\ 1 + w + w^2 = 0$

2 **a** If $w = \text{cis}\left(\frac{\pi}{2}\right)$, $w^2 = \text{cis}\,\pi$

and $w^3 = \text{cis}\left(\frac{3\pi}{2}\right)$ {De Moivre}

$\therefore\ w = i,\ w^2 = -1,$ and $w^3 = -i$

$\therefore\ 1, i, -1, -i$ can be written as

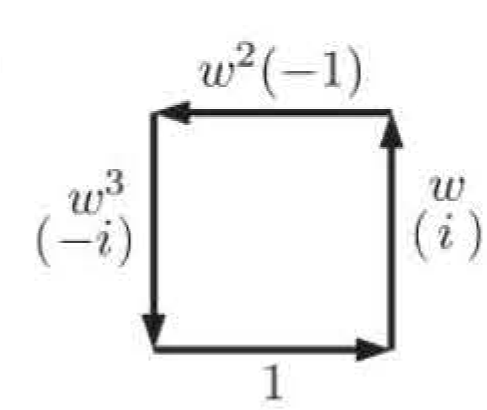
$1, w, w^2, w^3,$ where $w = \text{cis}\left(\frac{\pi}{2}\right)$

b $1 + w + w^2 + w^3 = 1 + (i) + (-1) + (-i)$

$= 1 + i - 1 - i$

$= 0$

c Adding these vectorially



3 **a** The 5th roots of unity are the solutions to $z^5 = 1$.

$\therefore\ z^5 = \text{cis}\,(0 + k2\pi)$

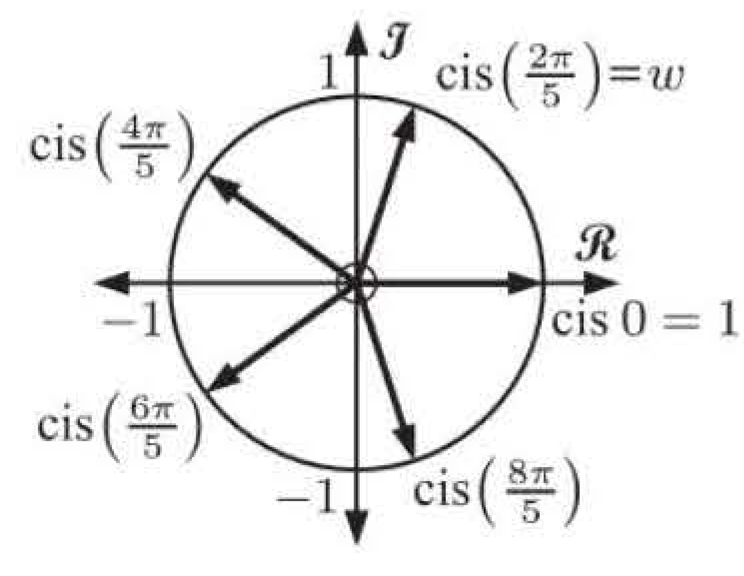
$\therefore\ z^5 = \text{cis}\,(k2\pi)$

$\therefore\ z = [\text{cis}\,(k2\pi)]^{\frac{1}{5}}$

$\therefore\ z = \text{cis}\left(\frac{k2\pi}{5}\right)$ {De Moivre}

$\therefore\ z = \text{cis}\,0 = 1,\ \text{cis}\left(\frac{2\pi}{5}\right),\ \text{cis}\left(\frac{4\pi}{5}\right),\ \text{cis}\left(\frac{6\pi}{5}\right),\ \text{cis}\left(\frac{8\pi}{5}\right)$

{when $k = 0, 1, 2, 3, 4$}



b If $\text{cis}\left(\frac{2\pi}{5}\right) = w$, then $\text{cis}\left(\frac{4\pi}{5}\right) = \left(\text{cis}\ \frac{2\pi}{5}\right)^2 = w^2$

$\text{cis}\left(\frac{6\pi}{5}\right) = \left(\text{cis}\ \frac{2\pi}{5}\right)^3 = w^3$

$\text{cis}\left(\frac{8\pi}{5}\right) = \left(\text{cis}\ \frac{2\pi}{5}\right)^4 = w^4$

Hence the five roots can be expressed as $1, w, w^2, w^3, w^4$ where $w = \text{cis}\left(\frac{2\pi}{5}\right)$

c $(1+w+w^2+w^3+w^4)(1-w)$

$= 1+w+w^2+w^3+w^4-w-w^2-w^3-w^4-w^5$

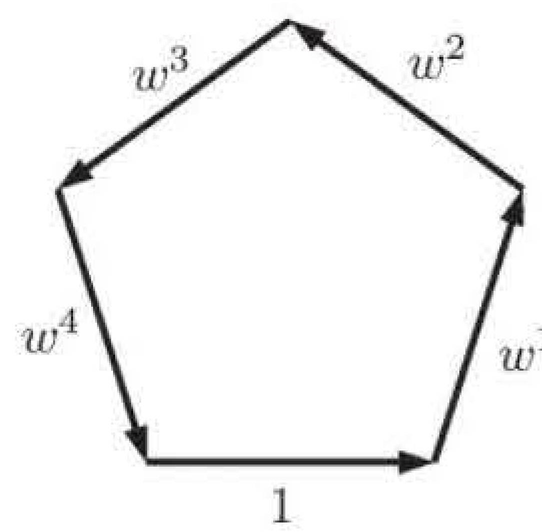
$= 1-w^5$

Since $w^5 = 1$, $1-w^5 = 0$

$\therefore\ (1+w+w^2+w^3+w^4)(1-w) = 0$

But $w \neq 1$, so $1+w+w^2+w^3+w^4 = 0$

d



4 **a** The nth roots of unity are the solutions to $z^n = 1$

$z^n = 1$

$\therefore\ z^n = \text{cis}\,(0 + k2\pi)$

$\therefore\ z^n = \text{cis}\,(k2\pi)$

$\therefore\ z = [\text{cis}\,(k2\pi)]^{\frac{1}{n}}$

$\therefore\ z = \text{cis}\left(\frac{k2\pi}{n}\right)$ {De Moivre}

The nth root of unity with smallest positive argument is $w = \text{cis}\left(\frac{2\pi}{n}\right)$, when $k = 1$.

b **i** The nth roots of unity are

$\text{cis}\ 0 = 1$ or

$\text{cis}\left(\frac{2\pi}{n}\right) = w$ or

$\text{cis}\left(\frac{4\pi}{n}\right) = \left(\text{cis}\left(\frac{2\pi}{n}\right)\right)^2 = w^2$ or

$\vdots$

$\text{cis}\left(\frac{2\pi}{n}(n-1)\right) = \left[\text{cis}\left(\frac{2\pi}{n}\right)\right]^{n-1} = w^{n-1}$ {letting $k = 0, 1, 2, 3, \ldots., n-1$}

$\therefore$ the n roots of $z^n = 1$ are $1, w, w^2, w^3, \ldots., w^{n-1}$ where $w = \text{cis}\left(\frac{2\pi}{n}\right)$

ii Now $(1+w+w^2+\ldots.+w^{n-1})(w-1) = w^n - 1$

But w is a solution to $z^n - 1 = 0$ so $w^n - 1 = 0$

$\therefore\ (1+w+w^2+\ldots.+w^{n-1})(w-1) = 0$

$\therefore$ since $w \neq 1$, $1+w+w^2+\ldots.+w^{n-1} = 0$

5 Let $\alpha = r\ \text{cis}\ \theta$

$\therefore\ z^n = r\ \text{cis}\ (\theta + k2\pi)$

$\therefore\ z = [r\ \text{cis}\ (\theta + k2\pi)]^{\frac{1}{n}}$

$\therefore\ z = r^{\frac{1}{n}}\ \text{cis}\left(\frac{\theta + k2\pi}{n}\right)$ {De Moivre}

$\therefore\ z = r^{\frac{1}{n}}\ \text{cis}\left(\frac{\theta}{n}\right)\ \text{cis}\left(\frac{k2\pi}{n}\right)$

$\therefore$ the n zeros of $z^n = \alpha$ are $r^{\frac{1}{n}}\ \text{cis}\left(\frac{\theta}{n}\right)$, $r^{\frac{1}{n}}\ \text{cis}\left(\frac{\theta}{n}\right)\ \text{cis}\left(\frac{2\pi}{n}\right)$, $r^{\frac{1}{n}}\ \text{cis}\left(\frac{\theta}{n}\right)\ \text{cis}\left(\frac{4\pi}{n}\right)$, ...,

$r^{\frac{1}{n}}\ \text{cis}\left(\frac{\theta}{n}\right)\text{cis}\left(\frac{2\pi}{n}(n-1)\right)$ {letting $k = 0, 1, 2, \ldots., n-1$}

$\therefore$ the sum of the n zeros of z is

$r^{\frac{1}{n}}\ \text{cis}\left(\frac{\theta}{n}\right) + r^{\frac{1}{n}}\ \text{cis}\left(\frac{\theta}{n}\right)\ \text{cis}\left(\frac{2\pi}{n}\right) + r^{\frac{1}{n}}\ \text{cis}\left(\frac{\theta}{n}\right)\ \text{cis}\left(\frac{4\pi}{n}\right) + \ldots. + r^{\frac{1}{n}}\ \text{cis}\left(\frac{\theta}{n}\right)\ \text{cis}\left(\frac{2\pi}{n}(n-1)\right)$

$= r^{\frac{1}{n}}\ \text{cis}\left(\frac{\theta}{n}\right) \underbrace{\left[1 + \text{cis}\left(\frac{2\pi}{n}\right) + \text{cis}\left(\frac{4\pi}{n}\right) + \ldots. + \text{cis}\left(\frac{2\pi}{n}(n-1)\right)\right]}_{\text{these are the } n\text{th roots of unity, whose sum } = 0 \ \{\text{using } \mathbf{4}\}}$

$= 0$

EXERCISE 16G

1 $z^2 - (2+i)z + (3+i) = 0$

$$\begin{aligned}
\therefore\ z &= \frac{2+i \pm \sqrt{(2+i)^2 - 4(1)(3+i)}}{2}\\
&= \frac{2+i \pm \sqrt{4+\cancel{4i}-1-12-\cancel{4i}}}{2}\\
&= \frac{2+i \pm \sqrt{-9}}{2}\\
&= \frac{2+i \pm 3i}{2}\\
&= \frac{2+4i}{2} \quad \text{or} \quad \frac{2-2i}{2}\\
&= 1+2i \quad \text{or} \quad 1-i
\end{aligned}$$

2 **a**

$$\begin{aligned}
z^* &= -iz\\
\therefore\ x - iy &= -i(x+iy)\\
\therefore\ x - iy &= -ix + y
\end{aligned}$$

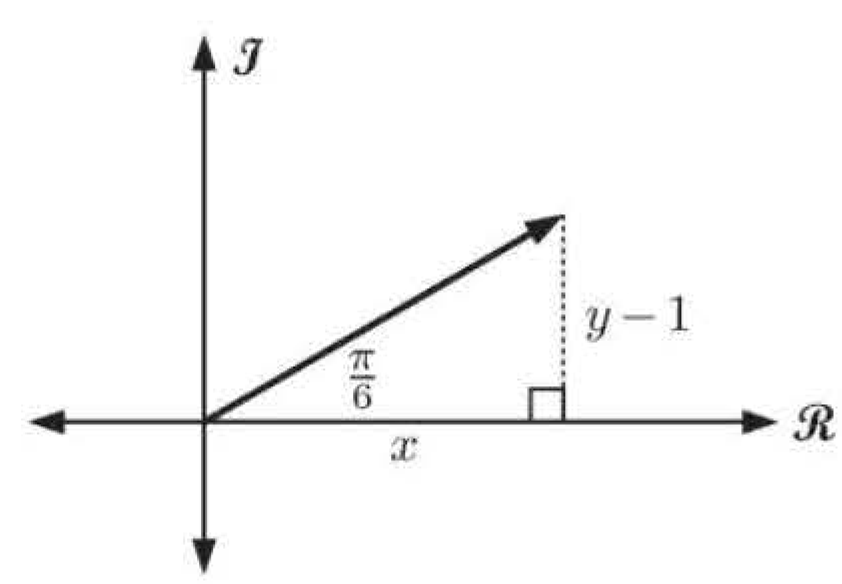
Equating real and imaginary parts,

$x = y$ and $-y = -x$

$\therefore\ y = x$

b

$$\begin{aligned}
\arg(z-i) &= \tfrac{\pi}{6}\\
\therefore\ \arg(x+iy-i) &= \tfrac{\pi}{6}\\
\therefore\ \arg(x+(y-1)i) &= \tfrac{\pi}{6}\\
\therefore\ \tan\tfrac{\pi}{6} = \tfrac{1}{\sqrt{3}} &= \frac{y-1}{x}\\
\therefore\ y-1 &= \frac{x}{\sqrt{3}}\\
\therefore\ y &= \frac{x}{\sqrt{3}} + 1, \quad x > 0
\end{aligned}$$

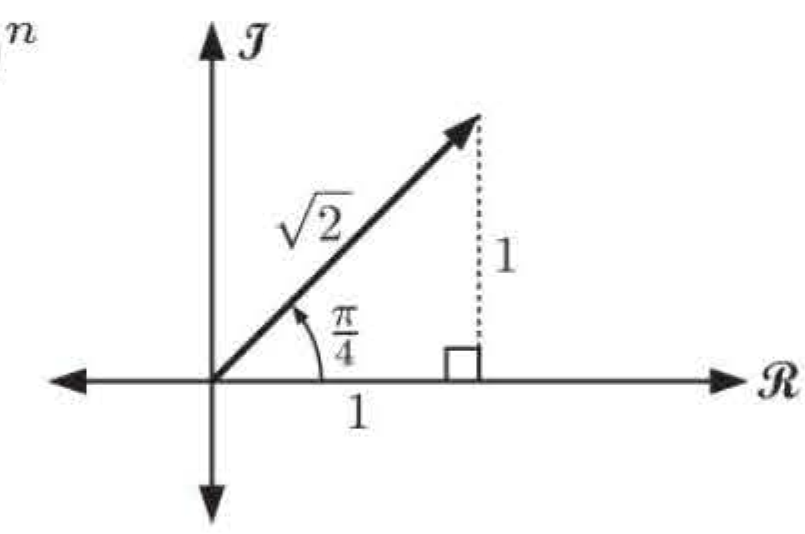


c

$$\begin{aligned}
|z+3| + |z-3| &= 8\\
\therefore\ |z+3| &= 8 - |z-3|\\
\therefore\ |z+3|^2 &= (8-|z-3|)^2\\
\therefore\ |z+3|^2 &= 64 - 16\,|z-3| + |z-3|^2\\
\therefore\ (z+3)(z+3)^* &= 64 - 16\,|z-3| + (z-3)(z-3)^* \qquad \{|z|^2 = zz^*\}\\
\therefore\ (z+3)(z^*+3) &= 64 - 16\,|z-3| + (z-3)(z^*-3) \qquad \{(z\pm w)^* = z^* \pm w^*\}\\
\therefore\ \cancel{zz^*} + 3z + 3z^* + \cancel{9} &= 64 - 16\,|(x-3)+yi| + \cancel{zz^*} - 3z - 3z^* + \cancel{9}\\
\therefore\ 6z + 6z^* &= 64 - 16\sqrt{(x-3)^2+y^2}\\
\therefore\ 6(x+yi) + 6(x-yi) &= 64 - 16\sqrt{(x-3)^2+y^2}\\
\therefore\ 12x - 64 &= -16\sqrt{(x-3)^2+y^2}\\
\therefore\ 3x - 16 &= -4\sqrt{(x-3)^2+y^2}\\
\therefore\ (3x-16)^2 &= 16\left[(x-3)^2+y^2\right]\\
\therefore\ 9x^2 - 96x + 256 &= 16(x^2 - 6x + 9 + y^2)\\
\therefore\ 9x^2 - \cancel{96x} + 256 &= 16x^2 - \cancel{96x} + 144 + 16y^2\\
\therefore\ 112 &= 7x^2 + 16y^2
\end{aligned}$$

3 $$(1+i)^{2n} = \binom{2n}{0}1^{2n}i^0 + \binom{2n}{1}1^{2n-1}i^1 + \binom{2n}{2}1^{2n-2}i^2 + \binom{2n}{3}1^{2n-3}i^3 + \ldots. + \binom{2n}{2n}1^0 i^{2n}$$
$$= \binom{2n}{0} + \binom{2n}{1}i - \binom{2n}{2} - \binom{2n}{3}i + \ldots. + \binom{2n}{2n}(-1)^n$$

But $$(1+i)^{2n} = \left(\sqrt{2}\text{ cis }\tfrac{\pi}{4}\right)^{2n}$$
$$= 2^n \text{ cis}\left(\tfrac{n\pi}{2}\right)$$
$$= 2^n\left[\cos\left(\tfrac{n\pi}{2}\right) + i\sin\left(\tfrac{n\pi}{2}\right)\right]$$



So, $\binom{2n}{0} + \binom{2n}{1}i - \binom{2n}{2} - \binom{2n}{3}i + \ldots. + \binom{2n}{2n}(-1)^n = 2^n\left[\cos\left(\tfrac{n\pi}{2}\right) + i\sin\left(\tfrac{n\pi}{2}\right)\right]$

Equating real parts,
$$\binom{2n}{0} - \binom{2n}{2} + \binom{2n}{4} - \binom{2n}{6} + \ldots. + (-1)^n\binom{2n}{2n} = 2^n\cos\left(\tfrac{n\pi}{2}\right),\quad n \in \mathbb{Z}^+$$

4 $$1 + \text{cis }\theta + \text{cis }2\theta + \text{cis }3\theta + \ldots. + \text{cis }n\theta$$
$$= 1 + (\cos\theta + i\sin\theta) + (\cos 2\theta + i\sin 2\theta) + (\cos 3\theta + i\sin 3\theta) + \ldots. + (\cos n\theta + i\sin n\theta)$$
$$= (1 + \cos\theta + \cos 2\theta + \cos 3\theta + \ldots. + \cos n\theta) + i(\sin\theta + \sin 2\theta + \sin 3\theta + \ldots. + \sin n\theta)$$
$$= \sum_{r=0}^{n}\cos r\theta + i\sum_{r=1}^{n}\sin r\theta$$

$\therefore$ $\mathcal{R}e\,(1 + \text{cis }\theta + \text{cis }2\theta + \text{cis }3\theta + \ldots. + \text{cis }n\theta) = \displaystyle\sum_{r=0}^{n}\cos r\theta$ .... (1)

Now $1 + \text{cis }\theta + \text{cis }2\theta + \text{cis }3\theta + \ldots. + \text{cis }n\theta = 1 + \text{cis }\theta + (\text{cis }\theta)^2 + (\text{cis }\theta)^3 + \ldots. + (\text{cis }\theta)^n$
which is a geometric series with $u_1 = 1,\ r = \text{cis }\theta$

$\therefore$ it has sum $S_n = \dfrac{u_1(r^n - 1)}{r-1}$

$$\underbrace{1 + \text{cis }\theta + \text{cis }2\theta + \ldots. + \text{cis }n\theta}_{n+1\text{ terms}} = S_{n+1}$$
$$= \frac{1((\text{cis }\theta)^{n+1} - 1)}{\text{cis }\theta - 1}$$
$$= \frac{\text{cis }(n+1)\theta - 1}{\text{cis }\theta - 1}$$
$$= \frac{\cos(n+1)\theta + i\sin(n+1)\theta - 1}{\cos\theta + i\sin\theta - 1}$$
$$= \left(\frac{\cos(n+1)\theta + i\sin(n+1)\theta - 1}{\cos\theta - 1 + i\sin\theta}\right)\left(\frac{\cos\theta - 1 - i\sin\theta}{\cos\theta - 1 - i\sin\theta}\right)$$

$\therefore$ $\mathcal{R}e\,(1 + \text{cis }\theta + \text{cis }2\theta + \ldots. + \text{cis }n\theta)$
$$= \frac{\cos(n+1)\theta\cos\theta - \cos(n+1)\theta + \sin(n+1)\theta\sin\theta - \cos\theta + 1}{(\cos\theta - 1)^2 + \sin^2\theta}$$
$$= \frac{[\cos(n+1)\theta\cos\theta + \sin(n+1)\theta\sin\theta] - \cos(n+1)\theta - \cos\theta + 1}{\cos^2\theta - 2\cos\theta + 1 + \sin^2\theta}$$
$$= \frac{\cos n\theta - \cos(n+1)\theta - \cos\theta + 1}{2 - 2\cos\theta} \quad \ldots. (2)$$

Equating (1) and (2) gives $\displaystyle\sum_{r=0}^{n}\cos r\theta = \frac{\cos n\theta - \cos(n+1)\theta - \cos\theta + 1}{2 - 2\cos\theta}$

5 $$2\cos\left(\tfrac{\theta}{2}\right)\text{ cis}\left(\tfrac{\theta}{2}\right) = 2\cos\left(\tfrac{\theta}{2}\right)\left[\cos\left(\tfrac{\theta}{2}\right) + i\sin\left(\tfrac{\theta}{2}\right)\right]$$
$$= 2\cos^2\left(\tfrac{\theta}{2}\right) + 2i\cos\left(\tfrac{\theta}{2}\right)\sin\left(\tfrac{\theta}{2}\right)$$
$$= \cos\theta + 1 + i\left(2\cos\left(\tfrac{\theta}{2}\right)\sin\left(\tfrac{\theta}{2}\right)\right)$$
$$= \cos\theta + 1 + i\sin\theta$$
$$= 1 + \text{cis }\theta$$

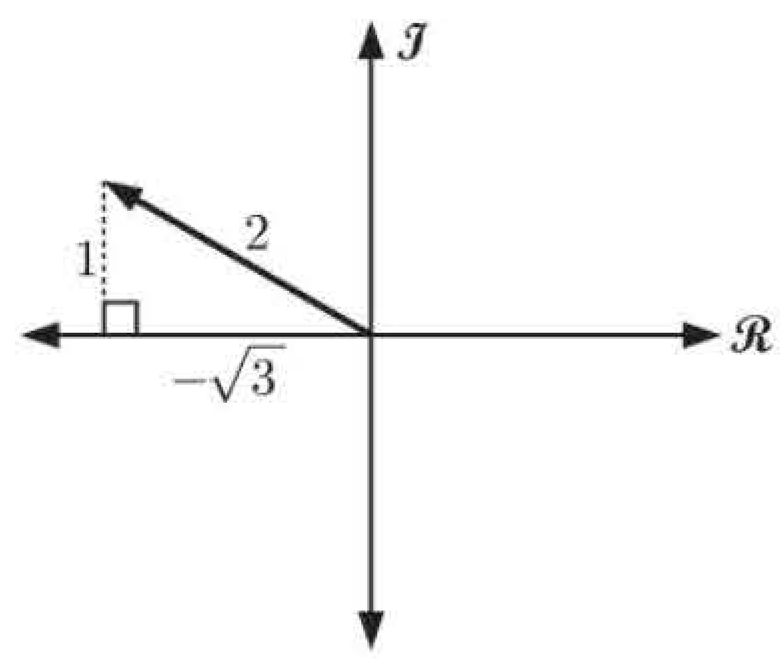
$$\left\{\begin{array}{r} \cos 2X = 2\cos^2 X - 1 \\ \therefore\ 2\cos^2 X = \cos 2X + 1 \end{array}\right\}$$

$$\text{Consider}\quad (1+\text{cis}\,\theta)^n = \binom{n}{0}1^n(\text{cis}\,\theta)^0 + \binom{n}{1}1^{n-1}(\text{cis}\,\theta)^1 + \binom{n}{2}1^{n-2}(\text{cis}\,\theta)^2 + ... + \binom{n}{n}1^0(\text{cis}\,\theta)^n$$
$$= \binom{n}{0} + \binom{n}{1}\text{cis}\,\theta + \binom{n}{2}\text{cis}\,2\theta + ... + \binom{n}{n}\text{cis}\,n\theta$$
$$\therefore\ \mathcal{R}e\,[(1+\text{cis}\,\theta)^n] = \binom{n}{0} + \binom{n}{1}\cos\theta + \binom{n}{2}\cos 2\theta + ... + \binom{n}{n}\cos n\theta$$
$$= \sum_{r=0}^{n}\binom{n}{r}\cos(r\theta)$$

$$\text{So}\quad \sum_{r=0}^{n}\binom{n}{r}\cos(r\theta) = \mathcal{R}e\,[(1+\text{cis}\,\theta)^n]$$
$$= \mathcal{R}e\left[\left(2\cos\left(\tfrac{\theta}{2}\right)\text{cis}\left(\tfrac{\theta}{2}\right)\right)^n\right]$$
$$= \mathcal{R}e\left[2^n\cos^n\left(\tfrac{\theta}{2}\right)\text{cis}\left(\tfrac{n\theta}{2}\right)\right]$$
$$= 2^n\cos^n\left(\tfrac{\theta}{2}\right)\cos\left(\tfrac{n\theta}{2}\right)$$

REVIEW SET 16A

1
$$(i-\sqrt{3})^5 = \left[2\,\text{cis}\left(\tfrac{5\pi}{6}\right)\right]^5$$
$$= 2^5\,\text{cis}\left(\tfrac{25\pi}{6}\right)$$
$$= 32\,\text{cis}\left(\tfrac{\pi}{6}\right)$$
$$= 32\left(\tfrac{\sqrt{3}}{2} + \tfrac{1}{2}i\right)$$
$$= 16\sqrt{3} + 16i$$



$\therefore$ real part is $16\sqrt{3}$, imaginary part is 16.

2 **a**
$$|z-i| = |z+1+i|$$
$$\text{Since}\ z = x+iy,\quad |x+iy-i| = |x+iy+1+i|$$
$$\therefore\ |x+i(y-1)| = |(x+1)+i(y+1)|$$
$$\therefore\ \sqrt{x^2+(y-1)^2} = \sqrt{(x+1)^2+(y+1)^2}$$
$$\therefore\ x^2+(y-1)^2 = (x+1)^2+(y+1)^2$$
$$\therefore\ \cancel{x^2}+\cancel{y^2}-2y+1 = \cancel{x^2}+2x+1+\cancel{y^2}+2y+1$$
$$\therefore\ 1-2y = 2x+2y+2$$
$$\therefore\ 2x+4y = -1$$

b
$$z^* - iz = 0$$
$$\text{Since}\ z = x+iy,$$
$$x-iy-i(x+iy) = 0$$
$$\therefore\ x-iy-ix+y = 0$$
$$\therefore\ (x+y)-i(x+y) = 0$$
$$\therefore\ x+y = 0$$
$$\therefore\ y = -x$$

3
$$|z+16| = 4\,|z+1|$$
$$\therefore\ |z+16|^2 = 16\,|z+1|^2$$
$$\therefore\ (z+16)(z+16)^* = 16(z+1)(z+1)^* \qquad \{|w|^2 = ww^*\}$$
$$\therefore\ (z+16)(z^*+16) = 16(z+1)(z^*+1) \qquad \{(z\pm w)^* = z^*\pm w^*\}$$
$$\therefore\ zz^*+16z+16z^*+256 = 16(zz^*+z+z^*+1)$$
$$\therefore\ |z|^2+\cancel{16z}+\cancel{16z^*}+256 = 16\,|z|^2+\cancel{16z}+\cancel{16z^*}+16$$
$$240 = 15\,|z|^2$$
$$\therefore\ |z|^2 = 16$$
$$\therefore\ |z| = 4\ \text{as}\ |z|\geqslant 0$$

4 **a**
$$z = r\,\text{cis}\,\theta$$
$$z^* = r\,\text{cis}\,(-\theta)$$
$\therefore$ reflection in the real axis.

b
$$z = r\,\text{cis}\,\theta$$
$$-z = -r\,\text{cis}\,\theta$$
$$= \text{cis}\,\pi \times r\,\text{cis}\,\theta$$
$$= r\,\text{cis}\,(\theta+\pi)$$
$\therefore$ rotation of π about O.

c
$$z = r\,\text{cis}\,\theta$$
$$iz = ir\,\text{cis}\,\theta$$
$$= \text{cis}\left(\tfrac{\pi}{2}\right) r\,\text{cis}\,\theta$$
$$= r\,\text{cis}\left(\theta+\tfrac{\pi}{2}\right)$$
$\therefore$ anticlockwise rotation of $\frac{\pi}{2}$ about O.

5 Let $z = a + bi$ and $w = c + di$ $\therefore$ $z + w = (a+c) + i(b+d)$, so $b + d = 0$ .... (1)

$$\begin{aligned} \text{and} \quad zw &= (a+bi)(c+di) \\ &= [ac - bd] + i[ad + bc] \end{aligned}$$

so, $ad + bc = 0$ .... (2)

From (1), $d = -b$ and in (2), $a(-b) + bc = 0$

$$\begin{aligned} &\therefore \ b(c-a) = 0 \\ &\therefore \ a = c \ \text{as} \ b \neq 0 \\ &\text{So} \ z = a + bi \ \text{and} \ w = a - bi \\ &\therefore \ z^* = a - bi = w \end{aligned}$$

6

$$\begin{aligned} (x+iy)^n &= X + Yi \\ \therefore \ |(x+yi)^n| &= |X + Yi| \\ \therefore \ |x+iy|^n &= |X + Yi| \\ \therefore \ \left(\sqrt{x^2+y^2}\right)^n &= \sqrt{X^2+Y^2} \end{aligned}$$

Squaring both sides, $X^2 + Y^2 = (x^2+y^2)^n$

7

$$\begin{aligned} |z-w|^2 + |z+w|^2 &= (z-w)(z-w)^* + (z+w)(z+w)^* \\ &= (z-w)(z^* - w^*) + (z+w)(z^* + w^*) \\ &= zz^* - \cancel{zw^*} - \cancel{wz^*} + ww^* + zz^* + \cancel{zw^*} + \cancel{wz^*} + ww^* \\ &= 2zz^* + 2ww^* \\ &= 2\,|z|^2 + 2\,|w|^2 \\ &= 2\left(|z|^2 + |w|^2\right) \end{aligned}$$

8 **a** Since 1 is a root of $z^5 - 1 = 0$, we find that

$z^5 - 1 = (z-1)(1 + z + z^2 + z^3 + z^4)$

$\therefore$ since α is a root,

$$(\alpha - 1)(1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4) = 0$$

But $\alpha \neq 1$, so $1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 = 0$

$$\begin{array}{c|ccccc|c} 1 & 1 & 0 & 0 & 0 & 0 & -1 \\ & 0 & 1 & 1 & 1 & 1 & 1 \\ \hline & 1 & 1 & 1 & 1 & 1 & 0 \end{array}$$

or Note that $1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4$ is the sum of a geometric series.

Then $S_n = \dfrac{u_1(1-r^n)}{1-r}$ where $u_1 = 1$, $r = \alpha$, $n = 5$

$\therefore$ $S_5 = \dfrac{1(1-\alpha^5)}{1-\alpha}$ and $\alpha^5 = \text{cis}\,(2\pi) = 1$

$$\therefore \ 1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 = \frac{1(1-1)}{1-\alpha} = 0$$

b Now if $\left(\dfrac{z+2}{z-1}\right)^5 = 1$ then $\dfrac{z+2}{z-1} = 1, \alpha, \alpha^2, \alpha^3,$ or α^4 where $\alpha = \text{cis}\left(\frac{2\pi}{5}\right)$

$$\begin{aligned} &\therefore \ \frac{z+2}{z-1} = \alpha^n \quad \text{where } n = 0, 1, 2, 3, 4 \\ &\therefore \ z + 2 = \alpha^n(z-1) \\ &\therefore \ z + 2 = \alpha^n z - \alpha^n \\ &\therefore \ z(\alpha^n - 1) = \alpha^n + 2 \quad \text{and so} \quad z = \frac{\alpha^n + 2}{\alpha^n - 1} \end{aligned}$$

$\therefore$ roots of $\left(\dfrac{z+2}{z-1}\right)^5 = 1$ are $\dfrac{\alpha+2}{\alpha-1}, \dfrac{\alpha^2+2}{\alpha^2-1}, \dfrac{\alpha^3+2}{\alpha^3-1}, \dfrac{\alpha^4+2}{\alpha^4-1}$, where $\alpha = \text{cis}\left(\frac{2\pi}{5}\right)$

Note: The case where $n = 0$ requires $\dfrac{z+2}{z-1} = 1$, and so $z + 2 = z - 1$, which has no solution.

9 Since $\left|\dfrac{z+1}{z-1}\right| = \dfrac{|z+1|}{|z-1|} = 1$, then $|z+1| = |z-1|$

Letting $z = x + iy$, $\therefore\ |(x+1)+iy| = |(x-1)+iy|$

$$\therefore\ \sqrt{(x+1)^2+y^2} = \sqrt{(x-1)^2+y^2}$$

Squaring both sides, we get $(x+1)^2 + \cancel{y^2} = (x-1)^2 + \cancel{y^2}$

$$\therefore\ x^2+2x+1 = x^2-2x+1$$
$$\therefore\ 4x = 0$$
$$\therefore\ x = 0$$

Therefore since $z \neq 0$, z is purely imaginary.

10
$$\begin{aligned} w &= \frac{1+z}{1+z^*} \\ &= \frac{1+\text{cis}\ \phi}{1+\text{cis}\ (-\phi)} \times \frac{\text{cis}\ \phi}{\text{cis}\ \phi} \\ &= \frac{(1+\text{cis}\ \phi)\ \text{cis}\ \phi}{\text{cis}\ \phi + \text{cis}\ 0} \\ &= \frac{(1+\text{cis}\ \phi)\ \text{cis}\ \phi}{1+\text{cis}\ \phi} \\ &= \text{cis}\ \phi \end{aligned}$$

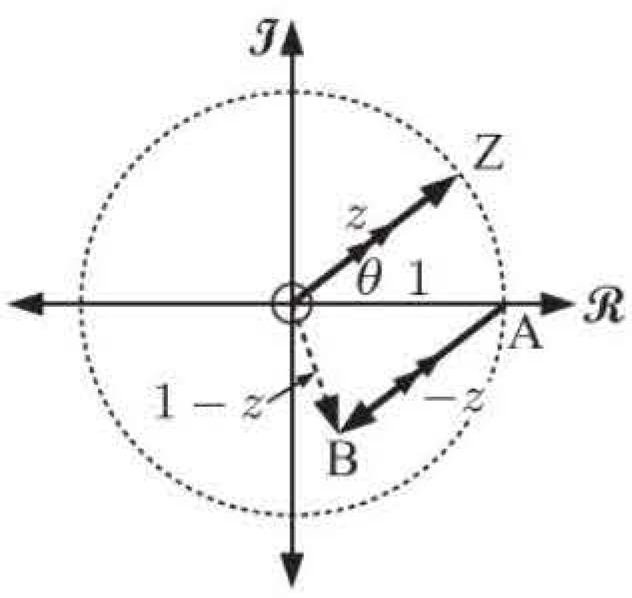
11 The 3 cube roots of $-64i$ are the solutions to $z^3 = -64i$

$\therefore\ z^3 = 64\ \text{cis}\,(-\frac{\pi}{2} + k2\pi)$ for integer k

$$\therefore\ z = \left[64\ \text{cis}\,(-\tfrac{\pi}{2} + k2\pi)\right]^{\frac{1}{3}}$$
$$\therefore\ z = 64^{\frac{1}{3}}\ \text{cis}\left(-\tfrac{\pi}{6} + \tfrac{k2\pi}{3}\right)$$
$$\therefore\ z = 4\ \text{cis}\left(-\tfrac{\pi}{6} + \tfrac{k4\pi}{6}\right)$$
$$\therefore\ z = 4\ \text{cis}\left(-\tfrac{5\pi}{6}\right),\ 4\ \text{cis}\left(-\tfrac{\pi}{6}\right),\ 4\ \text{cis}\left(\tfrac{\pi}{2}\right) \quad \{\text{letting } k = -1, 0, 1\}$$
$$\therefore\ z = 4(-\tfrac{\sqrt{3}}{2} - \tfrac{1}{2}i),\ 4(\tfrac{\sqrt{3}}{2} - \tfrac{1}{2}i),\ 4i$$
$$\therefore\ z = -2\sqrt{3} - 2i,\ 2\sqrt{3} - 2i,\ 4i$$

12 **a**
$$\begin{aligned} (2z)^{-1} &= (2\ \text{cis}\ \theta)^{-1} \\ &= 2^{-1}\ \text{cis}\,(-\theta) \end{aligned}$$
$$\therefore\ \left|(2z)^{-1}\right| = \tfrac{1}{2} \quad \text{and} \quad \arg\left[(2z)^{-1}\right] = -\theta$$

b
$$\begin{aligned} 1 - z &= 1 - \text{cis}\ \theta \\ &= (1-\cos\theta) - i\sin\theta \\ \therefore\ |1-z| &= \sqrt{(1-\cos\theta)^2 + \sin^2\theta} \\ &= \sqrt{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta} \\ &= \sqrt{2 - 2\cos\theta} \\ &= \sqrt{2 - 2\left(1 - 2\sin^2\left(\tfrac{\theta}{2}\right)\right)} \\ &= \sqrt{4\sin^2\left(\tfrac{\theta}{2}\right)} \\ &= 2\sin\left(\tfrac{\theta}{2}\right) \end{aligned}$$



$\triangle$OAB is isosceles since $|z| = 1$,

so we let $\text{A}\widehat{\text{O}}\text{B} = \text{A}\widehat{\text{B}}\text{O} = \phi$

Since [OZ] $\parallel$ [AB], $\text{O}\widehat{\text{A}}\text{B} = \theta$ {alternate $\measuredangle$s}

$$\therefore\ \phi + \phi + \theta = \pi$$
$$\therefore\ 2\phi = \pi - \theta$$
$$\phi = \tfrac{\pi}{2} - \tfrac{\theta}{2}$$

But $\arg(1-z) = -\phi$,

so $\arg(1-z) = -\left(\frac{\pi}{2} - \frac{\theta}{2}\right) = \frac{\theta}{2} - \frac{\pi}{2}$

13
$$-1 + i\sqrt{3} = 2\ \text{cis}\left(\tfrac{2\pi}{3}\right)$$
$$\therefore\ (-1+i\sqrt{3})^m = 2^m\ \text{cis}\left(\tfrac{m2\pi}{3}\right) \qquad \{\text{De Moivre}\}$$
$$= 2^m\left[\cos\left(\tfrac{m2\pi}{3}\right) + i\sin\left(\tfrac{m2\pi}{3}\right)\right]$$

This is real provided $\sin\left(\frac{m2\pi}{3}\right) = 0$

$$\therefore\ \tfrac{m2\pi}{3} = 0 + k\pi$$
$$\therefore\ m = \tfrac{3k}{2} \quad \text{where } k \text{ is any integer}$$

14 We first note that $\cos(A+B)+\cos(A-B)=2\cos A\cos B$

and $\sin(A+B)+\sin(A-B)=2\sin A\cos B$ .... $(*)$

Now
$$\begin{aligned}\text{cis }\theta+\text{cis }\phi &= (\cos\theta+i\sin\theta)+(\cos\phi+i\sin\phi)\\
&= (\cos\theta+\cos\phi)+i(\sin\theta+\sin\phi)\\
&= \left[\cos\left(\frac{\theta+\phi}{2}+\frac{\theta-\phi}{2}\right)+\cos\left(\frac{\theta+\phi}{2}-\frac{\theta-\phi}{2}\right)\right]\\
&\quad +i\left[\sin\left(\frac{\theta+\phi}{2}+\frac{\theta-\phi}{2}\right)+\sin\left(\frac{\theta+\phi}{2}-\frac{\theta-\phi}{2}\right)\right]\\
&= 2\cos\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right)+2i\sin\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right) \quad \{\text{using } (*)\}\\
&= 2\cos\left(\frac{\theta-\phi}{2}\right)\text{cis}\left(\frac{\theta+\phi}{2}\right) \quad \text{as required}\end{aligned}$$

Now $Z^5=1$ has solutions $Z=1,\ \alpha,\ \alpha^2,\ \alpha^3,\ \alpha^4$ where $\alpha=\text{cis}\left(\frac{2\pi}{5}\right)$

$\therefore\ Z=\text{cis}\left(\frac{2n\pi}{5}\right)$ where $n=0,\ 1,\ 2,\ 3,\ 4$

$\therefore$ if $\left(\dfrac{z+1}{z-1}\right)^5=1$, then $\dfrac{z+1}{z-1}=\text{cis}\left(\frac{2n\pi}{5}\right)$

$$\therefore\ z+1=\text{cis}\left(\tfrac{2n\pi}{5}\right)(z-1)$$

$$\therefore\ z\left(\text{cis}\left(\tfrac{2n\pi}{5}\right)-1\right)=\text{cis}\left(\tfrac{2n\pi}{5}\right)+1$$

$$\therefore\ z=\frac{\text{cis}\left(\frac{2n\pi}{5}\right)+1}{\text{cis}\left(\frac{2n\pi}{5}\right)-1}$$

$$\therefore\ z=\frac{\text{cis}\left(\frac{2n\pi}{5}\right)+\text{cis }0}{\text{cis}\left(\frac{2n\pi}{5}\right)+\text{cis }\pi}$$

$$\therefore\ z=\frac{2\cos\left(\dfrac{\frac{2n\pi}{5}-0}{2}\right)\text{cis}\left[\dfrac{\frac{2n\pi}{5}+0}{2}\right]}{2\cos\left(\dfrac{\frac{2n\pi}{5}-\pi}{2}\right)\text{cis}\left[\dfrac{\frac{2n\pi}{5}+\pi}{2}\right]} \quad \{\text{using the above identity}\}$$

$$\therefore\ z=\frac{\cos\frac{n\pi}{5}}{\cos\left(\frac{n\pi}{5}-\frac{\pi}{2}\right)}\text{cis}\left[\tfrac{n\pi}{5}-\left(\tfrac{n\pi}{5}+\tfrac{\pi}{2}\right)\right]$$

$$\therefore\ z=\frac{\cos\frac{n\pi}{5}}{\cos\left(\frac{\pi}{2}-\frac{n\pi}{5}\right)}\text{cis}\left(-\tfrac{\pi}{2}\right)$$

$$\therefore\ z=\frac{\cos\frac{n\pi}{5}}{\sin\frac{n\pi}{5}}(-i)$$

$\therefore\ z=-i\cot\left(\frac{n\pi}{5}\right),\quad n=1,\ 2,\ 3,\ 4$ {excluding $n=0$ since $\cot 0$ is undefined}

Now
$$\begin{aligned}\cot\left(\tfrac{n\pi}{5}\right) &= -\cot\left(\pi-\tfrac{n\pi}{5}\right)\\
&= -\cot\left(\tfrac{(5-n)\pi}{5}\right),\quad n=1,\ 2,\ 3,\ 4\\
&= -\cot\left(\tfrac{n\pi}{5}\right),\quad n=4,\ 3,\ 2,\ 1\end{aligned}$$

$$\therefore\ z=i\cot\left(\tfrac{n\pi}{5}\right),\quad n=1,\ 2,\ 3,\ 4$$

REVIEW SET 16B

1 $z_1 = \text{cis}\left(\frac{\pi}{6}\right)$ and $z_2 = \text{cis}\left(\frac{\pi}{4}\right)$

$$\begin{aligned}
\therefore \left(\frac{z_1}{z_2}\right)^3 &= \left[\frac{\text{cis}\left(\frac{\pi}{6}\right)}{\text{cis}\left(\frac{\pi}{4}\right)}\right]^3 \\
&= \left[\text{cis}\left(\tfrac{\pi}{6} - \tfrac{\pi}{4}\right)\right]^3 \\
&= \left[\text{cis}\left(-\tfrac{\pi}{12}\right)\right]^3 \\
&= \text{cis}\left(-\tfrac{3\pi}{12}\right) \qquad \{\text{De Moivre}\} \\
&= \text{cis}\left(-\tfrac{\pi}{4}\right) \\
&= \cos\left(-\tfrac{\pi}{4}\right) + i\sin\left(-\tfrac{\pi}{4}\right) \\
&= \tfrac{1}{\sqrt{2}} - \tfrac{1}{\sqrt{2}}i
\end{aligned}$$

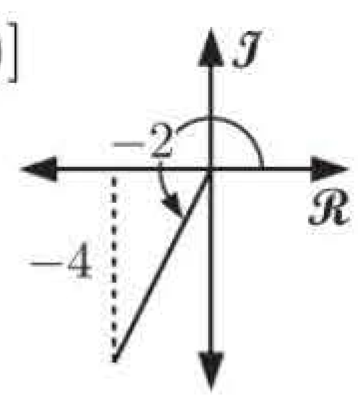
2 $z = 4 + i, \quad w = 2 - 3i$

a
$$\begin{aligned}
&2w^* - iz \\
&= 2(2+3i) - i(4+i) \\
&= 4 + 6i - 4i - i^2 \\
&= 5 + 2i
\end{aligned}$$

b
$$\begin{aligned}
|w - z^*| &= |(2-3i) - (4-i)| \\
&= |2 - 3i - 4 + i| \\
&= |-2 - 2i| \\
&= \sqrt{(-2)^2 + (-2)^2} \\
&= \sqrt{8} \\
&= 2\sqrt{2}
\end{aligned}$$

c
$$\begin{aligned}
\left|z^{10}\right| &= |z|^{10} \\
&= |4+i|^{10} \\
&= \left(\sqrt{16+1}\right)^{10} \\
&= \sqrt{17}^{10} \\
&= 17^5
\end{aligned}$$

d
$$\begin{aligned}
\arg(w - z) &= \arg[(2-3i) - (4+i)] \\
&= \arg[-2 - 4i] \\
&\approx -2.03
\end{aligned}$$



3 If $\dfrac{2-3i}{2a+bi} = 3 + 2i$, then $\dfrac{2-3i}{3+2i} = 2a + bi$

$$\begin{aligned}
\therefore \ 2a + bi &= \left(\frac{2-3i}{3+2i}\right)\left(\frac{3-2i}{3-2i}\right) \\
&= \frac{6 - 4i - 9i - 6}{9+4} \\
&= \frac{0 - 13i}{13} \\
&= 0 - i
\end{aligned}$$

$\therefore \ 2a = 0$ and $b = -1 \quad \therefore \ a = 0, \ b = -1$

4 **a** If $\arg z = \frac{\pi}{2}$, then we have a ray vertically upwards beginning at the origin.

If $\arg(z - i) = \frac{\pi}{2}$, the graph is translated $\binom{0}{1}$, and we have a ray vertically upwards beginning at i.

$\therefore \ x = 0$, and geometrically we require $y > 1$.

b
$$\left|\frac{z+2}{z-2}\right| = 2, \qquad \therefore \ \frac{|z+2|}{|z-2|} = 2$$

$$\therefore \ |z+2| = 2\,|z-2|$$

If $z = x + iy$, then $\sqrt{(x+2)^2 + y^2} = 2\sqrt{(x-2)^2 + y^2}$

$$\therefore\ (x+2)^2+y^2=4(x-2)^2+4y^2$$
$$\therefore\ x^2+4x+4+y^2=4x^2-16x+16+4y^2$$
$$\therefore\ 3x^2+3y^2-20x+12=0, \quad \text{which is a circle}$$

5
$$\begin{aligned}2-2\sqrt{3}i &= 4\left(\tfrac{1}{2}-\tfrac{\sqrt{3}}{2}i\right)\\ &= 4\text{ cis}\left(-\tfrac{\pi}{3}\right)\\ \therefore\ (2-2\sqrt{3}i)^n &= 4^n\text{ cis}\left(-\tfrac{n\pi}{3}\right) \qquad \{\text{De Moivre}\}\end{aligned}$$

This is real if $\sin\left(-\frac{n\pi}{3}\right)=0$

$\therefore\ -\frac{n\pi}{3}=k\pi,\quad k$ an integer

$\therefore\ n=3k\quad$ where k is an integer

6 The cube roots of -27 are the solutions to $z^3=-27$.

$$\begin{aligned}\therefore\ z^3 &= 27\text{ cis }(\pi+k2\pi)\\ \therefore\ z &= [27\text{ cis }(\pi+k2\pi)]^{\frac{1}{3}}\\ \therefore\ z &= 27^{\frac{1}{3}}\text{ cis}\left(\tfrac{\pi+k2\pi}{3}\right)\\ \therefore\ z &= 3\text{ cis}\left(\tfrac{\pi+k2\pi}{3}\right)\\ \therefore\ z &= 3\text{ cis}\left(-\tfrac{\pi}{3}\right),\ 3\text{ cis}\left(\tfrac{\pi}{3}\right),\ 3\text{ cis }\pi \qquad \{\text{letting } k=-1, 0, 1\}\\ \therefore\ z &= 3(\tfrac{1}{2}-\tfrac{\sqrt{3}}{2}i),\ 3(\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}i),\ -3\\ \therefore\ z &= \tfrac{3}{2}\pm\tfrac{3\sqrt{3}}{2}i\ \text{ or }\ -3\end{aligned}$$

7 **a**
$$\begin{aligned}z &= 4\text{ cis }\theta\\ z^3 &= (4\text{ cis }\theta)^3\\ &= 4^3\text{ cis }3\theta\\ \therefore\ \left|z^3\right| &= 64\\ \text{and}\quad \arg\left(z^3\right) &= 3\theta\end{aligned}$$

b
$$\begin{aligned}\frac{1}{z} &= z^{-1}\\ &= (4\text{ cis }\theta)^{-1}\\ &= 4^{-1}\text{ cis}\,(-\theta)\end{aligned}$$
$$\therefore\ \left|\frac{1}{z}\right|=\tfrac{1}{4}\ \text{ and }\ \arg\left(\frac{1}{z}\right)=-\theta$$

c
$$\begin{aligned}z &= 4\text{ cis }\theta\\ \therefore\ iz^* &= \left(\text{cis }\tfrac{\pi}{2}\right)(4\text{ cis}\,(-\theta))\\ &= 4\text{ cis}\left(\tfrac{\pi}{2}-\theta\right)\end{aligned}$$
$$\therefore\ |iz^*|=4\ \text{ and }\ \arg\,(iz^*)=\tfrac{\pi}{2}-\theta$$

8 **a**

Let $z=r\text{ cis }\theta$

$\therefore\ z^n=r^n\text{ cis }n\theta \quad \{\text{De Moivre}\}$

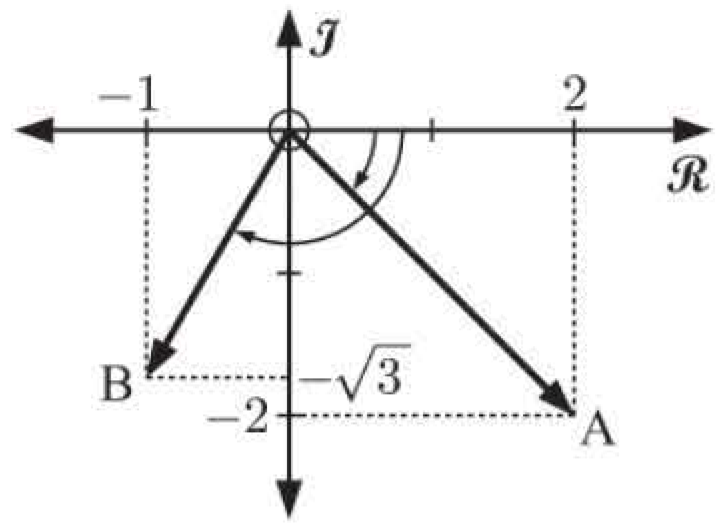
and so $\arg z^n=n\theta$

$\therefore\ \arg z^n=n\arg z\quad$ as required

b
$$\begin{aligned}\left(\frac{z}{w}\right)^* &= \left(\frac{a+bi}{c+di}\right)^*, \quad w\neq 0\\ &= \left(\frac{(a+bi)(c-di)}{(c+di)(c-di)}\right)^*\\ &= \left(\frac{(ac+bd)+i(bc-ad)}{c^2+d^2}\right)^*\\ &= \frac{(ac+bd)-i(bc-ad)}{c^2+d^2}\end{aligned}$$

and also
$$\begin{aligned}\frac{z^*}{w^*} &= \frac{a-bi}{c-di}, \quad w\neq 0\\ &= \frac{(a-bi)(c+di)}{(c-di)(c+di)}\\ &= \frac{(ac+bd)-i(bc-ad)}{c^2+d^2}\\ &= \left(\frac{z}{w}\right)^*\end{aligned}$$

9



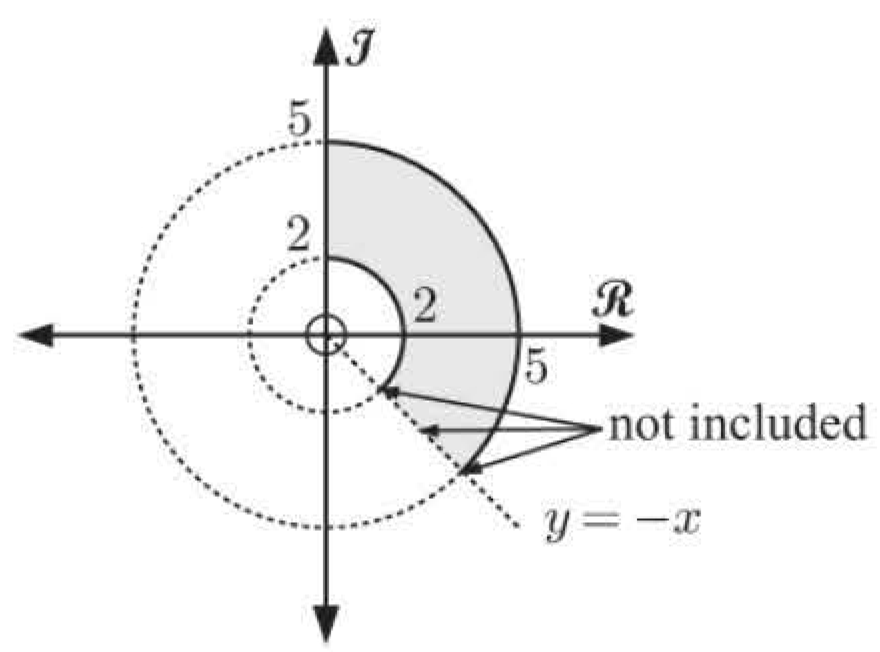
a

$$\arg \overrightarrow{OA} = -\tfrac{\pi}{4}$$
$$\arg \overrightarrow{OB} = -\tfrac{2\pi}{3}$$
$$\therefore \quad A\widehat{O}B = \tfrac{2\pi}{3} - \tfrac{\pi}{4} = \tfrac{5\pi}{12}$$

b

$$\begin{aligned} & zw \\ &= 2\sqrt{2}\text{ cis}\left(\tfrac{-\pi}{4}\right) \times 2\text{ cis}\left(\tfrac{-2\pi}{3}\right) \\ &= 4\sqrt{2}\text{ cis}\left(\tfrac{-\pi}{4} + \tfrac{-2\pi}{3}\right) \\ &= 4\sqrt{2}\text{ cis}\left(-\tfrac{11\pi}{12}\right) \end{aligned}$$
$$\therefore \quad \arg(zw) = -\tfrac{11\pi}{12}$$

10 $\{z \mid 2 \leqslant |z| \leqslant 5 \quad \text{and} \quad -\frac{\pi}{4} < \arg z \leqslant \frac{\pi}{2}\}$



11 If $z = r \text{ cis }\theta$, then $|z| = r$ and $\arg z = \theta$

Now $\dfrac{1}{z} = (r\text{ cis }\theta)^{-1} = r^{-1}\text{ cis}(-\theta)$
$= \dfrac{1}{r}\text{ cis}(-\theta)$

$\therefore \quad \left|\dfrac{1}{z}\right| = \dfrac{1}{r} = \dfrac{1}{|z|}$ (if $z \neq 0$),

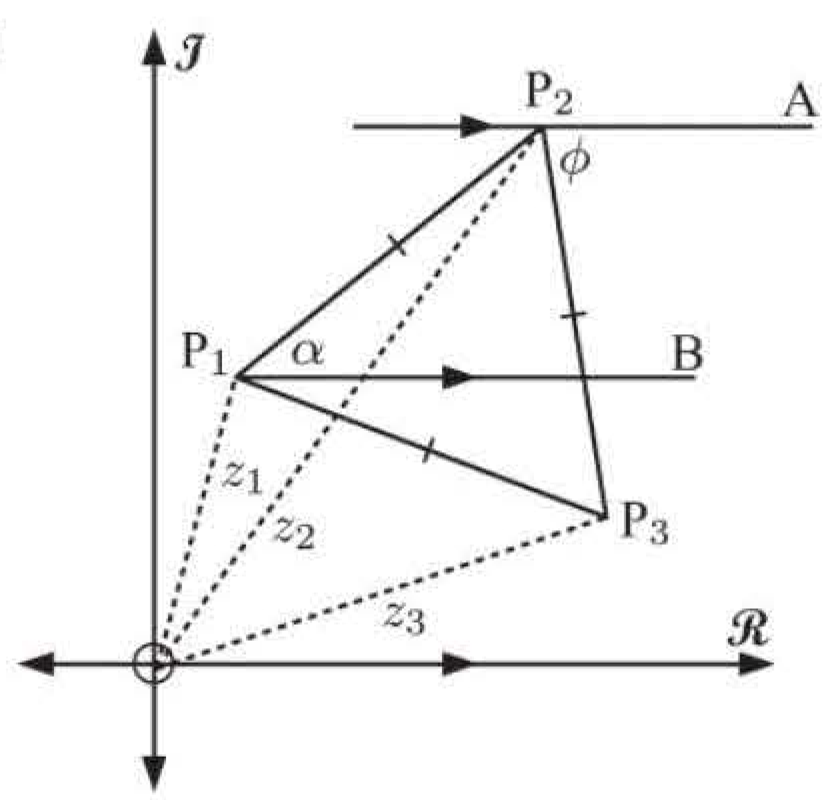
and $\arg\left(\dfrac{1}{z}\right) = -\theta$
$= -\arg z$

12

$$\begin{aligned} z &= \text{cis }\alpha \\ \therefore \quad 1 + z &= 1 + \text{cis }\alpha \\ &= 1 + \cos\alpha + i\sin\alpha \\ &= \left[1 + 2\cos^2\left(\tfrac{\alpha}{2}\right) - 1\right] + i\left[2\sin\left(\tfrac{\alpha}{2}\right)\cos\left(\tfrac{\alpha}{2}\right)\right] \\ &= 2\cos^2\left(\tfrac{\alpha}{2}\right) + i\left[2\sin\left(\tfrac{\alpha}{2}\right)\cos\left(\tfrac{\alpha}{2}\right)\right] \\ &= 2\cos\left(\tfrac{\alpha}{2}\right)\left[\cos\left(\tfrac{\alpha}{2}\right) + i\sin\left(\tfrac{\alpha}{2}\right)\right] \\ &= 2\cos\left(\tfrac{\alpha}{2}\right)\text{cis}\left(\tfrac{\alpha}{2}\right) \end{aligned}$$

$\therefore \quad |1 + z| = 2\cos\left(\frac{\alpha}{2}\right)$ and $\arg(1 + z) = \frac{\alpha}{2}$

13 **a**



Now $z_2 - z_1 = \overrightarrow{P_1P_2}$

so $\arg(z_2 - z_1) = \alpha$ as shown on the diagram alongside

$z_3 - z_2 = \overrightarrow{P_2P_3}$

so $\arg(z_3 - z_2) = -\phi$

Now $P_1\widehat{P_2}P_3 = \frac{\pi}{3}$ since the $\triangle$ is equilateral

$\therefore \quad \alpha + \frac{\pi}{3} + \phi = \pi$ $\{(P_1B) \parallel (P_2A)$, co-interior angles$\}$

$\therefore \quad \phi = -\alpha + \frac{2\pi}{3}$

$\therefore \quad \arg(z_3 - z_2) = \alpha - \frac{2\pi}{3}$ as required

b

$$\left|\frac{z_2 - z_1}{z_3 - z_2}\right| = \left|\frac{\overrightarrow{P_1P_2}}{\overrightarrow{P_2P_3}}\right| = \frac{|\overrightarrow{P_1P_2}|}{|\overrightarrow{P_2P_3}|} = 1 \quad \text{since the } \triangle \text{ is equilateral}$$

$$\begin{aligned} \arg\left(\frac{z_2 - z_1}{z_3 - z_2}\right) &= \arg(z_2 - z_1) - \arg(z_3 - z_2) \\ &= \alpha - \left(\alpha - \tfrac{2\pi}{3}\right) \quad \{\text{from } \textbf{a}\} \\ &= \alpha - \alpha + \tfrac{2\pi}{3} \\ &= \tfrac{2\pi}{3} \end{aligned}$$

14 The fifth roots of unity are $1, w, w^2, w^3$, and w^4, where $w = \text{cis}\left(\frac{2\pi}{5}\right)$.

a
$$\begin{aligned} (2z-1)^5 &= 32 \\ \therefore \frac{(2z-1)^5}{2^5} &= 1 \\ \therefore \left(\frac{2z-1}{2}\right)^5 &= 1 \\ \therefore \left(z-\tfrac{1}{2}\right)^5 &= 1 \\ \therefore z - \tfrac{1}{2} &= 1,\ w,\ w^2,\ w^3, \text{ or } w^4 \\ \therefore z &= \tfrac{3}{2},\ w+\tfrac{1}{2},\ w^2+\tfrac{1}{2}, \\ & \quad w^3+\tfrac{1}{2}, \text{ or } w^4+\tfrac{1}{2} \end{aligned}$$
where $w = \text{cis}\left(\frac{2\pi}{5}\right)$

b
$$\begin{aligned} z^5+5z^4+10z^3+10z^2+5z &= 0 \\ \therefore z^5+5z^4+10z^3+10z^2+5z+1 &= 1 \\ \therefore (z+1)^5 &= 1 \\ \therefore z+1 &= 1,\ w,\ w^2,\ w^3,\ w^4 \\ \therefore z &= 0,\ w-1,\ w^2-1, \\ & \quad w^3-1, \text{ or } w^4-1 \end{aligned}$$
where $w = \text{cis}\left(\frac{2\pi}{5}\right)$

c
$$\begin{aligned} (z+1)^5 &= (z-1)^5 \\ \therefore \frac{(z+1)^5}{(z-1)^5} &= 1, \quad z \neq 1 \\ \therefore \left(\frac{z+1}{z-1}\right)^5 &= 1 \\ \therefore \frac{z+1}{z-1} &= 1,\ w,\ w^2,\ w^3, \text{ or } w^4 \end{aligned}$$
$\dfrac{z+1}{z-1} = 1$ has no solutions as

$z+1 \neq z-1$ for any z

If $\dfrac{z+1}{z-1} = w^k$, $k = 1, 2, 3, 4$
$$\begin{aligned} z+1 &= w^k z - w^k \\ \therefore z(1-w^k) &= -w^k - 1 \\ \therefore z &= \frac{-w^k-1}{1-w^k} = \frac{w^k+1}{w^k-1} \\ \therefore z &= \frac{w+1}{w-1},\ \frac{w^2+1}{w^2-1},\ \frac{w^3+1}{w^3-1},\ \frac{w^4+1}{w^4-1} \end{aligned}$$
where $w = \text{cis}\left(\frac{2\pi}{5}\right)$

REVIEW SET 16C

1 **a** $-5i = 5 \text{ cis}\left(-\frac{\pi}{2}\right)$

b $2 - 2i\sqrt{3} = 4\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$
$= 4 \text{ cis}\left(-\frac{\pi}{3}\right)$

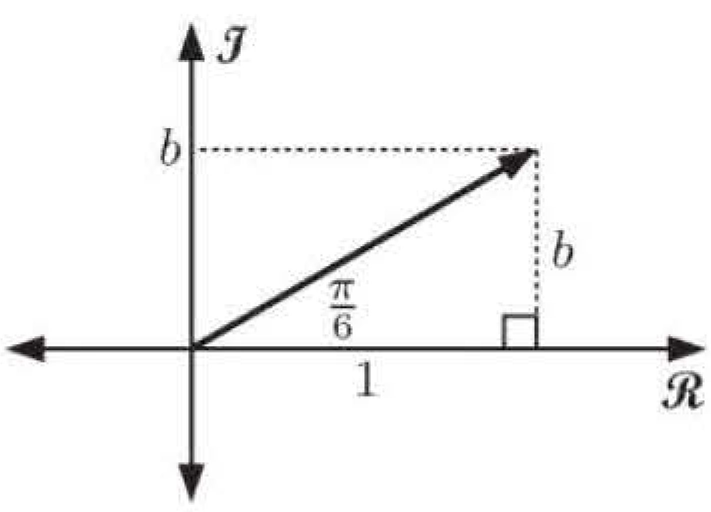
c $k - ki = -k\sqrt{2}\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)$
$= -k\sqrt{2} \text{ cis}\left(\frac{3\pi}{4}\right)$ which is in polar form since $k < 0$

2 $z = (1+bi)^2$ has argument $\frac{\pi}{3}$

$\therefore$ $1 + bi$ has argument $\frac{\pi}{6}$ $(b > 0)$

$\therefore \tan\left(\frac{\pi}{6}\right) = \dfrac{b}{1}$

$\therefore b = \frac{1}{\sqrt{3}}$



3 **a**
$$\begin{aligned} \text{cis } \theta \times \text{cis } \phi &= (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi) \\ &= (\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\sin\theta\cos\phi + \cos\theta\sin\phi) \\ &= \cos(\theta+\phi) + i\sin(\theta+\phi) \\ &= \text{cis}(\theta+\phi) \quad \text{as required} \end{aligned}$$

b
$$1 - i = \sqrt{2}\left(\tfrac{1}{\sqrt{2}} - \tfrac{1}{\sqrt{2}}i\right) = \sqrt{2} \text{ cis}\left(-\tfrac{\pi}{4}\right)$$
$$\begin{aligned} \therefore (1-i)z &= \sqrt{2} \text{ cis}\left(-\tfrac{\pi}{4}\right) \times 2\sqrt{2} \text{ cis } \alpha \\ &= 4 \text{ cis}\left(\alpha - \tfrac{\pi}{4}\right) \qquad \{\text{using } \textbf{a}\} \end{aligned}$$
$\therefore \arg[(1-i)z] = \alpha - \frac{\pi}{4}$

4 **a** $\left|\dfrac{{z_1}^2}{{z_2}^2}\right| = \dfrac{|z_1|^2}{|z_2|^2}$ But $|z_1| = |z_2|$ since the triangle is isosceles

$\therefore \left|\dfrac{{z_1}^2}{{z_2}^2}\right| = 1$

Also, $\arg\left(\dfrac{{z_1}^2}{{z_2}^2}\right) = \arg({z_1}^2) - \arg({z_2}^2)$

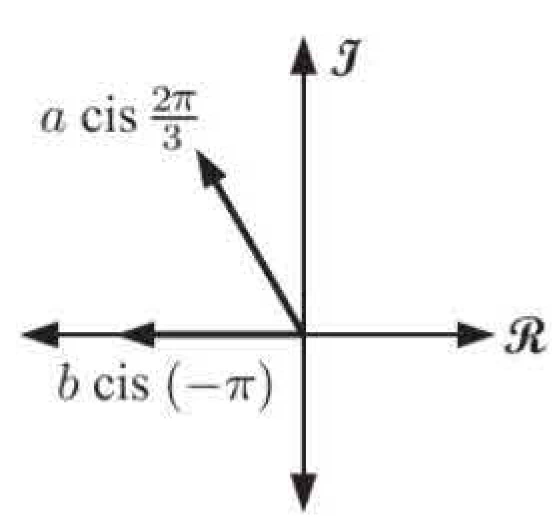
$$= 2\arg z_1 - 2\arg z_2$$
$$= 2(\arg z_1 - \arg z_2)$$
$$= 2 \times \tfrac{\pi}{2} \quad \text{since } z_1 \text{ and } z_2 \text{ are perpendicular}$$
$$= \pi$$

b $\dfrac{{z_1}^2}{{z_2}^2} = \text{cis}\,\pi = -1 \quad \therefore\ {z_1}^2 = -{z_2}^2 \quad \therefore\ {z_1}^2 + {z_2}^2 = 0$

5 $z = \sqrt[4]{a}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = \sqrt[4]{a}\ \text{cis}\ \frac{\pi}{6}$

$w = \sqrt[4]{b}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right) = \sqrt[4]{b}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right) = \sqrt[4]{b}\ \text{cis}\left(-\frac{\pi}{4}\right)$

$$\therefore \left(\frac{z}{w}\right)^4 = \frac{z^4}{w^4} = \frac{\left(\sqrt[4]{a}\ \text{cis}\ \frac{\pi}{6}\right)^4}{\left(\sqrt[4]{b}\ \text{cis}\left(-\frac{\pi}{4}\right)\right)^4}$$



$$= \frac{a\ \text{cis}\ \frac{2\pi}{3}}{b\ \text{cis}\ (-\pi)}$$
$$= \frac{a\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)}{b(-1)}$$
$$= \frac{a}{2b} - \frac{a\sqrt{3}}{2b}i \qquad \therefore\ \mathcal{R}e\left(\left(\frac{z}{w}\right)^4\right) = \frac{a}{2b},\quad \mathcal{I}m\left(\left(\frac{z}{w}\right)^4\right) = -\frac{a\sqrt{3}}{2b}$$

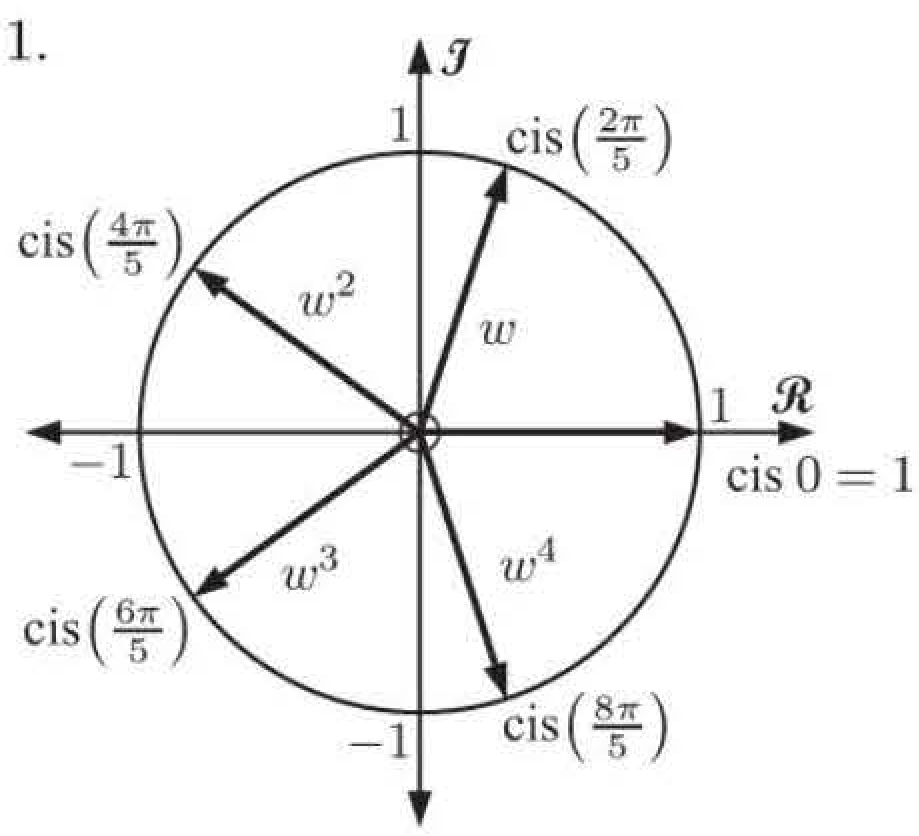
6 **a** The 5th roots of unity are the solutions to $z^5 = 1$.

$\therefore\ z^5 = \text{cis}\,(0 + k2\pi)$

$\therefore\ z^5 = \text{cis}\,(k2\pi)$

$\therefore\ z = [\text{cis}\,(k2\pi)]^{\frac{1}{5}}$

$\therefore\ z = \text{cis}\left(\frac{k2\pi}{5}\right)$ {De Moivre}



$\therefore\ z = \text{cis}\ 0 = 1$ or

$\text{cis}\left(\frac{2\pi}{5}\right) = w$ or

$\text{cis}\left(\frac{4\pi}{5}\right) = \left(\text{cis}\ \frac{2\pi}{5}\right)^2 = w^2$ or

$\text{cis}\left(\frac{6\pi}{5}\right) = \left(\text{cis}\ \frac{2\pi}{5}\right)^3 = w^3$ or

$\text{cis}\left(\frac{8\pi}{5}\right) = \left(\text{cis}\ \frac{2\pi}{5}\right)^4 = w^4$ {when $k = 0, 1, 2, 3, 4$}

Hence the five roots can be expressed as $1, w, w^2, w^3, w^4$ where $w = \text{cis}\left(\frac{2\pi}{5}\right)$

b $z^5 - 1 = (z-1)(z^4 + z^3 + z^2 + z + 1)$ .... (1)

Also, since $1, w, w^2, w^3$ and w^4 are the solutions to $z^5 = 1$,

$z^5 - 1 = (z-1)(z-w)(z-w^2)(z-w^3)(z-w^4)$ .... (2)

Equating (1) and (2),

$(z-1)(z^4 + z^3 + z^2 + z + 1) = (z-1)(z-w)(z-w^2)(z-w^3)(z-w^4)$

$\therefore\ z^4 + z^3 + z^2 + z + 1 = (z-w)(z-w^2)(z-w^3)(z-w^4)$

c $(2-w)(2-w^2)(2-w^3)(2-w^4) = 2^4+2^3+2^2+2+1$ {letting $z=2$}
$= 16+8+4+2+1$
$= 31$

7 The cube roots of $-8i$ are solutions to $z^3=-8i$

$z^3 = 8 \text{ cis}\left(-\frac{\pi}{2}+k2\pi\right)$ where k is an integer

$\therefore\ z = 8^{\frac{1}{3}} \text{ cis}\left(-\frac{\pi}{6}+\frac{k2\pi}{3}\right)$

$\therefore\ z = 2 \text{ cis}\left(-\frac{\pi}{6}+\frac{k4\pi}{6}\right)$

$\therefore\ z = 2 \text{ cis}\left(-\frac{5\pi}{6}\right),\ 2 \text{ cis}\left(-\frac{\pi}{6}\right),\ 2 \text{ cis}\left(\frac{\pi}{2}\right)$ {letting $k=-1, 0, 1$}

$\therefore\ z = -\sqrt{3}-i,\ \sqrt{3}-i,\ 2i$

8 **a** $\cos 3\theta + i\sin 3\theta = \text{cis } 3\theta = (\text{cis } \theta)^3$

b
$$\frac{1}{\cos 2\theta + i\sin 2\theta} = \frac{1}{\text{cis } 2\theta} = (\text{cis } 2\theta)^{-1} = [(\text{cis } \theta)^2]^{-1} = (\text{cis } \theta)^{-2}$$

c
$$\cos\theta - i\sin\theta = \cos(-\theta)+i\sin(-\theta) = \text{cis }(-\theta) = (\text{cis } \theta)^{-1}$$

9 The fifth roots of $2+2i$ are the solutions to $z^5 = 2+2i$

$\therefore\ z^5 = 2\sqrt{2} \text{ cis}\left(\frac{\pi}{4}+k2\pi\right)$

$\therefore\ z = \left[2^{\frac{3}{2}} \text{ cis}\left(\frac{\pi}{4}+k2\pi\right)\right]^{\frac{1}{5}}$

$\therefore\ z = 2^{0.3} \text{ cis}\left(\frac{\pi}{20}+\frac{k2\pi}{5}\right)$ {De Moivre}

$\therefore\ z = 2^{0.3} \text{ cis}\left(\frac{\pi}{20}+\frac{k8\pi}{20}\right)$

$\therefore\ z = 2^{0.3} \text{ cis}\left(-\frac{3\pi}{4}\right),\ 2^{0.3} \text{ cis}\left(-\frac{7\pi}{20}\right),\ 2^{0.3} \text{ cis}\left(\frac{\pi}{20}\right),\ 2^{0.3} \text{ cis}\left(\frac{9\pi}{20}\right),\ 2^{0.3} \text{ cis}\left(\frac{17\pi}{20}\right)$

{letting $k=-2, -1, 0, 1, 2$}

10 Let $z = x+iy$

$$\therefore\ z+\frac{1}{z} = (x+iy)+\frac{1}{x+iy}\times\frac{x-iy}{x-iy}$$
$$= (x+iy)+\frac{(x-iy)}{x^2+y^2}$$
$$= \frac{\left(x^2+y^2\right)(x+iy)+(x-iy)}{x^2+y^2}$$
$$= \frac{x^3+ix^2y+xy^2+y^3i+x-iy}{x^2+y^2}$$
$$= \frac{x(x^2+y^2+1)+i(x^2+y^2-1)y}{x^2+y^2}$$

which is real if $\dfrac{(x^2+y^2-1)y}{x^2+y^2} = 0$ $\quad\therefore\ x^2+y^2-1=0$ or $y=0$

$\therefore\ x^2+y^2=1$ or $y=0$

$\therefore\ |z|^2=1$ or $y=0$

$\therefore\ |z|=1$ or z is real

or Let $z = r \text{ cis } \theta$ $\quad \therefore \; z + \dfrac{1}{z} = r \text{ cis } \theta + \dfrac{1}{r} \text{ cis}(-\theta)$

$$= r\cos\theta + ir\sin\theta + \frac{1}{r}\cos(-\theta) + i \times \frac{1}{r}\sin(-\theta)$$

This is real if $\; r\sin\theta + \dfrac{1}{r}\sin(-\theta) = 0$

$$\therefore\; r\sin\theta - \frac{1}{r}\sin\theta = 0$$

$$\therefore\; \sin\theta\left(r - \frac{1}{r}\right) = 0$$

$$\therefore\; \sin\theta = 0 \quad \text{or} \quad r - \frac{1}{r} = 0$$

$$\therefore\; \theta = 0 \quad \text{or} \quad r = 1 \quad \{r \geqslant 0\}$$

$$\therefore\; z = r \text{ (which is real) or } |z| = 1$$

11 **a** If $z = \text{cis } \theta$

$$= \cos\theta + i\sin\theta$$

$$\therefore\; |z| = \sqrt{\cos^2\theta + \sin^2\theta} = \sqrt{1} = 1$$

b If $z = \text{cis } \theta$

then $z^* = \text{cis}(-\theta) = (\text{cis } \theta)^{-1} = z^{-1} = \dfrac{1}{z}$

c $z = \text{cis } \theta$

$\therefore\; z^4 = (\text{cis } \theta)^4$

$\therefore\; z^4 = \text{cis } 4\theta \qquad$ {De Moivre}

$\therefore\; z^4 = \cos 4\theta + i\sin 4\theta \quad$ .... (1)

Also, $z^4 = (\cos\theta + i\sin\theta)^4$

$\therefore\; z^4 = \cos^4\theta + 4\cos^3\theta i\sin\theta + 6\cos^2\theta\, i^2\sin^2\theta + 4\cos\theta\, i^3\sin^3\theta + i^4\sin^4\theta$

$\therefore\; z^4 = \cos^4\theta + 4i\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - 4i\cos\theta\sin^3\theta + \sin^4\theta \quad$ .... (2)

Equating real parts in (1) and (2) gives

$$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$$

$$\begin{aligned}\therefore\; \sin^4\theta &= \cos 4\theta - \cos^4\theta + 6\cos^2\theta\sin^2\theta \\ &= \cos 4\theta - (1-\sin^2\theta)^2 + 6(1-\sin^2\theta)\sin^2\theta \\ &= \cos 4\theta - (1 - 2\sin^2\theta + \sin^4\theta) + 6\sin^2\theta - 6\sin^4\theta \\ &= \cos 4\theta - 1 + 2\sin^2\theta - \sin^4\theta + 6\sin^2\theta - 6\sin^4\theta\end{aligned}$$

$$\begin{aligned}\therefore\; 8\sin^4\theta &= \cos 4\theta - 1 + 8\sin^2\theta \\ &= \cos 4\theta - 1 + 8\left(\tfrac{1}{2} - \tfrac{1}{2}\cos 2\theta\right) \\ &= \cos 4\theta - 1 + 4 - 4\cos 2\theta \\ &= \cos 4\theta - 4\cos 2\theta + 3\end{aligned}$$

$$\therefore\; \sin^4\theta = \tfrac{1}{8}(\cos 4\theta - 4\cos 2\theta + 3)$$

12 **a** If w is the root of $z^5 = 1$ with smallest positive argument, then $w = \text{cis}\left(\frac{2\pi}{5}\right)$ and $w^4 = \text{cis}\left(\frac{8\pi}{5}\right)$.

These have sum $= \cos\left(\frac{2\pi}{5}\right) + i\sin\left(\frac{2\pi}{5}\right) + \cos\left(\frac{8\pi}{5}\right) + i\sin\left(\frac{8\pi}{5}\right)$

$$= \cos\left(\tfrac{2\pi}{5}\right) + \cancel{i\sin\left(\tfrac{2\pi}{5}\right)} + \cos\left(\tfrac{2\pi}{5}\right) - \cancel{i\sin\left(\tfrac{2\pi}{5}\right)}$$

$$= 2\cos\left(\tfrac{2\pi}{5}\right)$$

and product $= \text{cis}\left(\frac{2\pi}{5}\right) \times \text{cis}\left(\frac{8\pi}{5}\right) = \text{cis}\left(\frac{10\pi}{5}\right) = \text{cis}2\pi = 1$

$\therefore\;$ a real quadratic with roots w, w^4 is $a\left(z^2 - 2\cos\left(\frac{2\pi}{5}\right)z + 1\right) = 0,\; a \neq 0$

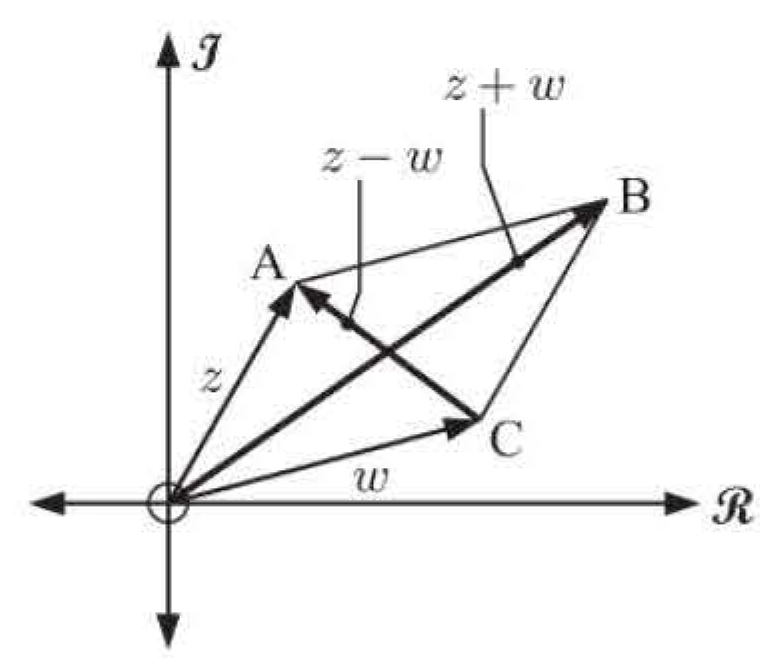
b Let $\alpha = w + w^4$ and $\beta = w^2 + w^3$

Now we know that $1 + w + w^2 + w^3 + w^4 = 0$ .... $(*)$

$$\begin{aligned}
1 + (w + w^4) + (w^2 + w^3) &= 0\\
1 + \alpha + \beta &= 0\\
\alpha + \beta &= -1\\
\text{and}\quad \alpha\beta &= (w + w^4)(w^2 + w^3)\\
&= w^3 + w^4 + w^6 + w^7\\
&= w^3 + w^4 + w + w^2 \quad \{\text{as } w^5 = 1\}\\
&= w + w^2 + w^3 + w^4\\
&= -1 \qquad \{\text{from } (*)\}
\end{aligned}$$

$\therefore$ the quadratic equation is $a(z^2 + z - 1) = 0,\quad a \neq 0$

13 Consider the diagram which shows vectors w, z, $z + w$, and $z - w$.



Clearly OABC is a parallelogram with $\overrightarrow{OB} = z + w$ and $\overrightarrow{CA} = z - w$

If $|z + w| = |z - w|$, the diagonals are equal in length.

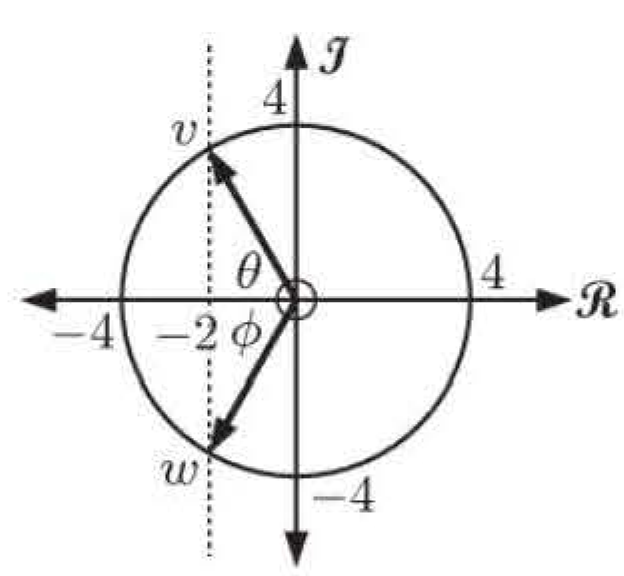
Hence, OABC is actually a rectangle and so $\widehat{COA}$ is a right angle $\therefore$ $\arg z$ and $\arg w$ differ by $\frac{\pi}{2}$.

14 **a**

$$\begin{aligned}
|z| &= |z + 4|\\
\therefore\quad |z|^2 &= |z + 4|^2\\
\text{Let}\quad z &= x + yi\\
\therefore\quad |x + yi|^2 &= |(x + 4) + yi|^2\\
\therefore\quad x^2 + y^2 &= (x + 4)^2 + y^2\\
\therefore\quad \cancel{x^2} + \cancel{y^2} &= \cancel{x^2} + 8x + 16 + \cancel{y^2}\\
\therefore\quad -8x &= 16\\
\therefore\quad x &= -2
\end{aligned}$$

$\therefore$ the real part of z is -2.

b **i**



ii In the diagram above, $\cos\theta = \frac{2}{4} = \frac{1}{2}$

$$\therefore\quad \theta = \arccos(\tfrac{1}{2}) = \tfrac{\pi}{3}$$

$$\therefore\quad \arg v = \pi - \theta = \tfrac{2\pi}{3}$$

iii $\cos\phi = \frac{2}{4} = \frac{1}{2}$

$$\therefore\quad \phi = \arccos(\tfrac{1}{2}) = \tfrac{\pi}{3}$$

$$\therefore\quad \arg w = -\pi + \phi = -\tfrac{2\pi}{3}$$

iv $v = 4\text{ cis}\left(\frac{2\pi}{3}\right),\quad w = 4\text{ cis}\left(-\frac{2\pi}{3}\right)$

$$\begin{aligned}
\therefore\quad \frac{v^m w}{i} &= \frac{\left(4\text{ cis}\left(\frac{2\pi}{3}\right)\right)^m 4\text{ cis}\left(-\frac{2\pi}{3}\right)}{\text{cis}\left(\frac{\pi}{2}\right)}\\
&= \frac{4^m\text{ cis}\left(\frac{2m\pi}{3}\right) 4\text{ cis}\left(-\frac{2\pi}{3}\right)}{\text{cis}\left(\frac{\pi}{2}\right)}\\
&= 4^{m+1}\text{ cis}\left(\tfrac{2m\pi}{3} - \tfrac{2\pi}{3} - \tfrac{\pi}{2}\right)
\end{aligned}$$

$$\begin{aligned}
\therefore\quad \arg\left(\frac{v^m w}{i}\right) &= \frac{2m\pi}{3} - \frac{2\pi}{3} - \frac{\pi}{2}\\
&= \frac{4m\pi}{6} - \frac{4\pi}{6} - \frac{3\pi}{6}\\
&= \frac{\pi(4m - 7)}{6}
\end{aligned}$$

v $\dfrac{v^m w}{i}$ is real when

$$\arg\left(\frac{v^m w}{i}\right) = 0 + k\pi,\quad k \in \mathbb{Z}$$

$$\therefore\quad \frac{\cancel{\pi}(4m - 7)}{6} = k\cancel{\pi}$$

$$\therefore\quad 4m - 7 = 6k$$

$$\therefore\quad m = \frac{7 + 6k}{4}$$

One such value of m is

$m = \frac{7}{4}$ $\{$when $k = 0\}$

Chapter 17

INTRODUCTION TO DIFFERENTIAL CALCULUS

EXERCISE 17A

1 **a** As $x \to 3$, $x + 4 \to 7$

$\therefore \lim\limits_{x \to 3} (x+4) = 7$

b As $x \to -1$, $5 - 2x \to 7$

$\therefore \lim\limits_{x \to -1} (5-2x) = 7$

c As $x \to 4$, $3x - 1 \to 11$

$\therefore \lim\limits_{x \to 4} (3x-1) = 11$

d As $x \to 2$, $5x^2 - 3x + 2 \to 5(4) - 3(2) + 2 = 16$

$\therefore \lim\limits_{x \to 2} (5x^2 - 3x + 2) = 16$

e As $h \to 0$, $h^2 \to 0$ and $1 - h \to 1$

$\therefore \lim\limits_{h \to 0} h^2(1-h) = 0 \times 1 = 0$

f As $x \to 0$, $x^2 + 5 \to 5$

$\therefore \lim\limits_{x \to 0} (x^2+5) = 5$

2 **a** $\lim\limits_{x \to 0} 5 = 5$

b $\lim\limits_{h \to 2} 7 = 7$

c $\lim\limits_{x \to 0} c = c$ (when c is a constant)

3 **a** $\lim\limits_{x \to 1} \dfrac{x^2 - 3x}{x} = \lim\limits_{x \to 1} \dfrac{x(x-3)}{x}$

$= \lim\limits_{x \to 1} (x-3)$ since $x \neq 0$

$= -2$

b $\lim\limits_{h \to 2} \dfrac{h^2 + 5h}{h} = \lim\limits_{h \to 2} \dfrac{h(h+5)}{h}$

$= \lim\limits_{h \to 2} (h+5)$ since $h \neq 0$

$= 7$

c $\dfrac{x-1}{x+1}$ can be made as close as we like to -1 by making x sufficiently close to 0.

$\therefore \lim\limits_{x \to 0} \dfrac{x-1}{x+1} = -1$

d $\lim\limits_{x \to 0} \dfrac{x}{x} = \lim\limits_{x \to 0} 1$ since $x \neq 0$

$= 1$

4 **a** $f(x) = \dfrac{1}{x}$ is not defined when $x = 0$

$\therefore f(x) = \dfrac{1}{x}$ is not continuous at $x = 0$.

b $f(x) = \dfrac{x^2 - x}{x}$ is not defined when $x = 0$

$\therefore f(x) = \dfrac{x^2 - x}{x}$ is not continuous at $x = 0$.

5 **a** $\lim\limits_{x \to 0} \dfrac{x^2 - 3x}{x} = \lim\limits_{x \to 0} \dfrac{x(x-3)}{x}$

$= \lim\limits_{x \to 0} (x-3)$ since $x \neq 0$

$= -3$

b $\lim\limits_{x \to 0} \dfrac{x^2 + 5x}{x} = \lim\limits_{x \to 0} \dfrac{x(x+5)}{x}$

$= \lim\limits_{x \to 0} (x+5)$ since $x \neq 0$

$= 5$

c $\lim\limits_{x \to 0} \dfrac{2x^2 - x}{x}$

$= \lim\limits_{x \to 0} \dfrac{x(2x-1)}{x}$

$= \lim\limits_{x \to 0} (2x-1)$ since $x \neq 0$

$= -1$

d $\lim\limits_{h \to 0} \dfrac{2h^2 + 6h}{h}$

$= \lim\limits_{h \to 0} \dfrac{2h(h+3)}{h}$

$= \lim\limits_{h \to 0} 2(h+3)$ since $h \neq 0$

$= 6$

e $\lim\limits_{h \to 0} \dfrac{3h^2 - 4h}{h}$

$= \lim\limits_{h \to 0} \dfrac{h(3h-4)}{h}$

$= \lim\limits_{h \to 0} (3h-4)$ since $h \neq 0$

$= -4$

f $\lim\limits_{h \to 0} \dfrac{h^3 - 8h}{h}$

$= \lim\limits_{h \to 0} \dfrac{h(h^2-8)}{h}$

$= \lim\limits_{h \to 0} (h^2-8)$ since $h \neq 0$

$= -8$

g $\lim\limits_{x\to1}\dfrac{x^2-x}{x-1}=\lim\limits_{x\to1}\dfrac{x(x-1)}{x-1}$
$=\lim\limits_{x\to1}x$ since $x\neq1$
$=1$

h $\lim\limits_{x\to2}\dfrac{x^2-2x}{x-2}=\lim\limits_{x\to2}\dfrac{x(x-2)}{x-2}$
$=\lim\limits_{x\to2}x$ since $x\neq2$
$=2$

i $\lim\limits_{x\to3}\dfrac{x^2-x-6}{x-3}=\lim\limits_{x\to3}\dfrac{(x+2)(x-3)}{x-3}$
$=\lim\limits_{x\to3}(x+2)$ since $x\neq3$
$=5$

EXERCISE 17B.1

1 As x gets larger and positive, $\dfrac{1}{x^2}$ gets smaller and closer to 0.

$\therefore\ \lim\limits_{x\to\infty}\dfrac{1}{x^2}=0$

2 **a** $\lim\limits_{x\to\infty}\dfrac{3x-2}{x+1}$
$=\lim\limits_{x\to\infty}\dfrac{3-\frac{2}{x}}{1+\frac{1}{x}}$
$=\frac{3}{1}=3$

b $\lim\limits_{x\to\infty}\dfrac{1-2x}{3x+2}$
$=\lim\limits_{x\to\infty}\dfrac{\frac{1}{x}-2}{3+\frac{2}{x}}$
$=-\frac{2}{3}$

c $\lim\limits_{x\to\infty}\dfrac{x}{1-x}$
$=\lim\limits_{x\to\infty}\dfrac{1}{\frac{1}{x}-1}$
$=\frac{1}{-1}=-1$

d $\lim\limits_{x\to\infty}\dfrac{x^2+3}{x^2-1}$
$=\lim\limits_{x\to\infty}\dfrac{1+\frac{3}{x^2}}{1-\frac{1}{x^2}}$
$=\frac{1}{1}$
$=1$

e $\lim\limits_{x\to\infty}\dfrac{x^2-2x+4}{x^2+x-1}$
$=\lim\limits_{x\to\infty}\dfrac{1-\frac{2}{x}+\frac{4}{x^2}}{1+\frac{1}{x}-\frac{1}{x^2}}$
$=\frac{1}{1}$
$=1$

EXERCISE 17B.2

1 **a** **i** As $x\to0^-$, $f(x)\to-\infty$
As $x\to0^+$, $f(x)\to\infty$
As $x\to\infty$, $f(x)\to0^+$
As $x\to-\infty$, $f(x)\to0^-$
The vertical asymptote is $x=0$.
The horizontal asymptote is $y=0$.

ii $\lim\limits_{x\to-\infty}f(x)=0$, $\lim\limits_{x\to\infty}f(x)=0$

b **i** As $x\to-3^-$, $f(x)\to\infty$
As $x\to-3^+$, $f(x)\to-\infty$
As $x\to\infty$, $f(x)\to3^-$
As $x\to-\infty$, $f(x)\to3^+$
The vertical asymptote is $x=-3$.
The horizontal asymptote is $y=3$.

ii $\lim\limits_{x\to-\infty}f(x)=3$, $\lim\limits_{x\to\infty}f(x)=3$

c **i** As $x\to-\frac{2}{3}^-$, $f(x)\to-\infty$
As $x\to-\frac{2}{3}^+$, $f(x)\to\infty$
As $x\to\infty$, $f(x)\to-\frac{2}{3}^+$
As $x\to-\infty$, $f(x)\to-\frac{2}{3}^-$
The vertical asymptote is $x=-\frac{2}{3}$.
The horizontal asymptote is $y=-\frac{2}{3}$.

ii $\lim\limits_{x\to-\infty}f(x)=-\frac{2}{3}$,
$\lim\limits_{x\to\infty}f(x)=-\frac{2}{3}$

d **i** As $x\to1^-$, $f(x)\to\infty$
As $x\to1^+$, $f(x)\to-\infty$
As $x\to\infty$, $f(x)\to-1^-$
As $x\to-\infty$, $f(x)\to-1^+$
The vertical asymptote is $x=1$.
The horizontal asymptote is $y=-1$.

ii $\lim\limits_{x\to-\infty}f(x)=-1$, $\lim\limits_{x\to\infty}f(x)=-1$

e **i** Since there are no real values of x that make $x^2+1=0$, $f(x)$ is defined for all $x \in \mathbb{R}$.

$\therefore$ there are no vertical asymptotes.

As $x \to \infty$, $f(x) \to 1^-$

As $x \to -\infty$, $f(x) \to 1^-$

The horizontal asymptote is $y=1$.

ii $\lim\limits_{x \to -\infty} f(x) = 1$, $\lim\limits_{x \to \infty} f(x) = 1$

f **i** Since there are no real values of x that make $x^2+1=0$, $f(x)$ is defined for all $x \in \mathbb{R}$.

$\therefore$ there are no vertical asymptotes.

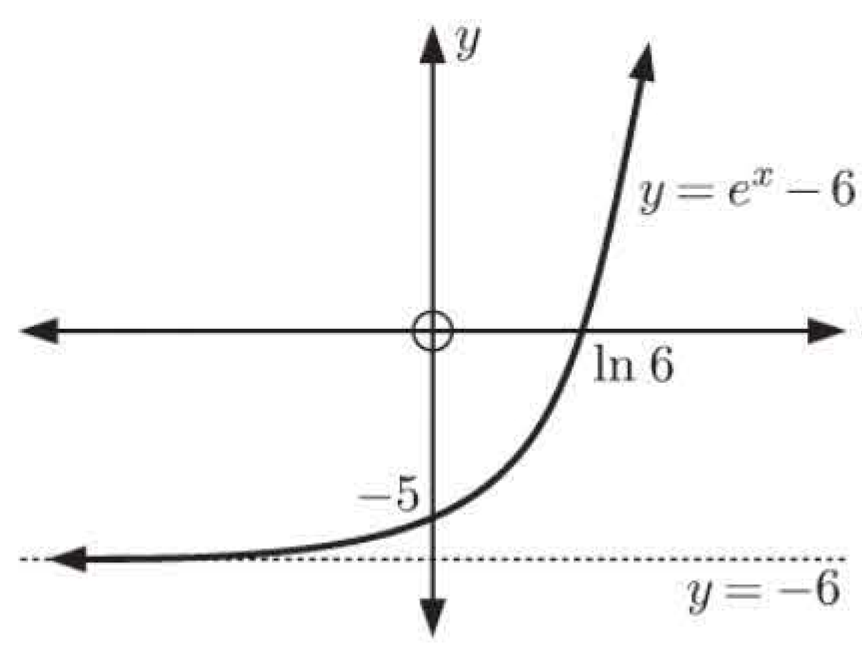
As $x \to \infty$, $f(x) \to 0^+$

As $x \to -\infty$, $f(x) \to 0^-$

The horizontal asymptote is $y=0$.

ii $\lim\limits_{x \to -\infty} f(x) = 0$, $\lim\limits_{x \to \infty} f(x) = 0$

2 **a**



b **i** As $x \to -\infty$, $e^x - 6 \to -6^+$

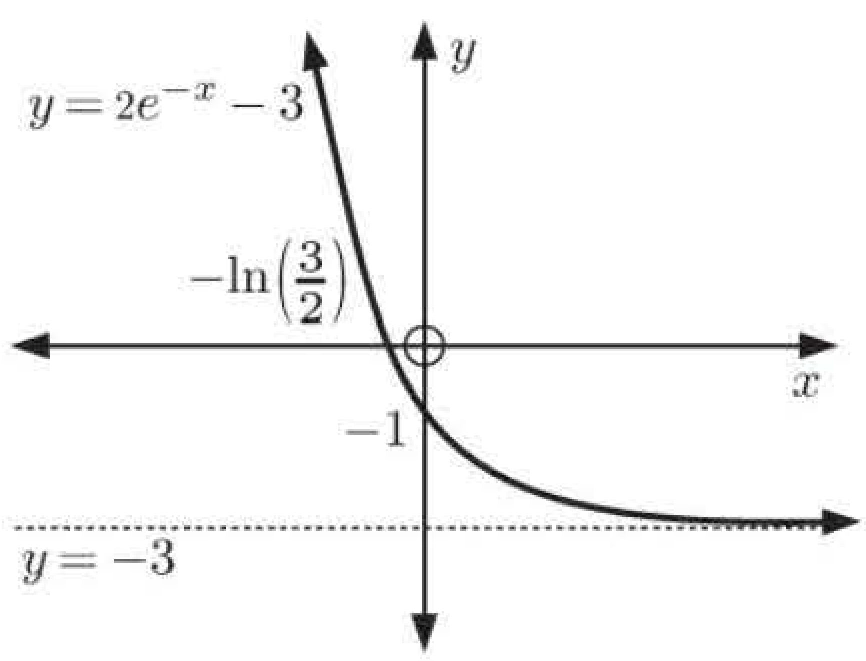
$\therefore \lim\limits_{x \to -\infty} (e^x - 6) = -6$

$\therefore$ the function has horizontal asymptote $y=-6$.

ii As $x \to \infty$, $e^x - 6 \to \infty$

$\therefore \lim\limits_{x \to \infty} (e^x - 6)$ does not exist.

3 We sketch the graph of $y = 2e^{-x} - 3$:



As $x \to -\infty$, $2e^{-x} - 3 \to \infty$

$\therefore \lim\limits_{x \to -\infty} (2e^{-x} - 3)$ does not exist.

As $x \to \infty$, $2e^{-x} - 3 \to -3^+$

$\therefore \lim\limits_{x \to \infty} (2e^{-x} - 3) = -3$.

4 **a** $f(x) = \ln x$

$f(x)$ is undefined when $x \leqslant 0$

$\therefore$ $x=0$ is a vertical asymptote.

As $x \to 0^+$, $y \to -\infty$

b $f(x) = e^{x - \frac{1}{x}}$

$f(x)$ is undefined when $x=0$

$\therefore$ $x=0$ is a vertical asymptote.

As $x \to -\infty$, $f(x) \to 0$

$\therefore$ $y=0$ is a horizontal asymptote.

As $x \to 0^-$, $f(x) \to \infty$

As $x \to 0^+$, $f(x) \to 0^+$

As $x \to \infty$, $f(x) \to \infty$

As $x \to -\infty$, $f(x) \to 0^+$

5 **a** $f(x) = x + \ln x$

$f(x)$ is undefined for $x \leqslant 0$

$\therefore$ $x=0$ is a vertical asymptote

As $x \to 0^+$, $f(x) \to -\infty$

b $f(x) = e^x - x$

$f(x)$ is defined for all $x \in \mathbb{R}$

$\therefore$ no vertical asymptotes exist.

As $x \to -\infty$, $f(x) \to -x$

$\therefore$ $y=-x$ is an oblique asymptote

As $x \to \infty$, $f(x) \to \infty$

As $x \to -\infty$, $f(x) \to (-x)^+$

c $f(x) = \dfrac{x^3-2}{x^2+1}$

$$\begin{array}{r|l} & \quad\quad\quad x \\ \hline x^2+1 & x^3+0x^2+0x-2 \\ & x^3 \quad\quad + \; x \\ \hline & \quad\quad -\; x-2 \end{array}$$

$\therefore \; f(x) = x - \dfrac{x+2}{x^2+1}$

x is defined for all $x \in \mathbb{R}$

$\therefore$ no vertical asymptotes exist.

As $|x| \to \infty$, $f(x) \to x$

$\therefore$ $y = x$ is an oblique asymptote

As $x \to \infty$, $f(x) \to x^-$

As $x \to -\infty$, $f(x) \to x^+$

d $f(x) = (x-2)e^{-x} = \dfrac{x-2}{e^x}$

$f(x)$ is defined for all $x \in \mathbb{R}$

$\therefore$ no vertical asymptotes exist.

As $x \to \infty$, $f(x) \to 0^+$

$\therefore$ $y = 0$ is a horizontal asymptote.

EXERCISE 17C

1 **a**
$$\begin{aligned} &\lim_{\theta\to 0} \frac{\sin 2\theta}{\theta} \\ &= \lim_{\theta\to 0} \frac{\sin 2\theta}{2\theta} \times 2 \\ &= 2 \times \lim_{2\theta\to 0} \frac{\sin 2\theta}{2\theta} \quad \{2\theta \to 0 \text{ as } \theta \to 0\} \\ &= 2 \times 1 \\ &= 2 \end{aligned}$$

b
$$\begin{aligned} &\lim_{\theta\to 0} \frac{\theta}{\sin\theta} \\ &= \lim_{\theta\to 0} \frac{1}{\frac{\sin\theta}{\theta}} \\ &= \frac{1}{\lim\limits_{\theta\to 0} \frac{\sin\theta}{\theta}} \\ &= \frac{1}{1} \\ &= 1 \end{aligned}$$

c
$$\begin{aligned} &\lim_{\theta\to 0} \frac{\tan\theta}{\theta} \\ &= \lim_{\theta\to 0} \frac{\sin\theta}{\theta} \times \frac{1}{\cos\theta} \\ &= 1 \times \frac{1}{1} \\ &= 1 \end{aligned}$$

d
$$\begin{aligned} &\lim_{\theta\to 0} \frac{\sin\theta\sin 4\theta}{\theta^2} \\ &= \lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right) \lim_{\theta\to 0}\left(\frac{\sin 4\theta}{\theta}\right) \\ &= 1 \times \lim_{\theta\to 0}\left(\frac{\sin 4\theta}{4\theta}\right) \times 4 \\ &= 4 \times \lim_{4\theta\to 0}\left(\frac{\sin 4\theta}{4\theta}\right) \quad \{4\theta \to 0 \text{ as } \theta \to 0\} \\ &= 4 \times 1 \\ &= 4 \end{aligned}$$

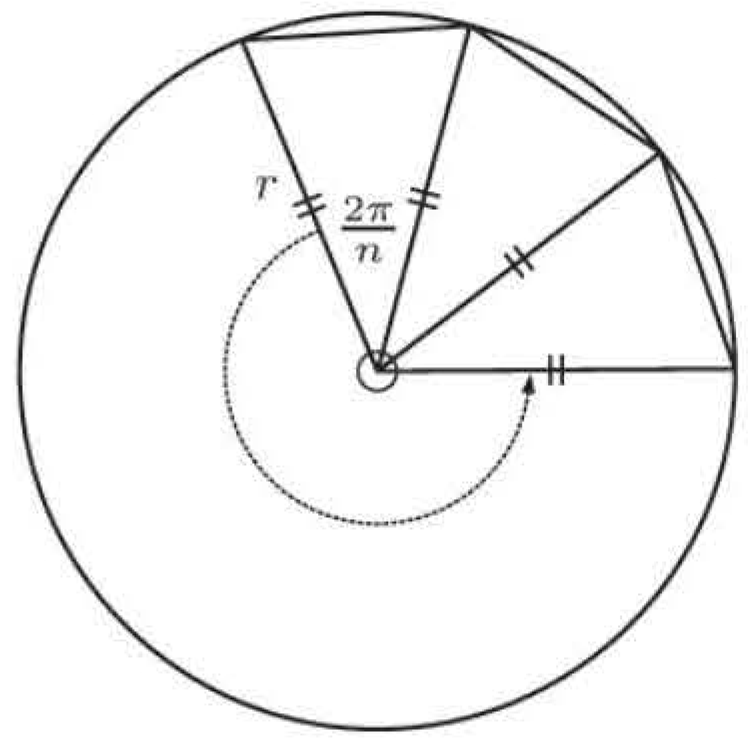
e
$$\begin{aligned} &\lim_{h\to 0} \frac{\sin\left(\frac{h}{2}\right)\cos h}{h} \\ &= \lim_{h\to 0} \cos h \lim_{h\to 0} \frac{\sin\left(\frac{h}{2}\right)}{h} \\ &= 1 \times \lim_{h\to 0} \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} \times \tfrac{1}{2} \\ &= \tfrac{1}{2} \times \lim_{\frac{h}{2}\to 0} \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} \quad \{\tfrac{h}{2} \to 0 \text{ as } h \to 0\} \\ &= \tfrac{1}{2} \times 1 \\ &= \tfrac{1}{2} \end{aligned}$$

f
$$\begin{aligned} &\lim_{n\to\infty} n\sin\left(\tfrac{2\pi}{n}\right) \\ &= \lim_{\frac{1}{n}\to 0^+} \frac{\sin\left(\frac{2\pi}{n}\right)}{\frac{1}{n}} \quad \{\tfrac{1}{n} \to 0^+ \text{ as } n \to \infty\} \\ &= \lim_{\frac{1}{n}\to 0^+} \frac{\sin\left(\frac{2\pi}{n}\right)}{\frac{2\pi}{n}} \times 2\pi \\ &= 2\pi \times \lim_{\frac{2\pi}{n}\to 0^+} \frac{\sin\left(\frac{2\pi}{n}\right)}{\frac{2\pi}{n}} \quad \left\{\begin{array}{l} \frac{2\pi}{n} \to 0^+ \\ \text{as } \frac{1}{n} \to 0^+ \end{array}\right\} \\ &= 2\pi \times 1 \\ &= 2\pi \end{aligned}$$

2 **a** The angle at the apex of each triangle $= \frac{2\pi}{n}$ {angles at a point}

$\therefore$ area of each triangle $= \frac{1}{2}r^2 \sin\left(\frac{2\pi}{n}\right)$

$\therefore$ area of the n triangles $S_n = \frac{1}{2}nr^2 \sin\left(\frac{2\pi}{n}\right)$



b **i** As the number of triangles increases, the triangles cover more of the circle. As $n \to \infty$, the triangles get closer to covering the whole circle.

$\therefore \lim\limits_{n\to\infty} S_n =$ area of the circle

ii $$\begin{aligned}\lim_{n\to\infty} S_n &= \lim_{n\to\infty} \tfrac{1}{2}nr^2 \sin\left(\tfrac{2\pi}{n}\right)\\ &= \tfrac{1}{2}r^2 \lim_{n\to\infty} n\sin\left(\tfrac{2\pi}{n}\right)\\ &= \tfrac{1}{2}r^2(2\pi) \qquad \{\text{using } \mathbf{1\ f}\}\\ &= \pi r^2\end{aligned}$$

c Area of circle $= \pi r^2$

3 **a** $$\begin{aligned}\cos(A+B) - \cos(A-B) &= \cos A\cos B - \sin A\sin B - (\cos A\cos B + \sin A\sin B)\\ &= \cancel{\cos A\cos B} - \sin A\sin B - \cancel{\cos A\cos B} - \sin A\sin B\\ &= -2\sin A\sin B\end{aligned}$$

b $$\begin{aligned}\cos S - \cos D &= \cos(A+B) - \cos(A-B)\\ &= -2\sin A\sin B \qquad \{\text{using } \mathbf{a}\}\end{aligned}$$

Now $S + D = A + B + A - B = 2A$ and $S - D = A + B - (A - B) = 2B$

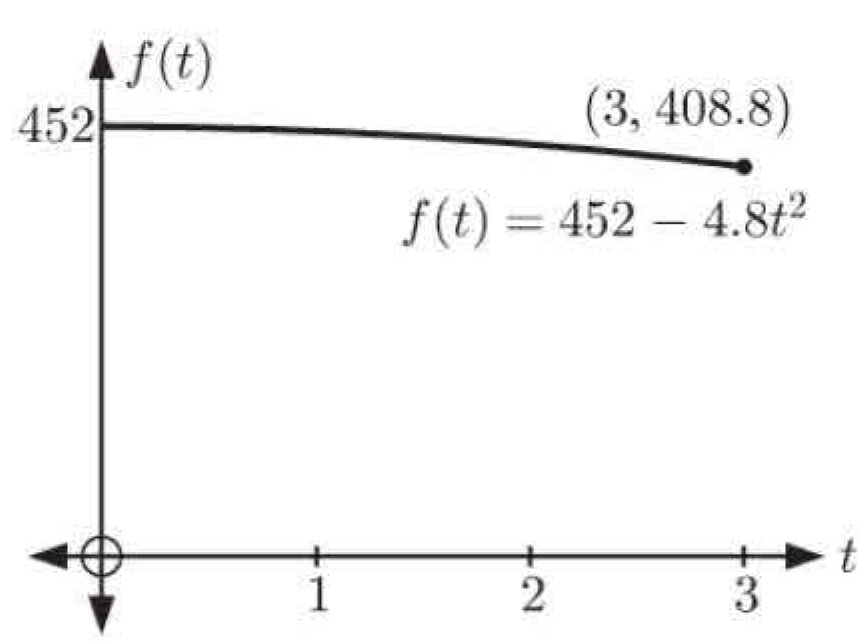
$\therefore A = \dfrac{S+D}{2}$ $\qquad$ $\therefore B = \dfrac{S-D}{2}$

So, $\cos S - \cos D = -2\sin\left(\frac{S+D}{2}\right)\sin\left(\frac{S-D}{2}\right)$

c $$\begin{aligned}\lim_{h\to0}\frac{\cos(x+h)-\cos x}{h} &= \lim_{h\to0}\frac{-2\sin\left[\frac{(x+h)+x}{2}\right]\sin\left[\frac{(x+h)-x}{2}\right]}{h} \qquad \{x+h=S,\ x=D\}\\ &= \lim_{h\to0}\frac{-2\sin\left(\frac{2x+h}{2}\right)\sin\left(\frac{h}{2}\right)}{h}\\ &= -2\lim_{h\to0}\frac{\sin\left(x+\frac{h}{2}\right)\sin\left(\frac{h}{2}\right)}{h}\\ &= -2\lim_{h\to0}\sin\left(x+\tfrac{h}{2}\right)\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\times\tfrac{1}{2}\\ &= -1\lim_{h\to0}\sin\left(x+\tfrac{h}{2}\right)\lim_{\frac{h}{2}\to0}\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} \qquad \{\tfrac{h}{2}\to0 \text{ as } h\to0\}\\ &= -1\lim_{h\to0}\sin\left(x+\tfrac{h}{2}\right)\\ &= -\sin x\end{aligned}$$

EXERCISE 17D

1 **a**

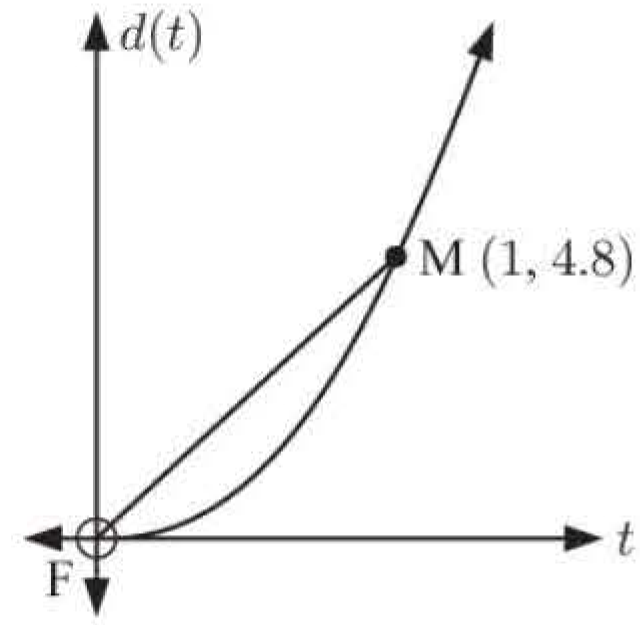


b The graph of $f(t)$ is not a straight line, so the jumper is not travelling with constant speed.

c If the *altitude* of the jumper is given by $f(t) = 452 - 4.8t^2$,
then the *distance* covered by the jumper $d(t) = 452 - f(t)$

$$\therefore \; d(t) = 452 - (452 - 4.8t^2)$$
$$\therefore \; d(t) = 452 - 452 + 4.8t^2$$
$$\therefore \; d(t) = 4.8t^2$$

i We choose a fixed point F on $d(t)$ when $t = 0$ seconds. This is the point $(0, 0)$.
We then choose another point M on the curve, for example the point $(1, 4.8)$.
The average speed in the interval $0 \leqslant t \leqslant 1$ is $\dfrac{4.8 - 0}{1 - 0} = 4.8 \text{ m s}^{-1}$.

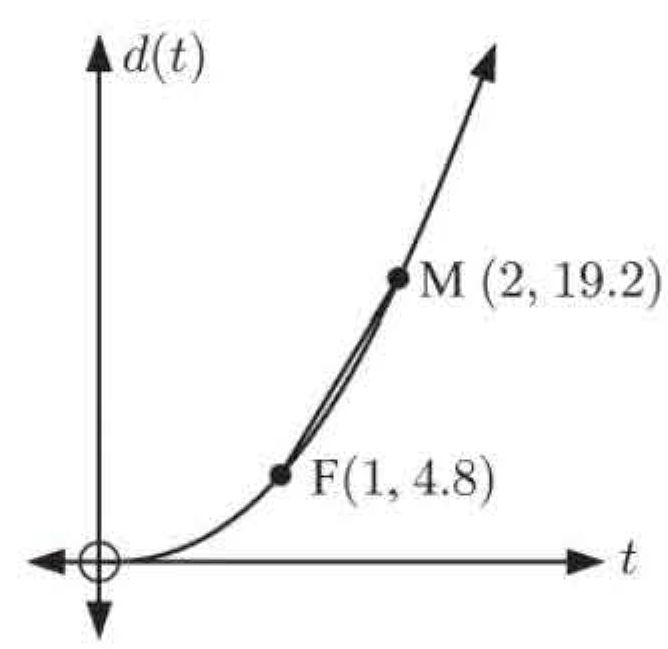


We repeat this process, moving M closer to F each time, and get the following results:

t	gradient of [FM]
1	4.8
0.5	2.4
0.1	0.48
0.01	0.048
0.001	0.0048

So, as M approaches F, the gradient of [FM] approaches 0.
$\therefore$ the speed of the jumper at $t = 0$ seconds is 0 m s^{-1}.

ii We now choose point F on $d(t)$ when $t = 1$ second. This is the point $(1, 4.8)$.
We then choose another point M on the curve, for example the point $(2, 19.2)$.
The average speed in the interval $1 \leqslant t \leqslant 2$ is $\dfrac{19.2 - 4.8}{2 - 1} = 14.4 \text{ m s}^{-1}$.



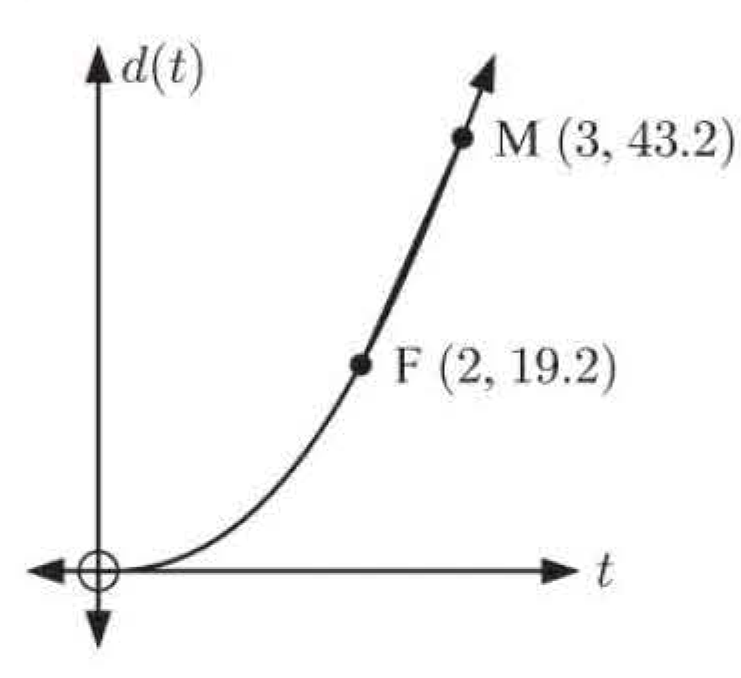
We repeat this process, moving M closer to F each time, and get the following results:

t	gradient of [FM]
2	14.4
1.5	12
1.1	10.08
1.01	9.648
1.001	9.6048
1.0001	9.600 48

t	gradient of [FM]
0	4.8
0.5	7.2
0.9	9.12
0.99	9.552
0.999	9.5952
0.9999	9.599 52

So, as M approaches F (from either direction), the gradient of [FM] approaches 9.6.
$\therefore$ the speed of the jumper at $t = 1$ second is 9.6 m s^{-1}.

iii We now choose point F on $d(t)$ when $t = 2$ seconds. This is the point $(2, 19.2)$.
We then choose another point M on the curve, for example the point $(3, 43.2)$.
The average speed in the interval $2 \leqslant t \leqslant 3$ is $\dfrac{43.2 - 19.2}{3 - 2} = 24 \text{ m s}^{-1}$.



We repeat this process, moving M closer to F each time, and get the following results:

t	gradient of [FM]
3	24
2.5	21.6
2.1	19.68
2.01	19.248
2.001	19.2048
2.0001	19.200 48

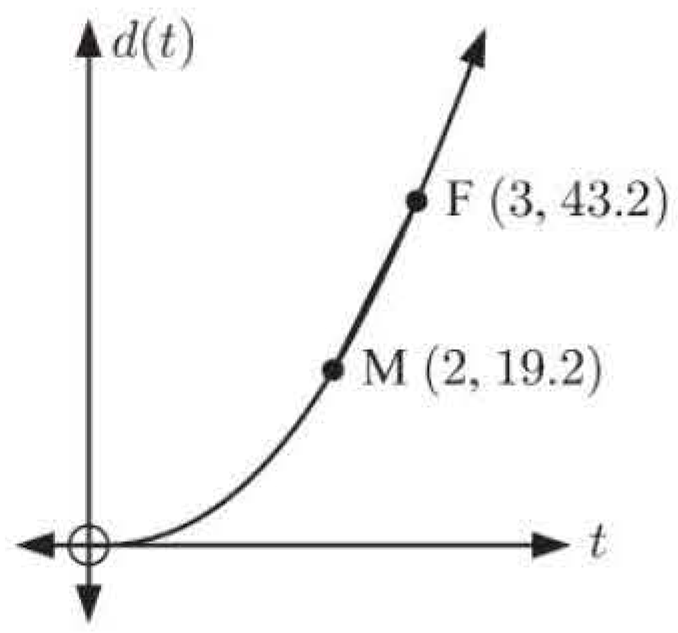
t	gradient of [FM]
1	14.4
1.5	16.8
1.9	18.72
1.99	19.152
1.999	19.1952
1.9999	19.199 52

So, as M approaches F (from either direction), the gradient of [FM] approaches 19.2.

$\therefore$ the speed of the jumper at $t = 2$ seconds is $19.2\ \text{m s}^{-1}$.

iv We choose a fixed point F on $d(t)$ when $t = 3$ seconds. This is the point (3, 43.2).
We then choose another point M on the curve, for example the point (2, 19.2).
The gradient of [MF] is

$$\frac{43.2 - 19.2}{3 - 2} = 24\ \text{m s}^{-1}.$$



We repeat this process, moving M closer to F each time, and get the following results:

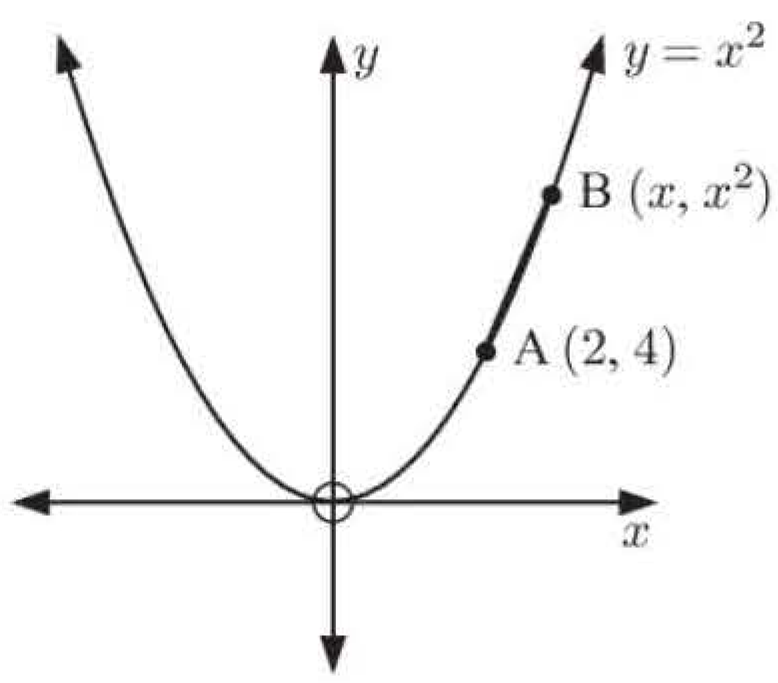
t	gradient of [MF]
2	24
2.5	26.4
2.9	28.32
2.99	28.752
2.999	28.7952
2.9999	28.799 52

So, as M approaches F, the gradient of [MF] approaches 28.8.

$\therefore$ the speed of the jumper at $t = 3$ seconds is $28.8\ \text{m s}^{-1}$.

2 **a** Suppose A is the point (2, 4) and B is a point on $y = x^2$ with coordinates (x, x^2).
The chord [AB] has gradient

$$\frac{x^2 - 4}{x - 2} \quad \left(\text{or } \frac{4 - x^2}{2 - x}\right).$$



As B moves closer to A (from either side), we get the following results:

x	Point B	Gradient of [AB]
0	(0, 0)	2
1	(1, 1)	3
1.5	(1.5, 2.25)	3.5
1.9	(1.9, 3.61)	3.9
1.99	(1.99, 3.9601)	3.99
1.999	(1.999, 3.996 001)	3.999

x	Point B	Gradient of [AB]
5	(5, 25)	7
3	(3, 9)	5
2.5	(2.5, 6.25)	4.5
2.1	(2.1, 4.41)	4.1
2.01	(2.01, 4.0401)	4.01
2.001	(2.001, 4.004 001)	4.001

So, as B approaches A, the gradient of [AB] approaches 4.

$\therefore$ the gradient of the tangent to $y = x^2$ at the point (2, 4) is 4.

b
$$\begin{aligned}\lim_{x \to 2} \frac{x^2 - 4}{x - 2} &= \lim_{x \to 2} \frac{(x+2)(x-2)}{x - 2} \\ &= \lim_{x \to 2} (x + 2) \quad \text{since } x \neq 2 \\ &= 4\end{aligned}$$

This is the gradient of the tangent to $y = x^2$ at the point where $x = 2$.

EXERCISE 17E

1 **a** $f(2) = 3$

b $f'(2)$ is the gradient of the tangent to $f(x)$ at the point where $x = 2$.
Since $f(x)$ is a straight line, this is the same as the gradient of $f(x)$ itself.
$f(x)$ is a horizontal line, and hence has gradient 0.
$\therefore\ f'(2) = 0$

2 **a** $f(0) = 4$

b $f'(0)$ is the gradient of the tangent to $f(x)$ at the point where $x = 0$.
Since $f(x)$ is a straight line, this is the same as the gradient of $f(x)$ itself.
$f(x)$ passes through (0, 4) and (4, 0), so it has gradient $= \dfrac{0-4}{4-0} = -1$
$\therefore\ f'(0) = -1$

3 The graph shows the tangent to the curve $y = f(x)$ at the point where $x = 2$.
The tangent passes through (0, 1) and (4, 5), so its gradient is $f'(2) = \dfrac{5-1}{4-0} = 1$.

The equation of the tangent is $\dfrac{y-1}{x-0} = 1$
$\therefore\ y = x + 1$

When $x = 2$, $y = 3$, so the point of contact is (2, 3).
$\therefore\ f(2) = 3$ and $f'(2) = 1$.

EXERCISE 17F

1 **a** **i** $f(x) = x$

$$\begin{aligned}\therefore\ f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)-x}{h} \\ &= \lim_{h\to 0} \frac{h}{h} \\ &= \lim_{h\to 0} 1 \quad \{\text{as } h \neq 0\} \\ &= 1\end{aligned}$$

ii $f(x) = 5$

$$\begin{aligned}\therefore\ f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{5-5}{h} \\ &= \lim_{h\to 0} \frac{0}{h} \\ &= \lim_{h\to 0} 0 \quad \{\text{as } h \neq 0\} \\ &= 0\end{aligned}$$

iii $f(x) = x^3$

$$\begin{aligned}\therefore\ & f'(x) \\ &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^3-x^3}{h} \\ &= \lim_{h\to 0} \frac{\cancel{x^3}+3x^2h+3xh^2+h^3-\cancel{x^3}}{h} \\ &= \lim_{h\to 0} \frac{3x^2h+3xh^2+h^3}{h} \\ &= \lim_{h\to 0} 3x^2+3xh+h^2 \quad \{\text{as } h \neq 0\} \\ &= 3x^2\end{aligned}$$

iv $f(x) = x^4$

$$\begin{aligned}\therefore\ & f'(x) \\ &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^4-x^4}{h} \\ &= \lim_{h\to 0} \frac{\cancel{x^4}+4x^3h+6x^2h^2+4xh^3+h^4-\cancel{x^4}}{h} \\ &= \lim_{h\to 0} \frac{4x^3h+6x^2h^2+4xh^3+h^4}{h} \\ &= \lim_{h\to 0} 4x^3+6x^2h+4xh^2+h^3 \quad \{\text{as } h \neq 0\} \\ &= 4x^3\end{aligned}$$

b From **a**, we predict that if $f(x) = x^n$, $f'(x) = nx^{n-1}$, $n \in \mathbb{N}$.

2 **a** $f(x) = 2x + 5$

$$\therefore \quad f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h \to 0} \frac{(2(x+h)+5) - (2x+5)}{h}$$
$$= \lim_{h \to 0} \frac{\cancel{2x} + 2h + \cancel{5} - \cancel{2x} - \cancel{5}}{h}$$
$$= \lim_{h \to 0} \frac{2h}{h}$$
$$= \lim_{h \to 0} 2 \quad \{\text{as } h \neq 0\}$$
$$= 2$$

b $f(x) = x^2 - 3x$

$$\therefore \quad f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h \to 0} \frac{[(x+h)^2 - 3(x+h)] - [x^2 - 3x]}{h}$$
$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{3x} - 3h - \cancel{x^2} + \cancel{3x}}{h}$$
$$= \lim_{h \to 0} \frac{2xh + h^2 - 3h}{h}$$
$$= \lim_{h \to 0} 2x + h - 3 \quad \{\text{as } h \neq 0\}$$
$$= 2x - 3$$

c $f(x) = -x^2 + 5x - 3$

$$\therefore \quad f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h \to 0} \frac{[-(x+h)^2 + 5(x+h) - 3] - [-x^2 + 5x - 3]}{h}$$
$$= \lim_{h \to 0} \frac{-\cancel{x^2} - 2xh - h^2 + \cancel{5x} + 5h - \cancel{3} + \cancel{x^2} - \cancel{5x} + \cancel{3}}{h}$$
$$= \lim_{h \to 0} \frac{-2xh - h^2 + 5h}{h}$$
$$= \lim_{h \to 0} -2x + 5 - h \quad \{\text{as } h \neq 0\}$$
$$= -2x + 5$$

3 **a** $y = f(x) = 4 - x$

$$\therefore \quad \frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h \to 0} \frac{[4 - (x+h)] - [4 - x]}{h}$$
$$= \lim_{h \to 0} \frac{\cancel{4} - \cancel{x} - h - \cancel{4} + \cancel{x}}{h}$$
$$= \lim_{h \to 0} \frac{-h}{h}$$
$$= \lim_{h \to 0} -1 \quad \{\text{as } h \neq 0\}$$
$$= -1$$

b $y = f(x) = 2x^2 + x - 1$

$$\therefore \quad \frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h \to 0} \frac{[2(x+h)^2 + (x+h) - 1] - [2x^2 + x - 1]}{h}$$
$$= \lim_{h \to 0} \frac{\cancel{2x^2} + 4xh + 2h^2 + \cancel{x} + h - \cancel{1} - \cancel{2x^2} - \cancel{x} + \cancel{1}}{h}$$
$$= \lim_{h \to 0} \frac{4xh + 2h^2 + h}{h}$$
$$= \lim_{h \to 0} 4x + 1 + 2h \quad \{\text{as } h \neq 0\}$$
$$= 4x + 1$$

c $y = f(x) = x^3 - 2x^2 + 3$

$$\therefore \quad \frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h \to 0} \frac{[(x+h)^3 - 2(x+h)^2 + 3] - [x^3 - 2x^2 + 3]}{h}$$
$$= \lim_{h \to 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{2x^2} - 4xh - 2h^2 + \cancel{3} - \cancel{x^3} + \cancel{2x^2} - \cancel{3}}{h}$$
$$= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3 - 4xh - 2h^2}{h}$$
$$= \lim_{h \to 0} 3x^2 + 3xh + h^2 - 4x - 2h \quad \{\text{as } h \neq 0\}$$
$$= 3x^2 - 4x$$

4 **a** $f(x) = x^3$

$\therefore \; f'(2)$

$= \lim\limits_{h\to 0} \dfrac{f(2+h) - f(2)}{h}$ where $f(2) = 2^3 = 8$

$= \lim\limits_{h\to 0} \dfrac{(2+h)^3 - 8}{h}$

$= \lim\limits_{h\to 0} \dfrac{\cancel{8} + 12h + 6h^2 + h^3 - \cancel{8}}{h}$

$= \lim\limits_{h\to 0} \dfrac{12h + 6h^2 + h^3}{h}$

$= \lim\limits_{h\to 0} 12 + 6h + h^2 \quad \{\text{as } h \neq 0\}$

$= 12$

b $f(x) = x^4$

$\therefore \; f'(3)$

$= \lim\limits_{h\to 0} \dfrac{f(3+h) - f(3)}{h}$ where $f(3) = 3^4 = 81$

$= \lim\limits_{h\to 0} \dfrac{(3+h)^4 - 81}{h}$

$= \lim\limits_{h\to 0} \dfrac{\cancel{81} + 108h + 54h^2 + 12h^3 + h^4 - \cancel{81}}{h}$

$= \lim\limits_{h\to 0} \dfrac{108h + 54h^2 + 12h^3 + h^4}{h}$

$= \lim\limits_{h\to 0} 108 + 54h + 12h^2 + h^3 \quad \{\text{as } h \neq 0\}$

$= 108$

5 **a** $f(x) = 3x + 5$

We need to find $f'(-2)$.

$f'(-2) = \lim\limits_{h\to 0} \dfrac{f(-2+h) - f(-2)}{h}$

where $f(-2) = 3(-2) + 5 = -1$

$= \lim\limits_{h\to 0} \dfrac{[3(-2+h)+5] - [-1]}{h}$

$= \lim\limits_{h\to 0} \dfrac{\cancel{-6} + 3h + \cancel{5} + \cancel{1}}{h}$

$= \lim\limits_{h\to 0} \dfrac{3h}{h}$

$= \lim\limits_{h\to 0} 3 \quad \{\text{as } h \neq 0\}$

$= 3$

$\therefore$ the gradient of the tangent to $f(x) = 3x + 5$ at $x = -2$ is 3.

b $f(x) = 5 - 2x^2$

We need to find $f'(3)$.

$f'(3) = \lim\limits_{h\to 0} \dfrac{f(3+h) - f(3)}{h}$

where $f(3) = 5 - 2(3)^2 = -13$

$= \lim\limits_{h\to 0} \dfrac{[5 - 2(3+h)^2] - [-13]}{h}$

$= \lim\limits_{h\to 0} \dfrac{5 - 2(9 + 6h + h^2) + 13}{h}$

$= \lim\limits_{h\to 0} \dfrac{\cancel{5} - \cancel{18} - 12h - 2h^2 + \cancel{13}}{h}$

$= \lim\limits_{h\to 0} \dfrac{-12h - 2h^2}{h}$

$= \lim\limits_{h\to 0} \dfrac{-2h(6+h)}{h}$

$= \lim\limits_{h\to 0} -2(6+h) \quad \{\text{as } h \neq 0\}$

$= -12$

c $f(x) = x^2 + 3x - 4$

We need to find $f'(3)$.

$f'(3) = \lim\limits_{h\to 0} \dfrac{f(3+h) - f(3)}{h}$ where $f(3) = 3^2 + 3(3) - 4 = 14$

$= \lim\limits_{h\to 0} \dfrac{[(3+h)^2 + 3(3+h) - 4] - 14}{h}$

$= \lim\limits_{h\to 0} \dfrac{\cancel{9} + 6h + h^2 + \cancel{9} + 3h - \cancel{4} - \cancel{14}}{h}$

$= \lim\limits_{h\to 0} \dfrac{9h + h^2}{h}$

$= \lim\limits_{h\to 0} \dfrac{h(9+h)}{h}$

$= \lim\limits_{h\to 0} (9+h) \quad \{\text{as } h \neq 0\}$

$= 9$

d $f(x) = 5 - 2x - 3x^2$

We need to find $f'(-2)$.

$$\begin{aligned}
f'(-2) &= \lim_{h\to 0} \frac{f(-2+h) - f(-2)}{h} \quad \text{where } f(-2) = 5 - 2(-2) - 3(-2)^2 = -3 \\
&= \lim_{h\to 0} \frac{[5 - 2(-2+h) - 3(-2+h)^2] - [-3]}{h} \\
&= \lim_{h\to 0} \frac{5 + 4 - 2h - 3(4 - 4h + h^2) + 3}{h} \\
&= \lim_{h\to 0} \frac{\cancel{5} + \cancel{4} - 2h - \cancel{12} + 12h - 3h^2 + \cancel{3}}{h} \\
&= \lim_{h\to 0} \frac{10h - 3h^2}{h} \\
&= \lim_{h\to 0} \frac{h(10 - 3h)}{h} \\
&= \lim_{h\to 0} (10 - 3h) \qquad \{\text{as } h \neq 0\} \\
&= 10
\end{aligned}$$

6 **a** $y = x^3 - 3x$

$$\begin{aligned}
\therefore\ \frac{dy}{dx} & \\
&= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{[(x+h)^3 - 3(x+h)] - [x^3 - 3x]}{h} \\
&= \lim_{h\to 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{3x} - 3h - \cancel{x^3} + \cancel{3x}}{h} \\
&= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3 - 3h}{h} \\
&= \lim_{h\to 0} \frac{h(3x^2 + 3xh + h^2 - 3)}{h} \\
&= \lim_{h\to 0} (3x^2 + 3xh + h^2 - 3) \qquad \{\text{as } h \neq 0\} \\
&= 3x^2 - 3
\end{aligned}$$

b The tangent has zero gradient when

$$\begin{aligned}
f'(x) &= 0 \\
\therefore\ 3x^2 - 3 &= 0 \\
\therefore\ 3x^2 &= 3 \\
\therefore\ x^2 &= 1 \\
\therefore\ x &= \pm 1
\end{aligned}$$

When $x = -1$,
$y = (-1)^3 - 3(-1) = 2$

When $x = 1$,
$y = (1)^3 - 3(1) = -2$

So, the points on the graph at which the tangent has zero gradient are $(-1, 2)$ and $(1, -2)$.

7 **a** $y = f(x) = \dfrac{4}{x}$

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \left(\frac{\frac{4}{x+h} - \frac{4}{x}}{h} \right) \\
&= \lim_{h\to 0} \frac{\left(\frac{4x - 4(x+h)}{x(x+h)} \right)}{h} \\
&= \lim_{h\to 0} \frac{4x - 4x - 4h}{xh(x+h)} \\
&= \lim_{h\to 0} \frac{-4\cancel{h}}{x\cancel{h}(x+h)} \\
&= \lim_{h\to 0} \frac{-4}{x(x+h)} \\
&= \frac{-4}{x^2}
\end{aligned}$$

b $y = f(x) = \dfrac{4x+1}{x-2}$

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\left(\frac{4(x+h)+1}{x+h-2} - \frac{4x+1}{x-2}\right)}{h} \\
&= \lim_{h\to 0} \frac{\left(\frac{4x+4h+1}{x+h-2} - \frac{4x+1}{x-2}\right)}{h} \\
&= \lim_{h\to 0} \frac{\left(\frac{(4x+4h+1)(x-2)-(4x+1)(x+h-2)}{(x+h-2)(x-2)}\right)}{h} \\
&= \lim_{h\to 0} \frac{4x^2-8x+4hx-8h+x-2-(4x^2+4hx-8x+x+h-2)}{h(x+h-2)(x-2)} \\
&= \lim_{h\to 0} \frac{\cancel{4x^2}-\cancel{7x}+\cancel{4hx}-8h-\cancel{2}-\cancel{4x^2}-\cancel{4hx}+\cancel{7x}-h+\cancel{2}}{h(x+h-2)(x-2)} \\
&= \lim_{h\to 0} \frac{-9\cancel{h}}{\cancel{h}(x+h-2)(x-2)} \\
&= \lim_{h\to 0} \frac{-9}{(x+h-2)(x-2)} \\
&= \frac{-9}{(x-2)^2}
\end{aligned}$$

8 **a** $f(x) = \dfrac{1}{x^2}$

$$\begin{aligned}
\therefore\ f(3) &= \frac{1}{3^2} = \frac{1}{9} \\
f'(3) &= \lim_{h\to 0} \frac{f(3+h)-f(3)}{h} \\
&= \lim_{h\to 0} \frac{\dfrac{1}{(3+h)^2} - \dfrac{1}{9}}{h} \\
&= \lim_{h\to 0} \frac{9-(3+h)^2}{9h(3+h)^2} \\
&= \lim_{h\to 0} \frac{\cancel{9}-\cancel{9}-6h-h^2}{9h(3+h)^2} \\
&= \lim_{h\to 0} \frac{-\cancel{h}(6+h)}{9\cancel{h}(3+h)^2} \\
&= \lim_{h\to 0} \frac{-(6+h)}{9(3+h)^2} \quad \{\text{as } h \neq 0\} \\
&= \tfrac{-6}{81} \\
&= -\tfrac{2}{27}
\end{aligned}$$

b $f(x) = \dfrac{3x}{x^2+1}$

$$\begin{aligned}
\therefore\ f(-4) &= -\tfrac{12}{17} \\
f'(-4) &= \lim_{x\to -4} \frac{f(x)-f(-4)}{x-(-4)} \\
&= \lim_{x\to -4} \frac{\dfrac{3x}{x^2+1} - \left(-\dfrac{12}{17}\right)}{x+4} \\
&= \lim_{x\to -4} \frac{51x+12(x^2+1)}{17(x^2+1)(x+4)} \\
&= \lim_{x\to -4} \frac{12x^2+51x+12}{17(x^2+1)(x+4)} \\
&= \lim_{x\to -4} \frac{\cancel{(x+4)}(12x+3)}{17(x^2+1)\cancel{(x+4)}} \\
&= \lim_{x\to -4} \frac{12x+3}{17(x^2+1)} \quad \{x \neq -4\} \\
&= -\frac{45}{17\times 17} \\
&= -\tfrac{45}{289}
\end{aligned}$$

c $f(x) = \sqrt{x}$ and $f(4) = \sqrt{4} = 2$

$$\begin{aligned} f'(4) &= \lim_{x \to 4} \frac{f(x) - f(4)}{x - 4} \\ &= \lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4} \\ &= \lim_{x \to 4} \frac{\cancel{\sqrt{x} - 2}}{(\sqrt{x} + 2)\cancel{(\sqrt{x} - 2)}} \\ &= \lim_{x \to 4} \frac{1}{(\sqrt{x} + 2)} \quad \{\text{as } x \neq 4\} \\ &= \frac{1}{2 + 2} \\ &= \tfrac{1}{4} \end{aligned}$$

d $f(x) = \dfrac{1}{\sqrt{x}}$

$\therefore\ f(1) = \dfrac{1}{\sqrt{1}} = 1$

$$\begin{aligned} f'(1) &= \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} \\ &= \lim_{h \to 0} \frac{\frac{1}{\sqrt{1+h}} - \frac{1}{1}}{h} \\ &= \lim_{h \to 0} \frac{1 - \sqrt{1+h}}{h\sqrt{1+h}} \\ &= \lim_{h \to 0} \frac{(1 - \sqrt{1+h})}{h\sqrt{1+h}} \left(\frac{1 + \sqrt{1+h}}{1 + \sqrt{1+h}} \right) \\ &= \lim_{h \to 0} \frac{1 - (1+h)}{h(\sqrt{1+h})(1 + \sqrt{1+h})} \\ &= \lim_{h \to 0} \frac{\overset{-1}{\cancel{-h}}}{\cancel{h}\sqrt{1+h}(1 + \sqrt{1+h})} \\ &= \lim_{h \to 0} \frac{-1}{\sqrt{1+h}(1 + \sqrt{1+h})} \quad \{h \neq 0\} \\ &= \frac{-1}{1(1+1)} = -\tfrac{1}{2} \end{aligned}$$

REVIEW SET 17A

1 **a** We can make $6x - 7$ as close as we like to -1 by making x sufficiently close to 1.

$\therefore\ \lim\limits_{x \to 1} (6x - 7) = -1$

b
$$\begin{aligned} \lim_{h \to 0} \frac{2h^2 - h}{h} &= \lim_{h \to 0} \frac{h(2h - 1)}{h} \\ &= \lim_{h \to 0} (2h - 1) \quad \{\text{as } h \neq 0\} \\ &= -1 \end{aligned}$$

c
$$\begin{aligned} \lim_{x \to 4} \frac{x^2 - 16}{x - 4} &= \lim_{x \to 4} \frac{(x+4)(x-4)}{x - 4} \\ &= \lim_{x \to 4} (x + 4) \quad \{\text{as } x \neq 0\} \\ &= 8 \end{aligned}$$

2 **a** $f(x) = x^2 + 2x$

$\therefore\ f'(x)$
$$\begin{aligned} &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{[(x+h)^2 + 2(x+h)] - [x^2 + 2x]}{h} \\ &= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 + \cancel{2x} + 2h - \cancel{x^2} - \cancel{2x}}{h} \\ &= \lim_{h \to 0} \frac{2xh + h^2 + 2h}{h} \\ &= \lim_{h \to 0} \frac{h(2x + h + 2)}{h} \\ &= \lim_{h \to 0} (2x + h + 2) \quad \{\text{as } h \neq 0\} \\ &= 2x + 2 \end{aligned}$$

b $y = f(x) = 4 - 3x^2$

$\therefore\ \dfrac{dy}{dx}$
$$\begin{aligned} &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{[4 - 3(x+h)^2] - [4 - 3x^2]}{h} \\ &= \lim_{h \to 0} \frac{4 - 3(x^2 + 2xh + h^2) - 4 + 3x^2}{h} \\ &= \lim_{h \to 0} \frac{\cancel{4} - \cancel{3x^2} - 6xh - 3h^2 - \cancel{4} + \cancel{3x^2}}{h} \\ &= \lim_{h \to 0} \frac{-3h(2x + h)}{h} \\ &= \lim_{h \to 0} -3(2x + h) \quad \{\text{as } h \neq 0\} \\ &= -6x \end{aligned}$$

3 **a** $f(x) = e^{x-2} - 3$

$f(x)$ is defined for all $x \in \mathbb{R}$

$\therefore$ no vertical asymptotes exist

As $x \to -\infty$, $f(x) \to -3$

$\therefore$ $y = -3$ is a horizontal asymptote

As $x \to \infty$, $f(x) \to \infty$

As $x \to -\infty$, $f(x) \to -3^+$

b $f(x) = \ln(x^2 + 3)$ has no asymptotes

c $f(x) = \ln(-x) + 2$

$f(x)$ is undefined for $x \geqslant 0$

$\therefore$ $x = 0$ is a vertical asymptote

As $x \to 0^-$, $f(x) \to -\infty$

4 **a**

$$\begin{aligned}
&\lim_{\theta\to 0} \frac{\sin 4\theta}{\theta}\\
&= \lim_{\theta\to 0} \frac{\sin 4\theta}{4\theta} \times 4\\
&= 4 \lim_{4\theta\to 0} \frac{\sin 4\theta}{4\theta} \qquad \{4\theta \to 0 \text{ as } \theta \to 0\}\\
&= 4 \times 1\\
&= 4
\end{aligned}$$

b

$$\begin{aligned}
&\lim_{\theta\to 0} \frac{2\theta}{\sin 3\theta}\\
&= \lim_{\theta\to 0} \frac{3\theta}{\sin 3\theta} \times \tfrac{2}{3}\\
&= \tfrac{2}{3} \lim_{\theta\to 0} \frac{1}{\frac{\sin 3\theta}{3\theta}}\\
&= \tfrac{2}{3} \frac{1}{\lim\limits_{3\theta\to 0} \frac{\sin 3\theta}{3\theta}} \qquad \{3\theta \to 0 \text{ as } \theta \to 0\}\\
&= \tfrac{2}{3} \times \tfrac{1}{1}\\
&= \tfrac{2}{3}
\end{aligned}$$

c

$$\begin{aligned}
&\lim_{n\to\infty} n \sin\left(\tfrac{\pi}{n}\right)\\
&= \lim_{\frac{1}{n}\to 0} \frac{\sin\left(\frac{\pi}{n}\right)}{\frac{1}{n}} \qquad \{\tfrac{1}{n} \to 0 \text{ as } n \to \infty\}\\
&= \lim_{\frac{1}{n}\to 0} \frac{\sin\left(\frac{\pi}{n}\right)}{\frac{\pi}{n}} \times \pi\\
&= \pi \times \lim_{\frac{\pi}{n}\to 0} \frac{\sin\left(\frac{\pi}{n}\right)}{\frac{\pi}{n}} \qquad \{\tfrac{\pi}{n} \to 0 \text{ as } \tfrac{1}{n} \to 0\}\\
&= \pi \times 1\\
&= \pi
\end{aligned}$$

5 $f(x) = 5x - x^2$

$$\begin{aligned}
f'(1) &= \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} \qquad \text{where } f(1) = 5(1) - (1)^2 = 4\\
&= \lim_{h\to 0} \frac{[5(1+h) - (1+h)^2] - 4}{h}\\
&= \lim_{h\to 0} \frac{5 + 5h - (1 + 2h + h^2) - 4}{h}\\
&= \lim_{h\to 0} \frac{\cancel{5} + 5h - \cancel{1} - 2h - h^2 - \cancel{4}}{h}\\
&= \lim_{h\to 0} \frac{3h - h^2}{h}\\
&= \lim_{h\to 0} \frac{h(3 - h)}{h}\\
&= \lim_{h\to 0} (3 - h) \qquad \{\text{as } h \neq 0\}\\
&= 3
\end{aligned}$$

6 **a** $f(t) = 452 - 4.8t^2$

$\therefore\ f'(t)$

$= \lim\limits_{h\to 0} \dfrac{f(t+h) - f(t)}{h}$

$= \lim\limits_{h\to 0} \dfrac{[452 - 4.8(t+h)^2] - [452 - 4.8t^2]}{h}$

$= \lim\limits_{h\to 0} \dfrac{\cancel{452} - 4.8(t^2 + 2th + h^2) - \cancel{452} + 4.8t^2}{h}$

$= \lim\limits_{h\to 0} \dfrac{\cancel{-4.8t^2} - 9.6th - 4.8h^2 + \cancel{4.8t^2}}{h}$

$= \lim\limits_{h\to 0} \dfrac{h(-9.6t - 4.8h)}{h}$

$= \lim\limits_{h\to 0} (-9.6t - 4.8h) \quad \{\text{as } h \neq 0\}$

$= -9.6t \text{ m s}^{-1}$

b To find the speed of the jumper at $t = 2$ seconds, we need to find $f'(2)$.

Now $f'(t) = -9.6t$ {from **a**}

$\therefore\ f'(2) = -9.6 \times 2$

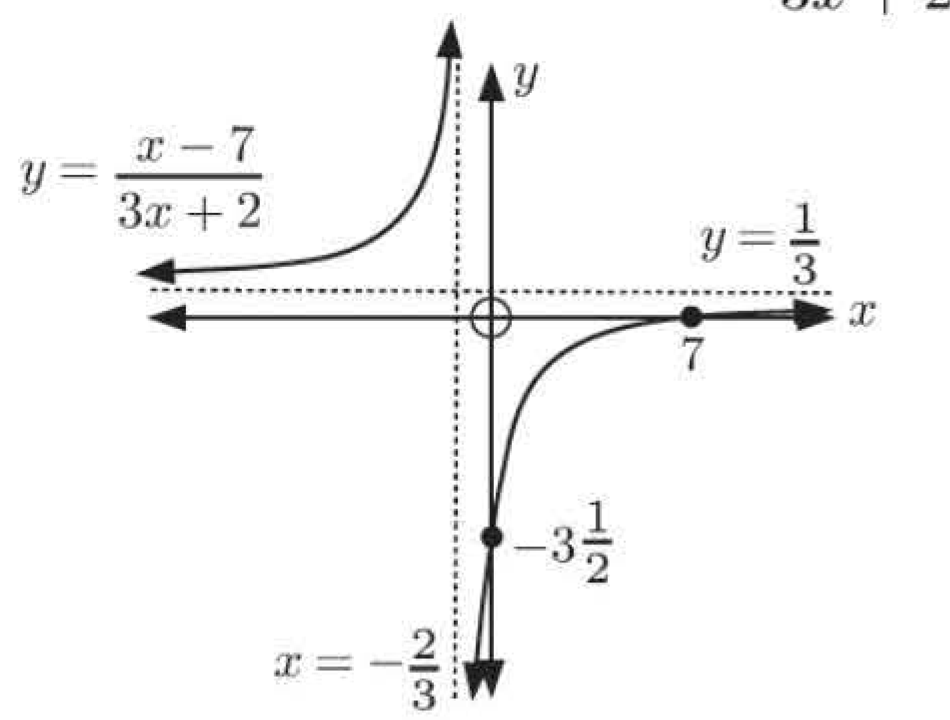
$= -19.2$

$\therefore$ the speed of the jumper at $t = 2$ seconds is 19.2 m s^{-1}.

(The $-$ sign indicates the jumper is moving downwards.)

REVIEW SET 17B

1 **a** We sketch the graph of $y = \dfrac{x-7}{3x+2}$:



As $x \to -\frac{2}{3}^-$, $y \to \infty$

As $x \to -\frac{2}{3}^+$, $y \to -\infty$

As $x \to \infty$, $y \to \frac{1}{3}^-$

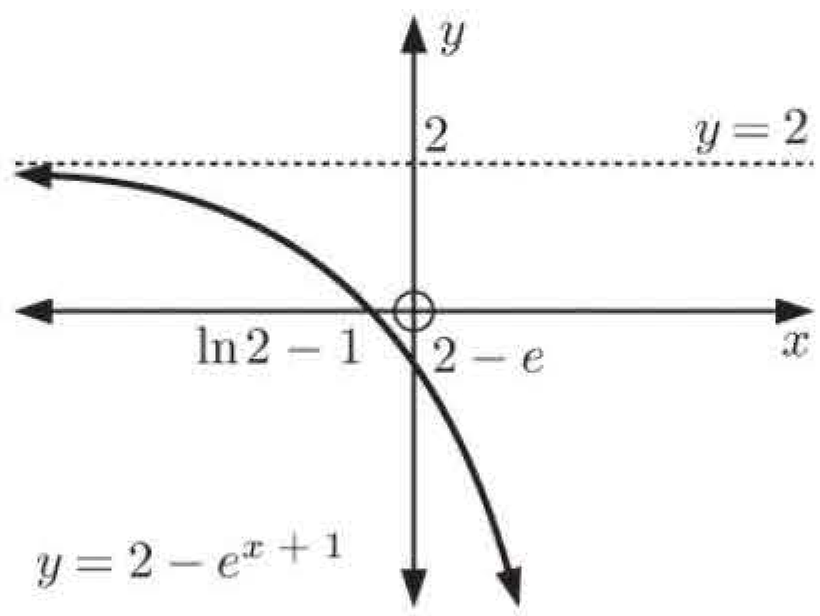
As $x \to -\infty$, $y \to \frac{1}{3}^+$

The vertical asymptote is $x = -\frac{2}{3}$.

The horizontal asymptote is $y = \frac{1}{3}$.

b $\lim\limits_{x\to -\infty} \left(\dfrac{x-7}{3x+2}\right) = \frac{1}{3}$, $\quad \lim\limits_{x\to \infty} \left(\dfrac{x-7}{3x+2}\right) = \frac{1}{3}$

2 **a**



b $\lim\limits_{x\to -\infty} (2 - e^{x+1}) = 2$,

$\lim\limits_{x\to \infty} (2 - e^{x+1})$ does not exist

c The horizontal asymptote is $y = 2$.

3 **a** $f(x) = \ln(x^2)$ is not defined when $x = 0$

$\therefore\ f(x) = \ln(x^2)$ is not continuous at $x = 0$.

b $f(x) = \dfrac{x^2 - 1}{1 - x}$ is not defined when $x = 1$

$\therefore\ f(x) = \dfrac{x^2 - 1}{1 - x}$ is not continuous at $x = 1$.

4 $\lim\limits_{h\to 0} \dfrac{2\cos\left(x + \frac{h}{2}\right)\sin\left(\frac{h}{2}\right)}{h} = 2 \lim\limits_{h\to 0} \cos\left(x + \frac{h}{2}\right) \lim\limits_{h\to 0} \dfrac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} \times \frac{1}{2}$

$= 1 \times \lim\limits_{h\to 0} \cos\left(x + \frac{h}{2}\right) \lim\limits_{\frac{h}{2}\to 0} \dfrac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} \quad \{\tfrac{h}{2} \to 0 \text{ as } h \to 0\}$

$= 1 \times \lim\limits_{h\to 0} \cos\left(x + \frac{h}{2}\right)$

$= \cos x$

It is assumed that x and h are in radians.

5 **a**
$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{2(x+h)^2-2x^2}{h}\\ &= \frac{2(x^2+2xh+h^2)-2x^2}{h}\\ &= \frac{\cancel{2x^2}+4xh+2h^2-\cancel{2x^2}}{h}\\ &= \frac{h(4x+2h)}{h}\\ &= 4x+2h \quad \text{provided } h \neq 0\end{aligned}$$

b If $x=3$ then
$$\begin{aligned}\frac{f(3+h)-f(3)}{h} &= 4(3)+2h \quad \{\text{using } \mathbf{a}\}\\ &= 12+2h\end{aligned}$$

i When $h=0.1$,
$$\begin{aligned}\frac{f(3+h)-f(3)}{h} &= 12+2(0.1)\\ &= 12+0.2\\ &= 12.2\end{aligned}$$

ii When $h=0.01$,
$$\begin{aligned}\frac{f(3+h)-f(3)}{h} &= 12+2(0.01)\\ &= 12+0.02\\ &= 12.02\end{aligned}$$

iii
$$\begin{aligned}\lim_{h\to 0}\frac{f(3+h)-f(3)}{h} &= \lim_{h\to 0} 12+2h\\ &= 12\end{aligned}$$

c The gradient of the tangent to $y=2x^2$ at the point $(3, 18)$ is 12.

6 $\displaystyle\lim_{x\to -\infty}\left(\frac{2x+3}{4-x}\right) = -2, \qquad \lim_{x\to\infty}\left(\frac{2x+3}{4-x}\right) = -2$

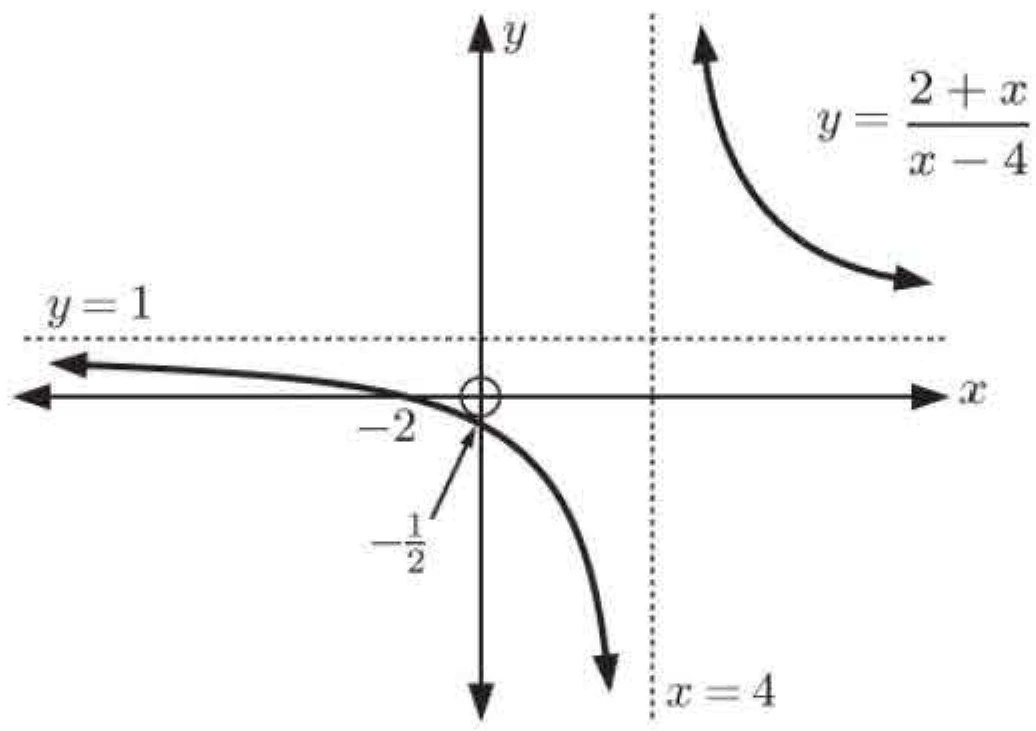
REVIEW SET 17C

1 **a**
$$\begin{aligned}\lim_{h\to 0}\frac{h^3-3h}{h} &= \lim_{h\to 0}\frac{h(h^2-3)}{h}\\ &= \lim_{h\to 0} h^2-3 \quad \{\text{as } h\neq 0\}\\ &= -3\end{aligned}$$

b
$$\begin{aligned}\lim_{x\to 1}\frac{3x^2-3x}{x-1} &= \lim_{x\to 1}\frac{3x(x-1)}{x-1}\\ &= \lim_{x\to 1} 3x \quad \{\text{as } x\neq 1\}\\ &= 3\end{aligned}$$

c
$$\begin{aligned}\lim_{x\to 2}\frac{x^2-3x+2}{2-x} &= \lim_{x\to 2}\frac{(x-1)(x-2)}{-(x-2)}\\ &= \lim_{x\to 2} -(x-1) \quad \{\text{as } x\neq 2\}\\ &= -1\end{aligned}$$

2 **a**



b As $x\to 4^-$, $\quad y\to -\infty$
As $x\to 4^+$, $\quad y\to \infty$
As $x\to \infty$, $\quad y\to 1^+$
As $x\to -\infty$, $\quad y\to 1^-$
The vertical asymptote is $x=4$.
The horizontal asymptote is $y=1$.

c $\displaystyle\lim_{x\to -\infty}\frac{2+x}{x-4} = 1, \quad \lim_{x\to\infty}\frac{2+x}{x-4} = 1$

3 $f(x) = x^4 - 2x$

$$\begin{aligned} f'(1) &= \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} \quad \text{where } f(1) = 1^4 - 2(1) = -1 \\ &= \lim_{h\to 0} \frac{[(1+h)^4 - 2(1+h)] - [-1]}{h} \\ &= \lim_{h\to 0} \frac{\cancel{1} + 4h + 6h^2 + 4h^3 + h^4 - \cancel{2} - 2h + \cancel{1}}{h} \\ &= \lim_{h\to 0} \frac{h^4 + 4h^3 + 6h^2 + 2h}{h} \\ &= \lim_{h\to 0} \frac{h(h^3 + 4h^2 + 6h + 2)}{h} \\ &= \lim_{h\to 0} (h^3 + 4h^2 + 6h + 2) \qquad \{\text{as } h \neq 0\} \\ &= 2 \end{aligned}$$

4 **a** $$\begin{aligned} \sin(A+B) - \sin(A-B) &= \sin A\cos B + \cos A \sin B - (\sin A \cos B - \cos A \sin B) \\ &= \cancel{\sin A\cos B} + \cos A\sin B - \cancel{\sin A \cos B} + \cos A \sin B \\ &= 2\cos A \sin B \end{aligned}$$

b $$\begin{aligned} \sin S - \sin D &= \sin(A+B) - \sin(A-B) \\ &= 2\cos A \sin B \qquad \{\text{using } \mathbf{a}\} \end{aligned}$$

Now $S + D = A + B + (A - B) = 2A$ and $S - D = A + B - (A - B) = 2B$

$\therefore\ A = \dfrac{S+D}{2}$ $\qquad \therefore\ B = \dfrac{S-D}{2}$

$\therefore\ \sin S - \sin D = 2\cos\left(\frac{S+D}{2}\right)\sin\left(\frac{S-D}{2}\right)$

c $$\begin{aligned} \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} &= \lim_{h\to 0} \frac{2\cos\left[\frac{(x+h)+x}{2}\right]\sin\left[\frac{(x+h)-x}{2}\right]}{h} \qquad \{x + h = S,\quad x = D\} \\ &= \lim_{h\to 0} \frac{2\cos\left(\frac{2x+h}{2}\right)\sin\left(\frac{h}{2}\right)}{h} \\ &= 2\lim_{h\to 0} \frac{\cos\left(x+\frac{h}{2}\right)\sin\left(\frac{h}{2}\right)}{h} \\ &= 2\lim_{h\to 0} \cos\left(x+\frac{h}{2}\right) \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} \times \frac{1}{2} \\ &= 1\lim_{h\to 0} \cos\left(x+\frac{h}{2}\right) \lim_{\frac{h}{2}\to 0} \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} \qquad \{\tfrac{h}{2} \to 0 \text{ as } h \to 0\} \\ &= \lim_{h\to 0} \cos\left(x+\frac{h}{2}\right) \\ &= \cos x \end{aligned}$$

d $\displaystyle\lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} = \cos x$ which is the gradient function of $f(x) = \sin x$.

5 **a**

$$y = 2x^2 - 1$$

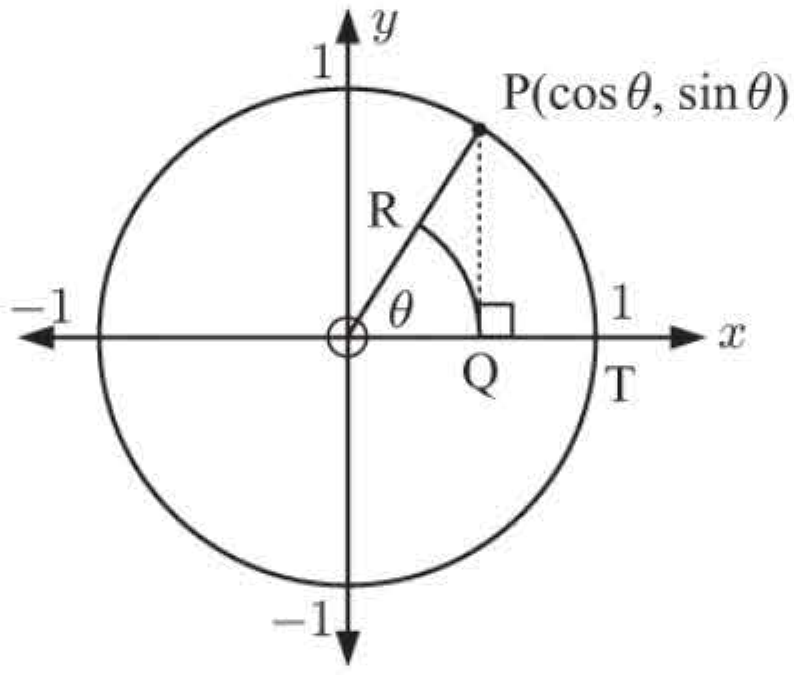
$$\begin{aligned}\therefore\ \frac{dy}{dx} &= \lim_{h\to 0} \frac{[2(x+h)^2 - 1] - [2x^2 - 1]}{h}\\ &= \lim_{h\to 0} \frac{2(x^2 + 2hx + h^2) - 1 - 2x^2 + 1}{h}\\ &= \lim_{h\to 0} \frac{\cancel{2x^2} + 4hx + 2h^2 - \cancel{1} - \cancel{2x^2} + \cancel{1}}{h}\\ &= \lim_{h\to 0} \frac{4hx + 2h^2}{h}\\ &= \lim_{h\to 0} \frac{h(4x + 2h)}{h}\\ &= \lim_{h\to 0} 4x + 2h \qquad \{\text{as } h \neq 0\}\\ &= 4x\end{aligned}$$

b The gradient of the tangent to $y = 2x^2 - 1$ at the point where $x = 4$ is $4 \times 4 = 16$.

c If the gradient of the tangent is equal to -12, then $4x = -12$

$\therefore\ x = -3$

6 **a** **i**



Suppose $P(\cos\theta, \sin\theta)$ lies on the unit circle in the first quadrant.

[PQ] is drawn perpendicular to the x-axis, and arc QR with centre O is drawn. Now,

area of sector OQR $\leqslant$ area $\triangle$OQP $\leqslant$ area sector OTP

$$\therefore\ \tfrac{1}{2}(\text{OQ})^2 \times \theta \leqslant \tfrac{1}{2}(\text{OQ})(\text{PQ}) \leqslant \tfrac{1}{2}(\text{OT})^2 \times \theta$$

$$\therefore\ \tfrac{1}{2}\theta\cos^2\theta \leqslant \tfrac{1}{2}\cos\theta\sin\theta \leqslant \tfrac{1}{2}\theta$$

Dividing throughout by $\frac{1}{2}\theta\cos\theta$, which is > 0, $\quad \cos\theta \leqslant \dfrac{\sin\theta}{\theta} \leqslant \dfrac{1}{\cos\theta}$

Now as $\theta \to 0$, both $\cos\theta \to 1$ and $\dfrac{1}{\cos\theta} \to 1$

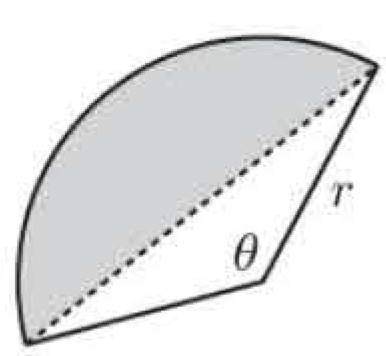
$\therefore$ as $\theta \to 0^+$, $\dfrac{\sin\theta}{\theta} \to 1$. So, if $\theta > 0$ then $\displaystyle\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$.

ii If $f(\theta) = \dfrac{\sin\theta}{\theta}$, $f(-\theta) = \dfrac{\sin(-\theta)}{-\theta} = \dfrac{-\sin\theta}{-\theta} = \dfrac{\sin\theta}{\theta} = f(\theta)$

$\therefore$ $\dfrac{\sin\theta}{\theta}$ is an even function, so as $\theta \to 0^-$, $\dfrac{\sin\theta}{\theta} \to 1$ also.

So, if $\theta < 0$, $\displaystyle\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$

b



Area of shaded segment

$= (\text{area of sector}) - (\text{area of triangle})$

$= \frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin\theta$

$= \frac{1}{2}r^2(\theta - \sin\theta)$

c As $\theta \to 0$, area of shaded segment $\to 0$

$$\begin{aligned}\therefore\ \tfrac{1}{2}r^2(\theta - \sin\theta) &\to 0\\ \therefore\ \theta - \sin\theta &\to 0\\ \therefore\ \theta &\to \sin\theta\\ \therefore\ \frac{\sin\theta}{\theta} &\to 1\\ \therefore\ \lim_{\theta\to 0} \frac{\sin\theta}{\theta} &= 1\end{aligned}$$

Chapter 18

RULES OF DIFFERENTIATION

EXERCISE 18A

1 **a** $f(x) = x^3$
$\therefore\ f'(x) = 3x^2$

b $f(x) = 2x^3$
$\therefore\ f'(x) = 2(3x^2)$
$= 6x^2$

c $f(x) = 7x^2$
$\therefore\ f'(x) = 7(2x)$
$= 14x$

d $f(x) = 6\sqrt{x} = 6x^{\frac{1}{2}}$
$\therefore\ f'(x) = 6\left(\frac{1}{2}x^{-\frac{1}{2}}\right)$
$= \dfrac{3}{\sqrt{x}}$

e $f(x) = 3\sqrt[3]{x} = 3x^{\frac{1}{3}}$
$\therefore\ f'(x) = 3\left(\frac{1}{3}x^{-\frac{2}{3}}\right)$
$= \dfrac{1}{\sqrt[3]{x^2}}$

f $f(x) = x^2 + x$
$\therefore\ f'(x) = 2x + 1$

g $f(x) = 4 - 2x^2$
$\therefore\ f'(x) = 0 - 2(2x)$
$= -4x$

h $f(x) = x^2 + 3x - 5$
$\therefore\ f'(x) = 2x + 3 - 0$
$= 2x + 3$

i $f(x) = \frac{1}{2}x^4 - 6x^2$
$\therefore\ f'(x) = \frac{1}{2}(4x^3) - 6(2x)$
$= 2x^3 - 12x$

j $f(x) = \dfrac{3x - 6}{x} = 3 - 6x^{-1}$
$\therefore\ f'(x) = 0 - 6(-1x^{-2})$
$= \dfrac{6}{x^2}$

k $f(x) = \dfrac{2x - 3}{x^2} = \dfrac{2x}{x^2} - \dfrac{3}{x^2}$
$= 2x^{-1} - 3x^{-2}$
$\therefore\ f'(x) = -2x^{-2} + 6x^{-3} = -\dfrac{2}{x^2} + \dfrac{6}{x^3}$

l $f(x) = \dfrac{x^3 + 5}{x} = x^2 + 5x^{-1}$
$\therefore\ f'(x) = 2x - 5x^{-2}$
$= 2x - \dfrac{5}{x^2}$

m $f(x) = \dfrac{x^3 + x - 3}{x}$
$= x^2 + 1 - 3x^{-1}$
$\therefore\ f'(x) = 2x + 0 + 3x^{-2}$
$= 2x + \dfrac{3}{x^2}$

n $f(x) = \dfrac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$
$\therefore\ f'(x) = -\frac{1}{2}x^{-\frac{3}{2}} = -\dfrac{1}{2x\sqrt{x}}$

o $f(x) = (2x - 1)^2 = 4x^2 - 4x + 1$
$\therefore\ f'(x) = 8x - 4$

p $f(x) = (x + 2)^3$
$= x^3 + 3x^2(2) + 3x(2^2) + 2^3$
$= x^3 + 6x^2 + 12x + 8$
$\therefore\ f'(x) = 3x^2 + 12x + 12$

2 **a** $y = 2.5x^3 - 1.4x^2 - 1.3$
$\therefore\ \dfrac{dy}{dx} = 7.5x^2 - 2.8x$

b $y = \pi x^2$
$\therefore\ \dfrac{dy}{dx} = 2\pi x$

c $y = \dfrac{1}{5x^2} = \frac{1}{5}x^{-2}$
$\therefore\ \dfrac{dy}{dx} = -\frac{2}{5}x^{-3} = -\dfrac{2}{5x^3}$

d $y = 100x$
$\therefore\ \dfrac{dy}{dx} = 100$

e $y = 10(x + 1)$
$= 10x + 10$
$\therefore\ \dfrac{dy}{dx} = 10$

f $y = 4\pi x^3$
$\therefore\ \dfrac{dy}{dx} = 12\pi x^2$

3 **a** $\dfrac{d}{dx}(6x+2)$
$= 6$

b $\dfrac{d}{dx}(x\sqrt{x})$
$= \dfrac{d}{dx}(x^{\frac{3}{2}})$
$= \frac{3}{2}x^{\frac{1}{2}}$
$= \dfrac{3\sqrt{x}}{2}$

c $\dfrac{d}{dx}(5-x)^2$
$= \dfrac{d}{dx}(25-10x+x^2)$
$= -10+2x$
$= 2x-10$

d $\dfrac{d}{dx}\left(\dfrac{6x^2-9x^4}{3x}\right)$
$= \dfrac{d}{dx}(2x-3x^3)$
$= 2-9x^2$

e $\dfrac{d}{dx}((x+1)(x-2))$
$= \dfrac{d}{dx}(x^2-x-2)$
$= 2x-1$

f $\dfrac{d}{dx}\left(\dfrac{1}{x^2}+6\sqrt{x}\right)$
$= \dfrac{d}{dx}\left(x^{-2}+6x^{\frac{1}{2}}\right)$
$= -2x^{-3}+3x^{-\frac{1}{2}}$
$= -\dfrac{2}{x^3}+\dfrac{3}{\sqrt{x}}$

g $\dfrac{d}{dx}\left(4x-\dfrac{1}{4x}\right)$
$= \dfrac{d}{dx}\left(4x-\frac{1}{4}x^{-1}\right)$
$= 4+\frac{1}{4}x^{-2}$
$= 4+\dfrac{1}{4x^2}$

h $\dfrac{d}{dx}(x(x+1)(2x-5))$
$= \dfrac{d}{dx}\left(x(2x^2-3x-5)\right)$
$= \dfrac{d}{dx}\left(2x^3-3x^2-5x\right)$
$= 6x^2-6x-5$

4 **a** Consider $y=x^2$ when $x=2$
Now $\dfrac{dy}{dx}=2x$
$\therefore$ when $x=2$,
$\dfrac{dy}{dx}=2(2)=4$
$\therefore$ the tangent has gradient 4.

b Consider $y=\dfrac{8}{x^2}$ at the point $(9, \frac{8}{81})$
Now $y=8x^{-2}$
$\therefore \dfrac{dy}{dx}=-16x^{-3}=-\dfrac{16}{x^3}$
$\therefore$ at $(9, \frac{8}{81})$, $x=9$ and so $\dfrac{dy}{dx}=-\frac{16}{729}$
$\therefore$ the tangent has gradient $-\frac{16}{729}$.

c Consider $y=2x^2-3x+7$ when $x=-1$
Now $\dfrac{dy}{dx}=4x-3$
$\therefore$ when $x=-1$,
$\dfrac{dy}{dx}=4(-1)-3=-7$
$\therefore$ the tangent has gradient -7.

d Consider $y=\dfrac{2x^2-5}{x}$ at the point $(2, \frac{3}{2})$
Now $y=2x-5x^{-1}$
$\therefore \dfrac{dy}{dx}=2+5x^{-2}=2+\dfrac{5}{x^2}$
$\therefore$ at $(2, \frac{3}{2})$, $x=2$ and so $\dfrac{dy}{dx}=2+\frac{5}{4}$
$= \frac{13}{4}$
$\therefore$ the tangent has gradient $\frac{13}{4}$.

e Consider $y=\dfrac{x^2-4}{x^2}$ at the point $(4, \frac{3}{4})$
Now $y=1-4x^{-2}$
$\therefore \dfrac{dy}{dx}=0+8x^{-3}=\dfrac{8}{x^3}$
$\therefore$ at $(4, \frac{3}{4})$, $x=4$ and so
$\dfrac{dy}{dx}=\dfrac{8}{4^3}=\frac{1}{8}$
$\therefore$ the tangent has gradient $\frac{1}{8}$.

f Consider $y=\dfrac{x^3-4x-8}{x^2}$ when $x=-1$
Now $y=x-4x^{-1}-8x^{-2}$
$\therefore \dfrac{dy}{dx}=1+4x^{-2}+16x^{-3}$
$= 1+\dfrac{4}{x^2}+\dfrac{16}{x^3}$
$\therefore$ when $x=-1$,
$\dfrac{dy}{dx}=1+4-16=-11$
$\therefore$ the tangent has gradient -11.

5 $f(x) = x^2 + (b+1)x + 2c, \quad f(2) = 4, \quad \text{and} \quad f'(-1) = 2$

$\therefore \; f'(x) = 2x + (b+1)$

But $f'(-1) = 2$, so $2(-1) + b + 1 = 2$

$\therefore \; -1 + b = 2$

$\therefore \; b = 3$

So, $f(x) = x^2 + (3+1)x + 2c$

$= x^2 + 4x + 2c$

But $f(2) = 4$, so $2^2 + 4(2) + 2c = 4$

$\therefore \; 2c = -8$

$\therefore \; c = -4$

6 **a** $f(x) = 4\sqrt{x} + x = 4x^{\frac{1}{2}} + x$

$\therefore \; f'(x) = 4\left(\frac{1}{2}x^{-\frac{1}{2}}\right) + 1$

$= \dfrac{2}{\sqrt{x}} + 1$

b $f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$

$\therefore \; f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$

$= \dfrac{1}{3\sqrt[3]{x^2}}$

c $f(x) = -\dfrac{2}{\sqrt{x}} = -2x^{-\frac{1}{2}}$

$\therefore \; f'(x) = -2\left(-\frac{1}{2}x^{-\frac{3}{2}}\right)$

$= x^{-\frac{3}{2}}$

$= \dfrac{1}{x\sqrt{x}}$

d $f(x) = 2x - \sqrt{x} = 2x - x^{\frac{1}{2}}$

$\therefore \; f'(x) = 2 - \frac{1}{2}x^{-\frac{1}{2}}$

$= 2 - \dfrac{1}{2\sqrt{x}}$

e $f(x) = \dfrac{4}{\sqrt{x}} - 5 = 4x^{-\frac{1}{2}} - 5$

$\therefore \; f'(x) = 4\left(-\frac{1}{2}x^{-\frac{3}{2}}\right)$

$= -2x^{-\frac{3}{2}} = -\dfrac{2}{x\sqrt{x}}$

f $f(x) = 3x^2 - x\sqrt{x} = 3x^2 - x^{\frac{3}{2}}$

$\therefore \; f'(x) = 6x - \frac{3}{2}x^{\frac{1}{2}}$

$= 6x - \frac{3}{2}\sqrt{x}$

g $f(x) = \dfrac{5}{x^2\sqrt{x}} = 5x^{-\frac{5}{2}}$

$\therefore \; f'(x) = 5\left(-\frac{5}{2}x^{-\frac{7}{2}}\right)$

$= -\frac{25}{2}x^{-\frac{7}{2}}$

$= \dfrac{-25}{2x^3\sqrt{x}}$

h $f(x) = 2x - \dfrac{3}{x\sqrt{x}} = 2x - 3x^{-\frac{3}{2}}$

$\therefore \; f'(x) = 2 - 3\left(-\frac{3}{2}x^{-\frac{5}{2}}\right)$

$= 2 + \frac{9}{2}x^{-\frac{5}{2}}$

$= 2 + \dfrac{9}{2x^2\sqrt{x}}$

7 **a** $y = 4x - \dfrac{3}{x} = 4x - 3x^{-1} \quad \therefore \; \dfrac{dy}{dx} = 4 + 3x^{-2} = 4 + \dfrac{3}{x^2}$

$\dfrac{dy}{dx}$ is the gradient function of $y = 4x - \dfrac{3}{x}$ from which the gradient at any point can be found.

b $S = 2t^2 + 4t$ m $\quad \therefore \; \dfrac{dS}{dt} = 4t + 4$ m s^{-1}

$\dfrac{dS}{dt}$ is the instantaneous rate of change in position at time t. It is the velocity function.

c $C = 1785 + 3x + 0.002x^2$ dollars.

$\dfrac{dC}{dx} = 3 + 0.002(2x) = 3 + 0.004x$ dollars per toaster

$\dfrac{dC}{dx}$ is the instantaneous rate of change in cost as the number of toasters changes.

EXERCISE 18B.1

1 **a** $g(x) = x^2, \quad f(x) = 2x + 7$
$\therefore \quad g(f(x)) = g(2x+7) = (2x+7)^2$

b $g(x) = 2x + 7, \quad f(x) = x^2$
$g(f(x)) = g(x^2) = 2x^2 + 7$

c $g(x) = \sqrt{x}, \quad f(x) = 3 - 4x$
$g(f(x)) = g(3-4x) = \sqrt{3-4x}$

d $g(x) = 3 - 4x, \quad f(x) = \sqrt{x}$
$g(f(x)) = g(\sqrt{x}) = 3 - 4\sqrt{x}$

e $g(x) = \dfrac{2}{x}, \quad f(x) = x^2 + 3$
$g(f(x)) = g(x^2+3) = \dfrac{2}{x^2+3}$

f $g(x) = x^2 + 3, \quad f(x) = \dfrac{2}{x}$
$g(f(x)) = g\left(\dfrac{2}{x}\right) = \left(\dfrac{2}{x}\right)^2 + 3 = \dfrac{4}{x^2} + 3$

2 **a** $g(f(x)) = (3x+10)^3 \qquad \therefore \quad g(x) = x^3, \quad f(x) = 3x + 10$

b $g(f(x)) = \dfrac{1}{2x+4} \qquad \therefore \quad g(x) = \dfrac{1}{x}, \quad f(x) = 2x + 4$

c $g(f(x)) = \sqrt{x^2 - 3x} \qquad \therefore \quad g(x) = \sqrt{x}, \quad f(x) = x^2 - 3x$

d $g(f(x)) = \dfrac{10}{(3x - x^2)^3} \qquad \therefore \quad g(x) = \dfrac{10}{x^3}, \quad f(x) = 3x - x^2$

{other answers are possible for **2**}

EXERCISE 18B.2

1 **a** $\dfrac{1}{(2x-1)^2}$
$= (2x-1)^{-2}$
$= u^{-2}$
where $\quad u = 2x - 1$

b $\sqrt{x^2 - 3x}$
$= (x^2 - 3x)^{\frac{1}{2}}$
$= u^{\frac{1}{2}}$
where $\quad u = x^2 - 3x$

c $\dfrac{2}{\sqrt{2 - x^2}}$
$= 2(2 - x^2)^{-\frac{1}{2}}$
$= 2u^{-\frac{1}{2}}$
where $\quad u = 2 - x^2$

d $\sqrt[3]{x^3 - x^2}$
$= (x^3 - x^2)^{\frac{1}{3}}$
$= u^{\frac{1}{3}}$
where $\quad u = x^3 - x^2$

e $\dfrac{4}{(3-x)^3}$
$= 4(3-x)^{-3}$
$= 4u^{-3}$
where $\quad u = 3 - x$

f $\dfrac{10}{x^2 - 3}$
$= 10(x^2 - 3)^{-1}$
$= 10u^{-1}$
where $\quad u = x^2 - 3$

2 **a** $y = (4x - 5)^2$
$\therefore \quad y = u^2 \quad$ where $\quad u = 4x - 5$

Now $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$
$= 2u(4)$
$= 8u$
$= 8(4x - 5)$

b $y = \dfrac{1}{5 - 2x}$
$\therefore \quad y = u^{-1} \quad$ where $\quad u = 5 - 2x$

Now $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$
$= -u^{-2}(-2)$
$= 2u^{-2}$
$= 2\,(5 - 2x)^{-2}$

c $y = \sqrt{3x - x^2}$
$\therefore \quad y = u^{\frac{1}{2}} \quad$ where $\quad u = 3x - x^2$

Now $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$
$= \frac{1}{2}u^{-\frac{1}{2}}(3 - 2x)$
$= \frac{1}{2}\left(3x - x^2\right)^{-\frac{1}{2}}(3 - 2x)$

d $y = (1 - 3x)^4$
$\therefore \quad y = u^4 \quad$ where $\quad u = 1 - 3x$

Now $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$
$= 4u^3(-3)$
$= -12u^3$
$= -12(1 - 3x)^3$

e $y = 6(5-x)^3$

$\therefore\ y = 6u^3$ where $u = 5-x$

Now $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$
$= 18u^2(-1)$
$= -18u^2$
$= -18(5-x)^2$

f $y = \sqrt[3]{2x^3 - x^2}$

$\therefore\ y = u^{\frac{1}{3}}$ where $u = 2x^3 - x^2$

Now $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$
$= \frac{1}{3}u^{-\frac{2}{3}}(6x^2 - 2x)$
$= \frac{1}{3}\left(2x^3 - x^2\right)^{-\frac{2}{3}}(6x^2 - 2x)$

g $y = \dfrac{6}{(5x-4)^2}$

$\therefore\ y = 6u^{-2}$ where $u = 5x-4$

Now $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$
$= -12u^{-3}(5)$
$= -60\,(5x-4)^{-3}$

h $y = \dfrac{4}{3x - x^2}$

$\therefore\ y = 4u^{-1}$ where $u = 3x - x^2$

Now $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$
$= -4u^{-2}(3-2x)$
$= -4\left(3x - x^2\right)^{-2}(3-2x)$

i $y = 2\left(x^2 - \dfrac{2}{x}\right)^3$

$\therefore\ y = 2u^3$ where $u = x^2 - 2x^{-1}$

Now $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$
$= 6u^2(2x + 2x^{-2})$
$= 6\left(x^2 - \dfrac{2}{x}\right)^2\left(2x + \dfrac{2}{x^2}\right)$

3 **a** $y = \sqrt{1-x^2}$ at $x = \frac{1}{2}$

$\therefore\ y = \sqrt{u}$ where $u = 1 - x^2$

Now $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = \frac{1}{2}u^{-\frac{1}{2}}(-2x)$
$= \dfrac{-x}{\sqrt{u}}$
$= \dfrac{-x}{\sqrt{1-x^2}}$

At $x = \frac{1}{2}$, $\dfrac{dy}{dx} = \dfrac{-\frac{1}{2}}{\sqrt{1-\frac{1}{4}}} = -\frac{1}{2}\left(\frac{2}{\sqrt{3}}\right)$

$\therefore$ gradient of tangent $= -\frac{1}{\sqrt{3}}$

b $y = (3x+2)^6$ at $x = -1$

$\therefore\ y = u^6$ where $u = 3x+2$

Now $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$
$= 6u^5(3)$
$= 18u^5$
$= 18(3x+2)^5$

At $x = -1$, $\dfrac{dy}{dx} = 18(-1)^5$

$\therefore$ gradient of tangent $= -18$

c $y = \dfrac{1}{(2x-1)^4}$ at $x = 1$

$\therefore\ y = u^{-4}$ where $u = 2x-1$

Now $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = -4u^{-5}(2)$
$= \dfrac{-8}{u^5}$
$= \dfrac{-8}{(2x-1)^5}$

At $x = 1$, $\dfrac{dy}{dx} = \dfrac{-8}{1^5}$

$\therefore$ gradient of tangent $= -8$

d $y = 6 \times \sqrt[3]{1-2x}$ at $x = 0$

$\therefore\ y = 6u^{\frac{1}{3}}$ where $u = 1-2x$

Now $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = 6(\frac{1}{3})u^{-\frac{2}{3}}(-2)$
$= 2u^{-\frac{2}{3}}(-2)$
$= \dfrac{-4}{\sqrt[3]{u^2}}$
$= \dfrac{-4}{\sqrt[3]{(1-2x)^2}}$

At $x = 0$, $\dfrac{dy}{dx} = \dfrac{-4}{\sqrt[3]{1^2}}$

$\therefore$ gradient of tangent $= -4$

e $y = \dfrac{4}{x+2\sqrt{x}}$ at $x = 4$

$\therefore\ y = 4u^{-1}$ where $u = x + 2x^{\frac{1}{2}}$

$$\begin{aligned}\text{Now}\quad \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx}\\ &= -4u^{-2}(1 + x^{-\frac{1}{2}})\\ &= -\frac{4}{u^2}\left(1+\frac{1}{\sqrt{x}}\right)\\ &= \frac{-4}{(x+2\sqrt{x})^2}\left(1+\frac{1}{\sqrt{x}}\right)\end{aligned}$$

At $x = 4$, $\dfrac{dy}{dx} = \dfrac{-4}{(4+4)^2}(1+\frac{1}{2}) = -\frac{6}{64}$

$\therefore$ gradient of tangent $= -\frac{3}{32}$

f $y = \left(x+\dfrac{1}{x}\right)^3$ at $x = 1$

$\therefore\ y = u^3$ where $u = x + x^{-1}$

$$\begin{aligned}\text{Now}\quad \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx}\\ &= 3u^2(1 - x^{-2})\\ &= 3\left(x+\frac{1}{x}\right)^2\left(1-\frac{1}{x^2}\right)\end{aligned}$$

At $x = 1$, $\dfrac{dy}{dx} = 3(1+1)^2(1-1)$

$\therefore$ gradient of tangent $= 0$

4 If $f(x) = (2x-b)^a$

$$\begin{aligned}\text{then}\quad f'(x) &= a(2x-b)^{a-1}\times 2\\ &= 2a(2x-b)^{a-1}\\ \text{but}\quad f'(x) &= 24x^2 - 24x + 6\\ &= 6(4x^2-4x+1)\\ &= 6(2x-1)^2\end{aligned}$$

Solving by inspection, $a - 1 = 2$ and $b = 1$

$\therefore\ a = 3$ and $b = 1$

(**Note:** We can also use $2a = 6$ when solving for a.)

5 $y = \dfrac{a}{\sqrt{1+bx}} = a\,(1+bx)^{-\frac{1}{2}}$

When $x = 3$, $y = 1$

$\therefore\ 1 = \dfrac{a}{\sqrt{1+3b}}$

$\therefore\ a = \sqrt{1+3b}$ .... (1)

$\dfrac{dy}{dx} = -\frac{1}{2}a\,(1+bx)^{-\frac{3}{2}}\times b$

When $x = 3$, $\dfrac{dy}{dx} = -\frac{1}{8}$

$\therefore\ -\frac{1}{8} = -\frac{1}{2}ab\,(1+3b)^{-\frac{3}{2}}$

$$\begin{aligned}\therefore\ a &= \frac{1}{4b}(1+3b)^{\frac{3}{2}} \quad .... (2)\\ &= \frac{1}{4b}(1+3b)\sqrt{1+3b}\end{aligned}$$

Equating the RHS of (1) and (2) gives:

$$\sqrt{1+3b} = \frac{1}{4b}(1+3b)\sqrt{1+3b}$$

$\therefore\ 1 = \dfrac{1}{4b}(1+3b)$ {since $b \neq 0$, $\sqrt{1+3b} \neq 0$}

$\therefore\ 4b = 1 + 3b$

$\therefore\ b = 1$

$$\begin{aligned}\therefore\ a &= \sqrt{1+3(1)}\\ &= \sqrt{4}\\ &= 2\end{aligned}$$

So, $a = 2$ and $b = 1$.

6 **a** $y = x^3 \quad \therefore \quad \dfrac{dy}{dx} = 3x^2$

$x = y^{\frac{1}{3}} \quad \therefore \quad \dfrac{dx}{dy} = \frac{1}{3}y^{-\frac{2}{3}}$

$$\begin{aligned}\frac{dy}{dx}\frac{dx}{dy} &= 3x^2\left(\tfrac{1}{3}\right)y^{-\frac{2}{3}} \\ &= x^2(y)^{-\frac{2}{3}} \\ &= x^2(x^3)^{-\frac{2}{3}} \quad \{\text{substituting } y = x^3\} \\ &= x^2(x^{-2}) \\ &= x^0 \\ &= 1 \quad \text{as required}\end{aligned}$$

b We know that $\dfrac{dy}{du}\dfrac{du}{dx} = \dfrac{dy}{dx}$ {chain rule}

Letting $x = y$, $\dfrac{dy}{du}\dfrac{du}{dy} = \dfrac{dy}{dy}$

$\therefore \quad \dfrac{dy}{du}\dfrac{du}{dy} = 1$

Letting $u = x$, $\dfrac{dy}{dx}\dfrac{dx}{dy} = 1$

EXERCISE 18C

1 **a** $f(x) = x(x-1)$ is the product of

$u = x$ and $v = x - 1$

$\therefore \; u' = 1$ and $v' = 1$

Now $$\begin{aligned}f'(x) &= u'v + uv' \quad \{\text{product rule}\} \\ &= 1 \times (x-1) + x \times 1 \\ &= x - 1 + x \\ &= 2x - 1\end{aligned}$$

b $f(x) = 2x(x+1)$ is the product of

$u = 2x$ and $v = x + 1$

$\therefore \; u' = 2$ and $v' = 1$

Now $$\begin{aligned}f'(x) &= u'v + uv' \quad \{\text{product rule}\} \\ &= 2(x+1) + 2x \times 1 \\ &= 2x + 2 + 2x \\ &= 4x + 2\end{aligned}$$

c $f(x) = x^2\sqrt{x+1}$ is the product of $u = x^2$ and $v = (x+1)^{\frac{1}{2}}$

$\therefore \; u' = 2x$ and $v' = \frac{1}{2}(x+1)^{-\frac{1}{2}}$

Now $$\begin{aligned}f'(x) &= u'v + uv' \quad \{\text{product rule}\} \\ &= 2x(x+1)^{\frac{1}{2}} + x^2 \times \tfrac{1}{2}(x+1)^{-\frac{1}{2}} \\ &= 2x(x+1)^{\frac{1}{2}} + \tfrac{1}{2}x^2(x+1)^{-\frac{1}{2}}\end{aligned}$$

2 **a** $y = x^2(2x-1)$ is the product of

$u = x^2$ and $v = 2x - 1$

$\therefore \; u' = 2x$ and $v' = 2$

Now $\dfrac{dy}{dx} = u'v + uv'$ {product rule}

$$\begin{aligned}\therefore \quad \frac{dy}{dx} &= 2x(2x-1) + x^2(2) \\ &= 2x(2x-1) + 2x^2\end{aligned}$$

b $y = 4x(2x+1)^3$ is the product of

$u = 4x$ and $v = (2x+1)^3$

$\therefore \; u' = 4$ and $v' = 3(2x+1)^2 \times 2$
$= 6(2x+1)^2$

Now $\dfrac{dy}{dx} = u'v + uv'$ {product rule}

$\therefore \quad \dfrac{dy}{dx} = 4(2x+1)^3 + 24x(2x+1)^2$

c $y = x^2\sqrt{3-x}$ is the product of

$u = x^2$ and $v = (3-x)^{\frac{1}{2}}$

$\therefore \; u' = 2x$ and $v' = \frac{1}{2}(3-x)^{-\frac{1}{2}}(-1)$
$= -\frac{1}{2}(3-x)^{-\frac{1}{2}}$

Now $\dfrac{dy}{dx} = u'v + uv'$ {product rule}

$$\begin{aligned}\therefore \quad \frac{dy}{dx} &= 2x(3-x)^{\frac{1}{2}} + x^2\left[-\tfrac{1}{2}(3-x)^{-\frac{1}{2}}\right] \\ &= 2x(3-x)^{\frac{1}{2}} - \tfrac{1}{2}x^2(3-x)^{-\frac{1}{2}}\end{aligned}$$

d $y = \sqrt{x}(x-3)^2$ is the product of

$u = x^{\frac{1}{2}}$ and $v = (x-3)^2$

$\therefore \; u' = \frac{1}{2}x^{-\frac{1}{2}}$ and $v' = 2(x-3)^1$

Now $\dfrac{dy}{dx} = u'v + uv'$ {product rule}

$\therefore \quad \dfrac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}}(x-3)^2 + 2\sqrt{x}(x-3)$

e $y = 5x^2(3x^2-1)^2$ is the product of $u = 5x^2$ and $v = (3x^2-1)^2$

$\therefore\ u' = 10x$ and $v' = 2(3x^2-1)^1(6x) = 12x(3x^2-1)$

Now $\dfrac{dy}{dx} = u'v + uv'$ {product rule}

$$\therefore\ \frac{dy}{dx} = 10x(3x^2-1)^2 + 5x^2(12x)(3x^2-1) = 10x(3x^2-1)^2 + 60x^3(3x^2-1)$$

f $y = \sqrt{x}(x-x^2)^3$ is the product of $u = x^{\frac{1}{2}}$ and $v = (x-x^2)^3$

$\therefore\ u' = \frac{1}{2}x^{-\frac{1}{2}}$ and $v' = 3(x-x^2)^2(1-2x)$

Now $\dfrac{dy}{dx} = u'v + uv'$ {product rule}

$$\therefore\ \frac{dy}{dx} = \tfrac{1}{2}x^{-\frac{1}{2}}(x-x^2)^3 + 3\sqrt{x}(x-x^2)^2(1-2x)$$

3 **a** $y = x^4(1-2x)^2$ is the product of $u = x^4$ and $v = (1-2x)^2$

$\therefore\ u' = 4x^3$ and $v' = 2(1-2x)^1(-2) = -4(1-2x)$

Now $\dfrac{dy}{dx} = u'v + uv'$ {product rule}

$$\therefore\ \frac{dy}{dx} = 4x^3(1-2x)^2 - 4x^4(1-2x)$$

At $x=-1$, $\dfrac{dy}{dx} = 4(-1)^3(3)^2 - 4(-1)^4(3) = -48$ $\therefore$ gradient of tangent $= -48$

b $y = \sqrt{x}(x^2-x+1)^2$ is the product of $u = x^{\frac{1}{2}}$ and $v = (x^2-x+1)^2$

$\therefore\ u' = \frac{1}{2}x^{-\frac{1}{2}}$ and $v' = 2(x^2-x+1)(2x-1)$

Now $\dfrac{dy}{dx} = u'v + uv'$ {product rule}

$$\therefore\ \frac{dy}{dx} = \tfrac{1}{2}x^{-\frac{1}{2}}(x^2-x+1)^2 + 2\sqrt{x}(x^2-x+1)(2x-1)$$

At $x=4$, $\dfrac{dy}{dx} = \frac{1}{2}(4)^{-\frac{1}{2}}(13)^2 + 2\sqrt{4}(13)(7) = 406\frac{1}{4}$ $\therefore$ gradient of tangent $= 406\frac{1}{4}$

c $y = x\sqrt{1-2x}$ is the product of $u = x$ and $v = (1-2x)^{\frac{1}{2}}$

$\therefore\ u' = 1$ and $v' = \frac{1}{2}(1-2x)^{-\frac{1}{2}}(-2) = -(1-2x)^{-\frac{1}{2}}$

Now $\dfrac{dy}{dx} = u'v + uv'$ {product rule}

$$\therefore\ \frac{dy}{dx} = \sqrt{1-2x} - \frac{x}{\sqrt{1-2x}}$$

At $x=-4$, $\dfrac{dy}{dx} = \sqrt{9} - \dfrac{(-4)}{\sqrt{9}} = 3 + \frac{4}{3} = \frac{13}{3}$ $\therefore$ gradient of tangent $= \frac{13}{3}$

d $y = x^3\sqrt{5-x^2}$ is the product of $u = x^3$ and $v = (5-x^2)^{\frac{1}{2}}$

$\therefore\ u' = 3x^2$ and $v' = \frac{1}{2}(5-x^2)^{-\frac{1}{2}}(-2x) = -x(5-x^2)^{-\frac{1}{2}}$

Now $\dfrac{dy}{dx} = u'v + uv'$ {product rule}

$$\therefore\ \frac{dy}{dx} = 3x^2\sqrt{5-x^2} - \frac{x^4}{\sqrt{5-x^2}}$$

At $x=1$, $\dfrac{dy}{dx} = 3(1)^2\sqrt{4} - \dfrac{1}{\sqrt{4}} = 6 - \frac{1}{2} = \frac{11}{2}$ $\therefore$ gradient of tangent $= \frac{11}{2}$

4 **a** $y = \sqrt{x}(3-x)^2$ is the product of

$u = x^{\frac{1}{2}}$ and $v = (3-x)^2$

$\therefore\ u' = \frac{1}{2}x^{-\frac{1}{2}}$ and $v' = 2(3-x)^1(-1)$
$= -2(3-x)$

Now $\dfrac{dy}{dx} = u'v + uv'$ {product rule}

$$\therefore\ \frac{dy}{dx} = \frac{1}{2\sqrt{x}}(3-x)^2 - 2\sqrt{x}(3-x)$$
$$= \frac{(3-x)^2 - (2\sqrt{x})(2\sqrt{x})(3-x)}{2\sqrt{x}}$$
$$= \frac{(3-x)\left[(3-x) - 4x\right]}{2\sqrt{x}}$$
$$= \frac{(3-x)(3-5x)}{2\sqrt{x}} \quad \text{as required}$$

b Tangents are horizontal when their gradients are 0.

$\dfrac{dy}{dx} = 0$ when $(3-x)(3-5x) = 0$

$\therefore\ 3 - x = 0$ or $3 - 5x = 0$

$\therefore\ x = 3$ or $x = \frac{3}{5}$

c $\dfrac{dy}{dx}$ is undefined when $2\sqrt{x} \leqslant 0$

$\therefore\ \sqrt{x} \leqslant 0$

$\therefore\ x \leqslant 0$

d $\dfrac{dy}{dx}$ is undefined when $x \leqslant 0$ {from **c**}

But when $x = 0$, $y = \sqrt{0}(3-0)^2$
$= 0$

So, when $x = 0$, y is defined but $\dfrac{dy}{dx}$ is not.

e As we approach the point $x = 0$ from the right, the curve has steeper and steeper gradient and approaches vertical.

5 $y = -2x^2(x+4)$ is the product of $u = -2x^2$ and $v = x + 4$

$\therefore\ u' = -4x$ and $v' = 1$

Now $\dfrac{dy}{dx} = u'v + uv'$ {product rule}
$= -4x(x+4) - 2x^2 \times 1$
$= -4x^2 - 16x - 2x^2$
$= -6x^2 - 16x$

If $\dfrac{dy}{dx} = 10$, $-6x^2 - 16x = 10$

$\therefore\ 6x^2 + 16x + 10 = 0$

$\therefore\ 3x^2 + 8x + 5 = 0$

$\therefore\ (3x+5)(x+1) = 0$

$\therefore\ x = -\frac{5}{3}$ and $x = -1$

EXERCISE 18D

1 **a** $y = \dfrac{1+3x}{2-x}$ is a quotient where

$u = 1 + 3x$ and $v = 2 - x$

$\therefore\ u' = 3$ and $v' = -1$

Now $\dfrac{dy}{dx} = \dfrac{u'v - uv'}{v^2}$ {quotient rule}

$$\therefore\ \frac{dy}{dx} = \frac{3(2-x) - (1+3x)(-1)}{(2-x)^2}$$
$$= \frac{7}{(2-x)^2}$$

b $y = \dfrac{x^2}{2x+1}$ is a quotient where

$u = x^2$ and $v = 2x + 1$

$\therefore\ u' = 2x$ and $v' = 2$

Now $\dfrac{dy}{dx} = \dfrac{u'v - uv'}{v^2}$ {quotient rule}

$$\therefore\ \frac{dy}{dx} = \frac{2x(2x+1) - x^2(2)}{(2x+1)^2}$$
$$= \frac{2x(2x+1) - 2x^2}{(2x+1)^2}$$

c $y = \dfrac{x}{x^2 - 3}$ is a quotient where

$u = x$ and $v = x^2 - 3$

$\therefore\ u' = 1$ and $v' = 2x$

Now $\dfrac{dy}{dx} = \dfrac{u'v - uv'}{v^2}$ {quotient rule}

$$\therefore\ \frac{dy}{dx} = \frac{1(x^2-3) - x(2x)}{(x^2-3)^2}$$
$$= \frac{(x^2-3) - 2x^2}{(x^2-3)^2}$$

d $y = \dfrac{\sqrt{x}}{1-2x}$ is a quotient where

$u = x^{\frac{1}{2}}$ and $v = 1 - 2x$

$\therefore\ u' = \frac{1}{2}x^{-\frac{1}{2}}$ and $v' = -2$

Now $\dfrac{dy}{dx} = \dfrac{u'v - uv'}{v^2}$ {quotient rule}

$$\therefore\ \frac{dy}{dx} = \frac{\frac{1}{2}x^{-\frac{1}{2}}(1-2x) - \sqrt{x}(-2)}{(1-2x)^2}$$
$$= \frac{\frac{1}{2}x^{-\frac{1}{2}}(1-2x) + 2\sqrt{x}}{(1-2x)^2}$$

e $y = \dfrac{x^2-3}{3x-x^2}$ is a quotient where $u = x^2 - 3$ and $v = 3x - x^2$

$\therefore\ u' = 2x$ and $v' = 3 - 2x$

Now $\dfrac{dy}{dx} = \dfrac{u'v - uv'}{v^2}$ {quotient rule}

$\therefore\ \dfrac{dy}{dx} = \dfrac{2x(3x-x^2)-(x^2-3)(3-2x)}{(3x-x^2)^2}$

f $y = \dfrac{x}{\sqrt{1-3x}}$ is a quotient where $u = x$ and $v = (1-3x)^{\frac{1}{2}}$

$\therefore\ u' = 1$ and $v' = -\frac{3}{2}(1-3x)^{-\frac{1}{2}}$

Now $\dfrac{dy}{dx} = \dfrac{u'v - uv'}{v^2}$ {quotient rule}

$\therefore\ \dfrac{dy}{dx} = \dfrac{(1-3x)^{\frac{1}{2}} - x\left(-\frac{3}{2}(1-3x)^{-\frac{1}{2}}\right)}{1-3x}$

$= \dfrac{(1-3x)^{\frac{1}{2}} + \frac{3}{2}x(1-3x)^{-\frac{1}{2}}}{1-3x}$

2 **a** $y = \dfrac{x}{1-2x}$ is a quotient where

$u = x$ and $v = 1 - 2x$

$\therefore\ u' = 1$ and $v' = -2$

Now $\dfrac{dy}{dx} = \dfrac{u'v - uv'}{v^2}$ {quotient rule}

$\therefore\ \dfrac{dy}{dx} = \dfrac{1(1-2x)-x(-2)}{(1-2x)^2}$

$= \dfrac{1}{(1-2x)^2}$

At $x = 1$, $\dfrac{dy}{dx} = \dfrac{1}{(1-2)^2} = \dfrac{1}{(-1)^2} = 1$

$\therefore$ the gradient of the tangent $= 1$

b $y = \dfrac{x^3}{x^2+1}$ is a quotient where

$u = x^3$ and $v = x^2 + 1$

$\therefore\ u' = 3x^2$ and $v' = 2x$

Now $\dfrac{dy}{dx} = \dfrac{u'v - uv'}{v^2}$ {quotient rule}

$\therefore\ \dfrac{dy}{dx} = \dfrac{3x^2(x^2+1)-x^3(2x)}{(x^2+1)^2}$

$= \dfrac{x^4+3x^2}{(x^2+1)^2}$

At $x = -1$, $\dfrac{dy}{dx} = \dfrac{1+3}{(1+1)^2} = \dfrac{4}{4} = 1$

$\therefore$ the gradient of the tangent $= 1$

c $y = \dfrac{\sqrt{x}}{2x+1}$ is a quotient where

$u = x^{\frac{1}{2}}$ and $v = 2x + 1$

$\therefore\ u' = \frac{1}{2}x^{-\frac{1}{2}}$ and $v' = 2$

Now $\dfrac{dy}{dx} = \dfrac{u'v - uv'}{v^2}$ {quotient rule}

$= \dfrac{\frac{1}{2\sqrt{x}}(2x+1) - \sqrt{x}(2)}{(2x+1)^2}$

At $x = 4$, $\dfrac{dy}{dx} = \dfrac{\frac{9}{4}-4}{81} = \dfrac{\left(\frac{9}{4}-4\right)}{81} \times \dfrac{4}{4}$

$= \dfrac{9-16}{324}$

$\therefore$ the gradient of the tangent $= -\frac{7}{324}$

d $y = \dfrac{x^2}{\sqrt{x^2+5}}$ is a quotient where

$u = x^2$ and $v = (x^2+5)^{\frac{1}{2}}$

$\therefore\ u' = 2x$ and $v' = \frac{1}{2}(x^2+5)^{-\frac{1}{2}}(2x)$

$= x(x^2+5)^{-\frac{1}{2}}$

Now $\dfrac{dy}{dx} = \dfrac{u'v - uv'}{v^2}$ {quotient rule}

$= \dfrac{2x\sqrt{x^2+5} - x^2\left(\dfrac{x}{\sqrt{x^2+5}}\right)}{(x^2+5)}$

At $x = -2$, $\dfrac{dy}{dx} = \dfrac{-4(3) - 4\left(\frac{-2}{3}\right)}{9}$

$= \dfrac{\left(-12+\frac{8}{3}\right)}{9} \times \dfrac{3}{3}$

$= \dfrac{-36+8}{27}$

$\therefore$ the gradient of the tangent $= -\frac{28}{27}$

3 **a** $y = \dfrac{2\sqrt{x}}{1-x}$ is a quotient where $u = 2x^{\frac{1}{2}}$ and $v = 1-x$

$\therefore\ u' = x^{-\frac{1}{2}}$ and $v' = -1$

Now $\dfrac{dy}{dx} = \dfrac{u'v - uv'}{v^2}$ {quotient rule}

$$\therefore\ \frac{dy}{dx} = \frac{\frac{1}{\sqrt{x}}(1-x) - 2\sqrt{x}(-1)}{(1-x)^2} \times \left(\frac{\sqrt{x}}{\sqrt{x}}\right)$$

$$= \frac{(1-x)+2x}{\sqrt{x}(1-x)^2}$$

$$= \frac{x+1}{\sqrt{x}(1-x)^2} \quad \text{as required}$$

b **i** $\dfrac{dy}{dx} = 0$ when $x+1=0$ $\therefore\ x = -1$.

However $\dfrac{dy}{dx}$ is not defined for $x \leqslant 0$ because of the $\sqrt{x}$ term. Hence $\dfrac{dy}{dx}$ never equals 0.

ii $\dfrac{dy}{dx}$ is undefined when $x \leqslant 0$ and when $x = 1$.

4 **a** $y = \dfrac{x^2-3x+1}{x+2}$ is a quotient where $u = x^2-3x+1$ and $v = x+2$

$\therefore\ u' = 2x-3$ and $v' = 1$

Now $\dfrac{dy}{dx} = \dfrac{u'v - uv'}{v^2}$ {quotient rule}

$$\therefore\ \frac{dy}{dx} = \frac{(2x-3)(x+2)-(x^2-3x+1)(1)}{(x+2)^2}$$

$$= \frac{2x^2+4x-3x-6-x^2+3x-1}{(x+2)^2}$$

$$= \frac{x^2+4x-7}{(x+2)^2} \quad \text{as required}$$

b **i** $\dfrac{dy}{dx} = 0$ when $x^2+4x-7=0$ $\therefore\ x = \dfrac{-4\pm\sqrt{44}}{2} = -2\pm\sqrt{11}$

ii $\dfrac{dy}{dx}$ is undefined when $(x+2)^2 = 0$

$\therefore\ x = -2$

c $\dfrac{dy}{dx}$ is zero when the tangent to the function is horizontal. This occurs at the function's turning points or points of horizontal inflection.

$\dfrac{dy}{dx}$ is undefined at vertical asymptotes of the function.

EXERCISE 18E

1 **a** $\dfrac{d}{dx}(2y) = 2\dfrac{dy}{dx}$ **b** $\dfrac{d}{dx}(-3y) = -3\dfrac{dy}{dx}$ **c** $\dfrac{d}{dx}(y^3) = 3y^2\dfrac{dy}{dx}$

d $\dfrac{d}{dx}\left(\dfrac{1}{y}\right) = \dfrac{d}{dx}(y^{-1}) = -y^{-2}\dfrac{dy}{dx}$ **e** $\dfrac{d}{dx}(y^4) = 4y^3\dfrac{dy}{dx}$

f $\dfrac{d}{dx}(\sqrt{y}) = \dfrac{d}{dx}(y^{\frac{1}{2}}) = \frac{1}{2}y^{-\frac{1}{2}}\dfrac{dy}{dx}$ **g** $\dfrac{d}{dx}\left(\dfrac{1}{y^2}\right) = \dfrac{d}{dx}(y^{-2}) = -2y^{-3}\dfrac{dy}{dx}$

h $\frac{d}{dx}(xy) = y + x\frac{dy}{dx}$ {product rule}

i $\frac{d}{dx}(x^2y) = 2xy + x^2\frac{dy}{dx}$ {product rule}

j $\frac{d}{dx}(xy^2) = y^2 + x(2y)\frac{dy}{dx} = y^2 + 2xy\frac{dy}{dx}$ {product rule}

2 **a** Differentiating both sides of $x^2 + y^2 = 25$ with respect to x,

$$2x + 2y\frac{dy}{dx} = 0$$
$$\therefore\ 2y\frac{dy}{dx} = -2x$$
$$\therefore\ \frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y}$$

b Differentiating both sides of $x^2 + 3y^2 = 9$ with respect to x,

$$2x + 6y\frac{dy}{dx} = 0$$
$$\therefore\ 6y\frac{dy}{dx} = -2x$$
$$\therefore\ \frac{dy}{dx} = -\frac{2x}{6y} = -\frac{x}{3y}$$

c Differentiating both sides of $y^2 - x^2 = 8$ with respect to x,

$$2y\frac{dy}{dx} - 2x = 0$$
$$\therefore\ 2y\frac{dy}{dx} = 2x$$
$$\therefore\ \frac{dy}{dx} = \frac{2x}{2y} = \frac{x}{y}$$

d Differentiating both sides of $x^2 - y^3 = 10$ with respect to x,

$$2x - 3y^2\frac{dy}{dx} = 0$$
$$\therefore\ 3y^2\frac{dy}{dx} = 2x$$
$$\therefore\ \frac{dy}{dx} = \frac{2x}{3y^2}$$

e Differentiating both sides of $x^2 + xy = 4$ with respect to x,

$$2x + \left(y + x\frac{dy}{dx}\right) = 0 \quad \text{\{product rule\}}$$
$$\therefore\ x\frac{dy}{dx} = -2x - y$$
$$\therefore\ \frac{dy}{dx} = \frac{-2x - y}{x}$$

f Differentiating both sides of $x^3 - 2xy = 5$ with respect to x,

$$3x^2 - \left(2y + 2x\frac{dy}{dx}\right) = 0 \quad \text{\{product rule\}}$$
$$\therefore\ 3x^2 - 2y = 2x\frac{dy}{dx}$$
$$\therefore\ \frac{dy}{dx} = \frac{3x^2 - 2y}{2x}$$

3 **a** Differentiating both sides of $x + y^3 = 4y$ with respect to x,

$$1 + 3y^2\frac{dy}{dx} = 4\frac{dy}{dx}$$

When $y = 1$, $1 + 3\frac{dy}{dx} = 4\frac{dy}{dx}$

$\therefore\ \frac{dy}{dx} = 1$ at this point, and the gradient of the tangent is 1.

b $x + y = 8xy$. Now when $x = \frac{1}{2}$, $\frac{1}{2} + y = 4y$ $\therefore\ y = \frac{1}{6}$

Thus the point of contact is $(\frac{1}{2}, \frac{1}{6})$.

Differentiating both sides of $x + y = 8xy$ with respect to x,

$$1 + \frac{dy}{dx} = 8y + 8x\frac{dy}{dx}$$

So, at the point $(\frac{1}{2}, \frac{1}{6})$, $1 + \frac{dy}{dx} = 8(\frac{1}{6}) + 8(\frac{1}{2})\frac{dy}{dx}$

$$\therefore\ 1 + \frac{dy}{dx} = \frac{4}{3} + 4\frac{dy}{dx}$$
$$\therefore\ -\frac{1}{3} = 3\frac{dy}{dx}$$
$$\therefore\ \frac{dy}{dx} = -\frac{1}{9}$$

$\therefore$ the gradient of the tangent is $-\frac{1}{9}$.

EXERCISE 18F

1 **a** $f(x) = e^{4x}$
$\therefore\ f'(x) = 4e^{4x}$

b $f(x) = e^x + 3$
$\therefore\ f'(x) = e^x + 0$
$= e^x$

c $f(x) = e^{-2x}$
$\therefore\ f'(x) = -2e^{-2x}$

d $f(x) = e^{\frac{x}{2}}$
$\therefore\ f'(x) = \frac{1}{2}e^{\frac{x}{2}}$

e $f(x) = 2e^{-\frac{x}{2}}$
$\therefore\ f'(x) = 2e^{-\frac{x}{2}}\left(-\frac{1}{2}\right)$
$= -e^{-\frac{x}{2}}$

f $f(x) = 1 - 2e^{-x}$
$\therefore\ f'(x) = 0 - 2e^{-x}(-1)$
$= 2e^{-x}$

g $f(x) = 4e^{\frac{x}{2}} - 3e^{-x}$
$\therefore\ f'(x) = 4e^{\frac{x}{2}}\left(\frac{1}{2}\right) - 3e^{-x}(-1)$
$= 2e^{\frac{x}{2}} + 3e^{-x}$

h $f(x) = \dfrac{e^x + e^{-x}}{2} = \frac{1}{2}(e^x + e^{-x})$
$\therefore\ f'(x) = \frac{1}{2}(e^x + e^{-x}(-1))$
$= \dfrac{e^x - e^{-x}}{2}$

i $f(x) = e^{-x^2}$
$\therefore\ f'(x) = e^{-x^2}(-2x)$
$= -2xe^{-x^2}$

j $f(x) = e^{\frac{1}{x}}$
$\therefore\ f'(x) = e^{\frac{1}{x}}\left(-\dfrac{1}{x^2}\right)$

k $f(x) = 10\left(1 + e^{2x}\right)$
$= 10 + 10e^{2x}$
$\therefore\ f'(x) = 0 + 10e^{2x}\,(2)$
$= 20e^{2x}$

l $f(x) = 20\left(1 - e^{-2x}\right)$
$= 20 - 20e^{-2x}$
$\therefore\ f'(x) = 0 - 20e^{-2x}\,(-2)$
$= 40e^{-2x}$

m $f(x) = e^{2x+1}$
$\therefore\ f'(x) = e^{2x+1}(2)$
$= 2e^{2x+1}$

n $f(x) = e^{\frac{x}{4}}$
$\therefore\ f'(x) = e^{\frac{x}{4}}\left(\frac{1}{4}\right)$
$= \frac{1}{4}e^{\frac{x}{4}}$

o $f(x) = e^{1-2x^2}$
$\therefore\ f'(x) = e^{1-2x^2}(-4x)$
$= -4xe^{1-2x^2}$

p $f(x) = e^{-0.02x}$
$\therefore\ f'(x) = e^{-0.02x} \times (-0.02)$
$= -0.02e^{-0.02x}$

2 **a** $f(x) = xe^x$
$\therefore\ f'(x) = 1e^x + xe^x$ {product rule}
$= e^x + xe^x$

b $f(x) = x^3e^{-x}$
$\therefore\ f'(x) = 3x^2e^{-x} + x^3(-e^{-x})$
{product rule}
$= 3x^2e^{-x} - x^3e^{-x}$

c $f(x) = \dfrac{e^x}{x}$
$\therefore\ f'(x) = \dfrac{e^x x - e^x\,(1)}{x^2}$ {quotient rule}
$= \dfrac{xe^x - e^x}{x^2}$

d $f(x) = \dfrac{x}{e^x}$
$\therefore\ f'(x) = \dfrac{1e^x - xe^x}{(e^x)^2}$ {quotient rule}
$= \dfrac{e^x(1-x)}{(e^x)^2} = \dfrac{1-x}{e^x}$

e $f(x) = x^2e^{3x}$
$\therefore\ f'(x) = 2xe^{3x} + 3x^2e^{3x}$ {product rule}

f $f(x) = \dfrac{e^x}{\sqrt{x}}$
$\therefore\ f'(x) = \dfrac{e^x\sqrt{x} - \dfrac{e^x}{2\sqrt{x}}}{(\sqrt{x})^2}$ {quotient rule}
$= \dfrac{xe^x - \frac{1}{2}e^x}{x\sqrt{x}}$

g $f(x) = \sqrt{x}e^{-x}$

$\therefore\ f'(x) = \frac{1}{2}x^{-\frac{1}{2}}e^{-x} - x^{\frac{1}{2}}e^{-x}$ {product rule}

h $f(x) = \dfrac{e^x + 2}{e^{-x} + 1}$

$\therefore\ f'(x) = \dfrac{e^x(e^{-x}+1) - (e^x+2)\left(-e^{-x}\right)}{(e^{-x}+1)^2}$ {quotient rule}

$= \dfrac{1 + e^x + 1 + 2e^{-x}}{(e^{-x}+1)^2}$

$= \dfrac{e^x + 2 + 2e^{-x}}{(e^{-x}+1)^2}$

3 **a** $f(x) = (e^x + 2)^4$

$= u^4$ where $u = e^x + 2$

$\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$ {chain rule}

$= 4u^3(e^x)$

$\therefore\ f'(x) = 4(e^x+2)^3(e^x)$

$= 4e^x(e^x+2)^3$

$\therefore\ f'(0) = 4\left(e^0+2\right)^3\left(e^0\right)$

$= 108$

$\therefore$ gradient of tangent $= 108$

b $f(x) = \dfrac{1}{2 - e^{-x}}$

$= u^{-1}$ where $u = 2 - e^{-x}$

$\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$ {chain rule}

$= -u^{-2}(e^{-x})$

$\therefore\ f'(x) = -\dfrac{e^{-x}}{(2-e^{-x})^2}$

$\therefore\ f'(0) = -\dfrac{e^0}{(2-e^0)^2}$

$= -1$

$\therefore$ gradient of tangent $= -1$

c $f(x) = \sqrt{e^{2x} + 10}$

$= u^{\frac{1}{2}}$ where $u = e^{2x} + 10$

$\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$ {chain rule}

$= \frac{1}{2}u^{-\frac{1}{2}}(2e^{2x})$

$\therefore\ f'(x) = \dfrac{e^{2x}}{\sqrt{e^{2x}+10}}$

$\therefore\ f'(\ln 3) = \dfrac{e^{2\ln 3}}{\sqrt{e^{2\ln 3}+10}} = \frac{9}{\sqrt{19}}$

$\therefore$ gradient of tangent $= \frac{9}{\sqrt{19}}$

4 **a** $f(x) = \dfrac{1}{(1-e^{3x})^2}$

$= u^{-2}$ where $u = 1 - e^{3x}$

$\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$ {chain rule}

$= -2u^{-3}(-3e^{3x}) = \dfrac{6e^{3x}}{u^3}$

$\therefore\ f'(x) = \dfrac{6e^{3x}}{(1-e^{3x})^3}$

b $f(x) = \dfrac{1}{\sqrt{1-e^{-x}}}$

$= u^{-\frac{1}{2}}$ where $u = 1 - e^{-x}$

$\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$ {chain rule}

$= -\frac{1}{2}u^{-\frac{3}{2}}(e^{-x})$

$= \dfrac{-e^{-x}}{2u^{\frac{3}{2}}}$

$\therefore\ f'(x) = \dfrac{-e^{-x}}{2(1-e^{-x})^{\frac{3}{2}}}$

c $f(x) = x\sqrt{1-2e^{-x}} = xu^{\frac{1}{2}}$ where $u = 1-2e^{-x}$

$\therefore\ f'(x) = 1u^{\frac{1}{2}} + x \times \frac{1}{2}u^{-\frac{1}{2}}\dfrac{du}{dx}$ {product rule and chain rule}

$= 1\sqrt{u} + x\,\frac{1}{2}u^{-\frac{1}{2}}2e^{-x}$

$= \dfrac{\sqrt{1-2e^{-x}}}{1} + \dfrac{xe^{-x}}{\sqrt{1-2e^{-x}}}$

$\therefore\ f'(x) = \dfrac{1-2e^{-x}+xe^{-x}}{\sqrt{1-2e^{-x}}}$

5 $f(x) = e^{kx} + x \quad \therefore\ f'(x) = ke^{kx} + 1$

Now $f'(0) = -8$, so $ke^0 + 1 = -8$

$\therefore\ k \times 1 = -9$

$\therefore\ k = -9$

6 **a** $y = 2^x$

$= (e^{\ln 2})^x$

$= e^{x\ln 2}$

$\therefore\ \dfrac{dy}{dx} = e^{x\ln 2} \times \ln 2$

$= 2^x \ln 2$

b $y = b^x$

$= (e^{\ln b})^x$

$= e^{x\ln b}$

$\therefore\ \dfrac{dy}{dx} = e^{x\ln b} \times \ln b$

$= b^x \times \ln b$

7 $f(x) = x^2e^{-x}$ is the product of $u = x^2$ and $v = e^{-x}$

$\therefore\ u' = 2x$ and $v' = -e^{-x}$

$\therefore\ f'(x) = u'v + uv'$

$= 2x\left(e^{-x}\right) + x^2\left(-e^{-x}\right)$

$= 2xe^{-x} - x^2e^{-x}$

Now $f'(x) = 0$ when $2xe^{-x} - x^2e^{-x} = 0$

$\therefore\ xe^{-x}(2-x) = 0$

$\therefore\ x = 0$ or $2 - x = 0$ $\{e^{-x} > 0$ for all $x\}$

$\therefore\ x = 0$ or $x = 2$

$f(0) = 0^2e^0$ and $f(2) = 2^2e^{-2}$

$= 0 \qquad\qquad = 4e^{-2}$ $\left(\text{or } \dfrac{4}{e^2}\right)$

So, the possible coordinates of P are $(0, 0)$ and $\left(2, \dfrac{4}{e^2}\right)$.

8 **a** $y = 2^x$

$\therefore\ \dfrac{dy}{dx} = 2^x \ln 2$

b $y = 5^x$

$\therefore\ \dfrac{dy}{dx} = 5^x \ln 5$

c $y = x\,2^x$

$\therefore\ \dfrac{dy}{dx} = \dfrac{d}{dx}(x)2^x + x\,\dfrac{d}{dx}(2^x)$

{product rule}

$= 2^x + x\,2^x \ln 2$

d $y = x^3 6^{-x} = \dfrac{x^3}{6^x}$

$\therefore\ \dfrac{dy}{dx} = \dfrac{\frac{d}{dx}(x^3)6^x - x^3\frac{d}{dx}(6^x)}{6^{2x}}$

{quotient rule}

$= \dfrac{3x^2 6^x - x^3 \times 6^x \ln 6}{6^{2x}}$

$= \dfrac{x^2(3 - x\ln 6)}{6^x}$

e $$y=\frac{2^x}{x}$$

$$\therefore\ \frac{dy}{dx}=\frac{\frac{d}{dx}(2^x)x-2^x\frac{d}{dx}(x)}{x^2}\quad\{\text{quotient rule}\}$$

$$=\frac{2^x\ln 2\times x-2^x}{x^2}$$

$$=\frac{2^x(x\ln 2-1)}{x^2}$$

f $$y=\frac{x}{3^x}$$

$$\therefore\ \frac{dy}{dx}=\frac{\frac{d}{dx}(x)3^x-x\frac{d}{dx}(3^x)}{3^{2x}}\quad\{\text{quotient rule}\}$$

$$=\frac{3^x-x\times 3^x\ln 3}{3^{2x}}$$

$$=\frac{1-x\ln 3}{3^x}$$

9 $x^3e^{3y}+4x^2y^3=27e^{-2x}$

$$\therefore\ 3x^2e^{3y}+x^3\left(3e^{3y}\frac{dy}{dx}\right)+8xy^3+4x^2\left(3y^2\frac{dy}{dx}\right)=27(-2e^{-2x})\quad\{\text{product rule}\}$$

$$\therefore\ \frac{dy}{dx}(3x^3e^{3y}+12x^2y^2)=-54e^{-2x}-3x^2e^{3y}-8xy^3$$

$$\therefore\ \frac{dy}{dx}=\frac{-54e^{-2x}-3x^2e^{3y}-8xy^3}{3x^3e^{3y}+12x^2y^2}$$

$$=\frac{-(54e^{-2x}+3x^2e^{3y}+8xy^3)}{3x^2(xe^{3y}+4y^2)}$$

EXERCISE 18G

1 **a** $$y=\ln(7x)$$

$$\therefore\ y=\ln 7+\ln x$$

$$\therefore\ \frac{dy}{dx}=0+\frac{1}{x}=\frac{1}{x}$$

or

$$y=\ln(7x)$$

$$\therefore\ \frac{dy}{dx}=\frac{7}{7x}\begin{array}{l}\leftarrow f'(x)\\ \leftarrow f(x)\end{array}$$

$$=\frac{1}{x}$$

b $$y=\ln(2x+1)$$

$$\therefore\ \frac{dy}{dx}=\frac{2}{2x+1}\begin{array}{l}\leftarrow f'(x)\\ \leftarrow f(x)\end{array}$$

c $$y=\ln(x-x^2)$$

$$\therefore\ \frac{dy}{dx}=\frac{1-2x}{x-x^2}\begin{array}{l}\leftarrow f'(x)\\ \leftarrow f(x)\end{array}$$

d $$y=3-2\ln x$$

$$\therefore\ \frac{dy}{dx}=0-2\left(\frac{1}{x}\right)$$

$$=-\frac{2}{x}$$

e $$y=x^2\ln x$$

$$\therefore\ \frac{dy}{dx}=2x\ln x+x^2\left(\frac{1}{x}\right)$$

$$=2x\ln x+x$$

f $$y=\frac{\ln x}{2x}$$

$$\therefore\ \frac{dy}{dx}=\frac{\left(\frac{1}{x}\right)2x-\ln x\times 2}{(2x)^2}$$

$$=\frac{2-2\ln x}{4x^2}$$

$$=\frac{1-\ln x}{2x^2}$$

g $$y=e^x\ln x$$

$$\therefore\ \frac{dy}{dx}=e^x\ln x+\frac{e^x}{x}$$

h $$y=(\ln x)^2$$

$$\therefore\ \frac{dy}{dx}=2(\ln x)^1\left(\frac{1}{x}\right)$$

$$=\frac{2\ln x}{x}$$

i $$y=\sqrt{\ln x}=(\ln x)^{\frac{1}{2}}$$

$$\therefore\ \frac{dy}{dx}=\tfrac{1}{2}(\ln x)^{-\frac{1}{2}}\left(\frac{1}{x}\right)$$

$$=\frac{1}{2x\sqrt{\ln x}}$$

j $$y=e^{-x}\ln x$$

$$\therefore\ \frac{dy}{dx}=-e^{-x}\ln x+e^{-x}\left(\frac{1}{x}\right)$$

$$=\frac{e^{-x}}{x}-e^{-x}\ln x$$

k
$$y = \sqrt{x}\ln(2x)$$
$$\therefore \quad \frac{dy}{dx} = \frac{1}{2\sqrt{x}}\ln(2x) + \sqrt{x}\left(\frac{1}{x}\right)$$
$$= \frac{\ln(2x)}{2\sqrt{x}} + \frac{1}{\sqrt{x}}$$

l
$$y = \frac{2\sqrt{x}}{\ln x}$$
$$\therefore \quad \frac{dy}{dx} = \frac{\frac{1}{\sqrt{x}}\ln x - 2\sqrt{x}\left(\frac{1}{x}\right)}{(\ln x)^2}$$
$$= \frac{\frac{1}{\sqrt{x}}\ln x - \frac{2}{\sqrt{x}}}{(\ln x)^2}$$
$$= \frac{\ln x - 2}{\sqrt{x}(\ln x)^2}$$

m
$$y = 3 - 4\ln(1-x)$$
$$\therefore \quad \frac{dy}{dx} = -\frac{4}{1-x} \times -1$$
$$= \frac{4}{1-x}$$

n
$$y = x\ln(x^2+1)$$
$$\therefore \quad \frac{dy}{dx} = \ln(x^2+1) + x\frac{2x}{x^2+1}$$
$$= \ln(x^2+1) + \frac{2x^2}{x^2+1}$$

2 **a**
$$y = x\ln 5$$
$$\therefore \quad \frac{dy}{dx} = \ln 5$$

b
$$y = \ln(x^3) = 3\ln x$$
$$\therefore \quad \frac{dy}{dx} = 3\left(\frac{1}{x}\right) = \frac{3}{x}$$

c
$$y = \ln(x^4 + x)$$
$$\therefore \quad \frac{dy}{dx} = \frac{4x^3+1}{x^4+x}$$

d
$$y = \ln(10-5x)$$
$$\therefore \quad \frac{dy}{dx} = \frac{-5}{10-5x} = \frac{1}{x-2}$$

e
$$y = [\ln(2x+1)]^3$$
$$\therefore \quad \frac{dy}{dx} = 3\,[\ln(2x+1)]^2 \times \frac{2}{2x+1}$$
$$= \frac{6}{2x+1}\,[\ln(2x+1)]^2$$

f
$$y = \frac{\ln(4x)}{x}$$
$$\therefore \quad \frac{dy}{dx} = \frac{\left(\frac{4}{4x}\right)x - \ln(4x) \times 1}{x^2}$$
$$= \frac{1-\ln(4x)}{x^2}$$

g
$$y = \ln\left(\frac{1}{x}\right)$$
$$= -\ln x$$
$$\therefore \quad \frac{dy}{dx} = -\frac{1}{x}$$

h
$$y = \ln(\ln x)$$
$$\therefore \quad \frac{dy}{dx} = \frac{\frac{1}{x}}{\ln x}$$
$$= \frac{1}{x\ln x}$$

i
$$y = \frac{1}{\ln x} = (\ln x)^{-1}$$
$$\therefore \quad \frac{dy}{dx} = -1(\ln x)^{-2} \times \frac{1}{x}$$
$$= \frac{-1}{x(\ln x)^2}$$

3 **a**
$$y = \ln\sqrt{1-2x}$$
$$= \ln(1-2x)^{\frac{1}{2}}$$
$$= \tfrac{1}{2}\ln(1-2x)$$
$$\therefore \quad \frac{dy}{dx} = \tfrac{1}{2} \times \frac{-2}{1-2x}$$
$$= \frac{-1}{1-2x}$$

b
$$y = \ln\left(\frac{1}{2x+3}\right)$$
$$= -\ln(2x+3)$$
$$\therefore \quad \frac{dy}{dx} = -\frac{2}{2x+3}$$

c
$$y = \ln(e^x\sqrt{x})$$
$$= \ln e^x + \ln x^{\frac{1}{2}}$$
$$= \ln e^x + \tfrac{1}{2}\ln x$$
$$= x + \tfrac{1}{2}\ln x$$
$$\therefore \quad \frac{dy}{dx} = 1 + \tfrac{1}{2}\left(\frac{1}{x}\right)$$
$$= 1 + \frac{1}{2x}$$

d
$$\begin{aligned} y &= \ln(x\sqrt{2-x}) \\ &= \ln x + \ln(2-x)^{\frac{1}{2}} \\ &= \ln x + \tfrac{1}{2}\ln(2-x) \\ \therefore\ \frac{dy}{dx} &= \frac{1}{x} + \tfrac{1}{2}\left(\frac{-1}{2-x}\right) \\ &= \frac{1}{x} - \frac{1}{2(2-x)} \end{aligned}$$

e
$$\begin{aligned} y &= \ln\left(\frac{x+3}{x-1}\right) \\ &= \ln(x+3) - \ln(x-1) \\ \therefore\ \frac{dy}{dx} &= \frac{1}{x+3} - \frac{1}{x-1} \end{aligned}$$

f
$$\begin{aligned} y &= \ln\left(\frac{x^2}{3-x}\right) \\ &= \ln x^2 - \ln(3-x) \\ &= 2\ln x - \ln(3-x) \\ \therefore\ \frac{dy}{dx} &= \frac{2}{x} - \frac{-1}{3-x} = \frac{2}{x} + \frac{1}{3-x} \end{aligned}$$

g
$$\begin{aligned} f(x) &= \ln\left((3x-4)^3\right) \\ &= 3\ln(3x-4) \\ \therefore\ f'(x) &= 3 \times \frac{3}{3x-4} = \frac{9}{3x-4} \end{aligned}$$

h
$$\begin{aligned} f(x) &= \ln\left(x(x^2+1)\right) \\ &= \ln x + \ln(x^2+1) \\ \therefore\ f'(x) &= \frac{1}{x} + \frac{2x}{x^2+1} \end{aligned}$$

i
$$\begin{aligned} f(x) &= \ln\left(\frac{x^2+2x}{x-5}\right) \\ &= \ln(x^2+2x) - \ln(x-5) \\ \therefore\ f'(x) &= \frac{2x+2}{x^2+2x} - \frac{1}{x-5} \end{aligned}$$

4 $y = x\ln x$ is the product of $u = x$ and $v = \ln x$

$\therefore\ u' = 1$ and $v' = \dfrac{1}{x}$

Now $\dfrac{dy}{dx} = u'v + uv'$ {product rule}

$$\begin{aligned} &= 1\ln x + x \times \frac{1}{x} \\ &= \ln x + 1 \end{aligned}$$

At $x = e$, $\dfrac{dy}{dx} = \ln e + 1 = 1 + 1 = 2$

$\therefore$ gradient of tangent $= 2$

5 $f(x) = a\ln(2x+b)$

Now $f(e) = 3$, $\therefore\ 3 = a\ln(2e+b)$

$\therefore\ a = \dfrac{3}{\ln(2e+b)}$

$f'(x) = a \times \dfrac{2}{2x+b}$

Now $f'(e) = \dfrac{6}{e}$ $\therefore\ \dfrac{6}{e} = \dfrac{2a}{2e+b}$

$$\begin{aligned} \therefore\ 6(2e+b) &= 2ae \\ \therefore\ 3(2e+b) &= ae \\ \therefore\ a &= \frac{3(2e+b)}{e} \\ &= \frac{6e+3b}{e} \\ &= 6 + \frac{3b}{e} \end{aligned}$$

Using technology we find $a = 3$, $b = -e$.

6 For this question, we remember that $\log_a x = \dfrac{\log_e x}{\log_e a} = \dfrac{\ln x}{\ln a}$

a $y = \log_2 x = \dfrac{\ln x}{\ln 2}$

$\therefore\ \dfrac{dy}{dx} = \dfrac{1}{x\ln 2}$

b $y = \log_{10} x = \dfrac{\ln x}{\ln 10}$

$\therefore\ \dfrac{dy}{dx} = \dfrac{1}{x\ln 10}$

c $y = x\log_3 x = \dfrac{x\ln x}{\ln 3}$

Since $\ln 3$ is a constant, $\dfrac{dy}{dx} = \dfrac{\frac{d}{dx}(x)\ln x + x\frac{d}{dx}(\ln x)}{\ln 3}$

$$= \frac{\ln x + x\left(\frac{1}{x}\right)}{\ln 3}$$

$$= \frac{1+\ln x}{\ln 3} = \frac{1}{\ln 3} + \log_3 x$$

7 $e^{2a}\ln b^2 - a^3b + \ln(ab) = 21$

$$\therefore \left(2e^{2a}\frac{da}{db}\right)\ln b^2 + e^{2a}\left(\frac{2b}{b^2}\right) - \left(3a^2\frac{da}{db}\right)b - a^3 + \frac{\frac{da}{db}\times b + a}{ab} = 0$$

$$\therefore 4abe^{2a}\ln b\frac{da}{db} + 2ae^{2a} - 3a^3b^2\frac{da}{db} - a^4b + b\frac{da}{db} + a = 0 \qquad \{\times ab\}$$

$$\therefore \frac{da}{db}\left(4abe^{2a}\ln b - 3a^3b^2 + b\right) = a^4b - 2ae^{2a} - a$$

$$\therefore \frac{da}{db} = \frac{a^4b - 2ae^{2a} - a}{4abe^{2a}\ln b - 3a^3b^2 + b}$$

EXERCISE 18H

1 **a** $y = \sin(2x)$

$$\therefore \frac{dy}{dx} = \cos(2x)\frac{d}{dx}(2x) = 2\cos(2x)$$

b $y = \sin x + \cos x$

$$\therefore \frac{dy}{dx} = \cos x - \sin x$$

c $y = \cos(3x) - \sin x$

$$\therefore \frac{dy}{dx} = -\sin(3x)\times 3 - \cos x = -3\sin(3x) - \cos x$$

d $y = \sin(x+1)$

$$\therefore \frac{dy}{dx} = \cos(x+1)\frac{d}{dx}(x+1) = 1\cos(x+1) = \cos(x+1)$$

e $y = \cos(3-2x)$

$$\therefore \frac{dy}{dx} = -\sin(3-2x)\times -2 = 2\sin(3-2x)$$

f $y = \tan(5x)$

$$\therefore \frac{dy}{dx} = \frac{1}{\cos^2(5x)}\times 5 = \frac{5}{\cos^2(5x)}$$

g $y = \sin\left(\frac{x}{2}\right) - 3\cos x$

$$\therefore \frac{dy}{dx} = \tfrac{1}{2}\cos\left(\tfrac{x}{2}\right) + 3\sin x$$

h $y = 3\tan(\pi x)$

$$\therefore \frac{dy}{dx} = 3\times\frac{1}{\cos^2(\pi x)}\times\pi = \frac{3\pi}{\cos^2(\pi x)}$$

i $y = 4\sin x - \cos(2x)$

$$\therefore \frac{dy}{dx} = 4\cos x + \sin(2x)\times 2 = 4\cos x + 2\sin(2x)$$

2 **a** $y = x^2 + \cos x$

$$\therefore \frac{dy}{dx} = 2x - \sin x$$

b $y = \tan x - 3\sin x$

$$\therefore \frac{dy}{dx} = \frac{1}{\cos^2 x} - 3\cos x$$

c $y = e^x \cos x$

$\therefore \dfrac{dy}{dx} = e^x \cos x + e^x(-\sin x)$

$= e^x \cos x - e^x \sin x$

d $y = e^{-x} \sin x$

$\therefore \dfrac{dy}{dx} = -e^{-x} \sin x + e^{-x} \cos x$

e $y = \ln(\sin x)$

$\therefore \dfrac{dy}{dx} = \dfrac{\cos x}{\sin x}$

f $y = e^{2x} \tan x$

$\therefore \dfrac{dy}{dx} = 2e^{2x} \tan x + \dfrac{e^{2x}}{\cos^2 x}$

g $y = \sin(3x)$

$\therefore \dfrac{dy}{dx} = 3\cos(3x)$

h $y = \cos(\frac{x}{2})$

$\therefore \dfrac{dy}{dx} = -\frac{1}{2}\sin(\frac{x}{2})$

i $y = 3\tan(2x)$

$\therefore \dfrac{dy}{dx} = \dfrac{3}{\cos^2(2x)} \times 2$

$= \dfrac{6}{\cos^2(2x)}$

j $y = x\cos x$

$\therefore \dfrac{dy}{dx} = 1 \times \cos x + x(-\sin x)$

$= \cos x - x\sin x$

k $y = \dfrac{\sin x}{x}$

$\therefore \dfrac{dy}{dx} = \dfrac{(\cos x)(x) - \sin x \times 1}{x^2}$

$= \dfrac{x\cos x - \sin x}{x^2}$

l $y = x\tan x$

$\therefore \dfrac{dy}{dx} = 1 \times \tan x + x \times \dfrac{1}{\cos^2 x}$

$= \tan x + \dfrac{x}{\cos^2 x}$

3 **a** $\dfrac{d}{dx}(\sec x) = \dfrac{d}{dx}\left(\dfrac{1}{\cos x}\right)$

$= \dfrac{d}{dx}(\cos x)^{-1}$

$= -(\cos x)^{-2} \times (-\sin x)$

$= \dfrac{\sin x}{(\cos x)^2}$

$= \dfrac{1}{\cos x} \times \dfrac{\sin x}{\cos x}$

$= \sec x \tan x$

b $\dfrac{d}{dx}(\cot x) = \dfrac{d}{dx}\left(\dfrac{1}{\tan x}\right)$

$= \dfrac{d}{dx}\left(\dfrac{\cos x}{\sin x}\right)$

$= \dfrac{(-\sin x) \times \sin x - \cos x \times \cos x}{(\sin x)^2}$

$= \dfrac{-(\sin^2 x + \cos^2 x)}{(\sin x)^2}$

$= \dfrac{-1}{(\sin x)^2}$ $\{$since $\sin^2 x + \cos^2 x = 1\}$

$= -\left(\dfrac{1}{\sin x}\right)^2$

$= -\csc^2 x$

4 **a** $y = \sin(x^2)$

$\therefore \dfrac{dy}{dx} = 2x\cos(x^2)$

b $y = \cos\left(\sqrt{x}\right) = \cos(x^{\frac{1}{2}})$

$\therefore \dfrac{dy}{dx} = -\sin(x^{\frac{1}{2}}) \times \frac{1}{2}x^{-\frac{1}{2}}$

$= -\dfrac{1}{2\sqrt{x}}\sin(\sqrt{x})$

c $y = \sqrt{\cos x} = (\cos x)^{\frac{1}{2}}$

$\therefore \dfrac{dy}{dx} = \frac{1}{2}(\cos x)^{-\frac{1}{2}} \times (-\sin x)$

$= -\dfrac{\sin x}{2\sqrt{\cos x}}$

d $y = \sin^2 x = (\sin x)^2$

$\therefore \dfrac{dy}{dx} = 2\sin x\cos x$

e $y = \cos^3 x = (\cos x)^3$

$\therefore \dfrac{dy}{dx} = 3\cos^2 x \times (-\sin x)$

$= -3\sin x\cos^2 x$

f $y = \cos x \sin(2x)$

$\therefore \dfrac{dy}{dx} = (-\sin x)\sin(2x) + \cos x(2\cos(2x))$

$= -\sin x \sin(2x) + 2\cos x\cos(2x)$

g
$$y = \cos(\cos x)$$
$$\therefore \frac{dy}{dx} = -\sin(\cos x) \times (-\sin x)$$
$$= \sin x \sin(\cos x)$$

h
$$y = \cos^3(4x) = (\cos(4x))^3$$
$$\therefore \frac{dy}{dx} = 3(\cos(4x))^2 \times (-4\sin(4x))$$
$$= -12\sin(4x)\cos^2(4x)$$

i
$$y = \csc(4x)$$
$$\therefore \frac{dy}{dx} = -\csc(4x)\cot(4x)\frac{d}{dx}(4x)$$
$$= -4\csc(4x)\cot(4x)$$

j
$$y = \sec(2x)$$
$$\therefore \frac{dy}{dx} = \sec(2x)\tan(2x)\frac{d}{dx}(2x)$$
$$= 2\sec(2x)\tan(2x)$$

k
$$y = \frac{2}{\sin^2(2x)} = 2(\sin(2x))^{-2}$$
$$\therefore \frac{dy}{dx} = -4(\sin(2x))^{-3} \times 2\cos(2x)$$
$$= -\frac{8\cos(2x)}{\sin^3(2x)}$$
or $-8\csc^2(2x)\cot(2x)$

l
$$y = 8\cot^3(\tfrac{x}{2}) = 8(\cot(\tfrac{x}{2}))^3$$
$$\therefore \frac{dy}{dx} = 24(\cot(\tfrac{x}{2}))^2 \times -\csc^2(\tfrac{x}{2}) \times \tfrac{1}{2}$$
$$= -12\cot^2(\tfrac{x}{2})\csc^2(\tfrac{x}{2})$$

5 **a**
$$f(x) = \sin^3 x$$
$$= (\sin x)^3$$
$$= u^3 \quad \text{where} \quad u = \sin x$$
$$f'(x) = \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \qquad \{\text{chain rule}\}$$
$$\therefore f'(x) = 3u^2 \times \frac{du}{dx}$$
$$= 3\sin^2 x \cos x$$
$$\therefore f'\left(\tfrac{2\pi}{3}\right) = 3\sin^2\left(\tfrac{2\pi}{3}\right)\cos\left(\tfrac{2\pi}{3}\right)$$
$$= -\tfrac{9}{8}$$

b $f(x) = \cos x \sin x$ is the product of
$$u = \cos x \quad \text{and} \quad v = \sin x$$
$$\therefore u' = -\sin x \quad \text{and} \quad v' = \cos x$$
Now $f'(x) = u'v + uv'$
$$= -\sin x \sin x + \cos x \cos x$$
$$= \cos^2 x - \sin^2 x$$
$$\therefore f'\left(\tfrac{\pi}{4}\right) = \cos^2\left(\tfrac{\pi}{4}\right) - \sin^2\left(\tfrac{\pi}{4}\right)$$
$$= 0$$

6 **a**
$$y = x\sec x$$
$$\therefore \frac{dy}{dx} = \sec x + x\sec x\tan x$$
{product rule}
$$= \sec x(x\tan x + 1)$$

b
$$y = e^x \cot x$$
$$\therefore \frac{dy}{dx} = e^x\cot x + e^x(-\csc^2 x)$$
{product rule}
$$= e^x(\cot x - \csc^2 x)$$

c
$$y = 4\sec(2x)$$
$$\therefore \frac{dy}{dx} = 4\sec(2x)\tan(2x)(2)$$
$$= 8\sec(2x)\tan(2x)$$

d
$$y = e^{-x}\cot\left(\tfrac{x}{2}\right)$$
$$\therefore \frac{dy}{dx} = -e^{-x}\cot\left(\tfrac{x}{2}\right) + e^{-x}\left(-\csc^2\left(\tfrac{x}{2}\right)\right)\left(\tfrac{1}{2}\right)$$
{product rule}
$$= -e^{-x}\left(\cot\left(\tfrac{x}{2}\right) + \tfrac{1}{2}\csc^2\left(\tfrac{x}{2}\right)\right)$$

e
$$y = x^2\csc x$$
$$\therefore \frac{dy}{dx} = 2x\csc x + x^2(-\csc x\cot x) \quad \{\text{product rule}\}$$
$$= x\csc x\,(2 - x\cot x)$$

f
$$y = x\sqrt{\csc x} = x(\csc x)^{\frac{1}{2}}$$
$$\therefore \frac{dy}{dx} = (\csc x)^{\frac{1}{2}} + \tfrac{1}{2}x(\csc x)^{-\frac{1}{2}}(-\csc x\cot x)$$
$$= (\csc x)^{\frac{1}{2}} - \tfrac{1}{2}x(\csc x)^{\frac{1}{2}}\cot x$$
$$= \sqrt{\csc x}\left(1 - \tfrac{1}{2}x\cot x\right)$$

g $y = \ln(\sec x)$

$\therefore \dfrac{dy}{dx} = \dfrac{\frac{d}{dx}(\sec x)}{\sec x}$

$= \dfrac{\sec x \tan x}{\sec x} = \tan x$

h $y = x\csc(x^2)$

$\therefore \dfrac{dy}{dx} = \csc(x^2) + x\left[-\csc(x^2)\cot(x^2)\right](2x)$ {product rule}

$= \csc(x^2)(1 - 2x^2\cot(x^2))$

i $y = \dfrac{\cot x}{\sqrt{x}} = x^{-\frac{1}{2}}\cot x$ $\quad\therefore \dfrac{dy}{dx} = -\frac{1}{2}x^{-\frac{3}{2}}\cot x + x^{-\frac{1}{2}}(-\csc^2 x)$ {product rule}

$= -\dfrac{\cot x + 2x\csc^2 x}{2x\sqrt{x}}$

$= -\dfrac{\cos x\sin x + 2x}{2x\sqrt{x}\sin^2 x}$

EXERCISE 18I

1 If $y = \arccos x$ then $x = \cos y$

$\therefore \dfrac{dx}{dy} = -\sin y = -\sqrt{1 - \cos^2 y} = -\sqrt{1 - x^2}$

$\therefore \dfrac{dy}{dx} = -\dfrac{1}{\sqrt{1-x^2}}, \quad x \in \,]-1, 1[$

2 If $y = \arctan x$ then $x = \tan y$

$\therefore 1 = \sec^2 y\,\dfrac{dy}{dx}$ {differentiating with respect to x}

$\therefore 1 = (1 + \tan^2 y)\,\dfrac{dy}{dx}$

$\therefore 1 = (1 + x^2)\,\dfrac{dy}{dx}$ and so $\dfrac{dy}{dx} = \dfrac{1}{1+x^2}, \quad x \in \mathbb{R}$

3 **a** $y = \arctan(2x)$

$\therefore \dfrac{dy}{dx} = 2 \times \dfrac{1}{1 + (2x)^2} = \dfrac{2}{1 + 4x^2}$

b $y = \arccos(3x)$

$\therefore \dfrac{dy}{dx} = 3 \times \left(\dfrac{-1}{\sqrt{1 - (3x)^2}}\right) = -\dfrac{3}{\sqrt{1 - 9x^2}}$

c $y = \arcsin\left(\frac{x}{4}\right)$

$\therefore \dfrac{dy}{dx} = \frac{1}{4} \times \dfrac{1}{\sqrt{1 - \left(\frac{x}{4}\right)^2}}$

$= \dfrac{1}{4\sqrt{1 - \frac{x^2}{16}}}$

$= \dfrac{1}{\sqrt{16 - x^2}}$

d $y = \arccos\left(\frac{x}{5}\right)$

$\therefore \dfrac{dy}{dx} = \frac{1}{5} \times \left(\dfrac{-1}{\sqrt{1 - \left(\frac{x}{5}\right)^2}}\right)$

$= -\dfrac{1}{5\sqrt{1 - \frac{x^2}{25}}}$

$= -\dfrac{1}{\sqrt{25 - x^2}}$

e $y = \arctan(x^2)$

$\therefore \dfrac{dy}{dx} = 2x \times \dfrac{1}{1 + (x^2)^2}$

$= \dfrac{2x}{1 + x^4}$

f $y = \arccos(\sin x)$

$\therefore \dfrac{dy}{dx} = \cos x \times \left(-\dfrac{1}{\sqrt{1 - \sin^2 x}}\right)$

$= -\dfrac{\cos x}{\cos x}$

$= -1$

4 **a** $y = x \arcsin x \quad \therefore \ \dfrac{dy}{dx} = \arcsin x + x\left(\dfrac{1}{\sqrt{1-x^2}}\right)$ {product rule}

$$= \arcsin x + \frac{x}{\sqrt{1-x^2}}$$

b $y = e^x \arccos x \quad \therefore \ \dfrac{dy}{dx} = e^x \arccos x + e^x\left(-\dfrac{1}{\sqrt{1-x^2}}\right)$ {product rule}

$$= e^x \arccos x - \frac{e^x}{\sqrt{1-x^2}}$$

c $y = e^{-x} \arctan x \quad \therefore \ \dfrac{dy}{dx} = -e^{-x} \arctan x + e^{-x}\left(\dfrac{1}{1+x^2}\right)$ {product rule}

$$= -e^{-x} \arctan x + \frac{e^{-x}}{1+x^2}$$

5 **a**

$$y = \arcsin\left(\tfrac{x}{a}\right)$$

$$\therefore \ \frac{dy}{dx} = \frac{1}{a}\frac{1}{\sqrt{1-\left(\dfrac{x}{a}\right)^2}}$$

$$= \frac{1}{a\sqrt{1-\dfrac{x^2}{a^2}}}$$

$$= \frac{1}{\sqrt{a^2-x^2}} \quad \text{as required,}$$

and this is defined for $x \in \,]-a, a[$.

b

$$y = \arctan\left(\tfrac{x}{a}\right)$$

$$\therefore \ \frac{dy}{dx} = \frac{1}{a} \times \frac{1}{1+\left(\dfrac{x}{a}\right)^2}$$

$$= \frac{a}{a^2} \times \frac{1}{1+\dfrac{x^2}{a^2}}$$

$$= \frac{a}{a^2+x^2} \quad \text{as required,}$$

and this is defined for $x \in \mathbb{R}$.

c

$$y = \arccos\left(\tfrac{x}{a}\right)$$

$$\therefore \ \frac{dy}{dx} = \frac{1}{a} \times -\frac{1}{\sqrt{1-\left(\dfrac{x}{a}\right)^2}} = -\frac{1}{a\sqrt{1-\dfrac{x^2}{a^2}}}$$

$$= -\frac{1}{\sqrt{a^2-x^2}}$$ and this is defined for $x \in \,]-a, a[$.

EXERCISE 18J

1 **a**

$$f(x) = 3x^2 - 6x + 2$$
$$\therefore \ f'(x) = 6x - 6$$
$$\therefore \ f''(x) = 6$$

b

$$f(x) = \frac{2}{\sqrt{x}} - 1 = 2x^{-\frac{1}{2}} - 1$$
$$\therefore \ f'(x) = -x^{-\frac{3}{2}}$$
$$\therefore \ f''(x) = \tfrac{3}{2}x^{-\frac{5}{2}}$$
$$= \frac{3}{2x^{\frac{5}{2}}}$$

c

$$f(x) = 2x^3 - 3x^2 - x + 5$$
$$\therefore \ f'(x) = 6x^2 - 6x - 1$$
$$\therefore \ f''(x) = 12x - 6$$

d

$$f(x) = \frac{2-3x}{x^2} = 2x^{-2} - 3x^{-1}$$
$$\therefore \ f'(x) = -4x^{-3} + 3x^{-2}$$
$$\therefore \ f''(x) = 12x^{-4} - 6x^{-3}$$
$$= \frac{12-6x}{x^4}$$

e
$$\begin{aligned} f(x) &= (1-2x)^3 \\ \therefore\ f'(x) &= 3(1-2x)^2(-2) \\ &= -6(1-2x)^2 \\ \therefore\ f''(x) &= -12(1-2x)^1(-2) \\ &= 24(1-2x) = 24-48x \end{aligned}$$

f $f(x)=\dfrac{x+2}{2x-1}$ is a quotient with $u=x+2$ and $v=2x-1$
$\therefore\ u'=1$ and $v'=2$

$$\begin{aligned} \therefore\ f'(x) &= \frac{1(2x-1)-2(x+2)}{(2x-1)^2} \quad \{\text{quotient rule}\} \\ &= \frac{-5}{(2x-1)^2} \\ &= -5(2x-1)^{-2} \\ \therefore\ f''(x) &= 10(2x-1)^{-3}(2) = \frac{20}{(2x-1)^3} \end{aligned}$$

2 **a**
$$\begin{aligned} y &= x-x^3 \\ \therefore\ \frac{dy}{dx} &= 1-3x^2 \\ \therefore\ \frac{d^2y}{dx^2} &= -6x \end{aligned}$$

b
$$\begin{aligned} y &= x^2-\frac{5}{x^2} \\ &= x^2-5x^{-2} \\ \therefore\ \frac{dy}{dx} &= 2x+10x^{-3} \\ \therefore\ \frac{d^2y}{dx^2} &= 2-30x^{-4} \\ &= 2-\frac{30}{x^4} \end{aligned}$$

c
$$\begin{aligned} y &= 2-\frac{3}{\sqrt{x}} \\ &= 2-3x^{-\frac{1}{2}} \\ \therefore\ \frac{dy}{dx} &= \tfrac{3}{2}x^{-\frac{3}{2}} \\ \therefore\ \frac{d^2y}{dx^2} &= -\tfrac{9}{4}x^{-\frac{5}{2}} \end{aligned}$$

d
$$\begin{aligned} y &= \frac{4-x}{x} \\ &= 4x^{-1}-1 \\ \therefore\ \frac{dy}{dx} &= -4x^{-2} \\ \therefore\ \frac{d^2y}{dx^2} &= 8x^{-3} \\ &= \frac{8}{x^3} \end{aligned}$$

e
$$\begin{aligned} y &= (x^2-3x)^3 \\ \therefore\ \frac{dy}{dx} &= 3(x^2-3x)^2(2x-3) \\ &= (6x-9)(x^2-3x)^2 \quad \text{which is a product where} \end{aligned}$$
$u=6x-9$ and $v=(x^2-3x)^2$
$\therefore\ u'=6$ and $v'=2(x^2-3x)^1(2x-3)$
$$\begin{aligned} \therefore\ \frac{d^2y}{dx^2} &= 6(x^2-3x)^2+(6x-9)(2)(x^2-3x)(2x-3) \\ &= 6(x^2-3x)\left[(x^2-3x)+(2x-3)^2\right] \\ &= 6(x^2-3x)(x^2-3x+4x^2-12x+9) \\ &= 6(x^2-3x)(5x^2-15x+9) \end{aligned}$$

f
$$\begin{aligned} y &= x^2-x+\frac{1}{1-x} \\ &= x^2-x+(1-x)^{-1} \\ \therefore\ \frac{dy}{dx} &= 2x-1+(-1)(1-x)^{-2}(-1) \\ &= 2x-1+(1-x)^{-2} \\ \therefore\ \frac{d^2y}{dx^2} &= 2-2(1-x)^{-3}(-1) \\ &= 2+\frac{2}{(1-x)^3} \end{aligned}$$

3 **a**
$$\begin{aligned} f(x) &= x^3-2x+5 \\ \therefore\ f(2) &= (2)^3-2(2)+5 \\ &= 9 \end{aligned}$$

b
$$\begin{aligned} f(x) &= x^3-2x+5 \\ \therefore\ f'(x) &= 3x^2-2 \\ \therefore\ f'(2) &= 3(2)^2-2 \\ &= 10 \end{aligned}$$

c $f'(x) = 3x^2 - 2$ {from **b**}

$\therefore\ f''(x) = 6x$

$\therefore\ f''(2) = 6(2)$

$= 12$

d $f''(x) = 6x$ {from **c**}

$\therefore\ f^{(3)}(x) = 6$

$\therefore\ f^{(3)}(2) = 6$

4 P_n is "if $y = Ae^{bx}$ where A and b are constants, then $\dfrac{d^n y}{dx^n} = b^n y$, for $n \in \mathbb{Z}^+$".

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, LHS $= \dfrac{dy}{dx} = Ae^{bx} \times b = bAe^{bx} = by$, RHS $= by$ $\therefore$ P_1 is true.

(2) If P_k is true, then $\dfrac{d^k y}{dx^k} = b^k y$ where $k \geqslant 1$, $k \in \mathbb{Z}^+$.

Now,
$$\begin{aligned}\frac{d^{k+1}y}{dx^{k+1}} &= \frac{d}{dx}\left(\frac{d^k y}{dx^k}\right)\\ &= \frac{d}{dx}(b^k y)\\ &= b^k \times \frac{d}{dx}(y) \qquad \{\text{since } b^k \text{ is a constant}\}\\ &= b^k \times bAe^{bx}\\ &= b^{k+1}Ae^{bx}\\ &= b^{k+1}y\end{aligned}$$

Thus P_{k+1} is true whenever P_k is true and P_1 is true

$\therefore$ P_n is true for all $n \geqslant 1$, $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

5 **a** $f(x) = 2x^3 - 6x^2 + 5x + 1$

$\therefore\ f'(x) = 6x^2 - 12x + 5$

$\therefore\ f''(x) = 12x - 12$

So, $f''(x) = 0$ when $12x - 12 = 0$

$\therefore\ 12x = 12$

$\therefore\ x = 1$

b $f(x) = \dfrac{x}{x^2+2}$ is a quotient where $u = x$ and $v = x^2 + 2$

$\therefore\ u' = 1$ and $v' = 2x$

$$\therefore\ f'(x) = \frac{1(x^2+2) - 2x^2}{(x^2+2)^2} \quad \{\text{quotient rule}\}$$
$$= \frac{2 - x^2}{(x^2+2)^2}$$

This is another quotient, this time with $u = 2 - x^2$ and $v = (x^2+2)^2$

$\therefore\ u' = -2x$ and $v' = 2(x^2+2)(2x)$

$$\begin{aligned}\therefore\ f''(x) &= \frac{-2x(x^2+2)^2 - 4x(x^2+2)(2-x^2)}{(x^2+2)^4}\\ &= \frac{-2x(x^2+2)[x^2+2+2(2-x^2)]}{(x^2+2)^4}\\ &= \frac{-2x(-x^2+6)}{(x^2+2)^3}\\ &= \frac{2x(x^2-6)}{(x^2+2)^3}\end{aligned}$$

So, $f''(x) = 0$ when $2x(x^2 - 6) = 0$

$\therefore\ x = 0$ or $x^2 - 6 = 0$

$\therefore\ x = 0$ or $x = \pm\sqrt{6}$

6 $f(x) = 2x^3 - x$

$\therefore\ f'(x) = 6x^2 - 1$

$\therefore\ f''(x) = 12x$

By substituting the various values of x into these three functions, we can fill in the table as follows:

x	-1	0	1
$f(x)$	$-$	0	$+$
$f'(x)$	$+$	$-$	$+$
$f''(x)$	$-$	0	$+$

7
$$\begin{aligned} f(x) &= \tfrac{2}{3}\sin 3x \\ \therefore\ f'(x) &= \tfrac{2}{3} \times (\cos 3x) \times 3 \\ &= 2\cos 3x \\ \therefore\ f''(x) &= 2 \times (-\sin 3x) \times 3 \\ &= -6\sin 3x \\ \therefore\ f^{(3)}(x) &= -6 \times (\cos 3x) \times 3 \\ &= -18\cos 3x \\ \therefore\ f^{(3)}\left(\tfrac{2\pi}{9}\right) &= -18\cos 3\left(\tfrac{2\pi}{9}\right) \\ &= -18\cos\left(\tfrac{2\pi}{3}\right) \\ &= 9 \end{aligned}$$

8 **a**
$$\begin{aligned} f(x) &= 2\sin^3 x - 3\sin x \\ &= 2(\sin x)^3 - 3\sin x \\ \therefore\ f'(x) &= 2 \times 3(\sin x)^2 \times (\cos x) - 3\cos x \\ &= -3\cos x(1 - 2\sin^2 x) \\ &= -3\cos x \cos 2x \end{aligned}$$

b
$$\begin{aligned} f''(x) &= -3(-\sin x \times \cos 2x + \cos x \times (-2)\sin 2x) \\ &= 3\sin x\cos 2x + 6\cos x \sin 2x \end{aligned}$$

9 **a**
$$\begin{aligned} y &= -\ln x \\ \therefore\ \frac{dy}{dx} &= -1 \times \frac{1}{x} \\ &= -x^{-1} \\ \therefore\ \frac{d^2y}{dx^2} &= -\left(-x^{-2}\right) \\ &= x^{-2} = \frac{1}{x^2} \end{aligned}$$

b $y = x\ln x$ is the product of $u = x$ and $v = \ln x$

$\therefore\ u' = 1$ and $v' = \dfrac{1}{x}$

$$\begin{aligned} \text{Now}\ \frac{dy}{dx} &= u'v + uv' \quad \{\text{product rule}\} \\ &= 1 \times \ln x + x \times \frac{1}{x} \\ &= \ln x + 1 \\ \therefore\ \frac{d^2y}{dx^2} &= \frac{1}{x} \end{aligned}$$

c $y = (\ln x)^2$

$\therefore\ \dfrac{dy}{dx} = 2(\ln x)\left(\dfrac{1}{x}\right) = \dfrac{2\ln x}{x}$ which is a quotient with $u = 2\ln x$ and $v = x$

$\therefore\ u' = \dfrac{2}{x}$ and $v' = 1$

$$\begin{aligned} \therefore\ \frac{d^2y}{dx^2} &= \frac{u'v - uv'}{v^2} \quad \{\text{quotient rule}\} \\ &= \frac{\frac{2}{x} \times x - 2\ln x \times 1}{x^2} \\ &= \frac{2 - 2\ln x}{x^2} = \frac{2}{x^2}(1 - \ln x) \end{aligned}$$

10 **a**
$$\begin{aligned} f(x) &= x^2 - \frac{1}{x} \\ \therefore\ f(1) &= (1)^2 - \frac{1}{1} \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

b
$$\begin{aligned} f(x) &= x^2 - \frac{1}{x} \\ &= x^2 - x^{-1} \\ \therefore\ f'(x) &= 2x - \left(-x^{-2}\right) \\ &= 2x + x^{-2} \\ &= 2x + \frac{1}{x^2} \\ \therefore\ f'(1) &= 2(1) + \frac{1}{1^2} \\ &= 2 + 1 = 3 \end{aligned}$$

c $f'(x) = 2x + x^{-2}$ {from **b**}

$\therefore\ f''(x) = 2 - 2x^{-3}$

$= 2 - \dfrac{2}{x^3}$

$\therefore\ f''(1) = 2 - \dfrac{2}{1^3}$

$= 2 - 2 = 0$

d $f''(x) = 2 - 2x^{-3}$ {from **c**}

$\therefore\ f^{(3)}(x) = 6x^{-4}$

$= \dfrac{6}{x^4}$

$\therefore\ f^{(3)}(1) = \dfrac{6}{1^4} = 6$

11 $y = 2e^{3x} + 5e^{4x}$ $\quad\therefore\ \dfrac{dy}{dx} = 6e^{3x} + 20e^{4x}$ and $\dfrac{d^2y}{dx^2} = 18e^{3x} + 80e^{4x}$

$$\begin{aligned}\text{Now}\quad \frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 12y &= \left(18e^{3x} + 80e^{4x}\right) - 7\left(6e^{3x} + 20e^{4x}\right) + 12\left(2e^{3x} + 5e^{4x}\right)\\ &= 18e^{3x} + 80e^{4x} - 42e^{3x} - 140e^{4x} + 24e^{3x} + 60e^{4x}\\ &= e^{3x}\left[18 - 42 + 24\right] + e^{4x}\left[80 - 140 + 60\right]\\ &= e^{3x}(0) + e^{4x}(0)\\ &= 0\end{aligned}$$

$\therefore\ \dfrac{d^2y}{dx^2} - 7\dfrac{dy}{dx} + 12y = 0$

12 If $y = \sin(2x+3)$, then $\dfrac{dy}{dx} = 2\cos(2x+3)$ and $\dfrac{d^2y}{dx^2} = -4\sin(2x+3)$

$\therefore\ \dfrac{d^2y}{dx^2} + 4y = -4\sin(2x+3) + 4\sin(2x+3) = 0$

13 $y = \sin x$

$\therefore\ \dfrac{dy}{dx} = \cos x$

$\therefore\ \dfrac{d^2y}{dx^2} = -\sin x$

$\therefore\ \dfrac{d^3y}{dx^3} = -\cos x$

$\therefore\ \dfrac{d^4y}{dx^4} = -(-\sin x)$

$= \sin x$

$= y$

14 If $y = 2\sin x + 3\cos x$, then $y' = 2\cos x - 3\sin x$ and $y'' = -2\sin x - 3\cos x$

$\therefore\ y'' + y = -2\sin x - 3\cos x + 2\sin x + 3\cos x = 0$

15 **a** $x^2 + y^2 = 25$

$\therefore\ 2x + 2y\dfrac{dy}{dx} = 0$

$\therefore\ \dfrac{dy}{dx} = -\dfrac{x}{y}$

$$\therefore\ \frac{d^2y}{dx^2} = \frac{(-1)y - (-x)(1)\frac{dy}{dx}}{y^2} = \frac{-y + x\left(-\frac{x}{y}\right)}{y^2} = \frac{-y - \frac{x^2}{y}}{y^2} = \frac{-y^2 - x^2}{y^3}$$

b $x^2 - y^2 = 10$

$\therefore\ 2x - 2y\dfrac{dy}{dx} = 0$

$\therefore\ \dfrac{dy}{dx} = \dfrac{x}{y}$

$$\therefore\ \frac{d^2y}{dx^2} = \frac{(1)y - x(1)\frac{dy}{dx}}{y^2} = \frac{y - x\left(\frac{x}{y}\right)}{y^2} = \frac{y - \frac{x^2}{y}}{y^2} = \frac{y^2 - x^2}{y^3}$$

c
$$x^3 + 2xy = 4$$
$$\therefore\ 3x^2 + 2y + 2x(1)\frac{dy}{dx} = 0$$
$$\therefore\ \frac{dy}{dx} = \frac{-3x^2 - 2y}{2x}$$
$$\therefore\ \frac{d^2y}{dx^2} = \frac{(-6x - 2\frac{dy}{dx})(2x) - (-3x^2 - 2y)(2)}{(2x)^2}$$
$$= \frac{-12x^2 - 2\left(\frac{-3x^2-2y}{\cancel{2x}}\right)(\cancel{2x}) - 2(-3x^2 - 2y)}{4x^2}$$
$$= \frac{-12x^2 + 6x^2 + 4y + 6x^2 + 4y}{4x^2}$$
$$= \frac{8y}{4x^2} = \frac{2y}{x^2}$$

16 **a**
$$3V^2 + 2q = 2Vq$$
Differentiating with respect to q,
$$\therefore\ 6V\frac{dV}{dq} + 2 = 2\frac{dV}{dq}q + 2V$$
$$\therefore\ \frac{dV}{dq}(6V - 2q) = 2V - 2$$
$$\therefore\ \frac{dV}{dq} = \frac{2V-2}{6V-2q} = \frac{V-1}{3V-q}$$

b
$$3V^2 + 2q = 2Vq$$
Differentiating with respect to V,
$$6V + 2\frac{dq}{dV} = 2q + 2V\frac{dq}{dV} \quad \text{.... (1)}$$
$$\therefore\ (2-2V)\frac{dq}{dV} = 2q - 6V$$
$$\therefore\ \frac{dq}{dV} = \frac{2q-6V}{2-2V} = \frac{q-3V}{1-V} \quad \text{.... (2)}$$
Now differentiating (1) with respect to V,
$$6 + 2\frac{d^2q}{dV^2} = 2\frac{dq}{dV} + 2\frac{dq}{dV} + 2V\frac{d^2q}{dV^2}$$
$$\therefore\ (2-2V)\frac{d^2q}{dV^2} = 4\frac{dq}{dV} - 6$$
$$= 4\left(\frac{q-3V}{1-V}\right) - 6 \quad \{\text{using (2)}\}$$
$$= \frac{4q - 12V - 6(1-V)}{1-V}$$
$$= \frac{4q - 6V - 6}{1-V}$$
$$\therefore\ \frac{d^2q}{dV^2} = \frac{4q-6V-6}{(2-2V)(1-V)} = \frac{2q-3V-3}{(1-V)^2}$$

17 $f(x) = e^{ax}(x+1), \quad a \in \mathbb{R}$

a $f'(x) = ae^{ax}(x+1) + e^{ax}(1)$ {product rule}
$$= e^{ax}(a[x+1]+1)$$

b $f''(x) = ae^{ax}(a[x+1]+1) + e^{ax}(a)$ {product rule}
$$= ae^{ax}(a[x+1]+1+1)$$
$$= ae^{ax}(a[x+1]+2)$$

c If $f^{(k)}(x) = a^{k-1}e^{ax}(a[x+1]+k)$

then $f^{(k+1)}(x) = a^{k-1}ae^{ax}(a[x+1]+k) + a^{k-1}e^{ax}(a)$ {product rule}
$$= a^k e^{ax}(a[x+1]+k) + a^k e^{ax}$$
$$= a^k e^{ax}(a[x+1]+[k+1])$$

18 $f(x) = e^{-x}(x+2)$

a **i** $f'(x) = -e^{-x}(x+2) + e^{-x}(1)$
$= -e^{-x}(x+2-1)$
$= -e^{-x}(x+1)$

ii $f''(x) = e^{-x}(x+1) - e^{-x}(1)$
$= e^{-x}(x+1-1)$
$= e^{-x}(x)$

iii $f'''(x) = -e^{-x}(x) + e^{-x}(1)$
$= -e^{-x}(x-1)$

iv $f^{(4)}(x) = e^{-x}(x-1) - e^{-x}(1)$
$= e^{-x}(x-1-1)$
$= e^{-x}(x-2)$

b $f^{(n)}(x) = (-1)^n e^{-x}\left(x-(n-2)\right)$
$= (-1)^n e^{-x}(x-n+2)$

c P_n is "for $f(x) = e^{-x}(x+2)$, $f^{(n)}(x) = (-1)^n e^{-x}(x-n+2)$", $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) For $n = 1$, $f'(x) = -e^{-x}(x+1)$ {using **a i**}
$= (-1)^1 e^{-x}(x-1+2)$ $\therefore$ P_1 is true.

(2) If P_k is true then $f^{(k)}(x) = (-1)^k e^{-x}(x-k+2)$

$\therefore$ $f^{(k+1)}(x) = (-1)^k(-1)e^{-x}(x-k+2) + (-1)^k e^{-x}(1)$ {product rule}
$= (-1)^{k+1} e^{-x}(x-k+2) + (-1)(-1)^k e^{-x}(-1)$
$= (-1)^{k+1} e^{-x}(x-k+2-1)$
$= (-1)^{k+1} e^{-x}(x-(k+1)+2)$

Thus P_{k+1} is true whenever P_k is true

$\therefore$ since P_1 is true, P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

REVIEW SET 18A

1 **a** $f(x) = 7 + x - 3x^2$
$\therefore$ $f(3) = 7 + 3 - 3(3)^2 = -17$

b $f'(x) = 1 - 6x$
$\therefore$ $f'(3) = 1 - 6(3) = -17$

c $f''(x) = -6$
$\therefore$ $f''(3) = -6$

2 **a** $y = 3x^2 - x^4$
$\therefore$ $\dfrac{dy}{dx} = 6x - 4x^3$

b $y = \dfrac{x^3 - x}{x^2} = x - x^{-1}$
$\therefore$ $\dfrac{dy}{dx} = 1 + x^{-2} = 1 + \dfrac{1}{x^2}$

3 $f(x) = \dfrac{x}{\sqrt{x^2+1}}$ is a quotient with $u = x$ and $v = \left(x^2+1\right)^{\frac{1}{2}}$

$\therefore$ $u' = 1$ and $v' = \frac{1}{2}\left(x^2+1\right)^{-\frac{1}{2}} \times 2x$
$= x\left(x^2+1\right)^{-\frac{1}{2}}$

Now $f'(x) = \dfrac{u'v - uv'}{v^2}$ {quotient rule}

$$= \frac{1 \times \left(x^2+1\right)^{\frac{1}{2}} - x \times x\left(x^2+1\right)^{-\frac{1}{2}}}{x^2+1}$$

$$= \frac{\sqrt{x^2+1} - \frac{x^2}{\sqrt{x^2+1}}}{x^2+1}$$

$$= \frac{(x^2+1) - x^2}{(x^2+1)\sqrt{x^2+1}}$$

$$= \frac{1}{(x^2+1)\sqrt{x^2+1}}$$

The tangent to $f(x)$ has gradient 1 when $f'(x) = 1$

$$\therefore \quad \frac{1}{(x^2+1)\sqrt{x^2+1}} = 1$$
$$\therefore \quad \left(x^2+1\right)^{\frac{3}{2}} = 1$$
$$\therefore \quad x^2+1 = 1$$
$$\therefore \quad x^2 = 0$$
$$\therefore \quad x = 0$$
$$\text{and} \quad f(0) = \frac{0}{\sqrt{0^2+1}} = 0$$

$\therefore$ the tangent to $f(x) = \dfrac{x}{\sqrt{x^2+1}}$ has gradient 1 at the point (0, 0).

4 **a**
$$\begin{aligned} y &= e^{x^3+2} \\ &= e^u \quad \text{where} \quad u = x^3+2 \\ \therefore \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \quad \{\text{chain rule}\} \\ &= e^u \times 3x^2 \\ &= 3x^2 e^u \\ &= 3x^2 e^{x^3+2} \end{aligned}$$

b
$$\begin{aligned} y &= \ln\left(\frac{x+3}{x^2}\right) \\ &= \ln(x+3) - \ln(x^2) \\ \therefore \frac{dy}{dx} &= \frac{1}{x+3} - \frac{2x}{x^2} \\ &= \frac{1}{x+3} - \frac{2}{x} \end{aligned}$$

c Consider $\ln(2y+1) = xe^y$

Differentiating with respect to x,
$$\begin{aligned} \frac{d}{dx}(\ln(2y+1)) &= \frac{d}{dx}(xe^y) \\ \therefore \frac{2}{2y+1}\frac{dy}{dx} &= \frac{d}{dx}(x)e^y + x\frac{d}{dx}(e^y) \\ \therefore \frac{2}{2y+1}\frac{dy}{dx} &= e^y + xe^y\frac{dy}{dx} \\ \therefore \frac{dy}{dx}\left(\frac{2}{2y+1} - xe^y\right) &= e^y \\ \therefore \frac{dy}{dx}(2 - xe^y(2y+1)) &= e^y(2y+1) \\ \therefore \frac{dy}{dx} &= \frac{e^y(2y+1)}{2-xe^y(2y+1)} \end{aligned}$$

5
$$\begin{aligned} y &= 3e^x - e^{-x} \\ \therefore \frac{dy}{dx} &= 3e^x - \left(-e^{-x}\right) \\ &= 3e^x + e^{-x} \end{aligned}$$
$$\begin{aligned} \therefore \frac{d^2y}{dx^2} &= 3e^x + \left(-e^{-x}\right) \\ &= 3e^x - e^{-x} \\ &= y \end{aligned}$$

6 **a**
$$\begin{aligned} \frac{d}{dx}(\sin(5x)\ln x) &= \frac{d}{dx}(\sin(5x))\ln x + \sin(5x)\frac{d}{dx}(\ln x) \quad \{\text{product rule}\} \\ &= 5\cos(5x)\ln x + \frac{\sin(5x)}{x} \end{aligned}$$

b
$$\begin{aligned} \frac{d}{dx}(\sin x\cos(2x)) &= \frac{d}{dx}(\sin x)\cos(2x) + \sin x\frac{d}{dx}(\cos(2x)) \quad \{\text{product rule}\} \\ &= \cos x\cos(2x) + \sin x(-2\sin(2x)) \\ &= \cos x\cos(2x) - 2\sin x\sin(2x) \end{aligned}$$

c
$$\begin{aligned} \frac{d}{dx}\left(e^{-2x}\tan x\right) &= \frac{d}{dx}\left(e^{-2x}\right)\tan x + e^{-2x}\frac{d}{dx}(\tan x) \quad \{\text{product rule}\} \\ &= -2e^{-2x}\tan x + \frac{e^{-2x}}{\cos^2 x} \end{aligned}$$

7 $y = \sin^2 x$

$= (\sin x)^2 = u^2$ where $u = \sin x$

$\therefore \dfrac{du}{dx} = \cos x$

Now $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$ {chain rule}

$= 2u\dfrac{du}{dx}$

$= 2\sin x \cos x$

When $x = \frac{\pi}{3}$, $\dfrac{dy}{dx} = 2\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{3}\right)$

$= \frac{\sqrt{3}}{2}$

$\therefore$ gradient of tangent $= \frac{\sqrt{3}}{2}$

8

$$x^2 + 2xy + y^2 = 4$$

$$\therefore 2x + 2y + 2x(1)\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = \frac{-x-y}{x+y} = \frac{-(x+y)}{x+y} = -1$$

$$\therefore \frac{d^2y}{dx^2} = 0$$

9 **a**

$$M = (t^2+3)^4$$

$$\therefore \frac{dM}{dt} = 4(t^2+3)^3(2t) = 8t(t^2+3)^3$$

b $A = \dfrac{\sqrt{t+5}}{t^2}$ is a quotient with

$u = (t+5)^{\frac{1}{2}}$ and $v = t^2$

$\therefore u' = \frac{1}{2}(t+5)^{-\frac{1}{2}}$, $v' = 2t$

$$\therefore \frac{dA}{dt} = \frac{\frac{1}{2}(t+5)^{-\frac{1}{2}}(t^2) - (t+5)^{\frac{1}{2}}(2t)}{t^4} = \frac{\frac{1}{2}t(t+5)^{-\frac{1}{2}} - 2(t+5)^{\frac{1}{2}}}{t^3}$$

10 **a**

$$y = \frac{4}{\sqrt{x}} - 3x = 4x^{-\frac{1}{2}} - 3x$$

$$\therefore \frac{dy}{dx} = -2x^{-\frac{3}{2}} - 3 = \frac{-2}{x\sqrt{x}} - 3$$

b

$$y = \sqrt{x^2-3x} = (x^2-3x)^{\frac{1}{2}}$$

$$\therefore \frac{dy}{dx} = \tfrac{1}{2}(x^2-3x)^{-\frac{1}{2}}(2x-3) = \frac{2x-3}{2\sqrt{x^2-3x}}$$

11 **a**

$$f(x) = 3x^2 - \frac{1}{x} = 3x^2 - x^{-1}$$

$$\therefore f'(x) = 6x + x^{-2}$$

$$\therefore f''(x) = 6 - 2x^{-3} = 6 - \frac{2}{x^3}$$

$$\therefore f''(2) = 6 - \frac{2}{2^3} = \frac{23}{4}$$

b

$$f(x) = \sqrt{x} = x^{\frac{1}{2}}$$

$$\therefore f'(x) = \tfrac{1}{2}x^{-\frac{1}{2}}$$

$$\therefore f''(x) = -\tfrac{1}{4}x^{-\frac{3}{2}}$$

$$\therefore f''(2) = -\tfrac{1}{4}(2^{-\frac{3}{2}}) = -\frac{1}{4\sqrt{2^3}} = -\frac{1}{8\sqrt{2}}$$

12

$$y = \left(1 - \tfrac{1}{3}x\right)^3$$

$$\therefore \frac{dy}{dx} = 3\left(1-\tfrac{1}{3}x\right)^2\left(-\tfrac{1}{3}\right) = -\left(1-\tfrac{1}{3}x\right)^2$$

$$\therefore \frac{d^2y}{dx^2} = -2\left(1-\tfrac{1}{3}x\right)\left(-\tfrac{1}{3}\right) = \tfrac{2}{3}\left(1-\tfrac{1}{3}x\right) = \tfrac{2}{3} - \tfrac{2}{9}x$$

$$\therefore \frac{d^3y}{dx^3} = -\tfrac{2}{9}$$

13 P_n is "for $y = \dfrac{1}{2x+1}$, $\dfrac{d^n y}{dx^n} = \dfrac{(-2)^n n!}{(2x+1)^{n+1}}$", $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) $y = \dfrac{1}{2x+1} = (2x+1)^{-1}$

$$\therefore \frac{dy}{dx} = -(2x+1)^{-2}(2) = \frac{-2}{(2x+1)^2}$$

$$= \frac{(-2)^1 1!}{(2x+1)^{1+1}} \qquad \therefore \; P_1 \text{ is true}$$

(2) If P_k is true then $\dfrac{d^k y}{dx^k} = \dfrac{(-2)^k k!}{(2x+1)^{k+1}}$

Now
$$\begin{aligned}\frac{d^{k+1}y}{dx^{k+1}} &= \frac{d}{dx}\left(\frac{d^k y}{dx^k}\right)\\ &= \frac{d}{dx}\left(\frac{(-2)^k k!}{(2x+1)^{k+1}}\right) \qquad \{\text{using } P_k\}\\ &= \frac{d}{dx}\left[(-2)^k k!\,(2x+1)^{-(k+1)}\right]\\ &= (-2)^k k!\,[-(k+1)]\,(2x+1)^{-(k+1)-1}\,(2)\\ &= -2(-2)^k (k+1)!\,(2x+1)^{-(k+2)}\\ &= \frac{(-2)^{k+1}(k+1)!}{(2x+1)^{(k+1)+1}}\end{aligned}$$

Thus P_{k+1} is true whenever P_k is true, and P_1 is true

$\therefore$ P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

REVIEW SET 18B

1 **a** $y = 5x - 3x^{-1}$

$\therefore \dfrac{dy}{dx} = 5 + 3x^{-2}$

b $y = (3x^2 + x)^4$

$\therefore \dfrac{dy}{dx} = 4(3x^2+x)^3(6x+1)$

c $y = (x^2+1)(1-x^2)^3$ is a product with $u = x^2 + 1$ and $v = (1-x^2)^3$

$\therefore u' = 2x$ and $v' = 3(1-x^2)^2(-2x)$
$= -6x(1-x^2)^2$

$$\therefore \frac{dy}{dx} = 2x(1-x^2)^3 + (x^2+1) \times -6x(1-x^2)^2 \qquad \{\text{product rule}\}$$
$$= 2x(1-x^2)^3 - 6x(x^2+1)(1-x^2)^2$$

2 $y = 2x^3 + 3x^2 - 10x + 3$

$\therefore \dfrac{dy}{dx} = 6x^2 + 6x - 10$

The gradient of the tangent is 2 where $6x^2 + 6x - 10 = 2$

$\therefore 6x^2 + 6x - 12 = 0$

$\therefore 6(x^2 + x - 2) = 0$

$\therefore 6(x+2)(x-1) = 0$

$\therefore x = -2$ or 1

$\therefore y = 2(-2)^3 + 3(-2)^2 - 10(-2) + 3$ *or* $y = 2(1)^3 + 3(1)^2 - 10(1) + 3$

$= -16 + 12 + 20 + 3$ $\qquad = 2 + 3 - 10 + 3$

$= 19$ $\qquad = -2$

So, the gradient of the tangent is 2 at $(-2, 19)$ and $(1, -2)$.

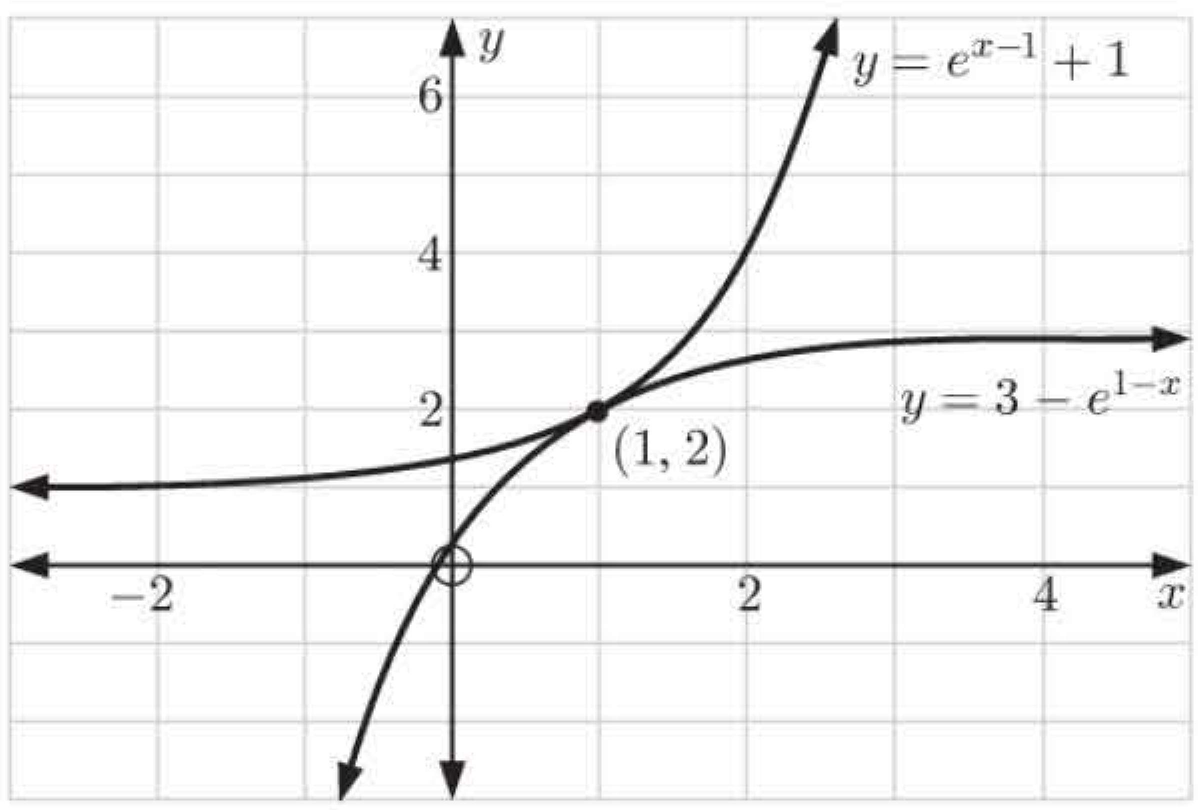
3 $y = \sqrt{5-4x} = (5-4x)^{\frac{1}{2}}$

a $\dfrac{dy}{dx} = \frac{1}{2}(5-4x)^{-\frac{1}{2}}(-4)$
$= -2(5-4x)^{-\frac{1}{2}}$

b $\dfrac{d^2y}{dx^2} = -2(-\frac{1}{2})(5-4x)^{-\frac{3}{2}}(-4)$
$= -4(5-4x)^{-\frac{3}{2}}$

c $\dfrac{d^3y}{dx^3} = -4(-\frac{3}{2})(5-4x)^{-\frac{5}{2}}(-4)$
$= -24(5-4x)^{-\frac{5}{2}}$

4 **a**



b Using technology, the point of intersection is $(1, 2)$.

c If $y = e^{x-1}+1$

then $\dfrac{dy}{dx} = e^{x-1}$

When $x = 1$, $\dfrac{dy}{dx} = e^{1-1}$
$= e^0$
$= 1$

$\therefore$ gradient of tangent $= 1$

If $y = 3 - e^{1-x}$

then $\dfrac{dy}{dx} = -e^{1-x} \times -1$
$= e^{1-x}$

When $x = 1$, $\dfrac{dy}{dx} = e^{1-1}$
$= e^0$
$= 1$

$\therefore$ gradient of tangent $= 1$

So, the tangents to each curve at $(1, 2)$ both have a gradient of 1.

d If the tangents of each curve at $(1, 2)$ have the same gradient, then they are in fact the same line. That is, the two curves have a *common tangent* at $(1, 2)$.

5 **a** $y = \ln(x^3 - 3x)$

$\therefore \dfrac{dy}{dx} = \dfrac{3x^2-3}{x^3-3x}$

b $y = \dfrac{e^x}{x^2}$

$\therefore \dfrac{dy}{dx} = \dfrac{e^x(x^2) - e^x(2x)}{x^4}$ {quotient rule}

$= \dfrac{e^x(x-2)}{x^3}$

c Consider $e^{x+y} = \ln(y^2+1)$.

Differentiating with respect to x, $\left(1 + \dfrac{dy}{dx}\right)e^{x+y} = \dfrac{2y}{y^2+1}\dfrac{dy}{dx}$

$$\therefore \left(1 + \frac{dy}{dx}\right)e^{x+y}(y^2+1) = 2y\frac{dy}{dx}$$

$$\therefore e^{x+y}(y^2+1) = \frac{dy}{dx}(2y - e^{x+y}(y^2+1))$$

$$\therefore \frac{dy}{dx} = \frac{e^{x+y}(y^2+1)}{2y - e^{x+y}(y^2+1)}$$

6 $f(x) = 2x^4 - 4x^3 - 9x^2 + 4x + 7$

$\therefore f'(x) = 8x^3 - 12x^2 - 18x + 4$

$\therefore f''(x) = 24x^2 - 24x - 18$

So, $f''(x) = 0$ where $24x^2 - 24x - 18 = 0$

$\therefore 4x^2 - 4x - 3 = 0$

$\therefore (2x+1)(2x-3) = 0$

$\therefore x = -\frac{1}{2}$ or $x = \frac{3}{2}$

7 **a** $f(x) = x - \cos x$
$\therefore\ f(\pi) = \pi - \cos\pi$
$= \pi - (-1)$
$= \pi + 1$

b $f(x) = x - \cos x$
$\therefore\ f'(x) = 1 - (-\sin x)$
$= 1 + \sin x$
$\therefore\ f'\left(\frac{\pi}{2}\right) = 1 + \sin\left(\frac{\pi}{2}\right)$
$= 1 + 1 = 2$

c $f'(x) = 1 + \sin x$ {from **b**}
$\therefore\ f''(x) = \cos x$
$\therefore\ f''\left(\frac{3\pi}{4}\right) = \cos\left(\frac{3\pi}{4}\right)$
$= -\frac{1}{\sqrt{2}}$ $\left(\text{or } -\frac{\sqrt{2}}{2}\right)$

8 $y = 3\sin bx - a\cos 2x$

$\therefore\ \dfrac{dy}{dx} = 3 \times (\cos bx) \times b - a \times (-\sin 2x) \times 2$
$= 3b\cos bx + 2a\sin 2x$

$\therefore\ \dfrac{d^2y}{dx^2} = 3b \times (-\sin bx) \times b + 2a \times (\cos 2x) \times 2$
$= -3b^2\sin bx + 4a\cos 2x$

Now $y + \dfrac{d^2y}{dx^2} = 6\cos 2x$

$\therefore\ 3\sin bx - a\cos 2x + 4a\cos 2x - 3b^2\sin bx = 6\cos 2x$
$\therefore\ 3\sin bx - 3b^2\sin bx + 3a\cos 2x = 6\cos 2x$
$\therefore\ (3 - 3b^2)\sin bx + 3a\cos 2x = 6\cos 2x$

$\therefore\ 3 - 3b^2 = 0$ and $3a = 6$
$\therefore\ 3(1 - b^2) = 0$ and $a = 2$
$\therefore\ 3(1 + b)(1 - b) = 0$
$\therefore\ 1 + b = 0$ or $1 - b = 0$
$\therefore\ b = -1$ or 1

So, $a = 2$ and $b = -1$ or 1.

9 **a** $\dfrac{d}{dx}(10x - \sin(10x)) = 10 - 10\cos(10x)$

b $\dfrac{d}{dx}\left(\ln\left(\dfrac{1}{\cos x}\right)\right) = \dfrac{1}{\left(\frac{1}{\cos x}\right)} \times \dfrac{d}{dx}\left(\dfrac{1}{\cos x}\right)$ {chain rule}

$= \cos x \times \dfrac{d}{dx}\left((\cos x)^{-1}\right)$

$= \cos x \times \left(-(\cos x)^{-2} \times (-\sin x)\right)$

$= \dfrac{\cos x \sin x}{\cos^2 x} = \tan x$

c $\dfrac{d}{dx}(\sin(5x)\ln(2x)) = \dfrac{d}{dx}(\sin(5x))\ln(2x) + \sin(5x)\dfrac{d}{dx}(\ln(2x))$ {product rule}

$= 5\cos(5x)\ln(2x) + \sin(5x) \times \dfrac{2}{2x}$

$= 5\cos(5x)\ln(2x) + \dfrac{\sin(5x)}{x}$

10 **a** $f(x) = \dfrac{(x+3)^3}{\sqrt{x}}$ is a quotient with $u = (x+3)^3$ and $v = x^{\frac{1}{2}}$

$\therefore\ u' = 3(x+3)^2$ and $v' = \frac{1}{2}x^{-\frac{1}{2}}$

$\therefore\ f'(x) = \dfrac{3(x+3)^2\sqrt{x} - \frac{1}{2}x^{-\frac{1}{2}}(x+3)^3}{x}$ {quotient rule}

b $f(x) = x^4\sqrt{x^2+3}$ is a product with $u = x^4$ and $v = (x^2+3)^{\frac{1}{2}}$

$\therefore\ u' = 4x^3$ and $v' = \frac{1}{2}(x^2+3)^{-\frac{1}{2}}(2x) = x(x^2+3)^{-\frac{1}{2}}$

$\therefore\ f'(x) = 4x^3\sqrt{x^2+3} + \dfrac{x^5}{\sqrt{x^2+3}}$ {product rule}

11 **a** $y = \dfrac{x}{\sqrt{\sec x}} = x(\cos x)^{\frac{1}{2}}$ $\quad \therefore \dfrac{dy}{dx} = (\cos x)^{\frac{1}{2}} + x \times \frac{1}{2}(\cos x)^{-\frac{1}{2}}(-\sin x)$ {product rule}

$$= \sqrt{\cos x} - \frac{x \sin x}{2\sqrt{\cos x}}$$

b $y = e^x \cot(2x)$ $\quad \therefore \dfrac{dy}{dx} = e^x \cot(2x) + e^x \times 2\left(-\csc^2(2x)\right)$ {product rule}

$$= e^x\left(\cot(2x) - 2\csc^2(2x)\right)$$

c $y = \arccos\left(\frac{x}{2}\right)$ $\quad \therefore \dfrac{dy}{dx} = \frac{1}{2} \times \left(\dfrac{-1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}\right) = -\dfrac{1}{2\sqrt{1-\frac{x^2}{4}}} = -\dfrac{1}{\sqrt{4-x^2}}$

12 $f(x) = 2x^3 + Ax + B$ $\quad \therefore f'(x) = 6x^2 + A$

Now as the gradient at $(-2, 33)$ is 10,

$\therefore f'(-2) = 10$

$\therefore 10 = 6(-2)^2 + A$

$\therefore A = -14$

$\therefore f(x) = 2x^3 - 14x + B$

Then, since $(-2, 33)$ lies on the curve,

$f(-2) = 33$

$\therefore 2(-2)^3 - 14(-2) + B = 33$

$\therefore -16 + 28 + B = 33$

$\therefore B = 21$

13
$$x^2 - 3y^2 = 0$$
$$\therefore 2x - 6y\frac{dy}{dx} = 0$$
$$\therefore \frac{dy}{dx} = \frac{x}{3y}$$

For any point (x, y) on $x^2 - 3y^2 = 0$, $\dfrac{dy}{dx}$ gives us the gradient of the tangent at that point.

REVIEW SET 18C

1 **a** $y = x^3\sqrt{1-x^2}$ is a product where $u = x^3$ and $v = \left(1-x^2\right)^{\frac{1}{2}}$

$\therefore u' = 3x^2$ and $v' = \frac{1}{2}\left(1-x^2\right)^{-\frac{1}{2}}(-2x)$

$= -x\left(1-x^2\right)^{-\frac{1}{2}}$

$$\therefore \frac{dy}{dx} = 3x^2\left(1-x^2\right)^{\frac{1}{2}} + x^3 \times -x\left(1-x^2\right)^{-\frac{1}{2}} \quad \text{\{product rule\}}$$
$$= 3x^2\left(1-x^2\right)^{\frac{1}{2}} - x^4\left(1-x^2\right)^{-\frac{1}{2}}$$

b $y = \dfrac{x^2 - 3x}{\sqrt{x+1}}$ is a quotient where $u = x^2 - 3x$ and $v = (x+1)^{\frac{1}{2}}$

$\therefore u' = 2x - 3$ and $v' = \frac{1}{2}(x+1)^{-\frac{1}{2}}$

$$\therefore \frac{dy}{dx} = \frac{(2x-3)(x+1)^{\frac{1}{2}} - \frac{1}{2}(x^2-3x)(x+1)^{-\frac{1}{2}}}{x+1} \quad \text{\{quotient rule\}}$$

2 **a**
$$y = 3x^4 - \frac{2}{x} = 3x^4 - 2x^{-1}$$
$$\therefore \frac{dy}{dx} = 12x^3 + 2x^{-2}$$
$$\therefore \frac{d^2y}{dx^2} = 36x^2 - 4x^{-3}$$
$$= 36x^2 - \frac{4}{x^3}$$

b
$$y = x^3 - x + \frac{1}{\sqrt{x}} = x^3 - x + x^{-\frac{1}{2}}$$
$$\therefore \frac{dy}{dx} = 3x^2 - 1 - \tfrac{1}{2}x^{-\frac{3}{2}}$$
$$\therefore \frac{d^2y}{dx^2} = 6x + \tfrac{3}{4}x^{-\frac{5}{2}}$$

3 $y = xe^x$ is the product of $u = x$ and $v = e^x$

$\therefore\ u' = 1$ and $v' = e^x$

Now $\dfrac{dy}{dx} = u'v + uv'$ {product rule}

$= 1 \times e^x + x \times e^x$

$= e^x + xe^x$

$= (1+x)e^x$

If $\dfrac{dy}{dx} = 2e$, then $(1+x)e^x = 2e$

Solving by inspection, we find $x = 1$. When $x = 1$, $y = 1 \times e^1 = e$.

$\therefore$ the gradient of $y = xe^x$ is $2e$ at the point $(1, e)$.

4 **a**
$$f(x) = \ln(e^x + 3)$$
$$\therefore\ f'(x) = \frac{e^x}{e^x + 3}$$

b
$$\begin{aligned} f(x) &= \ln\left[\frac{(x+2)^3}{x}\right] \\ &= \ln(x+2)^3 - \ln x \\ &= 3\ln(x+2) - \ln x \end{aligned}$$
$$\therefore\ f'(x) = \frac{3}{x+2} - \frac{1}{x}$$

c
$$f(x) = x^{x^2}$$
$$\therefore\ \ln y = \ln x^{x^2}$$
$$\therefore\ \ln y = x^2 \ln x$$

Differentiating with respect to x,
$$\frac{1}{y}\frac{dy}{dx} = 2x\ln x + x^2\left(\frac{1}{x}\right)$$
$$\begin{aligned} \therefore\ \frac{dy}{dx} &= y(2x\ln x + x) \\ &= x^{x^2}(2x\ln x + x) \\ &= x^{x^2+1}(2\ln x + 1) \end{aligned}$$

5
$$\begin{aligned} y &= \left(x - \frac{1}{x}\right)^4 \\ &= \left(x - x^{-1}\right)^4 \\ \therefore\ \frac{dy}{dx} &= 4\left(x - x^{-1}\right)^3(1 + x^{-2}) \\ &= 4\left(x - \frac{1}{x}\right)^3\left(1 + \frac{1}{x^2}\right) \end{aligned}$$

When $x = 1$,
$$\begin{aligned} \frac{dy}{dx} &= 4\left(1 - \frac{1}{1}\right)^3\left(1 + \frac{1}{1^2}\right) \\ &= 4 \times 0 \times 2 \\ &= 0 \end{aligned}$$

6 **a**
$$f(x) = x^{\frac{1}{2}}\cos(4x)$$
$$\begin{aligned} \therefore\ f'(x) &= \tfrac{1}{2}x^{-\frac{1}{2}}\cos(4x) + x^{\frac{1}{2}}(-4\sin(4x)) \qquad \text{\{product rule\}} \\ &= \tfrac{1}{2}x^{-\frac{1}{2}}\cos(4x) - 4x^{\frac{1}{2}}\sin(4x) \end{aligned}$$
and
$$\begin{aligned} f''(x) &= -\tfrac{1}{4}x^{-\frac{3}{2}}\cos(4x) + \tfrac{1}{2}x^{-\frac{1}{2}}(-4\sin(4x)) - \left[2x^{-\frac{1}{2}}\sin(4x) + 4x^{\frac{1}{2}} \times 4\cos(4x)\right] \\ &= -\tfrac{1}{4}x^{-\frac{3}{2}}\cos(4x) - 4x^{-\frac{1}{2}}\sin(4x) - 16x^{\frac{1}{2}}\cos(4x) \end{aligned}$$

b
$$f'(x) = \tfrac{1}{2}x^{-\frac{1}{2}}\cos(4x) - 4x^{\frac{1}{2}}\sin(4x) \qquad \text{\{from } \mathbf{a}\text{\}}$$
$$\therefore\ f'\left(\tfrac{\pi}{16}\right) = \tfrac{1}{2}\left(\tfrac{\pi}{16}\right)^{-\frac{1}{2}}\cos\left(\tfrac{\pi}{4}\right) - 4\left(\tfrac{\pi}{16}\right)^{\frac{1}{2}}\sin\left(\tfrac{\pi}{4}\right) \approx -0.455$$
$$f''(x) = -\tfrac{1}{4}x^{-\frac{3}{2}}\cos(4x) - 4x^{-\frac{1}{2}}\sin(4x) - 16x^{\frac{1}{2}}\cos(4x)$$
$$\begin{aligned} \therefore\ f''\left(\tfrac{\pi}{8}\right) &= -\tfrac{1}{4}\left(\tfrac{\pi}{8}\right)^{-\frac{3}{2}}\cos\left(\tfrac{\pi}{2}\right) - 4\left(\tfrac{\pi}{8}\right)^{-\frac{1}{2}}\sin\left(\tfrac{\pi}{2}\right) - 16\left(\tfrac{\pi}{8}\right)^{\frac{1}{2}}\cos\left(\tfrac{\pi}{2}\right) \\ &= 0 - 4\left(\tfrac{\pi}{8}\right)^{-\frac{1}{2}}(1) - 0 \qquad \text{\{since } \cos\left(\tfrac{\pi}{2}\right) = 0\text{\}} \\ &\approx -6.38 \end{aligned}$$

7
$$\begin{aligned}
y &= 3\sin 2x + 2\cos 2x \\
\therefore\ \frac{dy}{dx} &= 3 \times (\cos 2x) \times 2 + 2 \times (-\sin 2x) \times 2 \\
&= 6\cos 2x - 4\sin 2x \\
\therefore\ \frac{d^2y}{dx^2} &= 6 \times (-\sin 2x) \times 2 - 4 \times (\cos 2x) \times 2 \\
&= -12\sin 2x - 8\cos 2x
\end{aligned}$$

$$\begin{aligned}
\therefore\ 4y + \frac{d^2y}{dx^2} &= 4(3\sin 2x + 2\cos 2x) + (-12\sin 2x - 8\cos 2x) \\
&= 12\sin 2x + 8\cos 2x - 12\sin 2x - 8\cos 2x \\
&= 0
\end{aligned}$$

8 **a** $f(x) = \dfrac{6x}{3+x^2}$ Now, $f(x) = -\frac{1}{2}$ when $\dfrac{6x}{3+x^2} = -\frac{1}{2}$

$$\begin{aligned}
\therefore\ 12x &= -(3+x^2) \\
\therefore\ x^2 + 12x + 3 &= 0 \\
\therefore\ x &= \frac{-12 \pm \sqrt{12^2 - 4 \times 1 \times 3}}{2} \\
&= \frac{-12 \pm \sqrt{132}}{2} \approx -11.7 \text{ or } -0.255
\end{aligned}$$

b $f(x) = \dfrac{6x}{3+x^2}$ is a quotient where $u = 6x$ and $v = 3 + x^2$

$\therefore\ u' = 6$ and $v' = 2x$

$$\begin{aligned}
\therefore\ f'(x) &= \frac{u'v - uv'}{v^2} \quad \{\text{quotient rule}\} \\
&= \frac{6(3+x^2) - 6x(2x)}{(3+x^2)^2} \\
&= \frac{18 + 6x^2 - 12x^2}{(3+x)^2} \\
&= \frac{18 - 6x^2}{(3+x^2)^2}
\end{aligned}$$

Now, $f'(x) = 0$ when $\dfrac{18-6x^2}{(3+x^2)^2} = 0$

$$\begin{aligned}
\therefore\ 18 - 6x^2 &= 0 \quad \{\text{since } (3+x^2)^2 > 0 \text{ for all } x \in \mathbb{R}\} \\
\therefore\ 6(3 - x^2) &= 0 \\
\therefore\ x^2 &= 3 \\
\therefore\ x &= \pm\sqrt{3} \approx -1.73 \text{ or } 1.73
\end{aligned}$$

c $f'(x) = \dfrac{18-6x^2}{(3+x^2)^2}$ is a quotient where $u = 18 - 6x^2$ and $v = (3+x^2)^2$

$\therefore\ u' = -12x$ and $v' = 2(3+x^2) \times 2x$
$= 4x(3+x^2)$

$$\begin{aligned}
f''(x) &= \frac{u'v - uv'}{v^2} \quad \{\text{quotient rule}\} \\
&= \frac{-12x(3+x^2)^2 - (18-6x^2) \times 4x(3+x^2)}{(3+x^2)^4} \\
&= \frac{(3+x^2)[-12x(3+x^2) - 4x(18-6x^2)]}{(3+x^2)^4} \\
&= \frac{-36x - 12x^3 - 72x + 24x^3}{(3+x^2)^3} \\
&= \frac{12x^3 - 108x}{(3+x^2)^3}
\end{aligned}$$

Now, $f''(x) = 0$ when $\dfrac{12x^3 - 108x}{(3+x^2)^3} = 0$

$\therefore\ 12x^3 - 108x = 0$ $\{$since $3 + x^2 > 0$ for all $x \in \mathbb{R}\}$

$\therefore\ 12x(x^2 - 9) = 0$

$\therefore\ 12x(x+3)(x-3) = 0$

$\therefore\ x = 0, -3, \text{ or } 3$

9 **a** $f(x) = -10\sin 2x \cos 2x,\quad 0 \leqslant x \leqslant \pi$

$\therefore\ f(x) = -5\sin 4x$ $\{2\sin A\cos A = \sin 2A\}$

b $f'(x) = -20\cos 4x$

If $f'(x) = 0,\quad -20\cos 4x = 0$

$\therefore\ \cos 4x = 0$

$\therefore\ 4x = \frac{\pi}{2} + n\pi,\quad n$ any integer

$\therefore\ x = \frac{\pi}{8} + \frac{n\pi}{4}$

So, for the domain $0 \leqslant x \leqslant \pi,\quad x = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}$

10 **a** $e^x y - xy^2 = 1$

$\therefore\ e^x y + e^x(1)\dfrac{dy}{dx} - \left((1)y^2 + x(2y)\dfrac{dy}{dx}\right) = 0$

$\therefore\ e^x y + e^x\dfrac{dy}{dx} - y^2 - 2xy\dfrac{dy}{dx} = 0$

$\therefore\ \dfrac{dy}{dx} = \dfrac{y^2 - e^x y}{e^x - 2xy}$

b When $x = 0,\quad e^0 y - 0y^2 = 1$

$\therefore\ y = 1$

When $x = 0$ and $y = 1,\quad \dfrac{dy}{dx} = \dfrac{1^2 - e^0(1)}{e^0 - 2(0)(1)}$

$= \frac{0}{1}$

$= 0$

So, the gradient of the curve at $x = 0$ is 0.

11 P_n is "If $y = x^n$, then $\dfrac{dy}{dx} = nx^{n-1}$", $n \in \mathbb{Z}^+$.

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, then $y = x$. This has gradient 1, so $\dfrac{dy}{dx} = 1 = 1x^0$. $\therefore$ P_1 is true.

(2) If P_k is true, then $y = x^k$ implies that $\dfrac{dy}{dx} = kx^{k-1}$

If $y = x^{k+1} = x^k x$,

then $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(x^k\right)x + x^k\dfrac{d}{dx}(x)$ $\{$product rule$\}$

$= kx^{k-1}x + x^k \times 1$

$= kx^k + x^k$

$= (k+1)x^k$

Hence P_{k+1} is true whenever P_k is true, and P_1 is true

$\therefore$ P_n is true for all $n \in \mathbb{Z}^+$ $\{$Principle of mathematical induction$\}$

12 **a** $y = uv$

$\therefore\ \dfrac{dy}{dx} = \left(\dfrac{du}{dx}\right)v + u\left(\dfrac{dv}{dx}\right)$ {product rule}

$\therefore\ \dfrac{d^2y}{dx^2} = \left[\left(\dfrac{d^2u}{dx^2}\right)v + \dfrac{du}{dx}\dfrac{dv}{dx}\right] + \left[\dfrac{du}{dx}\dfrac{dv}{dx} + u\left(\dfrac{d^2v}{dx^2}\right)\right]$ {product rule}

$= \left(\dfrac{d^2u}{dx^2}\right)v + 2\,\dfrac{du}{dx}\dfrac{dv}{dx} + u\left(\dfrac{d^2v}{dx^2}\right)$

b $y = uvw = u(vw)$

$\therefore\ \dfrac{dy}{dx} = \dfrac{du}{dx}(vw) + u\left[\dfrac{d}{dx}(vw)\right]$ {product rule}

$= \dfrac{du}{dx}\,vw + u\left(\dfrac{dv}{dx}\,w + v\,\dfrac{dw}{dx}\right)$ {product rule}

$= \dfrac{du}{dx}\,vw + u\,\dfrac{dv}{dx}\,w + uv\,\dfrac{dw}{dx}$

13 **a** $f(x) = xe^{ax}$

$f'(x) = e^{ax} + x(ae^{ax})$
$= e^{ax}(ax+1)$

$f''(x) = ae^{ax}(ax+1) + e^{ax}(a)$
$= ae^{ax}(ax+1+1)$
$= ae^{ax}(ax+2)$

$f'''(x) = a^2e^{ax}(ax+2) + ae^{ax}(a)$
$= a^2e^{ax}(ax+2+1)$
$= a^2e^{ax}(ax+3)$

$f^{(4)}(x) = a^3e^{ax}(ax+3) + a^2e^{ax}(a)$
$= a^3e^{ax}(ax+3+1)$
$= a^3e^{ax}(ax+4)$

b $f^{(n)}(x) = a^{n-1}e^{ax}(ax+n)$

c P_n is "for $f(x) = xe^{ax}$, $f^{(n)}(x) = a^{n-1}e^{ax}(ax+n)$", $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) For $n = 1$, $f'(x) = e^{ax}(ax+1)$ {using **a**}
$= a^{1-1}e^{ax}(ax+1)$ $\therefore$ P_1 is true.

(2) If P_k is true then $f^{(k)}(x) = a^{k-1}e^{ax}(ax+k)$

$\therefore\ f^{(k+1)}(x) = a^{k-1}(a)e^{ax}(ax+k) + a^{k-1}e^{ax}(a)$ {product rule}
$= a^k e^{ax}(ax+k) + a^k e^{ax}$
$= a^{(k+1)-1}e^{ax}(ax+[k+1])$

Thus P_{k+1} is true whenever P_k is true

$\therefore$ since P_1 is true, P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

Chapter 19

PROPERTIES OF CURVES

EXERCISE 19A

1 **a** We seek the tangent to $y = x - 2x^2 + 3$ at $x = 2$.

When $x = 2$, $y = 2 - 2(2)^2 + 3 = -3$

$\therefore$ the point of contact is $(2, -3)$.

Now $\dfrac{dy}{dx} = 1 - 4x$, so at $x = 2$,

$\dfrac{dy}{dx} = 1 - 8 = -7$

$\therefore$ the tangent has equation

$$\frac{y - (-3)}{x - 2} = -7$$

$$\therefore\ y + 3 = -7(x - 2)$$

$$\therefore\ y = -7x + 14 - 3$$

$$\therefore\ y = -7x + 11$$

b We seek the tangent to

$y = \sqrt{x} + 1 = x^{\frac{1}{2}} + 1$ at $x = 4$.

When $x = 4$, $y = \sqrt{4} + 1 = 3$

$\therefore$ the point of contact is $(4, 3)$.

Now $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x}}$, so at $x = 4$,

$\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{4}} = \frac{1}{4}$

$\therefore$ the tangent has equation

$$\frac{y - 3}{x - 4} = \tfrac{1}{4}$$

$$\therefore\ 4y - 12 = x - 4$$

$$\therefore\ 4y = x + 8$$

c We seek the tangent to $y = x^3 - 5x$ at $x = 1$.

When $x = 1$, $y = 1^3 - 5(1) = -4$

$\therefore$ the point of contact is $(1, -4)$.

Now $\dfrac{dy}{dx} = 3x^2 - 5$, so at $x = 1$,

$\dfrac{dy}{dx} = 3 - 5 = -2$

$\therefore$ the tangent has equation

$$\frac{y - (-4)}{x - 1} = -2$$

$$\therefore\ y + 4 = -2x + 2$$

$$\therefore\ y = -2x - 2$$

d We seek the tangent to $y = \dfrac{4}{\sqrt{x}}$ at $(1, 4)$.

Now $y = \dfrac{4}{\sqrt{x}} = 4x^{-\frac{1}{2}}$

$\therefore$ $\dfrac{dy}{dx} = -2x^{-\frac{3}{2}}$ so at $x = 1$,

$\dfrac{dy}{dx} = -2\left(1^{-\frac{3}{2}}\right) = -2$

$\therefore$ the tangent has equation

$$\frac{y - 4}{x - 1} = -2$$

$$\therefore\ y - 4 = -2x + 2$$

$$\therefore\ y = -2x + 6$$

e We seek the tangent to

$y = \dfrac{3}{x} - \dfrac{1}{x^2} = 3x^{-1} - x^{-2}$ at $(-1, -4)$.

Now $\dfrac{dy}{dx} = -3x^{-2} + 2x^{-3}$

$= -\dfrac{3}{x^2} + \dfrac{2}{x^3}$ so at $x = -1$,

$\dfrac{dy}{dx} = -\dfrac{3}{(-1)^2} + \dfrac{2}{(-1)^3}$

$= -3 - 2$

$= -5$

$\therefore$ the tangent has equation

$$\frac{y - (-4)}{x - (-1)} = -5$$

$$\therefore\ y + 4 = -5x - 5$$

$$\therefore\ y = -5x - 9$$

f We seek the tangent to

$y = 3x^2 - \dfrac{1}{x} = 3x^2 - x^{-1}$ at $x = -1$.

When $x = -1$, $y = 3(-1)^2 - \frac{1}{(-1)} = 4$

$\therefore$ the point of contact is $(-1, 4)$.

Now $\dfrac{dy}{dx} = 6x + x^{-2}$

$= 6x + \dfrac{1}{x^2}$ so at $x = -1$,

$\dfrac{dy}{dx} = 6(-1) + \dfrac{1}{(-1)^2} = -5$

$\therefore$ the tangent has equation

$$\frac{y - 4}{x - (-1)} = -5$$

$$\therefore\ y - 4 = -5x - 5$$

$$\therefore\ y = -5x - 1$$

2 **a** We seek the normal to $y = x^2$ at $(3, 9)$.

Now $\dfrac{dy}{dx} = 2x$ so at $x = 3$,

$\dfrac{dy}{dx} = 2(3) = 6$

$\therefore$ the normal at $(3, 9)$ has gradient $-\frac{1}{6}$,
so the equation of the normal is

$$\frac{y-9}{x-3} = -\tfrac{1}{6}$$

$$\therefore\ 6y - 54 = -x + 3$$

$$\therefore\ 6y = -x + 57$$

b We seek the normal to $y = x^3 - 5x + 2$ at $x = -2$.

When $x = -2$, $y = (-2)^3 - 5(-2) + 2$
$= 4$

and so the point of contact is $(-2, 4)$.

Now $\dfrac{dy}{dx} = 3x^2 - 5$ so at $x = -2$,

$\dfrac{dy}{dx} = 3(-2)^2 - 5 = 7$

$\therefore$ the normal at $(-2, 4)$ has gradient $-\frac{1}{7}$,
so the equation of the normal is

$$\frac{y-4}{x-(-2)} = -\tfrac{1}{7}$$

$$\therefore\ 7y - 28 = -(x+2)$$

$$\therefore\ 7y = -x + 26$$

c We seek the normal to $y = \dfrac{5}{\sqrt{x}} - \sqrt{x}$ at $(1, 4)$.

Now $y = 5x^{-\frac{1}{2}} - x^{\frac{1}{2}}$

$\therefore\ \dfrac{dy}{dx} = -\frac{5}{2}x^{-\frac{3}{2}} - \frac{1}{2}x^{-\frac{1}{2}}$ so at $x = 1$,

$$\frac{dy}{dx} = -\tfrac{5}{2}\left(1^{-\frac{3}{2}}\right) - \tfrac{1}{2}\left(1^{-\frac{1}{2}}\right)$$

$$= -\tfrac{5}{2} - \tfrac{1}{2} = -3$$

$\therefore$ the normal at $(1, 4)$ has gradient $\frac{1}{3}$,
so the equation of the normal is

$$\frac{y-4}{x-1} = \tfrac{1}{3}$$

$$\therefore\ 3y - 12 = x - 1$$

$$\therefore\ 3y = x + 11$$

d We seek the normal to $y = 8\sqrt{x} - \dfrac{1}{x^2}$ at $x = 1$.

When $x = 1$, $y = 8\sqrt{1} - \dfrac{1}{1^2} = 7$

$\therefore$ the point of contact is $(1, 7)$.

Now $y = 8\sqrt{x} - \dfrac{1}{x^2} = 8x^{\frac{1}{2}} - x^{-2}$

$\therefore\ \dfrac{dy}{dx} = 4x^{-\frac{1}{2}} + 2x^{-3}$ so at $x = 1$,

$\dfrac{dy}{dx} = 4 + 2 = 6$

$\therefore$ the normal at $(1, 7)$ has gradient $-\frac{1}{6}$,
so the equation of the normal is

$$\frac{y-7}{x-1} = -\tfrac{1}{6}$$

$$\therefore\ 6y - 42 = -x + 1$$

$$\therefore\ 6y = -x + 43$$

3 **a**

$$y = 2x^3 + 3x^2 - 12x + 1$$

$$\therefore\ \frac{dy}{dx} = 6x^2 + 6x - 12$$

Horizontal tangents have gradient $= 0$

$$\text{so}\quad 6x^2 + 6x - 12 = 0$$

$$\therefore\ x^2 + x - 2 = 0$$

$$\therefore\ (x+2)(x-1) = 0$$

$$\therefore\ x = -2 \text{ or } x = 1$$

Now at $x = -2$,

$$y = 2(-2)^3 + 3(-2)^2 - 12(-2) + 1$$
$$= 21$$

and at $x = 1$,

$$y = 2(1)^3 + 3(1)^2 - 12(1) + 1$$
$$= -6$$

$\therefore$ the points of contact are $(-2, 21)$ and $(1, -6)$

$\therefore$ the tangents are $y = 21$ and $y = -6$.

b Now $y = 2\sqrt{x} + \dfrac{1}{\sqrt{x}} = 2x^{\frac{1}{2}} + x^{-\frac{1}{2}}$

$$\therefore\ \frac{dy}{dx} = x^{-\frac{1}{2}} - \tfrac{1}{2}x^{-\frac{3}{2}} = \frac{1}{\sqrt{x}} - \frac{1}{2x\sqrt{x}}$$

Horizontal tangents have gradient $= 0$

$$\therefore\ \frac{1}{\sqrt{x}} - \frac{1}{2x\sqrt{x}} = 0$$

$$\therefore\ \frac{2x-1}{2x\sqrt{x}} = 0$$

$$\therefore\ 2x - 1 = 0$$

$$\therefore\ x = \tfrac{1}{2}$$

Now at $x = \frac{1}{2}$,

$$y = 2\sqrt{\tfrac{1}{2}} + \frac{1}{\sqrt{\frac{1}{2}}} = \frac{2\left(\frac{1}{2}\right) + 1}{\sqrt{\frac{1}{2}}} = \frac{2}{\sqrt{\frac{1}{2}}}$$
$$= 2\sqrt{2}$$

$\therefore$ the only horizontal tangent touches at the curve at $(\frac{1}{2}, 2\sqrt{2})$.

c Now $y = 2x^3 + kx^2 - 3$

$\therefore \dfrac{dy}{dx} = 6x^2 + 2kx$

When $x = 2$, $\dfrac{dy}{dx} = 4$

$\therefore 6(2)^2 + 2k(2) = 4$

$\therefore 24 + 4k = 4$

$\therefore 4k = -20$

$\therefore k = -5$

d Now $y = 1 - 3x + 12x^2 - 8x^3$

$\therefore \dfrac{dy}{dx} = -3 + 24x - 24x^2$

When $x = 1$, $\dfrac{dy}{dx} = -3 + 24 - 24 = -3$

$\therefore$ the tangent at $(1, 2)$ has gradient -3

The tangents to the curve have gradient -3 when $-3 + 24x - 24x^2 = -3$

$\therefore 24x^2 - 24x = 0$

$\therefore 24x(x-1) = 0$

$\therefore$ when $x = 0$ or $x = 1$

So the other x-value for which the tangent to the curve has gradient -3 is $x = 0$, and when $x = 0$, $y = 1 - 0 + 0 - 0 = 1$

$\therefore$ the tangent to the curve at $(0, 1)$ is parallel to the tangent at $(1, 2)$.

This tangent has equation $\dfrac{y-1}{x-0} = -3$ or $y = -3x + 1$.

4 **a** Now $y = x^2 + ax + b$

$\therefore \dfrac{dy}{dx} = 2x + a$

At $x = 1$, $\dfrac{dy}{dx} = 2 + a$

$\therefore$ the gradient of the tangent to the curve at $x = 1$ will be $2 + a$.

However the equation of the tangent is $2x + y = 6$ or $y = -2x + 6$ and so the gradient of the tangent is -2.

$\therefore 2 + a = -2$

$\therefore a = -4$

So, the curve is $y = x^2 - 4x + b$.

We also know that the tangent contacts the curve when $x = 1$.

$\therefore 1^2 - 4(1) + b = -2(1) + 6$

$\therefore 1 - 4 + b = 4$

$\therefore b = 7$

$\therefore a = -4, \quad b = 7$

b Now $y = a\sqrt{x} + \dfrac{b}{\sqrt{x}} = ax^{\frac{1}{2}} + bx^{-\frac{1}{2}}$

$\therefore \dfrac{dy}{dx} = \dfrac{a}{2}x^{-\frac{1}{2}} - \dfrac{b}{2}x^{-\frac{3}{2}}$

$$\therefore \text{ at } x = 4, \ \frac{dy}{dx} = \frac{a}{2}\left(4^{-\frac{1}{2}}\right) - \frac{b}{2}\left(4^{-\frac{3}{2}}\right) = \frac{a}{2}\left(\frac{1}{2}\right) - \frac{b}{2}\left(\frac{1}{8}\right) = \frac{a}{4} - \frac{b}{16}$$

$\therefore$ the gradient of the tangent to the curve at $x = 4$ will be $\dfrac{a}{4} - \dfrac{b}{16} = \dfrac{4a - b}{16}$

However the equation of the *normal* is $4x + y = 22$ or $y = -4x + 22$.

$\therefore$ the normal has gradient -4.

$\therefore$ the tangent has gradient $\frac{1}{4}$, and so

$\dfrac{4a-b}{16} = \frac{1}{4}$

$\therefore 4a - b = 4$

$\therefore b = 4a - 4$ .... (1)

Also, at $x = 4$ the normal line intersects the curve. $\therefore a\sqrt{4} + \dfrac{b}{\sqrt{4}} = -4(4) + 22$

$\therefore 2a + \dfrac{b}{2} = 6$

Consequently, $2a + \dfrac{4a-4}{2} = 6$ {using (1)}

$\therefore 2a + 2a - 2 = 6$

$\therefore 4a = 8$

$\therefore a = 2$

and so $b = 4(2) - 4 = 4$ {from (1)}

$\therefore a = 2, \ b = 4$

c $y = 2x^2 - 1$

$\therefore \dfrac{dy}{dx} = 4x$

$\therefore$ at the point where $x = a$, $\dfrac{dy}{dx} = 4a$

$\therefore$ the gradient of the tangent at the point where $x = a$ is $4a$.

Also, at $x = a$, $y = 2a^2 - 1$.

$\therefore$ the tangent has equation

$\dfrac{y - (2a^2 - 1)}{x - a} = 4a$

$\therefore y - 2a^2 + 1 = 4a(x - a)$

$\therefore y - 2a^2 + 1 = 4ax - 4a^2$

$\therefore 4ax - y = 2a^2 + 1$

5 **a** $y = \sqrt{2x+1} = (2x+1)^{\frac{1}{2}}$

When $x = 4$, $y = \sqrt{2(4)+1} = 3$,

so the point of contact is $(4, 3)$.

Now $\dfrac{dy}{dx} = \frac{1}{2}(2x+1)^{-\frac{1}{2}}(2)$ {chain rule}

$= \dfrac{1}{\sqrt{2x+1}}$

$\therefore$ at $x = 4$, $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{2(4)+1}} = \frac{1}{3}$

$\therefore$ the tangent has equation $\dfrac{y-3}{x-4} = \frac{1}{3}$

$\therefore\ 3y - 9 = x - 4$

$\therefore\ 3y = x + 5$

b $y = \dfrac{1}{2-x} = (2-x)^{-1}$

$\therefore$ at $x = -1$, $y = \dfrac{1}{2-(-1)} = \frac{1}{3}$

So the point of contact is $(-1, \frac{1}{3})$.

Now $\dfrac{dy}{dx} = -1(2-x)^{-2}(-1)$ {chain rule}

$= \dfrac{1}{(2-x)^2}$

$\therefore$ at $x = -1$, $\dfrac{dy}{dx} = \dfrac{1}{(2-(-1))^2} = \frac{1}{9}$

$\therefore$ tangent has equation $\dfrac{y - \frac{1}{3}}{x-(-1)} = \frac{1}{9}$

$\therefore\ 9y - 3 = x + 1$

$\therefore\ 9y = x + 4$

c We seek the tangent to $f(x) = \dfrac{x}{1-3x}$ at $(-1, -\frac{1}{4})$.

$f(x)$ is a quotient where

$u = x$ and $v = 1 - 3x$

$\therefore\ u' = 1$ and $v' = -3$

Now $f'(x) = \dfrac{u'v - uv'}{v^2}$ {quotient rule}

$\therefore\ f'(x) = \dfrac{1(1-3x) - x(-3)}{(1-3x)^2}$

$= \dfrac{1}{(1-3x)^2}$

$\therefore\ f'(-1) = \dfrac{1}{(1-3(-1))^2} = \frac{1}{16}$

$\therefore$ tangent has equation $\dfrac{y - \left(-\frac{1}{4}\right)}{x-(-1)} = \frac{1}{16}$

$\therefore\ 16y + 4 = x + 1$

$\therefore\ 16y = x - 3$

d We seek the tangent to $f(x) = \dfrac{x^2}{1-x}$ at $(2, -4)$.

$f(x)$ is a quotient where

$u = x^2$ and $v = 1 - x$

$\therefore\ u' = 2x$ and $v' = -1$

Now $f'(x) = \dfrac{u'v - uv'}{v^2}$ {quotient rule}

$\therefore\ f'(x) = \dfrac{2x(1-x) - x^2(-1)}{(1-x)^2}$

$= \dfrac{2x - 2x^2 + x^2}{(1-x)^2} = \dfrac{2x - x^2}{(1-x)^2}$

$\therefore\ f'(2) = \dfrac{2(2) - 2^2}{(1-2)^2} = \dfrac{4-4}{1} = 0$

As the tangent has gradient 0, it is horizontal.

$\therefore$ its equation is $y = c$

Since the contact point is $(2, -4)$, the tangent has equation $y = -4$.

6 **a** We seek the normal to $y = \dfrac{1}{(x^2+1)^2}$ at $(1, \frac{1}{4})$.

As $y = (x^2+1)^{-2}$,

$\dfrac{dy}{dx} = -2(x^2+1)^{-3}(2x) = \dfrac{-4x}{(x^2+1)^3}$

$\therefore$ at $x = 1$, $\dfrac{dy}{dx} = \dfrac{-4}{(1+1)^3} = \frac{-4}{8} = -\frac{1}{2}$

$\therefore$ the normal at $(1, \frac{1}{4})$ has gradient 2.

So the equation of the normal is

$\dfrac{y - \frac{1}{4}}{x - 1} = 2$

$\therefore\ y - \frac{1}{4} = 2x - 2$

$\therefore\ y = 2x - \frac{7}{4}$

b $y = \dfrac{1}{\sqrt{3-2x}}$

$\therefore$ at $x = -3$, $y = \dfrac{1}{\sqrt{3-2(-3)}} = \frac{1}{3}$

$\therefore$ the point of contact is $(-3, \frac{1}{3})$

Now $y = (3-2x)^{-\frac{1}{2}}$

$\therefore\ \dfrac{dy}{dx} = -\frac{1}{2}(3-2x)^{-\frac{3}{2}}(-2) = (3-2x)^{-\frac{3}{2}}$

$\therefore$ at $x = -3$, $\dfrac{dy}{dx} = (3-2(-3))^{-\frac{3}{2}}$

$= 9^{-\frac{3}{2}} = 3^{-3} = \frac{1}{27}$

$\therefore$ the normal at $(-3, \frac{1}{3})$ has gradient -27.

So the equation of the normal is

$\dfrac{y - \frac{1}{3}}{x-(-3)} = -27$

$\therefore\ y - \frac{1}{3} = -27(x+3)$

$\therefore\ y = -27x - \frac{242}{3}$

c $$f(x) = \sqrt{x}(1-x)^2$$

Since $f(4) = \sqrt{4}(1-4)^2 = 18$,

the point of contact is $(4, 18)$.

Now $f(x)$ is a product where

$$u = x^{\frac{1}{2}} \quad \text{and} \quad v = (1-x)^2$$

$$\therefore \; u' = \tfrac{1}{2}x^{-\frac{1}{2}} \quad \text{and} \quad v' = 2(1-x)(-1) = -2(1-x)$$

Now $f'(x) = u'v + uv'$ {product rule}

$$\therefore \; f'(x) = \tfrac{1}{2}x^{-\frac{1}{2}}(1-x)^2 - x^{\frac{1}{2}}2(1-x)$$

$$\therefore \; f'(4) = \tfrac{1}{2\sqrt{4}}(1-4)^2 - \sqrt{4}(2)(1-4)$$

$$= \tfrac{1}{4}(9) - 2(2)(-3) = \tfrac{57}{4}$$

$\therefore$ the normal at $(4, 18)$ has gradient $-\frac{4}{57}$.

So, the equation of the normal is

$$\frac{y-18}{x-4} = -\tfrac{4}{57}$$

$$\therefore \; 57(y-18) = -4(x-4)$$

$$\therefore \; 57y = -4x + 1042$$

d $$f(x) = \frac{x^2-1}{2x+3}$$

Since $f(-1) = \dfrac{(-1)^2-1}{2(-1)+3} = \dfrac{0}{1} = 0$,

the point of contact is $(-1, 0)$.

Now $f(x)$ is a quotient where

$$u = x^2 - 1 \quad \text{and} \quad v = 2x+3$$

$$\therefore \; u' = 2x \quad \text{and} \quad v' = 2$$

Now $$f'(x) = \frac{u'v - uv'}{v^2} = \frac{2x(2x+3) - (x^2-1)(2)}{(2x+3)^2}$$

$$\therefore \; f'(-1) = \frac{2(-1)(-2+3) - ((-1)^2-1)(2)}{(2(-1)+3)^2} = \frac{-2(1)-(0)(2)}{(1)^2} = -2$$

$\therefore$ the normal at $(-1, 0)$ has gradient $\frac{1}{2}$.

So, the equation of the normal is

$$\frac{y-0}{x-(-1)} = \tfrac{1}{2}$$

or $2y = x + 1$

7 The tangent has equation $3x + y = 5$ or $y = -3x + 5$

$\therefore$ the tangent has gradient -3 .... (1)

Also, at $x = -1$, $y = -3(-1) + 5 = 8$

$\therefore$ the tangent contacts the curve at $(-1, 8)$ .... (2)

Now $y = a(1-bx)^{\frac{1}{2}}$, so $\dfrac{dy}{dx} = \tfrac{1}{2}a(1-bx)^{-\frac{1}{2}}(-b)$

$$\therefore \; -3 = \tfrac{1}{2}a(1+b)^{-\frac{1}{2}}(-b) \quad \{\text{using (1)}\}$$

$$\therefore \; 6 = \frac{ab}{\sqrt{1+b}} \quad .... \text{ (3)}$$

Using (2), $(-1, 8)$ must lie on the curve $y = a\sqrt{1-bx}$

$$\therefore \; 8 = a\sqrt{1+b} \quad .... \text{ (4)}$$

$$\therefore \; \frac{6\sqrt{1+b}}{b} = \frac{8}{\sqrt{1+b}} \quad \{\text{equating } a\text{s in (3) and (4)}\}$$

$$\therefore \; 6(1+b) = 8b$$

$$\therefore \; 6 + 6b = 8b$$

$$\therefore \; 6 = 2b$$

$$\therefore \; b = 3 \quad \text{and} \quad a = \tfrac{8}{\sqrt{4}} = 4$$

8 $f : x \mapsto \dfrac{x}{\sqrt{2-x}}$

a f is defined when $2 - x > 0$

$$\therefore \; x < 2$$

$\therefore$ domain of f is $\{x \mid x < 2\}$

b $f(x) = \dfrac{x}{\sqrt{2-x}}$

Now $f(x)$ is a quotient where

$u = x$ and $v = (2-x)^{\frac{1}{2}}$

$\therefore\ u' = 1$ and $v' = \frac{1}{2}(2-x)^{-\frac{1}{2}}(-1)$

Now $f'(x) = \dfrac{u'v - uv'}{v^2}$

$$= \frac{1(2-x)^{\frac{1}{2}} - x\frac{1}{2}(2-x)^{-\frac{1}{2}}(-1)}{(2-x)^{\frac{2}{2}}}$$

$$= \frac{\dfrac{2(2-x)}{2(2-x)^{\frac{1}{2}}} + \dfrac{x}{2(2-x)^{\frac{1}{2}}}}{(2-x)^{\frac{2}{2}}}$$

$$= \frac{2(2-x)+x}{2(2-x)^{\frac{3}{2}}}$$

$$= \frac{4-2x+x}{2(2-x)^{\frac{3}{2}}}$$

$$= \frac{4-x}{2(2-x)^{\frac{3}{2}}}$$

c $f(x) = \dfrac{x}{\sqrt{2-x}} = -1$

$\therefore\ x = -\sqrt{2-x}$

$\therefore\ x^2 = 2-x$

$\therefore\ x^2 + x - 2 = 0$

$\therefore\ (x+2)(x-1) = 0$

$\therefore\ x = -2$ or 1

but $x < 2$ $\therefore\ x = -2$

$\therefore$ the point of contact is $(-2,\ -1)$.

Now $f'(-2) = \dfrac{4-(-2)}{2(2-(-2))^{\frac{3}{2}}}$

$= \dfrac{4+2}{2(4)^{\frac{3}{2}}}$

$= \dfrac{6}{2(2)^3}$

$= \frac{3}{8}$

$\therefore$ the normal at $(-2,\ -1)$ has gradient $-\frac{8}{3}$.

So, the equation of the normal is

$\dfrac{y-(-1)}{x-(-2)} = -\frac{8}{3}$

$\therefore\ 3(y+1) = -8(x+2)$

$\therefore\ 3y + 3 = -8x - 16$

$\therefore\ 8x + 3y = -19$

9 **a** $f(x) = e^{-x}$

$\therefore\ f(1) = e^{-1}$

$\therefore$ the point of contact is $\left(1,\ \dfrac{1}{e}\right)$.

Now $f'(x) = -e^{-x}$

$\therefore\ f'(1) = -e^{-1} = -\dfrac{1}{e}$

So, the gradient of the tangent is $-\dfrac{1}{e}$

$\therefore$ the tangent has equation $\dfrac{y - \frac{1}{e}}{x-1} = -\dfrac{1}{e}$

$\therefore\ e\left(y - \dfrac{1}{e}\right) = -(x-1)$

$\therefore\ ey - 1 = -x + 1$

$\therefore\ x + ey = 2$

or $y = -\dfrac{1}{e}x + \dfrac{2}{e}$

b $y = \ln(2-x)$

so when $x = -1$, $y = \ln 3$

$\therefore$ the point of contact is $(-1,\ \ln 3)$.

Now $\dfrac{dy}{dx} = \dfrac{-1}{2-x}$

$\therefore$ when $x = -1$, $\dfrac{dy}{dx} = -\dfrac{1}{2+1} = -\frac{1}{3}$

So, the gradient of the tangent is $-\frac{1}{3}$.

$\therefore$ the tangent has equation $\dfrac{y - \ln 3}{x+1} = -\frac{1}{3}$

$\therefore\ 3(y - \ln 3) = -(x+1)$

$\therefore\ 3y - 3\ln 3 = -x - 1$

$\therefore\ x + 3y = 3\ln 3 - 1$

c $y = \ln\sqrt{x}$

$= \ln x^{\frac{1}{2}}$

$= \frac{1}{2}\ln x$

$\therefore$ when $y = -1$, $-1 = \frac{1}{2}\ln x$

$\therefore\ \ln x = -2$

$\therefore\ x = e^{-2}$

$\therefore\ x = \dfrac{1}{e^2}$

$\therefore$ the point of contact is $\left(\dfrac{1}{e^2},\ -1\right)$

Now $\dfrac{dy}{dx} = \frac{1}{2}\left(\dfrac{1}{x}\right) = \dfrac{1}{2x}$, so at the point of contact, $\dfrac{dy}{dx} = \dfrac{1}{2e^{-2}} = \dfrac{e^2}{2}$

$\therefore$ the tangent has gradient $\dfrac{e^2}{2}$ and the normal has gradient $-\dfrac{2}{e^2}$

$\therefore$ the normal has equation $\dfrac{y+1}{x - \frac{1}{e^2}} = -\dfrac{2}{e^2}$

$$\therefore\ e^2(y+1) = -2\left(x - \frac{1}{e^2}\right)$$

$$\therefore\ e^2y + e^2 = -2x + \frac{2}{e^2}$$

$$\therefore\ 2x + e^2y = \frac{2}{e^2} - e^2 \quad \text{or} \quad y = -\frac{2}{e^2}x + \frac{2}{e^4} - 1$$

10 $y = \dfrac{\cos x}{1 + \sin x}$

Now since y is a quotient where $u = \cos x$ and $v = 1 + \sin x$

$\therefore\ u' = -\sin x$ and $v' = \cos x$

$$\begin{aligned}
\text{So}\quad \frac{dy}{dx} &= \frac{u'v - uv'}{v^2} \\
&= \frac{(-\sin x)(1+\sin x) - \cos x(\cos x)}{(1+\sin x)^2} \\
&= \frac{-\sin x - \sin^2 x - \cos^2 x}{(1+\sin x)^2} \\
&= \frac{-1 - \sin x}{(1+\sin x)^2} \qquad \{\sin^2 x + \cos^2 x = 1\} \\
&= -\frac{(1+\sin x)}{(1+\sin x)^2} \\
&= \frac{-1}{1+\sin x}
\end{aligned}$$

Since $\dfrac{-1}{1+\sin x}$ never equals 0, there are no horizontal tangents.

11 **a** $y = \sin x$ $\quad\therefore\ \dfrac{dy}{dx} = \cos x$

When $x = 0$, $\dfrac{dy}{dx} = \cos 0 = 1$

$\therefore$ the tangent has equation $\dfrac{y-0}{x-0} = 1$

or $y = x$

b $y = \tan x$ $\quad\therefore\ \dfrac{dy}{dx} = \dfrac{1}{\cos^2 x}$

When $x = 0$, $\dfrac{dy}{dx} = \dfrac{1}{\cos^2 0} = 1$

$\therefore$ the tangent has equation $\dfrac{y-0}{x-0} = 1$

or $y = x$

c $y = \cos x$ $\quad\therefore\ \dfrac{dy}{dx} = -\sin x$

When $x = \frac{\pi}{6}$, $y = \frac{\sqrt{3}}{2}$

and $\dfrac{dy}{dx} = -\sin\left(\frac{\pi}{6}\right) = -\frac{1}{2}$

So, the normal has gradient 2,

and its equation is $\dfrac{y - \frac{\sqrt{3}}{2}}{x - \frac{\pi}{6}} = 2$

$$\therefore\ y - \tfrac{\sqrt{3}}{2} = 2x - \tfrac{\pi}{3}$$

$$\therefore\ 2x - y = \tfrac{\pi}{3} - \tfrac{\sqrt{3}}{2}$$

d
$$y = \frac{1}{\sin(2x)} = (\sin(2x))^{-1}$$

$$\begin{aligned}
\therefore\ \frac{dy}{dx} &= -1(\sin(2x))^{-2} \times 2\cos(2x) \\
&= -\frac{2\cos(2x)}{(\sin(2x))^2}
\end{aligned}$$

When $x = \frac{\pi}{4}$, $y = 1$

and $\dfrac{dy}{dx} = -\dfrac{2\cos(\frac{\pi}{2})}{\left(\sin(\frac{\pi}{2})\right)^2} = 0$

$\therefore$ the gradient of the normal is undefined, so the normal is $x = \frac{\pi}{4}$.

12 **a**
$$y = \sec x$$
$$\therefore \quad \frac{dy}{dx} = \sec x \tan x$$
When $x = \frac{\pi}{4}$, $y = \sec\left(\frac{\pi}{4}\right) = \sqrt{2}$

and $\frac{dy}{dx} = \sec\left(\frac{\pi}{4}\right)\tan\left(\frac{\pi}{4}\right) = \sqrt{2}$

$\therefore$ the gradient of the tangent at $(\frac{\pi}{4}, \sqrt{2})$ is $\sqrt{2}$ and the equation of the tangent is:
$$\frac{y-\sqrt{2}}{x-\frac{\pi}{4}} = \sqrt{2}$$
$$\therefore \quad y - \sqrt{2} = \sqrt{2}x - \frac{\sqrt{2}\pi}{4}$$
$$\therefore \quad \sqrt{2}x - y = \frac{\sqrt{2}\pi}{4} - \sqrt{2}$$
$$= \sqrt{2}(\tfrac{\pi}{4} - 1)$$

b
$$y = \cot(\tfrac{x}{2})$$
$$\therefore \quad \frac{dy}{dx} = -\tfrac{1}{2}\csc^2(\tfrac{x}{2})$$
When $x = \frac{\pi}{3}$, $y = \cot\left(\frac{\pi}{6}\right) = \sqrt{3}$

and $\frac{dy}{dx} = -\frac{1}{2}\csc^2\left(\frac{\pi}{6}\right) = -2$

$\therefore$ the gradient of the tangent at $(\frac{\pi}{3}, \sqrt{3})$ is -2 and the equation of the tangent is:
$$\frac{y-\sqrt{3}}{x-\frac{\pi}{3}} = -2$$
$$\therefore \quad y - \sqrt{3} = -2x + \frac{2\pi}{3}$$
$$\therefore \quad 2x + y = \frac{2\pi}{3} + \sqrt{3}$$

13 **a**
$$y = \csc x$$
$$\therefore \quad \frac{dy}{dx} = -\csc x \cot x$$
When $x = \frac{\pi}{6}$,
$$y = \csc\left(\tfrac{\pi}{6}\right) = \frac{1}{\sin\left(\frac{\pi}{6}\right)} = \frac{1}{\frac{1}{2}} = 2$$
and
$$\frac{dy}{dx} = -\csc\left(\tfrac{\pi}{6}\right)\cot\left(\tfrac{\pi}{6}\right) = -2\frac{\cos\left(\frac{\pi}{6}\right)}{\sin\left(\frac{\pi}{6}\right)} = -2\left(\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}\right) = -2\sqrt{3}$$

$\therefore$ the gradient of the normal at $(\frac{\pi}{6}, 2)$ is $\frac{1}{2\sqrt{3}}$
$$\therefore \quad \frac{y-2}{x-\frac{\pi}{6}} = \frac{1}{2\sqrt{3}}$$
$$\therefore \quad 2\sqrt{3}(y-2) = x - \tfrac{\pi}{6}$$
$$\therefore \quad 2\sqrt{3}y - 4\sqrt{3} = x - \tfrac{\pi}{6}$$
$$\therefore \quad x - 2\sqrt{3}y = \tfrac{\pi}{6} - 4\sqrt{3}$$

b
$$y = \sqrt{\sec(\tfrac{x}{3})} = \left(\sec(\tfrac{x}{3})\right)^{\frac{1}{2}}$$
$$\frac{dy}{dx} = \tfrac{1}{2}\left(\sec(\tfrac{x}{3})\right)^{-\frac{1}{2}}\frac{\sec(\frac{x}{3})\tan(\frac{x}{3})}{3}$$
$$= \tfrac{1}{6}\sec^{\frac{1}{2}}(\tfrac{x}{3})\tan(\tfrac{x}{3})$$

When $x = \pi$,
$$y = \sqrt{\sec(\tfrac{\pi}{3})} = \sqrt{\frac{1}{\cos(\frac{\pi}{3})}} = \sqrt{\frac{1}{\frac{1}{2}}} = \sqrt{2}$$
and
$$\frac{dy}{dx} = \tfrac{1}{6}\sec^{\frac{1}{2}}(\tfrac{\pi}{3})\tan(\tfrac{\pi}{3}) = \tfrac{1}{6}\sqrt{2}\frac{\sin(\frac{\pi}{3})}{\cos(\frac{\pi}{3})} = \frac{\sqrt{2}}{6}\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \frac{\sqrt{2}}{6}\sqrt{3} = \frac{1}{\sqrt{6}}$$

$\therefore$ the gradient of the normal at $(\pi, \sqrt{2})$ is $-\sqrt{6}$ and the equation of the normal is:
$$\frac{y-\sqrt{2}}{x-\pi} = -\sqrt{6}$$
$$\therefore \quad y - \sqrt{2} = -\sqrt{6}(x-\pi)$$
$$\therefore \quad y - \sqrt{2} = -\sqrt{6}x + \pi\sqrt{6}$$
$$\therefore \quad \sqrt{6}x + y = \pi\sqrt{6} + \sqrt{2}$$

14 **a** Consider the tangent to $y = x^3$ at $x = 2$.
When $x = 2$, $y = 2^3 = 8$ so the point of contact is $(2, 8)$

Now $\dfrac{dy}{dx} = 3x^2$ and so at $x = 2$,

$$\frac{dy}{dx} = 3(2)^2 = 12$$

$\therefore$ the tangent at $(2, 8)$ has gradient 12 and its equation is $\dfrac{y-8}{x-2} = 12$

$$\therefore\ y - 8 = 12x - 24$$
$$\therefore\ y = 12x - 16$$

$\therefore$ the tangent meets the curve where
$$12x - 16 = x^3$$
$\therefore\ x^3 - 12x + 16 = 0$

Because the tangent touches the curve at $x = 2$, there must be a repeated solution at this point.

$\therefore\ (x-2)^2$ must be a factor of this cubic

$\therefore\ (x-2)^2(x+4) = 0$

$\therefore$ the tangent meets the curve again when $x = -4$.

When $x = -4$, $y = (-4)^3 = -64$

$\therefore$ the tangent meets the curve again at $(-4, -64)$.

b Consider the tangent to $y = -x^3 + 2x^2 + 1$ at $x = -1$.

When $x = -1$, $y = -(-1)^3 + 2(-1)^2 + 1$
$= 4$

and so the point of contact is $(-1, 4)$.

Now $\dfrac{dy}{dx} = -3x^2 + 4x$ and so at $x = -1$,

$$\frac{dy}{dx} = -3(-1)^2 + 4(-1) = -7$$

$\therefore$ the tangent at $(-1, 4)$ has gradient -7 and its equation is

$$\frac{y-4}{x-(-1)} = -7$$
$$\therefore\ y - 4 = -7(x+1)$$
$$\therefore\ y = -7x - 3$$

$\therefore$ the tangent meets the curve where
$$-7x - 3 = -x^3 + 2x^2 + 1$$
$\therefore\ x^3 - 2x^2 - 7x - 4 = 0$

Because the tangent touches the curve at $x = -1$, there must be a repeated solution at this point.

$\therefore\ (x+1)^2$ must be a factor of this cubic

$\therefore\ (x+1)^2(x-4) = 0$

$\therefore$ the tangent meets the curve again when $x = 4$.

When $x = 4$, $y = -(4)^3 + 2(4)^2 + 1$
$= -64 + 32 + 1 = -31$

$\therefore$ the tangent meets the curve again at $(4, -31)$.

15 **a**
$$f(x) = x^2 + \frac{4}{x^2}$$
$$\therefore\ f'(x) = 2x - 2 \times \frac{4}{x^3}$$
$$\therefore\ f'(x) = 2x - \frac{8}{x^3}$$

b Horizontal tangents have gradient 0, so

$$2x - \frac{8}{x^3} = 0$$
$$\therefore\ 2x^4 = 8$$
$$\therefore\ x^4 = 4$$
$$\therefore\ x = \pm\sqrt{2}$$

c When $x = -\sqrt{2}$,

$$f(-\sqrt{2}) = \left(-\sqrt{2}\right)^2 + \frac{4}{\left(-\sqrt{2}\right)^2} = 2 + \frac{4}{2} = 4$$

$\therefore$ the horizontal tangent at $(-\sqrt{2}, 4)$ is $y = 4$.

When $x = \sqrt{2}$,

$$f(\sqrt{2}) = \left(\sqrt{2}\right)^2 + \frac{4}{\left(\sqrt{2}\right)^2} = 2 + \frac{4}{2} = 4$$

$\therefore$ the horizontal tangent at $(\sqrt{2}, 4)$ is $y = 4$.

$\therefore$ the tangents are the same line because they have the same equation.

16 $y = x^2e^x$ so when $x = 1$, $y = e$

$\therefore$ the point of contact is $(1, e)$.

Now $\dfrac{dy}{dx} = 2xe^x + x^2e^x$

$\therefore$ when $x = 1$, $\dfrac{dy}{dx} = 2e + e = 3e$

$\therefore$ the tangent has equation $\dfrac{y-e}{x-1} = 3e$

$$\therefore\ y - e = 3ex - 3e$$
$$\therefore\ y - 3ex = -2e$$
$$\therefore\ 3ex - y = 2e$$

The tangent cuts the x-axis when
$$y = 0$$
$$\therefore\ 3ex = 2e$$
$$\therefore\ x = \tfrac{2}{3}$$

and the y-axis when
$$x = 0$$
$$\therefore\ -y = 2e$$
$$\therefore\ y = -2e$$

So, A is $(\frac{2}{3}, 0)$ and B is $(0, -2e)$.

17 **a** Consider the tangent to $y = x^2 - x + 9$ at $x = a$.

When $x = a$, $y = a^2 - a + 9$, so the point of contact is $(a,\ a^2 - a + 9)$.

Now $\dfrac{dy}{dx} = 2x - 1$ and so at $x = a$, $\dfrac{dy}{dx} = 2a - 1$

$\therefore$ the gradient of the tangent at $(a,\ a^2 - a + 9)$ is $2a - 1$

$\therefore$ the equation of the tangent is $\dfrac{y - (a^2 - a + 9)}{x - a} = 2a - 1$

$$\therefore\ y - (a^2 - a + 9) = (2a - 1)(x - a)$$

$$\therefore\ y = (2a - 1)x - 2a^2 + \cancel{a} + a^2 - \cancel{a} + 9$$

$$\therefore\ y = (2a - 1)x - a^2 + 9 \quad .... \ (1)$$

But this tangent passes through $(0, 0)$, so $0 = a^2 - 9$

$$\therefore\ (a + 3)(a - 3) = 0$$

$$\therefore\ a = \pm 3$$

$\therefore$ the tangents are: At $a = 3$, $y = (2(3) - 1)x - 3^2 + 9$ {from (1)}

$\therefore\ y = 5x$, with contact at $(3,\ 15)$.

At $a = -3$, $y = (2(-3) - 1)x - (-3)^2 + 9$ {from (1)}

$\therefore\ y = -7x$, with contact at $(-3,\ 21)$.

b Let $(a,\ a^3)$ lie on $y = x^3$.

Now $\dfrac{dy}{dx} = 3x^2$, so at $x = a$, $\dfrac{dy}{dx} = 3a^2$

$\therefore$ the gradient of the tangent at $(a,\ a^3)$ is $3a^2$

$\therefore$ the equation of the tangent is $\dfrac{y - a^3}{x - a} = 3a^2$ or $y - a^3 = (3a^2)(x - a)$

But this tangent passes through $(-2,\ 0)$, so $0 - a^3 = 3a^2(-2 - a)$

$$\therefore\ -a^3 = -6a^2 - 3a^3$$

$$\therefore\ 2a^3 + 6a^2 = 0$$

$$\therefore\ 2a^2(a + 3) = 0$$

$$\therefore\ a = 0 \text{ or } -3$$

If $a = 0$, the tangent equation is $y = 0$, with contact point $(0,\ 0)$.

If $a = -3$, the tangent equation is $y - (-27) = 27(x + 3)$

$\therefore\ y = 27x + 54$, with contact point $(-3,\ -27)$.

c Let $(a,\ \sqrt{a})$ lie on $y = \sqrt{x}$.

Now $\dfrac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \dfrac{1}{2\sqrt{x}}$, so at $x = a$, $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{a}}$

$\therefore$ the gradient of the tangent at $(a,\ \sqrt{a})$ is $\dfrac{1}{2\sqrt{a}}$

and the gradient of the normal at this point is $-2\sqrt{a}$.

$\therefore$ the normal has equation $\dfrac{y - \sqrt{a}}{x - a} = -2\sqrt{a}$

or $y - \sqrt{a} = -2\sqrt{a}(x - a)$

But this normal passes through $(4,\ 0)$, so $0 - \sqrt{a} = -2\sqrt{a}(4 - a)$

$$\therefore\ 2\sqrt{a}(4 - a) - \sqrt{a} = 0$$

$$\therefore\ \sqrt{a}(8 - 2a - 1) = 0$$

$$\therefore\ \sqrt{a}(7 - 2a) = 0$$

$$\therefore\ a = 0 \text{ or } \tfrac{7}{2}$$

But $a = 0$ is the endpoint of the function, so there is no normal here.

When $a = \frac{7}{2}$, $\quad y - \sqrt{\frac{7}{2}} = -2\sqrt{\frac{7}{2}}\left(x - \frac{7}{2}\right)$

$\therefore\ \sqrt{2}y - \sqrt{7} = -2\sqrt{7}\left(x - \frac{7}{2}\right)$

$\therefore\ \sqrt{2}y + 2\sqrt{7}x = 7\sqrt{7} + \sqrt{7}$

$\therefore\ \sqrt{2}y + 2\sqrt{7}x = 8\sqrt{7}$

$\therefore\ y = -\sqrt{14}x + 4\sqrt{14}$ with contact point $\left(\frac{7}{2}, \sqrt{\frac{7}{2}}\right)$.

18 $y = e^x$ so when $x = a$, $y = e^a$

$\therefore$ the point of contact is (a, e^a).

Now $\dfrac{dy}{dx} = e^x$

$\therefore$ at the point (a, e^a), $\dfrac{dy}{dx} = e^a$

$\therefore$ the tangent has equation $\dfrac{y - e^a}{x - a} = e^a$

or $y - e^a = e^a(x - a)$ .... $(*)$

$\therefore\ y = e^a x + e^a - ae^a$

Since the tangent passes through the origin, $(0, 0)$ must satisfy $(*)$

$\therefore\ 0 - e^a = e^a(0 - a)$

$\therefore\ -e^a = -ae^a$

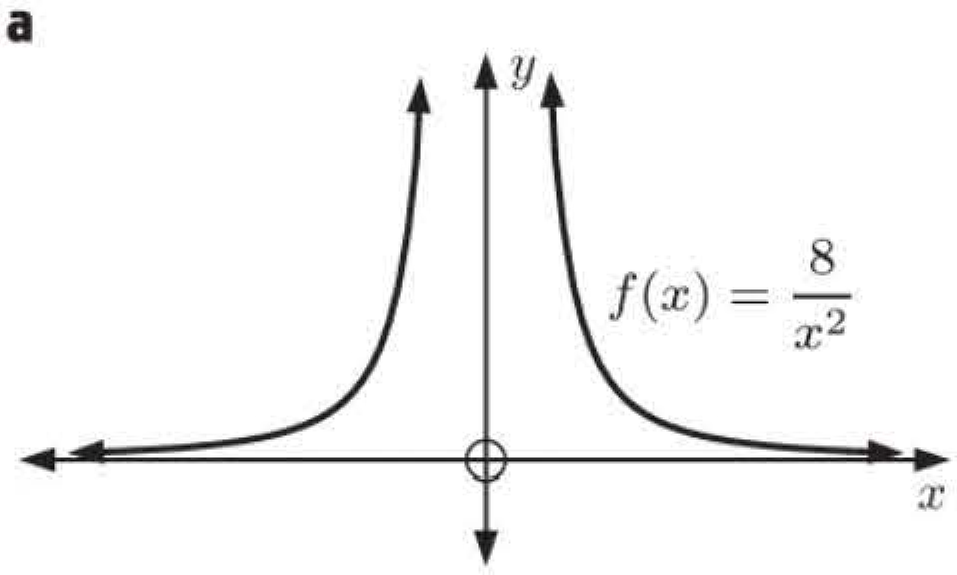
$\therefore\ e^a(a - 1) = 0$

$\therefore\ a = 1 \quad \{\text{as } e^a > 0\}$

So the equation of the tangent is

$y - e = ex - e$ or $y = ex$.

19 **a**



b Let $\left(a, \dfrac{8}{a^2}\right)$ lie on $f(x) = \dfrac{8}{x^2} = 8x^{-2}$

Now $f'(x) = -16x^{-3} = -\dfrac{16}{x^3}$

$\therefore\ f'(a) = -\dfrac{16}{a^3}$

$\therefore$ the gradient of the tangent at $\left(a, \dfrac{8}{a^2}\right)$ is $-\dfrac{16}{a^3}$

$\therefore$ the equation of the tangent is $\dfrac{y - \frac{8}{a^2}}{x - a} = -\dfrac{16}{a^3}$

$\therefore\ a^3y - 8a = -16x + 16a$

$\therefore\ 16x + a^3y = 24a$

c The tangent cuts the x-axis when $y = 0$

$\therefore\ 16x = 24a$

$\therefore\ x = \frac{3}{2}a$

$\therefore$ A is $(\frac{3}{2}a, 0)$.

The tangent cuts the y-axis when $x = 0$

$\therefore\ a^3y = 24a$

$\therefore\ y = \dfrac{24}{a^2}$

$\therefore$ B is $\left(0, \dfrac{24}{a^2}\right)$.

d Area of triangle OAB

$= \left|\frac{1}{2} \times \left(\frac{3}{2}a\right) \times \left(\dfrac{24}{a^2}\right)\right|$

$= \dfrac{18}{|a|}$ units^2

As $a \to \infty$, $\dfrac{18}{a} \to 0$

$\therefore$ area $\to 0$

20 $y_1 = \sqrt{x + a}$ and $y_2 = \sqrt{2x - x^2}$

$\therefore\ \dfrac{dy_1}{dx} = \frac{1}{2}(x + a)^{-\frac{1}{2}}$

$= \dfrac{1}{2\sqrt{x + a}}$

$\therefore\ \dfrac{dy_2}{dx} = \frac{1}{2}(2x - x^2)^{-\frac{1}{2}}(2 - 2x)$

$= \dfrac{1 - x}{\sqrt{2x - x^2}}$

y_1 intersects y_2 when $y_1 = y_2$

$\therefore\ \sqrt{x + a} = \sqrt{2x - x^2}$

$\therefore\ x + a = 2x - x^2$ .... (1)

But $\dfrac{dy_1}{dx} = \dfrac{dy_2}{dx}$ {the gradients are equal when y_1 and y_2 intersect}

$\therefore \dfrac{1}{2\sqrt{x+a}} = \dfrac{1-x}{\sqrt{2x-x^2}}$

$\therefore \dfrac{1}{2\cancel{\sqrt{2x-x^2}}} = \dfrac{1-x}{\cancel{\sqrt{2x-x^2}}}$ {using (1)}

$\therefore \frac{1}{2} = 1-x$

$\therefore x = 1 - \frac{1}{2} = \frac{1}{2}$

$\therefore \frac{1}{2} + a = 2(\frac{1}{2}) - (\frac{1}{2})^2$ {using (1)}

$\therefore a = 1 - \frac{1}{4} - \frac{1}{2}$

$= \frac{1}{4}$

$y = \sqrt{x+a}$

$= \sqrt{\frac{1}{2}+\frac{1}{4}}$

$= \sqrt{\frac{3}{4}}$

$= \frac{\sqrt{3}}{2}$

$\therefore a = \frac{1}{4}$ and the point of intersection is $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

21 P$(-2, 3)$, Q$(6, -3)$

$\therefore$ the gradient of line (PQ) is $\dfrac{y_2-y_1}{x_2-x_1} = \dfrac{-3-3}{6-(-2)} = -\frac{6}{8} = -\frac{3}{4}$

and the equation of line (PQ) is $\dfrac{y-3}{x-(-2)} = -\frac{3}{4}$

$\therefore 4y - 12 = -3x - 6$

$\therefore y = \dfrac{-3x+6}{4}$

Line (PQ) intersects $y = \dfrac{b}{(x+1)^2}$ when $\dfrac{-3x+6}{4} = \dfrac{b}{(x+1)^2}$

$\therefore b = \dfrac{3(2-x)(x+1)^2}{4}$ .... (1)

Now $y = \dfrac{b}{(x+1)^2} = b(x+1)^{-2}$

$\therefore \dfrac{dy}{dx} = b(-2)(x+1)^{-3} = \dfrac{-2b}{(x+1)^3}$

The line (PQ) is a tangent to $y = \dfrac{b}{(x+1)^2}$ so their gradients are the same at the point of contact.

$\therefore \dfrac{-2b}{(x+1)^3} = -\frac{3}{4}$

$\therefore b = \dfrac{3(x+1)^3}{8}$ .... (2)

$\therefore \dfrac{\cancel{3}(x+1)^{\cancel{3}}}{8} = \dfrac{\cancel{3}(2-x)\cancel{(x+1)^2}}{4}$ {equating (1) and (2)}

$\therefore (x+1) = 2(2-x)$

$\therefore x+1 = 4-2x$

$\therefore 3x = 3$

$\therefore x = 1$

$\therefore b = \dfrac{3(1+1)^3}{8}$

$= \dfrac{3 \times 2^3}{8}$

$= 3$

22 $y = 3e^{-x}$ and $y = 2 + e^x$ meet when $3e^{-x} = 2 + e^x$

$\therefore 3 = 2e^x + e^{2x}$ $\{\times e^x\}$

$\therefore e^{2x} + 2e^x - 3 = 0$

$\therefore (e^x+3)(e^x-1) = 0$

$\therefore e^x = -3$ or 1

$\therefore e^x = 1$ and so $x = 0$ {as $e^x > 0$}

Now when $x=0$, $y=3e^0=3$, so the graphs meet at $(0, 3)$.

For $y=2+e^x$, $\dfrac{dy}{dx}=e^x$,

so at the point $(0, 3)$, $\dfrac{dy}{dx}=e^0=1$

$\therefore$ the gradient of the tangent at this point is 1

$\therefore$ the tangent has direction vector $\binom{1}{1}$

For $y=3e^{-x}$, $\dfrac{dy}{dx}=-3e^{-x}$,

so at the point $(0, 3)$, $\dfrac{dy}{dx}=-3$

$\therefore$ the gradient of the tangent at this point is -3

$\therefore$ the tangent has direction vector $\binom{1}{-3}$

If θ is the acute angle between the tangents, then

$$\cos\theta=\frac{|1(1)+1(-3)|}{\sqrt{1^2+1^2}\sqrt{1^2+(-3)^2}}=\frac{|-2|}{\sqrt{2}\sqrt{10}}=\frac{2}{\sqrt{20}}=\frac{1}{\sqrt{5}}$$

$$\therefore\ \theta\approx 63.43^\circ$$

23 **a** $y=ax^2$, $a>0$ touches $y=\ln x$ when $ax^2=\ln x$

If the curves touch when $x=b$ then $ab^2=\ln b$ .... (1)

Now for $y=ax^2$, $\dfrac{dy}{dx}=2ax$ and for $y=\ln x$, $\dfrac{dy}{dx}=\dfrac{1}{x}$

$\therefore$ when $x=b$, $\dfrac{dy}{dx}=2ab$ $\qquad\therefore$ when $x=b$, $\dfrac{dy}{dx}=\dfrac{1}{b}$

Since the curves touch each other, they share a common tangent. $\therefore\ 2ab=\dfrac{1}{b}$ .... (2)

b Now $ab^2=\frac{1}{2}$ {from (2)}

and $ab^2=\ln b$ {from (1)}

$\therefore\ \ln b=\frac{1}{2}$

$\therefore\ b=e^{\frac{1}{2}}=\sqrt{e}$

When $x=b=\sqrt{e}$, $y=\ln x=\ln e^{\frac{1}{2}}=\frac{1}{2}$

$\therefore$ the point of contact is $(\sqrt{e}, \frac{1}{2})$.

c $a=\dfrac{1}{2b^2}$ {from (2)}

$\therefore\ a=\dfrac{1}{2(\sqrt{e})^2}=\dfrac{1}{2e}$

d The tangent has gradient $\dfrac{1}{b}=\dfrac{1}{\sqrt{e}}$ and passes through $(\sqrt{e}, \frac{1}{2})$

$\therefore$ the tangent is $\dfrac{y-\frac{1}{2}}{x-\sqrt{e}}=\dfrac{1}{\sqrt{e}}$ $\quad\therefore\ y-\frac{1}{2}=\dfrac{1}{\sqrt{e}}(x-\sqrt{e})$

$$\therefore\ y-\tfrac{1}{2}=\frac{1}{\sqrt{e}}x-1$$

$$\therefore\ y=e^{-\frac{1}{2}}x-\tfrac{1}{2}$$

24 **a** $x^2+y^2=1$

$\therefore\ y^2=1-x^2$

$\therefore\ y=\pm\sqrt{1-x^2}$

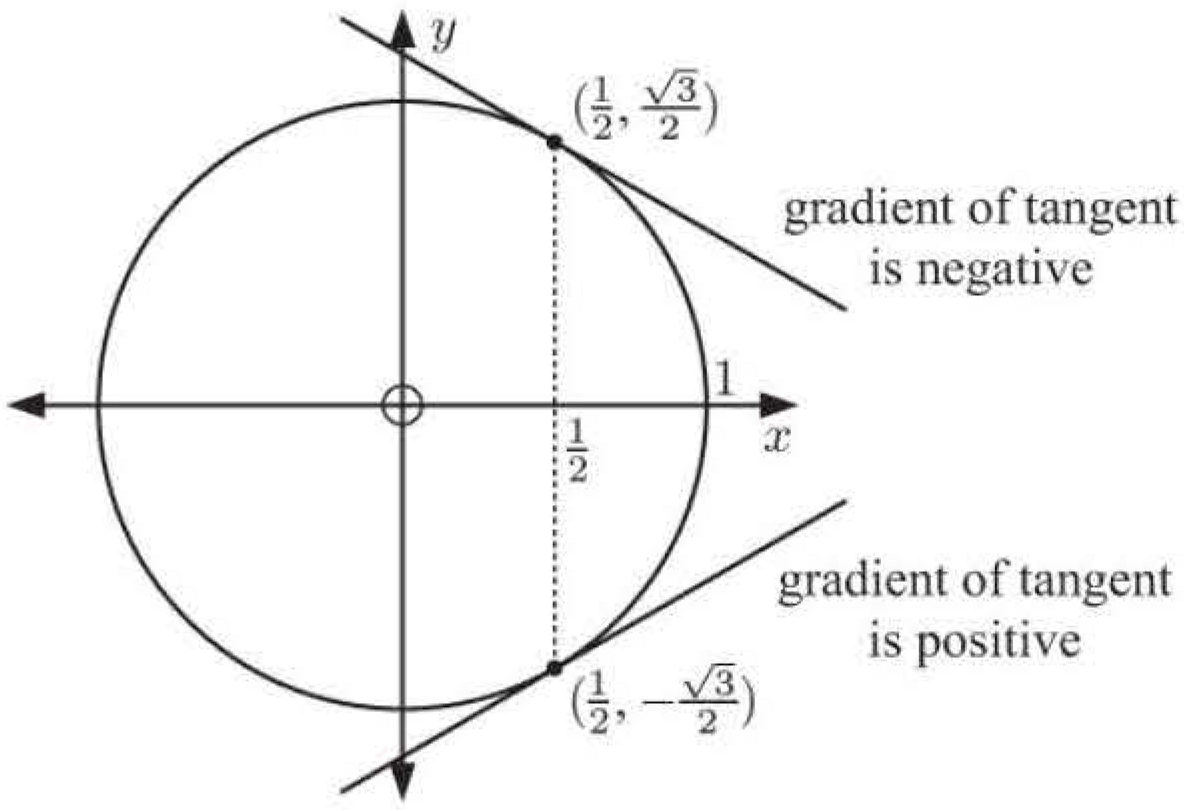
$=\pm(1-x^2)^{\frac{1}{2}}$

When $x=\frac{1}{2}$, then $y=\pm\sqrt{1-(\frac{1}{2})^2}$

$=\pm\sqrt{1-\frac{1}{4}}$

$=\pm\sqrt{\frac{3}{4}}$

$=\dfrac{\pm\sqrt{3}}{2}$



So we need to find the tangents at $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$.

Now $\dfrac{dy}{dx} = \pm\frac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x)$

$$= \frac{-x}{\pm\sqrt{1-x^2}}$$

$$= \mp\frac{x}{\sqrt{1-x^2}}$$

When $x = \frac{1}{2}$, then $\dfrac{dy}{dx} = \mp\dfrac{\frac{1}{2}}{\sqrt{1-(\frac{1}{2})^2}} = \mp\dfrac{\frac{1}{2}}{\sqrt{\frac{3}{4}}} = \mp\frac{1}{2}\times\frac{2}{\sqrt{3}} = \mp\frac{1}{\sqrt{3}}$

$\therefore$ the gradient of the tangent at $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ is $-\frac{1}{\sqrt{3}}$

and the gradient of the tangent at $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ is $\frac{1}{\sqrt{3}}$

$\therefore$ the equation of the tangent at $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ is $\dfrac{y-\frac{\sqrt{3}}{2}}{x-\frac{1}{2}} = -\frac{1}{\sqrt{3}}$

$$\therefore\ \sqrt{3}(y-\tfrac{\sqrt{3}}{2}) = -(x-\tfrac{1}{2})$$

$$\therefore\ \sqrt{3}y-\tfrac{3}{2} = -x+\tfrac{1}{2}$$

$$\therefore\ x+\sqrt{3}y = 2$$

and the equation of the tangent at $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ is $\dfrac{y-(-\frac{\sqrt{3}}{2})}{x-\frac{1}{2}} = \frac{1}{\sqrt{3}}$

$$\therefore\ \sqrt{3}(y+\tfrac{\sqrt{3}}{2}) = x-\tfrac{1}{2}$$

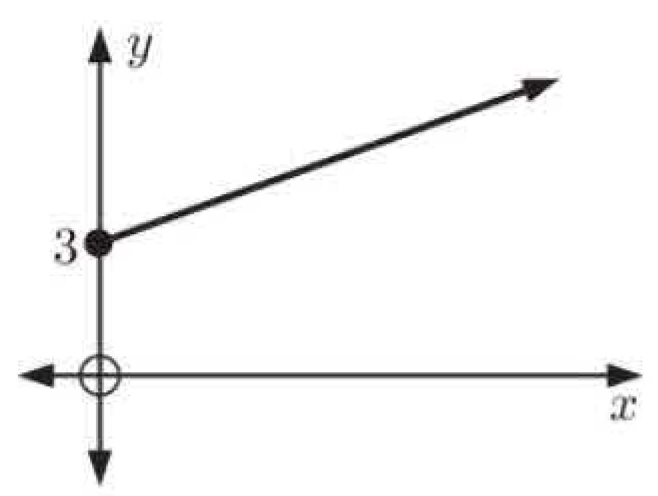
$$\therefore\ \sqrt{3}y+\tfrac{3}{2} = x-\tfrac{1}{2}$$

$$\therefore\ x-\sqrt{3}y = 2$$

b On the x-axis $y = 0$, $\therefore$ $x = 2$.
So, the tangents intersect on the x-axis at $(2, 0)$.

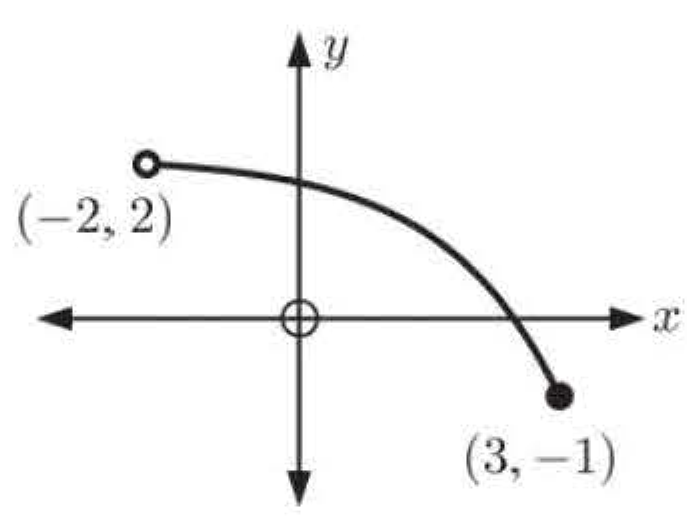
EXERCISE 19B

1 **a**



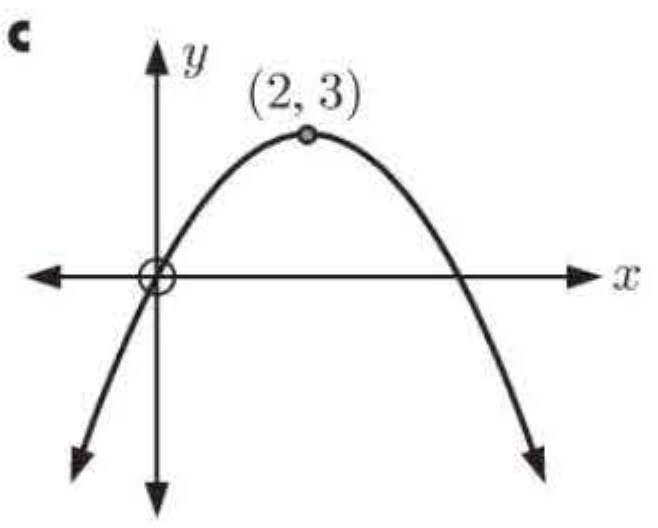
i $x \geqslant 0$ **ii** never

b



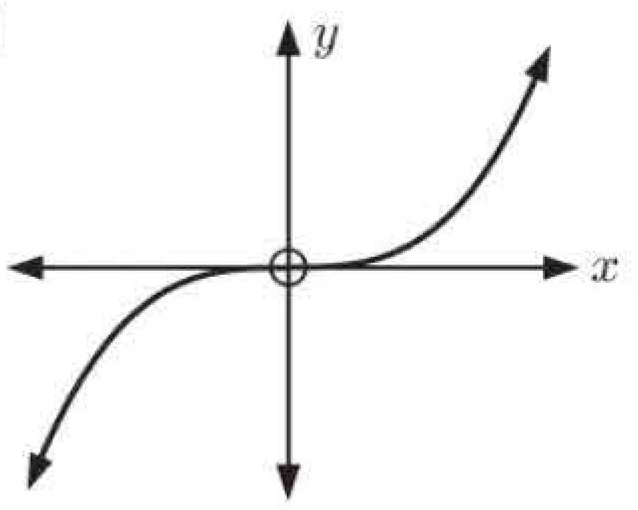
i never **ii** $-2 < x \leqslant 3$

c



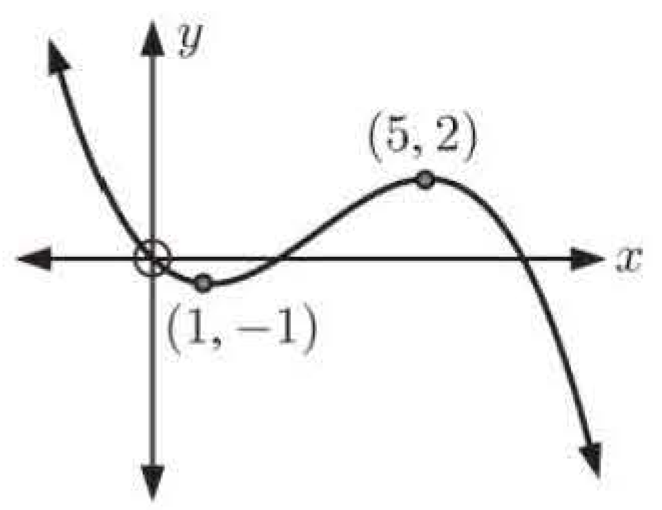
i $x \leqslant 2$ **ii** $x \geqslant 2$

d



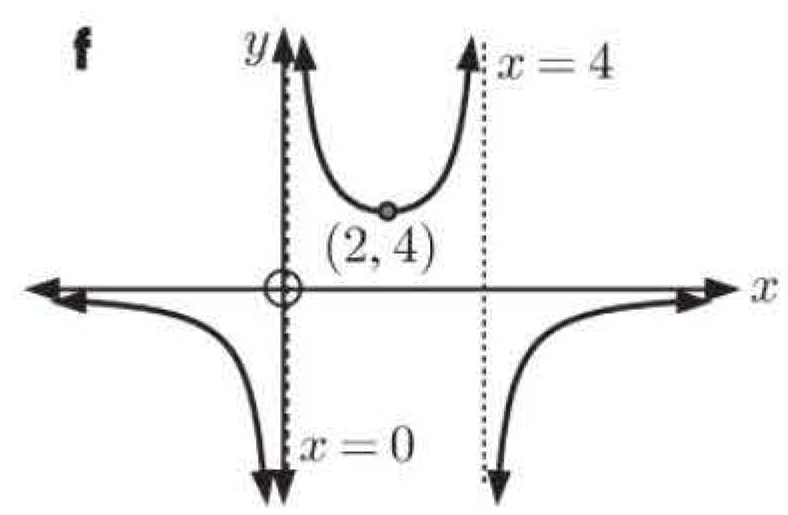
i all real x **ii** never

e



i $1 \leqslant x \leqslant 5$
ii $x \leqslant 1$, $x \geqslant 5$

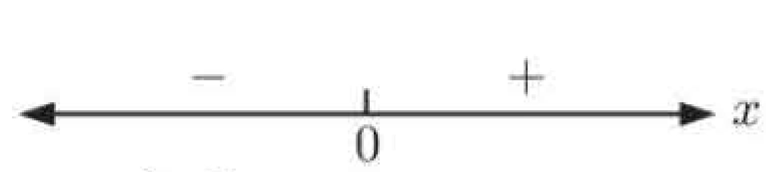
f



i $2 \leqslant x < 4$, $x > 4$
ii $x < 0$, $0 < x \leqslant 2$

2 **a** $f(x) = x^2$, $f'(x) = 2x$

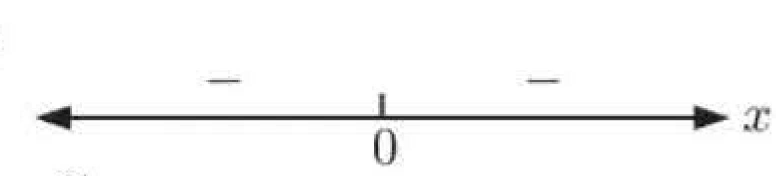
Sign diagram of $f'(x)$:



increasing when $x \geqslant 0$,
decreasing when $x \leqslant 0$

b $f(x) = -x^3$, $f'(x) = -3x^2$

Sign diagram of $f'(x)$:

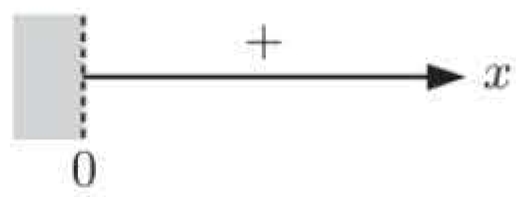


decreasing for all x

c $f(x) = \sqrt{x} = x^{\frac{1}{2}}$,

$f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \dfrac{1}{2\sqrt{x}}$

Sign diagram of $f'(x)$:



$f(x)$ is only defined when $x \geqslant 0$

increasing when $x \geqslant 0$, never decreasing

d $f(x) = 2x^2 + 3x - 4$, $f'(x) = 4x + 3$

Sign diagram of $f'(x)$:

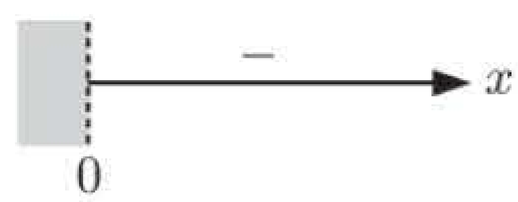
increasing when $x \geqslant -\frac{3}{4}$,

decreasing when $x \leqslant -\frac{3}{4}$

e $f(x) = \dfrac{2}{\sqrt{x}} = 2x^{-\frac{1}{2}}$

$f'(x) = -x^{-\frac{3}{2}} = \dfrac{-1}{x\sqrt{x}}$

Sign diagram of $f'(x)$:



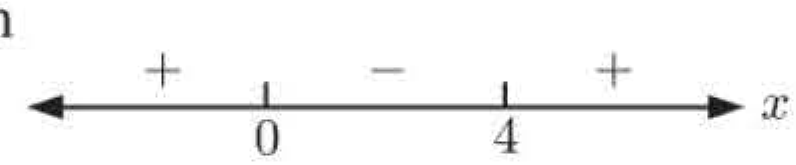
$f(x)$ is only defined for $x > 0$

never increasing, decreasing when $x > 0$

f $f(x) = x^3 - 6x^2$, $f'(x) = 3x^2 - 12x$

$= 3x(x - 4)$

Sign diagram of $f'(x)$:

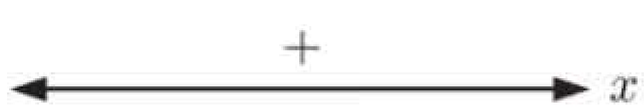


increasing when $x \leqslant 0$ or $x \geqslant 4$,

decreasing when $0 \leqslant x \leqslant 4$

g $f(x) = e^x$, $f'(x) = e^x$

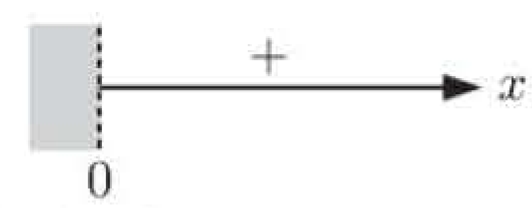
Sign diagram of $f'(x)$:



$f(x)$ is increasing for all x

h $f(x) = \ln x$, $f'(x) = \dfrac{1}{x}$

Sign diagram of $f'(x)$:



$f(x)$ is only defined when $x > 0$

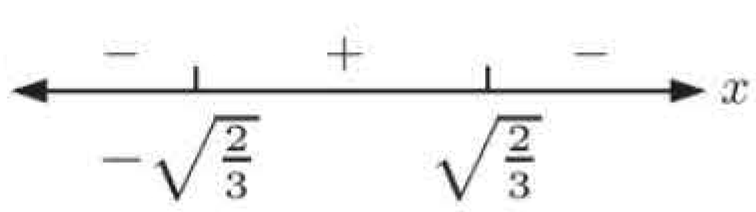
increasing when $x > 0$, never decreasing

i $f(x) = -2x^3 + 4x$

$f'(x) = -6x^2 + 4$

$= -2(3x^2 - 2)$

Sign diagram of $f'(x)$:

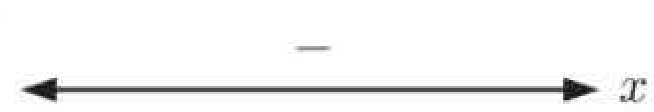


increasing for $-\sqrt{\frac{2}{3}} \leqslant x \leqslant \sqrt{\frac{2}{3}}$,

decreasing for $x \leqslant -\sqrt{\frac{2}{3}}$ or $x \geqslant \sqrt{\frac{2}{3}}$

j $f(x) = 3 + e^{-x}$, $f'(x) = -e^{-x}$

Sign diagram of $f'(x)$:

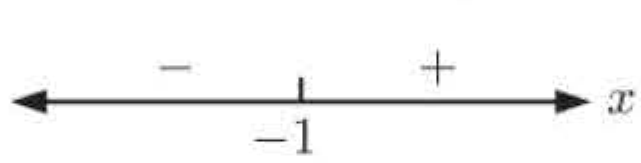


$f(x)$ is decreasing for all x.

k $f(x) = xe^x$, $f'(x) = e^x + xe^x$

$= e^x(1 + x)$

Sign diagram of $f'(x)$:



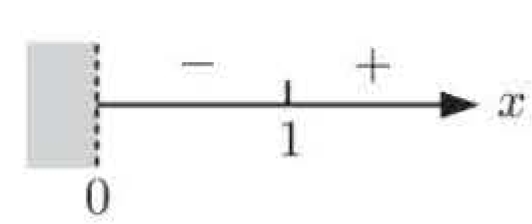
increasing when $x \geqslant -1$

decreasing when $x \leqslant -1$.

l $f(x) = x - 2\sqrt{x} = x - 2x^{\frac{1}{2}}$

$f'(x) = 1 - x^{-\frac{1}{2}} = 1 - \dfrac{1}{\sqrt{x}} = \dfrac{\sqrt{x} - 1}{\sqrt{x}}$

Sign diagram of $f'(x)$:



increasing when $x \geqslant 1$,

decreasing when $0 \leqslant x \leqslant 1$

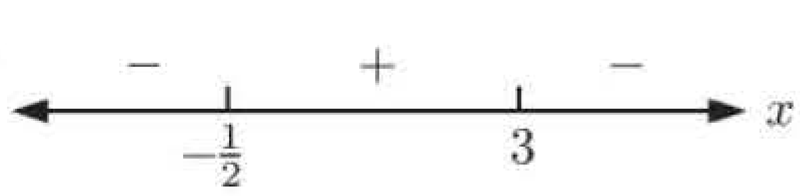
3 **a** $f(x) = -4x^3 + 15x^2 + 18x + 3$

$f'(x) = -12x^2 + 30x + 18$

$= -6(2x^2 - 5x - 3)$

$= -6(2x + 1)(x - 3)$

Sign diagram of $f'(x)$:



increasing when $-\frac{1}{2} \leqslant x \leqslant 3$,

decreasing when $x \leqslant -\frac{1}{2}$ or $x \geqslant 3$

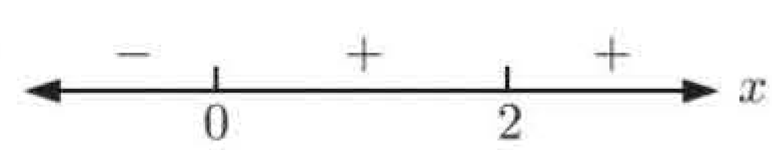
b $f(x) = 3x^4 - 16x^3 + 24x^2 - 2$,

$f'(x) = 12x^3 - 48x^2 + 48x$

$= 12x(x^2 - 4x + 4)$

$= 12x(x - 2)^2$

Sign diagram of $f'(x)$:



increasing when $x \geqslant 0$,

decreasing when $x \leqslant 0$

c $f(x) = 2x^3 + 9x^2 + 6x - 7,$

$f'(x) = 6x^2 + 18x + 6$
$\quad = 6(x^2 + 3x + 1)$

$f'(x) = 0$ when $x = \dfrac{-3 \pm \sqrt{9-4}}{2}$
$\quad = \dfrac{-3 \pm \sqrt{5}}{2}$

Sign diagram of $f'(x)$:

$+ \quad \dfrac{-3-\sqrt{5}}{2} \quad - \quad \dfrac{-3+\sqrt{5}}{2} \quad + \quad x$

increasing for $x \leqslant \dfrac{-3-\sqrt{5}}{2}$ or $x \geqslant \dfrac{-3+\sqrt{5}}{2}$,
decreasing for $\dfrac{-3-\sqrt{5}}{2} \leqslant x \leqslant \dfrac{-3+\sqrt{5}}{2}$

d $f(x) = x^3 - 6x^2 + 3x - 1,$

$f'(x) = 3x^2 - 12x + 3$
$\quad = 3(x^2 - 4x + 1)$

$f'(x) = 0$ when $x = \dfrac{4 \pm \sqrt{16-4}}{2}$
$\quad = 2 \pm \sqrt{3}$

Sign diagram of $f'(x)$:

$+ \quad 2-\sqrt{3} \quad - \quad 2+\sqrt{3} \quad + \quad x$

increasing when $x \leqslant 2-\sqrt{3}$ or $x \geqslant 2+\sqrt{3}$,
decreasing when $2-\sqrt{3} \leqslant x \leqslant 2+\sqrt{3}$

4 **a** $f(x) = \dfrac{4x}{x^2+1}$ is a quotient with

$u = 4x$ and $v = x^2 + 1$
$\therefore\ u' = 4$ and $v' = 2x$

$$\therefore\ f'(x) = \frac{4(x^2+1) - 4x \times 2x}{(x^2+1)^2} = \frac{4x^2 + 4 - 8x^2}{(x^2+1)^2} = \frac{4 - 4x^2}{(x^2+1)^2} = \frac{-4(x^2-1)}{(x^2+1)^2} = \frac{-4(x+1)(x-1)}{(x^2+1)^2}$$

Sign diagram of $f'(x)$:

$- \quad -1 \quad + \quad 1 \quad - \quad x$

b $f(x)$ is increasing for $-1 \leqslant x \leqslant 1$,
decreasing for $x \leqslant -1$ and $x \geqslant 1$

5 **a** $f(x) = \dfrac{4x}{(x-1)^2}$ is a quotient with

$u = 4x$ and $v = (x-1)^2$
$\therefore\ u' = 4$ and $v' = 2(x-1)$

$$\therefore\ f'(x) = \frac{4(x-1)^2 - 8x(x-1)}{(x-1)^4} = \frac{4(x-1)[(x-1) - 2x]}{(x-1)^4} = \frac{4(-1-x)}{(x-1)^3} = \frac{-4(x+1)}{(x-1)^3}$$

Sign diagram of $f'(x)$:

$- \quad -1 \quad + \quad 1 \quad - \quad x$

b $f(x)$ is increasing for $-1 \leqslant x < 1$,
decreasing for $x \leqslant -1$ and $x > 1$

6 **a** $f(x) = \dfrac{-x^2 + 4x - 7}{x-1}$ is a quotient with $u = -x^2 + 4x - 7$ and $v = x - 1$
$\therefore\ u' = -2x + 4$ and $v' = 1$

$$\therefore\ f'(x) = \frac{(-2x+4)(x-1) - (-x^2+4x-7)(1)}{(x-1)^2} = \frac{-2x^2 + 6x - 4 + x^2 - 4x + 7}{(x-1)^2} = \frac{-x^2 + 2x + 3}{(x-1)^2} = \frac{-(x^2 - 2x - 3)}{(x-1)^2} = \frac{-(x+1)(x-3)}{(x-1)^2}$$

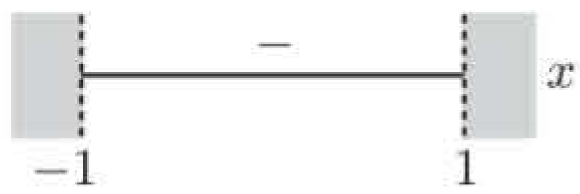
Sign diagram of $f'(x)$:

$- \quad -1 \quad + \quad 1 \quad + \quad 3 \quad - \quad x$

b $f(x)$ is increasing for $-1 \leqslant x < 1$ and $1 < x \leqslant 3$, and decreasing for $x \leqslant -1$ and $x \geqslant 3$.

7 **a** $y = \arccos x$, $\dfrac{dy}{dx} = \dfrac{-1}{\sqrt{1-x^2}}$, $x \in\,]-1, 1[$

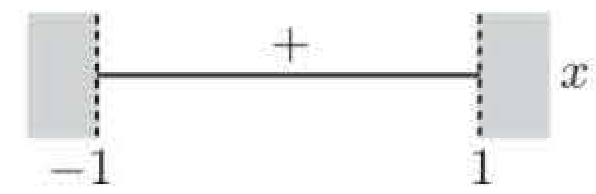
Sign diagram of y:



The function is decreasing for $-1 < x < 1$, never increasing.

b $y = \arcsin x$, $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-x^2}}$, $x \in\,]-1, 1[$

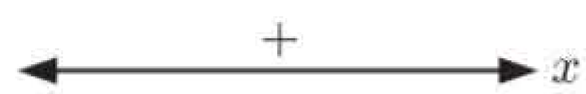
Sign diagram of y:



The function is increasing for $-1 < x < 1$, never decreasing.

c $y = \arctan x$, $\dfrac{dy}{dx} = \dfrac{1}{1+x^2}$, $x \in \mathbb{R}$

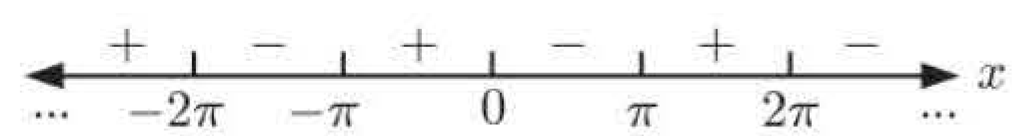
Sign diagram of y:



The function is increasing for all x, never decreasing.

d $y = \cos x$, $\dfrac{dy}{dx} = -\sin x$, $x \in \mathbb{R}$

Sign diagram of y:



The function is increasing for the interval $-\pi \leqslant x \leqslant 0$ but this is repeated periodically every 2π for the whole real number line.

$\therefore\; -\pi + 2k\pi \leqslant x \leqslant 0 + 2k\pi,\ k \in \mathbb{Z}$

$\therefore\; \pi(2k-1) \leqslant x \leqslant 2k\pi,\ k \in \mathbb{Z}$

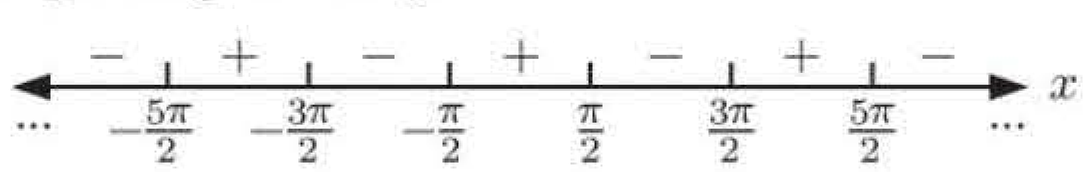
The function is decreasing for the interval $0 \leqslant x \leqslant \pi$ but this is repeated periodically every 2π for the whole real number line.

$\therefore\; 0 + 2k\pi \leqslant x \leqslant \pi + 2k\pi,\ k \in \mathbb{Z}$

$\therefore\; 2k\pi \leqslant x \leqslant \pi(1+2k),\ k \in \mathbb{Z}$

e $y = \sin x$, $\dfrac{dy}{dx} = \cos x$, $x \in \mathbb{R}$

Sign diagram of y:



The function is increasing for the interval $-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$ but this is repeated periodically every 2π for the whole real number line.

$\therefore\; -\frac{\pi}{2} + 2k\pi \leqslant x \leqslant \frac{\pi}{2} + 2k\pi,\ k \in \mathbb{Z}$

$\therefore\; \pi(2k - \frac{1}{2}) \leqslant x \leqslant \pi(2k + \frac{1}{2}),\ k \in \mathbb{Z}$

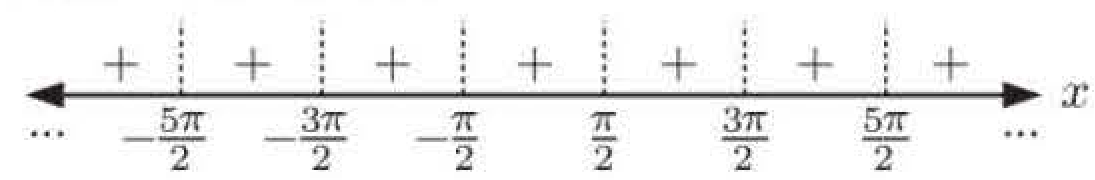
The function is decreasing for the interval $\frac{\pi}{2} \leqslant x \leqslant \frac{3\pi}{2}$ but this is repeated periodically every 2π for the whole real number line.

$\therefore\; \frac{\pi}{2} + 2k\pi \leqslant x \leqslant \frac{3\pi}{2} + 2k\pi,\ k \in \mathbb{Z}$

$\therefore\; \pi(\frac{1}{2} + 2k) \leqslant x \leqslant \pi(\frac{3}{2} + 2k),\ k \in \mathbb{Z}$

f $y = \tan x$, $\dfrac{dy}{dx} = \sec^2 x$, $x \in \mathbb{R}$, $x \neq \frac{\pi}{2} \pm k\pi$, $k \in \mathbb{Z}$

Sign diagram of y:



The function is increasing for the interval $-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$ but this is repeated periodically every π for the whole real number line.

$\therefore\; -\frac{\pi}{2} + k\pi \leqslant x \leqslant \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}$

$\therefore\; \pi(k - \frac{1}{2}) \leqslant x \leqslant \pi(\frac{1}{2} + k),\ k \in \mathbb{Z}$

It is never decreasing.

8 **a** $f(x) = \dfrac{x^3}{x^2-1}$ is a quotient with $u = x^3$ and $v = x^2 - 1$

$\therefore\; u' = 3x^2$ and $v' = 2x$

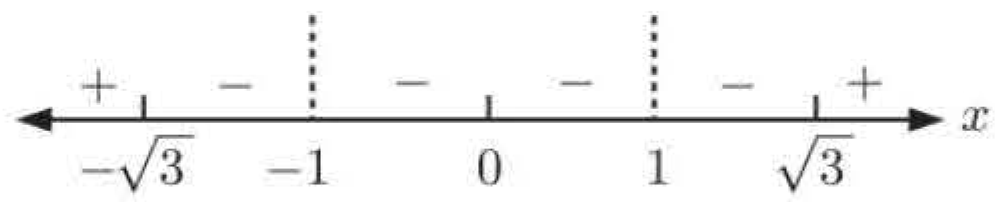
$$\therefore\; f'(x) = \frac{3x^2(x^2-1) - x^3 \times 2x}{(x^2-1)^2}$$

$$= \frac{3x^4 - 3x^2 - 2x^4}{(x^2-1)^2}$$

$$\therefore \ f'(x) = \frac{x^2(x^2-3)}{(x^2-1)^2}$$

$$= \frac{x^2(x+\sqrt{3})(x-\sqrt{3})}{(x-1)^2(x+1)^2}$$

Sign diagram of $f'(x)$:

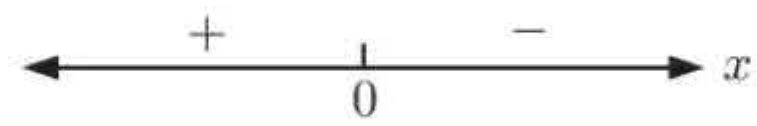


$\therefore$ $f(x)$ is increasing for $x \leqslant -\sqrt{3}$ and $x \geqslant \sqrt{3}$, and
decreasing for $-\sqrt{3} \leqslant x < -1$, $-1 < x < 1$, and $1 < x \leqslant \sqrt{3}$.

b $f(x) = e^{-x^2}$

$\therefore \ f'(x) = -2xe^{-x^2}$

Sign diagram of $f'(x)$:



$\therefore$ $f(x)$ is increasing for $x \leqslant 0$ and decreasing for $x \geqslant 0$.

c $f(x) = x^2 + \dfrac{4}{x-1} = x^2 + 4(x-1)^{-1}$

$$\therefore \ f'(x) = 2x - 4(x-1)^{-2} \times 1$$

$$= 2x - \frac{4}{(x-1)^2}$$

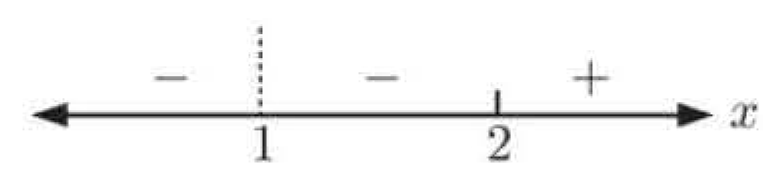
$$= \frac{2x(x-1)^2 - 4}{(x-1)^2}$$

$$= \frac{2x(x^2-2x+1) - 4}{(x-1)^2}$$

$$= \frac{2x^3 - 4x^2 + 2x - 4}{(x-1)^2}$$

$$= \frac{(x-2)(2x^2+2)}{(x-1)^2}$$

Sign diagram of $f'(x)$:



$\therefore$ $f(x)$ is increasing for $x \geqslant 2$, and decreasing for $x < 1$ and $1 < x \leqslant 2$.

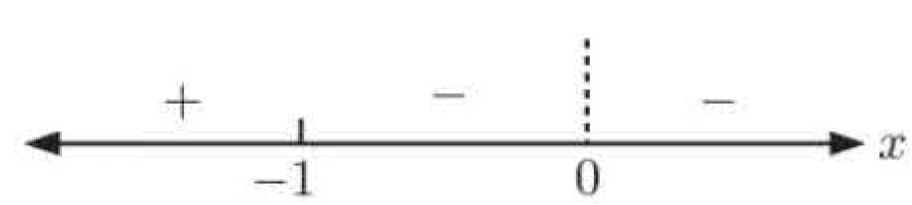
d $f(x) = \dfrac{e^{-x}}{x}$ is a quotient with $u = e^{-x}$ and $v = x$

$\therefore \ u' = -e^{-x}$ and $v' = 1$

$$\therefore \ f'(x) = \frac{-e^{-x}x - e^{-x} \times 1}{x^2}$$

$$= \frac{-e^{-x}(x+1)}{x^2}$$

Sign diagram of $f'(x)$:



$\therefore$ $f(x)$ is increasing for $x \leqslant -1$, and decreasing for $-1 \leqslant x < 0$ and $x > 0$.

e $f(x) = x + \cos x$

$\therefore \ f'(x) = 1 - \sin x$

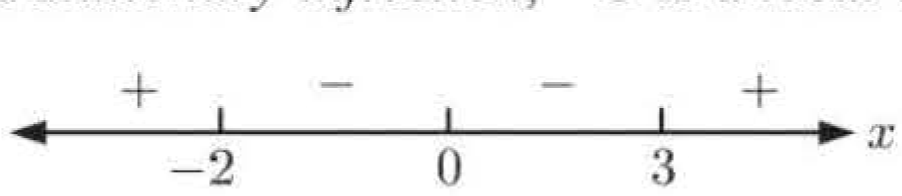
Sign diagram of $f'(x)$:

$\therefore$ $f(x)$ is increasing for all x, and never decreasing.

EXERCISE 19C

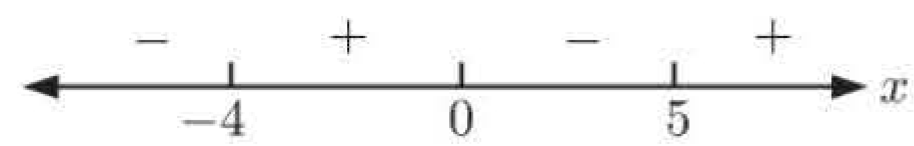
1 **a** A is a *local maximum*, B is a *stationary inflection*, C is a *local minimum*.

b $f'(x)$ has sign diagram:



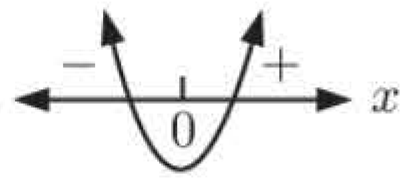
c **i** $f(x)$ is increasing for $x \leqslant -2$ and $x \geqslant 3$ **ii** $f(x)$ is decreasing for $-2 \leqslant x \leqslant 3$

d $f(x)$ has sign diagram:

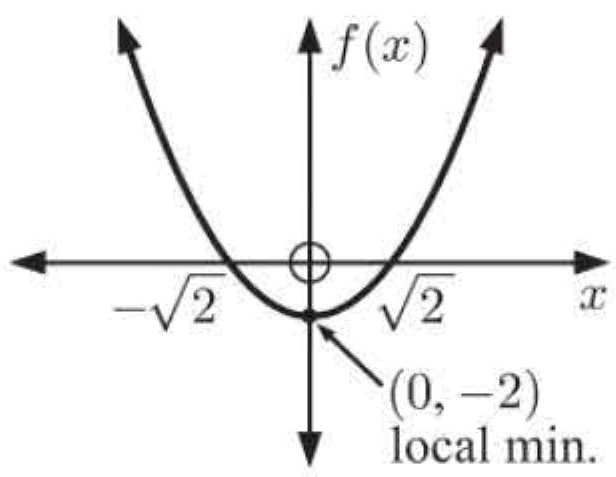


2 **a** $f(x) = x^2 - 2 \quad \therefore \quad f'(x) = 2x$

with sign diagram:

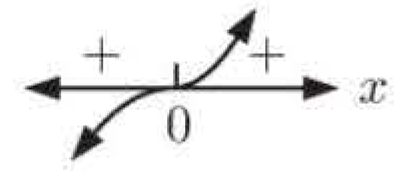


Now $f(0) = -2$,
so there is a local minimum at $(0, -2)$.

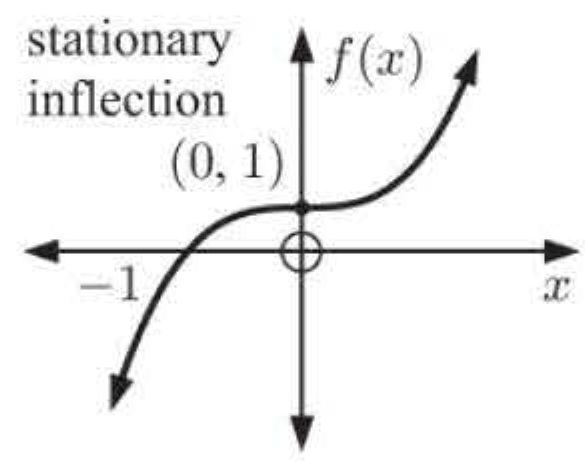


b $f(x) = x^3 + 1 \quad \therefore \quad f'(x) = 3x^2$

with sign diagram:



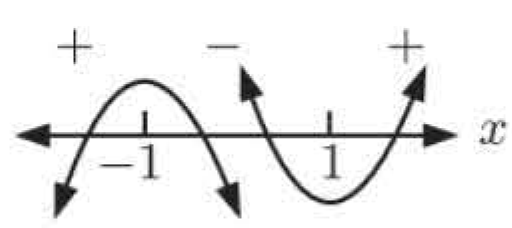
Now $f(0) = 1$,
so there is a stationary inflection at $(0, 1)$.



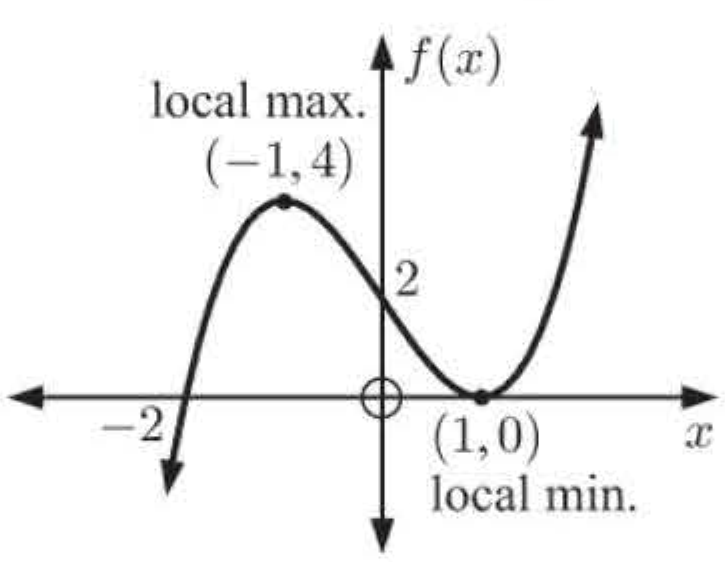
c $f(x) = x^3 - 3x + 2$

$$\begin{aligned} \therefore \quad f'(x) &= 3x^2 - 3 \\ &= 3(x^2 - 1) \\ &= 3(x + 1)(x - 1) \end{aligned}$$

with sign diagram:



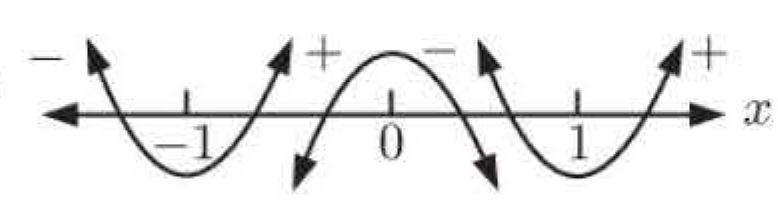
Now $f(-1) = 4$, $f(1) = 0$,
so there is a local maximum at $(-1, 4)$,
and a local minimum at $(1, 0)$.



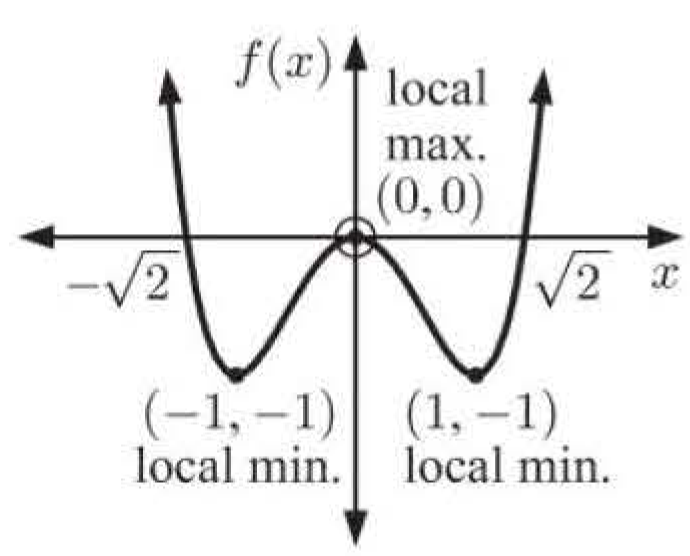
d $f(x) = x^4 - 2x^2$

$$\begin{aligned} \therefore \quad f'(x) &= 4x^3 - 4x \\ &= 4x(x^2 - 1) \\ &= 4x(x + 1)(x - 1) \end{aligned}$$

with sign diagram:



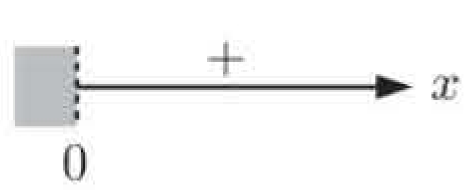
Now $f(-1) = -1$, $f(1) = -1$, $f(0) = 0$,
so there are local minima at $(-1, -1)$ and $(1, -1)$, and a local maximum at $(0, 0)$.



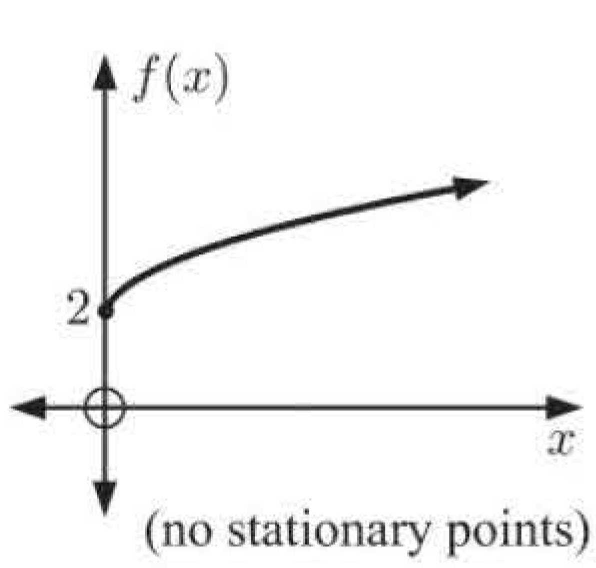
e $f(x) = \sqrt{x} + 2$

$$\begin{aligned} \therefore \quad f'(x) &= \tfrac{1}{2}x^{-\frac{1}{2}} \\ &= \frac{1}{2\sqrt{x}} \neq 0 \end{aligned}$$

with sign diagram:



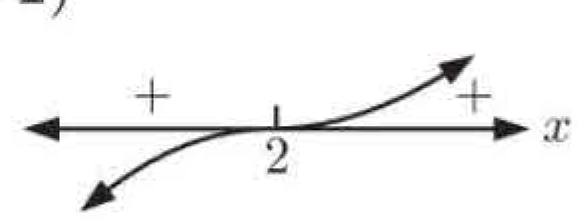
$\therefore$ no stationary points.



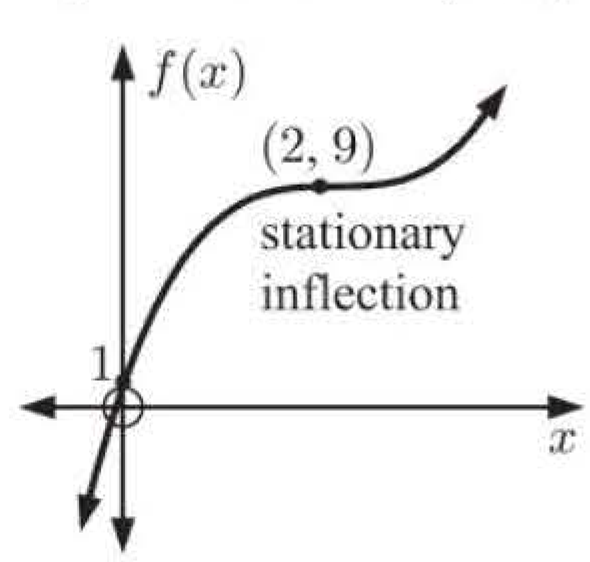
f $f(x) = x^3 - 6x^2 + 12x + 1$

$$\begin{aligned} \therefore \quad f'(x) &= 3x^2 - 12x + 12 \\ &= 3(x^2 - 4x + 4) \\ &= 3(x - 2)^2 \end{aligned}$$

with sign diagram:



Now $f(2) = 9$, so there is a stationary inflection at $(2, 9)$.

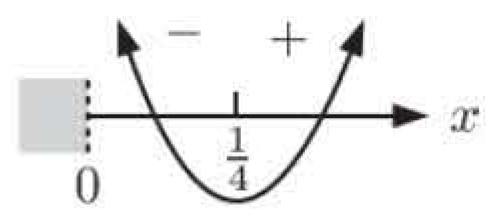


g $f(x) = x - \sqrt{x} = x - x^{\frac{1}{2}}$

$\therefore\ f'(x) = 1 - \frac{1}{2}x^{-\frac{1}{2}}$

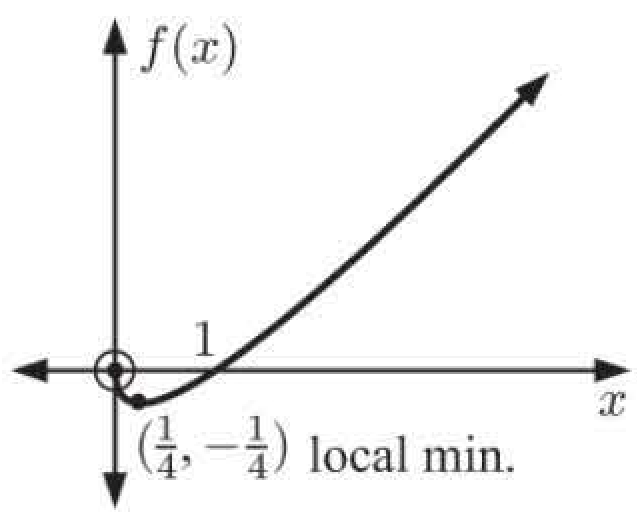
$= 1 - \dfrac{1}{2\sqrt{x}} = \dfrac{2\sqrt{x}-1}{2\sqrt{x}}$

with sign diagram:



$f(x)$ is defined for all $x \geqslant 0$

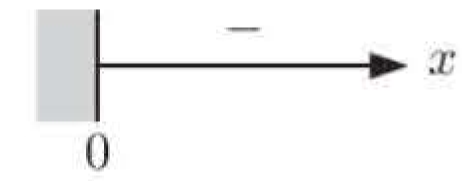
Now $f(\frac{1}{4}) = -\frac{1}{4}$, so there is a local minimum at $(\frac{1}{4}, -\frac{1}{4})$.



h $f(x) = 1 - x\sqrt{x} = 1 - x^{\frac{3}{2}}$

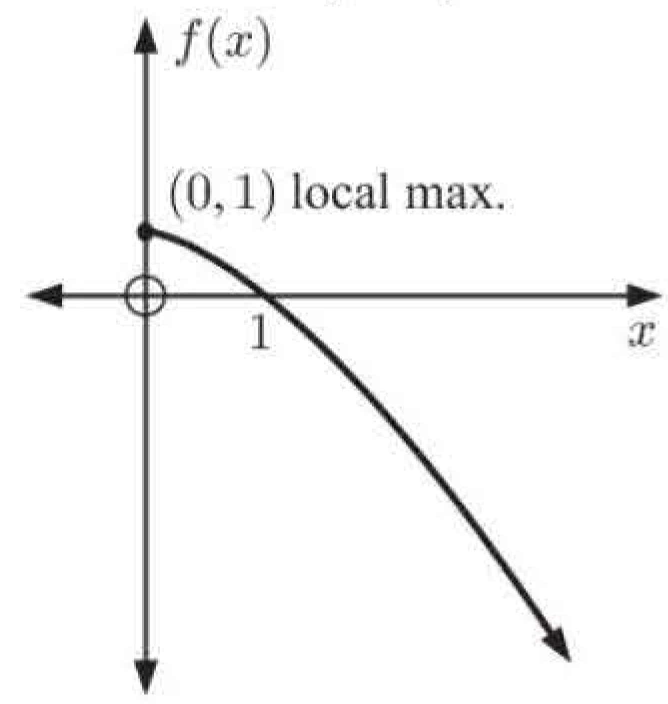
$\therefore\ f'(x) = -\frac{3}{2}x^{\frac{1}{2}} = \dfrac{-3\sqrt{x}}{2}$

with sign diagram:



$f(x)$ is only defined when $x \geqslant 0$

Now $f(0) = 1$, so there is a local maximum at $(0, 1)$.



i $f(x) = x^4 - 6x^2 + 8x - 3$

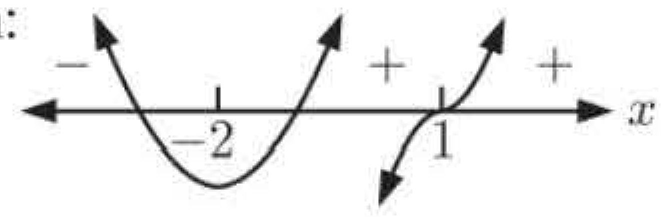
$\therefore\ f'(x) = 4x^3 - 12x + 8$

$= 4(x^3 - 3x + 2)$

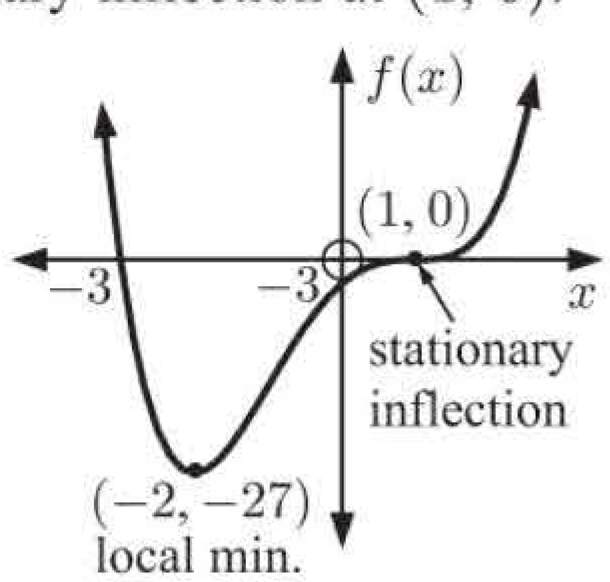
$= 4(x-1)(x^2 + x - 2)$

$= 4(x-1)(x+2)(x-1)$

with sign diagram:



Now $f(-2) = -27$, $f(1) = 0$, so there is a local minimum at $(-2, -27)$, and a stationary inflection at $(1, 0)$.

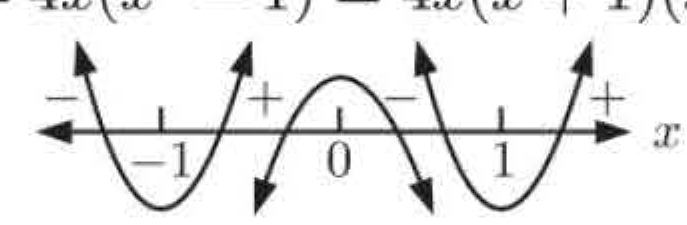


j $f(x) = x^4 - 2x^2 - 8$

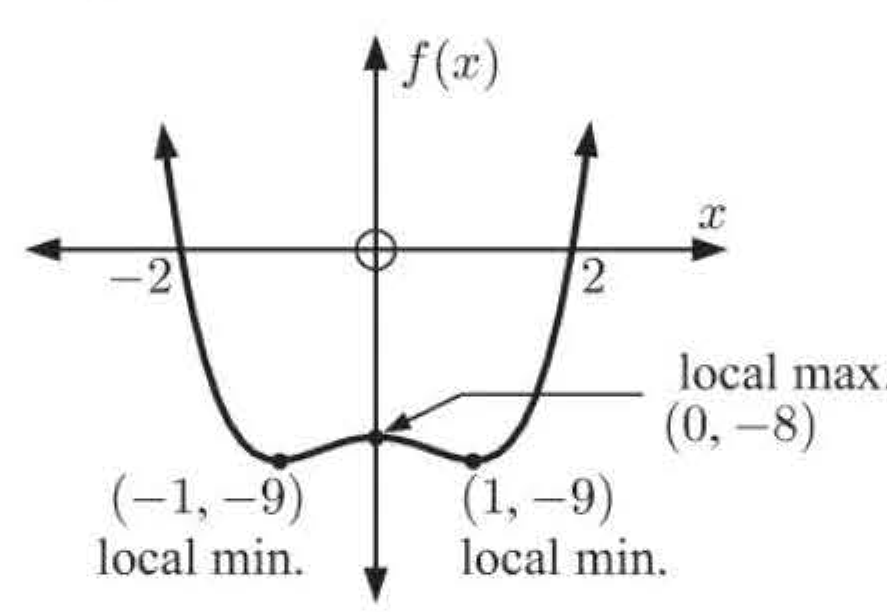
$\therefore\ f'(x) = 4x^3 - 4x$

$= 4x(x^2 - 1) = 4x(x+1)(x-1)$

with sign diagram:



Now $f(-1) = -9$, $f(1) = -9$, $f(0) = -8$, so there are local minima at $(-1, -9)$ and $(1, -9)$, and a local maximum at $(0, -8)$.

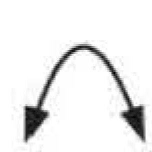
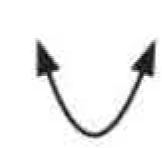


3 $f(x) = ax^2 + bx + c, \quad a \neq 0$

$\therefore\ f'(x) = 2ax + b$

$f(x)$ has a stationary point when $f'(x) = 0$

$\therefore\ x = -\dfrac{b}{2a}$

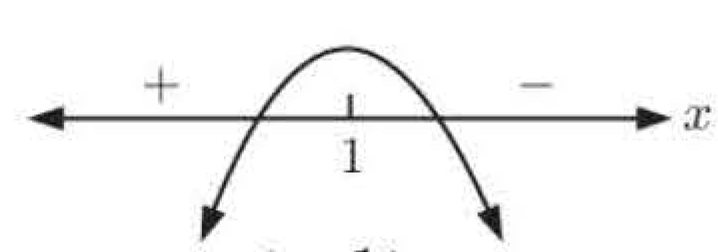
There is a local maximum when $a < 0$  and there is a local minimum when $a > 0$ 

4 **a** $y = xe^{-x}$

$\therefore\ \dfrac{dy}{dx} = 1e^{-x} - xe^{-x}$ {product rule}

$= e^{-x}(1-x)$

$= \dfrac{1-x}{e^x}$ which has sign diagram:

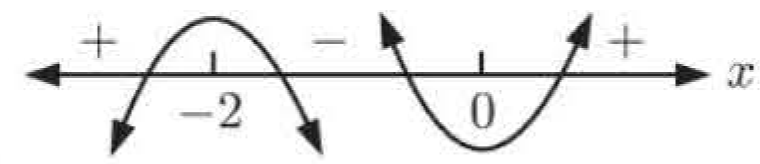


When $x = 1$, $y = 1e^{-1} = \dfrac{1}{e}$, so we have a local maximum at $\left(1, \dfrac{1}{e}\right)$.

b $y = x^2 e^x$

$\therefore \dfrac{dy}{dx} = 2xe^x + x^2e^x$ {product rule}

$= xe^x(2+x)$ which has sign diagram:



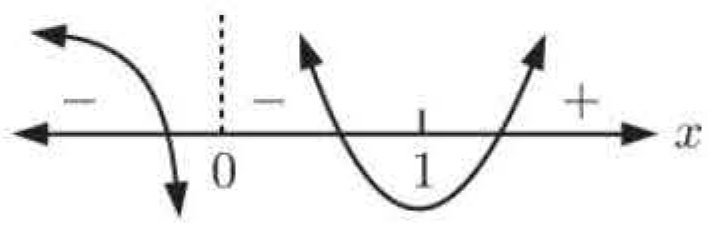
When $x = -2$, $y = 4e^{-2}$, and when $x = 0$, $y = 0$.

So, we have a local maximum at $\left(-2, \dfrac{4}{e^2}\right)$, and a local minimum at $(0, 0)$.

c $y = \dfrac{e^x}{x}$

$\therefore \dfrac{dy}{dx} = \dfrac{e^x x - e^x(1)}{x^2}$ {quotient rule}

$= \dfrac{e^x(x-1)}{x^2}$ which has sign diagram:



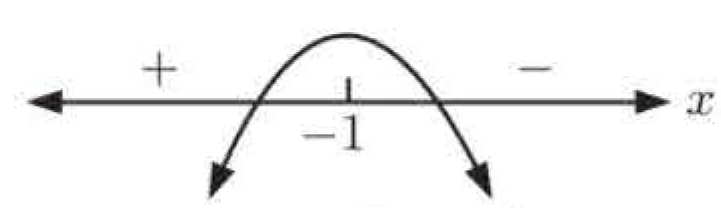
When $x = 1$, $y = \dfrac{e^1}{1} = e$, so we have a local minimum at $(1, e)$.

d $y = e^{-x}(x+2)$

$\therefore \dfrac{dy}{dx} = -e^{-x}(x+2) + e^{-x}$ {product rule}

$= e^{-x}(-x-2+1)$

$= e^{-x}(-x-1)$ which has sign diagram:



When $x = -1$, $y = e(-1+2) = e$, so we have a local maximum at $(-1, e)$.

5 $f(x) = 2x^3 + ax^2 - 24x + 1$

$\therefore f'(x) = 6x^2 + 2ax - 24$

But $f'(-4) = 0$, so $6(-4)^2 + 2a(-4) - 24 = 0$

$\therefore 96 - 8a - 24 = 0$

$\therefore 72 = 8a$

$\therefore a = 9$

6 **a** $f(x) = x^3 + ax + b$

$\therefore f'(x) = 3x^2 + a$

But $f'(-2) = 0$

$\therefore 3(-2)^2 + a = 0$

$\therefore 12 + a = 0$

$\therefore a = -12$

Also, $f(-2) = 3$

$\therefore (-2)^3 - 12(-2) + b = 3$

$\therefore -8 + 24 + b = 3$

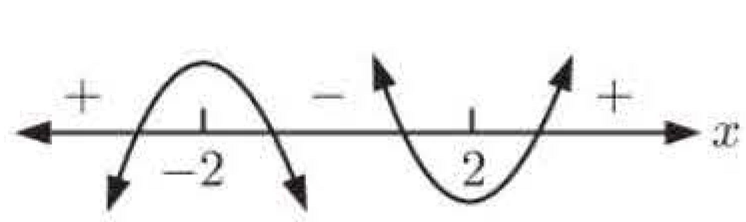
$\therefore b = -13$

b Now $f(x) = x^3 - 12x - 13$

$\therefore f'(x) = 3x^2 - 12$

$= 3(x^2 - 4)$

$= 3(x+2)(x-2)$ with sign diagram:



Now $f(2) = -29$, so there is a local minimum at $(2, -29)$ and a local maximum at $(-2, 3)$.

7 **a** $f(x)$ is defined when $\ln x$ is defined $\therefore$ $f(x)$ is defined for $x > 0$

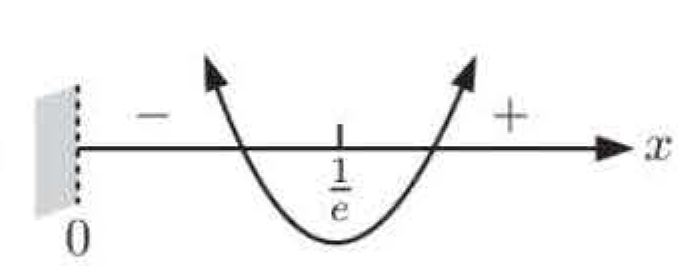
b $f'(x) = \ln x + \dfrac{x}{x}$ {product rule}

$= \ln x + 1$

which is 0 when $\ln x = -1$

$\therefore x = e^{-1} = \frac{1}{e}$

Sign diagram of $f'(x)$ is:



So, there is a local minimum at $\left(\frac{1}{e}, \frac{1}{e}\ln\frac{1}{e}\right)$

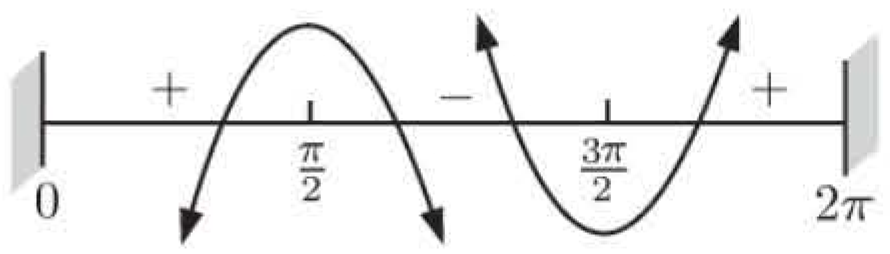
$\therefore$ the global minimum value of $f(x)$ is $\frac{1}{e}\ln e^{-1} = -\frac{1}{e}$

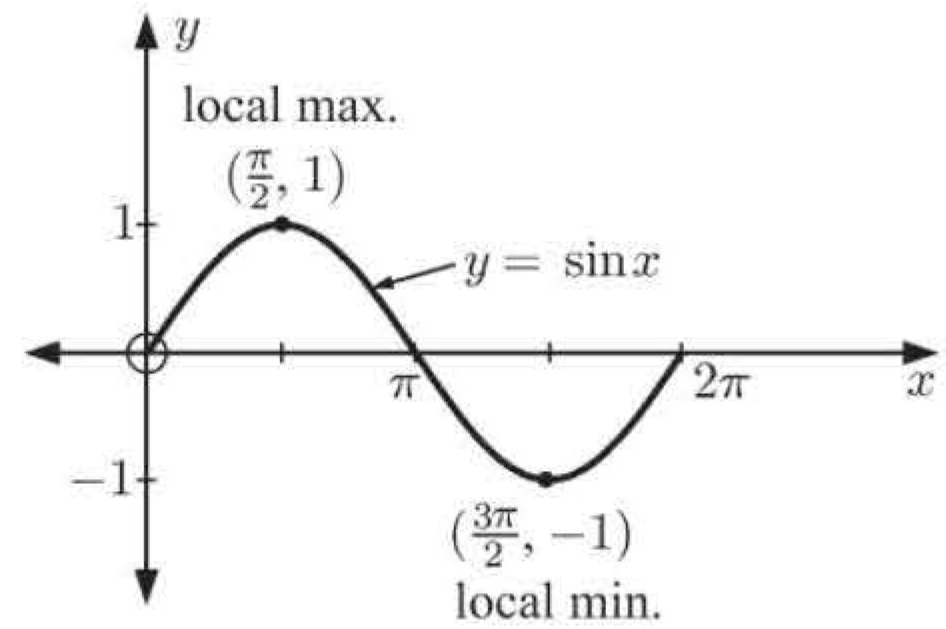
8 **a** If $f(x) = \sin x$ then $f'(x) = \cos x$

Stationary points occur when $f'(x) = 0$,

which is when $x = \frac{\pi}{2}, \frac{3\pi}{2}$

Sign diagram for $f'(x)$ is:





There is a local maximum at $(\frac{\pi}{2}, 1)$

and a local minimum at $(\frac{3\pi}{2}, -1)$.

b If $f(x) = \cos(2x)$ then $f'(x) = -2\sin(2x)$

$\therefore$ $f'(x) = 0$ when $-2\sin(2x) = 0$

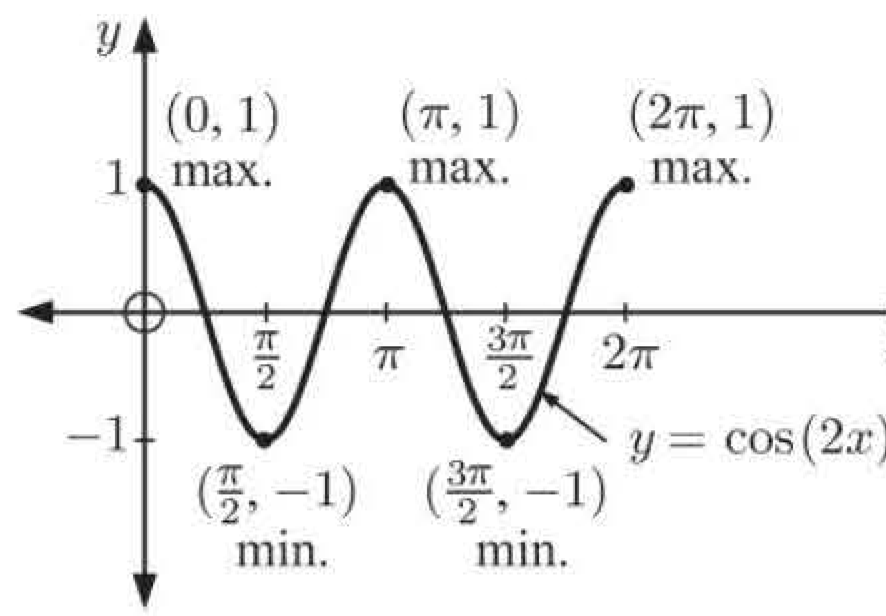
$\therefore$ $\sin(2x) = 0$

$\therefore$ $2x = k\pi$ for any integer k

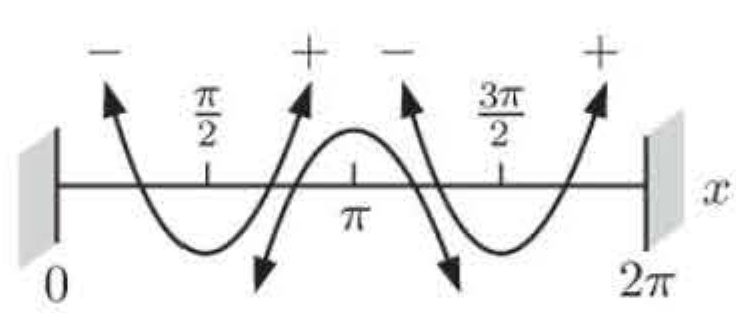
$\therefore$ $x = \dfrac{k\pi}{2}$

On the domain $0 \leqslant x \leqslant 2\pi$, $f'(x) = 0$

when $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, and 2π.



Sign diagram for $f'(x)$ is:



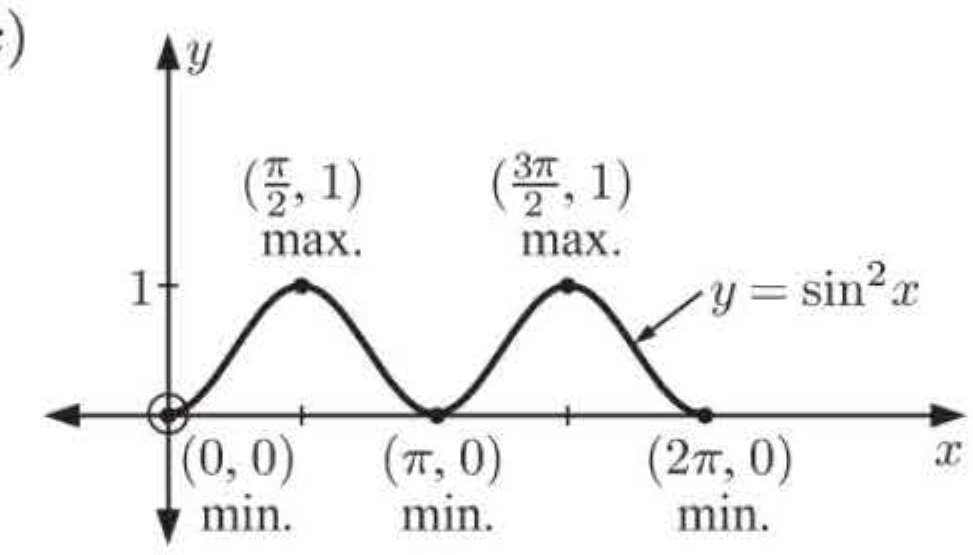
There are local maxima at $(0, 1)$, $(\pi, 1)$, $(2\pi, 1)$ and local minima at $(\frac{\pi}{2}, -1)$, $(\frac{3\pi}{2}, -1)$.

c If $f(x) = \sin^2 x$ then $f'(x) = 2\sin x \cos x = \sin(2x)$

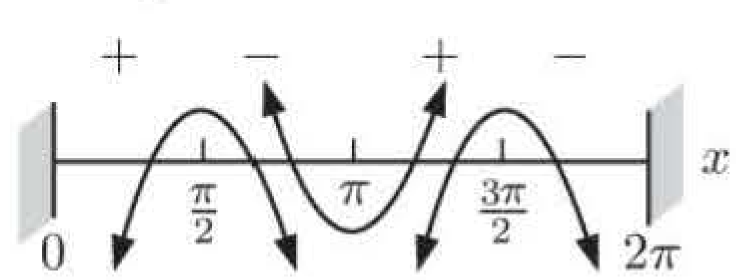
$\therefore$ $f'(x) = 0$ when $\sin(2x) = 0$

Using **b**, we know that on the domain $0 \leqslant x \leqslant 2\pi$

$f'(x) = 0$ when $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, and 2π.



Sign diagram for $f'(x)$ is:



There are local mimima at $(0, 0)$, $(\pi, 0)$, $(2\pi, 0)$ and local maxima at $(\frac{\pi}{2}, 1)$, $(\frac{3\pi}{2}, 1)$.

d If $f(x) = e^{\sin x}$ then $f'(x) = e^{\sin x} \times \cos x$

$\therefore$ $f'(x) = 0$ when $\cos x \, e^{\sin x} = 0$

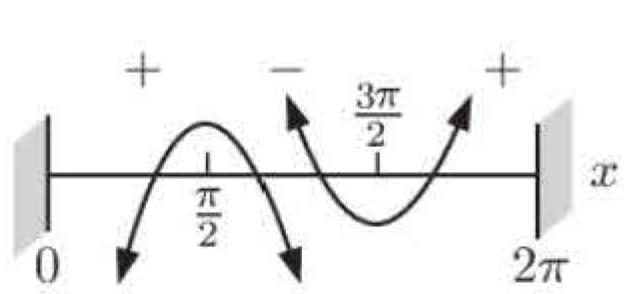
$\therefore$ $\cos x = 0$ $\{e^{\sin x} > 0$ for all $x\}$

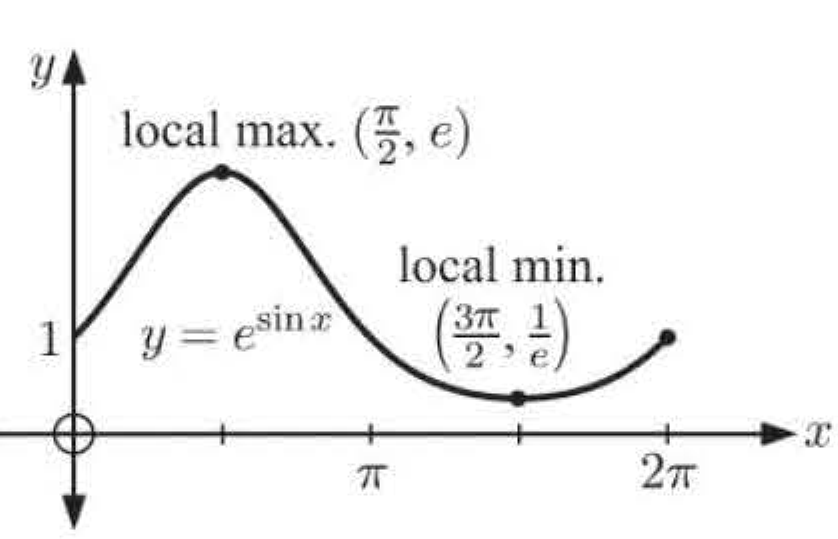
$\therefore$ $x = \frac{\pi}{2} + k\pi$, k an integer

On the domain $0 \leqslant x \leqslant 2\pi$, $f'(x) = 0$

when $x = \frac{\pi}{2}, \frac{3\pi}{2}$.

Sign diagram for $f'(x)$ is:





There is a local maximum at $(\frac{\pi}{2}, e)$

and a local minimum at $(\frac{3\pi}{2}, \frac{1}{e})$.

e If $f(x) = \sin(2x) + 2\cos x$ then $f'(x) = 2\cos(2x) - 2\sin x$

$\therefore$ $f'(x) = 0$ when $2\cos(2x) - 2\sin x = 0$

$\therefore$ $2(1 - 2\sin^2 x) - 2\sin x = 0$

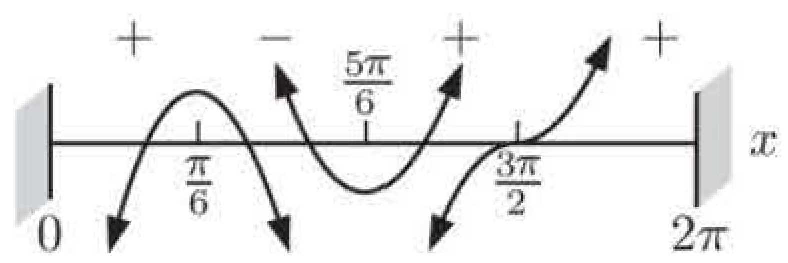
$\therefore$ $-2(2\sin^2 x + \sin x - 1) = 0$

$\therefore$ $-2(2\sin x - 1)(\sin x + 1) = 0$

$\therefore$ when $\sin x = \frac{1}{2}$ or $\sin x = -1$

On the domain $0 \leqslant x \leqslant 2\pi$, when $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$.

Sign diagram of $f'(x)$:



$f(\frac{\pi}{6}) = \sin(\frac{2\pi}{6}) + 2\cos(\frac{\pi}{6})$

$= \frac{\sqrt{3}}{2} + 2 \times \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$

$f(\frac{5\pi}{6}) = \sin(\frac{10\pi}{6}) + 2\cos(\frac{5\pi}{6})$

$= -\frac{\sqrt{3}}{2} + 2(-\frac{\sqrt{3}}{2}) = -\frac{3\sqrt{3}}{2}$

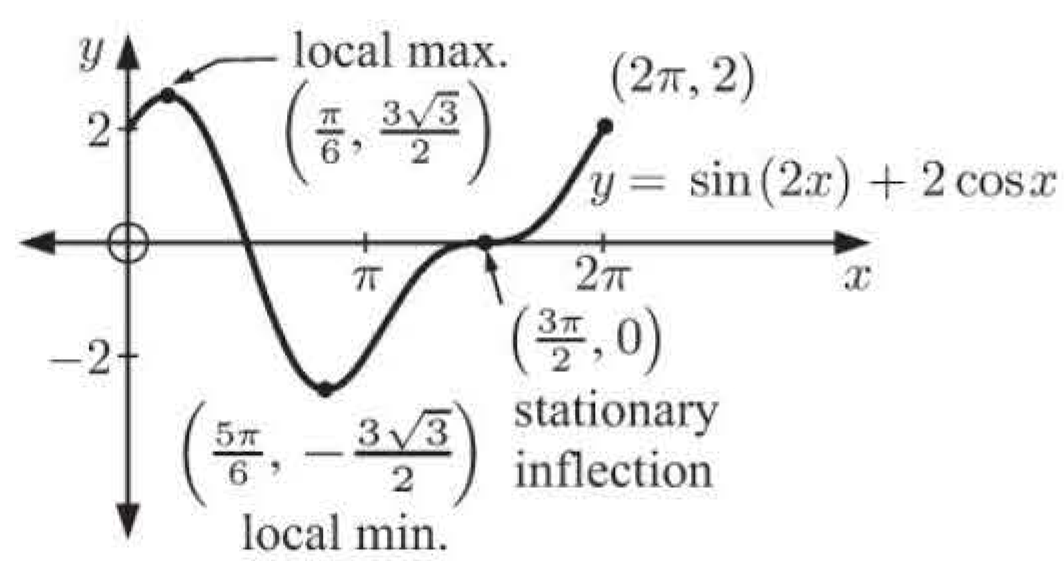
$f(\frac{3\pi}{2}) = \sin(3\pi) + 2\cos(\frac{3\pi}{2})$

$= 0 + 2 \times 0 = 0$

$\therefore$ there is a local maximum at $(\frac{\pi}{6}, \frac{3\sqrt{3}}{2})$,

a local minimum at $(\frac{5\pi}{6}, -\frac{3\sqrt{3}}{2})$

and a stationary point of inflection at $(\frac{3\pi}{2}, 0)$.



9 Let the cubic polynomial be

$P(x) = ax^3 + bx^2 + cx + d$

$\therefore$ $P'(x) = 3ax^2 + 2bx + c$ .... (1)

Now $(0, 2)$ lies on $P(x)$, so $P(0) = 2$

$\therefore$ $a(0) + b(0) + c(0) + d = 2$

$\therefore$ $d = 2$

The tangent at $(0, 2)$ is $y = 9x + 2$, so

$P'(0) = 9$

$\therefore$ $3a(0) + 2b(0) + c = 9$

$\therefore$ $c = 9$ .... (2)

There is a stationary point at $(-1, -7)$, so

$P'(-1) = 0$

$\therefore$ $3a(-1)^2 + 2b(-1) + c = 0$ {using (1)}

$\therefore$ $3a - 2b + c = 0$

So, using (2), $3a - 2b = -9$ .... (3)

Finally, $(-1, -7)$ lies on $P(x)$

$\therefore$ $a(-1)^3 + b(-1)^2 + c(-1) + d = -7$

$\therefore$ $-a + b - 9 + 2 = -7$

$\therefore$ $b - a = 0$

$\therefore$ $a = b$

So, using (3), $3a - 2a = -9$

$\therefore$ $a = -9$

$\therefore$ $a = b = -9$

$\therefore$ $P(x) = -9x^3 - 9x^2 + 9x + 2$

10 **a** $f(x) = x^3 - 12x - 2$, for $-3 \leqslant x \leqslant 5$

$\therefore$ $f'(x) = 3x^2 - 12$

$= 3(x + 2)(x - 2)$

which is 0 when $x = -2$ or 2

x	-3	-2	2	5
$f(x)$	7	14	-18	63

$\therefore$ the greatest value is 63 when $x = 5$, and the least value is -18 when $x = 2$.

b $f(x) = 4 - 3x^2 + x^3$, for $-2 \leqslant x \leqslant 3$

$\therefore$ $f'(x) = -6x + 3x^2$

$= 3x(x - 2)$

which is 0 when $x = 0$ or 2

x	-2	0	2	3
$f(x)$	-16	4	0	4

$\therefore$ greatest value is 4 when $x = 0$ or $x = 3$, least value is -16 when $x = -2$.

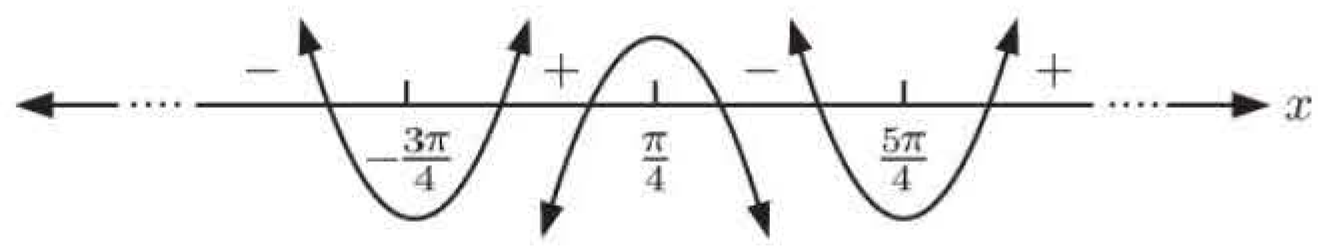
11 $y = 4e^{-x}\sin x$

$\therefore \; \dfrac{dy}{dx} = -4e^{-x}\sin x + 4e^{-x}\cos x$

$\therefore$ stationary points occur when $-4e^{-x}\sin x + 4e^{-x}\cos x = 0$

$$\therefore \; 4e^{-x}(\cos x - \sin x) = 0$$
$$\therefore \; \cos x - \sin x = 0 \quad \{e^{-x} > 0 \text{ for all } x\}$$
$$\therefore \; \sin x = \cos x$$
$$\therefore \; \tan x = 1$$
$$\therefore \; x = \tfrac{\pi}{4} + k\pi, \quad k \text{ an integer}$$

Sign diagram of $\dfrac{dy}{dx}$ is:



$\therefore \; y = 4e^{-x}\sin x$ has a local maximum when $x = \frac{\pi}{4}$.

12 $f(t) = ate^{bt^2}$

$$\therefore \; f'(t) = ae^{bt^2} + at \times 2bte^{bt^2}$$
$$= ae^{bt^2}(1 + 2bt^2)$$

There is a maximum of 1 when $t = 2$

$\therefore \; f'(2) = 0$ and $ae^{b2^2}(1 + 2b2^2) = 0$

$$\therefore \; ae^{4b}(1 + 8b) = 0 \quad .... \text{(1)}$$

Also the point $(2, 1)$ lies on $f(t)$

$$\therefore \; f(2) = a2e^{b2^2} = 1$$
$$\therefore \; ae^{4b} = \tfrac{1}{2} \quad .... \text{(2)}$$

Substituting this into (1) $\quad \therefore \; \frac{1}{2}(1 + 8b) = 0$

$$\therefore \; 1 + 8b = 0$$
$$\therefore \; b = -\tfrac{1}{8}$$

Substituting this into (2) $\quad \therefore \; ae^{4(-\frac{1}{8})} = \frac{1}{2}$

$$\therefore \; ae^{-\frac{1}{2}} = \tfrac{1}{2}$$
$$\therefore \; a = \tfrac{1}{2}e^{\frac{1}{2}} = \frac{\sqrt{e}}{2}$$

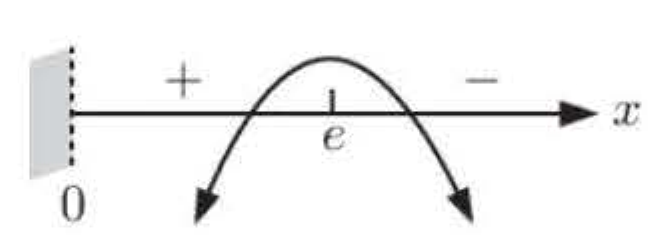
13 Consider $f(x) = \dfrac{\ln x}{x}$

$$\therefore \; f'(x) = \frac{\left(\frac{1}{x}\right)x - \ln x(1)}{x^2} = \frac{1 - \ln x}{x^2}$$

$\therefore \; f'(x) = 0$ when $1 - \ln x = 0$

$$\therefore \; \ln x = 1$$
$$\therefore \; x = e$$

Sign diagram of $f'(x)$ is:



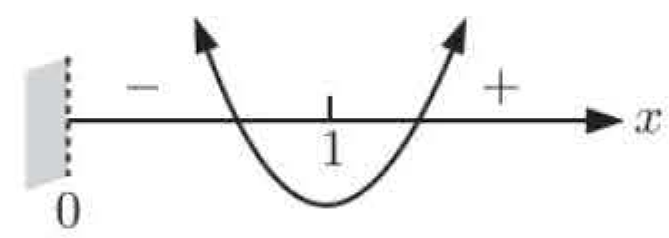
Now $f(e) = \dfrac{\ln e}{e} = \dfrac{1}{e}$

$\therefore$ there is a local maximum at $\left(e, \dfrac{1}{e}\right)$

$\therefore \; f(x) \leqslant \dfrac{1}{e}$ for all x, and so $\dfrac{\ln x}{x} \leqslant \dfrac{1}{e}$ for all $x > 0$

14 **a** $f(x) = x - \ln x$

$\therefore\ f'(x) = 1 - \dfrac{1}{x} = \dfrac{x-1}{x}$ and the sign diagram of $f'(x)$ is:



$\therefore\ f(x)$ has a local minimum at $(1, 1 - \ln 1)$ or $(1, 1)$. This is the only turning point.

b $f(x) \geqslant 1$ for all $x > 0$

$\therefore\ x - \ln x \geqslant 1$

$\therefore\ \ln x \leqslant x - 1$ for all $x > 0$

15 **a** $f(x) = \sec x = \dfrac{1}{\cos x}$

$\therefore\ f(x)$ is undefined when $\cos x = 0$.

But $\cos x = 0$ when $x = \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}$

$\therefore$ for $0 \leqslant x \leqslant 2\pi,\ f(x)$ is undefined when $x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$.

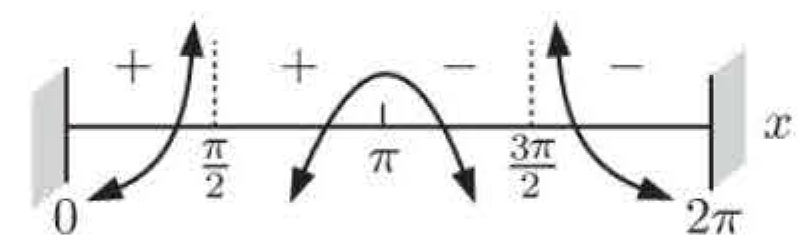
b
$$\begin{aligned} f'(x) &= \sec x \tan x \\ &= \frac{1}{\cos x} \times \frac{\sin x}{\cos x} \end{aligned}$$

$f'(x) = 0$ when $\sin x = 0$

$\therefore\ f'(x) = 0$ when $x = 0 + k\pi,\ k \in \mathbb{Z}$

$\therefore$ for $0 \leqslant x \leqslant 2\pi,\ x = 0, \pi, 2\pi$

$\therefore$ sign diagram of $f'(x)$ is:



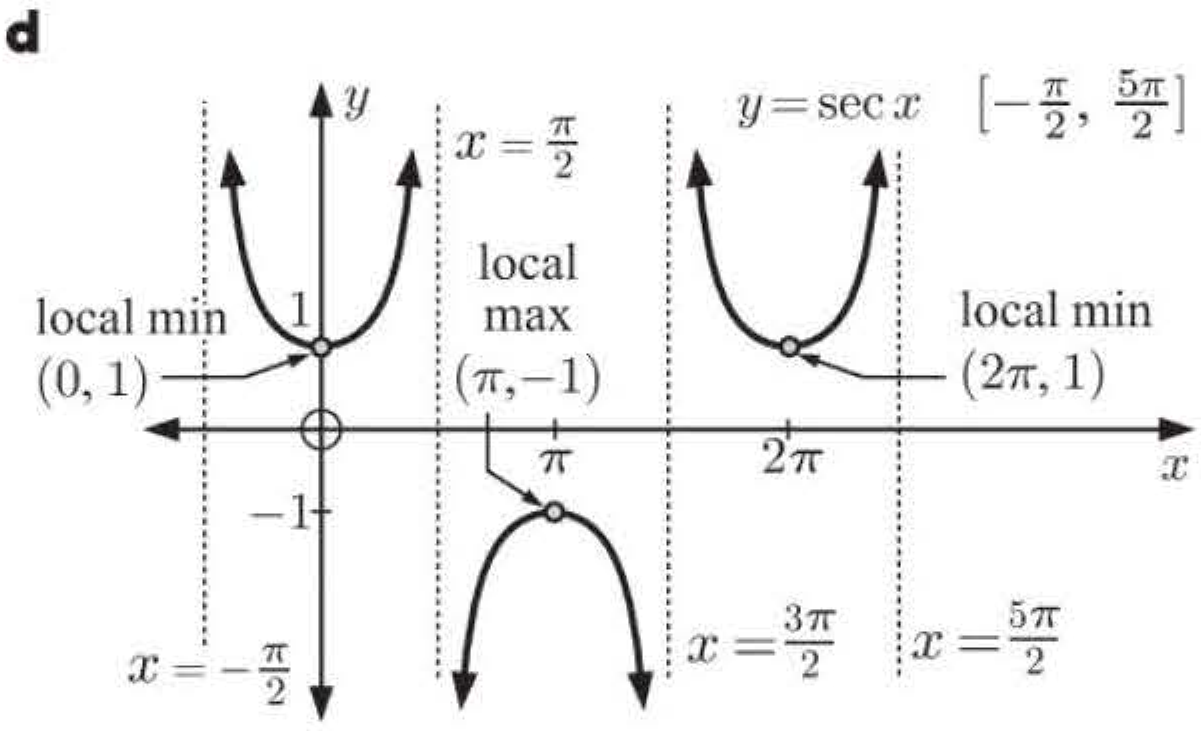
$$\begin{aligned} f(0) &= \sec 0 = 1 \\ f(\pi) &= \sec \pi = -1 \\ f(2\pi) &= \sec 2\pi = 1 \end{aligned}$$

$\therefore$ there is a local maximum at $(\pi, -1)$ and local minima at $(0, 1)$ and $(2\pi, 1)$.

c
$$\begin{aligned} f(x) &= \sec x \\ \therefore\ f(x + 2\pi) &= \sec(x + 2\pi) \\ &= \frac{1}{\cos(x + 2\pi)} \\ &= \frac{1}{\cos x \cos 2\pi - \sin x \sin 2\pi} \\ &= \frac{1}{\cos x \times 1 - \sin x \times 0} \\ &= \frac{1}{\cos x} \\ &= \sec x \\ &= f(x),\ x \in \mathbb{R} \end{aligned}$$

$\therefore$ the graph has a period of 2π.

d



EXERCISE 19D.1

1 **a**

Point	$f(x)$	$f'(x)$	$f''(x)$
A	$+$	$-$	$+$
B	$-$	0	$+$
C	$+$	$+$	0
D	$+$	0	$-$
E	0	$-$	$-$

b The turning points of $y = f(x)$ are point B, a local minimum, and point D, a local maximum.

c The inflection point of $y = f(x)$ is point C, a non-stationary point of inflection.

2 **a** $f(x) = x^2 + 3$

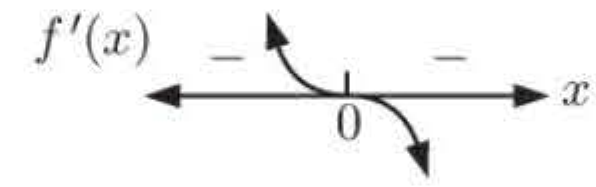
$\therefore \; f'(x) = 2x$

$\therefore \; f''(x) = 2$

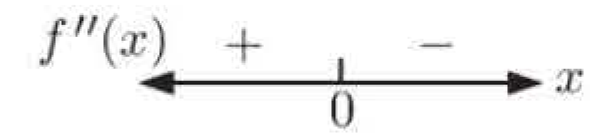
Since $f''(x) \neq 0$,
no points of inflection exist.

b $f(x) = 2 - x^3$

$\therefore \; f'(x) = -3x^2$



$\therefore \; f''(x) = -6x$

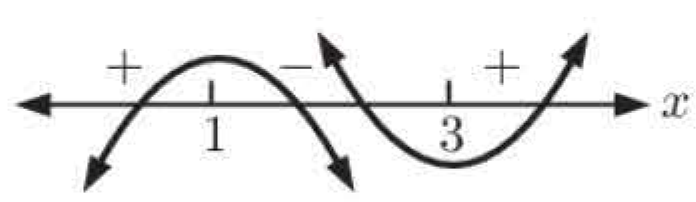


Now $f''(x) = 0$ when $x = 0$,
and $f'(0) = 0$

$\therefore$ there is a stationary inflection at (0, 2).

c $f(x) = x^3 - 6x^2 + 9x + 1$

$\therefore \; f'(x) = 3x^2 - 12x + 9$
$= 3(x^2 - 4x + 3)$
$= 3(x - 3)(x - 1)$



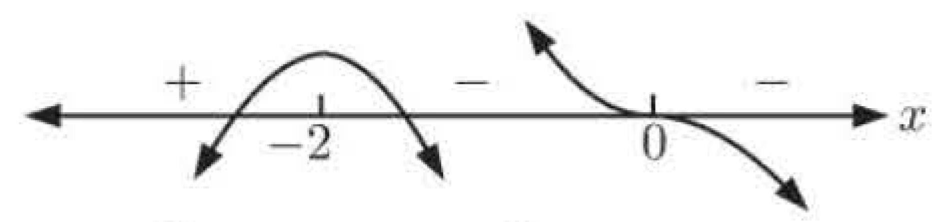
and $f''(x) = 6x - 12 = 6(x - 2)$

Now $f''(x) = 0$ when $x = 2$
and $f'(2) \neq 0$

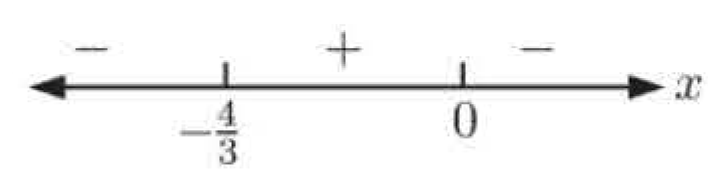
$\therefore$ there is a non-stationary inflection at $(2, f(2))$ which is $(2, 3)$.

d $f(x) = -3x^4 - 8x^3 + 2$

$\therefore \; f'(x) = -12x^3 - 24x^2$
$= -12x^2(x + 2)$



and $f''(x) = -36x^2 - 48x$
$= -12x(3x + 4)$



$\therefore$ there is a stationary inflection at (0, 2), and a non-stationary inflection at $(-\frac{4}{3}, f(-\frac{4}{3}))$, which is $\left(-\frac{4}{3}, \frac{310}{27}\right)$.

e $f(x) = 3 - \dfrac{1}{\sqrt{x}} = 3 - x^{-\frac{1}{2}}$

$\therefore \; f'(x) = \frac{1}{2}x^{-\frac{3}{2}}$

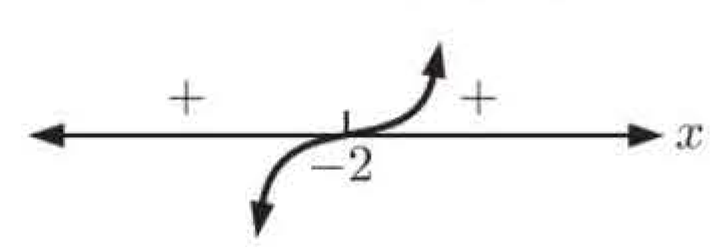
and $f''(x) = -\frac{3}{4}x^{-\frac{5}{2}} = \dfrac{-3}{4x^2\sqrt{x}}$

Now $f''(x) \neq 0$ for all x

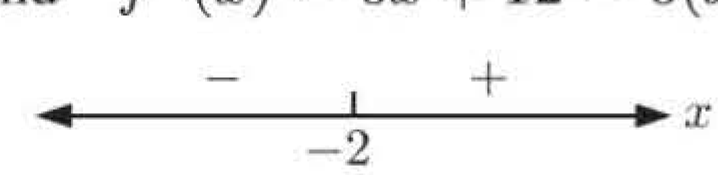
$\therefore$ there are no points of inflection.

f $f(x) = x^3 + 6x^2 + 12x + 5$

$\therefore \; f'(x) = 3x^2 + 12x + 12$
$= 3(x^2 + 4x + 4)$
$= 3(x + 2)^2$



and $f''(x) = 6x + 12 = 6(x + 2)$

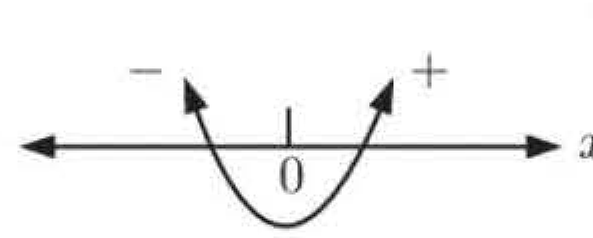


Now $f''(x) = 0$ when $x = -2$
and $f'(-2) = 0$

$\therefore$ there is a stationary inflection at $(-2, f(-2))$ which is $(-2, -3)$.

3 **a** $f(x) = x^2$

$\therefore \; f'(x) = 2x$ which has sign diagram:



and $f''(x) = 2$

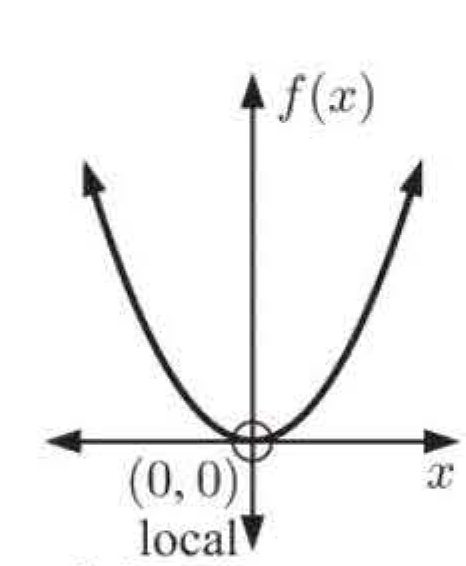
i There is a local minimum at (0, 0).

ii There are no points of inflection as $f''(x) \neq 0$.

iii $f(x)$ is increasing when $x \geqslant 0$, and decreasing when $x \leqslant 0$.

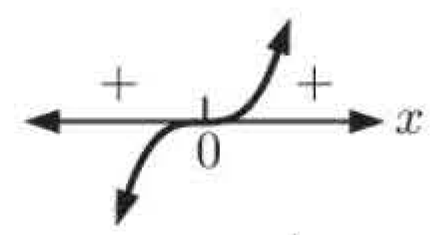
iv $f(x)$ is concave up for all x as $f''(x) > 0$ for all x.

v

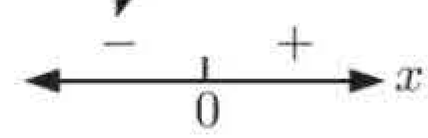


b $f(x) = x^3$

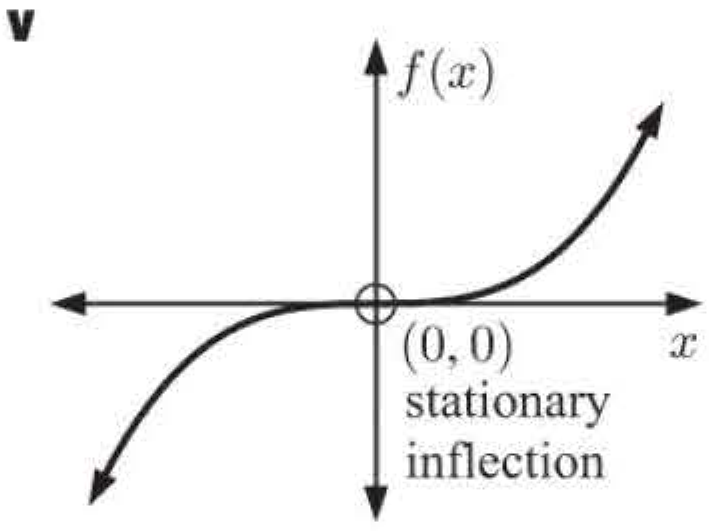
$\therefore\ f'(x) = 3x^2$ which has sign diagram:



and $f''(x) = 6x$ which has sign diagram:



v



i A stationary inflection at (0, 0).

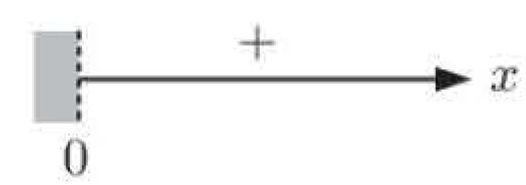
ii A stationary inflection at (0, 0).

iii $f(x)$ is increasing for all real x.

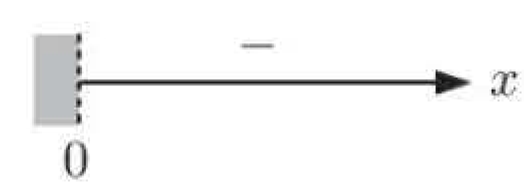
iv $f(x)$ is concave up when $x \geqslant 0$, and concave down when $x \leqslant 0$.

c $f(x) = \sqrt{x}$

$\therefore\ f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \dfrac{1}{2\sqrt{x}}$ which has sign diagram:



and $f''(x) = -\frac{1}{4}x^{-\frac{3}{2}} = \dfrac{-1}{4x\sqrt{x}}$ which has sign diagram:



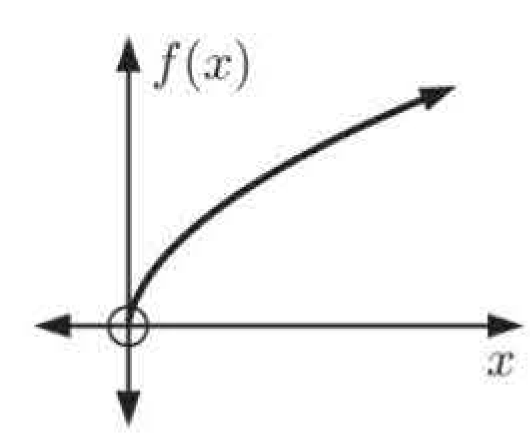
i There are no stationary points as $f'(x) \neq 0$.

ii There are no points of inflection as $f''(x) \neq 0$.

iii $f(x)$ is increasing for all $x \geqslant 0$, never decreasing.

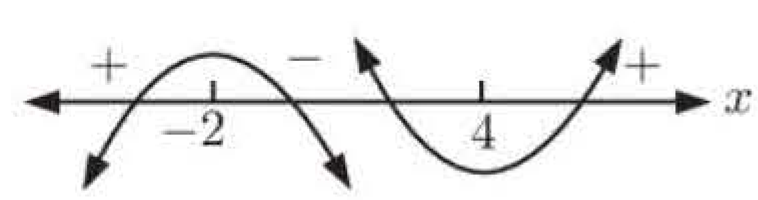
iv $f(x)$ is concave down for all $x \geqslant 0$ as $f''(x) < 0$ for all $x > 0$, never concave up.

v



d $f(x) = x^3 - 3x^2 - 24x + 1$

$$\begin{aligned}\therefore\ f'(x) &= 3x^2 - 6x - 24\\ &= 3(x^2 - 2x - 8)\\ &= 3(x-4)(x+2)\end{aligned}$$ which has sign diagram:



and $$\begin{aligned}f''(x) &= 6x - 6\\ &= 6(x-1)\end{aligned}$$ which has sign diagram:

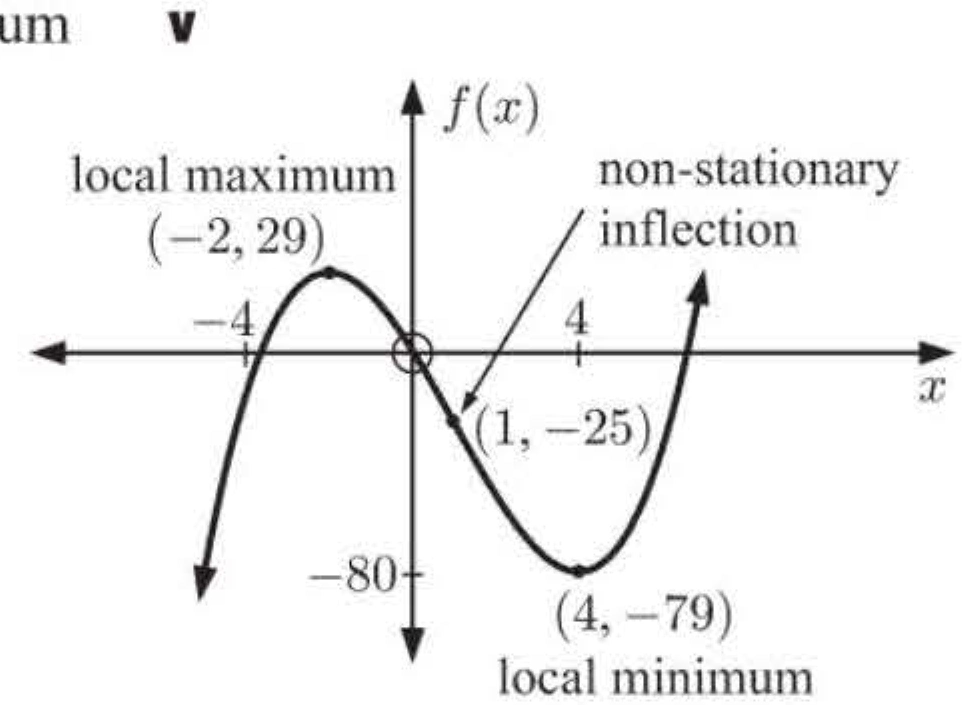
i $f(-2) = 29$, $f(4) = -79$, so there is a local maximum at $(-2, 29)$, and a local minimum at $(4, -79)$.

ii $f(1) = -25$, so there is a non-stationary inflection at $(1, -25)$.

iii $f(x)$ is increasing for $x \leqslant -2$ and $x \geqslant 4$, and decreasing for $-2 \leqslant x \leqslant 4$.

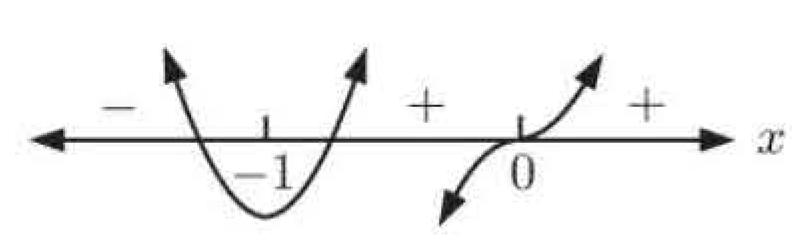
iv $f(x)$ is concave down for $x \leqslant 1$, and concave up for $x \geqslant 1$.

v

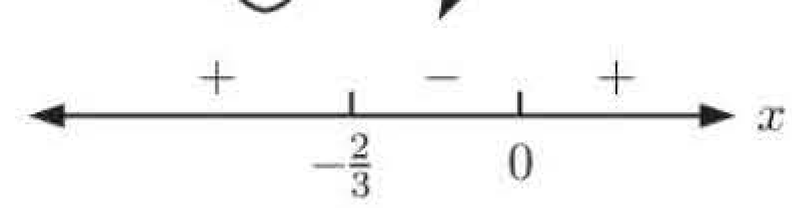


e $f(x) = 3x^4 + 4x^3 - 2$

$$\begin{aligned}\therefore\ f'(x) &= 12x^3 + 12x^2\\ &= 12x^2(x+1)\end{aligned}$$ which has sign diagram:



and $$\begin{aligned}f''(x) &= 36x^2 + 24x\\ &= 12x(3x+2)\end{aligned}$$ which has sign diagram:



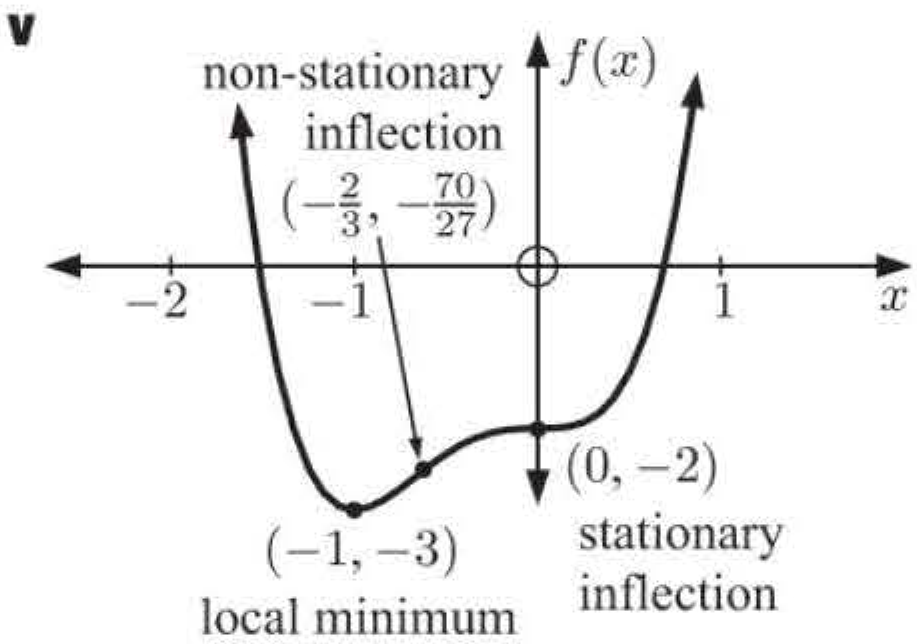
i There is a local minimum at $(-1, f(-1))$ which is $(-1, -3)$, and a stationary inflection at $(0, -2)$.

ii There is a non-stationary inflection at $(-\frac{2}{3}, f(-\frac{2}{3}))$ which is $(-\frac{2}{3}, -\frac{70}{27})$, and a stationary inflection at $(0, -2)$.

iii $f(x)$ is increasing for $x \geqslant -1$, and decreasing for $x \leqslant -1$.

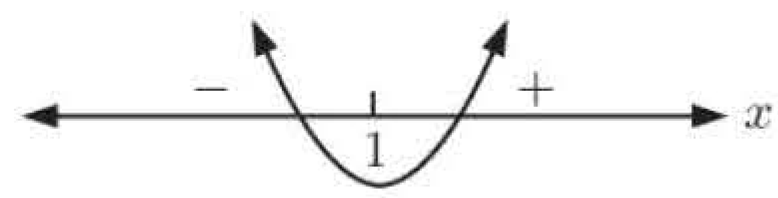
iv $f(x)$ is concave down for $-\frac{2}{3} \leqslant x \leqslant 0$, and concave up for $x \leqslant -\frac{2}{3}$ and $x \geqslant 0$.

v

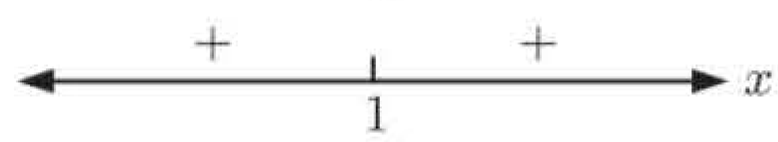


f $f(x) = (x-1)^4$

$\therefore$ $f'(x) = 4(x-1)^3$ which has sign diagram:



and $f''(x) = 12(x-1)^2$ which has sign diagram:



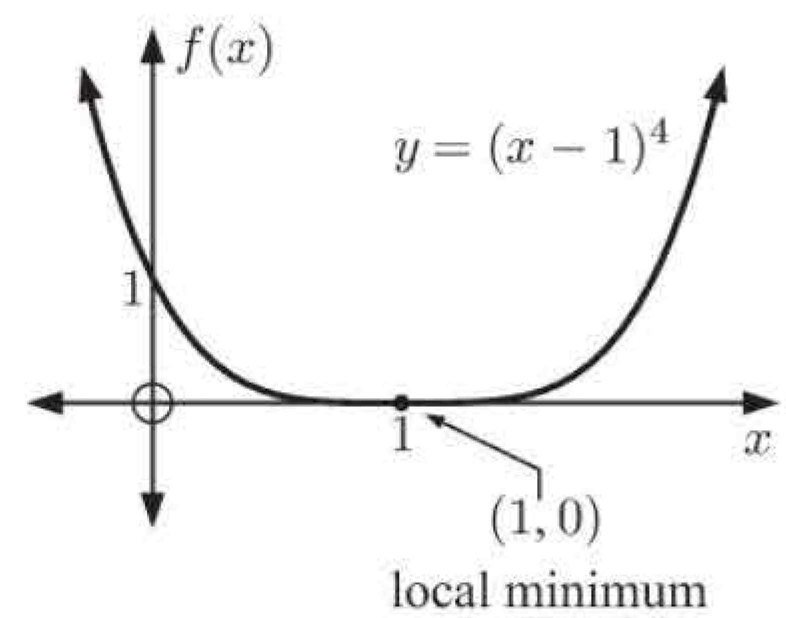
i There is a local minimum at $(1, 0)$.

ii Since there is no sign change in $f''(x)$ at $x = 1$, there are no points of inflection.

iii $f(x)$ is increasing for $x \geqslant 1$, and decreasing for $x \leqslant 1$.

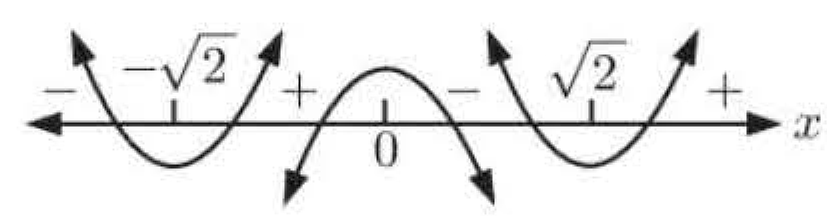
iv $f(x)$ is concave up for all x.

v

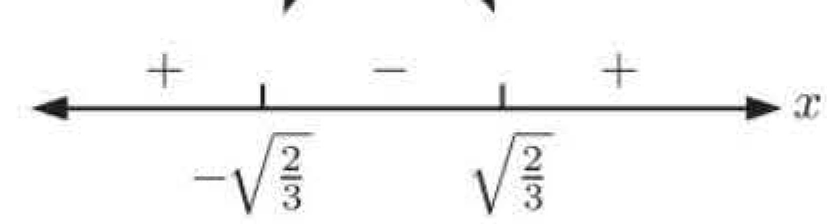


g $f(x) = x^4 - 4x^2 + 3$

$$\begin{aligned} f'(x) &= 4x^3 - 8x \\ &= 4x(x^2 - 2) \\ &= 4x(x + \sqrt{2})(x - \sqrt{2}) \end{aligned}$$ which has sign diagram:



$$\begin{aligned} f''(x) &= 12x^2 - 8 = 4(3x^2 - 2) \\ &= 4(\sqrt{3}x + \sqrt{2})(\sqrt{3}x - \sqrt{2}) \end{aligned}$$ which has sign diagram:



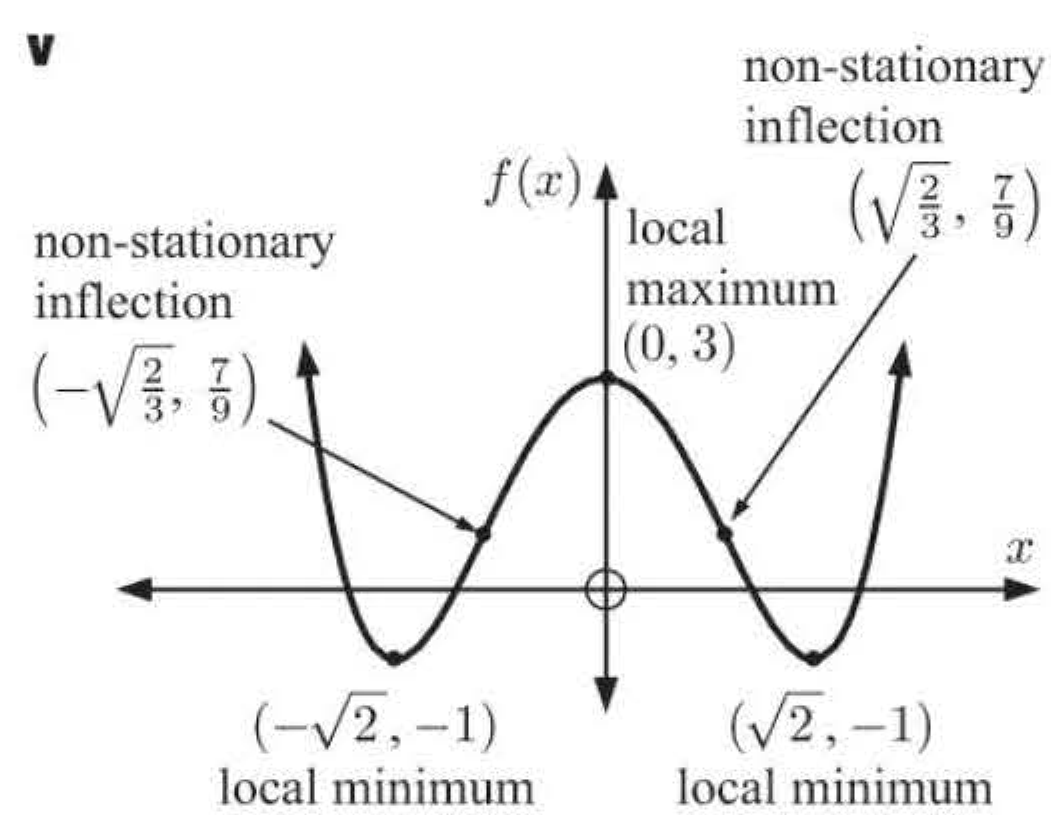
i There is a local maximum at $(0, 3)$, and $f(-\sqrt{2}) = f(\sqrt{2}) = -1$, so there are local minima at $(\sqrt{2}, -1)$ and $(-\sqrt{2}, -1)$.

ii $f\left(\sqrt{\frac{2}{3}}\right) = f\left(-\sqrt{\frac{2}{3}}\right) = \frac{7}{9}$, so there are non-stationary inflections at $(\sqrt{\frac{2}{3}}, \frac{7}{9})$ and $(-\sqrt{\frac{2}{3}}, \frac{7}{9})$.

iii $f(x)$ is increasing for $-\sqrt{2} \leqslant x \leqslant 0$ and $x \geqslant \sqrt{2}$, and decreasing for $x \leqslant -\sqrt{2}$ and $0 \leqslant x \leqslant \sqrt{2}$.

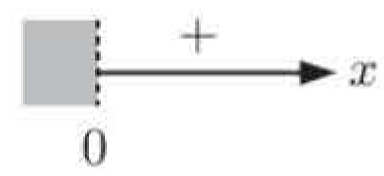
iv $f(x)$ is concave down for $-\sqrt{\frac{2}{3}} \leqslant x \leqslant \sqrt{\frac{2}{3}}$, and concave up for $x \leqslant -\sqrt{\frac{2}{3}}$ and $x \geqslant \sqrt{\frac{2}{3}}$.

v

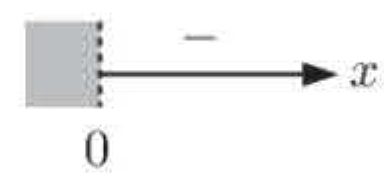


h $f(x) = 3 - \dfrac{4}{\sqrt{x}} = 3 - 4x^{-\frac{1}{2}}, \quad x > 0$

$\therefore \; f'(x) = 2x^{-\frac{3}{2}} = \dfrac{2}{x\sqrt{x}}$ with sign diagram:



and $f''(x) = -3x^{-\frac{5}{2}} = -\dfrac{3}{x^2\sqrt{x}}$ with sign diagram:



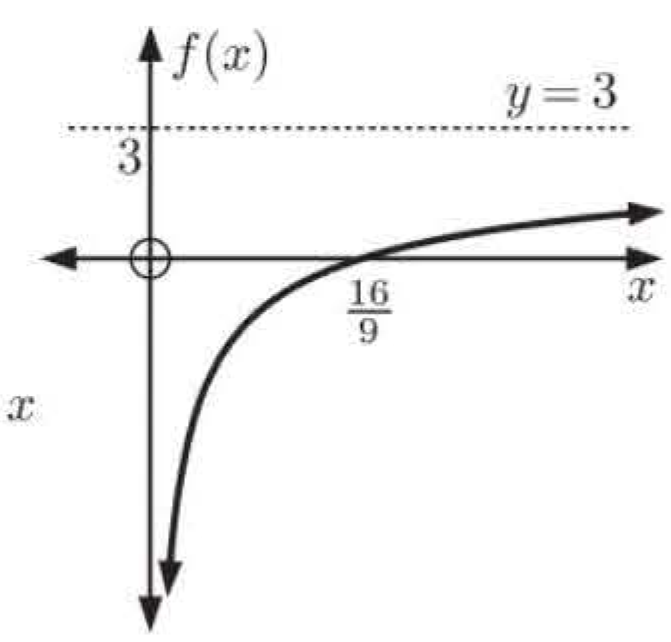
i There are no stationary points as $f'(x) \neq 0$.

ii There are no points of inflection as $f''(x) \neq 0$.

iii $f(x)$ is increasing for all $x > 0$ and never decreasing.

iv $f(x)$ is concave down for all $x > 0$ and never concave up.

v



4 **a** Consider $f(x) = e^{2x} - 3$

$f(x)$ cuts the x-axis at A when $f(x) = 0$

$$\therefore \; e^{2x} - 3 = 0$$
$$\therefore \; e^{2x} = 3$$
$$\therefore \; 2x = \ln 3$$
$$\therefore \; x = \frac{\ln 3}{2}$$
$$\therefore \; \text{A is } \left(\tfrac{\ln 3}{2}, 0\right)$$

$f(x)$ cuts the y-axis at B when $x = 0$

$$\therefore \; f(0) = e^{2\times 0} - 3$$
$$= e^0 - 3$$
$$= -2$$
$$\therefore \; \text{B is } (0, -2)$$

b $f(x) = e^{2x} - 3$

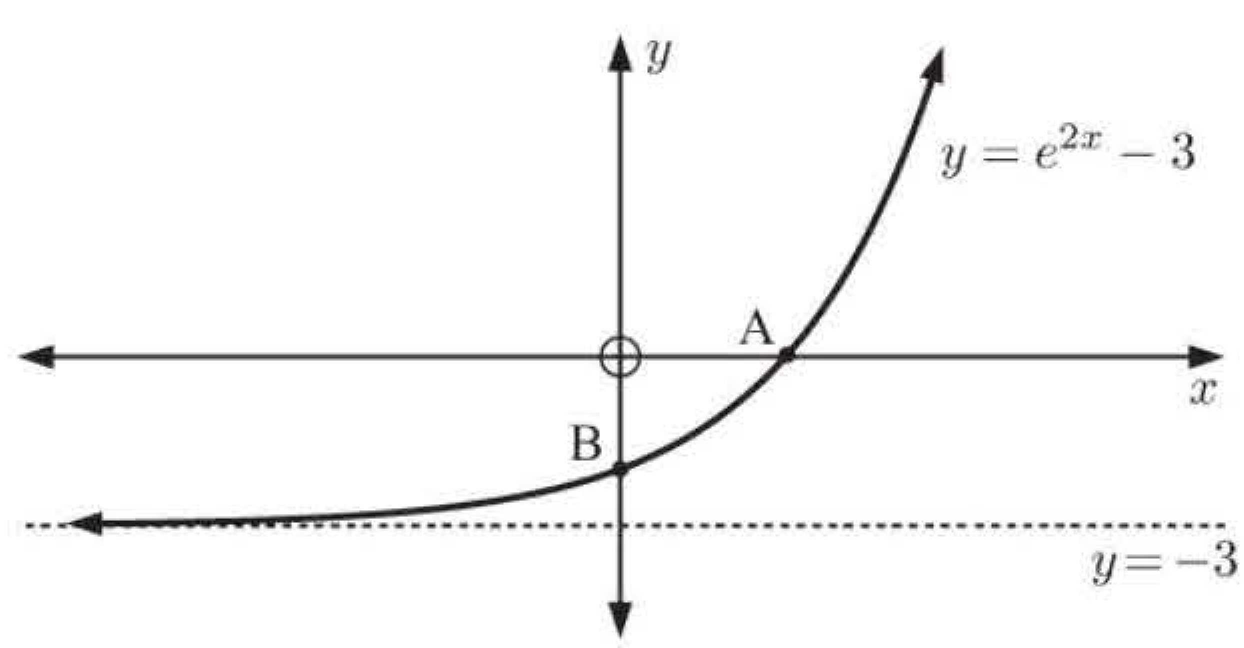
$\therefore \; f'(x) = 2e^{2x}$

Since $e^{2x} > 0$ for all x, $f'(x) > 0$ for all x, and hence $f(x)$ is increasing for all x.

c $f''(x) = 4e^{2x}$, which is always > 0.

$\therefore \; f(x)$ is concave up for all x.

d



e As $x \to -\infty$, $e^{2x} \to 0$, so $e^{2x} - 3 \to -3^+$

$\therefore \; y = -3$ is a horizontal asymptote.

5 **a** The x-intercepts occur when $y = 0$

For $f(x) = e^x - 3$, $e^x - 3 = 0$

$$\therefore \; e^x = 3$$
$$\therefore \; x = \ln 3$$

and for $g(x) = 3 - \dfrac{5}{e^x}$, $3 - \dfrac{5}{e^x} = 0$

$$\therefore \; \frac{3e^x - 5}{e^x} = 0$$
$$\therefore \; 3e^x - 5 = 0$$
$$\therefore \; e^x = \tfrac{5}{3}$$
$$\therefore \; x = \ln(\tfrac{5}{3})$$

$\therefore \; f(x)$ has x-intercept $\ln 3$

and $g(x)$ has x-intercept $\ln(\frac{5}{3})$.

The y-intercepts occur when $x = 0$

Now $f(0) = e^0 - 3 = -2$ and $g(0) = 3 - \dfrac{5}{e^0} = 3 - 5 = -2$

$\therefore$ both $f(x)$ and $g(x)$ have y-intercept -2.

b As $x \to \infty$, $f(x) \to \infty$ As $x \to \infty$, $g(x) \to 3^-$
$x \to -\infty$, $f(x) \to -3^+$ $x \to -\infty$, $g(x) \to -\infty$

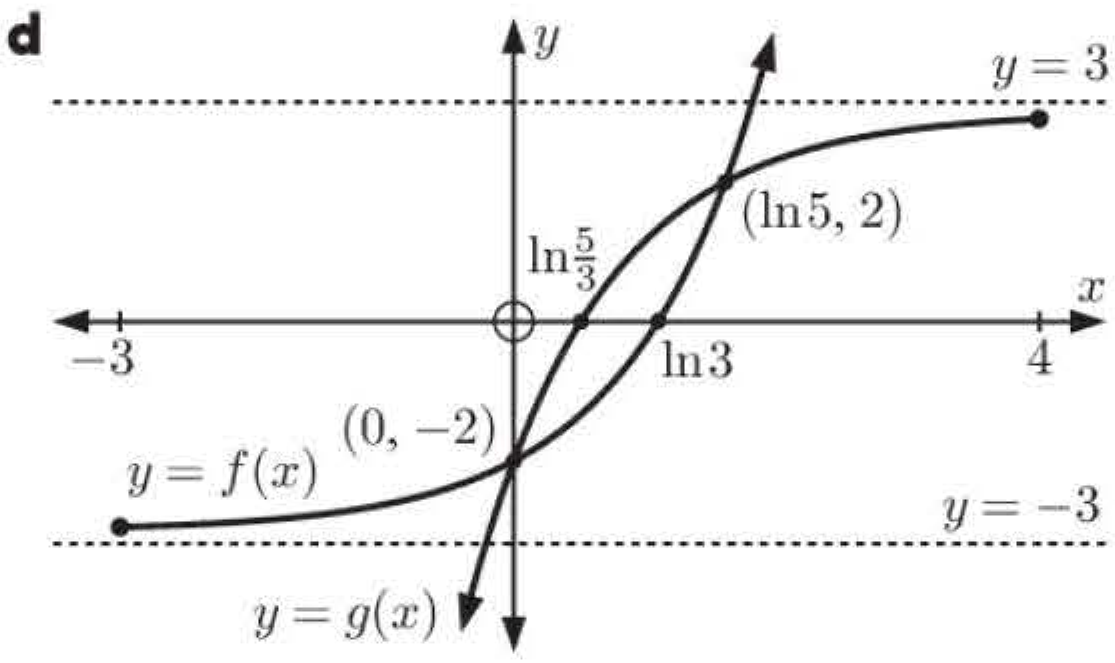
c $f(x)$ and $g(x)$ meet when

$$e^x - 3 = 3 - 5e^{-x}$$
$$\therefore\ e^{2x} - 3e^x = 3e^x - 5 \quad \{\times e^x\}$$
$$\therefore\ e^{2x} - 6e^x + 5 = 0$$
$$\therefore\ (e^x - 5)(e^x - 1) = 0$$
$$\therefore\ e^x = 5 \text{ or } 1$$
$$\therefore\ x = \ln 5 \text{ or } 0$$

Now $f(\ln 5) = e^{\ln 5} - 3 = 5 - 3 = 2$
and $f(0) = -2$

$\therefore$ $f(x)$ and $g(x)$ meet at $(\ln 5, 2)$ and $(0, -2)$.

d



6 **a** Consider $y = e^x - 3e^{-x}$

It cuts the x-axis at P when $y = 0$

$$\therefore\ e^x - 3e^{-x} = 0$$
$$\therefore\ e^{2x} - 3 = 0 \quad \{\times e^x\}$$
$$\therefore\ e^{2x} = 3$$
$$\therefore\ 2x = \ln 3$$
$$\therefore\ x = \tfrac{1}{2}\ln 3$$

It cuts the y-axis at Q when $x = 0$

$$\therefore\ y = e^0 - 3e^0 = 1 - 3 = -2$$

$\therefore$ P is $(\frac{1}{2}\ln 3, 0)$ and Q is $(0, -2)$.

b
$$\frac{dy}{dx} = e^x + 3e^{-x} = e^x + \frac{3}{e^x}$$

Since $e^x > 0$ for all x,

$\dfrac{dy}{dx} > 0$ for all x

$\therefore$ the function is increasing for all x

c
$$\frac{dy}{dx} = e^x + 3e^{-x}$$
$$\therefore\ \frac{d^2y}{dx^2} = e^x - 3e^{-x} = y$$

Above the x-axis $y > 0$ $\therefore$ $\dfrac{d^2y}{dx^2} > 0$

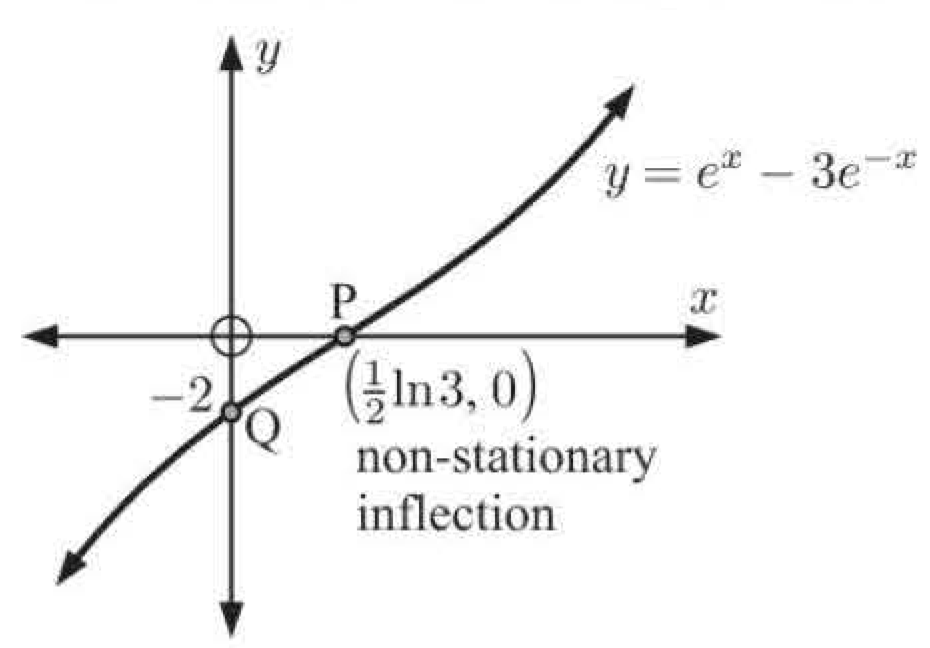
$\therefore$ the function is concave up

Below the x-axis $y < 0$ $\therefore$ $\dfrac{d^2y}{dx^2} < 0$

$\therefore$ the function is concave down

$\therefore$ a non-stationary inflection occurs when $y = 0$

d



7 $f(x) = \ln(2x - 1) - 3$

a $f(x) = 0$ when $\ln(2x - 1) = 3$

$$\therefore\ 2x - 1 = e^3$$
$$\therefore\ 2x = e^3 + 1$$
$$\therefore\ x = \frac{e^3 + 1}{2} \approx 10.5$$

$\therefore$ the x-intercept is $\dfrac{e^3 + 1}{2}$

b $f(0)$ cannot be found as $\ln(-1)$ is not defined. $\therefore$ there is no y-intercept.

c $f'(x) = \dfrac{2}{2x - 1}$ $\therefore$ $f'(1) = \dfrac{2}{2 - 1} = 2$ $\therefore$ gradient of tangent $= 2$

d $\ln(2x - 1)$ has meaning provided $2x - 1 > 0$ $\therefore$ $2x > 1$ and so $x > \frac{1}{2}$

$\therefore$ the domain of f is $\{x \mid x > \frac{1}{2}\}$

e $f'(x) = 2(2x-1)^{-1}$

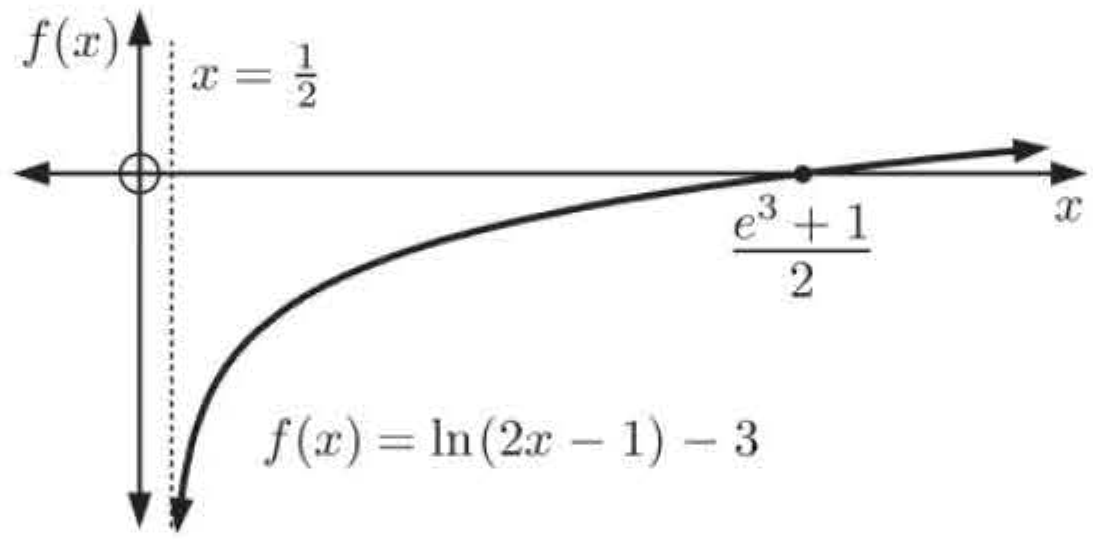
$\therefore\ f''(x) = -2(2x-1)^{-2}(2)$

$= \dfrac{-4}{(2x-1)^2}, \quad x > \frac{1}{2}$

$\therefore$ provided $x > \frac{1}{2}$, $f''(x) < 0$

$\therefore$ $f(x)$ is concave down for all x in the domain of f.

f



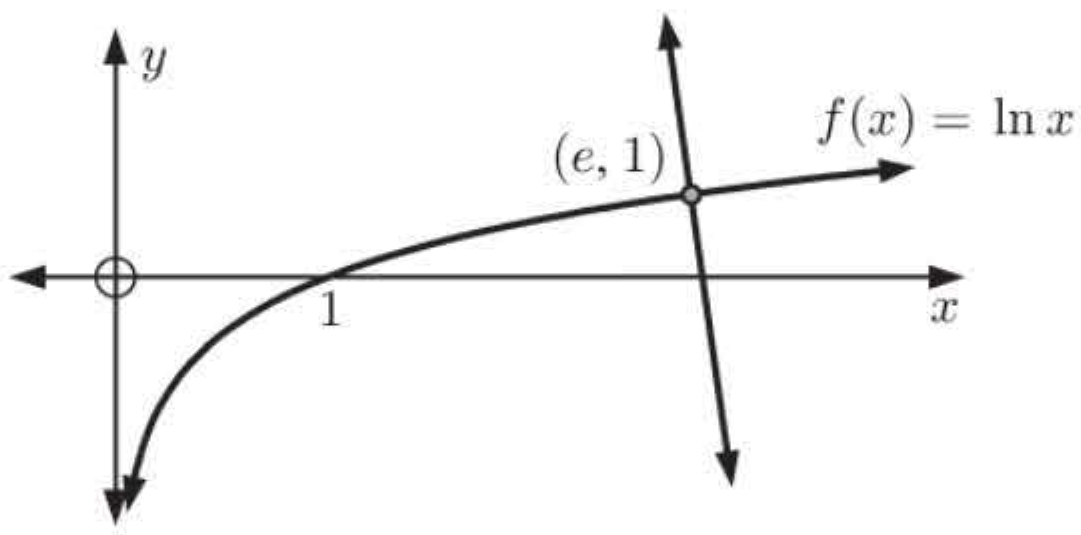
8 **a** $f(x) = \ln x$ is defined for all $x > 0$.

b $f'(x) = \dfrac{1}{x}$ which is > 0 for all $x > 0$

$\therefore$ $f(x)$ is increasing on $x > 0$; its gradient is always positive.

$f''(x) = -x^{-2} = \dfrac{-1}{x^2}$ which is < 0 for all $x > 0$ $\quad \therefore$ $f(x)$ is concave down on $x > 0$.

c



At $y = 1$, $1 = \ln x$

$\therefore\ x = e^1 = e$

$\therefore$ the point of contact is $(e, 1)$

Now $\dfrac{dy}{dx} = \dfrac{1}{x}$

$\therefore$ at $(e, 1)$, $\dfrac{dy}{dx} = \dfrac{1}{e}$

$\therefore$ the gradient of the tangent is $\dfrac{1}{e}$, and the gradient of the normal is $-e$

$\therefore$ the equation of the normal is $\dfrac{y-1}{x-e} = -e$ $\quad \therefore\ y - 1 = -e(x-e)$

$\therefore\ y - 1 = -ex + e^2$

$\therefore\ y = -ex + 1 + e^2$

9 Consider $f(x) = \dfrac{e^x}{x}$.

a $e^x \neq 0$ for all x, so $f(x) \neq 0$ and there is no x-intercept.

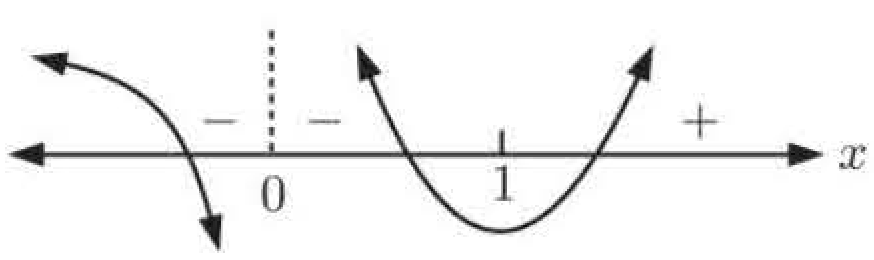
$f(0) = \dfrac{e^0}{0}$ is undefined, so there is also no y-intercept.

b As $x \to +\infty$ $f(x) \to \infty$, and as $x \to -\infty$, $f(x) \to 0^-$

$\left(\begin{array}{l} \text{As } x \to 0^+,\ y \to +\infty, \text{ and as } x \to 0^-,\ y \to -\infty \\ \therefore\ x = 0 \text{ is a vertical asymptote.} \end{array}\right)$

c Using the quotient rule, $f'(x) = \dfrac{e^x x - e^x(1)}{x^2} = \dfrac{e^x(x-1)}{x^2}$

with sign diagram:



$f(1) = \dfrac{e^1}{1} = e$, so there is a local minimum at $(1, e)$.

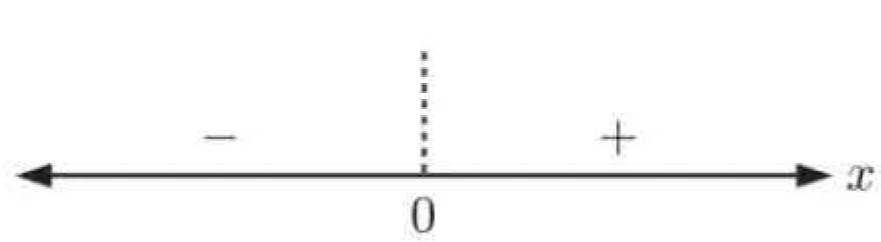
d Using the product and quotient rules,

$$f''(x) = \frac{[e^x(x-1) + e^x]x^2 - e^x(x-1)2x}{x^4}$$

$$= \frac{x^3e^x - \cancel{x^2e^x} + \cancel{x^2e^x} - 2x^2e^x + 2xe^x}{x^4}$$

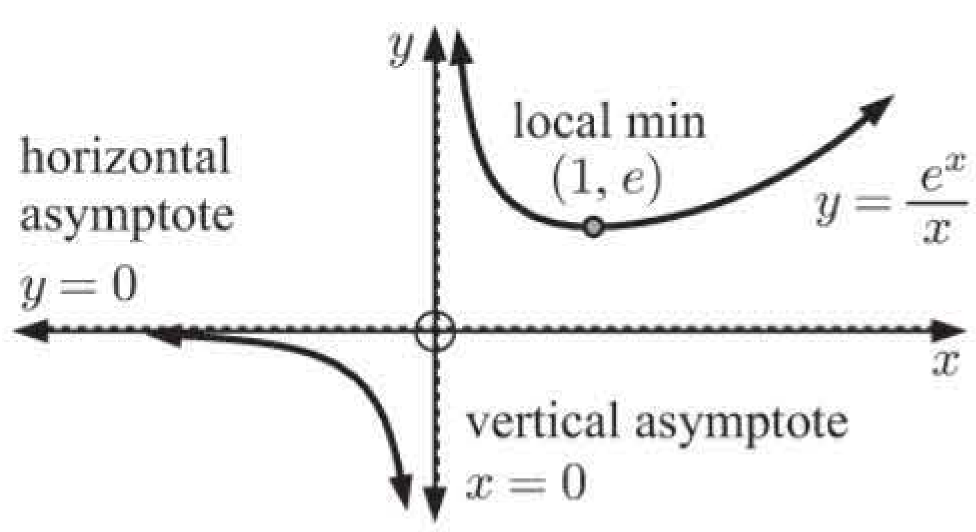
$$= \frac{e^x(x^2 - 2x + 2)}{x^3}$$

with sign diagram:



i $f(x)$ is concave up for $x > 0$. **ii** $f(x)$ is concave down for $x < 0$.

e



f Now $f'(x) = \dfrac{e^x(x-1)}{x^2}$

$\therefore\ f'(-1) = \dfrac{e^{-1}(-1-1)}{(-1)^2} = -\dfrac{2}{e}$

$\therefore$ the gradient of the tangent is $= -\dfrac{2}{e}$

When $x=-1$, $y = \dfrac{e^{-1}}{-1} = -\dfrac{1}{e}$

$\therefore$ the equation of the tangent is

$$\frac{y-\left(-\frac{1}{e}\right)}{x-(-1)} = -\frac{2}{e}$$

$$\therefore\ \frac{y+\frac{1}{e}}{x+1} = -\frac{2}{e}$$

$$\therefore\ e\left(y+\frac{1}{e}\right) = -2(x+1)$$

$$ey + 1 = -2x - 2$$

$$\therefore\ ey = -2x - 3$$

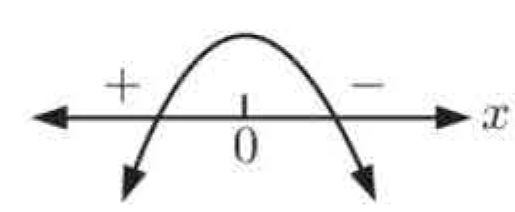
10 **a** $f(x) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2}$

$\therefore\ f'(x) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2}(-x)$

$= \dfrac{-x}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2}$

$\therefore\ f'(x) = 0$ when $x = 0$

$f'(x)$ has sign diagram:



Now $f(0) = \frac{1}{\sqrt{2\pi}}$

so there is a local maximum at $\left(0, \frac{1}{\sqrt{2\pi}}\right)$.

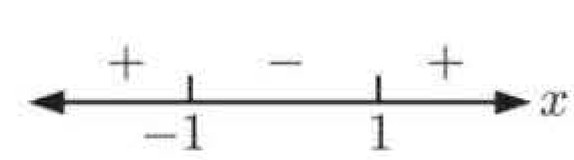
The function is increasing for $x \leqslant 0$ and decreasing for $x \geqslant 0$

b $f'(x) = \dfrac{-x}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2} = \frac{1}{\sqrt{2\pi}}\left(-xe^{-\frac{1}{2}x^2}\right)$

$\therefore\ f''(x) = \frac{1}{\sqrt{2\pi}}\left((-1)\,e^{-\frac{1}{2}x^2} + (-x)\,e^{-\frac{1}{2}x^2}(-x)\right)$ {product rule}

$= \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2}\left(x^2-1\right)$

$= \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2}(x+1)(x-1)$ which has sign diagram:



Now $f(1) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}} = \frac{1}{\sqrt{2e\pi}}$ and $f(-1) = \frac{1}{\sqrt{2e\pi}}$

$\therefore$ there are non-stationary points of inflection at $\left(1, \frac{1}{\sqrt{2e\pi}}\right)$ and $\left(-1, \frac{1}{\sqrt{2e\pi}}\right)$.

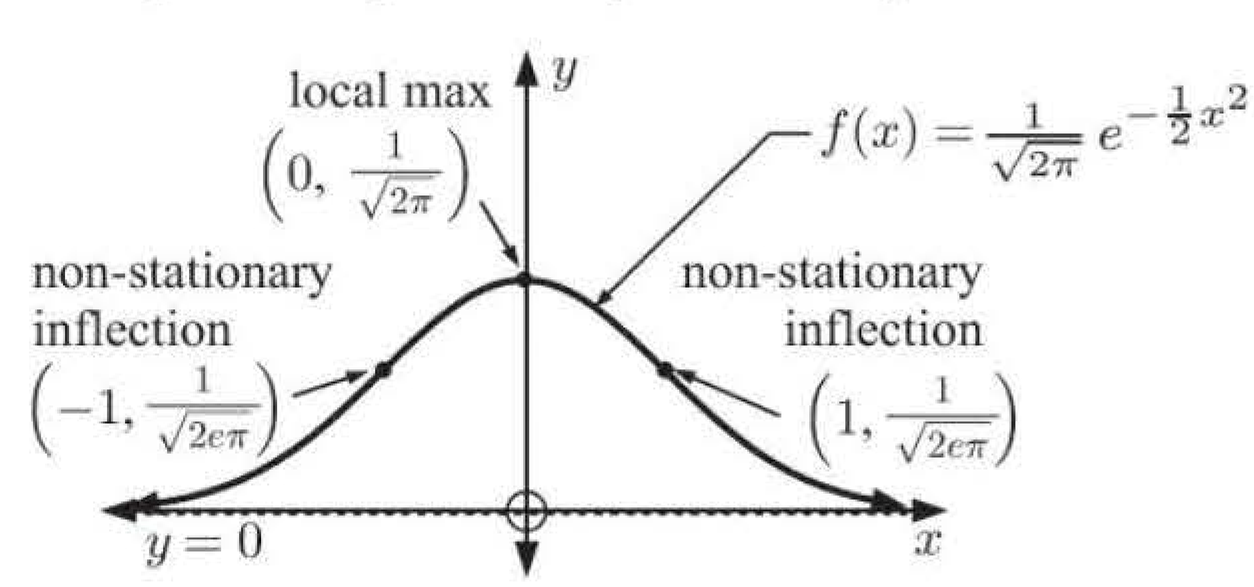
c As $x \to \infty$, $e^{-\frac{1}{2}x^2} \to 0^+$

$\therefore\ f(x) \to 0^+$

As $x \to -\infty$, $e^{-\frac{1}{2}x^2} \to 0^+$

$\therefore\ f(x) \to 0^+$

d



11 $y = 4^x - 2^x$

a When $x=0$, $y = 4^0 - 2^0 = 1 - 1 = 0$ $\therefore$ the y-intercept is 0.

When $y = 0$, $4^x - 2^x = 0$

$\therefore\ 4^x = 2^x$

$\therefore\ \dfrac{4^x}{2^x} = 1$

$\therefore\ \left(\frac{4}{2}\right)^x = 1$

$\therefore\ 2^x = 1$

$\therefore\ x = 0$ $\therefore$ the x-intercept is 0.

b As $x \to \infty$, $\quad y \to \infty$

As $x \to -\infty$, $\quad y \to 0^-$

c $\dfrac{dy}{dx} = 4^x \ln 4 - 2^x \ln 2 \qquad \{\dfrac{d}{dx}(a^x) = a^x \ln a\}$

When $\dfrac{dy}{dx} = 0$, $\quad 4^x \ln 4 - 2^x \ln 2 = 0$

$$\therefore \; 2^x \times 2^x \times 2\ln 2 - 2^x \ln 2 = 0 \qquad \{\ln a^x = x \ln a\}$$

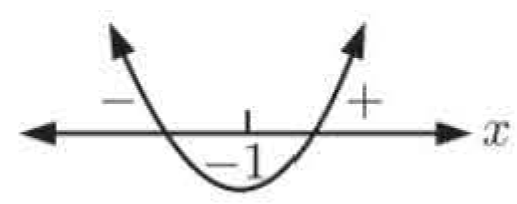
$$\therefore \; 2^x \ln 2(2^{x+1} - 1) = 0$$

$$\therefore \; 2^{x+1} = 1 \qquad \{\text{since } 2^x \ln 2 \neq 0\}$$

$$\therefore \; x + 1 = 0 \qquad \{\text{since } 2^0 = 1\}$$

$$\therefore \; x = -1$$

$\therefore$ sign diagram of $\dfrac{dy}{dx}$ is:



When $x = -1$, $\quad y = 4^{-1} - 2^{-1}$

$$= \tfrac{1}{4} - \tfrac{1}{2}$$

$$= -\tfrac{1}{4}$$

$\therefore$ there is a local minimum at $(-1, -\tfrac{1}{4})$.

$$\frac{d^2y}{dx^2} = 4^x(\ln 4)^2 - 2^x(\ln 2)^2 \qquad \{\frac{d}{dx}(a^x) = a^x \ln a\}$$

$$= 2^x \times 2^x(2\ln 2)^2 - 2^x(\ln 2)^2 \qquad \{\ln a^x = x \ln a\}$$

$$= 2^x(\ln 2)^2(2^x \times 2^2 - 1)$$

$$= 2^x(\ln 2)^2(2^{x+2} - 1)$$

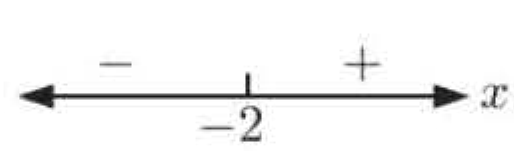
When $\dfrac{d^2y}{dx^2} = 0$, $\quad 2^x(\ln 2)^2(2^{x+2} - 1) = 0$

$$\therefore \; 2^{x+2} = 1 \qquad \{\text{since } 2^x(\ln 2)^2 \neq 0\}$$

$$\therefore \; x + 2 = 0 \qquad \{\text{since } 2^0 = 1\}$$

$$\therefore \; x = -2$$

$\therefore$ the sign diagram of $\dfrac{d^2y}{dx^2}$ is:



When $x = -2$, $\quad y = 4^{-2} - 2^{-2}$

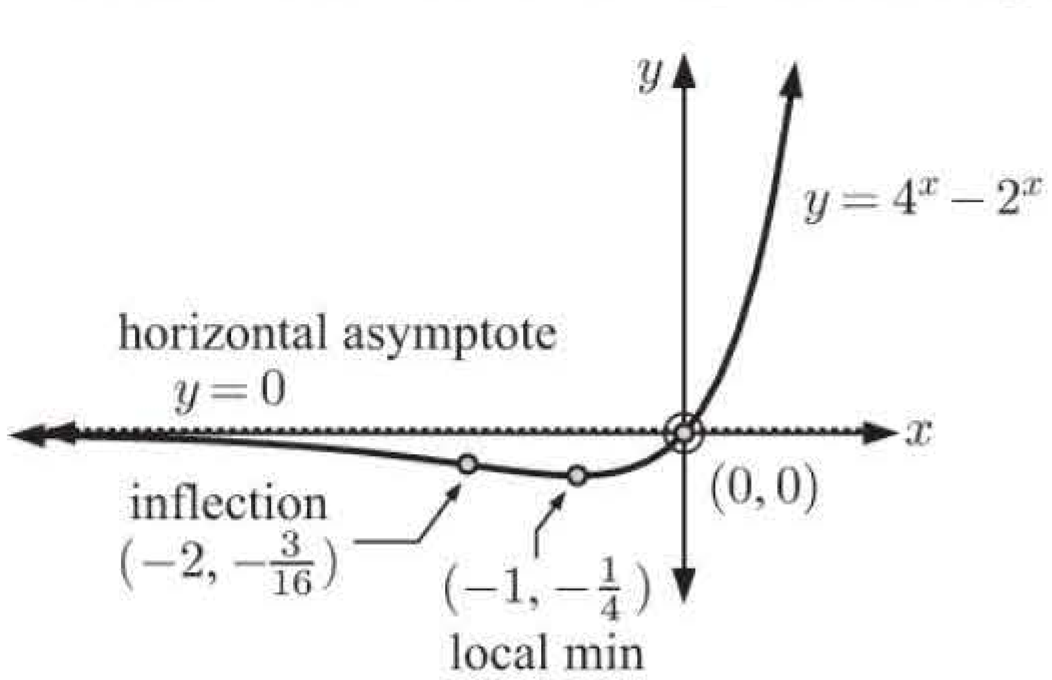
$$= \tfrac{1}{16} - \tfrac{1}{4}$$

$$= -\tfrac{3}{16}$$

$\therefore$ at $(-2, -\tfrac{3}{16})$ there is a non-stationary point of inflection.

d concave down for $x \leqslant -2$, concave up for $x \geqslant -2$

e

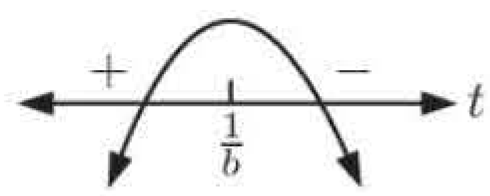


12 $f(t) = Ate^{-bt}, \quad t \geqslant 0, \quad A, b > 0$

a **i** $f'(t) = Ae^{-bt} - Abte^{-bt}$ {product rule}
$= Ae^{-bt}(1 - bt)$

When $f'(t) = 0$ then $Ae^{-bt}(1 - bt) = 0$ but $A > 0$ and $e^{-bt} > 0$

$\therefore \ 1 - bt = 0$ Sign diagram of $f'(t)$ is:



$\therefore \ -bt = -1$

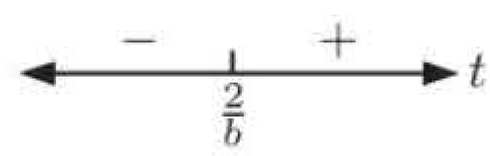
$\therefore \ t = \dfrac{1}{b}$

$\therefore \ t = \dfrac{1}{b}$ is a local maximum.

ii $f''(t) = -Abe^{-bt} - (Abe^{-bt} - Ab^2te^{-bt})$ {product rule}
$= -2Abe^{-bt} + Ab^2te^{-bt}$
$= Abe^{-bt}(bt - 2)$

When $f''(t) = 0$ then $Abe^{-bt}(bt - 2) = 0$ but $A, b > 0$ and $e^{-bt} > 0$

$\therefore \ bt - 2 = 0$ Sign diagram of $f''(x)$ is:

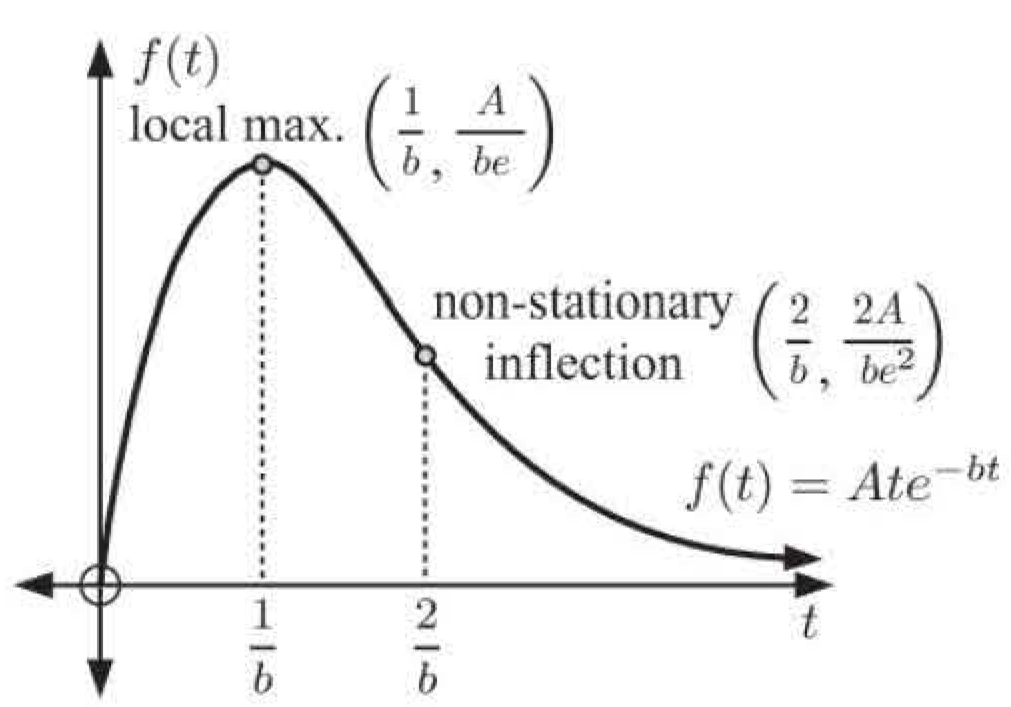


$\therefore \ bt = 2$

$\therefore \ t = \dfrac{2}{b}$

$\therefore \ t = \dfrac{2}{b}$ is a non-stationary point of inflection.

b



13 $f(t) = \dfrac{C}{1 + Ae^{-bt}}, \quad t \geqslant 0, \quad A, b, C > 0$

a When $t = 0$, then $f(0) = \dfrac{C}{1 + Ae^{-b0}}$
$= \dfrac{C}{1 + A}$

b **i** As $t \to \infty$, $Ae^{-bt} \to 0^+$ $\quad \therefore \ f(t) \to \dfrac{C}{1 + 0} = C^-$

$\therefore \ y = C$ is a horizontal asymptote.

ii $f(t) = C(1 + Ae^{-bt})^{-1}$

$f'(t) = C(-1)(1 + Ae^{-bt})^{-2}(-bAe^{-bt})$
$= AbCe^{-bt}(1 + Ae^{-bt})^{-2}$

$f''(t) = (-b)AbCe^{-bt}(1 + Ae^{-bt})^{-2} + AbCe^{-bt}(-2)(1 + Ae^{-bt})^{-3}(-bAe^{-bt})$
{product rule}

$= -\dfrac{Ab^2C}{e^{bt}(1 + Ae^{-bt})^2} + \dfrac{2A^2b^2C}{e^{2bt}(1 + Ae^{-bt})^3}$

When $f''(t) = 0$,

$$\frac{2A^{\cancel{2}}\cancel{b^2}\cancel{C}}{e^{\cancel{2}bt}(1+Ae^{-bt})^{\cancel{3}}} = \frac{\cancel{A}\cancel{b^2}\cancel{C}}{\cancel{e^{bt}}\cancel{(1+Ae^{-bt})^2}}$$

$$\therefore \quad \frac{2A}{e^{bt}(1+Ae^{-bt})} = 1$$

$$\therefore \quad 2A = e^{bt} + A\cancel{e^{-bt}}\cancel{e^{bt}}$$

$$\therefore \quad 2A = e^{bt} + A$$

$$\therefore \quad A = e^{bt}$$

$$\therefore \quad \ln A = bt$$

$$\therefore \quad t = \frac{\ln A}{b}, \quad \text{since } t \geqslant 0, \text{ then } \frac{\ln A}{b} \geqslant 0 \quad \therefore \quad b > 0, \; A > 1$$

$$f\left(\frac{\ln A}{b}\right) = \frac{C}{1 + Ae^{-b\left(\frac{\ln A}{b}\right)}}$$

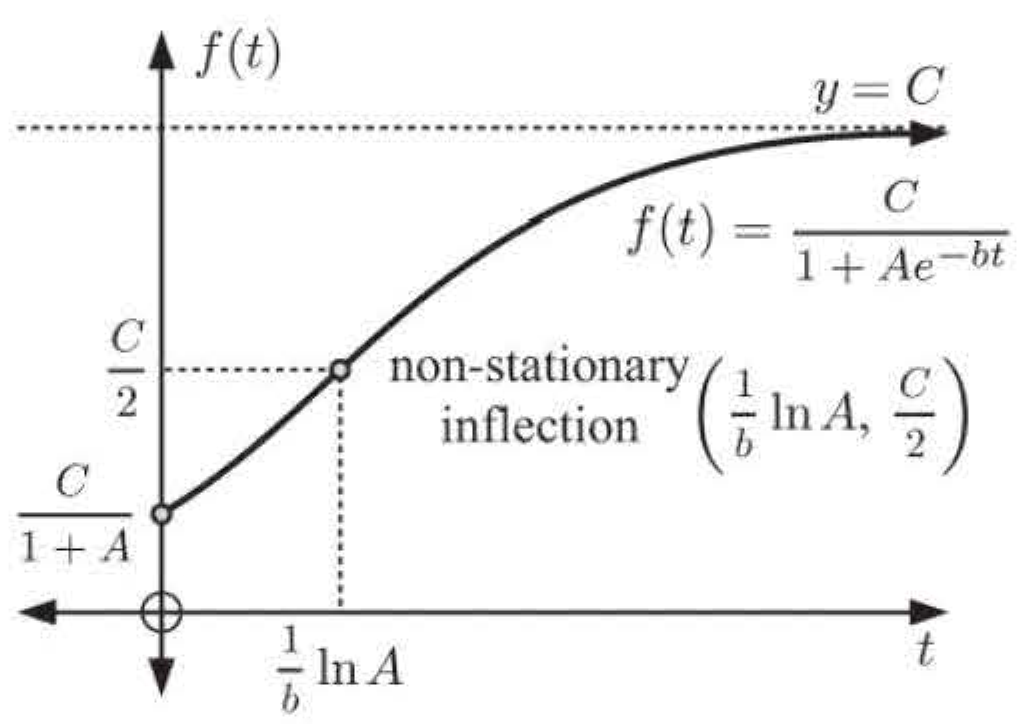
$$\therefore \quad = \frac{C}{1 + Ae^{-\ln A}}$$

$$= \frac{C}{1 + Ae^{\ln A^{-1}}}$$

$$= \frac{C}{1 + AA^{-1}}$$

$$= \frac{C}{1+1} = \frac{C}{2}$$

c



EXERCISE 19D.2

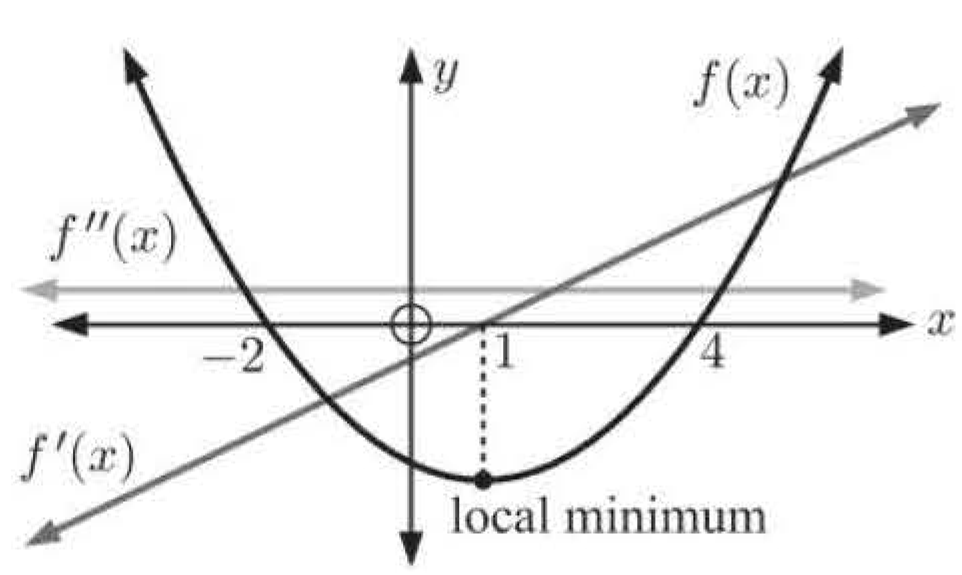
1 **a** $f(x)$ is quadratic, so $f'(x)$ will be linear and $f''(x)$ will be constant.

$f(x)$ is decreasing for $x \leqslant 1$ and increasing for $x \geqslant 1$

$\therefore \; f'(x) \leqslant 0$ for $x \leqslant 1$ and $f'(x) \geqslant 0$ for $x \geqslant 1$

$\therefore \; f'(x)$ is an increasing linear function which cuts the x-axis at 1.

As $f'(x)$ is increasing, $f''(x) > 0$.



b $f(x)$ is cubic, so $f'(x)$ will be quadratic and $f''(x)$ will be linear.

$f(x)$ has turning points at $x \approx \pm 1$

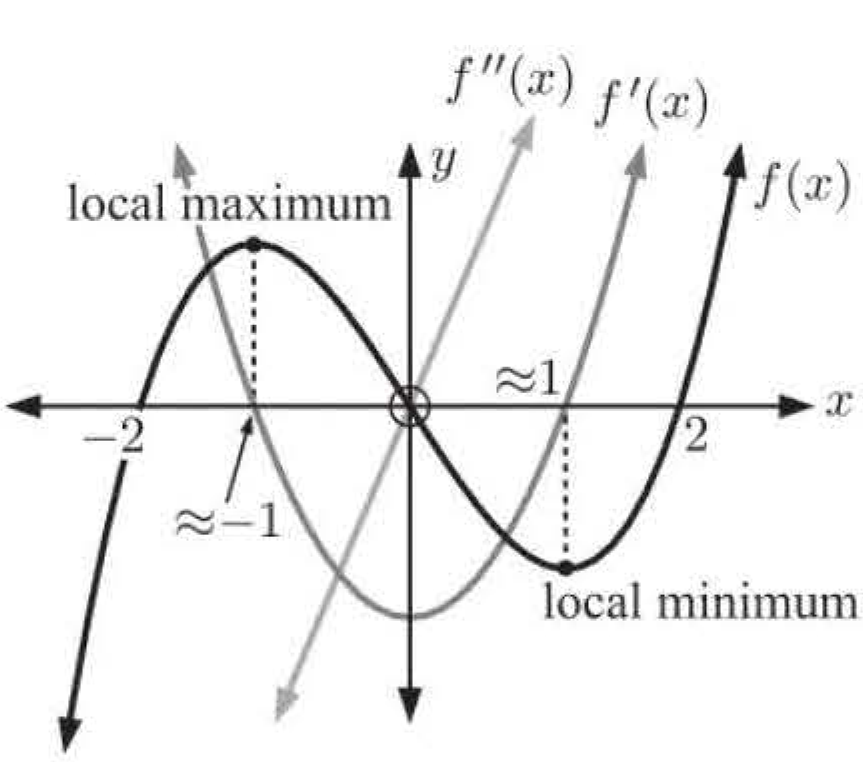
$\therefore \; f'(x)$ cuts the x-axis at these points.

$f(x)$ has a non-stationary inflection point at $x = 0$

$\therefore \; f'(x)$ has a turning point at $x = 0$, and $f''(0) = 0$.

$f(x)$ is concave down for $x \leqslant 0$ and concave up for $x \geqslant 0$

$\therefore \; f'(x)$ is decreasing for $x \leqslant 0$ and increasing for $x \geqslant 0$ and $f''(x) \leqslant 0$ for $x \leqslant 0$ and $\geqslant 0$ for $x \geqslant 0$.



c $f(x)$ is cubic, so $f'(x)$ will be quadratic and $f''(x)$ will be linear.

$f(x)$ has turning points at $x \approx 1$ and $x = 3$

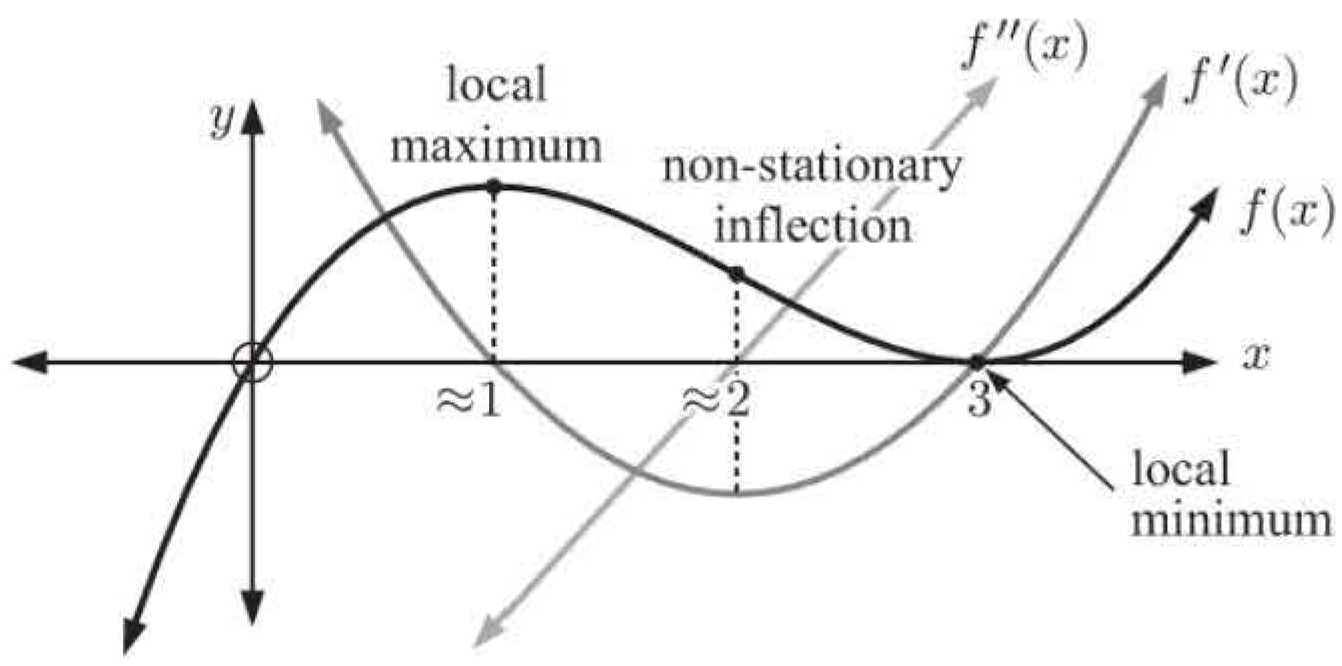
$\therefore$ $f'(x)$ cuts the x-axis at these points.

$f(x)$ has a non-stationary inflection point at $x \approx 2$

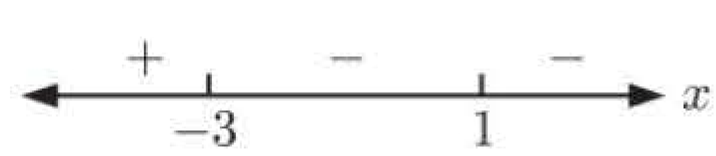
$\therefore$ $f'(x)$ has a turning point at $x \approx 2$, and $f''(2) = 0$

$f(x)$ is concave down for $x \leqslant 2$ and concave up for $x \geqslant 2$

$\therefore$ $f'(x)$ is decreasing for $x \leqslant 2$ and increasing for $x \geqslant 2$
and $f''(x) \leqslant 0$ for $x \leqslant 2$ and $\geqslant 0$ for $x \geqslant 2$.



2 **a** $f'(x)$ has sign diagram:



$\therefore$ $f(x)$ is increasing for $x \leqslant -3$, and decreasing for $x \geqslant -3$

$\therefore$ $f(x)$ has a local maximum at $x = -3$

$f'(x)$ has a turning point at $x \approx -1.7$.

At this point, $f''(x) = 0$, but $f'(x) \neq 0$

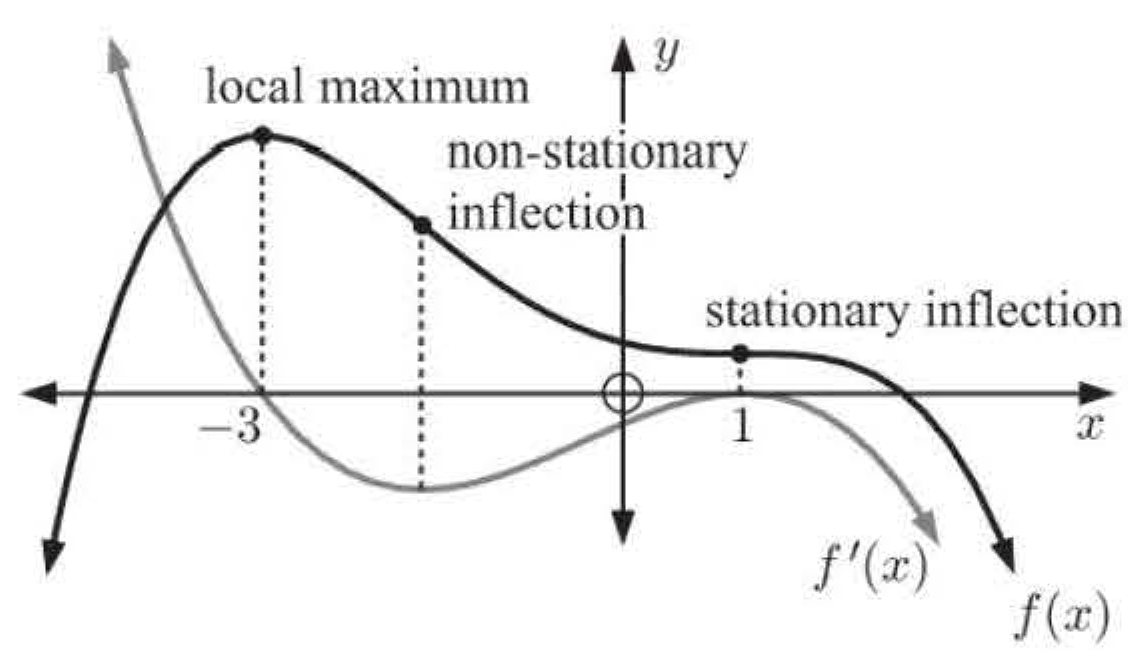
$\therefore$ $f(x)$ has a non-stationary inflection point here.

$f'(x)$ has another turning point at $x = 1$.

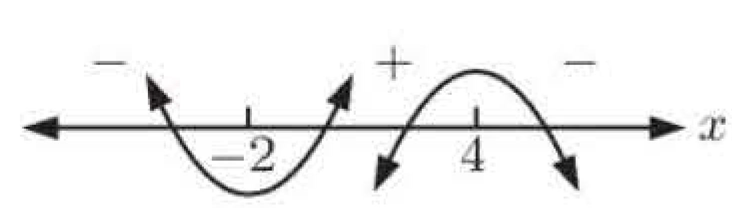
At this point, $f''(x) = 0$ and $f'(x) = 0$

$\therefore$ $f(x)$ has a stationary inflection point at $x = 1$.

A possible graph of $f(x)$ is shown alongside:



b $f'(x)$ has sign diagram:



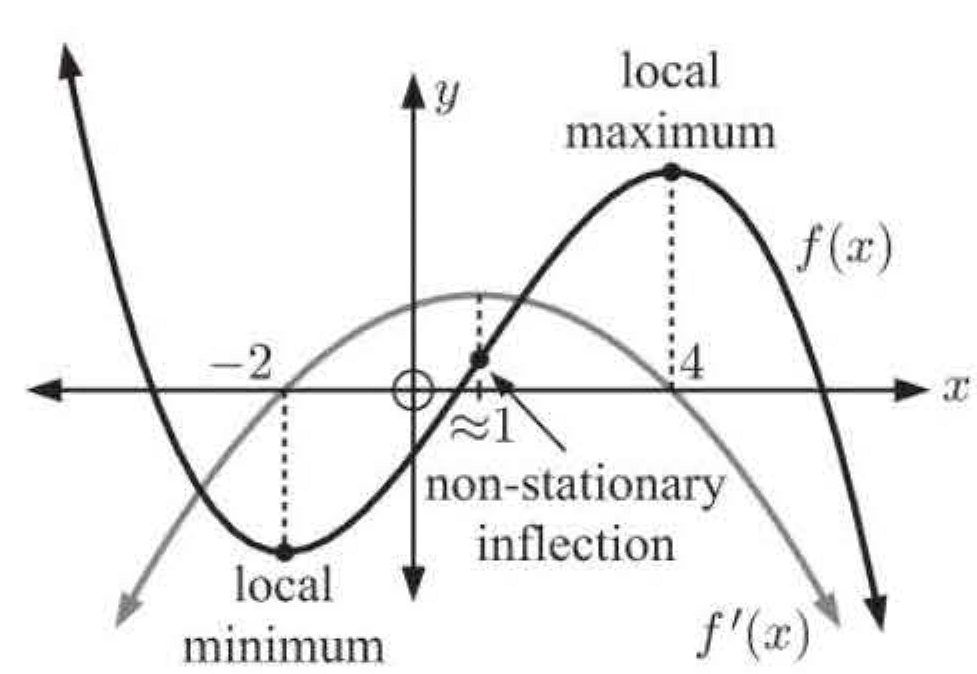
$\therefore$ $f(x)$ has a local minimum at $x = -2$ and a local maximum at $x = 4$

$f'(x)$ has a turning point at $x \approx 1$.

At this point, $f''(x) = 0$, but $f'(x) \neq 0$

$\therefore$ $f(x)$ has a non-stationary inflection point at $x \approx 1$.

A possible graph of $f(x)$ is shown alongside:



REVIEW SET 19A

1 Consider $y = -2x^2$.

When $x = -1$, $y = -2(-1)^2 = -2$, so the point of contact is $(-1, -2)$.

Now $\dfrac{dy}{dx} = -4x$

$\therefore$ at $x = -1$, $\dfrac{dy}{dx} = -4(-1) = 4$

$\therefore$ the tangent has equation $\dfrac{y - (-2)}{x - (-1)} = 4$

or $y = 4x + 2$

2 Consider $y = \dfrac{1-2x}{x^2}$.

When $x = 1$, $y = \dfrac{1-2(1)}{1^2} = -1$, so the point of contact is $(1, -1)$.

Since $y = \dfrac{1}{x^2} - \dfrac{2}{x}$, $\dfrac{dy}{dx} = -2x^{-3} + 2x^{-2} = -\dfrac{2}{x^3} + \dfrac{2}{x^2}$

$\therefore$ at $x = 1$, $\dfrac{dy}{dx} = -2 + 2 = 0$

So, the tangent is a horizontal line, and the normal must be a vertical line of the form $x = k$.
As the normal passes through $(1, -1)$, its equation must be $x = 1$.

3 **a** The vertical asymptote is $x + 3 = 0$ or $x = -3$.

b When $y = 0$, $\dfrac{3x-2}{x+3} = 0$

$\therefore\ 3x - 2 = 0$

$\therefore\ x = \frac{2}{3}$

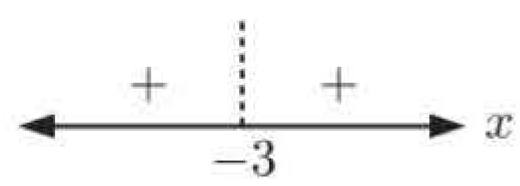
$\therefore$ the x-intercept is $\frac{2}{3}$.

When $x = 0$, $f(0) = \frac{-2}{3}$

$\therefore$ the y-intercept is $-\frac{2}{3}$.

c
$$f'(x) = \frac{3(x+3) - (3x-2)(1)}{(x+3)^2} \quad \{\text{quotient rule}\}$$
$$= \frac{3x + 9 - 3x + 2}{(x+3)^2}$$
$$= \frac{11}{(x+3)^2}$$

$f'(x)$ has sign diagram:



d $f'(x) \neq 0$ for any x, so $f(x)$ has no stationary points.

4 $y = e^{-x^2}$ so when $x = 1$, $y = e^{-1} = \dfrac{1}{e}$

$\therefore$ the point of contact is $\left(1, \dfrac{1}{e}\right)$

Now $\dfrac{dy}{dx} = -2xe^{-x^2}$

$\therefore$ when $x = 1$, $\dfrac{dy}{dx} = -2e^{-1}$

$\therefore$ the gradient of the tangent is $-\dfrac{2}{e}$

and the gradient of the normal is $\dfrac{e}{2}$

$\therefore$ the equation of the normal is $\dfrac{y - \frac{1}{e}}{x - 1} = \dfrac{e}{2}$

$\therefore\ 2\left(y - \dfrac{1}{e}\right) = e(x-1)$

$\therefore\ 2y - \dfrac{2}{e} = ex - e$

$\therefore\ 2y = ex + \dfrac{2}{e} - e$

$\therefore\ y = \dfrac{e}{2}x + \dfrac{1}{e} - \dfrac{e}{2}$

5 $y = x\tan x$

$\therefore\ \dfrac{dy}{dx} = 1 \times \tan x + x \times \left(\dfrac{1}{\cos^2 x}\right)$

$= \tan x + \dfrac{x}{\cos^2 x}$

Now $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ and $\tan\frac{\pi}{4} = 1$

$\therefore$ at $x = \frac{\pi}{4}$, $y = \frac{\pi}{4}$,

and $\dfrac{dy}{dx} = 1 + \dfrac{\frac{\pi}{4}}{\left(\frac{1}{\sqrt{2}}\right)^2} = 1 + \frac{\pi}{2}$

$\therefore$ the equation of the tangent is

$$\frac{y - \frac{\pi}{4}}{x - \frac{\pi}{4}} = 1 + \frac{\pi}{2}$$

$\therefore\ y - \frac{\pi}{4} = (1 + \frac{\pi}{2})(x - \frac{\pi}{4})$

$= x - \frac{\pi}{4} + \frac{\pi}{2}x - \frac{\pi^2}{8}$

$\therefore\ y = (1 + \frac{\pi}{2})x - \frac{\pi^2}{8}$

$\therefore\ 2y = (2 + \pi)x - \frac{\pi^2}{4}$

$\therefore\ (2 + \pi)x - 2y = \frac{\pi^2}{4}$ as required

6 $y = \dfrac{ax+b}{\sqrt{x}} = a\sqrt{x} + \dfrac{b}{\sqrt{x}} = ax^{\frac{1}{2}} + bx^{-\frac{1}{2}}$

$$\therefore \quad \frac{dy}{dx} = \frac{a}{2}x^{-\frac{1}{2}} - \frac{b}{2}x^{-\frac{3}{2}} = \frac{a}{2\sqrt{x}} - \frac{b}{2x\sqrt{x}}$$

The equation of the tangent at $x = 1$

is $2x - y = 1$

or $y = 2x - 1$

so the gradient of the tangent is 2

$\therefore$ at $x = 1$, $\dfrac{dy}{dx} = \dfrac{a}{2} - \dfrac{b}{2} = 2$

$\therefore \; a - b = 4$

$\therefore \; a = b + 4 \quad .... (1)$

Also at $x = 1$, the tangent touches the curve

$\therefore \; \dfrac{a(1)+b}{\sqrt{1}} = 2(1) - 1$

$\therefore \; a + b = 1$

$\therefore \; b + 4 + b = 1 \quad$ {using (1)}

$\therefore \; 2b = -3$

$\therefore \; b = -\frac{3}{2}$ and $a = 4 - \frac{3}{2} = \frac{5}{2}$

7 $f(x) = 4\ln(2x)$, $\quad$ P(1, $4\ln 2$)

$\therefore \; f'(x) = 4 \times \dfrac{2}{2x}$

$= \dfrac{4}{x}$

$\therefore$ at $x = 1$, $f'(1) = \dfrac{4}{1} = 4$

$\therefore$ the tangent has equation

$\dfrac{y - 4\ln 2}{x - 1} = 4$

$\therefore \; y - 4\ln 2 = 4x - 4$

$\therefore \; y = 4x + 4\ln 2 - 4$

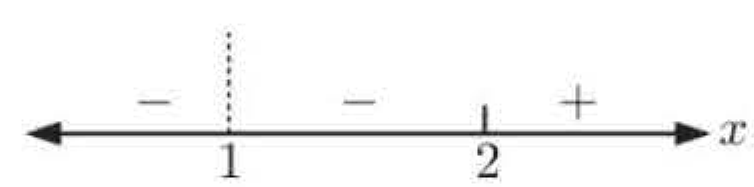
8 **a** $f(x) = \dfrac{e^x}{x-1}$

Now $f(0) = \dfrac{e^0}{-1} = -1$ so the y-intercept is -1.

b $f(x)$ is defined for all $x \neq 1$.

c $f'(x) = \dfrac{e^x(x-1) - e^x(1)}{(x-1)^2} \qquad$ {quotient rule}

$= \dfrac{e^x(x-2)}{(x-1)^2} \qquad$ and has sign diagram:



$\therefore \; f'(x) \leqslant 0$ for $x < 1$ and $1 < x \leqslant 2$ and $f'(x) \geqslant 0$ for $x \geqslant 2$

$\therefore \; f(x)$ is decreasing for $x < 1$ and $1 < x \leqslant 2$, and increasing for $x \geqslant 2$.

$$f''(x) = \frac{[e^x(x-2) + e^x(1)](x-1)^2 - e^x(x-2)[2(x-1)^1(1)]}{(x-1)^4} \qquad \text{\{product and quotient rules\}}$$

$$= \frac{[e^x(x-2+1)(x-1)^2] - 2e^x(x-2)(x-1)}{(x-1)^4}$$

$$= \frac{e^x(x-1)(x-1)^2 - 2e^x(x-2)(x-1)}{(x-1)^4}$$

$$= \frac{e^x(x-1)\left[(x-1)^2 - 2(x-2)\right]}{(x-1)^4}$$

$$= \frac{e^x(x-1)\left[x^2 - 2x + 1 - 2x + 4\right]}{(x-1)^4}$$

$$= \frac{e^x(x^2 - 4x + 5)}{(x-1)^3} \qquad \text{where the quadratic term has } \Delta < 0$$

The sign diagram of $f''(x)$ is:

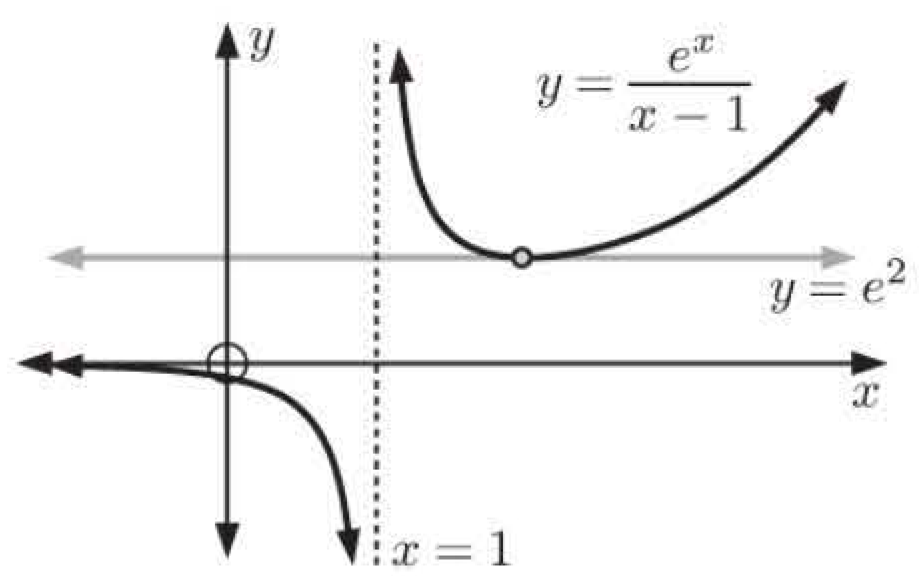
$\therefore \; f''(x) > 0$ for $x > 1$

and $f''(x) < 0$ for $x < 1$.

$\therefore \; f(x)$ is concave down for all $x < 1$

and concave up for all $x > 1$.

d



e Now $f(2) = \dfrac{e^2}{2-1} = e^2$

Using **c**, we have a local minimum at $(2,\ e^2)$

$\therefore$ the tangent at $x = 2$ is horizontal and is $y = e^2$.

9 $y = \dfrac{a}{(x+2)^2} = a(x+2)^{-2}$

The gradient of the line (AB) is

$\dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{8-4}{0-2} = \dfrac{4}{-2} = -2$

$\therefore$ the equation of the tangent is

$\dfrac{y-8}{x-0} = -2$ or $y = -2x + 8$

Now $\dfrac{dy}{dx} = -2a(x+2)^{-3}$, so for the given tangent, $-2a(x+2)^{-3} = -2$

$\therefore \dfrac{a}{(x+2)^3} = 1$

$\therefore a = (x+2)^3$ .... (1)

The line (AB) meets the curve where

$-2x + 8 = \dfrac{a}{(x+2)^2}$

$\therefore -2x + 8 = \dfrac{(x+2)^3}{(x+2)^2}$ {using (1)}

$\therefore -2x + 8 = x + 2$

$\therefore -3x = -6$

$\therefore x = 2$

and so $a = (2+2)^3 = 64$

10 $y = \dfrac{5}{\sqrt{x}} = 5x^{-\frac{1}{2}}$

$\therefore \dfrac{dy}{dx} = -\frac{5}{2}x^{-\frac{3}{2}}$

$\therefore$ the gradient of the tangent at the point $(1,\ 5)$ is $-\frac{5}{2}(1)^{-\frac{3}{2}} = -\frac{5}{2}$

$\therefore$ the equation of the tangent is

$\dfrac{y-5}{x-1} = -\dfrac{5}{2}$

$\therefore y - 5 = -\frac{5}{2}x + \frac{5}{2}$

$\therefore y = -\frac{5}{2}x + \frac{15}{2}$

Now, P and Q are the y- and x-intercepts, so:

P: $y = -\frac{5}{2}(0) + \frac{15}{2}$

$= \frac{15}{2}$

Q: $0 = -\frac{5}{2}x + \frac{15}{2}$

$\therefore \frac{5}{2}x = \frac{15}{2}$

$\therefore x = 3$

So P is $(0,\ 7.5)$ and Q is $(3,\ 0)$.

11 At $x = A$, $f'(x) = 0$ and $f''(x) = 0$

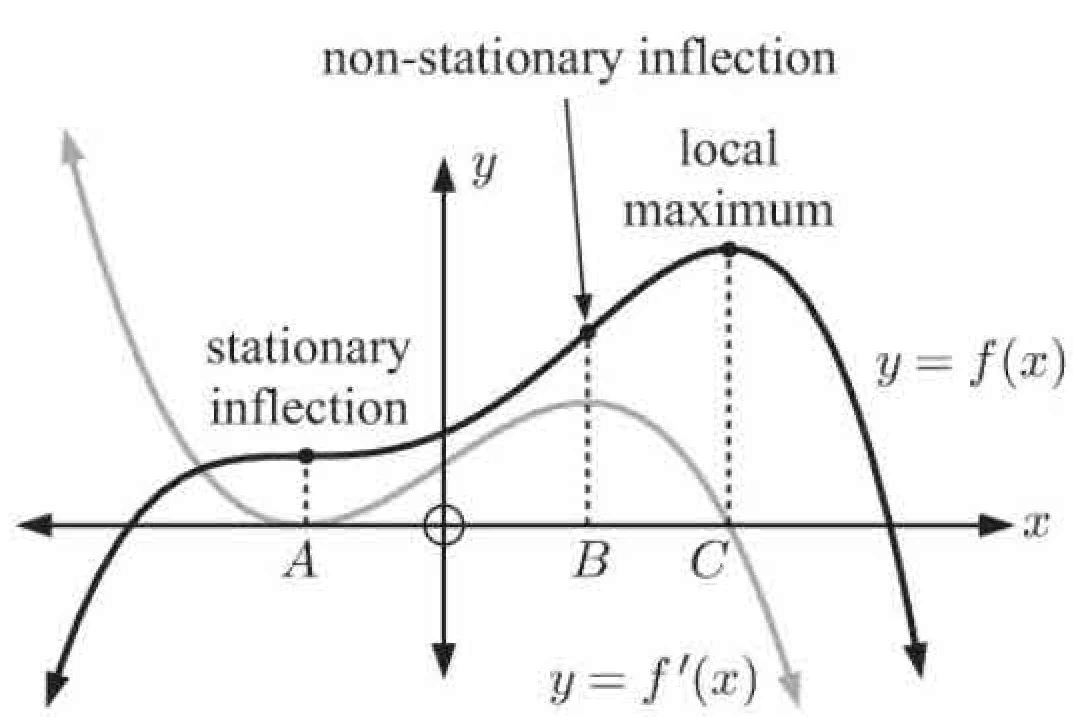
$\therefore$ $f(x)$ has a stationary inflection point at $x = A$.

At $x = B$, $f''(x) = 0$ but $f'(x) \neq 0$

$\therefore$ $f(x)$ has a non-stationary inflection point at $x = B$.

$f'(x)$ is above the x-axis for $x \leqslant C$, and below the x-axis for $x \geqslant C$

$\therefore$ $f(x)$ is increasing for $x \leqslant C$ and decreasing for $x \geqslant C$, so $f(x)$ has a local maximum at $x = C$.



12 $y = \ln(x^2 + 3)$ $\quad \therefore \dfrac{dy}{dx} = \dfrac{2x}{x^2+3}$

When $x = 0$, $\dfrac{dy}{dx} = 0$ so the gradient of the tangent at this point is 0.

But when $x = 0$, $y = \ln(0+3) = \ln 3$

$\therefore$ the tangent is $y = \ln 3$ which does not cut the x-axis.

13 **a** $y = \sec x = \dfrac{1}{\cos x}$

When $x = \frac{\pi}{3}$, then $y = \dfrac{1}{\cos\frac{\pi}{3}} = \dfrac{1}{\frac{1}{2}} = 2$

$\dfrac{dy}{dx} = \sec x \tan x = \dfrac{1}{\cos x}\,\dfrac{\sin x}{\cos x}$

When $x = \frac{\pi}{3}$, then $\dfrac{dy}{dx} = \dfrac{\sin\frac{\pi}{3}}{(\cos\frac{\pi}{3})^2}$

$$= \frac{\frac{\sqrt{3}}{2}}{(\frac{1}{2})^2}$$

$$= \tfrac{\sqrt{3}}{2} \times \tfrac{4}{1}$$

$$= 2\sqrt{3}$$

$\therefore$ the equation of the tangent at $(\frac{\pi}{3}, 2)$ is:

$$\frac{y-2}{x-\frac{\pi}{3}} = 2\sqrt{3}$$

$$\therefore\ y - 2 = 2\sqrt{3}x - \frac{2\sqrt{3}\pi}{3}$$

$$\therefore\ 2\sqrt{3}x - y = \frac{2\sqrt{3}\pi}{3} - 2$$

b $y = \arctan x$

When $x = \sqrt{3}$, then $y = \arctan\sqrt{3} = \frac{\pi}{3}$

$\dfrac{dy}{dx} = \dfrac{1}{1+x^2}$, $x \in \mathbb{R}$

When $x = \sqrt{3}$, then $\dfrac{dy}{dx} = \dfrac{1}{1+\sqrt{3}^2}$

$$= \tfrac{1}{1+3}$$

$$= \tfrac{1}{4}$$

$\therefore$ the gradient of the normal is -4

$\therefore$ the equation of the normal at $(\sqrt{3}, \frac{\pi}{3})$ is:

$$\frac{y-\frac{\pi}{3}}{x-\sqrt{3}} = -4$$

$$\therefore\ y - \tfrac{\pi}{3} = -4x + 4\sqrt{3}$$

$$\therefore\ 4x + y = 4\sqrt{3} + \tfrac{\pi}{3}$$

14 $x^2 + y^2 = 1$

$\therefore\ y^2 = 1 - x^2$

$\therefore\ y = (1-x^2)^{\frac{1}{2}} = \pm\sqrt{1-x^2}$, $x \in [-1, 1]$

For any point on the circle let $x = a$, then $y = \pm\sqrt{1-a^2}$, $a \in [-1, 1]$

If $y > 0$, $\dfrac{dy}{dx} = \frac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x) = \dfrac{-x}{\sqrt{1-x^2}}$, $x \in [-1, 1]$

$\therefore$ the gradient of the normal would be $\dfrac{\sqrt{1-x^2}}{x}$

$\therefore$ for the points $(a, \sqrt{1-a^2})$ the gradient is $\dfrac{\sqrt{1-a^2}}{a}$

$\therefore$ the equation of the normal at $(a, \sqrt{1-a^2})$ is:

$$\frac{y-\sqrt{1-a^2}}{x-a} = \frac{\sqrt{1-a^2}}{a}$$

$$\therefore\ ay - \cancel{a\sqrt{1-a^2}} = x\sqrt{1-a^2} - \cancel{a\sqrt{1-a^2}}$$

$$\therefore\ y = \frac{\sqrt{1-a^2}}{a}x$$

which for any $a \in [-1, 1]$ will pass through the origin.

If $y < 0$, $\dfrac{dy}{dx} = \dfrac{x}{\sqrt{1-x^2}}$, $x \in [-1, 1]$

$\therefore$ the gradient of the normal is $-\dfrac{\sqrt{1-x^2}}{x}$

$\therefore$ for the points $(a, -\sqrt{1-a^2})$ the gradient is $-\dfrac{\sqrt{1-a^2}}{a}$

$\therefore$ the equation of the normal at $(a, -\sqrt{1-a^2})$ is:

$$\frac{y+\sqrt{1-a^2}}{x-a} = -\frac{\sqrt{1-a^2}}{a}$$

$$\therefore\ y = -\frac{\sqrt{1-a^2}}{a}x$$ which will pass through the origin for any $a \in [-1, 1]$.

REVIEW SET 19B

1 $y = x^3 - 3x^2 - 9x + 2 \qquad \therefore \ \dfrac{dy}{dx} = 3x^2 - 6x - 9$

Horizontal tangents occur when $\dfrac{dy}{dx} = 0$
$$\therefore \ 3x^2 - 6x - 9 = 0$$
$$\therefore \ x^2 - 2x - 3 = 0$$
$$\therefore \ (x-3)(x+1) = 0$$
$$\therefore \ x = 3 \text{ or } x = -1$$

When $x = 3$, the horizontal tangent has equation $y = -25$.
When $x = -1$, the horizontal tangent has equation $y = 7$.

2 Consider the tangent to $y = x^2\sqrt{1-x}$ at $x = -3$.

When $x = -3$, $y = (-3)^2\sqrt{1-(-3)} = 9\sqrt{4} = 18 \quad \{y \geqslant 0\}$

$\therefore$ the point of contact is $(-3, 18)$.

Also, $y = x^2\sqrt{1-x}$ is a product with $u = x^2$ and $v = (1-x)^{\frac{1}{2}}$

$\therefore \ u' = 2x$ and $v' = \frac{1}{2}(1-x)^{-\frac{1}{2}}(-1)$

$$\therefore \ \frac{dy}{dx} = 2x(1-x)^{\frac{1}{2}} - x^2(\tfrac{1}{2})(1-x)^{-\frac{1}{2}}$$

$$\therefore \text{ at } x = -3, \ \frac{dy}{dx} = 2(-3)(1-(-3))^{\frac{1}{2}} - (-3)^2(\tfrac{1}{2})(1-(-3))^{-\frac{1}{2}}$$
$$= -6(2) - 9(\tfrac{1}{2})(\tfrac{1}{2})$$
$$= -\tfrac{57}{4}$$

$\therefore$ the tangent at $(-3, 18)$ has equation $\dfrac{y-18}{x-(-3)} = -\frac{57}{4}$
$$\therefore \ 4y - 72 = -57x - 171$$
$$\therefore \ 4y = -57x - 99$$

Now when $x = 0$, $y = -\frac{99}{4}$ and when $y = 0$, $x = -\frac{99}{57}$

$\therefore$ the area of $\triangle\text{OAB} = \frac{1}{2}\left(\frac{99}{4}\right)\left(\frac{99}{57}\right) = \frac{3267}{152} \approx 21.5 \text{ units}^2$

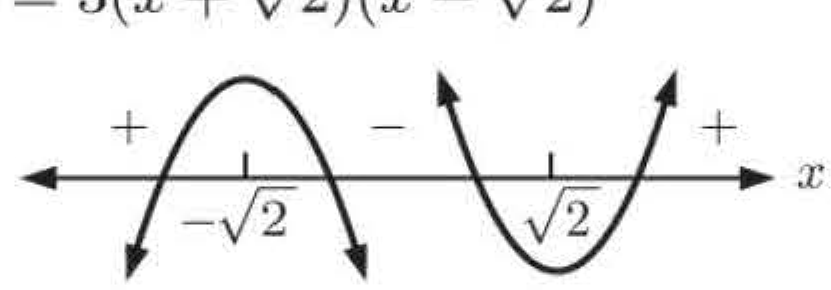
3 **a** $f(x) = x^3 + ax, \quad a < 0$

$\therefore \ f'(x) = 3x^2 + a$

$f(x)$ has a turning point at $x = \sqrt{2}$, so $f'(\sqrt{2}) = 0$
$$\therefore \ 3(\sqrt{2})^2 + a = 0$$
$$\therefore \ a = -6$$

b $f'(x) = 3x^2 - 6 = 3(x^2 - 2) = 3(x+\sqrt{2})(x-\sqrt{2})$

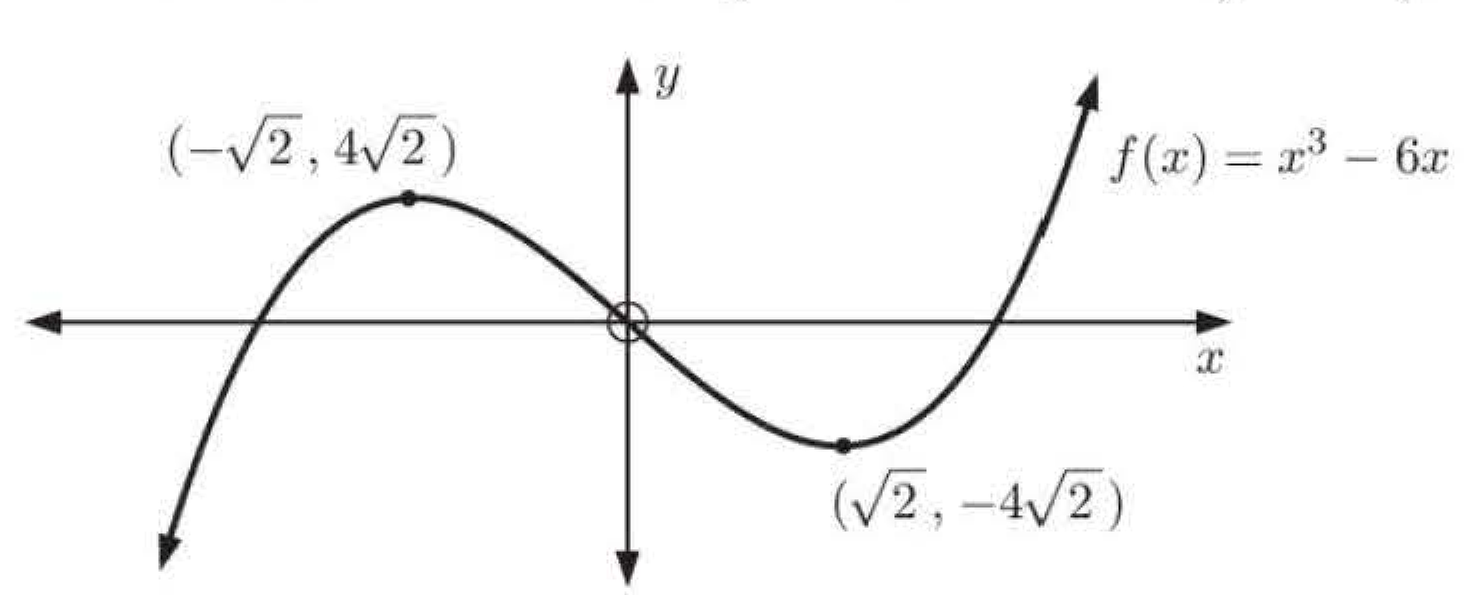
$\therefore \ f'(x)$ has sign diagram:



$\therefore \ f(x)$ has a local maximum at $\left(-\sqrt{2}, (-\sqrt{2})^3 - 6(-\sqrt{2})\right)$ or $\left(-\sqrt{2}, 4\sqrt{2}\right)$,

and a local minimum at $\left(\sqrt{2}, (\sqrt{2})^3 - 6\sqrt{2}\right)$ or $\left(\sqrt{2}, -4\sqrt{2}\right)$.

c



4 $\quad f(x) = e^{4x} + px + q$

$\therefore\ f'(x) = 4e^{4x} + p$

At the point where $x = 0$, the tangent to $f(x)$ has equation $y = 5x - 7$, so $f'(0) = 5$

$\therefore\ 4e^0 + p = 5$

$\therefore\ p = 1$

The tangent meets $f(x)$ when $x = 0$ and $y = 5(0) - 7 = -7$, so $(0, -7)$ must lie on $f(x)$ too.

$\therefore\ e^{4(0)} + p(0) + q = -7$

$\therefore\ 1 + q = -7$

$\therefore\ q = -8$

5 Consider the tangent to $y = 2x^3 + 4x - 1$ at $(1, 5)$.

$\dfrac{dy}{dx} = 6x^2 + 4 \qquad \therefore$ at $x = 1$, $\dfrac{dy}{dx} = 6(1)^2 + 4 = 10$

$\therefore$ the tangent has equation $\dfrac{y-5}{x-1} = 10$ or $y = 10x - 5$

Now the tangent meets the curve again where $10x - 5 = 2x^3 + 4x - 1$

$\therefore\ 2x^3 - 6x + 4 = 0$

$\therefore\ x^3 - 3x + 2 = 0$

We know that $(x-1)^2$ is a factor since the line is tangent to the curve at $x = 1$.

Consequently, $x^3 - 3x + 2 = (x-1)^2(x+2) = 0 \qquad$ {since the constant term is 2}

Thus $x = -2$ is the other solution and when $x = -2$, $y = 2(-2)^3 + 4(-2) - 1 = -25$

$\therefore$ the tangent meets the curve again at $(-2, -25)$.

6 Consider $y = 4(ax+1)^{-2}$.

When $x = 0$, $y = 4(0+1)^{-2} = 4$, so the point of contact is $(0, 4)$.

Now $\dfrac{dy}{dx} = -8(ax+1)^{-3}(a) = \dfrac{-8a}{(ax+1)^3} \qquad \therefore$ at $x = 0$, $\dfrac{dy}{dx} = -8a$

$\therefore$ the tangent has equation $\dfrac{y-4}{x-0} = -8a$ or $y - 4 = -8ax$

This tangent passes through $(1, 0)$, so $0 - 4 = -8a(1) \qquad \therefore\ a = \frac{1}{2}$

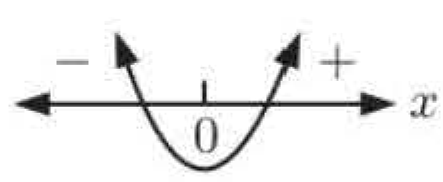
7 $f(x) = e^x - x$

a $f'(x) = e^x - 1$

so $f'(x) = 0$ when $e^x = 1$

$\therefore\ x = 0$

Sign diagram of $f'(x)$ is:



Now $f(0) = e^0 - 0 = 1$

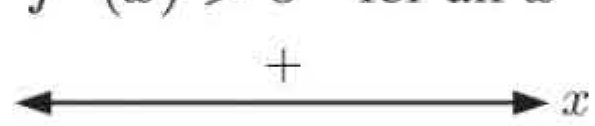
$\therefore$ there is a local minimum at $(0, 1)$.

b As $x \to \infty$, $e^x \to \infty$ faster than x

$\therefore\ f(x) \to \infty$

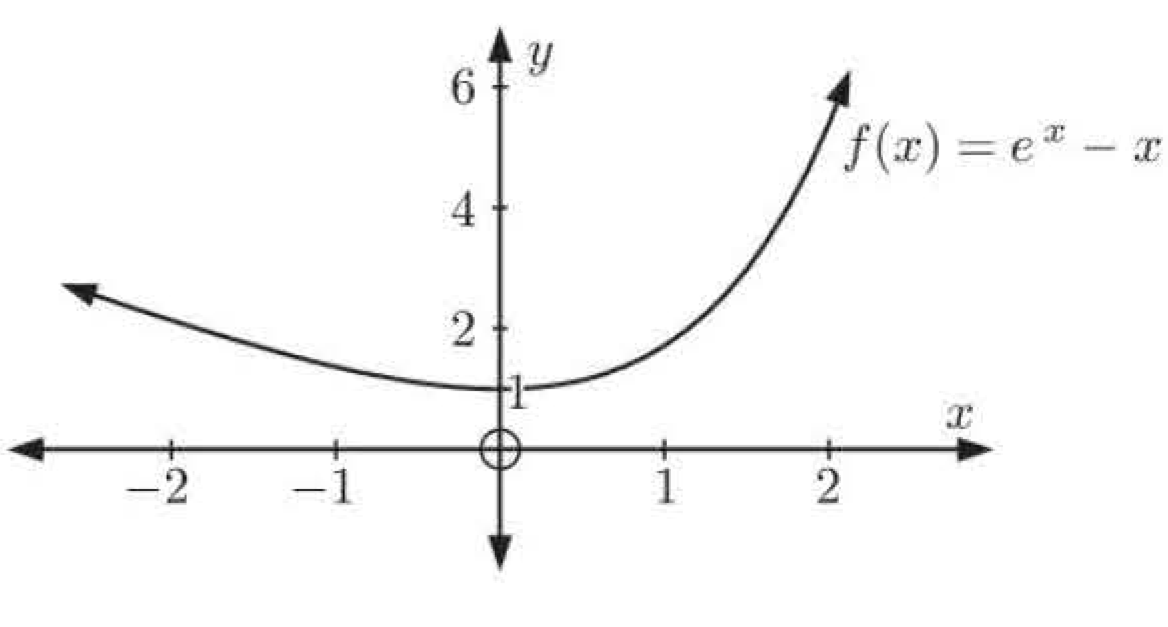
c $\quad f''(x) = e^x$

$\therefore\ f''(x) > 0$ for all x



$\therefore\ f(x)$ is concave up for all x

d



e Since a local minimum exists at $(0, 1)$,

$f(x) \geqslant 1$ for all x

$\therefore\ e^x - x \geqslant 1$

$\therefore\ e^x \geqslant x + 1$ for all x

8 Consider $y = \dfrac{x+1}{x^2-2}$.

When $x = 1$, $y = \dfrac{1+1}{1^2-2} = -2$

$\therefore$ the point of contact is $(1, -2)$.

$y = \dfrac{x+1}{x^2-2}$ is a quotient with $u = x+1$ and $v = x^2-2$

$\therefore$ $u' = 1$ and $v' = 2x$

$$\frac{dy}{dx} = \frac{1(x^2-2)-(x+1)(2x)}{(x^2-2)^2} \quad \{\text{quotient rule}\}$$

$$\therefore \text{ at } x = 1, \quad \frac{dy}{dx} = \frac{(1-2)-2(1+1)}{(1-2)^2} = \frac{-1-4}{1} = -5$$

$\therefore$ the normal at $(1, -2)$ has gradient $\frac{1}{5}$.

So the normal has equation $\dfrac{y-(-2)}{x-1} = \frac{1}{5}$

$$\therefore\ 5y + 10 = x - 1$$

$$\therefore\ y = \tfrac{1}{5}x - \tfrac{11}{5} \quad (\text{or} \quad x - 5y = 11)$$

9 $$y = 2 - \frac{7}{1+2x} = 2 - 7(1+2x)^{-1}$$

$$\therefore\ \frac{dy}{dx} = 7(1+2x)^{-2} \times 2 \quad \{\text{chain rule}\}$$

$$= \frac{14}{(1+2x)^2}$$

The tangent is horizontal when the gradient $\dfrac{dy}{dx} = 0$.

But $\dfrac{14}{(1+2x)^2}$ is never 0, so

$y = 2 - \dfrac{7}{1+2x}$ has no horizontal tangents.

10 Let $g(x) = ax^2 + bx + c$.

$g(x)$ has y-intercept $(0, 3)$, so $g(0) = a(0)^2 + b(0) + c = 3$

$$\therefore\ c = 3$$

$\therefore$ $g(x) = ax^2 + bx + 3$

The point $(2, 7)$ lies on $g(x)$, so $g(2) = a(2)^2 + b(2) + 3 = 7$

$$\therefore\ 4a + 2b = 4 \quad .... (1)$$

Also, $g'(x) = 2ax + b$

$\therefore$ $g'(2) = 2a(2) + b = 4a + b$

$\therefore$ the gradient of the tangent to $g(x)$ at $(2, 7)$ is $4a + b$.

But, the tangent at $(2, 7)$ passes through $(0, 11)$, so the gradient $= \dfrac{7-11}{2-0} = -2$

$$\therefore\ 4a + b = -2 \quad .... (2)$$

Solving (1) and (2) simultaneously,

$$\begin{aligned} 4a + 2b &= 4 \\ 4a + b &= -2 \\ \hline \text{subtracting:} \quad b &= 6 \end{aligned}$$

Using (2), $4a + 6 = -2$, so $a = -2$

So, $g(x) = -2x^2 + 6x + 3$

11 **a** $f(x)=\sqrt{\cos x},\quad 0 \leqslant x \leqslant 2\pi$

$f(x)$ is defined when $\cos x \geqslant 0$,

which is when $0 \leqslant x \leqslant \frac{\pi}{2}$

and $\frac{3\pi}{2} \leqslant x \leqslant 2\pi$

b $f(x)=(\cos x)^{\frac{1}{2}}$

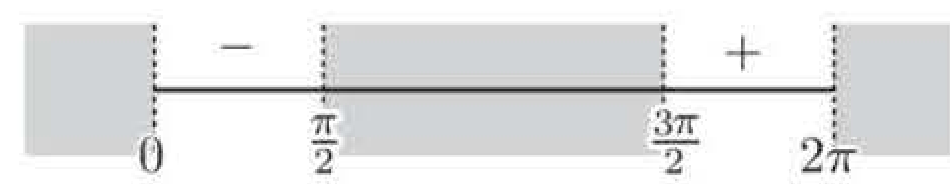
$\therefore\ f'(x)=\frac{1}{2}(\cos x)^{-\frac{1}{2}}(-\sin x)$

$=\dfrac{-\sin x}{2\sqrt{\cos x}}$

$\therefore\ f'(x)=0$ when $-\sin x=0$

For $0 \leqslant x \leqslant 2\pi$, this is when $x=0,\ \pi,\ 2\pi$.

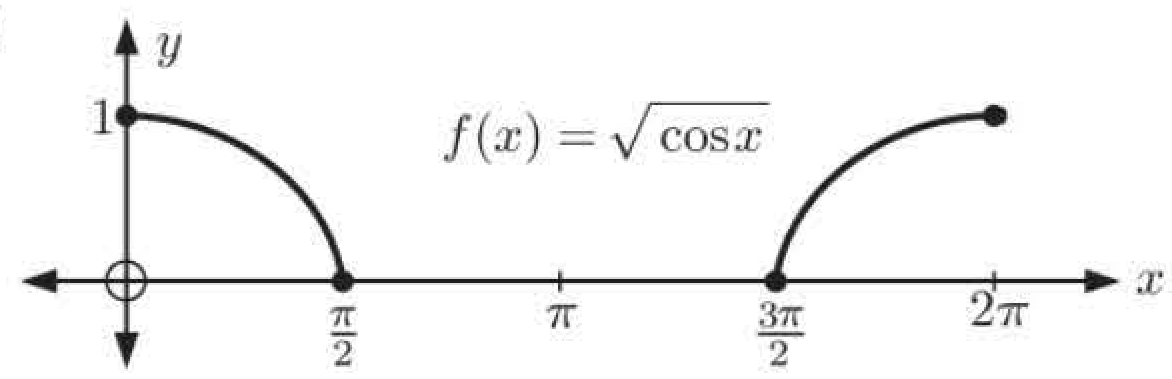
Sign diagram for $f'(x)$ is:



$f(x)$ is increasing for $\frac{3\pi}{2} \leqslant x \leqslant 2\pi$

and decreasing for $0 \leqslant x \leqslant \frac{\pi}{2}$.

c



12 $f(x)$ has a turning point at $x=0$

$\therefore\ f'(0)=0$

$f(x)$ is increasing for $x \geqslant 0$,

except at the asymptote,

so $f'(x)$ is positive for $x \geqslant 0$.

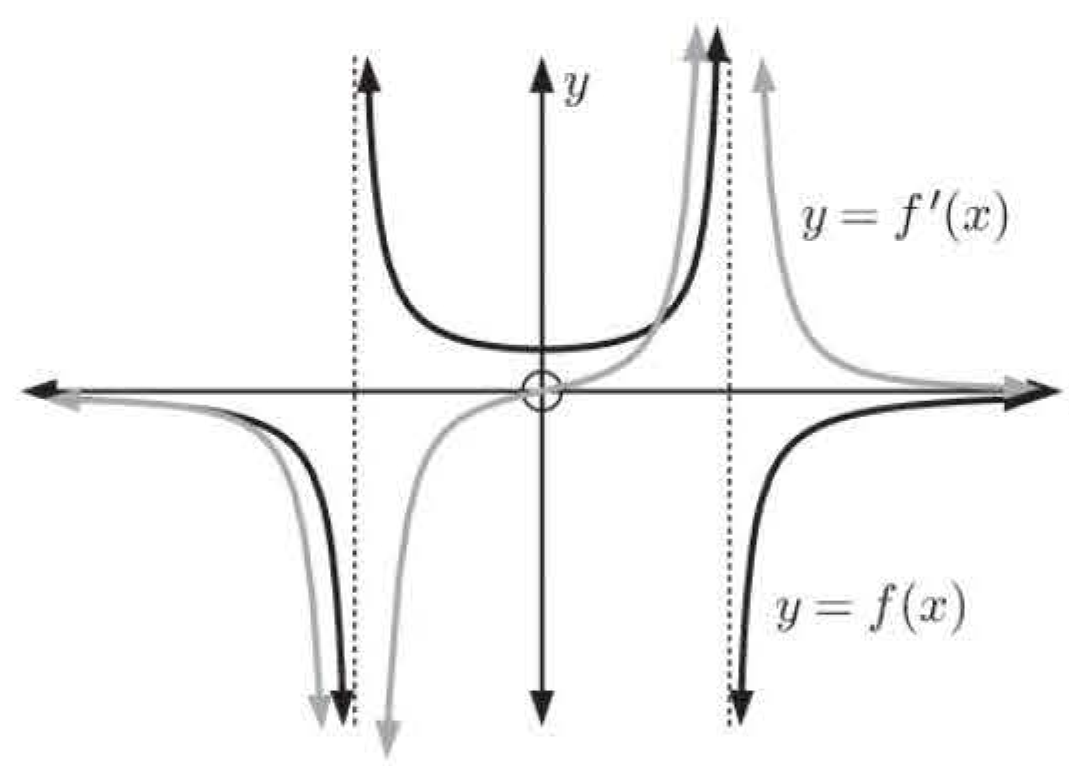
$f(x)$ is decreasing for $x \leqslant 0$,

except at the asymptote,

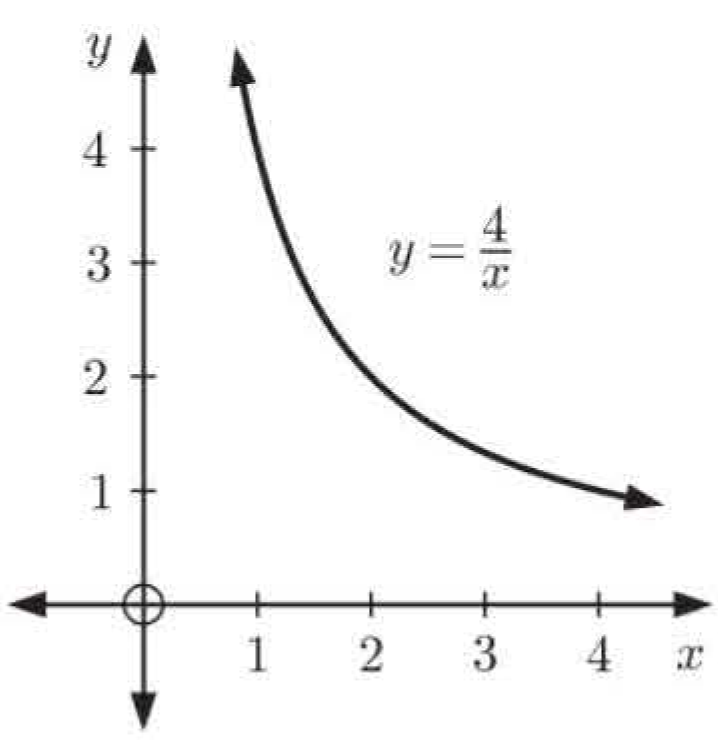
so $f'(x)$ is negative for $x \leqslant 0$.

As $x \to \pm\infty$, $f(x)$ becomes

closer to horizontal so $f'(x) \to 0$.



13 **a**



b For $f(x)=\dfrac{4}{x}=4x^{-1}$,

$f'(x)=-4x^{-2}=-\dfrac{4}{x^2}$ and $f'(k)=-\dfrac{4}{k^2},\quad k>0$

$\therefore$ the gradient of the tangent to $f(x)$ at $\left(k,\ \dfrac{4}{k}\right)$ is $-\dfrac{4}{k^2}$

$\therefore$ the equation of the tangent is $\dfrac{y-\frac{4}{k}}{x-k}=-\dfrac{4}{k^2}$

$\therefore\ yk^2-4k=-4x+4k$

$\therefore\ k^2y=-4x+8k$

$\therefore\ y=-\dfrac{4}{k^2}x+\dfrac{8}{k}$

c $y=-\dfrac{4}{k^2}x+\dfrac{8}{k}$ cuts the x-axis when $y=0$

$\therefore\ -\dfrac{4}{k^2}x+\dfrac{8}{k}=0$

$\therefore\ \dfrac{4}{k^2}x=\dfrac{8}{k}$

$\therefore\ x=2k$ $\quad\therefore$ A is at $(2k,\ 0)$

$y=-\dfrac{4}{k^2}x+\dfrac{8}{k}$ cuts the y-axis when $x=0$

$\therefore\ y=\dfrac{8}{k}$ $\quad\therefore$ B is at $\left(0,\ \dfrac{8}{k}\right)$

d Area of triangle OAB $=\frac{1}{2}(2k)\left(\dfrac{8}{k}\right)=8$ units^2

e The gradient of the tangent to $f(x)$ at $\left(k, \frac{4}{k}\right)$ is $-\frac{4}{k^2}$

$\therefore$ the gradient of the normal to $f(x)$ at $\left(k, \frac{4}{k}\right)$ is $\frac{k^2}{4}$

$\therefore$ the equation of the normal is $\dfrac{y-\frac{4}{k}}{x-k} = \dfrac{k^2}{4}$

$$\therefore \; 4y - \frac{16}{k} = k^2x - k^3$$

$$\therefore \; 4ky - k^3x = 16 - k^4$$

This normal passes through $(1, 1)$, so $4k - k^3 = 16 - k^4$

$$\therefore \; k^4 - k^3 + 4k - 16 = 0$$

$$\therefore \; (k-2)(k+2)(k^2-k+4) = 0 \quad \{\text{using technology}\}$$

$$\therefore \; k = \pm 2$$

But $k > 0$, so $k = 2$

14 $y = \dfrac{x}{\sqrt{1-x}} = x(1-x)^{-\frac{1}{2}}$

When $x = -3$, $y = \dfrac{-3}{\sqrt{1-(-3)}} = \frac{-3}{\sqrt{4}} = -\frac{3}{2}$

$$\frac{dy}{dx} = x(-\tfrac{1}{2})(1-x)^{-\frac{3}{2}}(-1) + (1-x)^{-\frac{1}{2}}$$
$$= \frac{x}{2(1-x)^{\frac{3}{2}}} + \frac{1}{(1-x)^{\frac{1}{2}}}$$

When $x = -3$, $\dfrac{dy}{dx} = \dfrac{-3}{2(1-(-3))^{\frac{3}{2}}} + \dfrac{1}{(1-(-3))^{\frac{1}{2}}}$

$$= \frac{-3}{2 \times 4^{\frac{3}{2}}} + \frac{1}{4^{\frac{1}{2}}}$$
$$= \tfrac{-3}{16} + \tfrac{8}{16}$$
$$= \tfrac{5}{16}$$

$\therefore$ the equation of the tangent at $(-3, -\frac{3}{2})$ is: $\quad \therefore \; \dfrac{y-(-\frac{3}{2})}{x-(-3)} = \frac{5}{16}$

$$\therefore \; 16(y + \tfrac{3}{2}) = 5(x+3)$$
$$\therefore \; 16y + 24 = 5x + 15$$
$$\therefore \; 5x - 16y = 9$$

Given $5x + by = a$ $\quad \therefore \; a = 9$ and $b = -16$

REVIEW SET 19C

1 Consider the normal to the curve $y = \dfrac{1}{\sqrt{x}}$ at $x = 4$.

When $x = 4$, $y = \frac{1}{\sqrt{4}} = \frac{1}{2}$, so the point of contact is $(4, \frac{1}{2})$.

Now $\dfrac{dy}{dx} = -\frac{1}{2}x^{-\frac{3}{2}}$ $\quad \therefore$ at $x = 4$, $\dfrac{dy}{dx} = -\frac{1}{2}\left(4^{-\frac{3}{2}}\right) = -\frac{1}{2}\left(\frac{1}{8}\right) = -\frac{1}{16}$

$\therefore$ the normal at $(4, \frac{1}{2})$ has gradient 16.

So the equation is $\dfrac{y-\frac{1}{2}}{x-4} = 16$

$$\therefore \; y - \tfrac{1}{2} = 16x - 64$$
$$\therefore \; y = 16x - \tfrac{127}{2}$$

2 **a** The tangent shown on the graph passes through $(0, 5)$ and $(5, 0)$.

$\therefore$ the gradient of the tangent is

$\dfrac{0-5}{5-0} = -1$, so $f'(3) = -1$.

Also, since the tangent passes through $(0, 5)$,

it has equation $\dfrac{y-5}{x-0} = -1$

$\therefore\ y - 5 = -x$

$\therefore\ y = -x + 5$

So when $x = 3$, $y = -3 + 5 = 2$

$\therefore$ the point of contact is $(3, 2)$, and hence $f(3) = 2$.

b $f(x)$ has the form $f(x) = ax^2 + bx + c$

The y-intercept is 14 $\therefore\ f(0) = 14$.

$\therefore\ a(0)^2 + b(0) + c = 14$

$\therefore\ c = 14$

$f(3) = 2$

$\therefore\ a(3)^2 + b(3) + 14 = 2$

$\therefore\ 9a + 3b = -12$ .... (1)

$f'(3) = -1$

and $f'(x) = 2ax + b$

$\therefore\ 2a(3) + b = -1$

$\therefore\ 6a + b = -1$

$\therefore\ b = -6a - 1$ .... (2)

Substituting (2) into (1),

$9a + 3(-6a - 1) = -12$

$\therefore\ 9a - 18a - 3 = -12$

$\therefore\ -9a = -9$

$\therefore\ a = 1$

Using (2), $b = -6(1) - 1$

$\therefore\ b = -7$

So, $f(x) = x^2 - 7x + 14$

3 $y = x^3 + ax + b$ $\quad\therefore\ \dfrac{dy}{dx} = 3x^2 + a$

$\therefore$ at $x = 1$, $\dfrac{dy}{dx} = 3 + a$

The equation of the tangent at $x = 1$ is $y = 2x$, so the gradient is 2.

$\therefore\ 3 + a = 2$ and so $a = -1$

Also at $x = 1$, the tangent touches the curve.

$\therefore\ x^3 + ax + b = 2x$ when $x = 1$

$\therefore\ (1)^3 + (-1)(1) + b = 2(1)$

$\therefore\ 1 - 1 + b = 2$

$\therefore\ b = 2$

4 **a** $y = x^3 + ax^2 - 4x + 3$ $\quad\therefore\ \dfrac{dy}{dx} = 3x^2 + 2ax - 4$

The tangent at $x = 1$ is parallel to $y = 3x$, so when $x = 1$, $\dfrac{dy}{dx} = 3$

$\therefore\ 3 = 3(1)^2 + 2a(1) - 4$

$\therefore\ 2a = 4$

$\therefore\ a = 2$

When $x = 1$, $y = 1^3 + 2(1)^2 - 4(1) + 3 = 2$

The contact point is $(1, 2)$ and since the gradient is 3, the tangent at $(1, 2)$ has equation

$\dfrac{y-2}{x-1} = 3$ $\quad\therefore\ y - 2 = 3x - 3$

$\therefore\ y = 3x - 1$

b The tangent meets the curve where $x^3 + 2x^2 - 4x + 3 = 3x - 1$

$\therefore\ x^3 + 2x^2 - 7x + 4 = 0$

Since the line touches the curve at $x = 1$, $(x-1)^2$ must be a factor.

Consequently, $x^3 + 2x^2 - 7x + 4 = (x-1)^2(x+4) = 0$ $\quad$ {since the constant term is 4}

$\therefore$ the curve cuts the tangent when $x = -4$.

When $x = -4$, $y = (-4)^3 + 2(-4)^2 - 4(-4) + 3 = -13$

$\therefore$ the curve cuts the tangent at $(-4, -13)$.

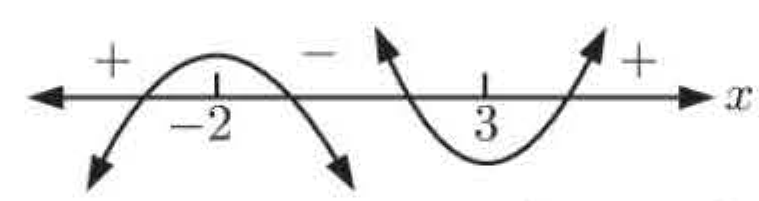
5 $$y = \ln(x^4+3)$$
$$\therefore \quad \frac{dy}{dx} = \frac{4x^3}{x^4+3}$$

$\therefore$ when $x = 1$, $\dfrac{dy}{dx} = \dfrac{4(1)^3}{1^4+3} = 1$ and $y = \ln(1^4+3) = \ln 4$

$\therefore$ the tangent has equation $\dfrac{y-\ln 4}{x-1} = 1$ or $y = x - 1 + \ln 4$

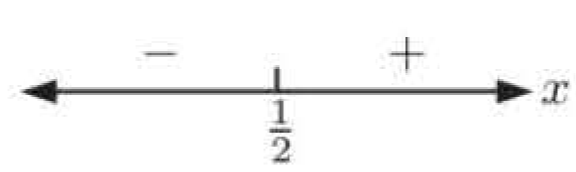
Now when $x = 0$, $y = \ln 4 - 1$, so the tangent cuts the y-axis at $(0, \ln 4 - 1)$.

6 **a** $$f(x) = 2x^3 - 3x^2 - 36x + 7$$
$$\begin{aligned}\therefore \quad f'(x) &= 6x^2 - 6x - 36\\ &= 6(x^2 - x - 6)\\ &= 6(x-3)(x+2)\end{aligned}$$
with sign diagram:



Now $f(-2) = 51$, $f(3) = -74$, so there is a local maximum at $(-2, 51)$, and a local minimum at $(3, -74)$.

$$\begin{aligned}f''(x) &= 12x - 6\\ &= 6(2x-1)\end{aligned}$$
with sign diagram:

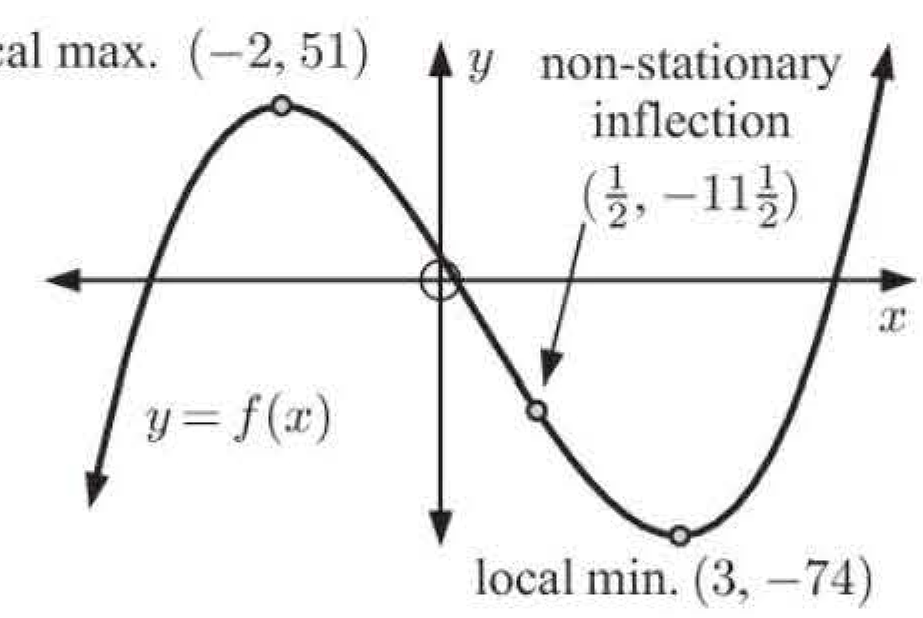


Now $f(\frac{1}{2}) = -\frac{23}{2}$, so there is a non-stationary inflection at $(\frac{1}{2}, -\frac{23}{2})$.

b $f(x)$ is increasing when $x \leqslant -2$ or $x \geqslant 3$, and decreasing when $-2 \leqslant x \leqslant 3$.

c $f(x)$ is concave up when $x \geqslant \frac{1}{2}$, and concave down when $x \leqslant \frac{1}{2}$.

d



7 Consider the normal to $f(x) = \dfrac{3x}{1+x}$ at $(2, 2)$.

$f(x)$ is a quotient with $u = 3x$ and $v = 1 + x$
$\therefore$ $u' = 3$ and $v' = 1$

$\therefore$ $f'(x) = \dfrac{3(1+x) - 1(3x)}{(1+x)^2} = \dfrac{3}{(1+x)^2}$ {quotient rule}

$\therefore$ $f'(2) = \frac{3}{9} = \frac{1}{3}$

$\therefore$ the normal at $(2, 2)$ has gradient -3

So, the equation of the normal is $\dfrac{y-2}{x-2} = -3$
$$\therefore \quad y - 2 = -3(x-2)$$
$$\therefore \quad y = -3x + 8$$

When $x = 0$, $y = 8$ and when $y = 0$, $x = \frac{8}{3}$

$\therefore$ B and C are at $(0, 8)$ and $(\frac{8}{3}, 0)$,

and the distance BC $= \sqrt{\left(0 - \frac{8}{3}\right)^2 + (8-0)^2} = \sqrt{\frac{64}{9} + 64} = \sqrt{\frac{640}{9}} = \frac{8\sqrt{10}}{3}$ units

8 $$\begin{aligned}f(x) &= x^3 - 4x^2 + 4x\\ &= x(x^2 - 4x + 4)\\ &= x(x-2)^2\end{aligned}$$

a $f(0) = 0$, so the y-intercept is 0.

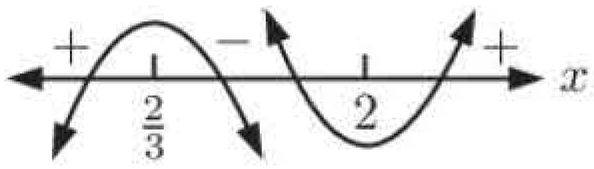
$f(x)$ cuts the x-axis when $y = 0$ $\quad\therefore$ $x(x-2)^2 = 0$
$\therefore$ $x = 0$ or 2

$\therefore$ the x-intercepts are 0 and 2.

b $f'(x) = 3x^2 - 8x + 4$
$\quad = (3x-2)(x-2)$

which is 0 when $x = \frac{2}{3}$ or 2

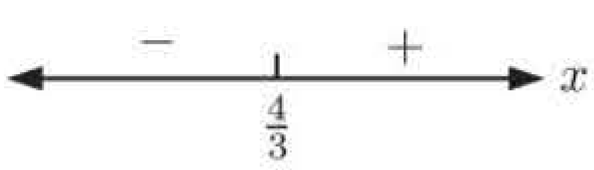
Sign diagram of $f'(x)$ is:



Now $f(\frac{2}{3}) = \frac{32}{27}$, so there is a local maximum at $(\frac{2}{3}, \frac{32}{27})$, and a local minimum at $(2, 0)$.

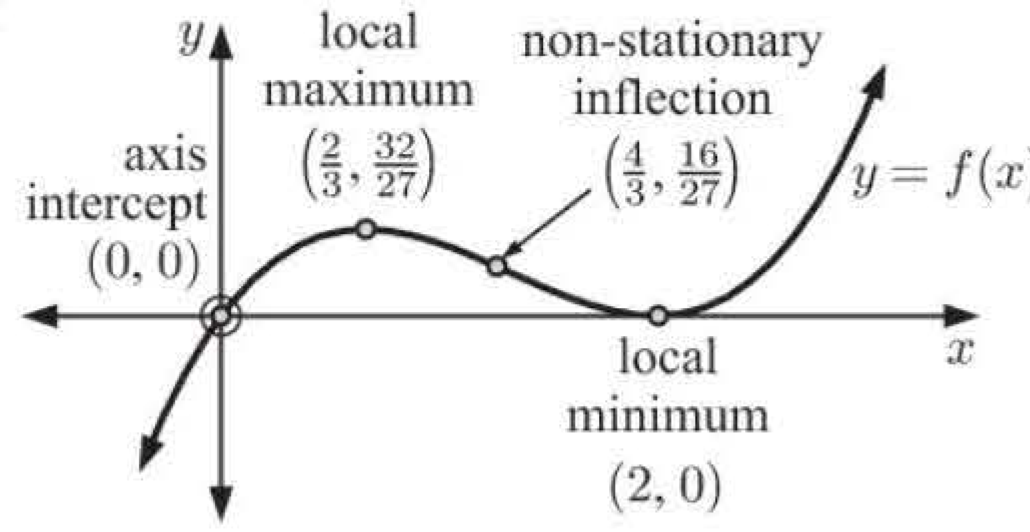
$f''(x) = 6x - 8 = 2(3x-4)$

Sign diagram of $f''(x)$ is:



Now $f(\frac{4}{3}) = \frac{16}{27}$, so there is a non-stationary inflection at $(\frac{4}{3}, \frac{16}{27})$.

c



9 **a** $y = \dfrac{1}{\sin x} = (\sin x)^{-1}$

$\therefore \dfrac{dy}{dx} = -(\sin x)^{-2}(\cos x)$
$\quad = -\dfrac{\cos x}{\sin^2 x}$

When $x = \frac{\pi}{3}$, $y = \dfrac{1}{\sin(\frac{\pi}{3})} = \frac{2}{\sqrt{3}}$

and $\dfrac{dy}{dx} = -\dfrac{\cos(\frac{\pi}{3})}{\sin^2(\frac{\pi}{3})} = -\dfrac{\frac{1}{2}}{(\frac{\sqrt{3}}{2})^2} = -\frac{2}{3}$

$\therefore$ the tangent has equation $\dfrac{y - \frac{2}{\sqrt{3}}}{x - \frac{\pi}{3}} = -\frac{2}{3}$ which is $3y - 2\sqrt{3} = -2x + \frac{2\pi}{3}$
or $2x + 3y = 2\sqrt{3} + \frac{2\pi}{3}$

b $y = \cos(\frac{x}{2})$

$\therefore \dfrac{dy}{dx} = -\frac{1}{2}\sin(\frac{x}{2})$

When $x = \frac{\pi}{2}$, $y = \cos(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$

and $\dfrac{dy}{dx} = -\frac{1}{2}\sin(\frac{\pi}{4}) = -\frac{1}{2\sqrt{2}}$

$\therefore$ the normal has gradient $2\sqrt{2}$, and its equation is $\dfrac{y - \frac{1}{\sqrt{2}}}{x - \frac{\pi}{2}} = 2\sqrt{2}$

$\therefore y - \frac{1}{\sqrt{2}} = 2\sqrt{2}x - \pi\sqrt{2}$

$\therefore y - 2\sqrt{2}x = \frac{1}{\sqrt{2}} - \pi\sqrt{2}$

or $\sqrt{2}y - 4x = 1 - 2\pi$

10 $f(x) = 3x^3 + ax^2 + b$

$\therefore f'(x) = 9x^2 + 2ax$

Since the tangent at $(-2, 14)$ has gradient 0, $f'(-2) = 0$
$\therefore 36 - 4a = 0$
$\therefore a = 9$

As the point $(-2, 14)$ lies on the curve, $14 = 3(-2)^3 + 9(-2)^2 + b$
$\therefore b = 14 + 24 - 36$
$\therefore b = 2$

$\therefore f'(x) = 9x^2 + 18x$

$\therefore f''(x) = 18x + 18$ and so $f''(-2) = -36 + 18 = -18$

11 The curves $y = \sqrt{3x+1}$ and $y = \sqrt{5x - x^2}$ meet when $\sqrt{3x+1} = \sqrt{5x - x^2}$

Squaring both sides, $3x + 1 = 5x - x^2$
$\therefore x^2 - 2x + 1 = 0$
$\therefore (x-1)^2 = 0$
$\therefore x = 1$

When $x = 1$, $y = \sqrt{3+1} = 2$, so the curves meet at $(1, 2)$.

Now for $y = \sqrt{3x+1} = (3x+1)^{\frac{1}{2}}$

$\dfrac{dy}{dx} = \frac{1}{2}(3x+1)^{-\frac{1}{2}}(3)$

$\therefore$ at $(1, 2)$, $\dfrac{dy}{dx} = \dfrac{3}{2(3+1)^{\frac{1}{2}}} = \frac{3}{4}$

Check: $y = \sqrt{5x-x^2} = (5x-x^2)^{\frac{1}{2}}$

$\dfrac{dy}{dx} = \frac{1}{2}(5x-x^2)^{-\frac{1}{2}}(5-2x) = \dfrac{5-2x}{2\sqrt{5x-x^2}}$

$\therefore$ at $(1, 2)$, $\dfrac{dy}{dx} = \dfrac{5-2}{2\sqrt{5-1}} = \frac{3}{4}$ ✓

$\therefore$ the curves have a common tangent at their point of intersection.

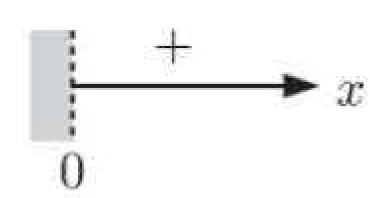
The equation of the common tangent at $(1, 2)$ is $\dfrac{y-2}{x-1} = \frac{3}{4}$

$\therefore \ 4(y-2) = 3(x-1)$

$\therefore \ 4y = 3x + 5$

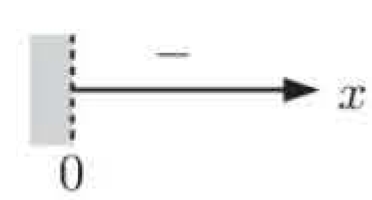
12 **a** $f(x) = x + \ln x$ is defined when $x > 0$

b $f'(x) = 1 + \dfrac{1}{x} = \dfrac{x+1}{x}$ which has sign diagram:



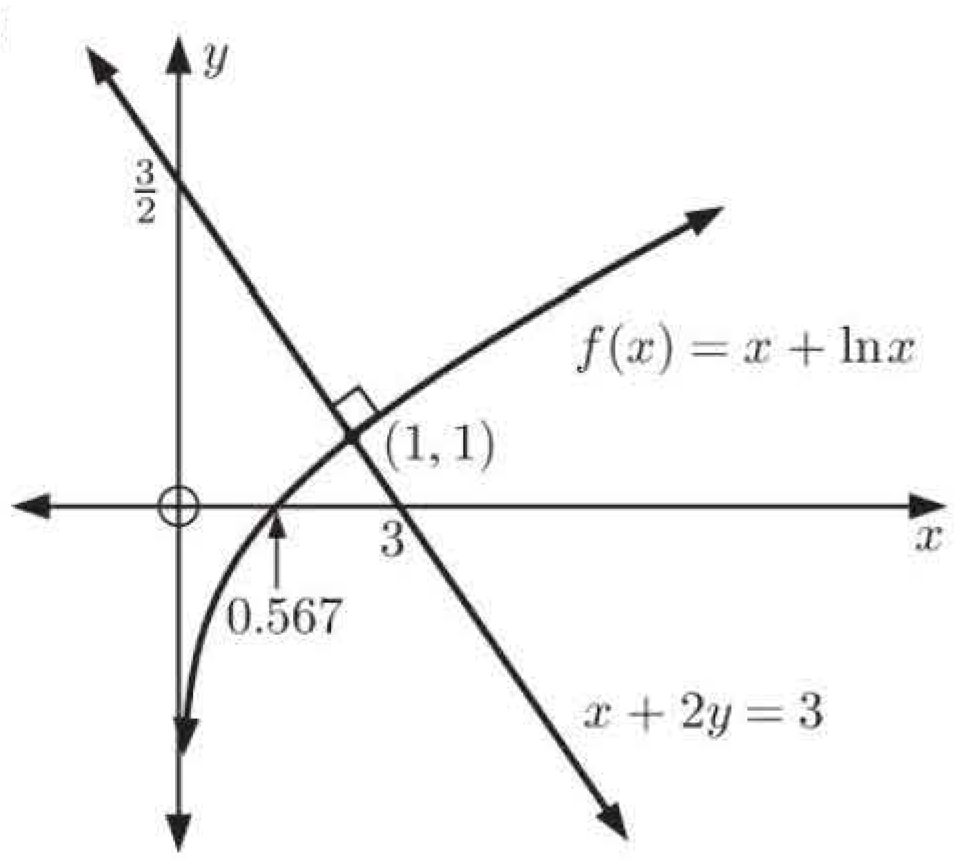
$\therefore$ $f(x)$ is increasing for all $x > 0$.

$f''(x) = -\dfrac{1}{x^2}$ which has sign diagram:



$\therefore$ $f(x)$ is concave down for all $x > 0$.

c



d $f(1) = 1 + \ln(1) = 1$

$\therefore$ $(1, 1)$ is the point of contact.

$f'(1) = \dfrac{1+1}{1} = 2$

$\therefore$ the tangent at $x = 1$ has gradient 2,

so the normal has gradient $-\frac{1}{2}$

$\therefore$ the normal has equation $\dfrac{y-1}{x-1} = -\frac{1}{2}$

$\therefore \ 2y - 2 = -x + 1$

$\therefore \ x + 2y = 3$

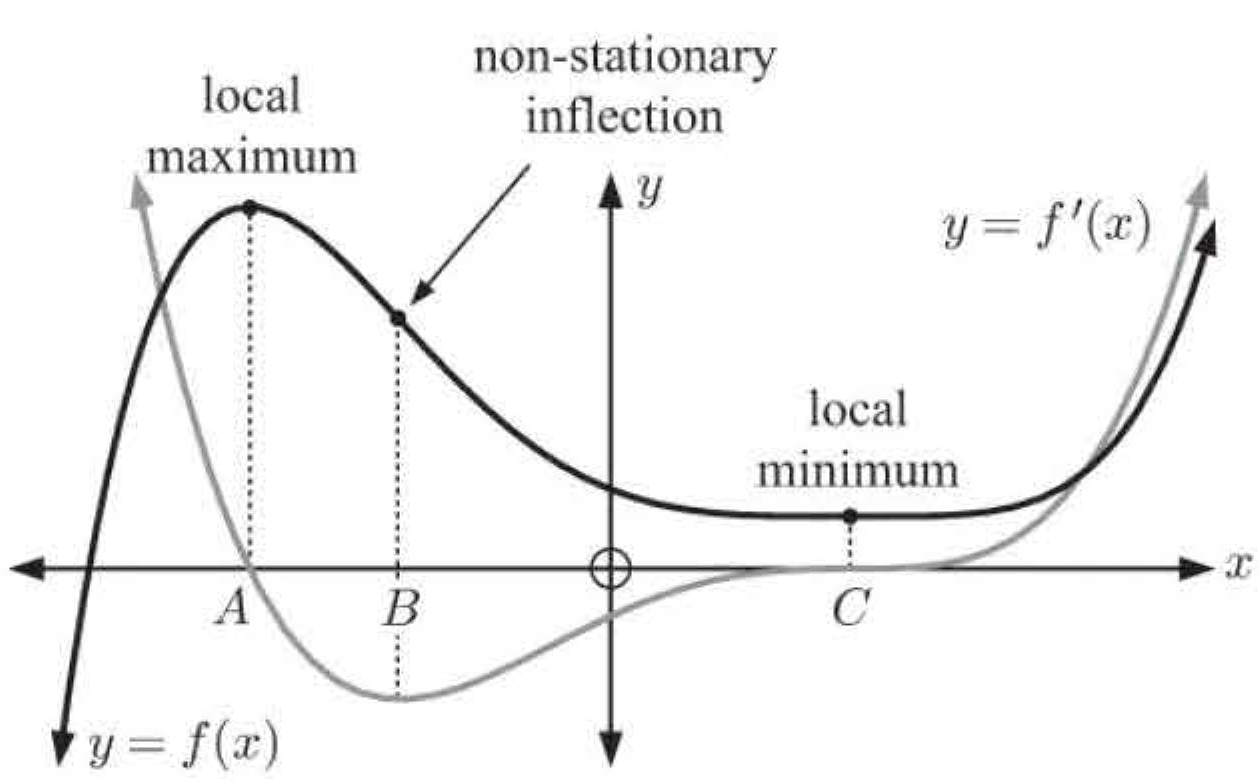
13 At $x = B$, $f''(x) = 0$ but $f'(x) \neq 0$

$\therefore$ $f(x)$ has a non-stationary inflection point at $x = B$.

$f'(x)$ is above the x-axis for $x \leqslant A$ and $x \geqslant C$, and below the x-axis for $A \leqslant x \leqslant C$

$\therefore$ $f(x)$ is increasing for $x \leqslant A$, decreasing for $A \leqslant x \leqslant C$, then increasing for $x \geqslant C$

$\therefore$ $f(x)$ has a local maximum at $x = A$ and a local minimum at $x = C$.



14 $f(x) = \arcsin x + \arccos x$

$$f'(x) = \frac{1}{\sqrt{1-x^2}} + \left(\frac{-1}{\sqrt{1-x^2}}\right), \quad x \in]-1, 1[$$
$$= 0$$

$\therefore$ the gradient of the function is zero.

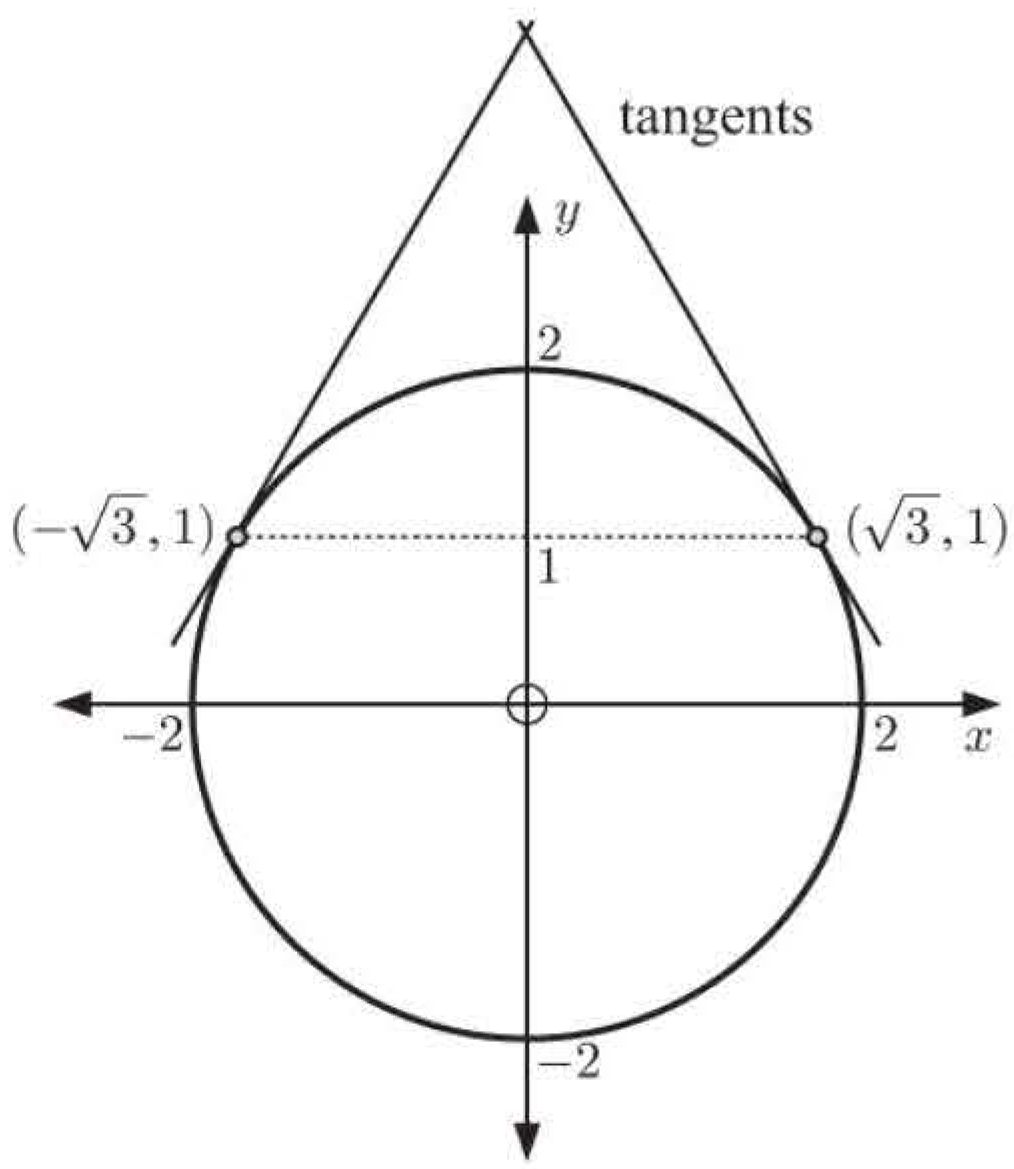
$$f(0) = \arcsin(0) + \arccos(0)$$
$$= 0 + \tfrac{\pi}{2}$$

$\therefore \; f(x) = \frac{\pi}{2}$, a constant.

15 **a** $x^2 + y^2 = 4$

$$\therefore \; y = \pm\sqrt{4-x^2} = \pm(4-x^2)^{\frac{1}{2}}$$

$$\therefore \; \frac{dy}{dx} = \pm\tfrac{1}{2}(4-x^2)^{-\frac{1}{2}}(-2x)$$
$$= \frac{\mp x}{\sqrt{4-x^2}}$$



$\therefore$ when $y = (4-x^2)^{\frac{1}{2}}$, $\quad \dfrac{dy}{dx} = \dfrac{-x}{\sqrt{4-x^2}}$

and when $y = -(4-x^2)^{\frac{1}{2}}$, $\quad \dfrac{dy}{dx} = \dfrac{x}{\sqrt{4-x^2}}$

When $y = 1$, then $x^2 + 1^2 = 4$

$$\therefore \; x^2 = 3$$
$$\therefore \; x = \pm\sqrt{3}$$

At the point $(\sqrt{3}, 1)$, $\quad \dfrac{dy}{dx} = \dfrac{-\sqrt{3}}{\sqrt{4-\sqrt{3}^2}} = \frac{-\sqrt{3}}{1} = -\sqrt{3}$

At the point $(-\sqrt{3}, 1)$, $\quad \dfrac{dy}{dx} = \dfrac{-(-\sqrt{3})}{\sqrt{4-\left(-\sqrt{3}\right)^2}} = \frac{\sqrt{3}}{1} = \sqrt{3}$

$\therefore$ the equation of the tangent at $(\sqrt{3}, 1)$, is:

$$\frac{y-1}{x-\sqrt{3}} = -\sqrt{3}$$
$$\therefore \; y - 1 = -\sqrt{3}x + 3$$
$$\therefore \; \sqrt{3}x + y = 4$$

$\therefore$ the equation of the tangent at $(-\sqrt{3}, 1)$ is:

$$\frac{y-1}{x-(-\sqrt{3})} = \sqrt{3}$$
$$\therefore \; y - 1 = \sqrt{3}(x + \sqrt{3})$$
$$\therefore \; y - 1 = \sqrt{3}x + 3$$
$$\therefore \; \sqrt{3}x - y = -4$$

b The tangents intersect when $\sqrt{3}x = 4 - y$, $\sqrt{3}x = -4 + y$

$$\therefore \; 4 - y = -4 + y$$
$$\therefore \; 8 = 2y$$
$$\therefore \; y = 4$$

When $y = 4$, then $\sqrt{3}x = 4 - 4$

$$\therefore \; \sqrt{3}x = 0$$
$$\therefore \; x = 0$$

$\therefore$ the tangents intersect at $(0, 4)$.

Chapter 20

APPLICATIONS OF DIFFERENTIAL CALCULUS

EXERCISE 20A.1

1 **a** $s(t) = t^2 + 3t - 2, \quad t \geqslant 0$

$$\begin{aligned}\text{Average velocity} &= \frac{s(t_2)-s(t_1)}{t_2-t_1} \\ &= \frac{s(3)-s(1)}{3-1} \\ &= \frac{16-2}{2} \\ &= 7 \text{ m s}^{-1}\end{aligned}$$

b $$\begin{aligned}\text{Average velocity} &= \frac{s(t_2)-s(t_1)}{t_2-t_1} \\ &= \frac{s(1+h)-s(1)}{(1+h)-1} \\ &= \frac{(1+h)^2+3(1+h)-2-2}{h} \\ &= \frac{2h+h^2+3h}{h} \\ &= (h+5) \text{ m s}^{-1}, \quad h \neq 0\end{aligned}$$

c $$\begin{aligned}&\lim_{h\to 0} \frac{s(1+h)-s(1)}{h} \\ &= \lim_{h\to 0} 5+h \\ &= 5 \text{ m s}^{-1}\end{aligned}$$

This is the instantaneous velocity at $t = 1$ second, or $s'(1)$.

d Average velocity

$$\begin{aligned}&= \frac{s(t_2)-s(t_1)}{t_2-t_1} \\ &= \frac{s(t+h)-s(t)}{(t+h)-t} \\ &= \frac{\left[(t+h)^2+3(t+h)-2\right]-\left[t^2+3t-2\right]}{h} \\ &= \frac{2ht+h^2+3h}{h} \\ &= (2t+h+3) \text{ m s}^{-1}, \quad h \neq 0\end{aligned}$$

Now $$\begin{aligned}\lim_{h\to 0} \frac{s(t+h)-s(t)}{h} &= \lim_{h\to 0} (2t+h+3) \\ &= (2t+3) \text{ m s}^{-1}\end{aligned}$$

This is the instantaneous velocity at t seconds.

2 **a** $s(t) = 5 - 2t^2$ cm

$$\begin{aligned}\text{Average velocity} &= \frac{s(t_2)-s(t_1)}{t_2-t_1} \\ &= \frac{s(5)-s(2)}{5-2} \\ &= \frac{(-45)-(-3)}{3} \\ &= -14 \text{ cm s}^{-1}\end{aligned}$$

b $$\begin{aligned}\text{Average velocity} &= \frac{s(t_2)-s(t_1)}{t_2-t_1} \\ &= \frac{s(2+h)-s(2)}{(2+h)-2} \\ &= \frac{5-2(2+h)^2+3}{h} \\ &= \frac{-8h-2h^2}{h} \\ &= (-8-2h) \text{ cm s}^{-1}, h \neq 0\end{aligned}$$

c $$\begin{aligned}\lim_{h\to 0} \frac{s(2+h)-s(2)}{h} &= \lim_{h\to 0} (-8-2h) \\ &= -8 \text{ cm s}^{-1}\end{aligned}$$

This is the instantaneous velocity when $t = 2$ seconds, or $s'(2)$.

d $$\begin{aligned}&\lim_{h\to 0} \frac{s(t+h)-s(t)}{h} \\ &= \lim_{h\to 0} \frac{\left[5-2(t+h)^2\right]-\left[5-2t^2\right]}{h} \\ &= \lim_{h\to 0} \frac{-4th-2h^2}{h} \\ &= \lim_{h\to 0} (-4t-2h) \\ &= -4t \text{ cm s}^{-1}\end{aligned}$$

This is the instantaneous velocity at t seconds.

3 $v(t) = 2\sqrt{t} + 3 \text{ cm s}^{-1}, \ t \geqslant 0$

a Average acceleration

$$= \frac{v(t_2) - v(t_1)}{t_2 - t_1}$$
$$= \frac{v(4) - v(1)}{4 - 1}$$
$$= \frac{7 - 5}{3}$$
$$= \tfrac{2}{3} \text{ cm s}^{-2}$$

b Average acceleration

$$= \frac{v(t_2) - v(t_1)}{t_2 - t_1}$$
$$= \frac{v(1+h) - v(1)}{(1+h) - 1}$$
$$= \frac{[2\sqrt{1+h} + 3] - [2\sqrt{1} + 3]}{h}$$
$$= \frac{2\sqrt{1+h} - 2}{h} \text{ cm s}^{-2}$$

c

$$\lim_{h \to 0} \frac{v(1+h) - v(1)}{(1+h) - 1}$$
$$= \lim_{h \to 0} \frac{2\sqrt{1+h} - 2}{h}$$
$$= \lim_{h \to 0} \frac{2\left(\sqrt{1+h} - 1\right)}{h} \times \frac{\sqrt{1+h} + 1}{\sqrt{1+h} + 1}$$
$$= \lim_{h \to 0} \frac{2(1 + h - 1)}{h\left(\sqrt{1+h} + 1\right)}$$
$$= \lim_{h \to 0} \frac{2h}{h\left(\sqrt{1+h} + 1\right)}$$
$$= \lim_{h \to 0} \frac{2}{\sqrt{1+h} + 1}$$
$$= \tfrac{2}{2}$$
$$= 1 \text{ cm s}^{-2}$$

This is the instantaneous acceleration when $t = 1$ second.

d

$$\lim_{h \to 0} \frac{v(t+h) - v(t)}{h}$$
$$= \lim_{h \to 0} \frac{2\sqrt{t+h} - 2\sqrt{t}}{h}$$
$$= \lim_{h \to 0} \frac{2(\sqrt{t+h} - \sqrt{t})}{h} \times \frac{\sqrt{t+h} + \sqrt{t}}{\sqrt{t+h} + t}$$
$$= \lim_{h \to 0} \frac{2h}{h(\sqrt{t+h} + \sqrt{t})}$$
$$= \frac{2}{2\sqrt{t}}$$
$$= \frac{1}{\sqrt{t}} \text{ cm s}^{-2}$$

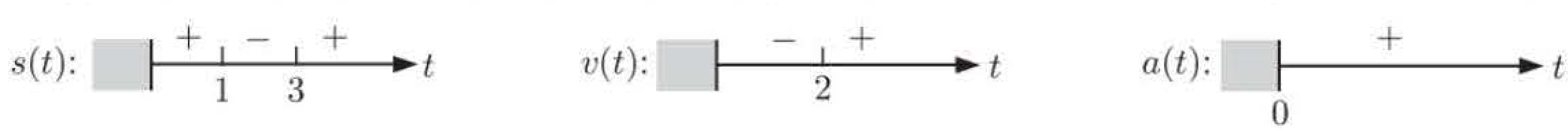
This is the instantaneous acceleration at t seconds.

4 **a** This is the instantaneous velocity at $t = 4$ seconds.

b This is the instantaneous acceleration at $t = 4$ seconds.

EXERCISE 20A.2

1 **a** $s(t) = t^2 - 4t + 3$ cm, $\ t \geqslant 0 \quad \therefore \ v(t) = 2t - 4 \text{ cm s}^{-1}$ and $a(t) = 2 \text{ cm s}^{-2}$.



b When $t = 0$, $s(0) = 3$ cm
$v(0) = -4 \text{ cm s}^{-1}$
$a(0) = 2 \text{ cm s}^{-2}$

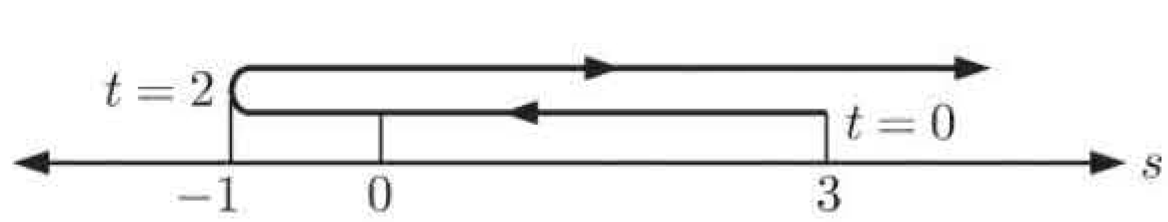
$\therefore$ the object is 3 cm right of O and is moving to the left with a velocity of 4 cm s^{-1} and slowing down, its acceleration being 2 cm s^{-2} to the right.

c When $t = 2$, $s(2) = -1$ cm
$v(2) = 0 \text{ cm s}^{-1}$
$a(2) = 2 \text{ cm s}^{-2}$

$\therefore$ the object is 1 cm left of O, instantaneously stationary and accelerating to the right at 2 cm s^{-2}.

d The object reverses direction when $v(t) = 0$, which occurs at $t = 2$ seconds.
At $t = 2$, the particle is 1 cm left of O.

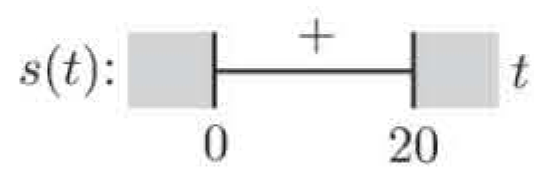
e

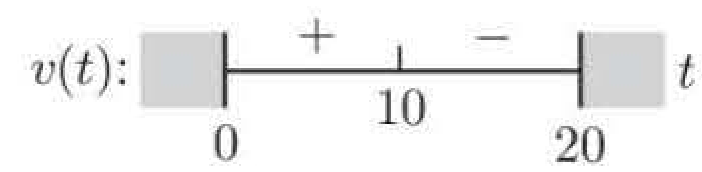


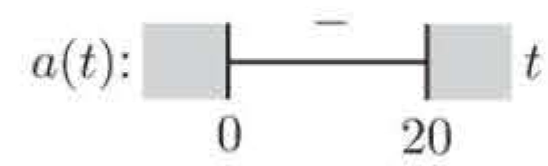
f Speed decreases when $v(t)$ and $a(t)$ have opposite signs, which is when $0 \leqslant t \leqslant 2$.

2 $s(t) = 98t - 4.9t^2$ m, $t \geqslant 0$

a $v(t) = 98 - 9.8t$ m s^{-1}
$a(t) = -9.8$ m s^{-2}







b When $t = 0$, $s(0) = 0$ m, $v(0) = 98$ m s^{-1} skyward

c When $t = 5$, $s(5) = 367.5$ m
$v(5) = 49$ m s^{-1}
$a(5) = -9.8$ m s^{-2}

The stone is 367.5 m above the ground and moving skyward at 49 m s^{-1}. Its speed is decreasing.

When $t = 12$, $s(12) = 470.4$ m
$v(12) = -19.6$ m s^{-1}
$a(12) = -9.8$ m s^{-2}

The stone is 470.4 m above the ground and moving groundward at 19.6 m s^{-1}. Its speed is increasing.

d The maximum height is reached when $v(t) = 0$ m s^{-1}

$\therefore\ 98 - 9.8t = 0$
$\therefore\ 9.8t = 98$
$\therefore\ t = 10$ seconds

$\therefore$ the maximum height is
$s(10) = 98(10) - 4.9\,(100)$
$= 980 - 490$
$= 490$ m

e The stone is at ground level when $s(t) = 0$ which is when $98t - 4.9t^2 = 0$
$\therefore\ 4.9t(20 - t) = 0$
$\therefore\ t = 0$ or 20 seconds

$\therefore$ it hits the ground after 20 seconds.

3 $s(t) = 1.2 + 28.1t - 4.9t^2$ metres

a When released, $t = 0$ and $s(0) = 1.2$ m $\quad\therefore$ it is released 1.2 m above the ground.

b $s'(t) = 28.1 - 9.8t$ m s^{-1} is the instantaneous velocity of the ball at the time t seconds after release.

c When $s'(t) = 0$, $28.1 - 9.8t = 0 \quad \therefore\ t = \dfrac{28.1}{9.8} \approx 2.87$ seconds

So, after 2.87 seconds the ball has reached its maximum height, and is instantaneously at rest.

d $s(2.867) = 1.2 + 28.1 \times 2.867 - 4.9 \times 2.867^2 \approx 41.5$ m

So, the maximum height reached is about 41.5 m.

e **i** $s'(0) = 28.1$ m s^{-1}

ii $s'(2) = 28.1 - 19.6$
$= 8.5$ m s^{-1}

iii $s'(5) = 28.1 - 49$
$= -20.9$ m s^{-1}
$\therefore$ speed $= 20.9$ m s^{-1}

If $s'(t) \geqslant 0$, the ball is travelling upwards. If $s'(t) \leqslant 0$, the ball is travelling downwards.

f $s(t) = 0$ when $1.2 + 28.1t - 4.9t^2 = 0$
$\therefore\ 4.9t^2 - 28.1t - 1.2 = 0$

$$\therefore\ t = \frac{28.1 \pm \sqrt{28.1^2 - 4(4.9)(-1.2)}}{9.8} \approx -0.0424 \text{ or } 5.777$$

But $t > 0$, so the ball hits the ground after 5.78 seconds.

g $s''(t) = -9.8$ m s^{-2} and is the rate of change in $s'(t)$

$\therefore$ the instantaneous acceleration is constant at -9.8 m s^{-2} for the entire motion.

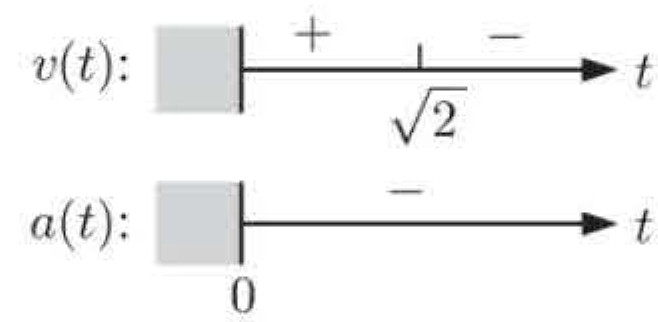
4 **a** $s(t) = bt - 4.9t^2$
$s'(t) = b - 9.8t$
$\therefore\ s'(0) = b$
$\therefore$ the initial velocity is b m s^{-1} upwards.

b Since $s(14.2) = 0$,
$b(14.2) - 4.9(14.2)^2 = 0$
$\therefore\ 14.2\,[b - 4.9 \times 14.2] = 0$
$\therefore\ b = 4.9 \times 14.2$
$\therefore\ b = 69.58$
$\therefore$ the initial velocity is 69.6 m s^{-1}.

5 **a** $s(t) = 12t - 2t^3 - 1$ cm, $t \geqslant 0$

$\therefore\ v(t) = 12 - 6t^2$ cm s^{-1}

and $a(t) = -12t$ cm s$^{-2$}



b When $t = 0$, $s(0) = -1$ cm, $v(0) = 12$ cm s^{-1}, $a(0) = 0$ cm s^{-2}

The particle was 1 cm left of O and was moving right at a constant speed of 12 cm s^{-1}.

c The particle reverses direction when $v(t) = 0$ which is when $12 - 6t^2 = 0$

$\therefore\ t^2 = 2$

$\therefore\ t = \sqrt{2}\quad \{t > 0\}$

When $t = \sqrt{2}$, $s(\sqrt{2}) = 12\sqrt{2} - 2(2\sqrt{2}) - 1$

$= 8\sqrt{2} - 1$

$\therefore$ the particle is $(8\sqrt{2} - 1)$ cm to the right of O.

d **i** From the sign diagrams in **a**, the speed increases for $t \geqslant \sqrt{2}$ seconds.

ii The velocity of the particle never increases $\{a(t) \leqslant 0\}$.

6 **a** $x(t) = t^3 - 9t^2 + 24t$ m, $t \geqslant 0$

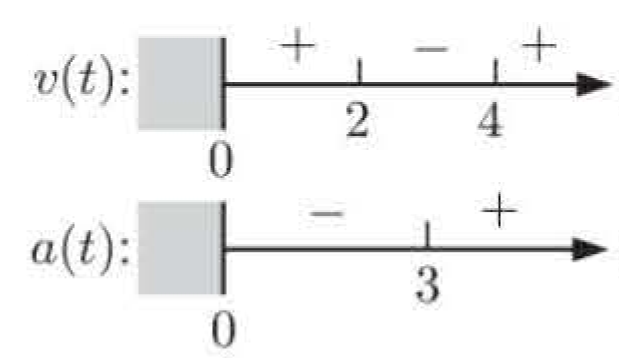
$v(t) = 3t^2 - 18t + 24$

$= 3(t^2 - 6t + 8)$

$= 3(t-4)(t-2)$ m s^{-1}

and $a(t) = 6t - 18$

$= 6(t-3)$ m s^{-2}



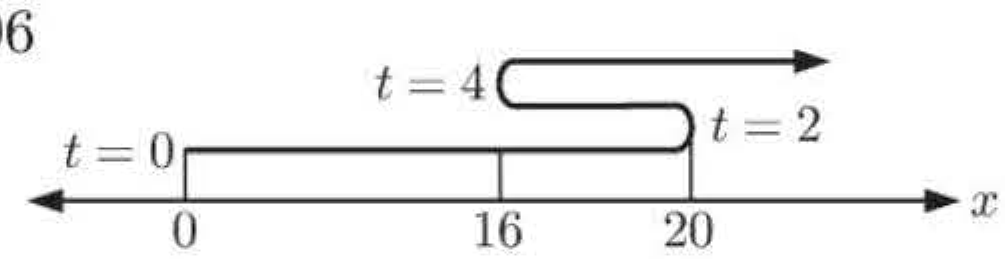
b The particle reverses direction when $v(t) = 0$, which occurs at $t = 2$ and $t = 4$ seconds.

$x(2) = 8 - 36 + 48$ m

$= 20$ m

and $x(4) = 64 - 144 + 96$

$= 16$ m



c **i** The speed decreases when $v(t)$ and $a(t)$ have the opposite sign, which is when $0 \leqslant t \leqslant 2$ and $3 \leqslant t \leqslant 4$.

ii The velocity decreases when $a(t) \leqslant 0$, which is when $0 \leqslant t \leqslant 3$.

d When $t = 5$, $x(5) = 5^3 - 9(5)^2 + 24(5)$

$= 125 - 225 + 120$

$= 20$ m

$\therefore$ distance travelled $= 20 + 4 + 4$ m

$= 28$ m

7 **a** $s(t) = 100t + 200e^{-\frac{t}{5}}$ cm, $t \geqslant 0$

$v(t) = 100 - 40e^{-\frac{t}{5}}$ cm s^{-1}

$a(t) = 8e^{-\frac{t}{5}}$ cm s^{-2}

b When $t = 0$, $s(0) = 200$ cm (right of the origin)

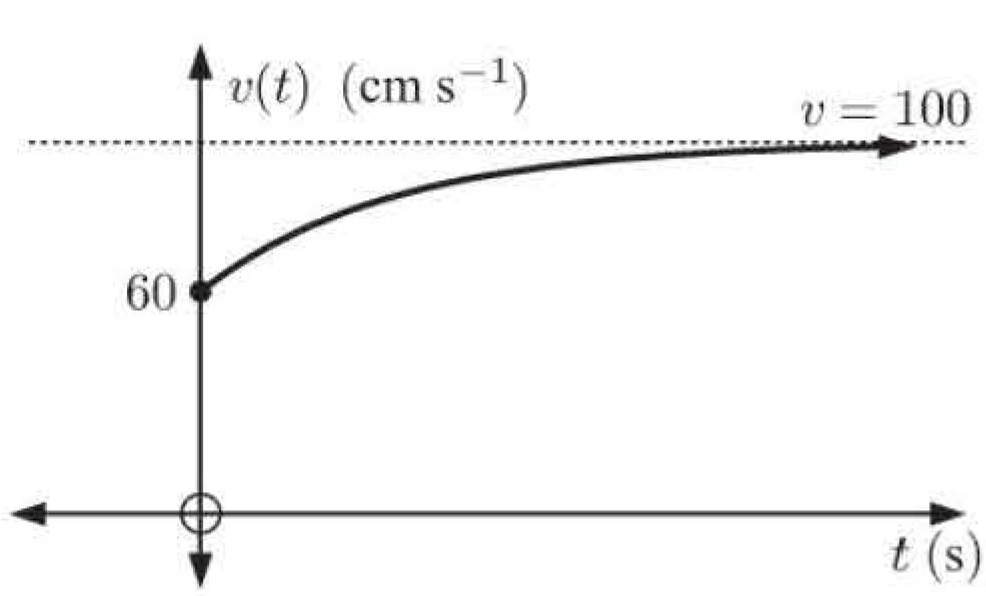
$v(0) = 60$ cm s^{-1}

$a(0) = 8$ cm s^{-2}

c As $t \to \infty$, $e^{-\frac{t}{5}} \to 0$,

$\therefore\ v(t) \to 100$ cm s^{-1} (below)

d



e When $v(t) = 80$ cm s^{-1},

$100 - 40e^{-\frac{t}{5}} = 80$

$\therefore\ -40e^{-\frac{t}{5}} = -20$

$\therefore\ e^{-\frac{t}{5}} = 0.5$

$\therefore\ -\dfrac{t}{5} = \ln 0.5$

$\therefore\ t = -5\ln 0.5 \approx 3.47$ s

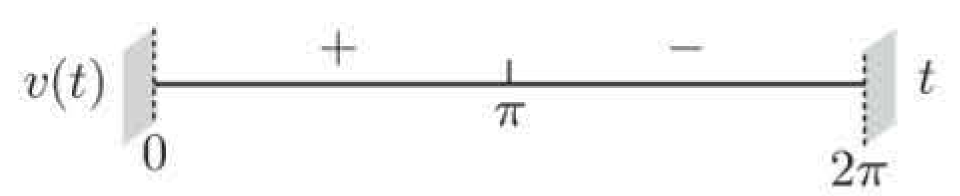
8 $x(t) = 1 - 2\cos t$ cm
$\therefore\ v(t) = x'(t) = 2\sin t$
$\therefore\ a(t) = v'(t) = 2\cos t$

a When $t = 0$,
$$\begin{aligned} x(0) &= 1 - 2\cos 0 \\ &= -1 \text{ cm} \\ v(0) &= 2\sin 0 \\ &= 0 \text{ cm s}^{-1} \\ a(0) &= 2\cos 0 \\ &= 2 \text{ cm s}^{-2} \end{aligned}$$

b When $t = \frac{\pi}{4}$,
$$\begin{aligned} x(\tfrac{\pi}{4}) &= 1 - \tfrac{2}{\sqrt{2}} \\ &= 1 - \sqrt{2} \text{ cm} \\ v(\tfrac{\pi}{4}) &= \tfrac{2}{\sqrt{2}} = \sqrt{2} \text{ cm s}^{-1} \\ a(\tfrac{\pi}{4}) &= \tfrac{2}{\sqrt{2}} = \sqrt{2} \text{ cm s}^{-2} \end{aligned}$$

The particle is $(\sqrt{2} - 1)$ cm left of the origin, moving right at $\sqrt{2}$ cm s^{-1} with increasing speed.

c We need to look for the points where the velocity equals zero.
If $v(t) = 2\sin t = 0$
then $\sin t = 0$
$\therefore\ t = \pi \quad (0 < t < 2\pi)$



The particle reverses direction when $t = \pi$.
At $t = \pi$, $x(\pi) = 3$ cm.

d The particle's speed is increasing when $v(t) = 2\sin t$ and $a(t) = 2\cos t$ have the same sign.
If $a(t) = 2\cos t = 0$
then $\cos t = 0$
$t = \frac{\pi}{2}, \frac{3\pi}{2} \quad (0 \leqslant t \leqslant 2\pi)$

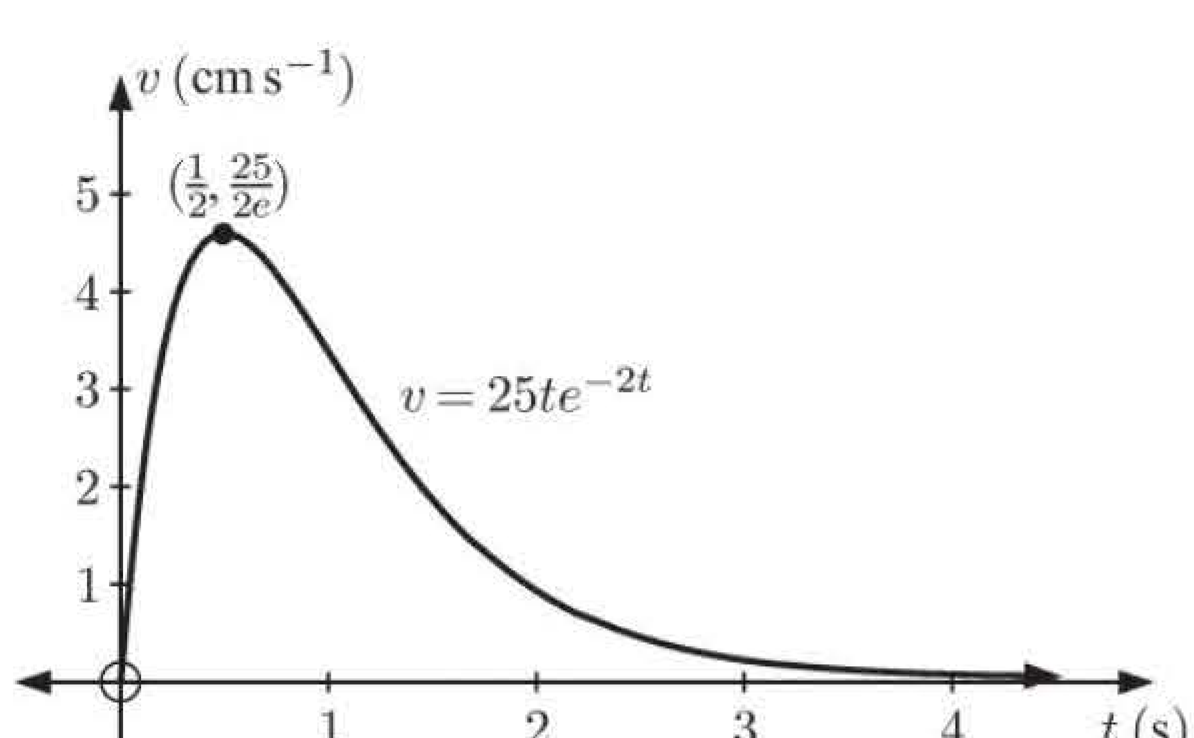
$\therefore$ the particle's speed is increasing when $0 \leqslant t \leqslant \frac{\pi}{2}$ and $\pi \leqslant t \leqslant \frac{3\pi}{2}$.

9 **a** Let the equation be $s(t) = at^2 + bt + c$
$\therefore\ v(t) = 2at + b$
and $a(t) = 2a = g$ {gravitational acceleration}
$\therefore\ a = \frac{1}{2}g$ and $v(t) = gt + b$

But when $t = 0$, $v(0) = g \times 0 + b = b$
$\therefore$ the initial velocity is b
$\therefore\ v(t) = v(0) + gt$ as required

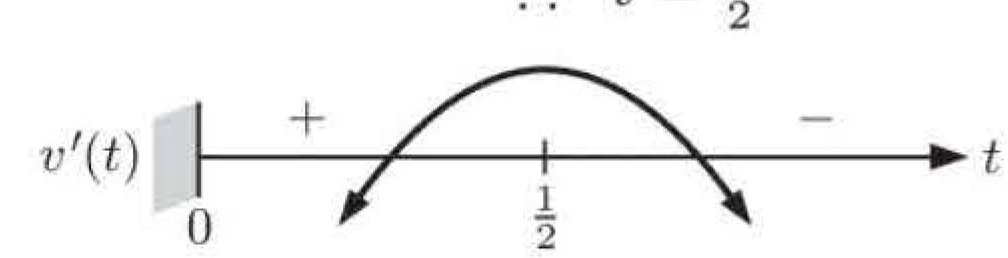
b Now when $t = 0$, $s(0) = 0$
$\therefore\ a \times 0^2 + b \times 0 + c = 0$
$\therefore\ c = 0$
and so $s(t) = \left(\frac{1}{2}g\right)t^2 + v(0)t$
$\therefore\ s(t) = v(0) \times t + \frac{1}{2}gt^2$ as required

10 **a**



b $a(t) = v'(t)$
$= 25 \times e^{-2t} + 25t(-2)e^{-2t}$
$= 25(1 - 2t)e^{-2t}$ cm s^{-2}

c $v'(t) = 0$ when $1 - 2t = 0$
$\therefore\ t = \frac{1}{2}$



$\therefore$ the velocity is increasing for $0 \leqslant t \leqslant \frac{1}{2}$

EXERCISE 20B

1 $P(t) = 2t^2 - 12t + 118$ thousand dollars, $t \geqslant 0$

a $P(0) = \$118\,000$ is the current annual profit

b $\dfrac{dP}{dt} = 4t - 12$ thousand dollars per year

c $\dfrac{dP}{dt}$ is the rate of change in profit with time.

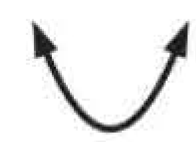
d **i** The profit decreases when $\dfrac{dP}{dt} \leqslant 0$, which occurs when $4t - 12 \leqslant 0$

$\therefore\ 4t \leqslant 12$

$\therefore\ t \leqslant 3$

But $t \geqslant 0$, so $0 \leqslant t \leqslant 3$ years.

ii The profit increases on the previous year when $\dfrac{dP}{dt} \geqslant 0$, which is for $t > 3$ years.

e The profit function is a quadratic with $a > 0$ $\therefore$ the shape is 

So, a minimum profit occurs when $\dfrac{dP}{dt} = 0$, which is when $t = 3$ years

and $P(3) = 18 - 36 + 118 = 100$ thousand dollars or \$100 000.

f When $t = 4$, $\dfrac{dP}{dt} = 4$ thousand dollars per year.

So, the profit is increasing at \$4000 per year after 4 years.

When $t = 10$, $\dfrac{dP}{dt} = 28$ thousand dollars per year.

So, the profit is increasing at \$28 000 per year after 10 years.

When $t = 25$, $\dfrac{dP}{dt} = 88$ thousand dollars per year.

So, the profit is increasing at \$88 000 per year after 25 years.

2 $V = 200(50 - t)^2$ m^3

a average rate on $0 \leqslant t \leqslant 5$

$$= \frac{V(5) - V(0)}{5 - 0}$$

$$= \frac{200(45)^2 - 200(50)^2}{5}$$

$= -19\,000$ m^3 per minute

$\therefore$ leaving at 19 000 m^3 per minute

b $V'(t) = 400(50 - t)^1 \times (-1)$

$\therefore\ V'(5) = 400 \times 45 \times -1$

$= -18\,000$ m^3 per minute

$\therefore$ leaving at 18 000 m^3 per minute

3 $Q = 100 - 10\sqrt{t}, \quad t \geqslant 0$

a **i** At $t = 0$, $Q = 100$ units

ii At $t = 25$, $Q = 50$ units

iii At $t = 100$, $Q = 0$ units

b $\dfrac{dQ}{dt} = -5t^{-\frac{1}{2}} = -\dfrac{5}{\sqrt{t}}$

i At $t = 25$, $\dfrac{dQ}{dt} = -1$ unit per year

$\therefore$ decreasing at 1 unit per year

ii At $t = 50$, $\dfrac{dQ}{dt} = -\frac{5}{\sqrt{50}}$

$= -\frac{1}{\sqrt{2}}$ units per year

$\therefore$ decreasing at $\frac{1}{\sqrt{2}}$ units per year

c $\dfrac{dQ}{dt} = -\dfrac{5}{\sqrt{t}}$ $\therefore$ the skin *loses* the chemical at the rate $R = \dfrac{5}{\sqrt{t}} = 5t^{-\frac{1}{2}}$ units per year.

Now $\dfrac{dR}{dt} = -\frac{5}{2}t^{-\frac{3}{2}} = -\dfrac{5}{2t\sqrt{t}}$

Since $2t\sqrt{t} > 0$ for all $t > 0$, $\dfrac{dR}{dt} < 0$ for all $t > 0$.

$\therefore$ the rate at which the skin loses the chemical is decreasing for all $t > 0$.

4 $H = 20 - \dfrac{97.5}{t+5}$ m, $t \geqslant 0$

a At planting, $t = 0$ $\quad \therefore \quad H(0) = 20 - \dfrac{97.5}{0+5} = 0.5$ m

b $H(4) = 20 - \dfrac{97.5}{4+5} \approx 9.17$ m

$H(8) = 20 - \dfrac{97.5}{8+5} = 12.5$ m

$H(12) = 20 - \dfrac{97.5}{12+5} \approx 14.3$ m

c Now $\dfrac{dH}{dt} = 97.5(t+5)^{-2} = \dfrac{97.5}{(t+5)^2}$

When $t = 0$, $\dfrac{dH}{dt} = \frac{97.5}{25} = 3.9$ m year^{-1}

When $t = 5$, $\dfrac{dH}{dt} = \frac{97.5}{100} = 0.975$ m year^{-1}

When $t = 10$, $\dfrac{dH}{dt} = \frac{97.5}{225} \approx 0.433$ m year^{-1}

d Now $\dfrac{dH}{dt} = \dfrac{97.5}{(t+5)^2}$

Since $(t+5)^2 > 0$ for all $t \geqslant 0$, $\dfrac{dH}{dt} > 0$ for all $t \geqslant 0$

$\therefore$ the height of the tree is always increasing, which means that the tree is always growing.

5 **a** $C(x) = 0.0003x^3 + 0.02x^2 + 4x + 2250$

$\therefore \quad C'(x) = 0.0009x^2 + 0.04x + 4$ dollars per pair

b $C'(220) = 0.0009(220)^2 + 0.04(220) + 4 = \$56.36$ per pair

This estimates the cost of making the 221st pair of jeans if 220 pairs are currently being made.

c $C(221) - C(220) \approx \$7348.98 - \$7292.40 \approx \$56.58$

This is the actual cost to make the extra pair of jeans (221 instead of 220).

d $C''(x) = 0.0018x + 0.04$

$C''(x) = 0$ when $0.0018x + 0.04 = 0$

$$\therefore \quad x = -\frac{0.04}{0.0018} \approx -22.2$$

This is the point when the rate of change is a minimum. However, it is out of the bounds of our model, as we cannot make a negative quantity of jeans.

6 **a** $C(v) = \frac{1}{5}v^2 + 200\,000v^{-1}$ euros

i At $v = 50$ km h^{-1}, $\quad C = \frac{1}{5}(50)^2 + \dfrac{200\,000}{50} = €4500$

ii At $v = 100$ km h^{-1}, $\quad C = \frac{1}{5}(100)^2 + \dfrac{200\,000}{100} = €4000$

b $\dfrac{dC}{dv} = \frac{2}{5}v - 200\,000v^{-2} = \frac{2}{5}v - \dfrac{200\,000}{v^2}$

i At $v = 30$ km h^{-1},

$\dfrac{dC}{dv} = \frac{2}{5}(30) - \dfrac{200\,000}{30^2}$

$\approx -€210.22$ per km h^{-1}

So, a decrease of €210.22 per km h^{-1}.

ii At $v = 90$ km h^{-1},

$\dfrac{dC}{dv} = \frac{2}{5}(90) - \dfrac{200\,000}{90^2}$

$\approx €11.31$ per km h^{-1}

So, an increase of €11.31 per km h^{-1}.

c The cost is a minimum when $\dfrac{dC}{dv} = 0$, which occurs when $\frac{2}{5}v - \dfrac{200\,000}{v^2} = 0$

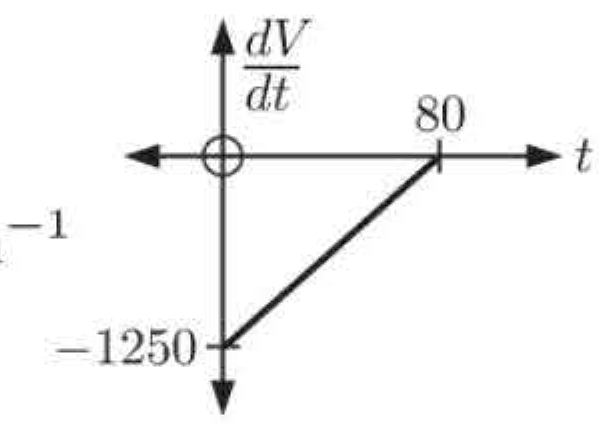
$$\therefore \quad \tfrac{2}{5}v = \frac{200\,000}{v^2}$$

$$\therefore \quad v^3 = 500\,000$$

$$\therefore \quad v \approx 79.4 \text{ km h}^{-1}$$

7 **a** $V = 50\,000\left(1 - \frac{t}{80}\right)^2, \quad 0 \leqslant t \leqslant 80$

$\therefore \quad \frac{dV}{dt} = 2 \times 50\,000\left(1 - \frac{t}{80}\right)^1 \times \left(-\frac{1}{80}\right) = -1250\left(1 - \frac{t}{80}\right) \text{ L min}^{-1}$



b The outflow was fastest when $t = 0$, when the tap was first opened.

c $\frac{dV}{dt} = -1250 + \frac{1250}{80}t \qquad \therefore \quad \frac{d^2V}{dt^2} = \frac{1250}{80} = \frac{125}{8} \text{ L min}^{-2}$

Since $\frac{d^2V}{dt^2}$ is constant and positive, $\frac{dV}{dt}$ is constantly increasing

$\therefore$ the outflow is decreasing at a constant rate.

8 $y = \frac{1}{10}x(x-2)(x-3) = \frac{1}{10}(x^3 - 5x^2 + 6x)$

a When $y = 0, \quad x = 0, 2,$ or 3

$\therefore$ the lake is between 2 and 3 km from the shoreline.

b $\frac{dy}{dx} = \frac{1}{10}(3x^2 - 10x + 6)$

$\quad = \frac{3}{10}x^2 - x + \frac{3}{5}$

When $x = \frac{1}{2}, \quad \frac{dy}{dx} = \frac{7}{40} = 0.175 \qquad \therefore$ land is sloping upwards.

When $x = 1\frac{1}{2}, \quad \frac{dy}{dx} = -\frac{9}{40} = -0.225 \qquad \therefore$ land is sloping downwards.

This means the top of the hill is between $x = \frac{1}{2}$ and $x = 1\frac{1}{2}$.

c The deepest point of the lake occurs when the slope of the land is 0, which is when $\frac{dy}{dx} = 0$

$\therefore \quad \frac{1}{10}(3x^2 - 10x + 6) = 0$

$\therefore \quad 3x^2 - 10x + 6 = 0$

$$\therefore \quad x = \frac{10 \pm \sqrt{100 - 72}}{6} = \frac{5 \pm \sqrt{7}}{3}$$

but it must be the value between 2 and 3 km, so $x = \frac{5 + \sqrt{7}}{3} \approx 2.55$ km from the sea

The depth at this point is $y(2.549) \approx \frac{1}{10}(2.549)(0.549)(-0.451)$

≈ -0.06311 km

≈ 63.1 m

9 **a** $\frac{dP}{dt} = aP\left(1 - \frac{P}{b}\right) - \left(\frac{c}{100}\right)P$ and when $\frac{dP}{dt} = 0$, the rate of change of population is zero, so the population is not changing and is stable.

b If $a = 0.06, \ b = 24\,000, \ c = 5$ then

$\frac{dP}{dt} = 0.06P\left(1 - \frac{P}{24\,000}\right) - \frac{5}{100}P$

$= 0.06P - 0.05P - \frac{0.06P^2}{24\,000}$

$= P\left(0.01 - \frac{P}{400\,000}\right)$

Now for a stable population, $\frac{dP}{dt} = 0$

$\therefore \quad P = 0$ or $\frac{P}{400\,000} = 0.01$

$\therefore \quad P = 0$ or 4000

$\therefore$ the stable population is 4000 fish.

c If the harvest rate is 4%, then $\frac{dP}{dt} = 0.06P\left(1 - \frac{P}{24\,000}\right) - \frac{4}{100}P = P\left(0.02 - \frac{0.06P}{24\,000}\right)$

For a stable population, $\frac{dP}{dt} = 0$ and so $0 = P\left(0.02 - \frac{0.06P}{24\,000}\right)$

$\therefore\ P = 0$ or $\dfrac{0.06P}{24\,000} = 0.02$

$\therefore\ P = 0$ or $\dfrac{0.02 \times 24\,000}{0.06}$

$\therefore\ P = 0$ or 8000

$\therefore$ the stable population is 8000 fish.

10 **a** $W = 20e^{-kt}$ so when $t = 50$ hours, $W = 10$ g

$\therefore\ 20e^{-50k} = 10$

$\therefore\ e^{-50k} = \frac{1}{2}$

$\therefore\ -50k = \ln\frac{1}{2} = -\ln 2 \qquad \therefore\ k = \frac{1}{50}\ln 2 \approx 0.0139$

b **i** When $t = 0$,
$W = 20e^0$
$= 20$ g

ii When $t = 24$,
$W = 20e^{-24k}$
$= 20e^{-24\frac{\ln 2}{50}}$
≈ 14.3 g

iii When $t = 1$ week
$= 7 \times 24$ hours
$= 168$ hours
$W = 20e^{-168\frac{\ln 2}{50}}$
≈ 1.95 g

c When $W = 1$ g, $\quad 20e^{-\frac{\ln 2}{50}\times t} = 1$

$\therefore\ e^{-\frac{\ln 2}{50}\times t} = 0.05$

$\therefore\ -\frac{\ln 2}{50} \times t = \ln 0.05$

$\therefore\ t = \dfrac{-50\ln 0.05}{\ln 2} = 216.096\,404\,7$

≈ 216 hours or 9 days and 6 minutes

d $\dfrac{dW}{dt}$

$= 20e^{-kt}(-k)$

$= \left(-20\dfrac{\ln 2}{50}\right) \times e^{-\frac{\ln 2}{50}t}$

i When $t = 100$ hours,
$\dfrac{dW}{dt} = \left(\dfrac{-20\ln 2}{50}\right)e^{-2\ln 2}$
$\approx -0.0693\ \text{g h}^{-1}$

ii When $t = 1000$ hours,
$\dfrac{dW}{dt} = \left(\dfrac{-20\ln 2}{50}\right)e^{-20\ln 2}$
$\approx -2.64 \times 10^{-7}\ \text{g h}^{-1}$

e $\dfrac{dW}{dt} = -k(20e^{-kt}) = -kW \qquad \therefore\ \dfrac{dW}{dt} \propto W$

11 $T = 5 + 95e^{-kt}$ °C

a $T = 20$°C when $t = 15$

$\therefore\ 20 = 5 + 95e^{-15k}$

$\therefore\ 15 = 95e^{-15k}$

$\therefore\ e^{15k} = \frac{95}{15}$

$\therefore\ 15k = \ln\left(\frac{19}{3}\right)$

$\therefore\ k = \frac{1}{15}\ln\left(\frac{19}{3}\right) \approx 0.123$

b When $t = 0$,
$T = 5 + 95e^0$
$= 5 + 95$
$= 100$°C

c $\dfrac{dT}{dt} = 0 + 95e^{-kt}(-k)$
$= -(95e^{-kt})k$
$= c(T - 5)$ where $c = -k$
$\therefore\ c \approx -0.123$

d $\dfrac{dT}{dt} = -95e^{-kt} \times k \approx -11.6902e^{-0.1231t}$

i When $t = 0$, $\dfrac{dT}{dt} \approx -11.69$, so the temperature is decreasing at 11.7°C min^{-1}.

ii When $t = 10$, $\dfrac{dT}{dt} \approx -11.6902e^{-1.231} \approx -3.415$,
so the temperature is decreasing at 3.42°C min^{-1}.

iii When $t = 20$, $\dfrac{dT}{dt} \approx -11.6902e^{-2.461} \approx -0.998$,
so the temperature is decreasing at 0.998°C min^{-1}.

12 $H(t) = 20\ln(3t+2) + 30$ cm, $t \geqslant 0$

a The shrubs were planted when $t = 0$. $H(0) = 20\ln(2) + 30 \approx 43.9$ cm

b When $H = 1$ m $= 100$ cm,

$$\begin{aligned} 20\ln(3t+2) + 30 &= 100 \\ \therefore\ 20\ln(3t+2) &= 70 \\ \therefore\ \ln(3t+2) &= 3.5 \\ \therefore\ 3t+2 &= e^{3.5} \\ \therefore\ 3t &= e^{3.5} - 2 \\ \therefore\ t &= \frac{e^{3.5}-2}{3} \text{ years} \\ \therefore\ t &\approx 10.4 \text{ years} \end{aligned}$$

c $\dfrac{dH}{dt} = 20 \times \dfrac{3}{(3t+2)} = \dfrac{60}{3t+2}$ cm year^{-1}

i When $t = 3$, $\dfrac{dH}{dt} = \frac{60}{11} \approx 5.4545$

$\therefore$ it is growing at 5.45 cm year^{-1}

ii When $t = 10$, $\dfrac{dH}{dt} = \frac{60}{32} = 1.875$

$\therefore$ it is growing at 1.88 cm year^{-1}

13 **a** $A = s(1 - e^{-kt})$, $t \geqslant 0$

When $t = 0$, $A = s(1 - e^0)$
$= s(1-1) = 0$

b **i** When $t = 3$, $A = 5$, and $s = 10$,

$$\begin{aligned} 5 &= 10(1 - e^{-3k}) \\ \therefore\ 0.5 &= 1 - e^{-3k} \\ \therefore\ e^{-3k} &= 0.5 \\ \therefore\ e^{3k} &= 2 \\ \therefore\ 3k &= \ln 2 \\ \therefore\ k &= \frac{\ln 2}{3} \approx 0.231 \end{aligned}$$

ii $\dfrac{dA}{dt} = ske^{-kt}$

$\therefore$ when $t = 5$ and $s = 10$,

$$\begin{aligned} \frac{dA}{dt} &= 10\left(\tfrac{1}{3}\ln 2\right)\left(e^{-\frac{5}{3}\ln 2}\right) \\ &\approx 0.728 \text{ litres per hour} \end{aligned}$$

c
$$\begin{aligned} A &= s\left(1 - e^{-kt}\right) \\ \therefore\ A &= s - se^{-kt} \\ \therefore\ A - s &= -se^{-kt} \end{aligned}$$

Now,
$$\begin{aligned} \frac{dA}{dt} &= ske^{-kt} \\ &= k\left(se^{-kt}\right) \\ &= -k\left(-se^{-kt}\right) \\ &= -k(A-s) \\ \therefore\ \frac{dA}{dt} &\propto (A-s) \end{aligned}$$

14 **a** When $t = 0$,

$$B(0) = \frac{C}{1+0.5e^0} = \frac{C}{1.5} = \frac{2C}{3} \text{ bees}$$

b When $t = 1$,
$$\begin{aligned} B(1) &= \frac{C}{1+0.5e^{-1.73}} \\ &\approx \frac{C}{1.089} \approx 0.919C \end{aligned}$$

$$\begin{aligned} \therefore\ \%\text{ increase} &= \left(\frac{0.919C - \frac{2}{3}C}{\frac{2}{3}C}\right) \times 100\% \\ &\approx 37.8\% \text{ increase} \end{aligned}$$

c As $t \to \infty$, $e^{-1.73t} \to 0$, and so

$B(t)$ approaches $\dfrac{C}{1+0} = C$

$\therefore$ the population is limited to C bees.

d
$$\begin{aligned} B(2) = \frac{C}{1+0.5e^{-1.73\times 2}} &= 4500 \\ \therefore\ \frac{C}{1.016} &= 4500 \\ \therefore\ C &\approx 4570.7 \end{aligned}$$

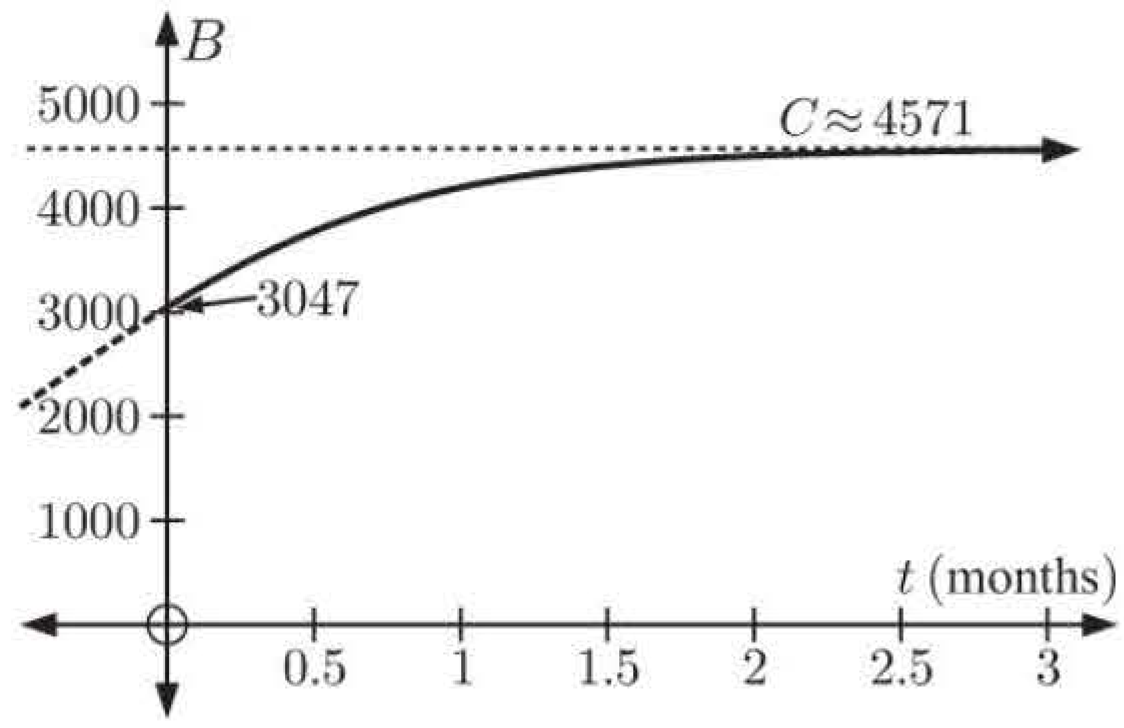
$$\begin{aligned} \therefore\ \text{from } \textbf{a}\text{, initial population} &= \frac{2C}{3} \\ &\approx 3047 \text{ bees} \end{aligned}$$

e $$B(t) = C(1 + 0.5e^{-1.73t})^{-1}$$
$$\therefore \quad B'(t) = -C(1 + 0.5e^{-1.73t})^{-2} \times (0.5(-1.73)e^{-1.73t})$$
$$= \frac{0.865Ce^{-1.73t}}{(1 + 0.5e^{-1.73t})^2}$$
$$= \frac{0.865C}{e^{1.73t}(1 + 0.5e^{-1.73})^2}$$

Since $C > 0$, $B'(t) > 0$ for all t

$\therefore$ $B(t)$ is increasing over time.

f



15 Triangle PQR has area $A = \frac{1}{2} \times 6 \times 7 \times \sin\theta$
$$\therefore \quad A = 21\sin\theta \ \text{cm}^2$$
$$\therefore \quad \frac{dA}{d\theta} = 21\cos\theta \ \text{cm}^2 \text{ per radian}$$
When $\theta = 45° = \frac{\pi}{4}$, $$\frac{dA}{d\theta} = 21\cos(\tfrac{\pi}{4}) \ \text{cm}^2 \text{ per radian}$$
$$= 21 \times \tfrac{1}{\sqrt{2}} \ \text{cm}^2 \text{ per radian}$$
$$\approx \tfrac{21}{\sqrt{2}} \ \text{cm}^2 \text{ per radian}$$

16 $$d = 9.3 + 6.8\cos(0.507t) \text{ m}$$
$$\therefore \quad \frac{dd}{dt} = -6.8\sin(0.507t) \times 0.507$$
$$= -3.4476\sin(0.507t)$$

a When $t = 8$, $$\frac{dd}{dt} = -3.4476\sin(0.507 \times 8)$$
$$\approx 2.73$$

$\therefore$ the rate of change in the depth of water is 2.73 m per hour.

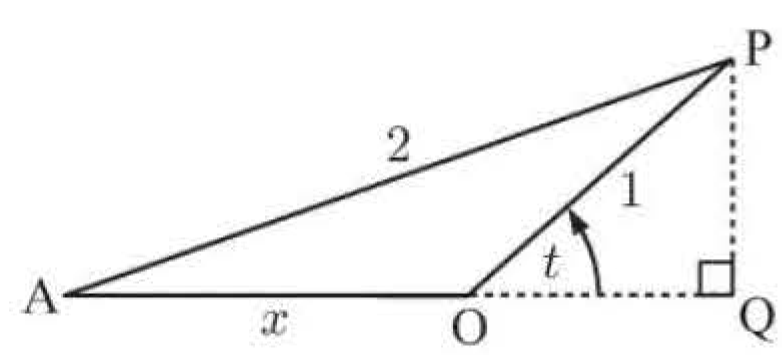
b When $t = 8$, $\dfrac{dd}{dt} \approx 2.73 > 0$

$\therefore$ the tide is rising.

17 **a** $$V(t) = 340\sin(100\pi t)$$
$$\therefore \quad V'(t) = 340\cos(100\pi t) \times 100\pi$$
$$= 34\,000\pi\cos(100\pi t)$$
When $t = 0.01$,
$$V'(0.01) = 34\,000\pi \times \cos\pi$$
$$= -34\,000\pi \text{ units per second}$$

b When $V(t)$ is a maximum, $V'(t)$ must be 0 units per second.

18 **a** The distance from $A(-x, 0)$ to $P(\cos t, \sin t)$ is fixed at 2 m.



$$\cos t = \frac{OQ}{1} = OQ$$
$$\therefore \quad (\cos t + x)^2 + \sin^2 t = 2^2 \quad \{\text{Pythagoras in triangle APQ}\}$$
$$\therefore \quad (\cos t + x)^2 = 4 - \sin^2 t$$
$$\therefore \quad x + \cos t = \pm\sqrt{4 - \sin^2 t}$$
$\therefore$ since $x > 0$, $x = \sqrt{4 - \sin^2 t} - \cos t$

b Now $$\frac{dx}{dt} = \tfrac{1}{2}(4 - \sin^2 t)^{-\frac{1}{2}}(-2\sin t\cos t) + \sin t$$
$$= \frac{-\sin t\cos t}{\sqrt{4 - \sin^2 t}} + \sin t$$

i When $t = 0$,
$\sin t = 0$ and $\cos t = 1$
$\therefore \dfrac{dx}{dt} = 0 + 0$
$= 0 \text{ m s}^{-1}$

ii When $t = \frac{\pi}{2}$,
$\sin t = 1$ and $\cos t = 0$
$\therefore \dfrac{dx}{dt} = 0 + \sin(\frac{\pi}{2})$
$= 1 \text{ m s}^{-1}$

iii When $t = \frac{2\pi}{3}$,
$\sin t = \frac{\sqrt{3}}{2}$ and $\cos t = -\frac{1}{2}$
$\therefore \dfrac{dx}{dt} = \dfrac{-\frac{\sqrt{3}}{2}(-\frac{1}{2})}{\sqrt{4 - \frac{3}{4}}} + \dfrac{\sqrt{3}}{2}$
$\approx 1.11 \text{ m s}^{-1}$

EXERCISE 20C

1 $C(x) = 720 + 4x + 0.02x^2$, $p(x) = 15 - 0.002x$

Revenue $R(x) = xp(x) = 15x - 0.002x^2$

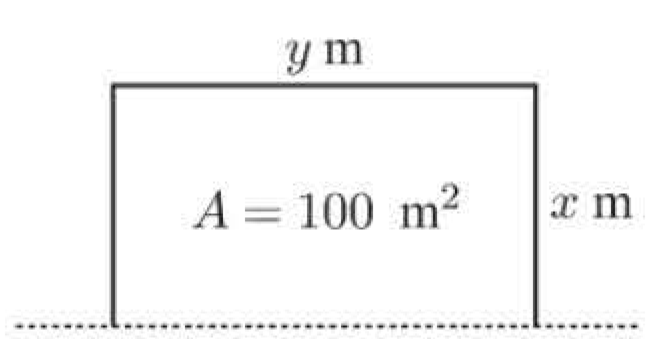
Profit $P(x) = \text{revenue} - \text{cost}$
$= (15x - 0.002x^2) - (720 + 4x + 0.02x^2)$
$= -0.022x^2 + 11x - 720$

$\therefore P'(x) = -0.044x + 11$

Now $P'(x) = 0$ when $x = \dfrac{11}{0.044} = 250$

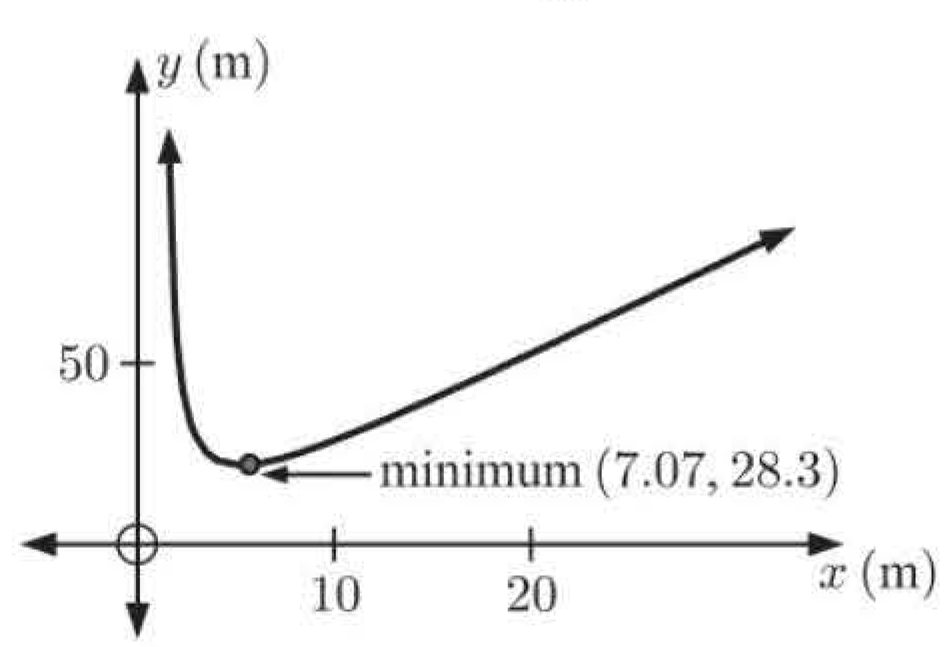
$\therefore$ as $P''(x) = -0.044 < 0$, P is maximised when 250 items are produced.

2 **a**



Now $xy = 100$
$\therefore y = \dfrac{100}{x}$
$\therefore L = 2x + y$
$\therefore L = 2x + \dfrac{100}{x}$

b



c $\dfrac{dL}{dx} = 2 - 100x^{-2} = 2 - \dfrac{100}{x^2}$

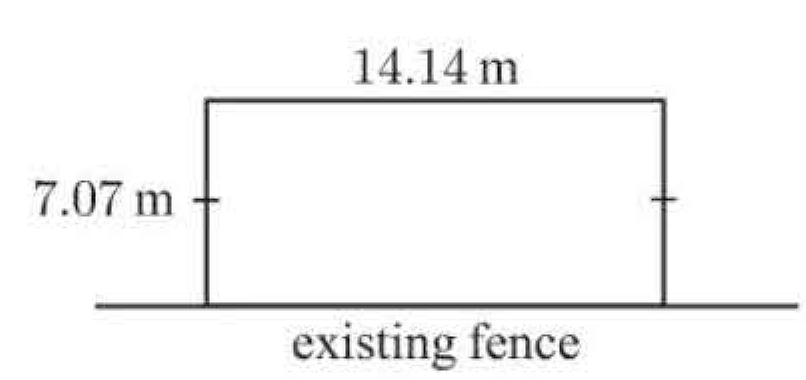
which is 0 when $\dfrac{100}{x^2} = 2$
$\therefore x^2 = 50$
$\therefore x = \sqrt{50}$ $\{x > 0\}$

$\dfrac{d^2L}{dt^2} = 200x^{-3} = \dfrac{200}{x^3} > 0$ for $x > 0$

$\therefore L_{\min} = 2\sqrt{50} + \dfrac{100}{\sqrt{50}}$
$= 2\sqrt{50} + 2\sqrt{50}$
$= 4\sqrt{50}$
$= 20\sqrt{2}$ m when $x = 5\sqrt{2}$ m

$\therefore$ min $L \approx 28.3$ m when $x \approx 7.07$ m

d



3 Suppose x fittings are produced daily.

$\therefore C(x) = 1000 + 2x + \dfrac{5000}{x}$
$= 1000 + 2x + 5000x^{-1}$ euros

$\therefore C'(x) = 2 - \dfrac{5000}{x^2}$

Now $C'(x) = 0$ when $x^2 = 2500$
$\therefore x = 50$ $\{\text{as } x > 0\}$

Also, $C''(x) = 10\,000x^{-3} = \dfrac{10\,000}{x^3}$
which is > 0 when $x > 0$.

$\therefore$ the cost is minimised when 50 fittings are produced.

4 $C(x) = \frac{1}{4}x^2 + 8x + 20$
$p(x) = 23 - \frac{1}{2}x$

Revenue $R(x) = xp(x) = 23x - \frac{1}{2}x^2$

Profit $P(x) = \text{revenue} - \text{cost}$
$= (23x - \frac{1}{2}x^2) - (\frac{1}{4}x^2 + 8x + 20)$
$= -\frac{3}{4}x^2 + 15x - 20$

$\therefore P'(x) = -\frac{3}{2}x + 15$

Now $P'(x) = 0$ when $x = \dfrac{15}{\frac{3}{2}} = 10$

$\therefore$ as $P''(x) = -\frac{3}{2} < 0$, P is maximised when 10 blankets per day are produced.

5 Cost per hour $= \dfrac{v^2}{10} + 22$

Now, cost per km $= \dfrac{\text{cost per hour}}{\text{km per hour}}$

$$\therefore \quad C(v) = \frac{\frac{v^2}{10} + 22}{v} = 0.1v + 22v^{-1}$$

$$\therefore \quad C'(v) = 0.1 - 22v^{-2}$$

Now $C'(v) = 0$ when $0.1 = \dfrac{22}{v^2}$

$$\therefore \quad v^2 = 220$$

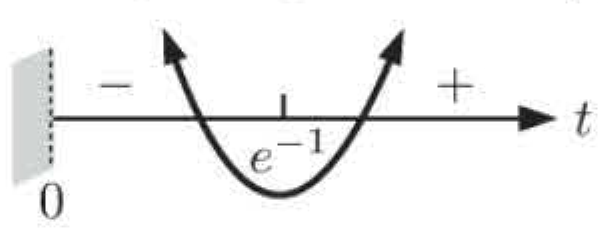
$$\therefore \quad v \approx 14.8 \text{ km h}^{-1}$$

6 **a** $A(t) = t\ln t + 1, \quad 0 < t \leqslant 5$

$$\therefore \quad A'(t) = \ln t + t \times \frac{1}{t} + 0 \quad \{\text{product rule}\}$$
$$= \ln t + 1$$

$$\therefore \quad A'(t) = 0 \text{ when } \ln t = -1$$
$$\therefore \quad t = e^{-1}$$

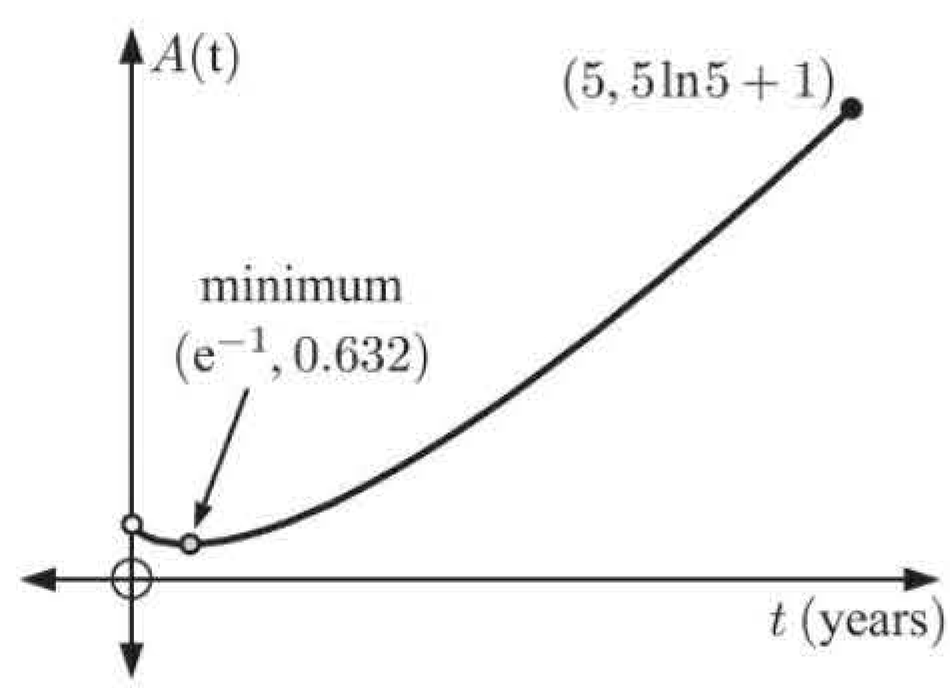
and the sign diagram of $A'(t)$ is:



$$\therefore \quad A(t) \text{ is a minimum when } t = \frac{1}{e}$$
$$\approx 0.3679 \text{ years}$$
$$\approx 4.41 \text{ months}$$

$\therefore$ the child's memorising ability is a minimum at 4.41 months old.

b



7 $C(x) = 0.0007x^3 - 0.1796x^2 + 14.663x + 160$ for $50 \leqslant x \leqslant 150$

$C'(x) = 0.0021x^2 - 0.3592x + 14.663$

$C'(x) = 0$ when $0.0021x^2 - 0.3592x + 14.663 = 0$

Using technology, $x \approx 103.74$ or $x \approx 67.30$

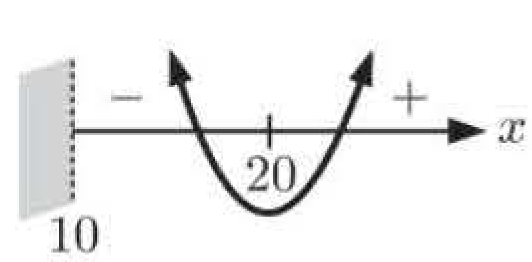
x	50	67.30	103.74	150
$C(x)$	531.65	546.73	529.80	680.95

$\therefore$ the maximum hourly cost is \$680.95 when 150 hinges are made per hour. The minimum hourly cost is \$529.80 when 104 hinges are made per hour.

8 $C(x) = 4\ln x + \left(\dfrac{30-x}{10}\right)^2, \quad x \geqslant 10$

$$\therefore \quad C'(x) = \frac{4}{x} + 2\left(\frac{30-x}{10}\right)\left(-\tfrac{1}{10}\right)$$
$$= \frac{4}{x} - \frac{30-x}{50}$$
$$= \frac{200 - x(30-x)}{50x}$$
$$= \frac{200 - 30x + x^2}{50x}$$
$$= \frac{(x-10)(x-20)}{50x}$$

$C'(x)$ has sign diagram:



$\therefore$ the minimum cost occurs when $x = 20$ or when 20 kettles per day are produced.

9 **a** Inner length of box $= 2x$ cm

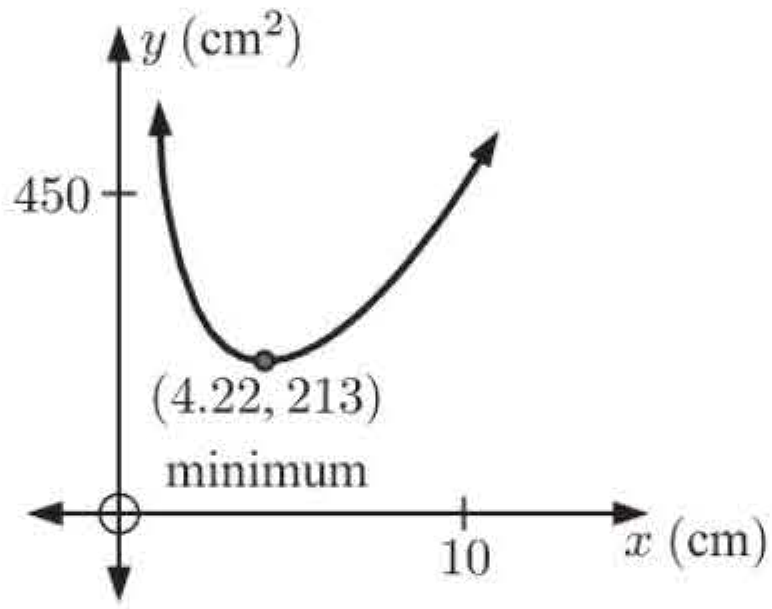
b $$\text{Volume} = 200 \text{ cm}^3$$
$$\therefore\ x \times 2x \times h = 200$$
$$2x^2 h = 200$$
$$\therefore\ x^2 h = 100$$

c From **b**, $h = \dfrac{100}{x^2}$

Area of inner surface is

$$\begin{aligned} A(x) &= 2(2x \times x) + 2(2x \times h) \\ &\quad + 2(x \times h) \\ &= 4x^2 + 4xh + 2xh \\ &= 4x^2 + 6xh \\ &= 4x^2 + \frac{600}{x} \text{ cm}^2 \end{aligned}$$

d



e
$$\begin{aligned} A(x) &= 4x^2 + 600x^{-1} \\ \therefore\ A'(x) &= 8x - 600x^{-2} \\ &= 8x - \frac{600}{x^2} \end{aligned}$$

$\therefore\ A'(x) = 0$ when $8x = \dfrac{600}{x^2}$

$$\therefore\ 8x^3 = 600$$
$$\therefore\ x^3 = 75$$
$$\therefore\ x \approx 4.217 \text{ cm}$$

$$\begin{aligned} A''(x) &= 8 + 1200x^{-3} \\ &= 8 + \frac{1200}{x^3} \end{aligned}$$

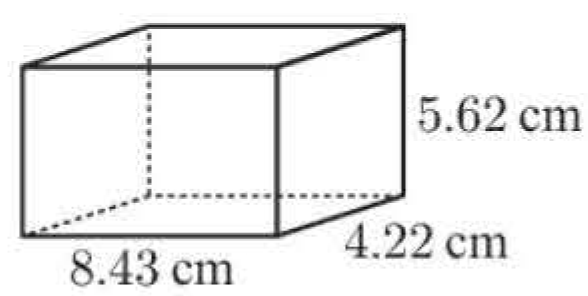
$\therefore\ A''(x) > 0 \quad \{\text{as } x > 0\}$

$\therefore$ area is minimised when $x \approx 4.22$ cm

$$\begin{aligned} \therefore\ A_{\min} &\approx 4(4.217)^2 + \frac{600}{(4.217)} \\ &\approx 213 \text{ cm}^2 \end{aligned}$$

f Height $h \approx \dfrac{100}{(4.217)^2}$

≈ 5.62 cm



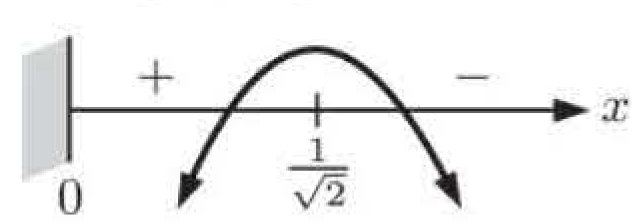
10 Let coordinates of D be $(x, 0)$ where $x > 0$.

$\therefore$ the coordinates of C are (x, e^{-x^2}).

$\therefore$ area ABCD $= 2xe^{-x^2}$

$$\begin{aligned} \therefore\ \frac{dA}{dx} &= 2e^{-x^2} + 2xe^{-x^2}(-2x) \quad \{\text{product rule}\} \\ &= 2e^{-x^2}\left(1 - 2x^2\right) \\ &= 2e^{-x^2}(1 + \sqrt{2}x)(1 - \sqrt{2}x) \end{aligned}$$

$\dfrac{dA}{dx}$ has sign diagram:



$\therefore$ the area is a maximum when $x = \frac{1}{\sqrt{2}}$

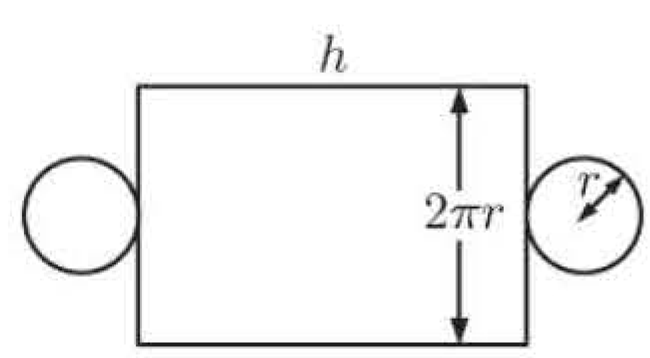
and so C is $\left(\frac{1}{\sqrt{2}}, e^{-\frac{1}{2}}\right)$.

11 **a** Volume of can $= \pi r^2 h$

$\therefore\ 1000 = \pi r^2 h$ (in cm)

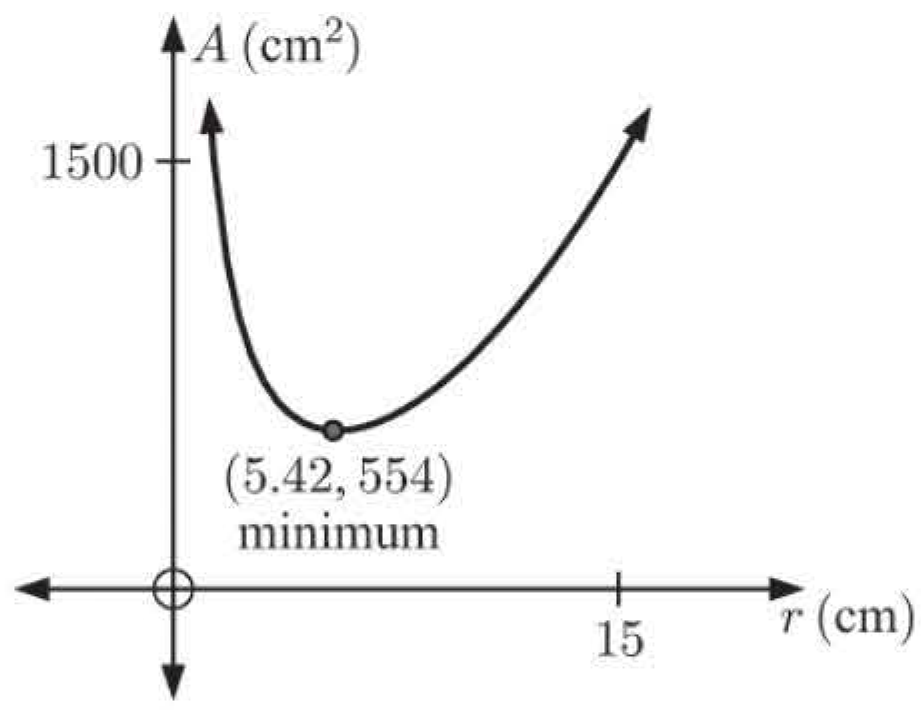
$\therefore\ h = \dfrac{1000}{\pi r^2}$ cm

b Opening the can up we get:



$$\begin{aligned} \therefore\ A(r) &= \pi r^2 + \pi r^2 + 2\pi rh \\ &= 2\pi r^2 + 2\pi rh \\ &= 2\pi r^2 + \frac{2000}{r} \text{ cm}^2 \end{aligned}$$

c



d $A(r) = 2\pi r^2 + 2000r^{-1}$

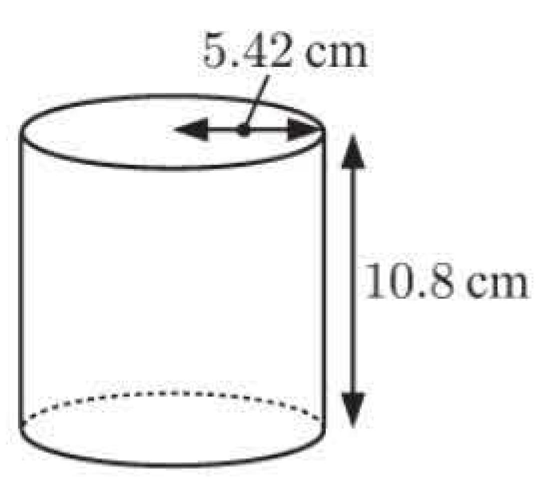
$$A'(r) = 4\pi r - 2000r^{-2} = 4\pi r - \frac{2000}{r^2}$$

So, $A'(r) = 0$ when $4\pi r = \dfrac{2000}{r^2}$

$$r^3 = \frac{2000}{4\pi}$$

$$r = \sqrt[3]{\frac{500}{\pi}} \approx 5.42 \text{ cm}$$

e



$$A''(r) = 4\pi + 4000r^{-3} = 4\pi + \frac{4000}{r^3}$$

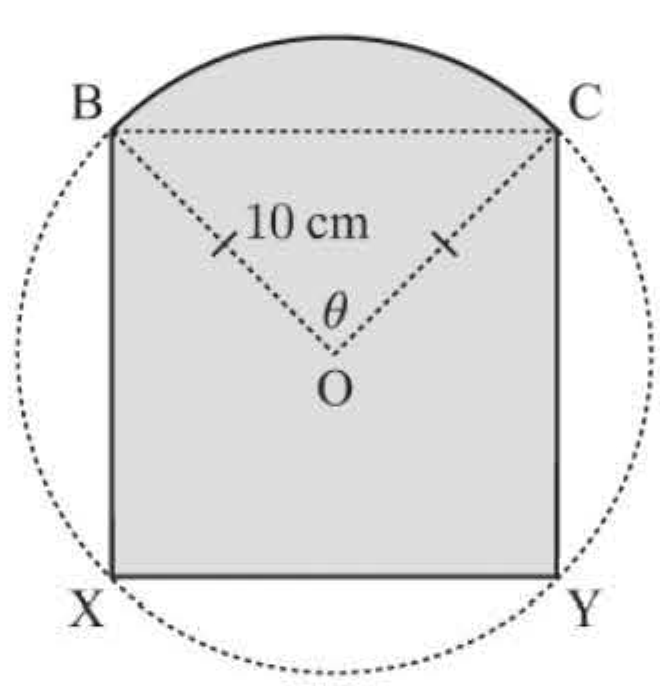
and as $r > 0$, $A''(r) > 0$

$\therefore$ area is a minimum when $r \approx 5.42$ cm

and $h = \dfrac{1000}{\pi r^2} \approx 10.8$ cm

$$A_{\min} = 2\pi r^2 + 2\pi rh \approx 554 \text{ cm}^2$$

12



a Using the cosine rule in $\triangle$BCO,

$$\text{BC}^2 = 10^2 + 10^2 - 2 \times 10 \times 10\cos\theta$$

$$\therefore \text{BC} = \sqrt{200 - 200\cos\theta}$$

$\therefore$ $\text{XY} = \sqrt{200 - 200\cos\theta}$ also

Now $\text{BY}^2 = \text{BX}^2 + \text{XY}^2$ {Pythagoras}

$$\therefore 400 = \text{BX}^2 + (200 - 200\cos\theta)$$

$$\therefore \text{BX}^2 = 200 + 200\cos\theta$$

$$\therefore \text{BX} = \sqrt{200 + 200\cos\theta}$$

The shaded area is equal to the area of the sector plus $\frac{3}{4}$ of the area of BCYX.

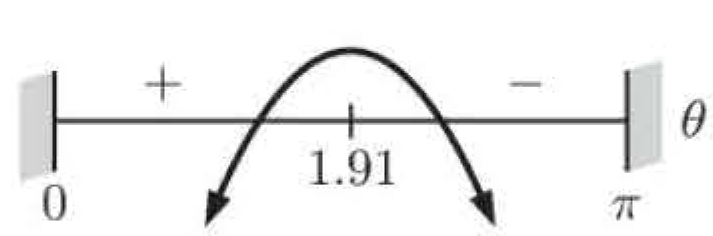
$$\begin{aligned}
\therefore A &= \tfrac{1}{2}(10)^2\theta + \tfrac{3}{4}(\text{BX} \times \text{BC}) \\
&= 50\theta + \tfrac{3}{4}\sqrt{200 + 200\cos\theta}\sqrt{200 - 200\cos\theta} \\
&= 50\theta + \tfrac{3}{4} \times 200\sqrt{1 + \cos\theta}\sqrt{1 - \cos\theta} \\
&= 50\theta + 150\sqrt{1 - \cos^2\theta} \\
&= 50\theta + 150\sin\theta \\
&= 50(\theta + 3\sin\theta) \quad \text{as required}
\end{aligned}$$

b $\dfrac{dA}{d\theta} = 50 + 150\cos\theta = 50(1 + 3\cos\theta)$

which is zero when $\cos\theta = -\frac{1}{3}$

$\therefore$ $\theta \approx 1.91$

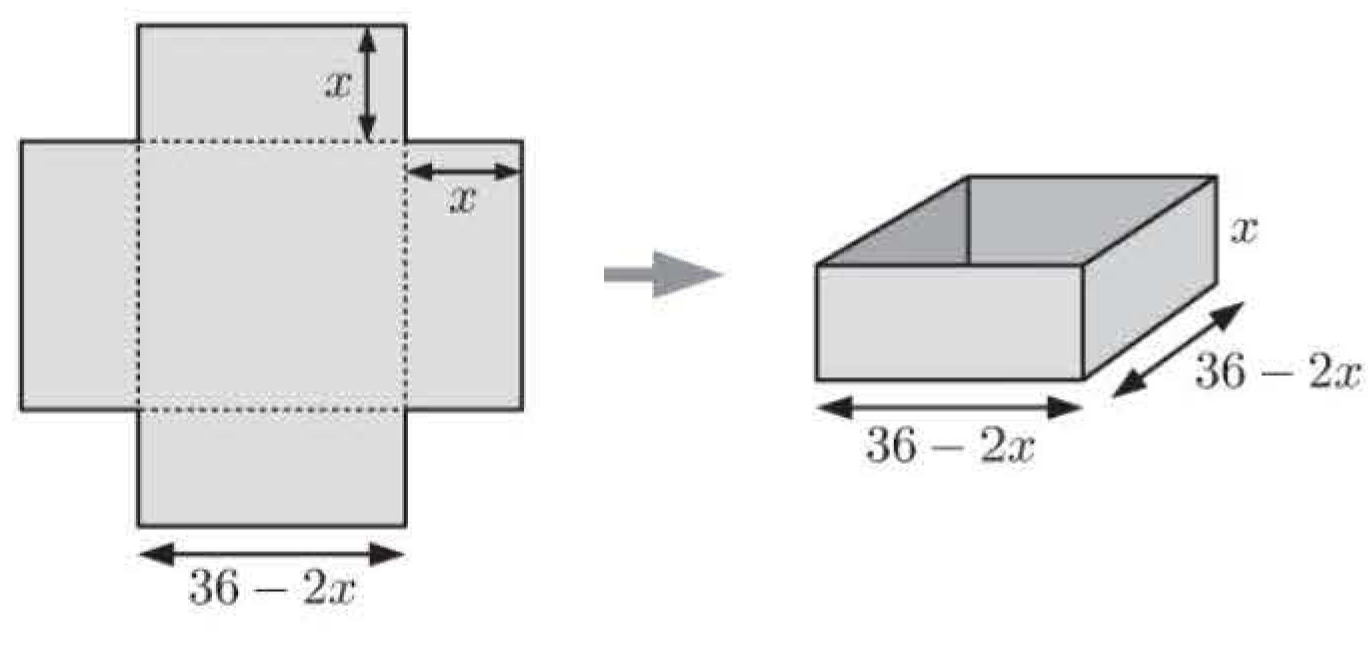
The sign diagram of $\dfrac{dA}{d\theta}$ is:



Since $0 < \theta < \pi$, A is maximised when $\theta \approx 1.91$.

When $\theta \approx 1.91$, $A \approx 237$ $\therefore$ the maximum area is 237 cm^2.

13 **a**



The volume of the container is

$$\begin{aligned}
V &= \text{length} \times \text{width} \times \text{depth} \\
&= x(36 - 2x)(36 - 2x)
\end{aligned}$$

$$\therefore V = x(36 - 2x)^2 \text{ cm}^3$$

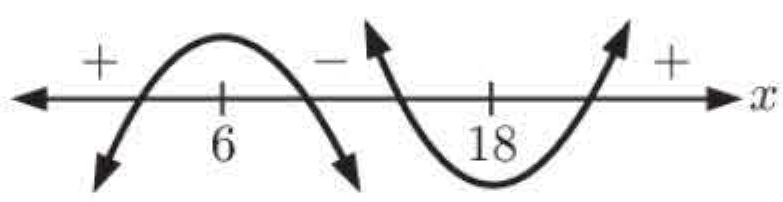
b The container will have the greatest capacity when $V(x)$ is maximised.

Using the product rule,

$$\begin{aligned} V'(x) &= (36-2x)^2 - 4x(36-2x) \\ &= (36-2x)[(36-2x)-4x] \\ &= (36-2x)(36-6x) \end{aligned}$$

$\therefore\ V'(x) = 0$ when $x = 6$ or $x = 18$

Sign diagram of $V'(x)$ is:



$\therefore$ the volume is maximised when $x = 6$ cm $\{0 \leqslant x < 18\}$

So, 6 cm × 6 cm squares should be cut out to maximise the capacity.

c The metal sheet is a cm by a cm and squares which are x cm by x cm are cut out.

$$\begin{aligned} V &= \text{length} \times \text{width} \times \text{depth} \\ &= (a-2x) \times (a-2x) \times x \end{aligned}$$

$\therefore\ V = x(a-2x)^2$

$$\begin{aligned} \text{Now}\quad V'(x) &= (a-2x)^2 - 4x(a-2x) \\ &= (a-2x)[(a-2x)-4x] \\ &= (a-2x)(a-6x) \end{aligned}$$

$\therefore\ V'(x) = 0$ when $x = \dfrac{a}{2}$ or $x = \dfrac{a}{6}$

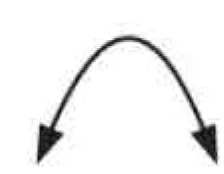
But $a - 2x > 0$ $\therefore\ x < \dfrac{a}{2}$

So $x = \dfrac{a}{6}$ is the only value in $0 < x < \dfrac{a}{2}$ with $V'(x) = 0$.

Second derivative test:

$$\begin{aligned} V''(x) &= -2(a-6x) + (a-2x)(-6) \\ &= -2a + 12x - 6a + 12x \\ &= 24x - 8a \end{aligned}$$

$\therefore\ V''\left(\dfrac{a}{6}\right) = 4a - 8a = -4$ which is < 0

$\therefore$ the shape is  and we have a local maximum.

$\therefore$ the volume is maximised when $x = \dfrac{a}{6}$.

14 **a**

$$\begin{aligned} P &= 2\pi r + 2l \\ \therefore\ 400 &= 2\pi(x) + 2l \\ \therefore\ 200 &= \pi x + l \\ \therefore\ l &= 200 - \pi x \end{aligned}$$

Now clearly $x \geqslant 0$ and $l \geqslant 0$

$\therefore\ \pi x \leqslant 200$

$\therefore\ x \leqslant \frac{200}{\pi}$

So, $0 \leqslant x \leqslant \frac{200}{\pi}$

or $0 \leqslant x \leqslant 63.7$

b Area of shaded rectangle $A = 2xl$ m^2

$$\begin{aligned} \therefore\ A &= 2x(200 - \pi x) \text{ m}^2 \\ &= 400x - 2\pi x^2 \text{ m}^2 \end{aligned}$$

Now $\dfrac{dA}{dx} = 400 - 4\pi x$

which is 0 when $4\pi x = 400$

$\therefore\ x = \frac{100}{\pi} \approx 31.83$ m

and so $l = 200 - \pi\left(\frac{100}{\pi}\right)$
$= 100$ m

So, the maximum area of the rectangle is

$2 \times \frac{100}{\pi} \times 100 \approx 6366$ m^2.

15

$$\begin{aligned} P(t) &= \frac{50\,000}{1 + 1000e^{-0.5t}}, \quad 0 \leqslant t \leqslant 25 \\ &= 50\,000(1 + 1000e^{-0.5t})^{-1} \\ \therefore\ P'(t) &= -50\,000(1 + 1000e^{-0.5t})^{-2}\left(-500e^{-0.5t}\right) \\ &= 2.5 \times 10^7 e^{-0.5t}(1 + 1000e^{-0.5t})^{-2} \end{aligned}$$

The wasp population is growing the fastest when $\dfrac{dP}{dt}$ is a maximum.

Using technology, the graph of $P'(t)$ can be drawn and the maximum obtained.

The maximum occurs when $t \approx 13.8$ weeks.

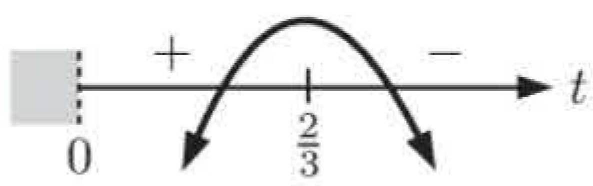
16
$$\begin{aligned} E(t) &= 750te^{-1.5t} \\ \therefore\ E'(t) &= 750e^{-1.5t} + 750t \times -1.5e^{-1.5t} \\ &= 750e^{-1.5t} - 1125te^{-1.5t} \\ &= e^{-1.5t}(750 - 1125t) \end{aligned}$$

Now $E'(t) = 0$ when $750 - 1125t = 0$ $\{$since $e^{-1.5t} > 0$ for all $t \in \mathbb{R}\}$

$\therefore\ 1125t = 750$

$\therefore\ t = \frac{2}{3}$ hours (or 40 minutes)

Sign diagram for $E'(t)$:



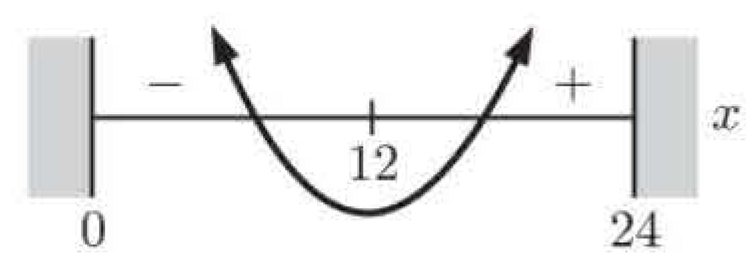
So, the drug is most effective 40 minutes after the injection.

17 **a** $\text{AB} = x$ m

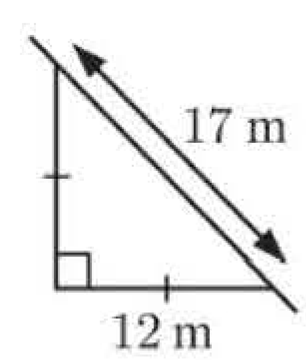
$\therefore\ \text{BC} = (24 - x)$ m $\qquad \therefore\ D(x) = \sqrt{x^2 + (24 - x)^2}$ $\{$Pythagoras$\}$

b
$$\begin{aligned} [D(x)]^2 &= x^2 + (24 - x)^2 \\ &= x^2 + 576 - 48x + x^2 \\ &= 2x^2 - 48x + 576 \\ \therefore\ \frac{d[D(x)]^2}{dx} &= 4x - 48 \\ \therefore\ \frac{d[D(x)]^2}{dx} &= 0 \text{ when } x = 12 \end{aligned}$$

Sign diagram for $\dfrac{d[D(x)]^2}{dx}$:



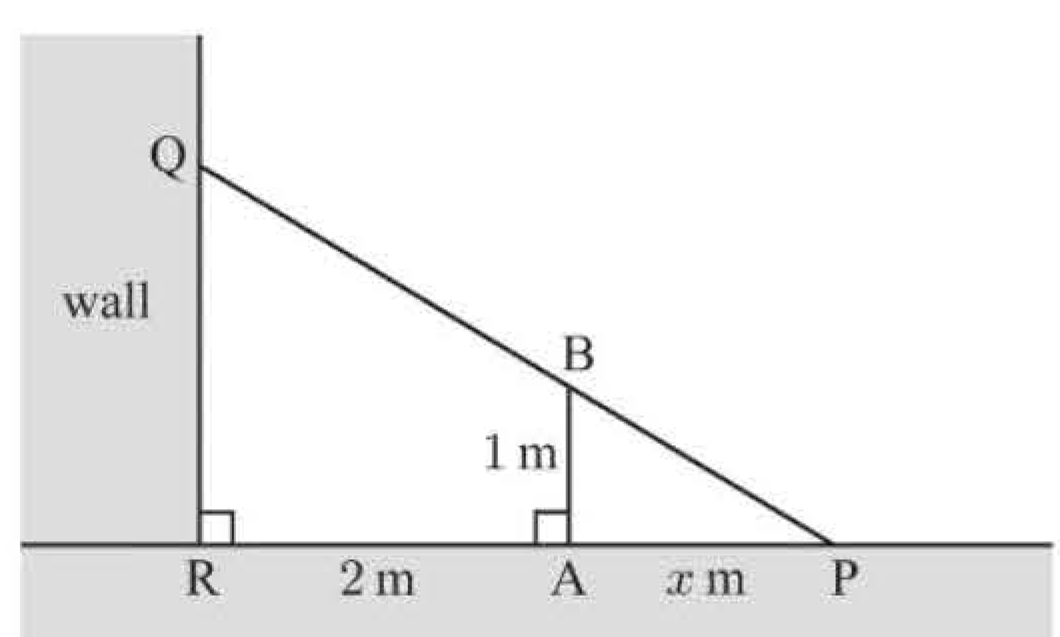
c When $\text{AB} = \text{BC} = 12$ m, $D(x)$ is a minimum, and the minimum $D(x) = 12\sqrt{2}$ m ≈ 17.0 m.



18 **a** $\triangle$s PAB and PRQ are similar.

$$\therefore\ \frac{\text{PA}}{\text{PR}} = \frac{\text{PB}}{\text{PQ}} = \frac{\text{AB}}{\text{RQ}}$$

$$\therefore\ \frac{x}{x+2} = \frac{1}{\text{QR}} \quad \text{and} \quad \therefore\ \text{QR} = \frac{x+2}{x}$$



b Now $[L(x)]^2 = \text{RP}^2 + \text{QR}^2$ $\{$Pythagoras$\}$

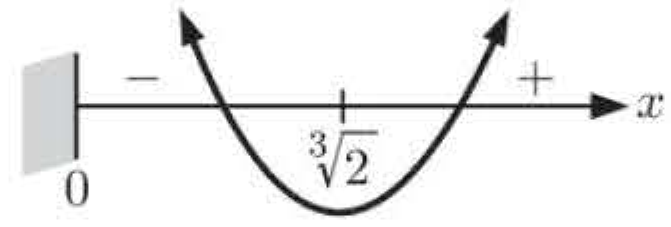
$$\begin{aligned} &= (x+2)^2 + \left(\frac{x+2}{x}\right)^2 \\ &= (x+2)^2 \times 1 + (x+2)^2 \times \frac{1}{x^2} \end{aligned}$$

$$\therefore\ [L(x)]^2 = (x+2)^2\left(1 + \frac{1}{x^2}\right) \quad \text{as required}$$

c
$$\begin{aligned} [L(x)]^2 &= (x+2)^2(1 + x^{-2}) \\ \therefore\ \frac{d[L(x)]^2}{dx} &= 2(x+2)(1 + x^{-2}) \ + \ (x+2)^2(-2x^{-3}) \quad \{\text{product rule}\} \\ &= 2(x+2)(1 + x^{-2} - (x+2)x^{-3}) \\ &= 2(x+2)(1 + x^{-2} - x^{-2} - 2x^{-3}) \\ &= 2(x+2)\left(1 - \frac{2}{x^3}\right) \\ &= \frac{2(x+2)(x^3 - 2)}{x^3} \end{aligned}$$

$$\therefore\ \frac{d[L(x)]^2}{dx} = 0 \text{ when } x = \sqrt[3]{2} \approx 1.2599 \qquad \{\text{as } x > 0 \text{ and } L(x) > 0\}$$

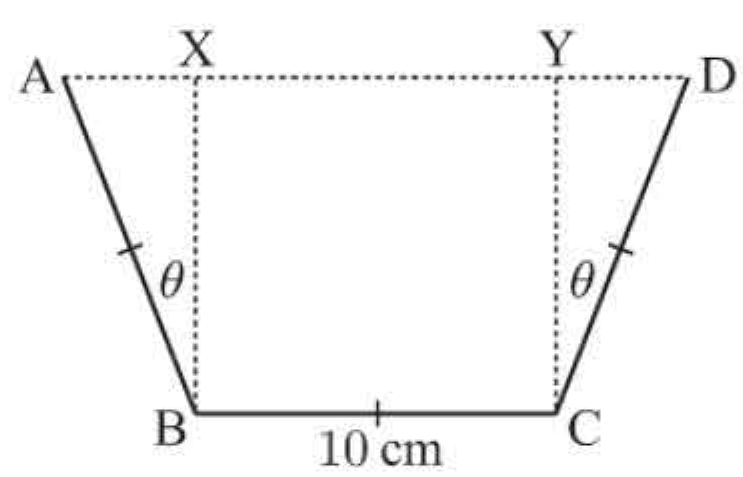
d Sign diagram of $\dfrac{d\,[L(x)]^2}{dx}$ is:



$\therefore$ the ladder is shortest when $x = \sqrt[3]{2}$ m and

its length at this time is $L = \sqrt{(x+2)^2\left(1+\dfrac{1}{x^2}\right)} = \sqrt{(\sqrt[3]{2}+2)^2\left(1+\dfrac{1}{2^{\frac{2}{3}}}\right)} \approx 4.16$ m

19 **a**



The triangles have height $10\cos\theta$ and width $10\sin\theta$.

$\therefore$ area A
$= $ area of $\triangle$s $+$ area of rectangle
$= 2 \times \frac{1}{2} \times 10\cos\theta \times 10\sin\theta + 10 \times 10\cos\theta$
$= 100\sin\theta\cos\theta + 100\cos\theta$
$= 100\cos\theta(1+\sin\theta)$

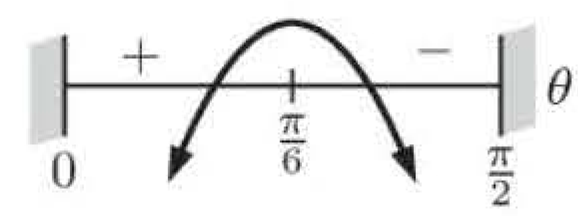
b $\dfrac{dA}{d\theta} = 100(-\sin\theta(1+\sin\theta) + \cos\theta \times \cos\theta)$
$= 100(-\sin\theta - \sin^2\theta + \cos^2\theta)$
$= 100(-\sin\theta - \sin^2\theta + 1 - \sin^2\theta)$
$= -100(2\sin^2\theta + \sin\theta - 1)$
$= -100(2\sin\theta - 1)(\sin\theta + 1)$

$\therefore \dfrac{dA}{d\theta} = 0$ when $2\sin\theta - 1 = 0$ or $\sin\theta + 1 = 0$
$\therefore \sin\theta = \frac{1}{2}$ or $\sin\theta = -1$

c The maximum carrying capacity occurs when A is maximised.

Using **b**, $\dfrac{dA}{d\theta} = 0$ when $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$, or $\frac{3\pi}{2}$.

But $0 \leqslant \theta \leqslant \frac{\pi}{2}$, so the sign diagram for $\dfrac{dA}{d\theta}$ is:



So, the maximum area occurs when $\theta = \frac{\pi}{6} = 30°$

When $\theta = 30°$, $A = 100\cos 30°(1+\sin 30°)$
$= 100 \times \frac{\sqrt{3}}{2} \times \frac{3}{2}$
$= 75\sqrt{3}$
≈ 130 cm^2

$\therefore$ the maximum cross-sectional area is 130 cm^2.

20 **a** Arc AC $= \dfrac{\theta}{360} \times (2\pi r_{\text{sector}})$
$= \dfrac{\theta}{360}(2 \times \pi \times 10)$
$= \dfrac{\theta\pi}{18}$

b Now arc AC forms the base of the cone.

$\therefore 2\pi r = \dfrac{\theta\pi}{18}$ {from **a**}

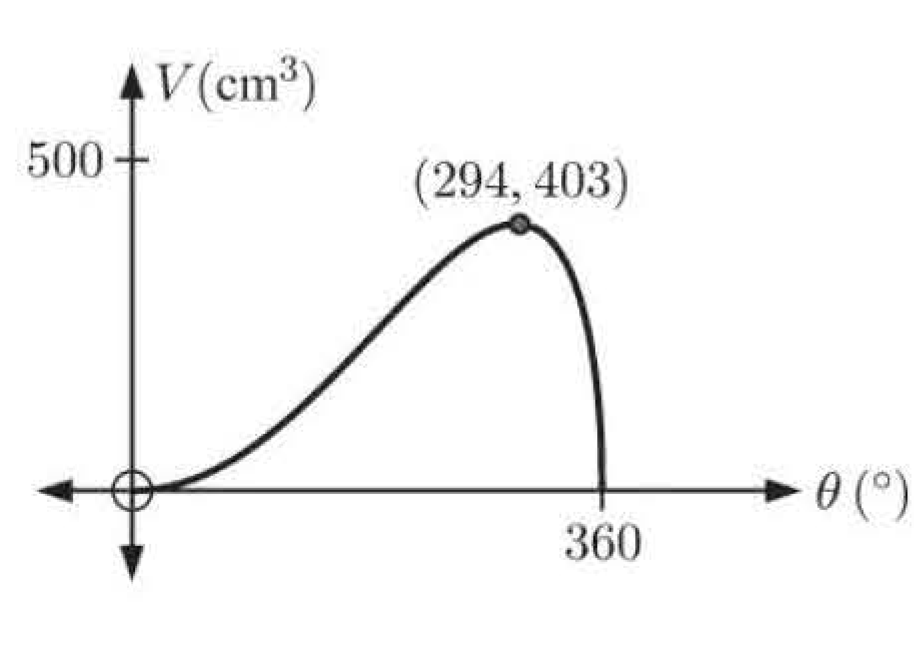
$\therefore r = \dfrac{\theta}{36}$

c Height of cone $= \sqrt{10^2 - r^2}$ {Pythagoras}

$\therefore h = \sqrt{100 - \left(\dfrac{\theta}{36}\right)^2}$

d $V = \frac{1}{3}\pi r^2 h$
$= \frac{1}{3}\pi\left(\dfrac{\theta}{36}\right)^2 \sqrt{100 - \left(\dfrac{\theta}{36}\right)^2}$

e



f $V(\theta) = \frac{1}{3}\pi\left(\frac{\theta}{36}\right)^2\sqrt{100 - \left(\frac{\theta}{36}\right)^2}$

$$= \frac{\pi\theta^2}{3 \times 36^2}\sqrt{\frac{100 \times 36^2 - \theta^2}{36^2}}$$

$$= \frac{\pi\theta^2}{3888} \times \frac{1}{36}\sqrt{129\,600 - \theta^2}$$

$$= \frac{\pi\theta^2}{139\,968}\sqrt{129\,600 - \theta^2}$$

Now $V'(\theta) = \frac{2\pi\theta}{139\,968}\left(129\,600 - \theta^2\right)^{\frac{1}{2}} + \frac{\pi\theta^2}{139\,968}\left(\frac{1}{2}\right)(129\,600 - \theta^2)^{-\frac{1}{2}}(-2\theta)$ {product rule}

$$= \frac{\pi\theta}{139\,968}\left(\frac{2\sqrt{129\,600 - \theta^2}}{1} - \frac{\theta^2}{\sqrt{129\,600 - \theta^2}}\right)$$

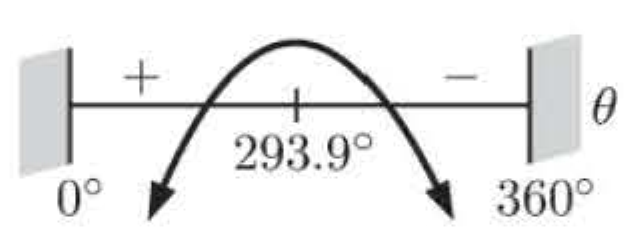
$$= \frac{\pi\theta}{139\,968}\left(\frac{2(129\,600 - \theta^2) - \theta^2}{\sqrt{129\,600 - \theta^2}}\right)$$

and $V'(\theta) = 0$ when $\theta = 0$ or $2(129\,600 - \theta^2) = \theta^2$

$259\,200 - 2\theta^2 = \theta^2$

$\therefore\ 3\theta^2 = 259\,200$

Sign diagram of $V'(\theta)$ is:

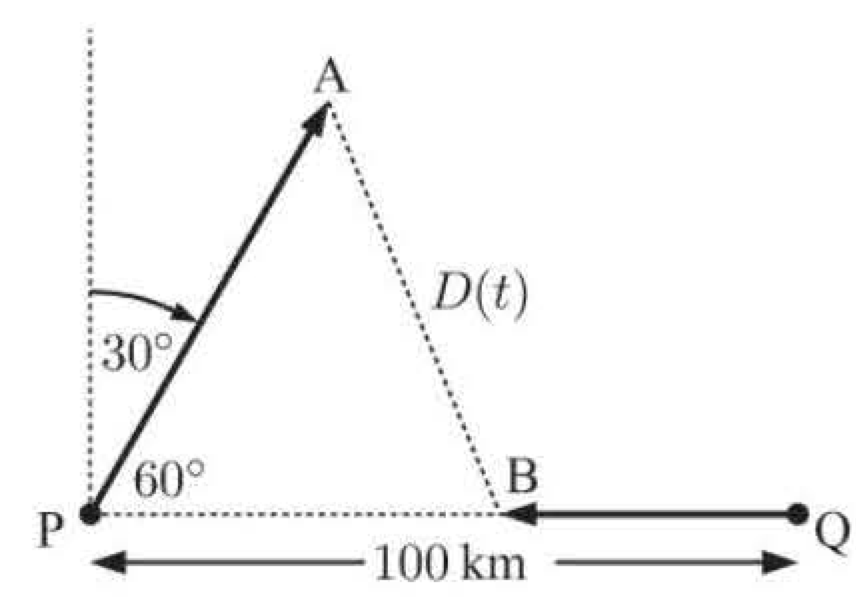


$\therefore\ \theta = \sqrt{86\,400}$ {as $\theta > 0$}

$\therefore\ \theta \approx 293.9°$

$\therefore$ maximum V occurs when $\theta \approx 294°$.

21 **a** Consider each boat's position t hours after 1:00 pm.



PA $= 12t$ and QB $= 8t$

$\therefore$ PB $= 100 - 8t$

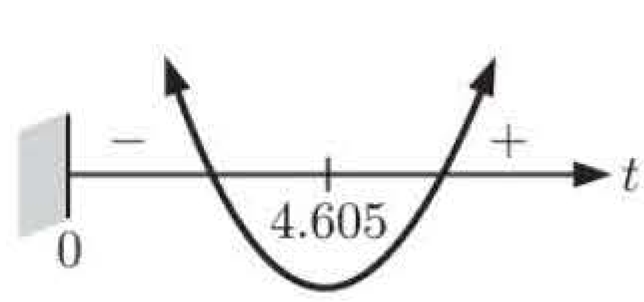
Using the cosine rule in $\triangle$PAB,

$$\begin{aligned} D(t)^2 &= \text{AP}^2 + \text{BP}^2 - 2\text{AP} \times \text{BP}\cos 60° \\ &= (12t)^2 + (100 - 8t)^2 - 2(12t)(100 - 8t)\tfrac{1}{2} \\ &= 144t^2 + (100 - 8t)^2 - 12t(100 - 8t) \\ &= 144t^2 + 10\,000 - 1600t + 64t^2 - 1200t + 96t^2 \\ &= 304t^2 - 2800t + 10\,000 \end{aligned}$$

$\therefore\ D(t) = \sqrt{304t^2 - 2800t + 10\,000}$

b Now $\frac{d[D(t)]^2}{dt} = 608t - 2800$

$\therefore\ \frac{d[D(t)]^2}{dt} = 0$ when $t = \frac{2800}{608} \approx 4.605\,26$



$\therefore\ D(t)$ is a minimum when $t \approx 4.605\,26$ hours after 1:00 pm

and $[D(t)]^2_{\text{min}} \approx 304\,(4.6053)^2 - 2800\,(4.6053) + 10\,000$

$\therefore\ [D(t)]^2_{\text{min}} \approx 3553\ \text{km}^2$

c The ships are closest when $t = 4.605\,26$ hours which occurs when the time is 4 hours 36 minutes after 1:00 pm.
So, the ships are closest at approximately 5:36 pm.

22 **a** X must lie either between A and C or else at one of the two points.
If $x = 0$, then he rows straight to the shore and runs to C.
If $x = 6$, then he rows straight to C. $\therefore\ 0 \leqslant x \leqslant 6$

b Now $\text{XC} = 6 - x$

$\therefore$ the time to row from B to X $= \dfrac{\text{BX}}{8} = \dfrac{\sqrt{5^2 + x^2}}{8}$

and the time to run from X to C $= \dfrac{\text{XC}}{17} = \dfrac{6 - x}{17}$

$\therefore$ the total time $T(x) = \dfrac{\sqrt{25 + x^2}}{8} + \dfrac{6 - x}{17}$ hours, $0 \leqslant x \leqslant 6$

$$= \tfrac{1}{8}(25 + x^2)^{\frac{1}{2}} + \tfrac{6}{17} - \frac{x}{17}$$

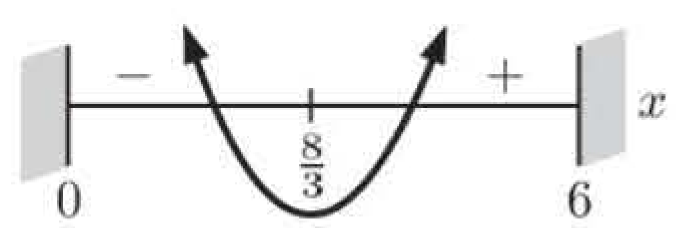
c $\dfrac{dT}{dx} = \tfrac{1}{16}(25 + x^2)^{-\frac{1}{2}}(2x) - \tfrac{1}{17}$

$$= \frac{x}{8\sqrt{25 + x^2}} - \tfrac{1}{17}$$

So, $\dfrac{dT}{dx} = 0$ when $\dfrac{x}{8\sqrt{25 + x^2}} = \tfrac{1}{17}$

$$17x = 8\sqrt{25 + x^2}$$
$$\therefore\ 289x^2 = 64(25 + x^2)$$
$$\therefore\ 225x^2 = 1600$$
$$\therefore\ x^2 = \tfrac{1600}{225}$$
$$\therefore\ x = \tfrac{40}{15}$$
$$= \tfrac{8}{3} \approx 2.67 \text{ km}$$

Sign diagram of $\dfrac{dT}{dx}$:



The time taken is minimised if Peter aims for X such that $x \approx 2.67$ km.

23 Let $\text{MX} = x$ km, so $\text{XN} = 5 - x$ km

$\therefore$ $\text{AX} = \sqrt{4 + x^2}$ km and $\text{XB} = \sqrt{1 + (5 - x)^2}$ km {Pythagoras}

Now $P = \text{AX} + \text{XB}$

$$= (4 + x^2)^{\frac{1}{2}} + (26 - 10x + x^2)^{\frac{1}{2}}$$

$$\therefore\ \frac{dP}{dx} = \tfrac{1}{2}(4 + x^2)^{-\frac{1}{2}}(2x) + \tfrac{1}{2}(26 - 10x + x^2)^{-\frac{1}{2}}(2x - 10)$$

$$= \frac{x}{\sqrt{4 + x^2}} + \frac{x - 5}{\sqrt{x^2 - 10x + 26}}$$

Now $\dfrac{dP}{dx} = 0$ when $\dfrac{x}{\sqrt{4 + x^2}} = \dfrac{5 - x}{\sqrt{x^2 - 10x + 26}}$

$$\therefore\ \frac{x^2}{4 + x^2} = \frac{(5 - x)^2}{x^2 - 10x + 26} \qquad \text{\{squaring both sides\}}$$

$$\therefore\ x^2(x^2 - 10x + 26) = (4 + x^2)(25 - 10x + x^2)$$

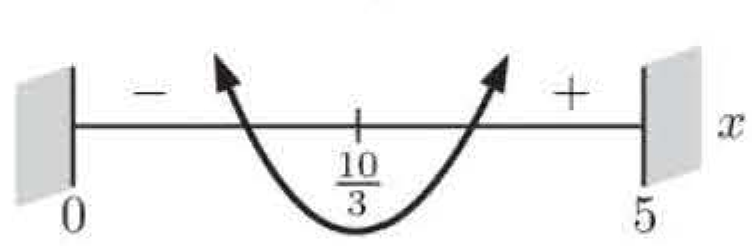
$$\therefore\ x^4 - 10x^3 + 26x^2 = 100 - 40x + 4x^2 + 25x^2 - 10x^3 + x^4$$

$$\therefore\ 3x^2 - 40x + 100 = 0$$

$$\therefore\ (3x - 10)(x - 10) = 0$$

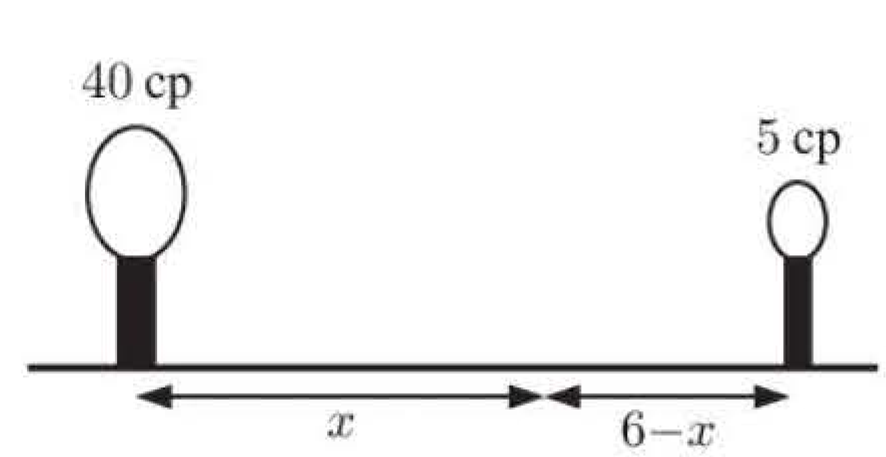
$$\therefore\ x = \tfrac{10}{3} \approx 3.33 \qquad \text{\{as } x \text{ cannot be 10\}}$$

Sign diagram of $\dfrac{dP}{dx}$ is:



$\therefore$ the minimum length pipeline occurs when $x \approx 3.33$ km.

24 Now $I \propto \dfrac{s}{d^2}$ where s is the power of the source and d is the distance from it



$\therefore$ $I = \dfrac{ks}{d^2}$ where k is a constant

So, the intensity due to the 40 cp bulb $= \dfrac{40k}{x^2}$

and the intensity due to the 5 cp bulb $= \dfrac{5k}{(6 - x)^2}$

The total intensity $I = \dfrac{40k}{x^2} + \dfrac{5k}{(6-x)^2}$
$= k[40x^{-2} + 5(6-x)^{-2}]$

$\therefore \dfrac{dI}{dx} = k[-80x^{-3} - 10(6-x)^{-3}(-1)]$
$= k\left[\dfrac{-80}{x^3} + \dfrac{10}{(6-x)^3}\right]$

$\therefore \dfrac{dI}{dx} = 0$ when $\dfrac{80}{x^3} = \dfrac{10}{(6-x)^3}$

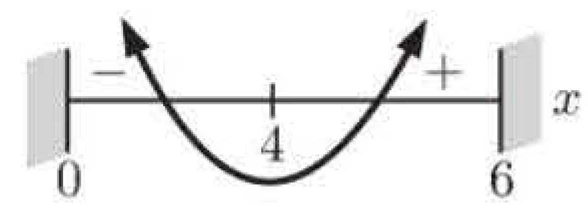
$\therefore 8(6-x)^3 = x^3$

$\therefore 2(6-x) = x$ {finding cube roots}

$\therefore 12 - 2x = x$

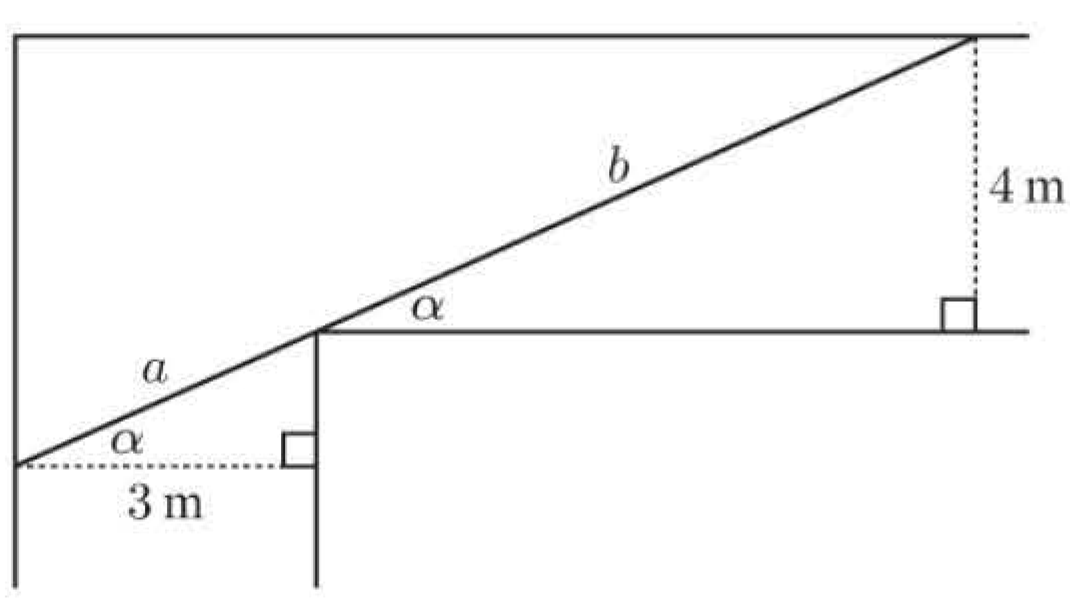
$\therefore x = 4$

Sign diagram of $\dfrac{dI}{dx}$ is:



$\therefore$ the darkest point occurs 4 m from the 40 cp lamp.

25



$\cos\alpha = \dfrac{3}{a}$ and $\sin\alpha = \dfrac{4}{b}$

$\therefore a = \dfrac{3}{\cos\alpha}$ and $b = \dfrac{4}{\sin\alpha}$

Now $L = a + b$

$\therefore L = \dfrac{3}{\cos\alpha} + \dfrac{4}{\sin\alpha}$
$= 3(\cos\alpha)^{-1} + 4(\sin\alpha)^{-1}$

$\therefore \dfrac{dL}{d\alpha} = -3(\cos\alpha)^{-2} \times (-\sin\alpha) - 4(\sin\alpha)^{-2} \times \cos\alpha$
$= \dfrac{3\sin\alpha}{\cos^2\alpha} - \dfrac{4\cos\alpha}{\sin^2\alpha}$
$= \dfrac{3\sin^3\alpha - 4\cos^3\alpha}{\cos^2\alpha\sin^2\alpha}$

$\therefore \dfrac{dL}{d\alpha} = 0$

when $3\sin^3\alpha - 4\cos^3\alpha = 0$

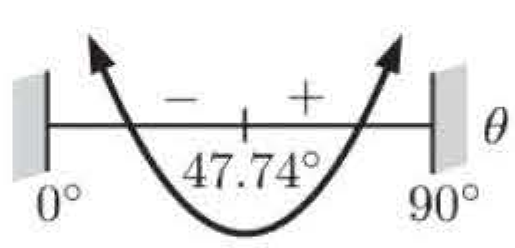
$\therefore 3\sin^3\alpha = 4\cos^3\alpha$

$\therefore \tan^3\alpha = \frac{4}{3}$

$\therefore \tan\alpha = \sqrt[3]{\frac{4}{3}}$

$\therefore \alpha \approx 47.74°$

Sign diagram of $\dfrac{dL}{d\alpha}$ is:



$\therefore$ L is minimised when $\alpha \approx 47.74°$ and $L = \dfrac{3}{\cos\alpha} + \dfrac{4}{\sin\alpha} \approx 9.87$ m

So, the maximum possible length of the X-ray screen is 9.87 m.

26 **a** $\tan\theta = \dfrac{2}{\text{AX}}$ and $\sin\theta = \dfrac{2}{\text{BX}}$

$\therefore \text{AX} = \dfrac{2}{\tan\theta} = \dfrac{2\cos\theta}{\sin\theta}$ and $\text{BX} = \dfrac{2}{\sin\theta}$

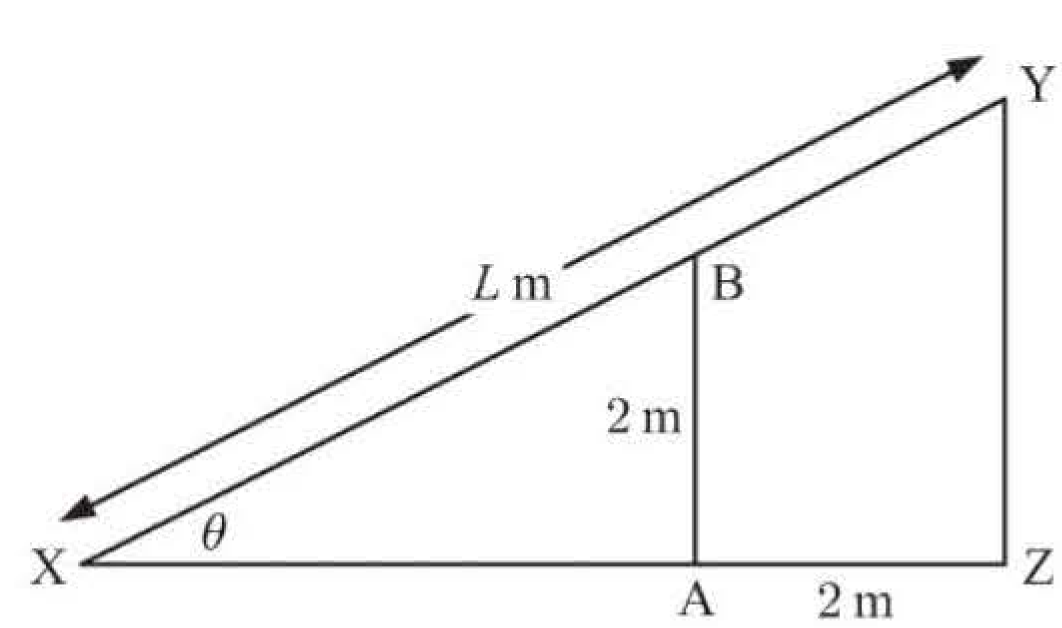
From the similar $\triangle$s, $\dfrac{L}{\text{BX}} = \dfrac{\text{AX}+2}{\text{AX}} = 1 + \dfrac{2}{\text{AX}}$

$\therefore L = \text{BX} + \dfrac{2\text{BX}}{\text{AX}}$

$= \dfrac{2}{\sin\theta} + \dfrac{2\left(\dfrac{2}{\sin\theta}\right)}{\left(\dfrac{2\cos\theta}{\sin\theta}\right)}$

$= \dfrac{2}{\sin\theta} + 2\left(\dfrac{2}{\sin\theta}\right)\left(\dfrac{\sin\theta}{2\cos\theta}\right)$

$= 2\sec\theta + 2\csc\theta$, as required



b Now $L = 2\sec\theta + 2\csc\theta$

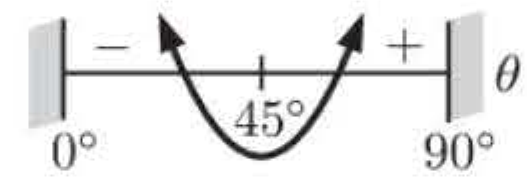
$$\therefore \frac{dL}{d\theta} = 2\sec\theta\tan\theta - 2\csc\theta\cot\theta$$
$$= \frac{2\sin\theta}{\cos^2\theta} - \frac{2\cos\theta}{\sin^2\theta}$$
$$= \frac{2\sin^3\theta - 2\cos^3\theta}{\sin^2\theta\cos^2\theta}$$

c Now $\dfrac{dL}{d\theta} = 0$ when $2\sin^3\theta - 2\cos^3\theta = 0$

$$\therefore 2\sin^3\theta = 2\cos^3\theta$$
$$\therefore \tan^3\theta = 1$$
$$\therefore \tan\theta = 1$$

$\therefore$ since $0 < \theta < 90°$, $\theta = 45°$

Sign diagram of $\dfrac{dL}{d\theta}$ is:

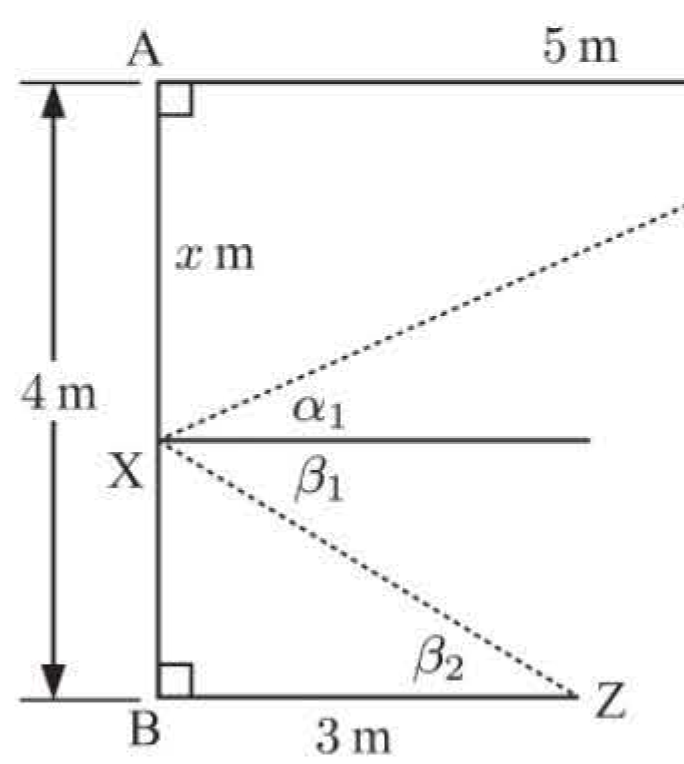


$\therefore$ the ladder is shortest when $\theta = 45°$

$\therefore$ $\sec\theta = \sqrt{2}$ and $\csc\theta = \sqrt{2}$

$\therefore$ $L_{\min} = 2\sqrt{2} + 2\sqrt{2} = 4\sqrt{2}$ m

27



Now $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$ {alternate angles}

$\therefore$ $\theta = \alpha + \beta$

Let AX $= x$ m

$\therefore$ XB $= (4 - x)$ m

$\therefore$ $\tan\alpha = \dfrac{x}{5}$ and $\tan\beta = \dfrac{4-x}{3}$

Now $\tan\theta = \tan(\alpha + \beta)$

$$= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$
$$= \frac{\frac{x}{5} + \frac{4-x}{3}}{1 - \frac{x}{5}\left(\frac{4-x}{3}\right)} \times \frac{15}{15}$$
$$= \frac{3x + 20 - 5x}{15 - x(4-x)} = \frac{20 - 2x}{x^2 - 4x + 15}$$

Differentiating both sides with respect to x

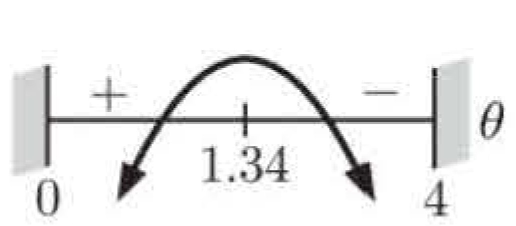
$$\sec^2\theta\,\frac{d\theta}{dx} = \frac{-2(x^2 - 4x + 15) - (20 - 2x)(2x - 4)}{(x^2 - 4x + 15)^2} \quad \text{\{chain and quotient rules\}}$$
$$= \frac{-2x^2 + 8x - 30 - 40x + 80 + 4x^2 - 8x}{(x^2 - 4x + 15)^2}$$
$$= \frac{2x^2 - 40x + 50}{(x^2 - 4x + 15)^2}$$
$$\therefore \frac{d\theta}{dx} = 2\cos^2\theta\,\frac{x^2 - 20x + 25}{(x^2 - 4x + 15)^2}$$

Now by inspection, $\theta < 90°$, so $\dfrac{d\theta}{dx} = 0$ when $x^2 - 20x + 25 = 0$

$\therefore$ $x \approx 1.3397$ or $x \approx 18.660$ (where 18.660 is not physically possible)

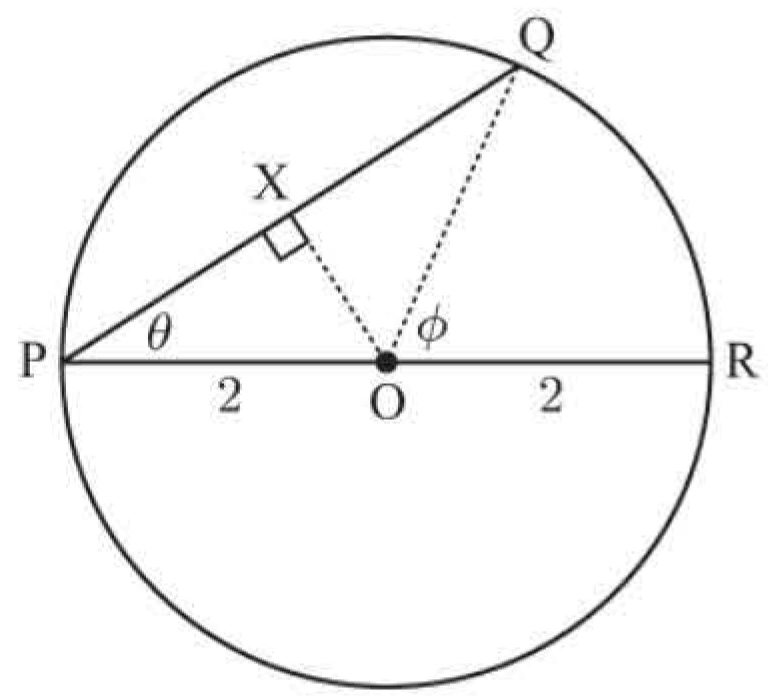
$\therefore$ $x \approx 1.34$ m from A

Sign diagram for $\dfrac{d\theta}{dx}$ is:



$\therefore$ θ is a maximum when $x \approx 1.34$ m from A.

28



$\dfrac{\text{PX}}{2} = \cos\theta \quad \therefore \ \text{PQ} = 2\text{PX} = 4\cos\theta$

$\therefore$ the time taken to row from P to Q is $\dfrac{4\cos\theta}{3}$ hours

Now $\phi = 2\theta$ {angle at the centre}

But, arc length $\text{QR}_{\text{arc}} = 2\phi$

$\therefore \ \text{QR}_{\text{arc}} = 4\theta$

and the time taken to walk from Q to R is $\dfrac{4\theta}{5}$

$\therefore$ the total time from P to R, $T = \frac{4}{3}\cos\theta + \dfrac{4\theta}{5}$

$\therefore \ \dfrac{dT}{d\theta} = -\frac{4}{3}\sin\theta + \frac{4}{5}$

$\therefore \ \dfrac{dT}{d\theta} = 0$ when $-\frac{4}{3}\sin\theta = -\frac{4}{5}$

$\therefore \ \sin\theta = \frac{3}{5}$

$\therefore \ \theta \approx 0.6435$ radians

$\therefore \ \theta \approx 36.87°$

and the sign diagram of $\dfrac{dT}{d\theta}$ is:

So the maximum time occurs when $\theta \approx 36.9°$

and the maximum time is $\frac{4}{3}\cos 0.6435 + \frac{4}{5} \times 0.6435 \approx 1.581$ hours

≈ 1 hour 34 min 53 s

29 **a** $\tan\alpha = \dfrac{2}{x}$ and $\tan(\alpha+\theta) = \dfrac{3}{x}$

b Now $\theta = (\alpha+\theta) - \alpha = \arctan\left(\frac{3}{x}\right) - \arctan\left(\frac{2}{x}\right)$

c $$\frac{d\theta}{dx} = \left(-\frac{3}{x^2}\right) \times \frac{1}{1+\left(\frac{3}{x}\right)^2} - \left(-\frac{2}{x^2}\right) \times \frac{1}{1+\left(\frac{2}{x}\right)^2}$$

$$= -\frac{3}{x^2+9} + \frac{2}{x^2+4}$$

$$= \frac{2}{x^2+4} - \frac{3}{x^2+9} \quad \text{as required}$$

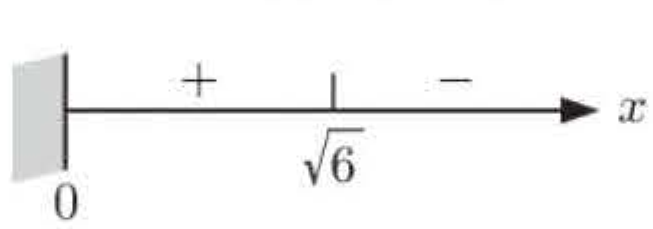
So, $\dfrac{d\theta}{dx} = 0$ when $2(x^2+9) - 3(x^2+4) = 0$

$\therefore \ 2x^2 + 18 - 3x^2 - 12 = 0$

$\therefore \ x^2 = 6$

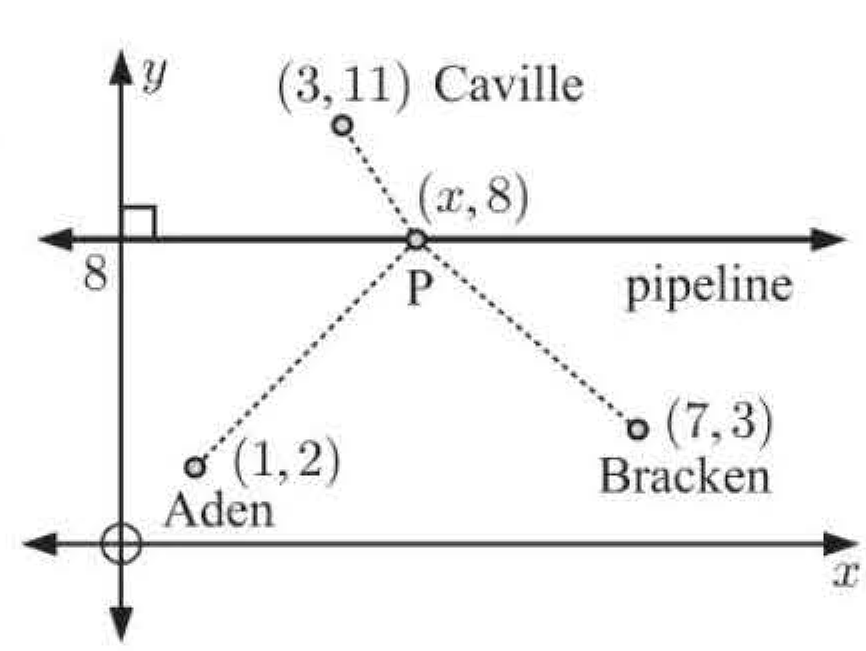
$\therefore \ x = \sqrt{6} \quad \{x > 0\}$

and $\dfrac{d\theta}{dx}$ has sign diagram:



d The maximum viewing angle occurs when $x = \sqrt{6}$, which is when Sonia is $\sqrt{6}$ m from the wall.

30 Suppose P has coordinates $(x, 8)$



$\therefore \ \text{CP} = \sqrt{(x-3)^2 + (8-11)^2}$
$= \sqrt{x^2 - 6x + 9 + 9}$
$= \sqrt{x^2 - 6x + 18}$

$\text{AP} = \sqrt{(x-1)^2 + (8-2)^2}$
$= \sqrt{x^2 - 2x + 1 + 36}$
$= \sqrt{x^2 - 2x + 37}$

$\text{BP} = \sqrt{(x-7)^2 + (8-3)^2}$
$= \sqrt{x^2 - 14x + 49 + 25}$
$= \sqrt{x^2 - 14x + 74}$

$\therefore$ the length of pipeline $L = \sqrt{x^2 - 6x + 18} + \sqrt{x^2 - 2x + 37} + \sqrt{x^2 - 14x + 74}$

We use technology to graph L and to find its minimum. This occurs when $x \approx 3.543\,66$, and $L \approx 15.6$

$\therefore$ P is at $(3.54, 8)$ and the shortest total length of pipe required is 15.6 km.

31 Suppose $\text{PN} = x$ m

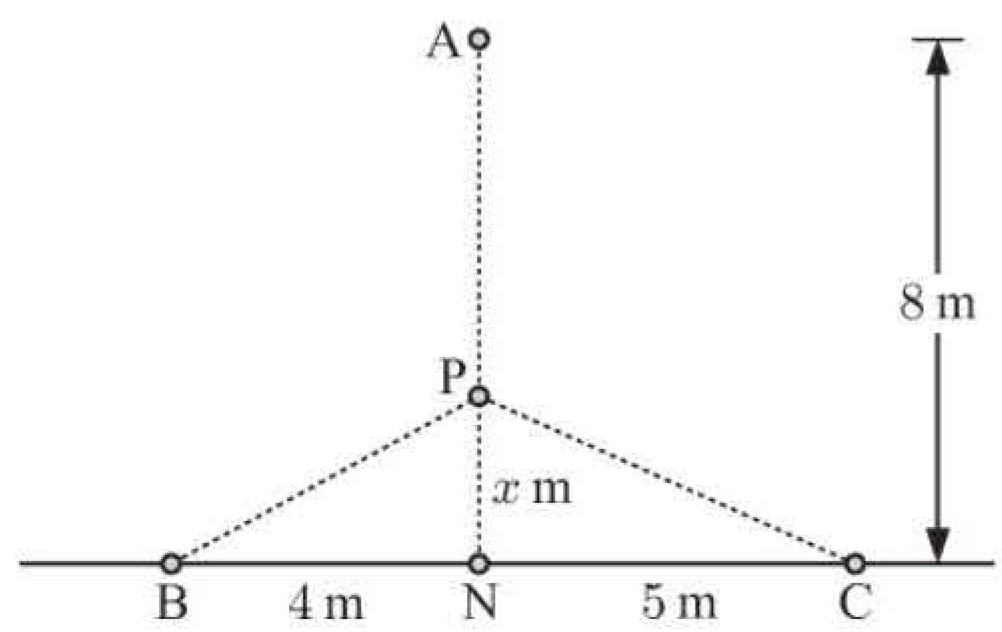
$\therefore$ length of cable $= \text{PA} + \text{PB} + \text{PC}$

$$\therefore \quad L = 8 - x + \sqrt{x^2+16} + \sqrt{x^2+25}$$

Using technology we graph and find the minimum value of the function.

$\therefore$ the minimum length occurs when $x \approx 2.577\,98$

$\therefore$ P should be ≈ 2.58 m from N.



32

$$r^2 + h^2 = s^2 \quad \{\text{Pythagoras}\}$$
$$\therefore \quad h^2 = s^2 - r^2$$
$$\therefore \quad h = \sqrt{s^2 - r^2}$$

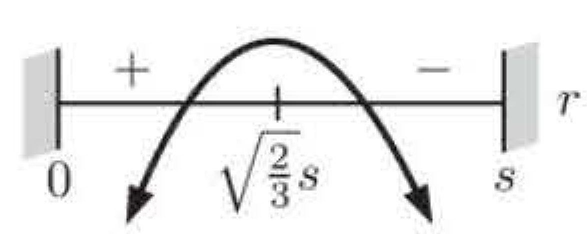
But $V = \frac{1}{3}\pi r^2 h$

$$\therefore \quad V = \tfrac{1}{3}\pi r^2 \sqrt{s^2 - r^2}$$
$$\therefore \quad V^2 = \frac{\pi^2}{9} r^4 (s^2 - r^2)$$
$$= \frac{\pi^2}{9}\left(r^4 s^2 - r^6\right)$$
$$\therefore \quad \frac{d(V^2)}{dr} = \frac{\pi^2}{9}\left(4r^3 s^2 - 6r^5\right)$$
$$= \frac{\pi^2}{9} 2r^3 (2s^2 - 3r^2)$$

$\dfrac{d(V^2)}{dr} = 0$ when $2s^2 - 3r^2 = 0 \quad \{\text{as } r > 0\}$

$$\therefore \quad 2s^2 = 3r^2$$
$$\therefore \quad \frac{s^2}{r^2} = \tfrac{3}{2}$$
$$\therefore \quad \frac{s}{r} = \sqrt{\tfrac{3}{2}}$$
$$\therefore \quad s : r = \sqrt{\tfrac{3}{2}} : 1$$

Sign diagram of $\dfrac{d(V^2)}{dr}$ is:



$\therefore$ V is a maximum when $s : r = \sqrt{\frac{3}{2}} : 1 = \sqrt{3} : \sqrt{2}$

33 **a**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$
$$\therefore \quad x^2 b^2 + y^2 a^2 = a^2 b^2 \quad \{\times \text{ by } a^2 b^2\}$$
$$\therefore \quad a^2 y^2 = a^2 b^2 - x^2 b^2$$
$$\therefore \quad y^2 = \frac{a^2 b^2 - x^2 b^2}{a^2}$$
$$\therefore \quad y = \pm\sqrt{\frac{a^2 b^2 - x^2 b^2}{a^2}}$$

Since A lies in Q_1, $y > 0$

$$\therefore \quad y = \sqrt{\left(\frac{b^2}{a^2}\right)(a^2 - x^2)}$$
$$\therefore \quad y = \frac{b}{a}\sqrt{a^2 - x^2}$$

b The seating area is $A = 2x \times 2y$

$$= 4xy$$
$$= 4x\left(\frac{b}{a}\sqrt{a^2 - x^2}\right)$$
$$\therefore \quad A(x) = \frac{4bx}{a}\sqrt{a^2 - x^2} \quad \text{as required}$$
$$\therefore \quad A^2 = \frac{16b^2 x^2}{a^2}(a^2 - x^2)$$
$$= \frac{16b^2}{a^2}(a^2 x^2 - x^4)$$

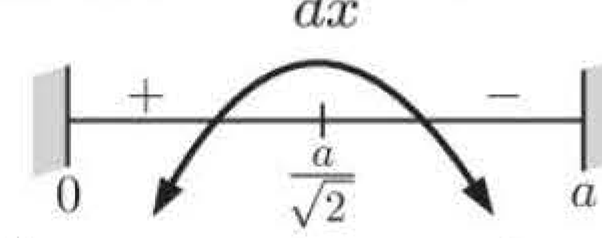
c

$$\frac{d(A^2)}{dx} = \frac{16b^2}{a^2}(2a^2 x - 4x^3)$$

which is 0 when

$$2a^2 x - 4x^3 = 0$$
$$\therefore \quad 2x(a^2 - 2x^2) = 0$$
$$\therefore \quad 2x^2 = a^2 \quad \{\text{as } x > 0\}$$
$$\therefore \quad x = \pm\frac{a}{\sqrt{2}}$$
$$\therefore \quad x = \frac{a}{\sqrt{2}} \quad \{\text{as } x \text{ is in } \text{Q}_1\}$$

Sign diagram of $\dfrac{d(A^2)}{dx}$ is:



$\therefore$ maximum area occurs when $x = \dfrac{a}{\sqrt{2}}$

$$\text{Max. area} = \frac{4b}{a} \times \frac{a}{\sqrt{2}}\sqrt{a^2 - \left(\frac{a}{\sqrt{2}}\right)^2}$$
$$= \frac{4b}{\sqrt{2}} \times \sqrt{\frac{a^2}{2}}$$
$$= \frac{4b}{\sqrt{2}} \times \frac{a}{\sqrt{2}} = 2ab$$

d $\%$ occupied $= \dfrac{2ab}{\pi ab} \times 100\% = 63.7\%$

EXERCISE 20D

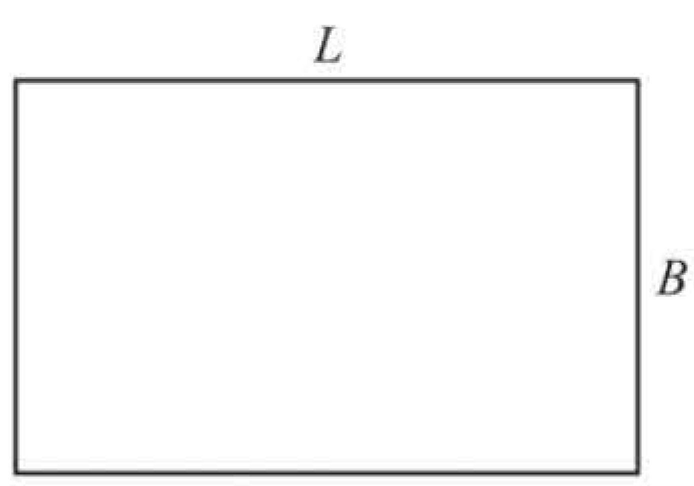
1 $ab^3 = 40 \quad \therefore \dfrac{da}{dt} b^3 + a(3b^2)\dfrac{db}{dt} = 0$

Particular case: When $a = 5$, $b = 2$, and $\dfrac{db}{dt} = +1$, so $8\dfrac{da}{dt} + 5(12)(1) = 0$

$$\therefore \frac{da}{dt} = -\tfrac{60}{8} = -7.5$$

$\therefore$ a decreases at 7.5 units per second

2



$A = LB = 100 \text{ cm}^2 \quad \therefore \dfrac{dL}{dt} B + L\dfrac{dB}{dt} = 0$

Particular case:

When a square, $L = B = 10$ cm and $\dfrac{dL}{dt} = -1$

$$\therefore 10\frac{dB}{dt} + 10(-1) = 0$$

$$\therefore \frac{dB}{dt} = 1 \text{ cm min}^{-1}$$

$\therefore$ the breadth is increasing at 1 cm min^{-1}.

3 $A = \pi r^2 \quad \therefore \dfrac{dA}{dt} = 2\pi r\dfrac{dr}{dt} = 2\pi r \quad \left\{\text{since } \dfrac{dr}{dt} = 1 \text{ m s}^{-1}\right\}$

Particular cases:

a When $t = 2$ and $r = 2$, $\dfrac{dA}{dt} = 2\pi(2) = 4\pi \text{ m}^2$ per second

b When $t = 4$ and $r = 4$, $\dfrac{dA}{dt} = 2\pi(4) = 8\pi \text{ m}^2$ per second

4 $V = \frac{4}{3}\pi r^3 \quad \therefore \dfrac{dV}{dt} = 4\pi r^2\dfrac{dr}{dt} = 6\pi \text{ m}^3 \text{ min}^{-1}$

$$\therefore \frac{dr}{dt} = \frac{6\pi}{4\pi r^2} = \frac{3}{2r^2} \text{ m min}^{-1}$$

Now $A = 4\pi r^2$

$$\therefore \frac{dA}{dt} = 8\pi r\frac{dr}{dt} = 8\pi r \times \frac{3}{2r^2}$$

Particular case: When $r = 2$, $\dfrac{dA}{dt} = \dfrac{8\pi \times 2 \times 3}{2 \times 4} \text{ m}^2 \text{ min}^{-1} = 6\pi \text{ m}^2 \text{ min}^{-1}$

$\therefore$ the surface area is increasing at $6\pi \text{ m}^2$ per minute.

5 $pV^{\frac{3}{2}} = 400 \quad \therefore \dfrac{dp}{dt} V^{\frac{3}{2}} + \frac{3}{2}pV^{\frac{1}{2}}\dfrac{dV}{dt} = 0$

Particular case: When $p = 50 \text{ Nm}^{-2}$, $V^{\frac{3}{2}} = 8$ and so $V = 4$

$$\therefore 3(8) + \tfrac{3}{2}(50)(2)\frac{dV}{dt} = 0 \qquad \left\{\text{as } \frac{dp}{dt} = +3 \text{ Nm}^{-2}\right\}$$

$$\therefore \frac{dV}{dt} = -\tfrac{24}{150} \text{ m}^3 \text{ min}^{-1}$$

$\therefore$ the volume is decreasing at 0.16 m^3 per minute.

6 $V = \frac{1}{3}\pi r^2 h$ and $r = 3h$

$\therefore\ V = \frac{1}{3}\pi(3h)^2 h = 3\pi h^3$ .... $(*)$

Particular case: After 1 min, the volume $V = 3\pi(20)^3\ \text{cm}^3 = 24\,000\pi\ \text{cm}^3$

$$\therefore\ \frac{dV}{dt} = 24\,000\pi\ \text{cm}^3\ \text{min}^{-1}$$

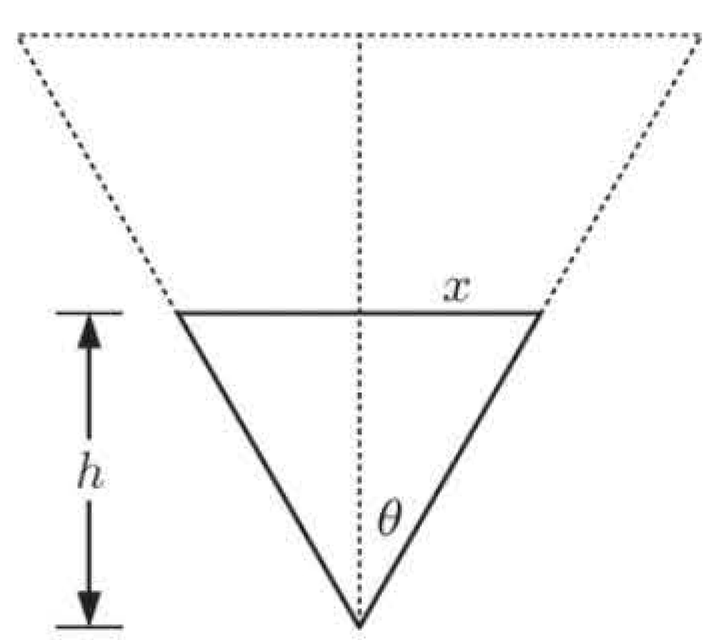
$$\text{But}\ \frac{dV}{dt} = 9\pi h^2\,\frac{dh}{dt} \qquad \{\text{from } (*)\}$$

$$\therefore\ \text{when } h = 20,\quad 24\,000\pi = 9\pi \times (20^2) \times \frac{dh}{dt}$$

$$\therefore\ \frac{dh}{dt} = \frac{24\,000\pi}{400 \times 9\pi} = \tfrac{20}{3}\ \text{cm min}^{-1}$$

$\therefore$ the height is rising at $\frac{20}{3}$ cm per minute.

7



$$\theta = 30^\circ \qquad \therefore\ \frac{x}{h} = \tan 30^\circ$$

$$\therefore\ x = h\tan 30^\circ = \frac{h}{\sqrt{3}}$$

$$\therefore\ V = \frac{h}{\sqrt{3}} \times h \times 600 = 200\sqrt{3}h^2\ \text{cm}^3$$

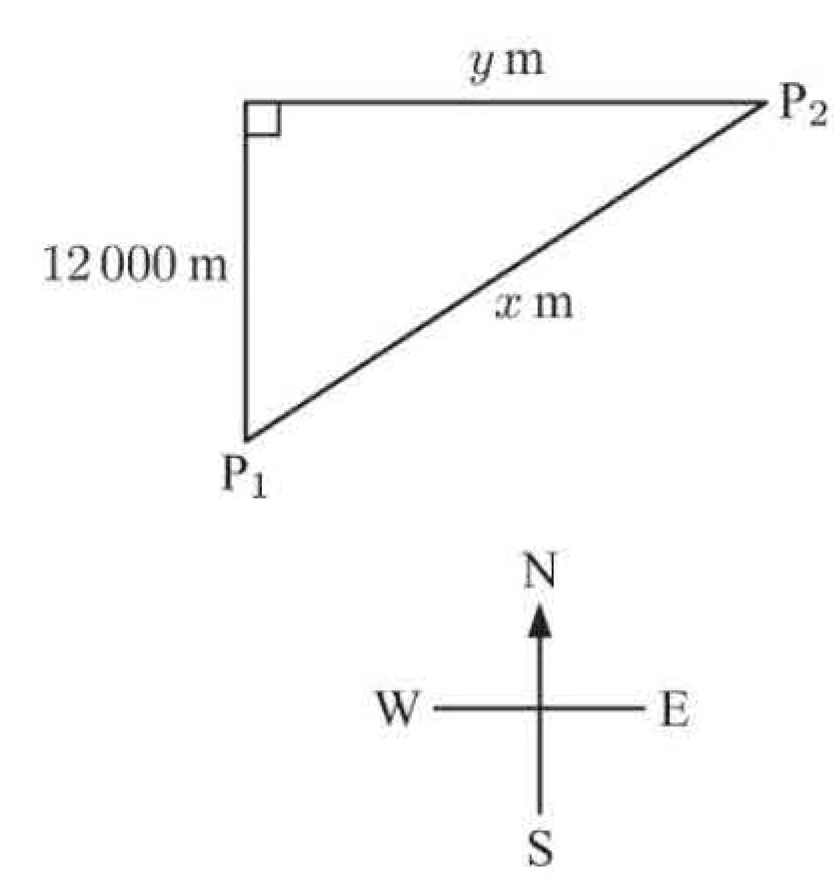
$$\therefore\ \frac{dV}{dt} = 400\sqrt{3}h\,\frac{dh}{dt}$$

Particular case: When $h = 20$, $-100\,000 = 400\sqrt{3}(20)\,\dfrac{dh}{dt}$ $\left\{\dfrac{dV}{dt} = -0.1\ \text{m}^3 = -100\,000\ \text{cm}^3\right\}$

$$\therefore\ \frac{dh}{dt} = \frac{-100\,000}{400\sqrt{3} \times 20} = -\tfrac{25}{6}\sqrt{3}\ \text{cm min}^{-1}$$

$\therefore$ the water level is falling at $\frac{25\sqrt{3}}{6} \approx 7.22$ cm per minute

8 Let P_1 in the diagram be the faster jet and P_2 be the slower jet. Let y m be the distance that P_2 is ahead of P_1, and x m be the distance between them.



Now $x^2 = y^2 + (12\,000)^2$ {Pythagoras}

$$\therefore\ 2x\,\frac{dx}{dt} = 2y\,\frac{dy}{dt}$$

Particular case:

As P_1 is behind P_2, it is catching up at a rate of $50\ \text{m s}^{-1}$.

$$\therefore\ \frac{dy}{dt} = -50\ \text{m s}^{-1}$$

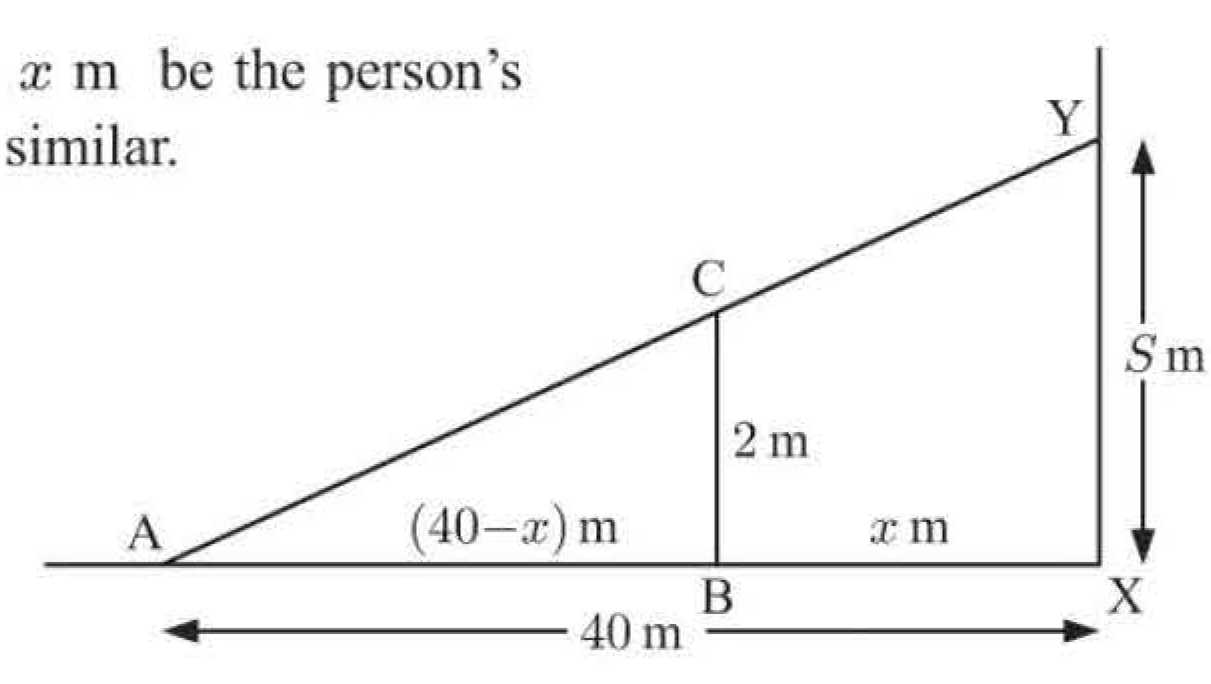
When $y = 5000$, $x = 13\,000$

$$\therefore\ 26\,000 \times \frac{dx}{dt} = 10\,000 \times (-50)$$

$$\frac{dx}{dt} = \tfrac{10}{26} \times (-50) = -\tfrac{250}{13}\ \text{m s}^{-1}$$

$\therefore$ their separation is decreasing at $\frac{250}{13} \approx 19.2\ \text{m s}^{-1}$.

9 Let S m be the height of the person's shadow and x m be the person's distance from the building. $\triangle$ABC and $\triangle$AXY are similar.



$$\therefore\ \frac{AB}{AX} = \frac{BC}{XY}$$

$$\therefore\ \frac{40 - x}{40} = \frac{2}{S}$$

$$\therefore\ S = \frac{80}{40 - x} = 80(40 - x)^{-1}$$

$$\therefore \quad \frac{dS}{dt} = -80(40-x)^{-2}(-1)\frac{dx}{dt} = \frac{80}{(40-x)^2}\frac{dx}{dt}$$

But $\dfrac{dx}{dt} = -1 \text{ m s}^{-1}$, so $\dfrac{dS}{dt} = -\dfrac{80}{(40-x)^2}$

Particular cases:

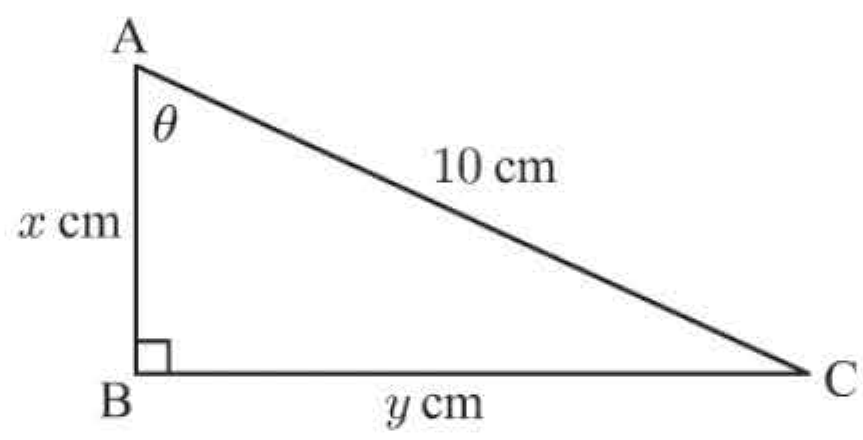
a When $x = 20$ m, $\dfrac{dS}{dt} = -\dfrac{80}{(40-20)^2} = -\frac{80}{400} = -0.2$

$\therefore$ the person's shadow is shortening at 0.2 m s^{-1}

b When $x = 10$ m, $\dfrac{dS}{dt} = -\dfrac{80}{(40-10)^2} = -\frac{80}{900} = -\frac{8}{90}$

$\therefore$ the person's shadow is shortening at $\frac{8}{90} \text{ m s}^{-1}$

10



$$\cos\theta = \frac{x}{10}$$

$$\therefore \quad -\sin\theta\,\frac{d\theta}{dt} = \frac{1}{10}\frac{dx}{dt} \qquad \{\text{differentiating with respect to } t\}$$

If the length AB increases at 0.1 cm s^{-1}, $\dfrac{dx}{dt} = 0.1 \text{ cm s}^{-1}$

Particular case: When ABC is isosceles, $\theta = 45°$

$\therefore \sin\theta = \frac{1}{\sqrt{2}}$

$\therefore -\frac{1}{\sqrt{2}}\dfrac{d\theta}{dt} = \frac{1}{10} \times 0.1$

$\therefore \dfrac{d\theta}{dt} = -\frac{\sqrt{2}}{100}$ radians s^{-1}

$\therefore$ $\widehat{CAB}$ is decreasing at $\frac{\sqrt{2}}{100}$ radians per second.

11

$$\tan E = \frac{5000}{x} = 5000x^{-1}$$

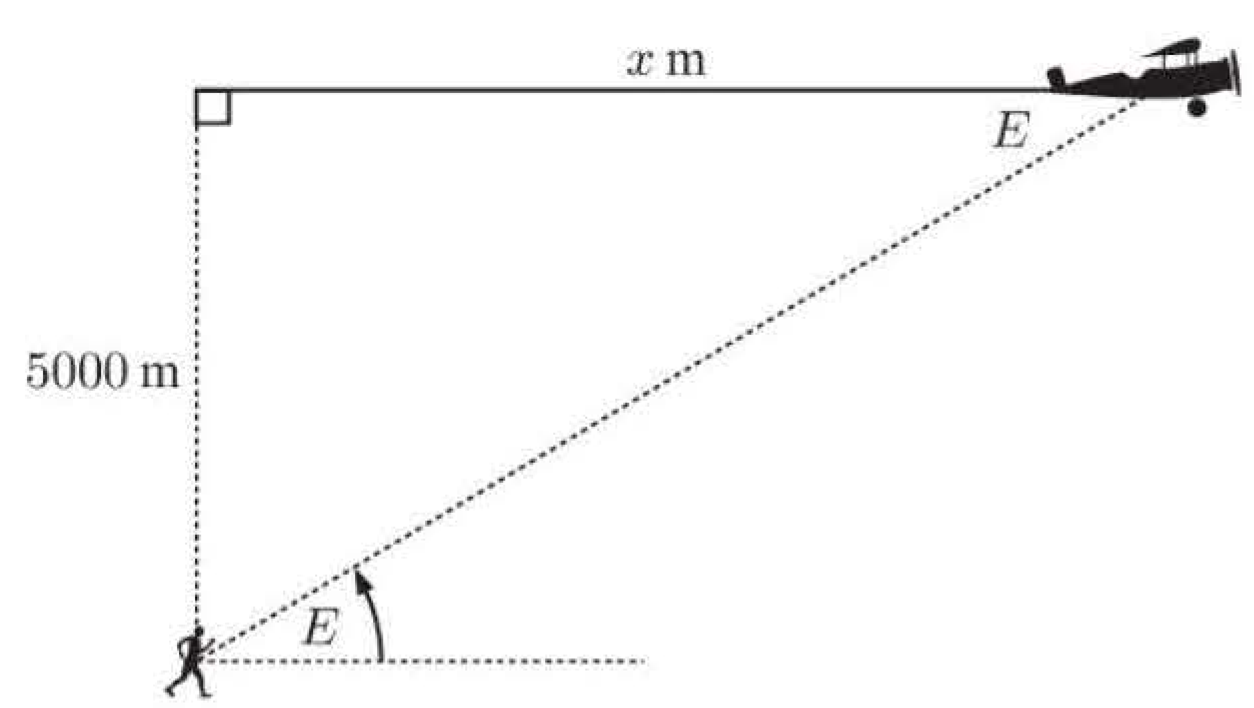
Differentiating with respect to t,

$$\sec^2 E\,\frac{dE}{dt} = -5000x^{-2}\frac{dx}{dt}$$

Now $\dfrac{dx}{dt} = 200 \text{ m s}^{-1}$,

$$\therefore \quad \frac{1}{\cos^2 E}\frac{dE}{dt} = -5000 \times \frac{1}{x^2} \times 200$$

$$\therefore \quad \frac{dE}{dt} = -1\,000\,000 \times \frac{\cos^2 E}{x^2}$$



Particular cases:

a When $E = 60°$, $\cos E = \frac{1}{2}$

and $\tan E = \sqrt{3} = \dfrac{5000}{x}$

$\therefore x = \frac{5000}{\sqrt{3}}$

$$\therefore \quad \frac{dE}{dt} = -1\,000\,000 \times \frac{\left(\frac{1}{2}\right)^2}{\left(\frac{5000}{\sqrt{3}}\right)^2}$$

$= -0.03$

$\therefore$ the angle of elevation is decreasing at 0.03 radians per second

b When $E = 30°$, $\cos E = \frac{\sqrt{3}}{2}$

and $\tan E = \frac{1}{\sqrt{3}} = \dfrac{5000}{x}$

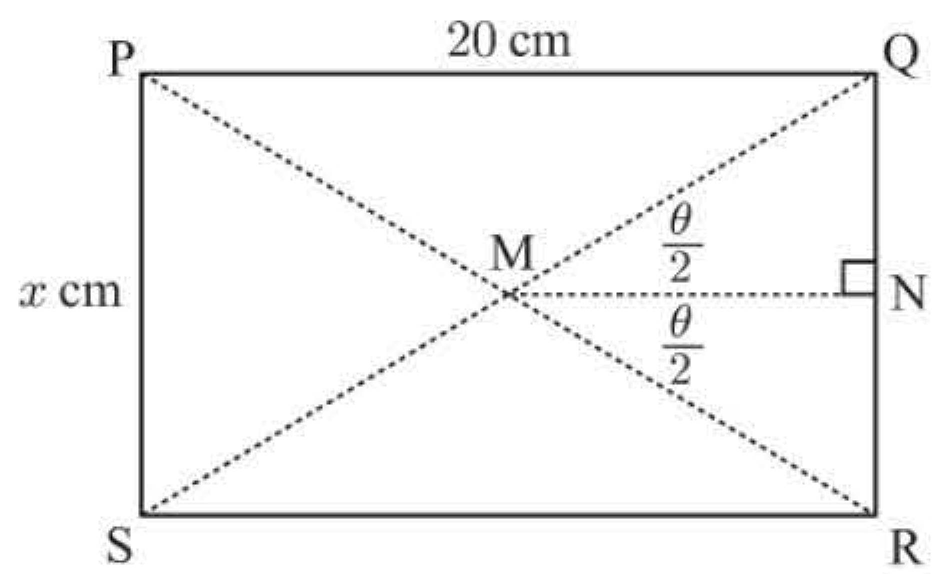
$\therefore x = 5000\sqrt{3}$

$$\therefore \quad \frac{dE}{dt} = -1\,000\,000 \times \frac{\left(\frac{\sqrt{3}}{2}\right)^2}{\left(5000\sqrt{3}\right)^2}$$

$= -0.01$

$\therefore$ the angle of elevation is decreasing at 0.01 radians per second

12



Let N be the midpoint of [QR] in isosceles triangle QMR.

$\therefore$ MN $= 10$ cm

Let QR $= x$ cm and let $Q\widehat{M}R = \theta$.

In triangle MNQ, $\tan\left(\frac{\theta}{2}\right) = \dfrac{\text{QN}}{\text{MN}} = \dfrac{\frac{x}{2}}{10} = \dfrac{x}{20}$

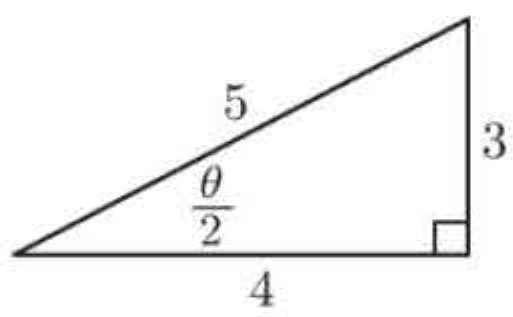
$$\therefore \ \tfrac{1}{2}\sec^2\left(\tfrac{\theta}{2}\right)\frac{d\theta}{dt} = \frac{1}{20}\frac{dx}{dt}$$

$$\therefore \ \frac{d\theta}{dt} = \tfrac{1}{10}\cos^2\left(\tfrac{\theta}{2}\right)\frac{dx}{dt}$$

where $\dfrac{dx}{dt} = 2\ \text{cm s}^{-1}$

Particular case: When $x = 15$ cm, $\tan\left(\frac{\theta}{2}\right) = \frac{15}{20} = \frac{3}{4}$

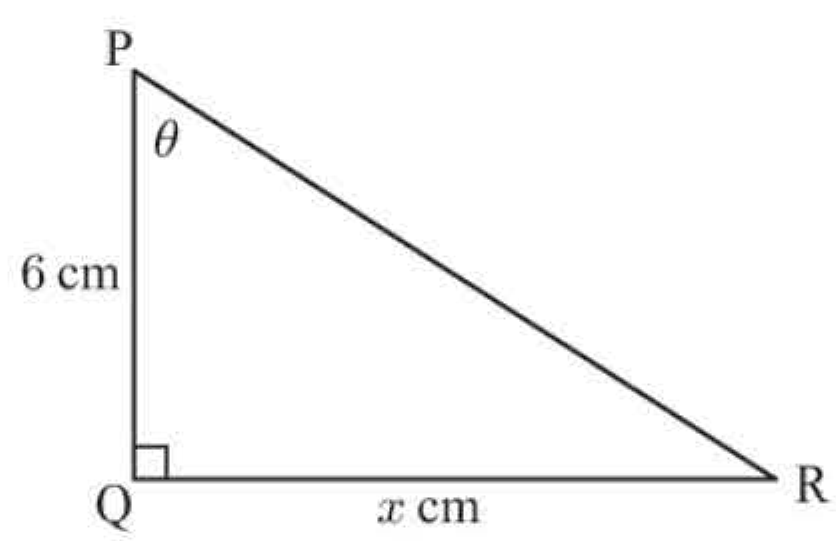
$$\therefore \ \cos\left(\tfrac{\theta}{2}\right) = \tfrac{4}{5}$$



$$\therefore \ \frac{d\theta}{dt} = \tfrac{1}{10}\left(\tfrac{4}{5}\right)^2 2 = 0.128$$

$\therefore$ θ is increasing at 0.128 radians per second

13



Let QR $= x$ cm and the angle at P be θ.

Then $\tan\theta = \dfrac{x}{6}$

$$\therefore \ \sec^2\theta\,\frac{d\theta}{dt} = \tfrac{1}{6}\frac{dx}{dt}$$

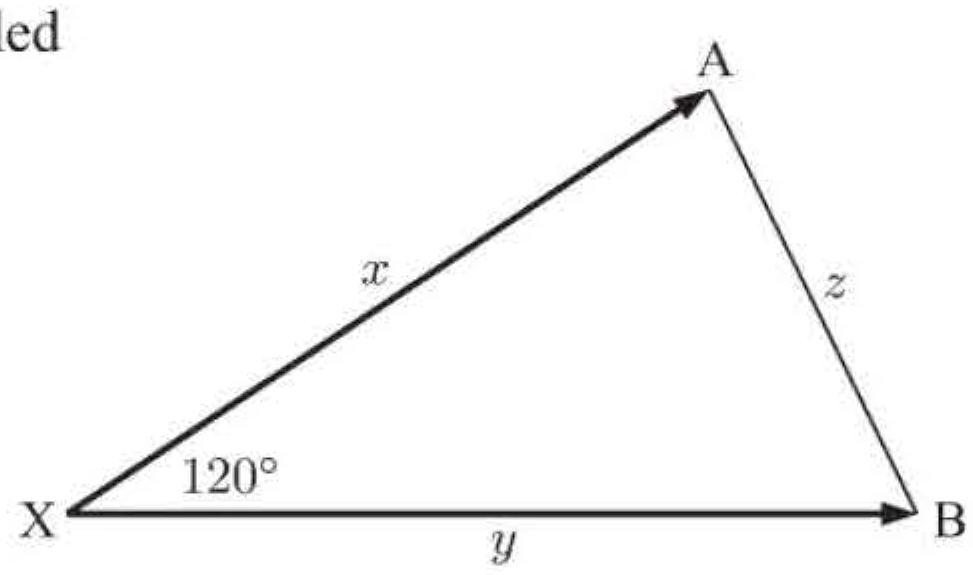
$$\therefore \ \frac{d\theta}{dt} = \frac{\cos^2\theta}{6}\frac{dx}{dt} \quad \text{where} \quad \frac{dx}{dt} = 2\ \text{cm min}^{-1}$$

Particular case: When $x = 8$ cm, PR $= 10$ cm

Now $\cos\theta = \frac{6}{10}$, so $\dfrac{d\theta}{dt} = \left(\frac{6}{10}\right)^2 \times \frac{1}{6} \times 2 = 0.12$

$\therefore$ the angle at P is increasing at a rate of 0.12 radians per minute.

14 Let x and y be the distances the cyclists A and B have travelled respectively at time t, and let z be the distance between them.



So, $z^2 = x^2 + y^2 - 2xy\cos 120°$ {cosine rule}

$\therefore$ $z^2 = x^2 + y^2 + xy$ .... (1)

$$\therefore \ 2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} + \frac{dx}{dt}y + x\frac{dy}{dt} \quad \text{.... (2)}$$

Particular case:

After 2 minutes, $t = 120$ s, $x = 1440$ m, $y = 1920$ m, $\dfrac{dx}{dt} = 12\ \text{m s}^{-1}$, and $\dfrac{dy}{dt} = 16\ \text{m s}^{-1}$.

Using (1), $z^2 = 1440^2 + 1920^2 + 1440 \times 1920 = 8\,524\,800$

$$\therefore \ z = \sqrt{8\,524\,800} = 480\sqrt{37}$$

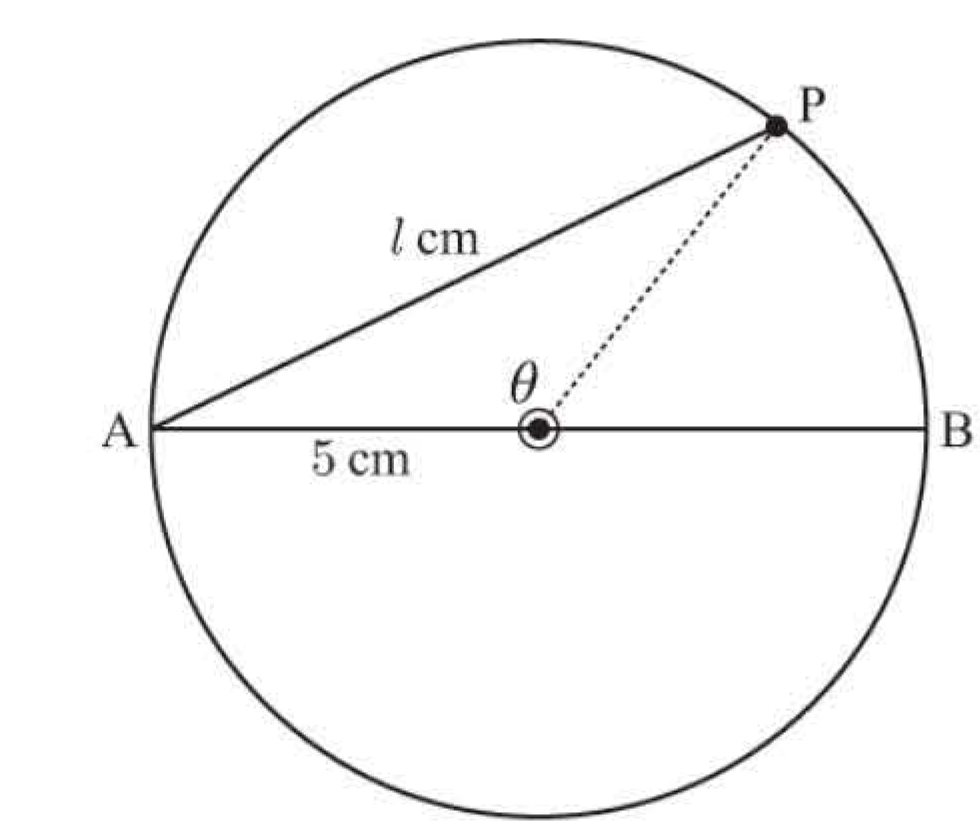
$$\therefore \ 2\left(480\sqrt{37}\right)\frac{dz}{dt} = 2880(12) + 3840(16) + (12)1920 + 1440(16) \qquad \{\text{using (2)}\}$$

$$\therefore \ 960\sqrt{37}\,\frac{dz}{dt} = 142\,080$$

$$\therefore \ \frac{dz}{dt} = \frac{148}{\sqrt{37}} = 4\sqrt{37} \approx 24.33$$

$\therefore$ the distance between the cyclists is increasing at $24.3\ \text{m s}^{-1}$.

15



Let $\text{AP} = l$ cm and let $\text{A}\widehat{\text{O}}\text{P} = \theta$

$\therefore\ l^2 = 5^2 + 5^2 - 2 \times 5 \times 5\cos\theta$ {cosine rule}

$\therefore\ l^2 = 50 - 50\cos\theta$

$\therefore\ 2l\,\dfrac{dl}{dt} = 50\sin\theta\,\dfrac{d\theta}{dt}$

$\therefore\ \dfrac{dl}{dt} = \dfrac{25\sin\theta}{l}\,\dfrac{d\theta}{dt}$

Now the point moves at one revolution every 10 seconds.

$\therefore\ \dfrac{d\theta}{dt} = \frac{2\pi}{10} = \frac{\pi}{5}$ radians per second

Particular cases:

a If $\text{AP} = l = 5$ cm, $\dfrac{dl}{dt} > 0$,

then $\theta = \frac{\pi}{3}$ {$\triangle$APO is equilateral}

$\therefore\ \dfrac{dl}{dt} = \dfrac{25\sin(\frac{\pi}{3})}{5} \times \frac{\pi}{5}$

$= \frac{\sqrt{3}}{2}\pi\ \text{cm s}^{-1}$

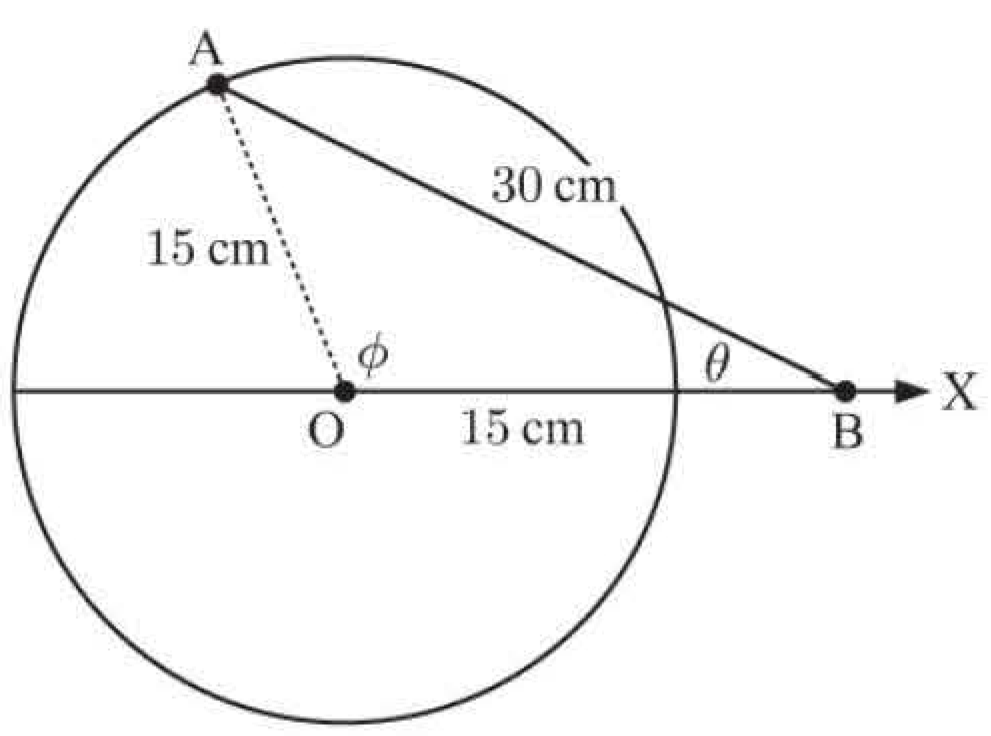
b If P is at B, then $l = 10$ cm

and $\theta = \pi$

$\therefore\ \dfrac{dl}{dt} = \dfrac{25\sin\pi}{10} \times \frac{\pi}{5}$

$= 0\ \text{cm s}^{-1}$

16



Let $\text{A}\widehat{\text{O}}\text{B} = \phi$ and $\text{A}\widehat{\text{B}}\text{O} = \theta$

Now $\dfrac{d\phi}{dt} = -100$ revolutions per second
{negative for clockwise rotation}

$\therefore\ \dfrac{d\phi}{dt} = -200\pi$ radians per second

Also, $\dfrac{30}{\sin\phi} = \dfrac{15}{\sin\theta}$ {sine rule}

$\therefore\ \sin\phi = 2\sin\theta$

$\therefore\ \cos\phi\,\dfrac{d\phi}{dt} = 2\cos\theta\,\dfrac{d\theta}{dt}$

$\therefore\ \dfrac{d\theta}{dt} = \frac{1}{2}\,\dfrac{\cos\phi}{\cos\theta}\,\dfrac{d\phi}{dt}$

where $\dfrac{d\phi}{dt} = -200\pi$ radians s^{-1}

Particular cases:

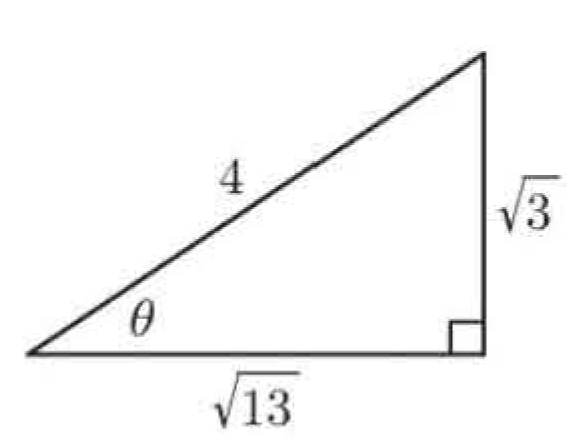
a When $\text{A}\widehat{\text{O}}\text{X} = 120°$, $\phi = \frac{2\pi}{3}$

$\therefore\ \cos\phi = -\frac{1}{2}$ and $\sin\phi = \frac{\sqrt{3}}{2}$

$\therefore\ \sin\theta = \frac{1}{2}\sin\phi$

$= \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)$

$= \frac{\sqrt{3}}{4}$



$\therefore\ \cos\theta = \frac{\sqrt{13}}{4}$

$\therefore\ \dfrac{d\theta}{dt} = \frac{1}{2} \times \dfrac{-\frac{1}{2}}{\frac{\sqrt{13}}{4}} \times (-200\pi) = \frac{200\pi}{\sqrt{13}}$

$\therefore$ $\text{A}\widehat{\text{B}}\text{O}$ is increasing at $\frac{200\pi}{\sqrt{13}}$ radians per second.

b When $\text{A}\widehat{\text{O}}\text{X} = 180°$, $\phi = \pi$,

$\therefore\ \cos\phi = -1$ and $\sin\phi = 0$

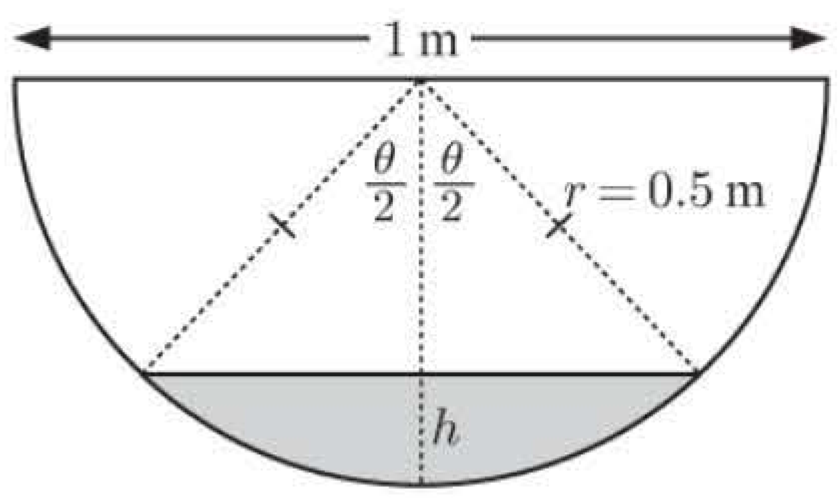
$\therefore\ \sin\theta = 0$ and $\cos\theta = 1$

$\therefore\ \dfrac{d\theta}{dt} = \frac{1}{2} \times \frac{-1}{1} \times (-200\pi)$

$= 100\pi$

$\therefore$ $\text{A}\widehat{\text{B}}\text{O}$ is increasing at 100π radians per second.

17



Denote the radius of the semi-circle $r = \frac{1}{2}$ m.

Let h be the depth and V be the volume of water in the trough at time t.

a The cross-sectional area of water in the trough

$= $ area of sector $-$ area of triangle

$= \frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin\theta$

$= \frac{1}{2}r^2(\theta - \sin\theta)$

$= \frac{1}{8}(\theta - \sin\theta)$

$\therefore$ the volume of water,

$V = $ area of water $\times$ length of trough

$= \frac{1}{8}(\theta - \sin\theta) \times 8$

$= \theta - \sin\theta$ as required

b Now $\dfrac{dV}{dt} = \dfrac{d\theta}{dt} - \cos\theta\,\dfrac{d\theta}{dt}$

$\therefore \dfrac{dV}{dt} = \dfrac{d\theta}{dt}(1 - \cos\theta)$

But $\dfrac{dV}{dt} = 0.1\ \text{m}^3\,\text{min}^{-1}$

$\therefore \dfrac{d\theta}{dt} = \dfrac{0.1}{1 - \cos\theta}$ .... (1)

Also, $\cos\left(\frac{\theta}{2}\right) = \dfrac{\frac{1}{2} - h}{\frac{1}{2}} = 1 - 2h$

Differentiating with respect to t,

$-\sin\left(\frac{\theta}{2}\right) \times \frac{1}{2}\,\dfrac{d\theta}{dt} = -2\,\dfrac{dh}{dt}$

$\therefore \dfrac{dh}{dt} = \frac{1}{4}\sin\left(\frac{\theta}{2}\right)\dfrac{d\theta}{dt}$ .... (2)

Particular case:

When $h = 0.25$ m, $\cos\left(\frac{\theta}{2}\right) = \dfrac{r - h}{r} = \dfrac{\frac{1}{2} - \frac{1}{4}}{\frac{1}{2}} = \dfrac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$

$\therefore \sin\left(\frac{\theta}{2}\right) = \frac{\sqrt{3}}{2}$ and $\cos\theta = 2\cos^2\left(\frac{\theta}{2}\right) - 1 = 2\left(\frac{1}{2}\right)^2 - 1 = -\frac{1}{2}$

Using (1), $\dfrac{d\theta}{dt} = \dfrac{0.1}{1 - \left(-\frac{1}{2}\right)} = \frac{1}{15}$ $\therefore$ θ is increasing at $\frac{1}{15}$ radians per minute.

Using (2), $\dfrac{dh}{dt} = \frac{1}{4}\sin\left(\frac{\theta}{2}\right)\dfrac{d\theta}{dt}$

$= \frac{1}{4} \times \frac{\sqrt{3}}{2} \times \frac{1}{15}$

$= \frac{\sqrt{3}}{120}$ $\therefore$ h is increasing at $\frac{\sqrt{3}}{120}$ metres per minute.

REVIEW SET 20A

1 **a** $x(t) = 3 + \sin(2t)$ cm, $t \geqslant 0$ s

$v(t) = x'(t) = 0 + 2\cos(2t)\ \text{cm s}^{-1}$

$a(t) = v'(t) = -4\sin(2t)\ \text{cm s}^{-2}$

$\therefore x(0) = 3$ cm

$v(0) = 2\ \text{cm s}^{-1}$

$a(0) = 0\ \text{cm s}^{-2}$

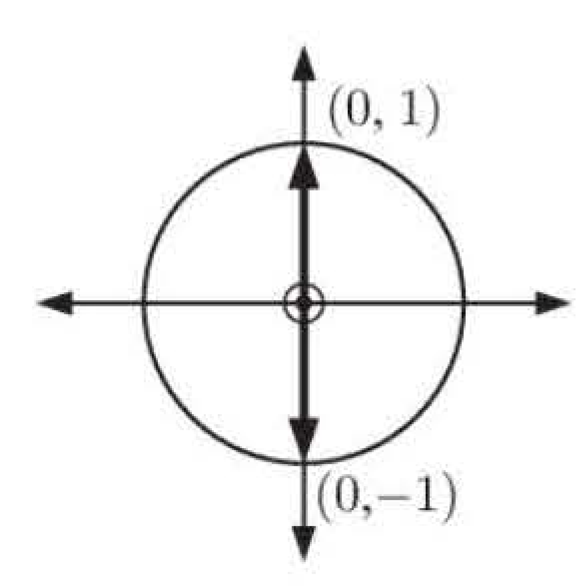
$\therefore$ initially the particle is 3 cm right of O, moving right at a speed of $2\ \text{cm s}^{-1}$.

b $x'(t) = 0$ when $2\cos(2t) = 0$

$\therefore \cos(2t) = 0$

$\therefore 2t = \frac{\pi}{2} + k\pi$

For the interval $0 \leqslant t \leqslant \pi$, $t = \frac{\pi}{4}$ or $\frac{3\pi}{4}$

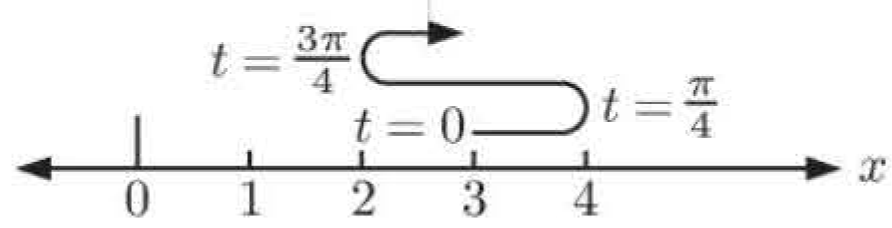


$\therefore$ the particle reverses direction at $t = \frac{\pi}{4}$ s, $\frac{3\pi}{4}$ s.

c $x(0) = 3, \quad x\left(\frac{\pi}{4}\right) = 3 + \sin\left(\frac{\pi}{2}\right) = 4,$

$x\left(\frac{3\pi}{4}\right) = 3 + \sin\left(\frac{3\pi}{2}\right) = 3 - 1 = 2,$

$x(\pi) = 3 + \sin(2\pi) = 3$



$\therefore$ the total distance travelled $= 1 + 2 + 1 = 4$ cm.

2 **a** $s(t) = 2t^3 - 9t^2 + 12t - 5$ cm, $\quad t \geqslant 0$

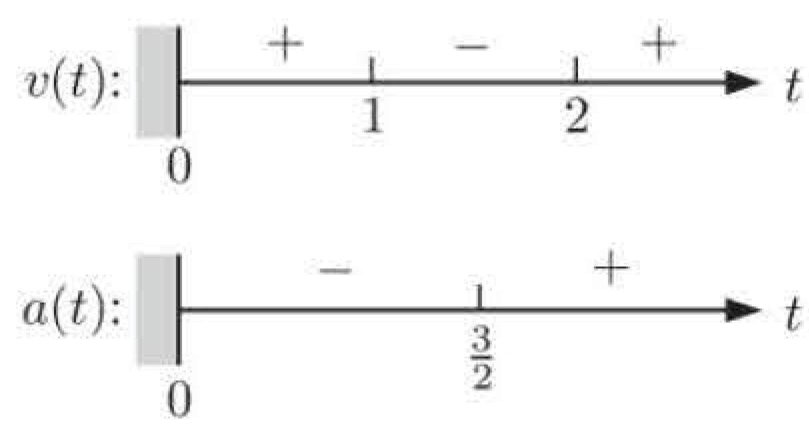
$v(t) = 6t^2 - 18t + 12$

$= 6(t^2 - 3t + 2)$

$= 6(t-2)(t-1)$ cm s^{-1}

and $a(t) = 12t - 18$

$= 6(2t - 3)$ cm s^{-2}



b When $t = 0$, $s(0) = -5$ cm

$v(0) = 12$ cm s^{-1}

$a(0) = -18$ cm s^{-2}

Initially, the particle is 5 cm to the left of O, moving at 12 cm s^{-1} towards the origin and decreasing in speed.

c When $t = 2$, $s(2) = -1$ cm

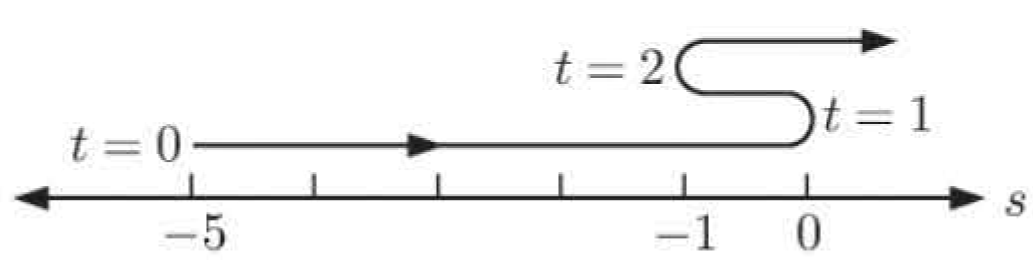
$v(2) = 0$ cm s^{-1}

$a(2) = 6$ cm s^{-2}

When $t = 2$, the particle is 1 cm to the left of O, instantaneously at rest and increasing in speed towards O.

d The particle changes direction when $t = 1$ and $t = 2$, at $s(1) = 0$ cm, $s(2) = -1$ cm.

e



f The speed is increasing when $1 \leqslant t \leqslant \frac{3}{2}$ and $t \geqslant 2$

$\{v(t)$ and $a(t)$ have the same sign$\}$

3 **a** Now if OD $= x$, the coordinates of C are $(x, k - x^2)$.

$\therefore$ the area of ABCD $= 2x \times (k - x^2)$

$\therefore \; A = 2kx - 2x^3, \quad x > 0$

b Now $\dfrac{dA}{dx} = 2k - 6x^2$

But $\dfrac{dA}{dx} = 0$ when AD $= 2\sqrt{3}$, and this occurs when $x = \sqrt{3}$

$\therefore \; 2k - 6(\sqrt{3})^2 = 0$

$\therefore \; 2k - 18 = 0$

$\therefore \; 2k = 18$

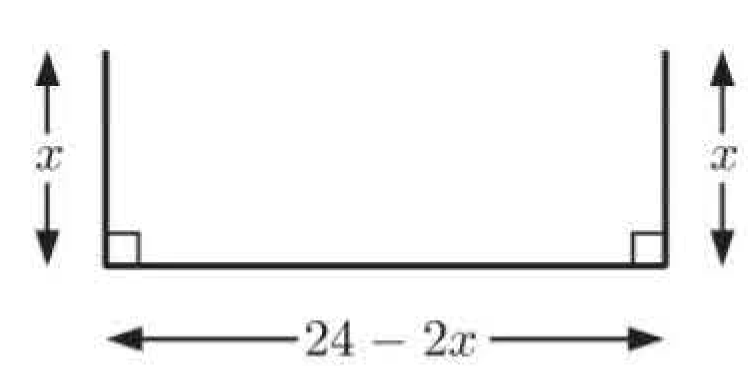
$\therefore \; k = 9$

Check: $\dfrac{dA}{dx} = 18 - 6x^2$

$= 6(3 - x^2)$

$= 6(\sqrt{3} + x)(\sqrt{3} - x)$

4 Suppose the sheet is bent x cm from each end. To maximise the water carried we need to maximise the area of cross-section.

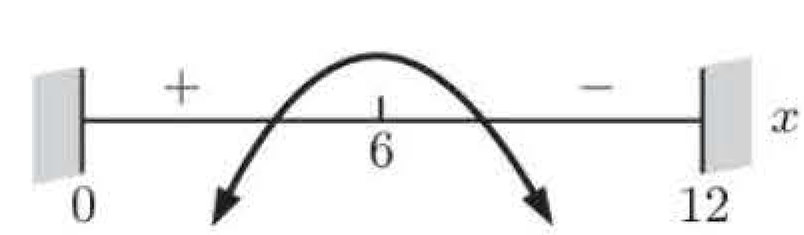


$A = x(24 - 2x), \quad 0 \leqslant x \leqslant 12$

$= 24x - 2x^2$

$\therefore \; \dfrac{dA}{dx} = 24 - 4x$

So, $\dfrac{dA}{dx} = 0$ when $x = 6$, and $\dfrac{dA}{dx}$ has sign diagram:



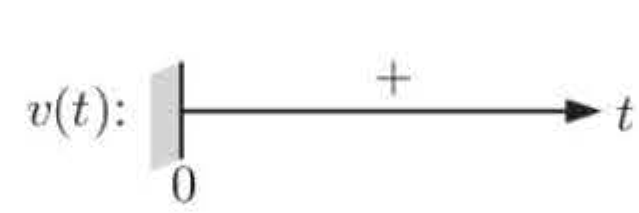
The maximum water is held when $x = 6$ cm

$\therefore$ the bends must be made 6 cm from each end.

5 **a** $s(t) = 2t - \dfrac{4}{t+1} = 2t - 4(t+1)^{-1}$

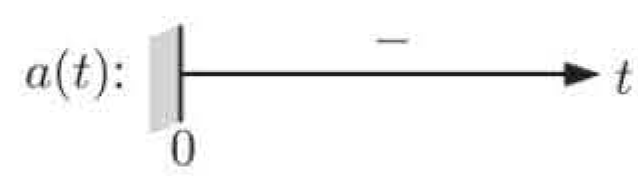
$\therefore \; v(t) = 2 + 4(t+1)^{-2}$

$= 2 + \dfrac{4}{(t+1)^2}$ cm s^{-1}



$\therefore\ a(t) = -8(t+1)^{-3}$

$= -\dfrac{8}{(t+1)^3}\ \text{cm s}^{-2}$



b $s(1) = 2(1) - \dfrac{4}{(1+1)} = 0$ cm

$v(1) = 2 + \dfrac{4}{(1+1)^2} = 3\ \text{cm s}^{-1}$

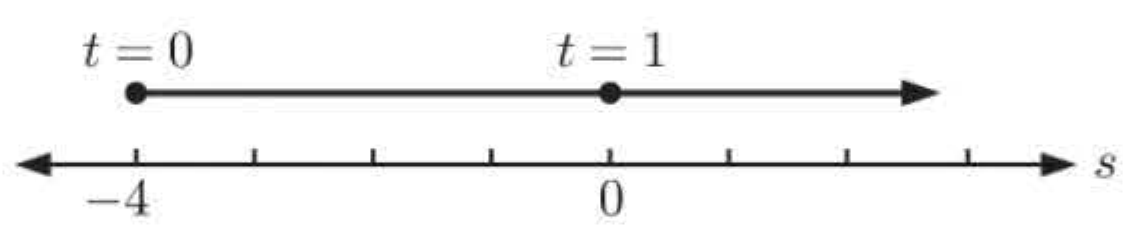
$a(1) = -\dfrac{8}{(1+1)^3} = -1\ \text{cm s}^{-2}$

$\therefore$ the particle is at the origin and is moving to the right with velocity $3\ \text{cm s}^{-1}$ and slowing down, its acceleration being $1\ \text{cm s}^{-2}$ to the left.

c $v(t) = 2 + \dfrac{4}{(t+1)^2} = \dfrac{2(t+1)^2 + 4}{(t+1)^2}$

$\therefore\ v(t) \neq 0$ for any real t, so the particle never changes direction.

d



e **i** The velocity is never increasing {acceleration is negative for all $t > 0$}.

ii The speed is never increasing, as $v(t)$ and $a(t)$ have different signs for all $t > 0$.

6 When the box is manufactured its base is $(2k - 2x)$ by $(k - 2x)$ and its height is x cm.

$\therefore\ V = x(2k - 2x)(k - 2x)$

$\therefore\ V = x(2k^2 - 4kx - 2xk + 4x^2)$

$= 2k^2x - 6kx^2 + 4x^3$

$\therefore\ \dfrac{dV}{dx} = 2k^2 - 12kx + 12x^2$

$= 2(6x^2 - 6kx + k^2)$

So, $\dfrac{dV}{dx} = 0$ when $x = \dfrac{6k \pm \sqrt{36k^2 - 4(6)k^2}}{12}$

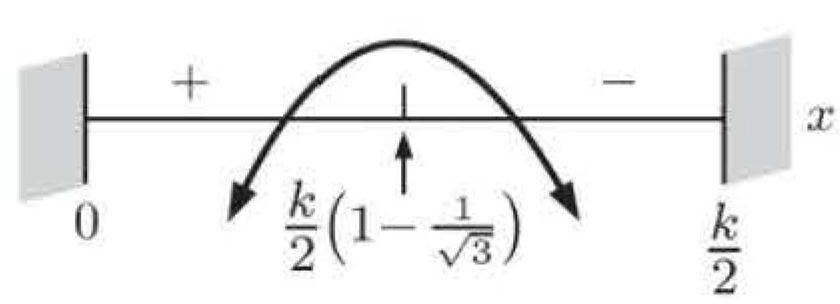
$= \dfrac{6k \pm k\sqrt{12}}{12}$

$= \dfrac{k}{2} \pm \dfrac{k}{\sqrt{12}}$

$= \dfrac{k}{2} - \dfrac{k}{2\sqrt{3}}$ $\quad\{\text{as } x \leqslant \frac{k}{2}\}$

$= \dfrac{k}{2}\left(1 - \frac{1}{\sqrt{3}}\right)$

The sign diagram of $\dfrac{dV}{dx}$ is:



$\therefore$ the maximum capacity occurs when $x = \dfrac{k}{2}\left(1 - \frac{1}{\sqrt{3}}\right)$.

7 **a** $s(t) = 30 + \cos(\pi t)$ cm, $\quad t \geqslant 0$

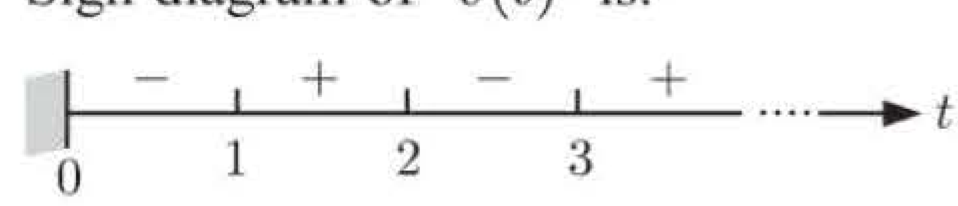
$\therefore\ v(t) = s'(t) = -\pi \sin(\pi t)$

So, $v(0) = 0\ \text{cm s}^{-1}$, $v(\frac{1}{2}) = -\pi\ \text{cm s}^{-1}$,

$v(1) = 0\ \text{cm s}^{-1}$, $v(\frac{3}{2}) = \pi\ \text{cm s}^{-1}$,

$v(2) = 0\ \text{cm s}^{-1}$

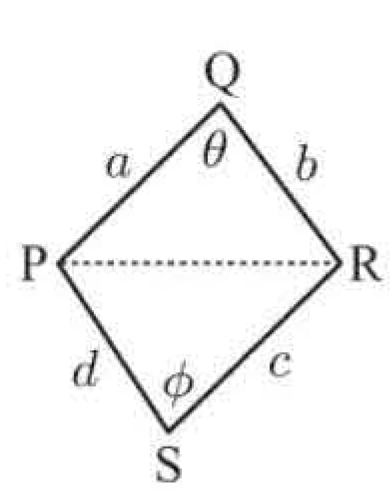
Sign diagram of $v(t)$ is:



b The cork is falling when $v(t) \leqslant 0$, which is for $0 \leqslant t \leqslant 1$, $2 \leqslant t \leqslant 3$, $4 \leqslant t \leqslant 5$, ...

$\therefore$ the cork is falling for $\quad 2n \leqslant t \leqslant 2n+1, \quad n \in \{0, 1, 2, 3, ...\}$

8 **a**



Using the cosine rule: in $\triangle$PQR, $\text{PR}^2 = a^2 + b^2 - 2ab\cos\theta$

in $\triangle$PSR, $\text{PR}^2 = c^2 + d^2 - 2cd\cos\phi$

$\therefore\ a^2 + b^2 - 2ab\cos\theta = c^2 + d^2 - 2cd\cos\phi$

b Now a, b, c, and d are constants, so differentiating with respect to ϕ,

$$\therefore\quad 2ab\sin\theta\,\frac{d\theta}{d\phi} = 2cd\sin\phi$$

$$\therefore\quad \frac{d\theta}{d\phi} = \frac{2cd\sin\phi}{2ab\sin\theta} = \frac{cd\sin\phi}{ab\sin\theta} \quad \text{as required}$$

c Area of quadrilateral, $A =$ area of $\triangle$PQR + area of $\triangle$PSR

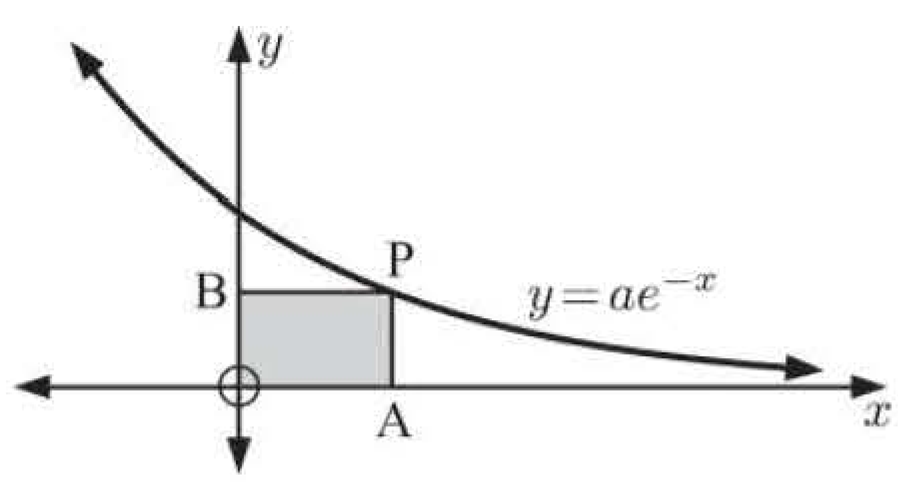
$$\begin{aligned} &= \tfrac{1}{2}ab\sin\theta + \tfrac{1}{2}cd\sin\phi \\ \therefore\quad \frac{dA}{d\phi} &= \tfrac{1}{2}ab\cos\theta\,\frac{d\theta}{d\phi} + \tfrac{1}{2}cd\cos\phi \\ &= \tfrac{1}{2}ab\cos\theta\left(\frac{cd\sin\phi}{ab\sin\theta}\right) + \tfrac{1}{2}cd\cos\phi \quad \{\text{using } \mathbf{a}\} \\ &= \tfrac{1}{2}cd\left[\frac{\cos\theta\sin\phi}{\sin\theta} + \cos\phi\right] \\ &= \frac{cd}{2\sin\theta}(\sin\phi\cos\theta + \cos\phi\sin\theta) \\ &= \frac{cd}{2\sin\theta}\sin(\phi+\theta) \end{aligned}$$

$$\therefore\quad \frac{dA}{d\phi} = 0 \text{ when } \sin(\phi+\theta) = 0, \quad \text{which is when } \phi+\theta = \pi$$

$\therefore$ the area of PQRS is a maximum when the opposite angles are supplementary, which occurs when PQRS is a cyclic quadrilateral.

9 Let the coordinates of A be $(x, 0)$

$\therefore$ the coordinates of P are (x, ae^{-x})



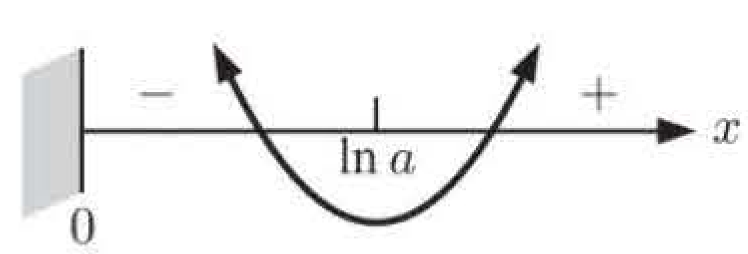
OAPB has perimeter $P = 2(x + ae^{-x})$

$$\begin{aligned} &= 2x + 2ae^{-x} \\ \therefore\quad \frac{dP}{dx} &= 2 - 2ae^{-x} \\ &= 2\left(1 - \frac{a}{e^x}\right) \end{aligned}$$

Now $\dfrac{dP}{dx} = 0$ when $1 = \dfrac{a}{e^x}$

$$\therefore\quad e^x = a$$

$$\therefore\quad x = \ln a$$

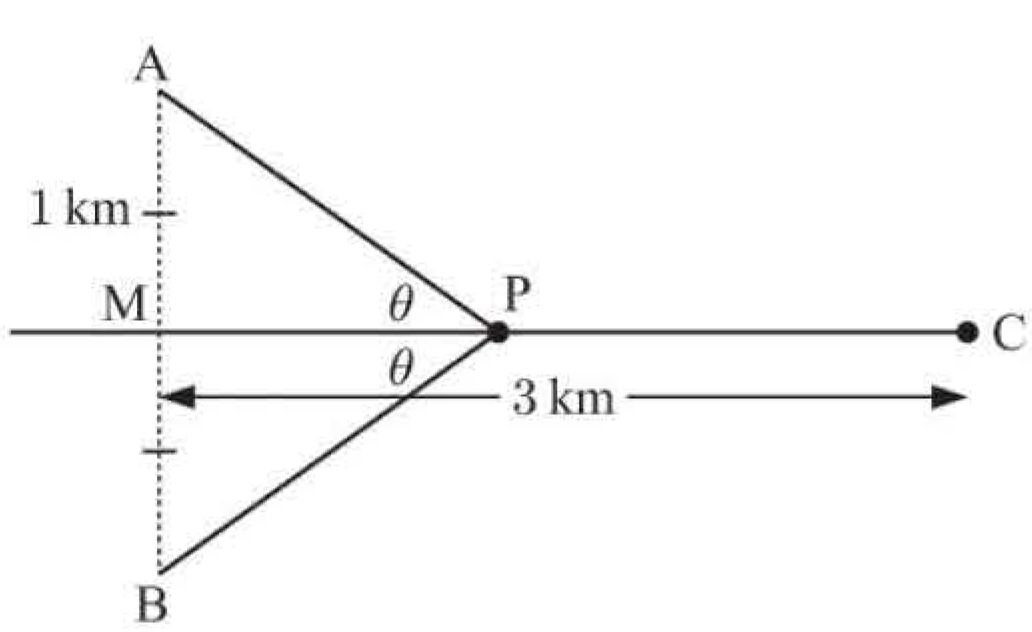


$\therefore$ there is a local minimum when $x = \ln a$

When $x = \ln a$, $y = ae^{-(\ln a)} = \dfrac{a}{e^{\ln a}} = 1$

$\therefore$ rectangle OAPB has minimum perimeter when P is at $(\ln a, 1)$.

10



a Length of cable required = (PA + PB + PC) km

i If P is at M, then PA = PB = 1 km and PC = 3 km
$\therefore$ 5 km of cable is required

ii If P is at C, then
PA = PB = $\sqrt{1^2+3^2} = \sqrt{10}$ km
and PC = 0 km
$\therefore$ $2\sqrt{10}$ km of cable is required

b Now $\sin\theta = \dfrac{1}{\text{AP}} = \dfrac{1}{\text{BP}}$ and $\tan\theta = \dfrac{1}{\text{MP}}$

$$\therefore\quad \text{AP} = \text{BP} = \frac{1}{\sin\theta} \quad \text{and} \quad \text{MP} = \frac{1}{\tan\theta} = \cot\theta$$

$\therefore$ AP + BP + CP $= \dfrac{2}{\sin\theta} + (\text{CM} - \text{MP})$ $\quad\therefore\ L(\theta) = 2\csc\theta + 3 - \cot\theta$ as required

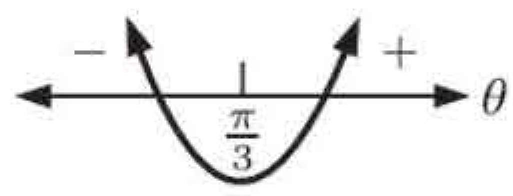
c Since $L(\theta) = 2\csc\theta + 3 - \cot\theta$, $\quad \dfrac{dL}{d\theta} = 2(-\csc\theta\cot\theta) - (-\csc^2\theta)$

$$= \frac{-2\cos\theta}{\sin^2\theta} + \frac{1}{\sin^2\theta}$$

$$= \frac{1-2\cos\theta}{\sin^2\theta} \quad \text{as required}$$

$\therefore \dfrac{dL}{d\theta} = 0$ if $\cos\theta = \frac{1}{2}$

$\therefore \theta = \frac{\pi}{3}$ and sign diagram of $\dfrac{dL}{d\theta}$ is:



$\therefore$ the minimum length of cable is required when $\theta = \frac{\pi}{3}$

When $\theta = \frac{\pi}{3}$, $\quad \sin\theta = \frac{\sqrt{3}}{2}$ and $\tan\theta = \sqrt{3}$,

$\therefore \csc\theta = \frac{2}{\sqrt{3}}$ and $\cot\theta = \frac{1}{\sqrt{3}}$

so $L_{\min} = \frac{4}{\sqrt{3}} + 3 - \frac{1}{\sqrt{3}} = (3+\sqrt{3})$ km as required

REVIEW SET 20B

1 $H(t) = 60 + 40\ln(2t+1)$ cm, $t \geqslant 0$

a When first planted, $t = 0$ $\quad \therefore H(0) = 60 + 40\ln(1) = 60 + 40(0) = 60$ cm.

b **i** When $H(t) = 150$ cm,

$\therefore 60 + 40\ln(2t+1) = 150$

$\therefore 40\ln(2t+1) = 90$

$\therefore \ln(2t+1) = \frac{90}{40} = 2.25$

$\therefore 2t+1 = e^{2.25}$

$\therefore 2t = e^{2.25} - 1$

$\therefore t = \frac{1}{2}(e^{2.25} - 1)$

$\therefore t \approx 4.24$ years

ii When $H(t) = 300$ cm,

$\therefore 60 + 40\ln(2t+1) = 300$

$\therefore 40\ln(2t+1) = 240$

$\therefore \ln(2t+1) = 6$

$\therefore 2t+1 = e^6$

$\therefore 2t = e^6 - 1$

$\therefore t = \frac{1}{2}(e^6 - 1)$

$\therefore t \approx 201$ years

c $H'(t) = 40\left(\dfrac{2}{2t+1}\right) = \dfrac{80}{2t+1}$ cm per year

i When $t = 2$, $\quad H'(2) = \frac{80}{5} = 16$ cm per year

ii When $t = 20$, $\quad H'(20) = \frac{80}{41} \approx 1.95$ cm per year

2 $s(t) = 80e^{-\frac{t}{10}} - 40t$ metres, $t \geqslant 0$

a $v(t) = s'(t) = -8e^{-\frac{t}{10}} - 40$ m s^{-1}

$a(t) = v'(t) = \frac{4}{5}e^{-\frac{t}{10}}$ m s^{-2}

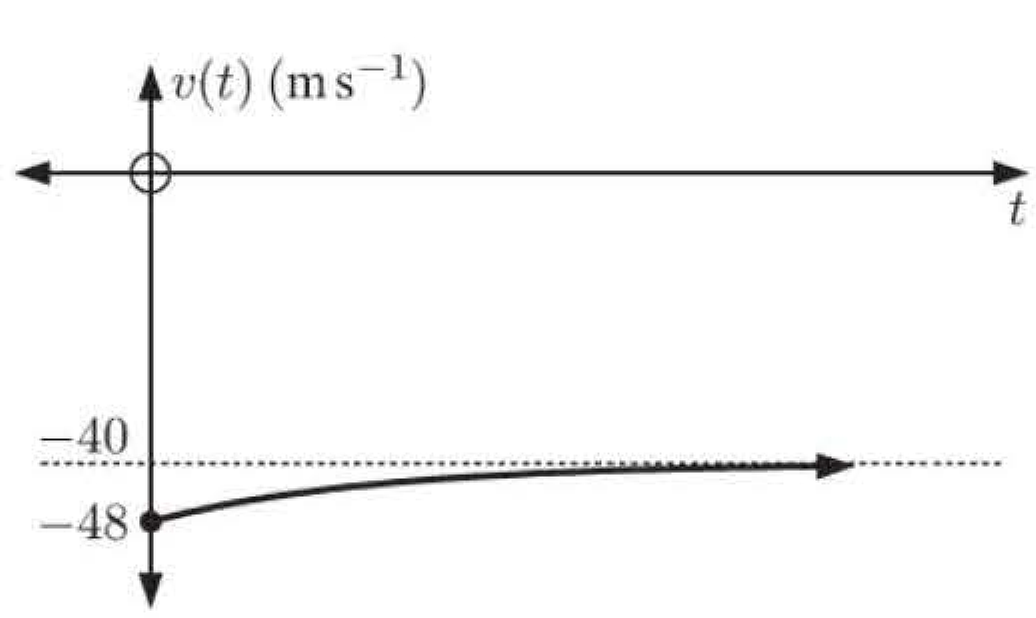
b When $t = 0$, $\quad s(0) = 80$ m

$v(0) = -48$ m s^{-1}

$a(0) = 0.8$ m s^{-2}

c As $t \to \infty$, $e^{-\frac{t}{10}} \to 0$ $\quad \therefore v(t) \to -40$ m s^{-1} (below)

d



e When $v(t) = -44$ m s^{-1}

$\therefore -8e^{-\frac{t}{10}} - 40 = -44$

$\therefore -8e^{-\frac{t}{10}} = -4$

$\therefore e^{-\frac{t}{10}} = 0.5$

$\therefore -\frac{t}{10} = \ln 0.5$

$\therefore t = -10\ln(2^{-1})$

$\therefore t = 10\ln 2$ seconds

3 $C(v) = \dfrac{v^2}{30} + \dfrac{9000}{v}$ dollars per hour

a **i** For $t = 2$ hours at $v = 45\ \text{km h}^{-1}$,

$$\text{cost} = \left(\frac{45^2}{30} + \frac{9000}{45}\right) \times 2 \text{ dollars}$$
$$= \$535.00$$

ii For $t = 5$ hours at $v = 64\ \text{km h}^{-1}$,

$$\text{cost} = \left(\frac{64^2}{30} + \frac{9000}{64}\right) \times 5 \text{ dollars}$$
$$\approx \$1385.79$$

b $C'(v) = \dfrac{2v}{30} - 9000v^{-2} = \dfrac{v}{15} - \dfrac{9000}{v^2}$

i For $v = 50\ \text{km h}^{-1}$

$$\therefore\ C'(50) = \frac{50}{15} - \frac{9000}{50^2}$$
$$\approx -\$0.267 \text{ per km h}^{-1}$$

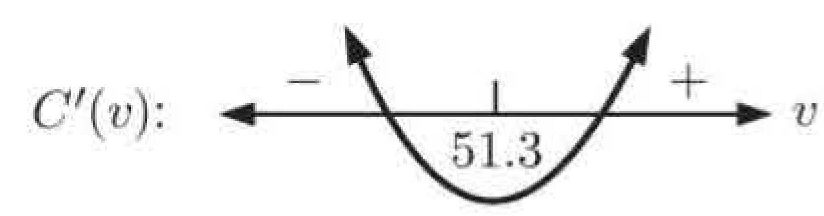
ii For $v = 66\ \text{km h}^{-1}$

$$\therefore\ C'(66) = \frac{66}{15} - \frac{9000}{66^2}$$
$$= \$2.33 \text{ per km h}^{-1}$$

c Now $C'(v) = \dfrac{v}{15} - \dfrac{9000}{v^2} = \dfrac{v^3 - 135\,000}{15v^2}$

$\therefore\ C'(v) = 0$ when $v^3 = 135\,000$

$\therefore\ v \approx 51.3$

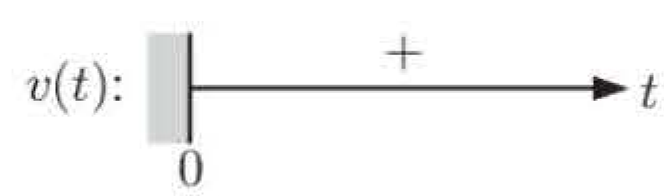


$\therefore$ the minimum cost occurs when $v \approx 51.3\ \text{km h}^{-1}$

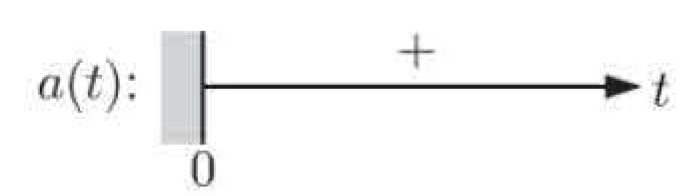
4 **a** $x(t) = 3t - \sqrt{t+1} = 3t - (t+1)^{\frac{1}{2}}$ cm, $t \geqslant 0$

$\therefore\ v(t) = 3 - \frac{1}{2}(t+1)^{-\frac{1}{2}} = 3 - \dfrac{1}{2\sqrt{t+1}}$

since $t \geqslant 0$, $\sqrt{t+1}$ exists and is > 0



and $a(t) = \frac{1}{4}(t+1)^{-\frac{3}{2}} = \dfrac{1}{4(t+1)^{\frac{3}{2}}}$ which is always positive



b $x(0) = 3(0) - \sqrt{0+1} = -1$ cm

$v(0) = 3 - \dfrac{1}{2\sqrt{0+1}} = 2.5\ \text{cm s}^{-1}$

$a(0) = \dfrac{1}{4(0+1)^{\frac{3}{2}}} = 0.25\ \text{cm s}^{-2}$

The particle is 1 cm to the left of the origin, is travelling to the right at $2.5\ \text{cm s}^{-1}$, and accelerating at $0.25\ \text{cm s}^{-2}$.

c $x(8) = 3(8) - \sqrt{8+1} = 21$ cm

$v(8) = 3 - \dfrac{1}{2\sqrt{8+1}} \approx 2.83\ \text{cm s}^{-1}$

$a(8) = \dfrac{1}{4(8+1)^{\frac{3}{2}}} \approx 0.009\,26\ \text{cm s}^{-2}$

The particle is 21 cm to the right of the origin, is travelling to the right at $2.83\ \text{cm s}^{-1}$, and accelerating at $0.009\,26\ \text{cm s}^{-2}$.

d Since $v(t)$ is > 0 for all $t \geqslant 0$, the particle never changes direction.

e $v(t)$ and $a(t)$ have the same sign for all $t \geqslant 0$, so the speed of the particle is always increasing.

$\therefore$ the speed of the particle is never decreasing.

5 **a** At time $t = 0$, $V = 20\,000e^{-0.4\times 0}$

$= 20\,000$ dollars

$\therefore$ the purchase price of the car was \$20 000.

b $V' = -0.4(20\,000)e^{-0.4t}$

$= -8000e^{-0.4t}$

At time $t = 10$, $V' = -8000e^{-0.4\times 10}$

≈ -146.53 dollars year^{-1}

$\therefore$ after 10 years, the car is decreasing in value at \$146.53 per year.

6 $P(x) = I(x) - C(x)$

$$= \left[200\ln\left(1+\frac{x}{100}\right)+1000\right] - \left[(x-100)^2+200\right]$$

$$= 200\ln(1+0.01x) - (x-100)^2 + 800$$

$$\therefore\ \frac{dP}{dx} = 200\left(\frac{0.01}{1+0.01x}\right) - 2(x-100)^1$$

$$= \frac{2}{1+0.01x} - \frac{2(x-100)}{1}$$

$$= \frac{2-2(x-100)(1+0.01x)}{1+0.01x}$$

$$= \frac{2-2(x+0.01x^2-100-x)}{1+0.01x}$$

$$= \frac{2-0.02x^2+200}{1+0.01x}$$

$$= \frac{202-0.02x^2}{1+0.01x}$$

$\therefore\ \dfrac{dP}{dx} = 0$ when $0.02x^2 = 202$

$\therefore\ x^2 = 10\,100$

$\therefore\ x = \sqrt{10\,100}$ $\{x > 0\}$

$\therefore\ x \approx 100.49$

and the sign diagram of $\dfrac{dP}{dx}$ is:

$\therefore$ the maximum profit occurs when $x \approx 100.49$

Now $P(100) \approx \$938.63$ and $P(101) \approx \$938.63$

$\therefore$ the maximum daily profit is \$938.63 when 100 or 101 shirts are made.

7 **a** $P = 200$ m

But $P = 2x + 2y + \pi x$

$\therefore\ 200 = 2x + 2y + \pi x$

$\therefore\ 2y = 200 - 2x - \pi x$

$\therefore\ y = 100 - x - \frac{\pi}{2}x$

b Area of lawn $= 2x \times y + \frac{1}{2}\pi x^2$

$= 2x(100 - x - \frac{\pi}{2}x) + \frac{1}{2}\pi x^2$

$= 200x - 2x^2 - \pi x^2 + \frac{1}{2}\pi x^2$

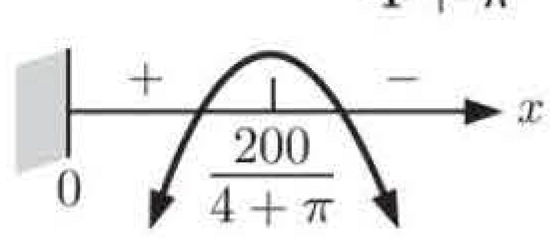
$\therefore\ A = 200x - 2x^2 - \frac{1}{2}\pi x^2$

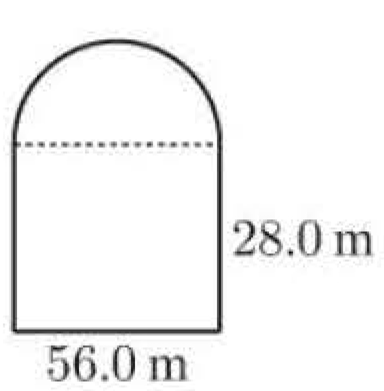
c $A = 200x - 2x^2 - \frac{1}{2}\pi x^2 = 200x - (2+\frac{\pi}{2})x^2$ m^2

$\therefore\ \dfrac{dA}{dx} = 200 - 2\left(2+\frac{\pi}{2}\right)x = 200 - (4+\pi)x$

$\therefore\ \dfrac{dA}{dx} = 0$ when $(4+\pi)x = 200$ $\quad\therefore\ x = \dfrac{200}{4+\pi}$

and the sign diagram for $\dfrac{dA}{dx}$ is:





$\therefore$ the maximum area occurs when $x = \dfrac{200}{4+\pi} \approx 28.0$ m

and $y = 100 - x - \frac{\pi}{2}x \approx 28.0$ m

8 **a** $\triangle$s LQX and XPM are similar.

$$\therefore\ \frac{\text{LQ}}{\text{XP}} = \frac{\text{LX}}{\text{XM}} = \frac{\text{QX}}{\text{PM}}$$

$$\therefore\ \frac{\text{LQ}}{1} = \frac{8}{\text{PM}}$$

$$\therefore\ \text{LQ} = \frac{8}{\text{PM}}$$

$$\therefore\ \text{LQ} = \frac{8}{x}\text{ km}$$

b $L^2 = (\text{LQ}+\text{QB})^2 + (\text{BP}+\text{PM})^2$

$$= \left(\frac{8}{x}+1\right)^2 + (8+x)^2$$

$$= \left(\frac{8}{x}+1\right)^2 + \left[x\left(\frac{8}{x}+1\right)\right]^2$$

$$= \left(\frac{8}{x}+1\right)^2 + x^2\left(\frac{8}{x}+1\right)^2$$

$$= \left(\frac{8}{x}+1\right)^2(1+x^2)$$

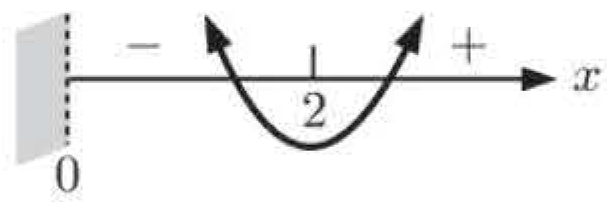
$$= (x^2+1)\left(\frac{8}{x}+1\right)^2$$

$$\therefore\ L(x) = \sqrt{x^2+1}\left(\frac{8}{x}+1\right)$$

c $$\begin{aligned}\frac{d[L(x)]^2}{dx} &= 2x\left(1+\frac{8}{x}\right)^2 + (x^2+1)2\left(1+\frac{8}{x}\right)\left(-\frac{8}{x^2}\right) \quad \{\text{product rule}\}\\ &= 2\left(1+\frac{8}{x}\right)\left[x\left(1+\frac{8}{x}\right) - (x^2+1)\left(\frac{8}{x^2}\right)\right]\\ &= 2\left(1+\frac{8}{x}\right)\left[x+8-8-\frac{8}{x^2}\right]\\ &= 2\left(\frac{x+8}{x}\right)\left(\frac{x^3-8}{x^2}\right)\end{aligned}$$

d $\dfrac{d[L(x)]^2}{dx} = 0$ when $x=-8$ or $x^3=8$, but $x>0$ so $\dfrac{d[L(x)]^2}{dx}=0$ when $x=2$

The sign diagram for $\dfrac{d[L(x)]^2}{dx}$ is:



$\therefore$ minimum $L(x)$ occurs when $x=2$ and the shortest length is $\sqrt{2^2+1}\left(1+\frac{8}{2}\right) = 5\sqrt{5}$
≈ 11.2 km

9 **a** $(3\cos\theta, 2\sin\theta)$ lies on the curve

$\therefore$ $x = 3\cos\theta$ and $y = 2\sin\theta$

$\therefore$ $x^2 = 9\cos^2\theta$ and $y^2 = 4\sin^2\theta$

$\therefore$ $\dfrac{x^2}{9}+\dfrac{y^2}{4} = \cos^2\theta + \sin^2\theta = 1$

$\therefore$ the curve has equation $\dfrac{x^2}{9}+\dfrac{y^2}{4}=1$

b $$\frac{x^2}{9}+\frac{y^2}{4}=1$$

$$\therefore \quad \frac{2x}{9}+\frac{2y}{4}\frac{dy}{dx}=0$$

$$\therefore \quad \frac{dy}{dx} = \left(\frac{2}{y}\right)\left(-\frac{2x}{9}\right) = -\frac{4x}{9y} = -\frac{4\times 3\cos\theta}{9\times 2\sin\theta} = -\frac{2\cos\theta}{3\sin\theta}$$

c The tangent has gradient $-\dfrac{2\cos\theta}{3\sin\theta}$ and passes through $(3\cos\theta, 2\sin\theta)$.

$\therefore$ the tangent has equation $\dfrac{y-2\sin\theta}{x-3\cos\theta} = -\dfrac{2\cos\theta}{3\sin\theta}$

$$\therefore \quad 3y\sin\theta - 6\sin^2\theta = -2x\cos\theta + 6\cos^2\theta$$

$$\therefore \quad 2x\cos\theta + 3y\sin\theta = 6 \qquad \{\sin^2\theta+\cos^2\theta=1\}$$

The tangent meets the x-axis when $y=0$

$\therefore$ $2x\cos\theta = 6$

$\therefore$ $x = \dfrac{3}{\cos\theta}$

$\therefore$ A is at $\left(\dfrac{3}{\cos\theta}, 0\right)$

The tangent meets the y-axis when $x=0$

$\therefore$ $3y\sin\theta = 6$

$\therefore$ $y=\dfrac{2}{\sin\theta}$

$\therefore$ B is at $\left(0, \dfrac{2}{\sin\theta}\right)$

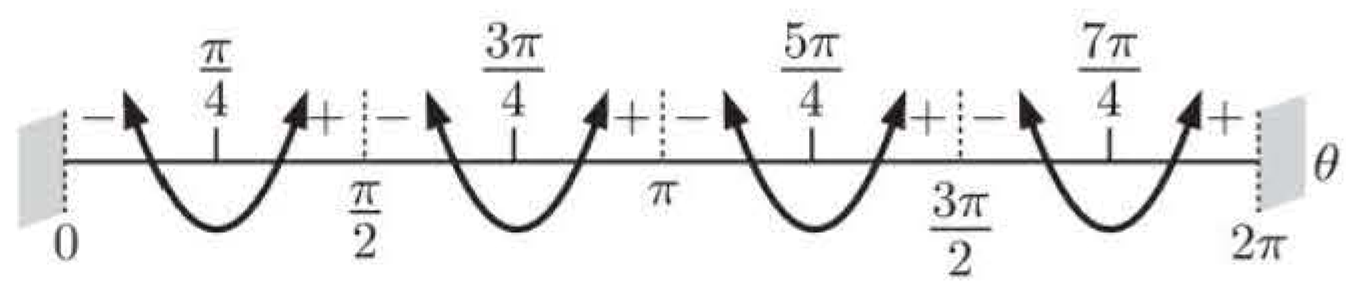
$\therefore$ triangle OAB has area $A = \left|\frac{1}{2}\left(\dfrac{3}{\cos\theta}\right)\left(\dfrac{2}{\sin\theta}\right)\right| = \left|\dfrac{6}{\sin 2\theta}\right|$ .... $(*)$

$$\therefore \quad A^2 = 36(\sin 2\theta)^{-2}$$

$$\therefore \quad \frac{d(A^2)}{d\theta} = -72(\sin 2\theta)^{-3}\times 2\cos 2\theta = -\frac{144\cos 2\theta}{\sin^3 2\theta}$$

Since $0 \leqslant \theta \leqslant 2\pi$, $0 \leqslant 2\theta \leqslant 4\pi$

$\therefore$ $\dfrac{d(A^2)}{d\theta} = 0$ when $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$, and $\frac{7\pi}{4}$,
and is undefined when $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, and 2π.

Sign diagram for $\dfrac{d(A^2)}{d\theta}$ is: 

$\therefore$ there are local minima at $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

For all of these values of θ, $\sin 2\theta = -1$ or 1

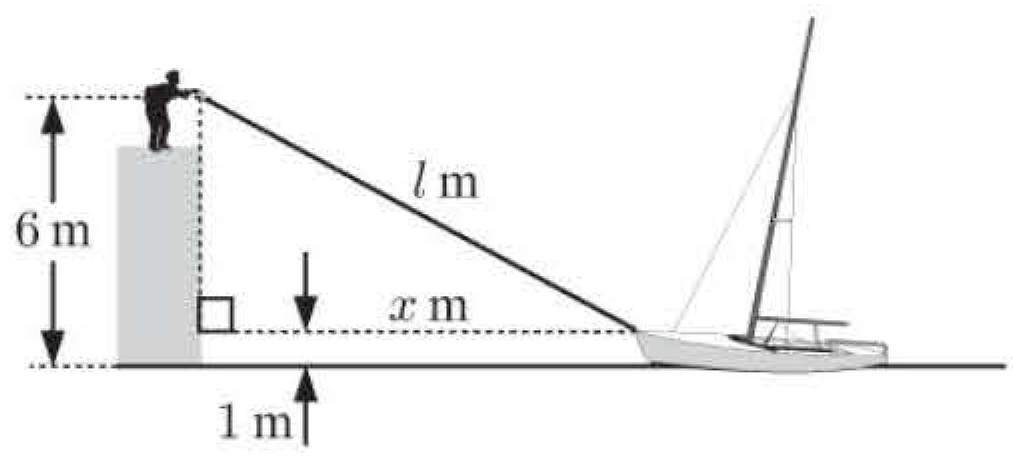
$$\therefore \quad A = 6 \quad \{\text{using } (*)\}$$

$\therefore$ the smallest area of triangle OAB is 6 units^2, and this occurs when $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$, or $\frac{7\pi}{4}$.

10 Let l m be the length of rope and x m be the distance of the boat from the jetty.

Then $x^2 + 5^2 = l^2$

$$\therefore \quad 2x \frac{dx}{dt} = 2l \frac{dl}{dt}$$



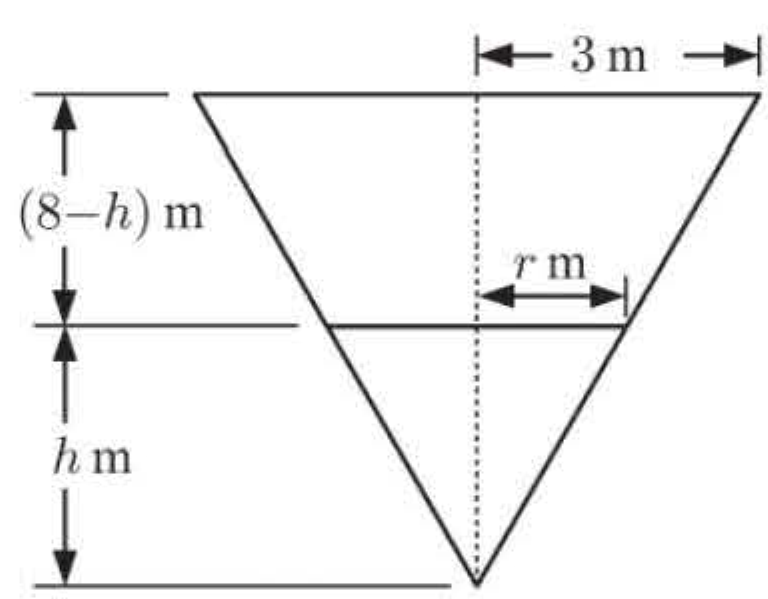
Particular case: When $x = 15$ m, $l = \sqrt{15^2 + 5^2} = \sqrt{250}$ and $\dfrac{dl}{dt} = -20$ m min^{-1}

$$\therefore \quad 2(15) \frac{dx}{dt} = 2\sqrt{250}(-20)$$

$$\therefore \quad \frac{dx}{dt} = -\frac{40\sqrt{250}}{30} \approx -21.1$$

$\therefore$ the boat is approaching the jetty at 21.1 metres per minute.

11



a Volume $V = \frac{1}{3}\pi r^2 h$

Using similar triangles $\dfrac{h}{r} = \frac{8}{3}$

$$\therefore \quad h = \frac{8r}{3}$$

$$\therefore \quad V(r) = \tfrac{1}{3}\pi r^2 \left(\frac{8r}{3}\right) = \tfrac{8\pi}{9} r^3 \ \text{m}^3$$

b *Particular case*: When $h = 5$, $r = \dfrac{3h}{8} = \frac{15}{8}$ and $\dfrac{dV}{dt} = -0.2 = -\frac{1}{5}$ m^3 min^{-1}

Now $\dfrac{dV}{dt} = \frac{8\pi}{3} r^2 \dfrac{dr}{dt}$ $\quad \therefore \quad -\frac{1}{5} = \frac{8\pi}{3}\left(\frac{15}{8}\right)^2 \dfrac{dr}{dt}$

$$\therefore \quad -\tfrac{1}{5} = \tfrac{225}{24}\pi \frac{dr}{dt}$$

$$\therefore \quad \frac{dr}{dt} = -\tfrac{8}{375\pi}$$

$$\therefore \quad \frac{dr}{dt} = -0.006\,79$$

$\therefore$ the radius is decreasing at 0.006 79 m per minute.

REVIEW SET 20C

1 **a** Volume = length $\times$ width $\times$ depth

$$\therefore \quad x^2 y = 1$$

$$\therefore \quad y = \frac{1}{x^2}, \quad x > 0$$

b area $= x^2 + 4xy$

$$\therefore \quad \text{cost} = (x^2 + 4xy) \times 2$$

$$\therefore \quad C = 2x^2 + 8xy$$

$$= 2x^2 + \frac{8}{x} \text{ dollars} \qquad \{\text{using } \mathbf{a}\}$$

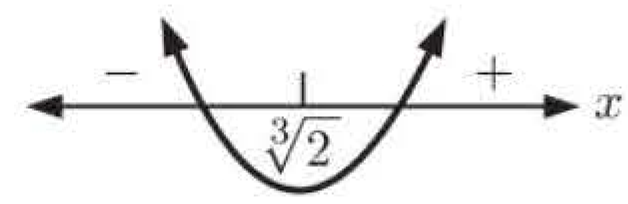
c $\dfrac{dC}{dx} = 4x - 8x^{-2}$

$= 4x - \dfrac{8}{x^2}$

$= \dfrac{4(x^3-2)}{x^2}$

So, $\dfrac{dC}{dx} = 0$ when $x = \sqrt[3]{2}$ m

$\dfrac{dC}{dx}$ has sign diagram:



The minimum cost is when $x = \sqrt[3]{2} \approx 1.26$ m

$\therefore\ y = \dfrac{1}{x^2} \approx 0.630$

and the box is 1.26 m by 1.26 m by 0.630 m.

2 **a** $s(t) = 15t - \dfrac{60}{(t+1)^2}$ cm, $t \geqslant 0$

$= 15t - 60(t+1)^{-2}$ cm

$\therefore\ v(t) = 15 + 120(t+1)^{-3}$ cm s^{-1}

$\therefore\ a(t) = -360(t+1)^{-4}$ cm s^{-2}

b When $t = 3$, $s(t) = 41.25$ cm

$v(t) \approx 16.88$ cm s^{-1}

$a(t) \approx -1.41$ cm s^{-2}

The particle is 41.25 cm right of O, travelling right at 16.88 cm s^{-1}, and is slowing down (decelerating) at 1.41 cm s^{-2}.

c $v(t) = 15 + \dfrac{120}{(t+1)^3}$ cm s^{-1}

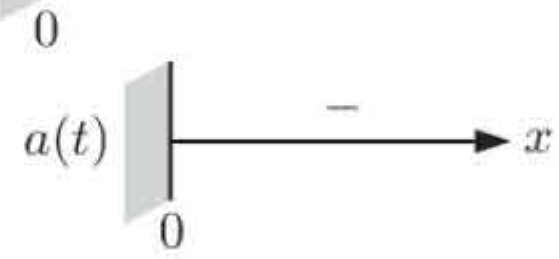
$v(t) = 0$ when $15 + \dfrac{120}{(t+1)^3} = 0$

$\therefore\ 15(t+1)^3 + 120 = 0$

$\therefore\ (t+1)^3 = -8$

$\therefore\ t = -3$

$a(t) = -360(t+1)^{-4} = \dfrac{-360}{(t+1)^4}$ cm s^{-2}



where $(t+1)^4$ is always positive. $\therefore\ a(t) < 0$ for all $t > 0$

Since $v(t) > 0$ and $a(t) < 0$ for all $t > 0$, $v(t)$ is always decreasing.

$\therefore$ the particle's speed is never increasing.

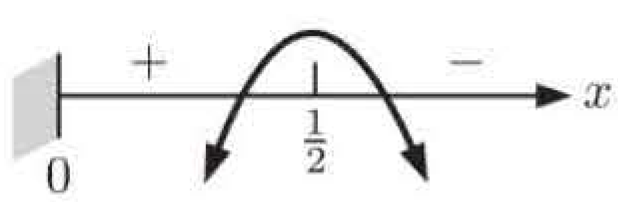
3 Let the coordinates of B be $(x, 0)$, so the coordinates of A are (x, e^{-2x}).

$\therefore$ the area OBAC is $A = xe^{-2x}$

$\therefore\ \dfrac{dA}{dx} = (1)e^{-2x} + x(-2e^{-2x})$ {product rule}

$= e^{-2x}(1-2x)$

$= \dfrac{1-2x}{e^{2x}}$ and has sign diagram:



So, the maximum area occurs when $x = \frac{1}{2}$ and $y = e^{-2(\frac{1}{2})} = e^{-1} = \frac{1}{e}$

$\therefore$ the coordinates of A are $\left(\frac{1}{2}, \frac{1}{e}\right)$.

4 **a** The tree was $H(0) = 6\left(1 - \frac{2}{3}\right) = 2$ metres tall when first planted.

b $t = 3$: $H(3) = 6\left(1 - \frac{2}{3+3}\right) = 4$ metres

$t = 6$: $H(6) = 6\left(1 - \frac{2}{6+3}\right) = 4\frac{2}{3}$ metres

$t = 9$: $H(9) = 6\left(1 - \frac{2}{9+3}\right) = 5$ metres

c $H(t) = 6\left(1 - \dfrac{2}{t+3}\right)$

$= 6 - 12(t+3)^{-1}$

$\therefore\ H'(t) = 12(t+3)^{-2}$

$= \dfrac{12}{(t+3)^2}$

So, $t = 0$: $H'(0) = \dfrac{12}{3^2} = \frac{4}{3}$ m year^{-1}

$t = 3$: $H'(3) = \dfrac{12}{6^2} = \frac{1}{3}$ m year^{-1}

$t = 6$: $H'(6) = \dfrac{12}{9^2} = \frac{4}{27}$ m year^{-1}

$t = 9$: $H'(9) = \dfrac{12}{12^2} = \frac{1}{12}$ m year^{-1}

d $$H'(t) = \frac{12}{(t+3)^2},$$

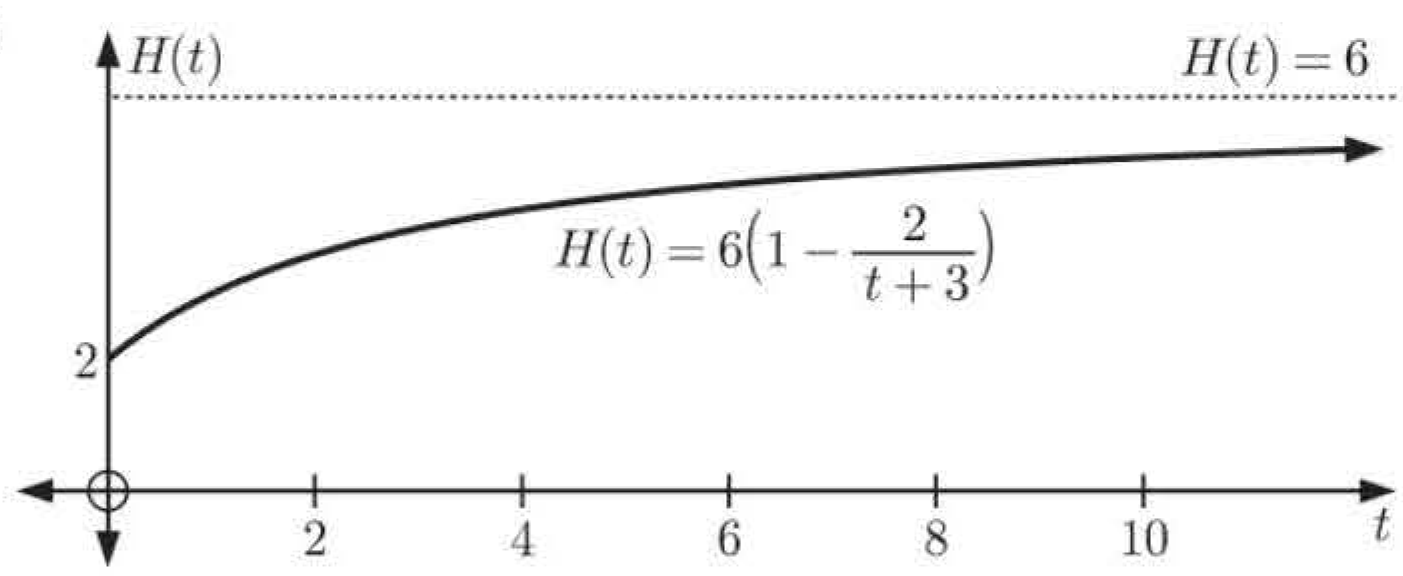
and $(t+3)^2 > 0$ for all $t \geqslant 0$

$\therefore\ \frac{12}{(t+3)^2} > 0$

$\therefore\ H'(t) > 0$ for all $t \geqslant 0$

This means that the height of the tree is always increasing.

e



5 **a** $s(t) = 25t - 10\ln t$ cm, $t \geqslant 1$

$\therefore\ v(t) = 25 - \frac{10}{t}$ cm min^{-1}

$\therefore\ a(t) = 10t^{-2} = \frac{10}{t^2}$ cm min^{-2}

b When $t = e$,

$s(e) = 25e - 10\ln e = (25e - 10)$ cm
≈ 58.0 cm

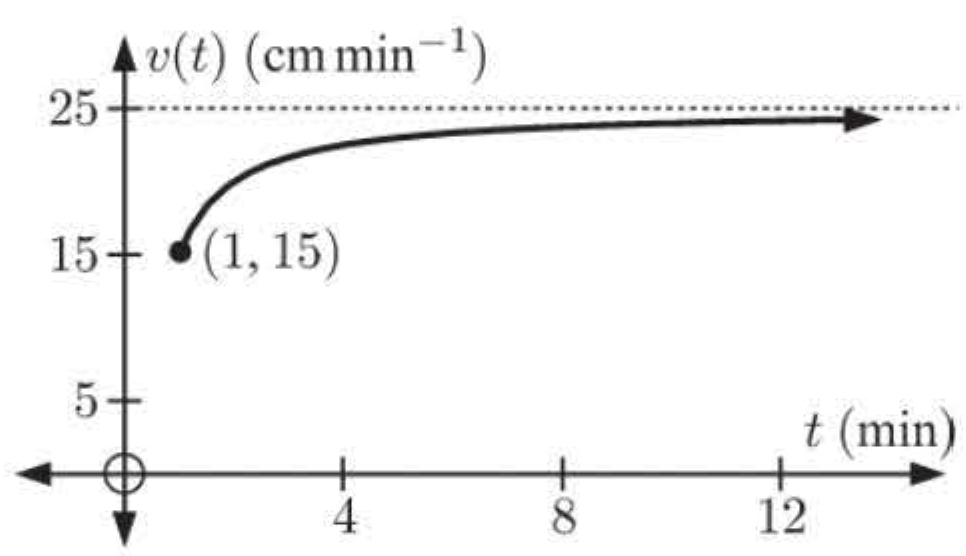
$v(e) = \left(25 - \frac{10}{e}\right)$ cm min^{-1}
≈ 21.3 cm min^{-1}

$a(e) = \frac{10}{e^2}$ cm min^{-2} ≈ 1.35 cm min^{-2}

c As $t \to \infty$, $\frac{10}{t} \to 0$

$\therefore\ v(t) \to 25$ cm min^{-1} (below)

d



e When $v(t) = 20$ cm min^{-1},

$25 - \frac{10}{t} = 20$

$\therefore\ \frac{10}{t} = 5$

$\therefore\ t = 2$ minutes

6 **a** AC $= 2x$ m

Now ABC is an isosceles triangle.

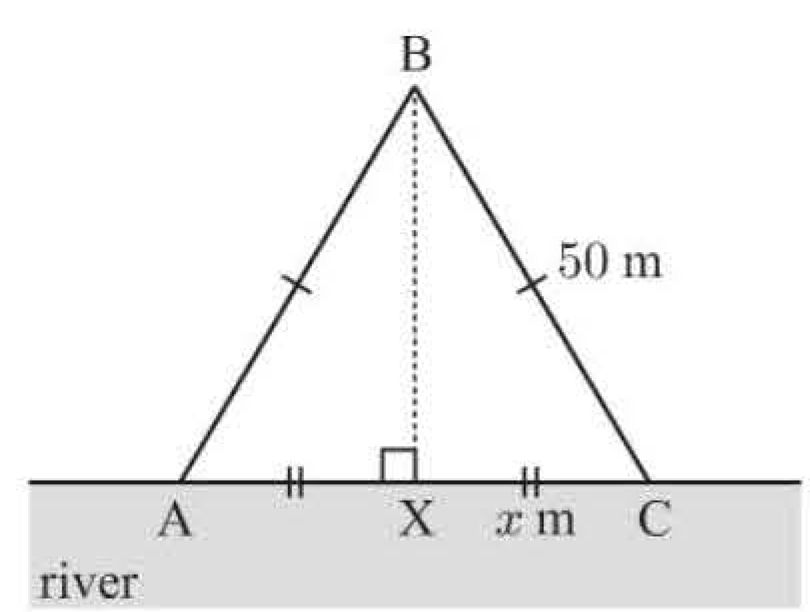
$\therefore$ XC $= x$

But BC^2 = BX^2 + XC^2 {Pythagoras}

$\therefore\ 2500 = \text{BX}^2 + x^2$

$\therefore\ \text{BX} = \sqrt{2500 - x^2}$

$\therefore\ A(x) = \frac{1}{2}(2x)\sqrt{2500 - x^2} = x\sqrt{2500 - x^2}$

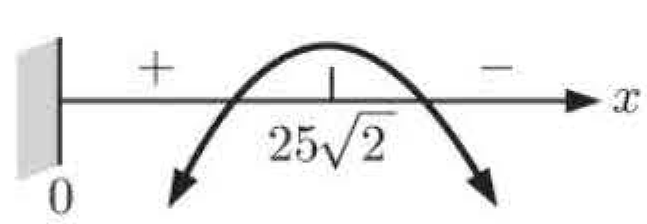


b Now $[A(x)]^2 = x^2(2500 - x^2)$

$\therefore\ A^2 = 2500x^2 - x^4$

$$\therefore\ \frac{d[(A(x)]^2}{dx} = 5000x - 4x^3$$
$$= 4x(1250 - x^2)$$
$$= 4x(\sqrt{1250} + x)(\sqrt{1250} - x)$$

c Sign diagram for $\frac{d\,[A(x)]^2}{dx}$ is:



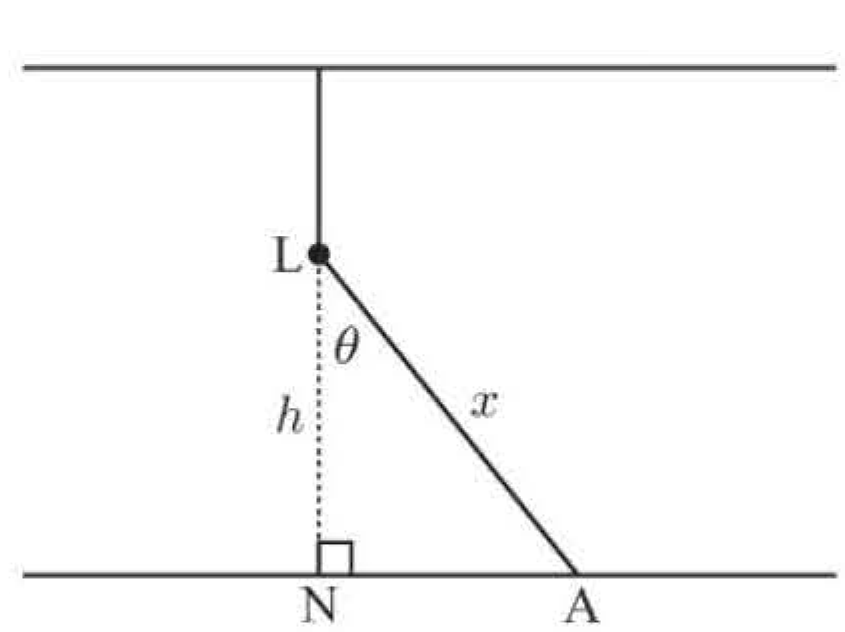
$\therefore$ the maximum area occurs when $x = 25\sqrt{2}$ m ≈ 35.4 m

The corresponding maximum area $= \sqrt{1250} \times \sqrt{1250} = 1250$ m^2.

7 **a** $\sin\theta = \frac{\text{NA}}{x} = \frac{1}{x}$

$\therefore\ \frac{1}{x^2} = \sin^2\theta$

$\therefore$ at A, $I = \frac{\sqrt{8}\cos\theta}{x^2} = \sqrt{8}\cos\theta\sin^2\theta$



b $\dfrac{dI}{d\theta} = \sqrt{8}(-\sin\theta)\sin^2\theta + \sqrt{8}\cos\theta(2\sin\theta\cos\theta)$

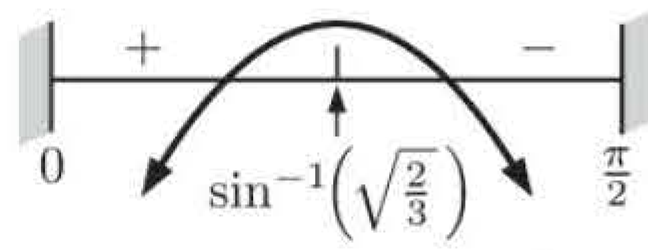
$= \sqrt{8}\sin\theta[2\cos^2\theta - \sin^2\theta]$

$= \sqrt{8}\sin\theta[2(1-\sin^2\theta) - \sin^2\theta]$

$= \sqrt{8}\sin\theta[2 - 3\sin^2\theta]$

$\dfrac{dI}{d\theta} = 0$ when $\sin\theta = \sqrt{\frac{2}{3}}$, $0 < \theta < \frac{\pi}{2}$

and the sign diagram of $\dfrac{dI}{d\theta}$ is:

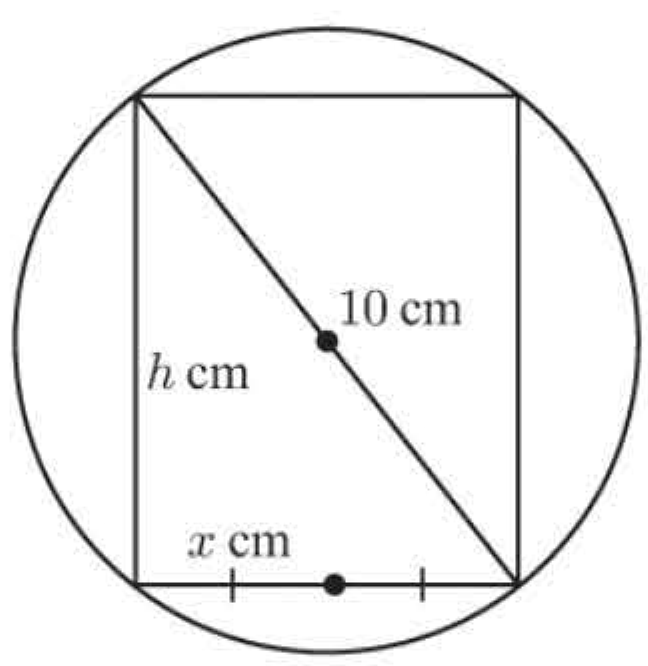


$\therefore$ the maximum illumination at A is obtained when $\sin\theta = \sqrt{\frac{2}{3}}$

$\therefore$ $x = \dfrac{1}{\sin\theta} = \sqrt{\frac{3}{2}}$ and $h = \sqrt{x^2 - \text{NA}^2} = \sqrt{\frac{3}{2} - 1} = \frac{1}{\sqrt{2}}$

$\therefore$ the bulb is $\frac{1}{\sqrt{2}}$ m above the floor.

8 **a**



Let the height of the cylinder be h cm.

$\therefore$ $(2x)^2 + h^2 = 10^2$ {Pythagoras}

$\therefore$ $h = \sqrt{100 - 4x^2}$

$\therefore$ $V(x) =$ area of base $\times$ height

$= \pi x^2 \times \sqrt{100 - 4x^2}$

So, $V(x) = \pi x^2\sqrt{100 - 4x^2}$ cm^3

b Now $V^2 = \pi^2 x^4(100 - 4x^2)$

$= \pi^2(100x^4 - 4x^6)$

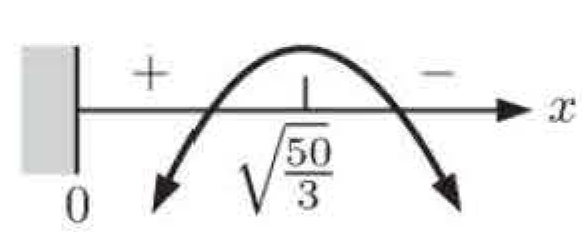
$\therefore$ $\dfrac{d(V^2)}{dx} = \pi^2(400x^3 - 24x^5)$

$= 8\pi^2x^3(50 - 3x^2)$

$= 8\pi^2x^3(\sqrt{50} + \sqrt{3}x)(\sqrt{50} - \sqrt{3}x)$

$\therefore$ $\dfrac{d(V^2)}{dx} = 0$ when $x = \sqrt{\frac{50}{3}}$ {as $x > 0$}

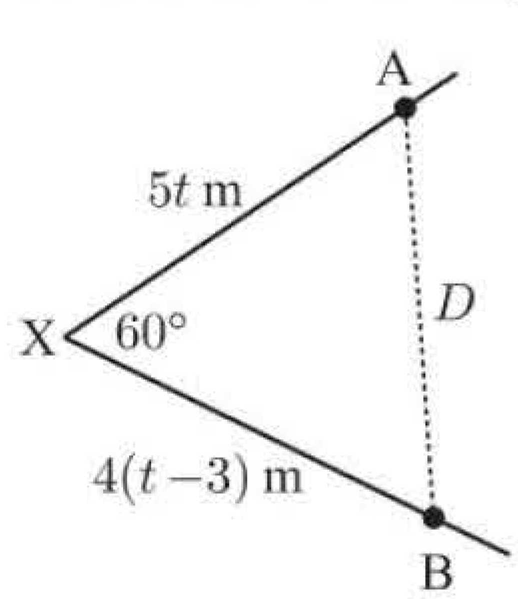
and $\dfrac{d(V^2)}{dx}$ has sign diagram:



$\therefore$ maximum V occurs when $x = \sqrt{\frac{50}{3}} \approx 4.08$

$\therefore$ radius ≈ 4.08 cm, height $= \sqrt{100 - 4\left(\frac{50}{3}\right)} \approx 5.77$ cm

9 Let t be the number of seconds after A passes through X. In this time, A travels $5t$ m.
B passes through X when $t = 3$.
$\therefore$ for $t > 3$, B is $4(t-3)$ m from X.



Using the cosine rule,

$D^2 = 25t^2 + 16(t-3)^2 - 2 \times 5t \times 4(t-3) \times \cos 60°$

$= 25t^2 + 16(t-3)^2 - 20t(t-3)$

$\therefore$ $2D\dfrac{dD}{dt} = 50t + 32(t-3) - 20(t-3) - 20t$

When $5t = 20$, $t = 4$ and $D^2 = 25 \times 16 + 16 - 20 \times 4 = 336$

$\therefore$ $2\sqrt{336}\,\dfrac{dD}{dt} = 200 + 32 - 20 - 80 = 132$

$\therefore$ $\dfrac{dD}{dt} = \dfrac{66}{\sqrt{336}} \approx 3.60$ m s^{-1}

10 **a** Using the cosine rule for $\triangle$BPO,

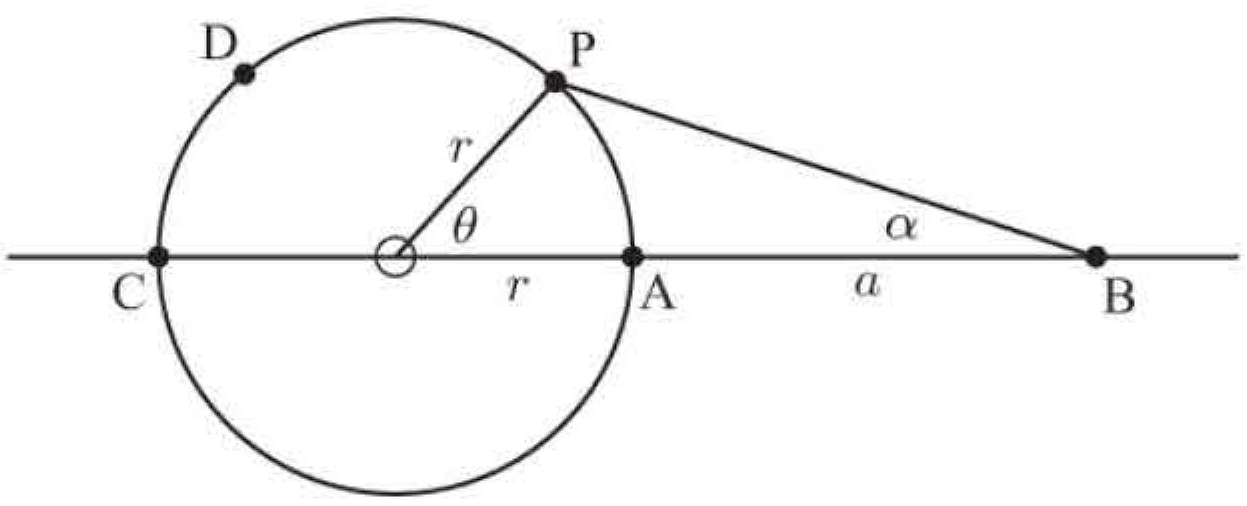
$$\mathrm{BP}^2 = r^2 + (a+r)^2 - 2r(a+r)\cos\theta$$

$$\therefore \quad \mathrm{BP} = \sqrt{r^2 + (a+r)^2 - 2r(a+r)\cos\theta}$$

$\therefore$ time taken to travel from B to P

$$= \frac{\text{distance}}{\text{speed}}$$

$$= \frac{\sqrt{r^2 + (a+r)^2 - 2r(a+r)\cos\theta}}{v}$$



Now arc AP $= r\theta$

$$\therefore \quad \text{arc PC} = (\text{perimeter of semi-circle}) - \text{arc AP}$$
$$= \tfrac{1}{2} \times 2\pi r - r\theta$$
$$= r(\pi - \theta)$$

$$\therefore \quad \text{the time taken to travel from P to C} = \frac{\text{distance}}{\text{speed}} = \frac{r(\pi - \theta)}{w}$$

The total time for the journey $T = \dfrac{\sqrt{r^2 + (a+r)^2 - 2r(a+r)\cos\theta}}{v} + \dfrac{r(\pi-\theta)}{w}$

b

$$T = \frac{w\left[r^2 + (a+r)^2 - 2r(a+r)\cos\theta\right]^{\frac{1}{2}} + rv(\pi - \theta)}{vw}$$

$$\therefore \quad \frac{dT}{d\theta} = \frac{\frac{1}{2}w\left[r^2 + (a+r)^2 - 2r(a+r)\cos\theta\right]^{-\frac{1}{2}}(2r(a+r)\sin\theta) - rv}{vw}$$

$$= \frac{2r(a+r)\sin\theta}{2v\sqrt{r^2 + (a+r)^2 - 2r(a+r)\cos\theta}} - \frac{rv}{vw}$$

$$= \frac{r(a+r)\sin\theta}{v \times \mathrm{BP}} - \frac{rv}{vw}$$

Now $\dfrac{\mathrm{BP}}{\sin\theta} = \dfrac{r}{\sin\alpha}$ {sine rule}

$$\therefore \quad \mathrm{BP} = \frac{r\sin\theta}{\sin\alpha}$$

$$\therefore \quad \frac{dT}{d\theta} = \frac{r(a+r)\sin\theta}{v \times \dfrac{r\sin\theta}{\sin\alpha}} - \frac{rv}{vw}$$

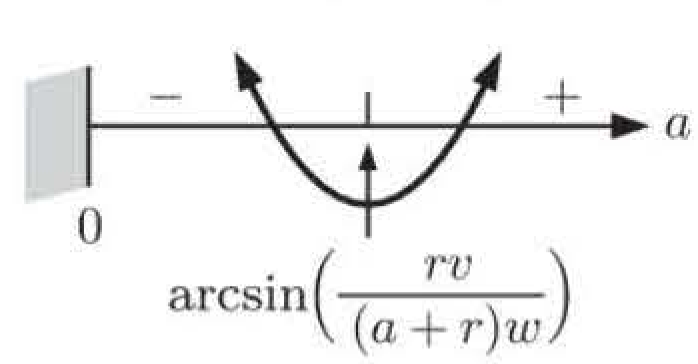
$$= \frac{a+r}{v}\sin\alpha - \frac{rv}{vw}$$

$$= \frac{a+r}{v}\left(\sin\alpha - \frac{rv}{(a+r)w}\right)$$

c Now $a + r \neq 0$ so $\dfrac{dT}{d\theta} = 0$ when $\sin\alpha - \dfrac{rv}{(a+r)w} = 0$

$$\therefore \quad \sin\alpha = \frac{rv}{(a+r)w}$$

Sign diagram for $\dfrac{dT}{d\theta}$ is:

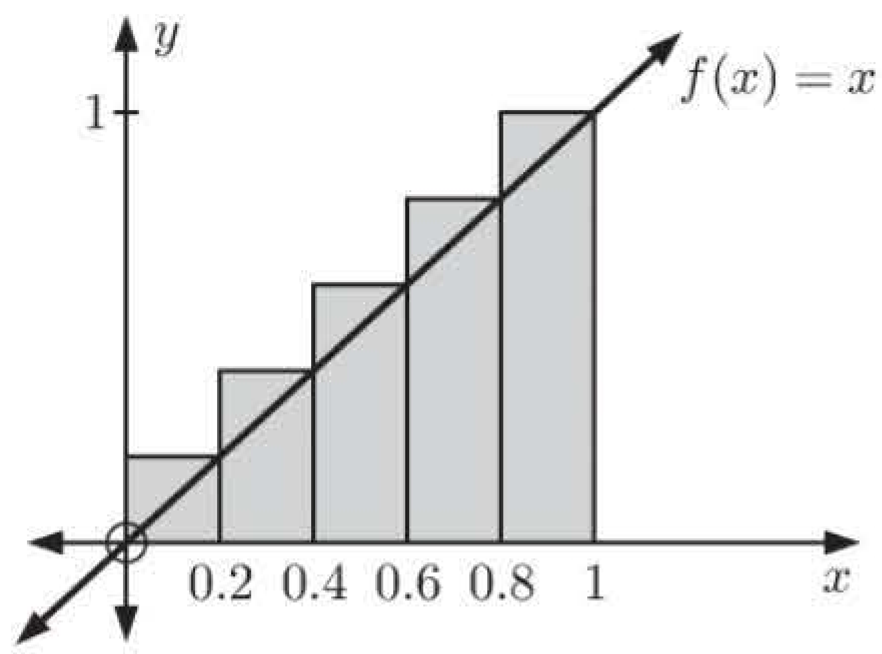


$\therefore$ T is minimised when $\sin\alpha = \dfrac{rv}{(a+r)w}$

Chapter 21
INTEGRATION

EXERCISE 21A.1

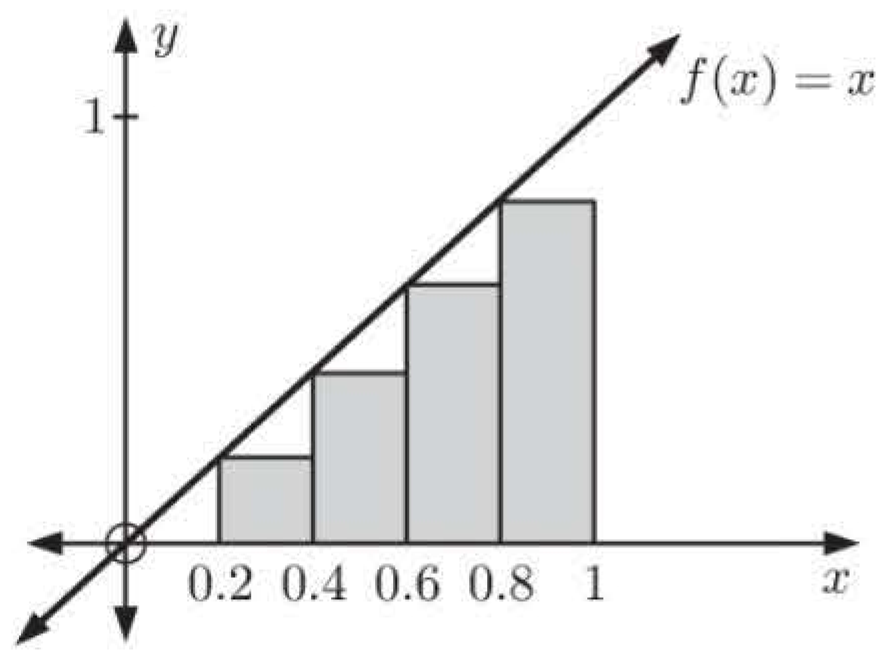
1 a i



The rectangles are $\frac{1}{5} = 0.2$ units wide.

$$\begin{aligned} A_U &= 0.2 \times f(0.2) + 0.2 \times f(0.4) + 0.2 \times f(0.6) \\ &\quad + 0.2 \times f(0.8) + 0.2 \times f(1) \\ &= 0.2 \times 0.2 + 0.2 \times 0.4 + 0.2 \times 0.6 \\ &\quad + 0.2 \times 0.8 + 0.2 \times 1 \\ &= 0.6 \text{ units}^2 \end{aligned}$$

ii



$$\begin{aligned} A_L &= 0.2 \times f(0) + 0.2 \times f(0.2) + 0.2 \times f(0.4) \\ &\quad + 0.2 \times f(0.6) + 0.2 \times f(0.8) \\ &= 0.2 \times 0 + 0.2 \times 0.2 + 0.2 \times 0.4 \\ &\quad + 0.2 \times 0.6 + 0.2 \times 0.8 \\ &= 0.4 \text{ units}^2 \end{aligned}$$

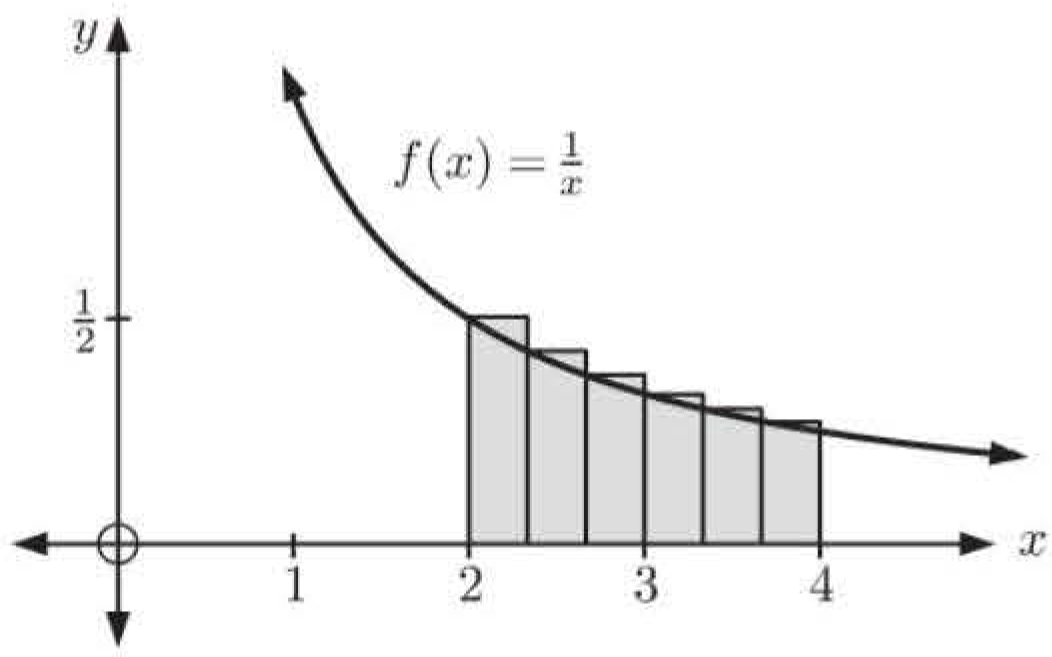
b The area between $y = x$ and the x-axis from $x = 0$ to $x = 1$ is a triangle.

$$\begin{aligned} \therefore \quad \text{area} &= \tfrac{1}{2} \times \text{base} \times \text{height} \\ &= \tfrac{1}{2} \times 1 \times 1 \\ &= \tfrac{1}{2} \text{ unit}^2 \end{aligned}$$

$\therefore \quad A_L < \text{area} < A_U$, and both A_L and A_U are within 0.1 unit^2, or 20%, of the actual area.

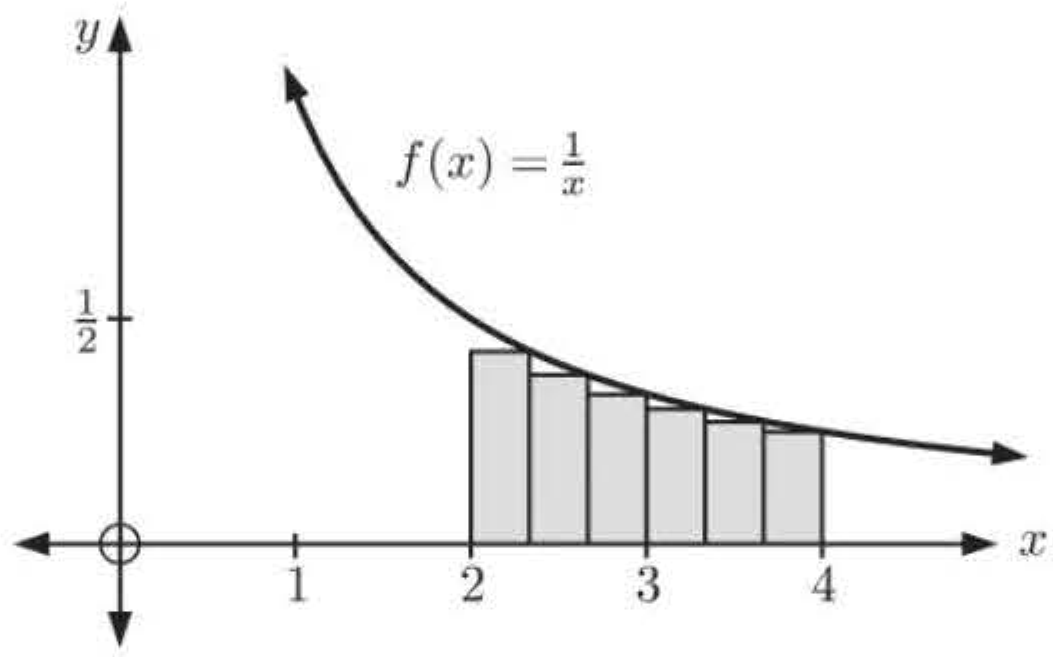
2 The rectangles are $\frac{2}{6} = \frac{1}{3}$ units wide.

a



$$\begin{aligned} A_U &= \tfrac{1}{3}f(2) + \tfrac{1}{3}f(\tfrac{7}{3}) + \tfrac{1}{3}f(\tfrac{8}{3}) + \tfrac{1}{3}f(3) \\ &\quad + \tfrac{1}{3}f(\tfrac{10}{3}) + \tfrac{1}{3}f(\tfrac{11}{3}) \\ &= \tfrac{1}{3} \times \tfrac{1}{2} + \tfrac{1}{3} \times \tfrac{3}{7} + \tfrac{1}{3} \times \tfrac{3}{8} + \tfrac{1}{3} \times \tfrac{1}{3} \\ &\quad + \tfrac{1}{3} \times \tfrac{3}{10} + \tfrac{1}{3} \times \tfrac{3}{11} \\ &\approx 0.737 \text{ units}^2 \end{aligned}$$

b



$$\begin{aligned} A_L &= \tfrac{1}{3}f(\tfrac{7}{3}) + \tfrac{1}{3}f(\tfrac{8}{3}) + \tfrac{1}{3}f(3) + \tfrac{1}{3}f(\tfrac{10}{3}) \\ &\quad + \tfrac{1}{3}f(\tfrac{11}{3}) + \tfrac{1}{3}f(4) \\ &= \tfrac{1}{3} \times \tfrac{3}{7} + \tfrac{1}{3} \times \tfrac{3}{8} + \tfrac{1}{3} \times \tfrac{1}{3} + \tfrac{1}{3} \times \tfrac{3}{10} \\ &\quad + \tfrac{1}{3} \times \tfrac{3}{11} + \tfrac{1}{3} \times \tfrac{1}{4} \\ &\approx 0.653 \text{ units}^2 \end{aligned}$$

3 Using provided software,

n	A_L	A_U
10	2.1850	2.4850
25	2.2736	2.3936
50	2.3034	2.3634
100	2.3184	2.3484
500	2.3303	2.3363

A_L and A_U converge to $\frac{7}{3}$

4 **a** **i**

n	A_L	A_U
5	0.160 00	0.360 00
10	0.202 50	0.302 50
50	0.240 10	0.260 10
100	0.245 03	0.255 03
500	0.249 00	0.251 00
1000	0.249 50	0.250 50
10 000	0.249 95	0.250 05

ii

n	A_L	A_U
5	0.400 00	0.600 00
10	0.450 00	0.550 00
50	0.490 00	0.510 00
100	0.495 00	0.505 00
500	0.499 00	0.501 00
1000	0.499 50	0.500 50
10 000	0.499 95	0.500 05

iii

n	A_L	A_U
5	0.549 74	0.749 74
10	0.610 51	0.710 51
50	0.656 10	0.676 10
100	0.661 46	0.671 46
500	0.665 65	0.667 65
1000	0.666 16	0.667 16
10 000	0.666 62	0.666 72

iv

n	A_L	A_U
5	0.618 67	0.818 67
10	0.687 40	0.787 40
50	0.738 51	0.758 51
100	0.744 41	0.754 41
500	0.748 93	0.750 93
1000	0.749 47	0.750 47
10 000	0.749 95	0.750 05

b **i** A_L and A_U converge to $0.25 = \frac{1}{4} = \frac{1}{3+1}$

ii A_L and A_U converge to $0.5 = \frac{1}{2} = \frac{1}{1+1}$

iii A_L and A_U converge to $0.6\overline{6} = \frac{2}{3} = \frac{1}{\frac{1}{2}+1}$

iv A_L and A_U converge to $0.75 = \frac{3}{4} = \frac{1}{\frac{1}{3}+1}$

c From **b**, it appears that the area between the graph of $y = x^a$ and the x-axis for $0 \leqslant x \leqslant 1$ is $\dfrac{1}{a+1}$.

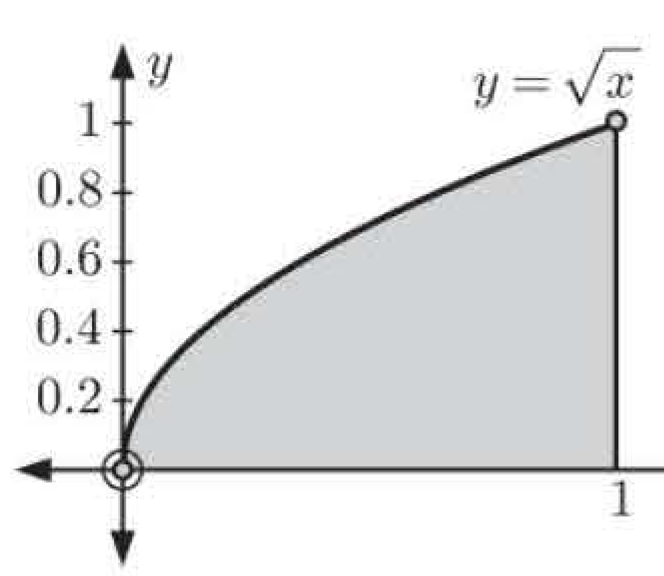
5 **a**

n	*Rational bounds for* π
10	$2.9045 < \pi < 3.3045$
50	$3.0983 < \pi < 3.1783$
100	$3.1204 < \pi < 3.1604$
200	$3.1312 < \pi < 3.1512$
1000	$3.1396 < \pi < 3.1436$
10 000	$3.1414 < \pi < 3.1418$

b $3\frac{10}{71} < \pi < 3\frac{1}{7}$ is approximately
$3.1408 < \pi < 3.1429$
From **a**, this is a better approximation than our estimate using $n = 10, 50, 100, 200, 1000$.
Only $n = 10\,000$ gives us a better estimate.

EXERCISE 21A.2

1 **a**

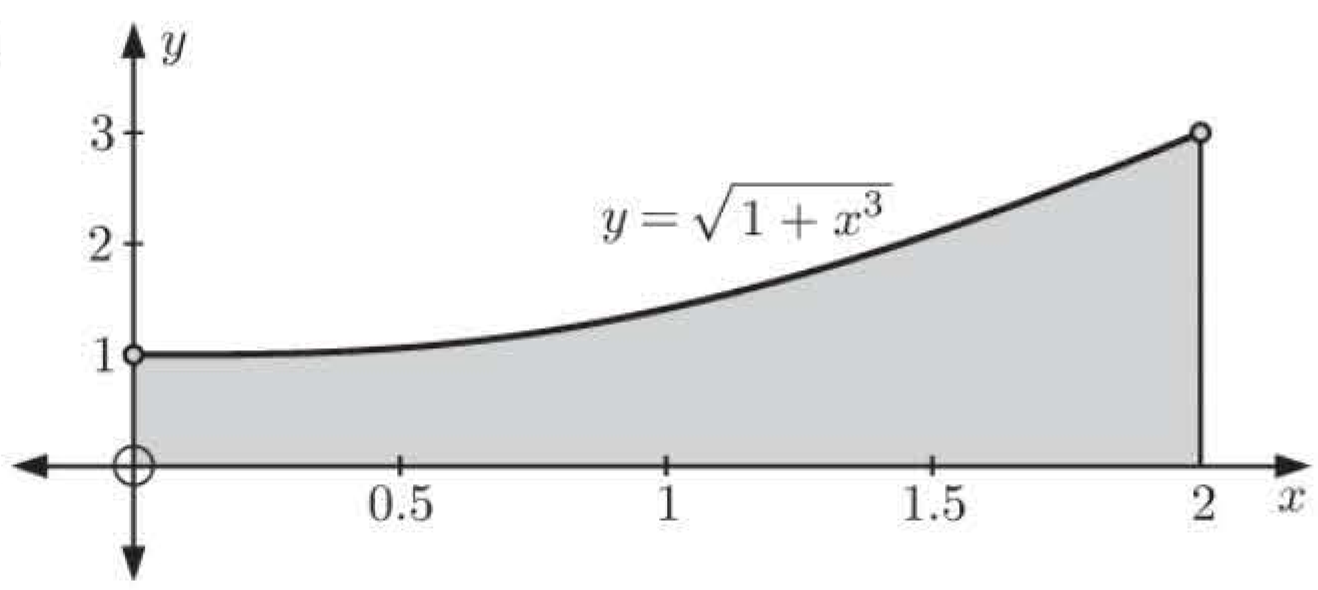


b

n	A_L	A_U
5	0.5497	0.7497
10	0.6105	0.7105
50	0.6561	0.6761
100	0.6615	0.6715
500	0.6656	0.6676

c $\int_0^1 \sqrt{x}\, dx \approx 0.67$

2 **a**



b The rectangles will have width $\dfrac{2-0}{n}=\dfrac{2}{n}$.

The lower rectangle sum will be

$$\begin{aligned} A_L &= \frac{2}{n}\times\sqrt{1+{x_0}^3}+\frac{2}{n}\times\sqrt{1+{x_1}^3}+....+\frac{2}{n}\times\sqrt{1+x_{n-1}^{\ 3}} \\ &= \frac{2}{n}\sum_{i=0}^{n-1}\sqrt{1+{x_i}^3} \end{aligned}$$

The upper rectangle sum will be

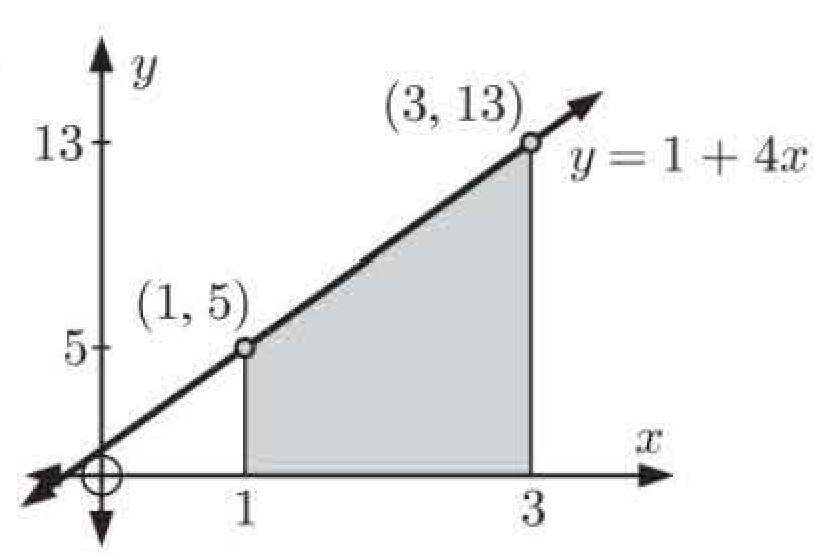
$$\begin{aligned} A_U &= \frac{2}{n}\sqrt{1+{x_1}^3}+\frac{2}{n}\sqrt{1+{x_2}^3}+....+\frac{2}{n}\sqrt{1+{x_n}^3} \\ &= \frac{2}{n}\sum_{i=1}^{n}\sqrt{1+{x_i}^3} \end{aligned}$$

c

n	A_L	A_U
50	3.2016	3.2816
100	3.2214	3.2614
500	3.2373	3.2453

d $\int_0^2 \sqrt{1+x^3}\,dx \approx 3.24$

3 **a**



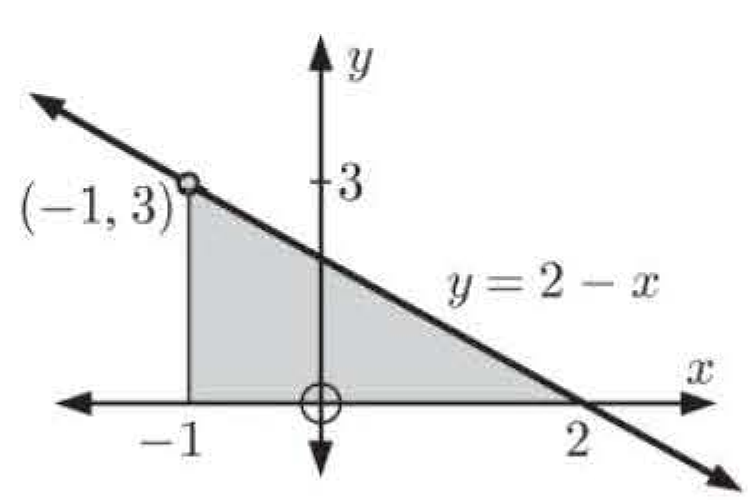
$\int_1^3 (1+4x)\ dx$

$=$ area of the shaded trapezium

$=\left(\dfrac{5+13}{2}\right)\times 2$

$=18$

b



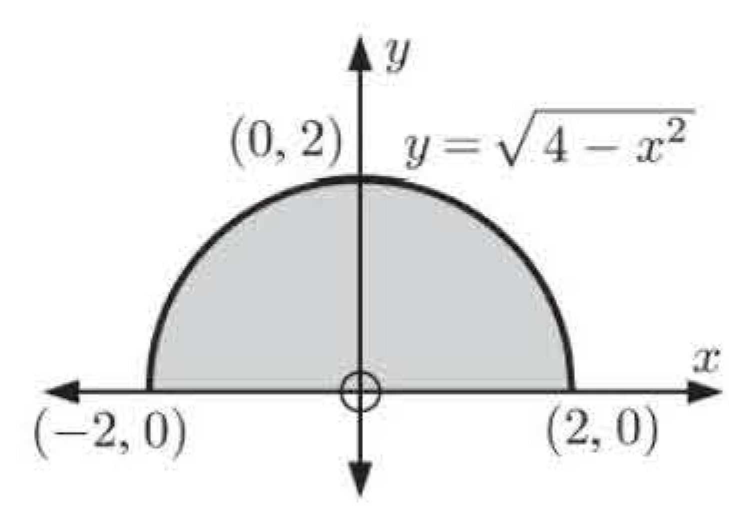
$\int_{-1}^2 (2-x)\ dx$

$=$ area of shaded triangle

$=\frac{1}{2}(3\times 3)$

$=4.5$

c



$\int_{-2}^2 \sqrt{4-x^2}\,dx$

$=$ area of semi-circle, radius 2

$=\frac{1}{2}\left(\pi\times 2^2\right)$

$=2\pi$

EXERCISE 21B

1 **a** **i** $\dfrac{d}{dx}(x^2)=2x$

$\therefore\ \dfrac{d}{dx}\left(\frac{1}{2}x^2\right)=x$

$\therefore$ the antiderivative of x is $\frac{1}{2}x^2$

ii $\dfrac{d}{dx}(x^3)=3x^2$

$\therefore\ \dfrac{d}{dx}\left(\frac{1}{3}x^3\right)=x^2$

$\therefore$ the antiderivative of x^2 is $\frac{1}{3}x^3$

iii $\dfrac{d}{dx}(x^6)=6x^5$

$\therefore\ \dfrac{d}{dx}\left(\frac{1}{6}x^6\right)=x^5$

$\therefore$ the antiderivative of x^5 is $\frac{1}{6}x^6$

iv $\dfrac{d}{dx}(x^{-1})=-x^{-2}$

$\therefore\ \dfrac{d}{dx}(-x^{-1})=x^{-2}$

$\therefore$ the antiderivative of x^{-2} is $-x^{-1}$ or $-\dfrac{1}{x}$

v $\dfrac{d}{dx}(x^{-3}) = -3x^{-4}$

$\therefore \dfrac{d}{dx}\left(-\frac{1}{3}x^{-3}\right) = x^{-4}$

$\therefore$ the antiderivative of x^{-4} is $-\frac{1}{3}x^{-3} = -\dfrac{1}{3x^3}$

vi $\dfrac{d}{dx}\left(x^{\frac{4}{3}}\right) = \frac{4}{3}x^{\frac{1}{3}}$

$\therefore \dfrac{d}{dx}\left(\frac{3}{4}x^{\frac{4}{3}}\right) = x^{\frac{1}{3}}$

$\therefore$ the antiderivative of $x^{\frac{1}{3}}$ is $\frac{3}{4}x^{\frac{4}{3}}$

vii $\dfrac{d}{dx}(x^{\frac{1}{2}}) = \frac{1}{2}x^{-\frac{1}{2}}$

$\therefore \dfrac{d}{dx}(2x^{\frac{1}{2}}) = x^{-\frac{1}{2}}$

$\therefore$ the antiderivative of $x^{-\frac{1}{2}}$ is $2x^{\frac{1}{2}} = 2\sqrt{x}$

b The antiderivative of x^n is $\dfrac{x^{n+1}}{n+1}$ $(n \neq -1)$.

2 **a** **i** $\dfrac{d}{dx}(e^{2x}) = 2e^{2x}$

$\therefore \dfrac{d}{dx}\left(\frac{1}{2}e^{2x}\right) = e^{2x}$

$\therefore$ the antiderivative of e^{2x} is $\frac{1}{2}e^{2x}$

ii $\dfrac{d}{dx}(e^{5x}) = 5e^{5x}$

$\therefore \dfrac{d}{dx}\left(\frac{1}{5}e^{5x}\right) = e^{5x}$

$\therefore$ the antiderivative of e^{5x} is $\frac{1}{5}e^{5x}$

iii $\dfrac{d}{dx}(e^{\frac{1}{2}x}) = \frac{1}{2}e^{\frac{1}{2}x}$

$\therefore \dfrac{d}{dx}\left(2e^{\frac{1}{2}x}\right) = e^{\frac{1}{2}x}$

$\therefore$ the antiderivative of $e^{\frac{1}{2}x}$ is $2e^{\frac{1}{2}x}$

iv $\dfrac{d}{dx}(e^{0.01x}) = 0.01e^{0.01x}$

$\therefore \dfrac{d}{dx}\left(100e^{0.01x}\right) = e^{0.01x}$

$\therefore$ the antiderivative of $e^{0.01x}$ is $100e^{0.01x}$

v $\dfrac{d}{dx}(e^{\pi x}) = \pi e^{\pi x}$

$\therefore \dfrac{d}{dx}\left(\frac{1}{\pi}e^{\pi x}\right) = e^{\pi x}$

$\therefore$ the antiderivative of $e^{\pi x}$ is $\frac{1}{\pi}e^{\pi x}$

vi $\dfrac{d}{dx}\left(e^{\frac{x}{3}}\right) = \frac{1}{3}e^{\frac{x}{3}}$

$\therefore \dfrac{d}{dx}\left(3e^{\frac{x}{3}}\right) = e^{\frac{x}{3}}$

$\therefore$ the antiderivative of $e^{\frac{x}{3}}$ is $3e^{\frac{x}{3}}$

b The antiderivative of e^{kx} is $\dfrac{1}{k}e^{kx}$.

3 **a** $\dfrac{d}{dx}(x^3 + x^2) = 3x^2 + 2x$

$\therefore \dfrac{d}{dx}\left(2x^3 + 2x^2\right) = 6x^2 + 4x$

$\therefore$ the antiderivative of $6x^2 + 4x$ is $2x^3 + 2x^2$

b $\dfrac{d}{dx}(e^{3x+1}) = 3e^{3x+1}$

$\therefore \dfrac{d}{dx}\left(\frac{1}{3}e^{3x+1}\right) = e^{3x+1}$

$\therefore$ the antiderivative of e^{3x+1} is $\frac{1}{3}e^{3x+1}$

c $\dfrac{d}{dx}(x\sqrt{x}) = \dfrac{d}{dx}(x^{\frac{3}{2}}) = \frac{3}{2}x^{\frac{1}{2}}$
$= \frac{3}{2}\sqrt{x}$

$\therefore \dfrac{d}{dx}\left(\frac{2}{3}x\sqrt{x}\right) = \sqrt{x}$

$\therefore$ the antiderivative of $\sqrt{x}$ is $\frac{2}{3}x\sqrt{x}$

d $\dfrac{d}{dx}((2x+1)^4) = 4(2x+1)^3 \times 2$
$= 8(2x+1)^3$

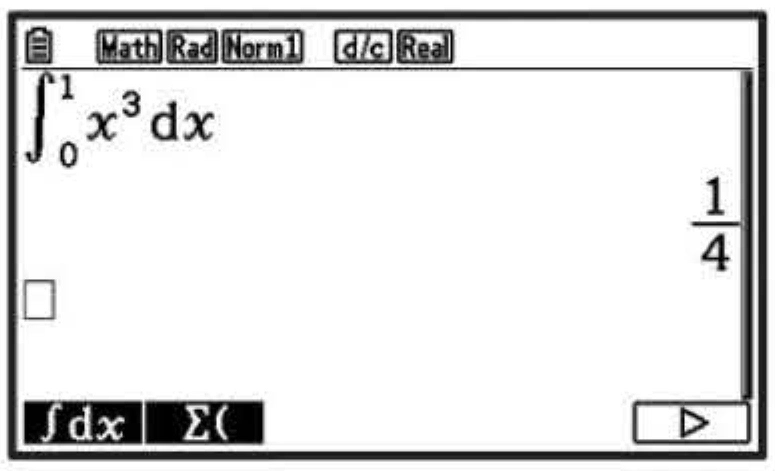
$\therefore \dfrac{d}{dx}\left(\frac{1}{8}(2x+1)^4\right) = (2x+1)^3$

$\therefore$ the antiderivative of $(2x+1)^3$ is $\frac{1}{8}(2x+1)^4$

EXERCISE 21C

1 **a** $f(x) = x^3$ has antiderivative $F(x) = \dfrac{x^4}{4}$

$\therefore$ area $= \int_0^1 x^3\,dx$
$= F(1) - F(0)$
$= \frac{1}{4} - 0 = \frac{1}{4}$ units^2

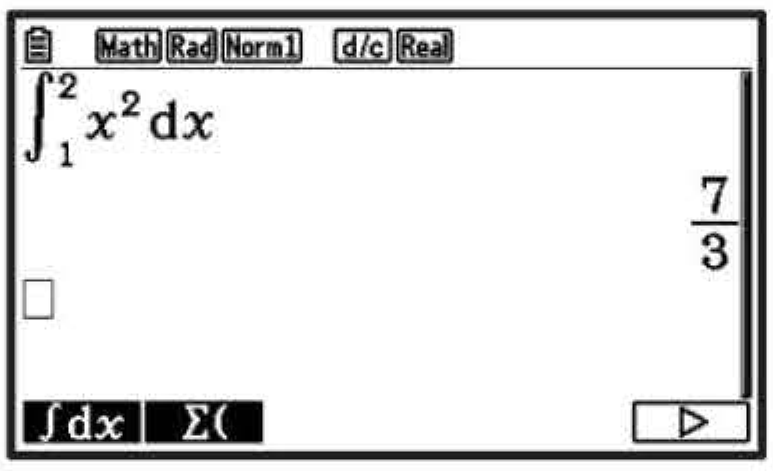


b $f(x) = x^2$ has antiderivative $F(x) = \dfrac{x^3}{3}$

$\therefore$ area $= \int_1^2 x^2\, dx$

$= F(2) - F(1)$

$= \frac{8}{3} - \frac{1}{3} = 2\frac{1}{3}$ units^2



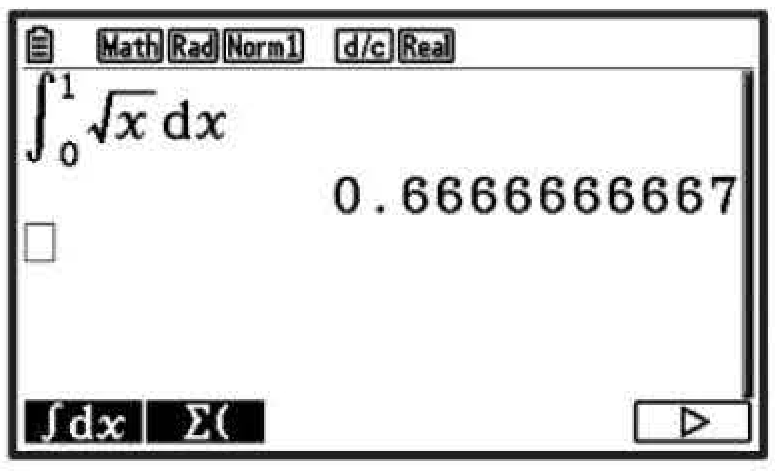
c $f(x) = \sqrt{x} = x^{\frac{1}{2}}$ has antiderivative

$F(x) = \dfrac{x^{\frac{3}{2}}}{\frac{3}{2}} = \frac{2}{3}x\sqrt{x}$

$\therefore$ area $= \int_0^1 \sqrt{x}\, dx$

$= F(1) - F(0)$

$= \frac{2}{3} \times 1\sqrt{1} - 0 = \frac{2}{3}$ units^2



2 **a** $\int_a^a f(x)\, dx = F(a) - F(a) = 0$

$\int_a^a f(x)\, dx =$ area of the strip between $x = a$ and $x = a$.

This strip has 0 width, so its area $= 0$.

b The antiderivative of c is cx.

$\therefore \int_a^b c\, dx = F(b) - F(a)$

$= cb - ca$

$= c(b-a)$

c $\int_b^a f(x)\, dx = F(a) - F(b)$

$= -[F(b) - F(a)]$

$= -\int_a^b f(x)\, dx$

d If $\dfrac{d}{dx} F(x) = f(x)$ then

$\dfrac{d}{dx}\, c\,F(x) = c\,f(x)$

$\therefore \int_a^b c\,f(x)\, dx = c\,F(b) - c\,F(a)$

$= c[F(b) - F(a)]$

$= c\int_a^b f(x)\, dx$

e $\int_a^b (f(x) + g(x))\, dx$

$= [F(b) + G(b)] - [F(a) + G(a)]$

$= [F(b) - F(a)] + [G(b) - G(a)]$

$= \int_a^b f(x)\, dx + \int_a^b g(x)\, dx$

3 **a** $f(x) = x^3$ has antiderivative $F(x) = \dfrac{x^4}{4}$

$\therefore$ area $= \int_1^2 x^3\, dx$

$= F(2) - F(1)$

$= \frac{16}{4} - \frac{1}{4}$

$= 3\frac{3}{4}$ units^2

b $f(x) = x^2 + 3x + 2$ has antiderivative

$F(x) = \dfrac{x^3}{3} + \dfrac{3x^2}{2} + 2x$

$\therefore$ area $= \int_1^3 (x^2 + 3x + 2)\, dx$

$= F(3) - F(1)$

$= (\frac{27}{3} + \frac{27}{2} + 6) - (\frac{1}{3} + \frac{3}{2} + 2)$

$= 24\frac{2}{3}$ units^2

c $f(x) = \sqrt{x} = x^{\frac{1}{2}}$ has antiderivative

$F(x) = \dfrac{x^{\frac{3}{2}}}{\frac{3}{2}} = \frac{2}{3}x\sqrt{x}$

$\therefore$ area $= \int_1^2 \sqrt{x}\, dx$

$= F(2) - F(1)$

$= \frac{2}{3}\,2\sqrt{2} - \frac{2}{3}\,1\sqrt{1}$

$= \frac{4\sqrt{2}}{3} - \frac{2}{3}$

$= \frac{-2+4\sqrt{2}}{3}$ units^2

d $f(x) = e^x$ has antiderivative $F(x) = e^x$

$\therefore$ area $= \int_0^{1.5} e^x\, dx$

$= F(1.5) - F(0)$

$= e^{1.5} - e^0$

$= e^{1.5} - 1$

≈ 3.48 units^2

e $f(x) = \dfrac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$ has antiderivative

$F(x) = \dfrac{x^{\frac{1}{2}}}{\frac{1}{2}} = 2\sqrt{x}$

$\therefore$ area $= \int_1^4 \dfrac{1}{\sqrt{x}}\, dx$

$= F(4) - F(1)$

$= 2\sqrt{4} - 2\sqrt{1} = 2$ units^2

f $f(x) = x^3 + 2x^2 + 7x + 4$ has antiderivative

$F(x) = \dfrac{x^4}{4} + \dfrac{2x^3}{3} + \dfrac{7x^2}{2} + 4x$

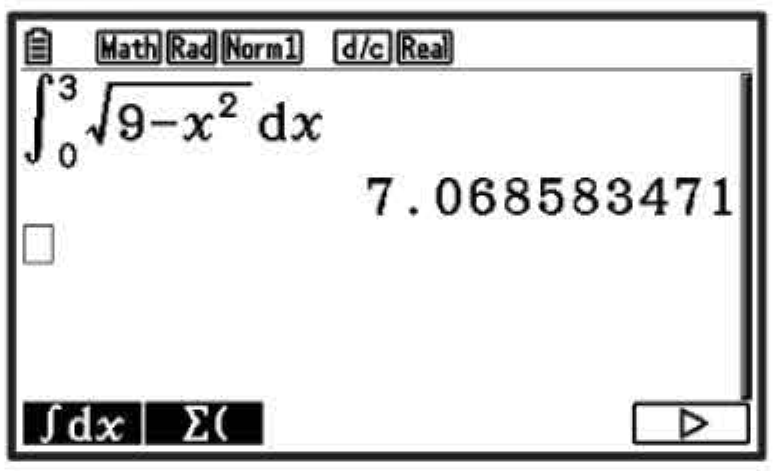
$\therefore$ area $= \int_1^{1.25} \left(x^3 + 2x^2 + 7x + 4\right)\, dx$

$= F(1.25) - F(1)$

$= [12.381\,18 - 8.416\,67]$

≈ 3.96 units^2

4



$\therefore$ using technology, area $= \int_0^3 \sqrt{9 - x^2}\, dx \approx 7.07$ units^2

Check: The area is a quarter circle with radius 3 units.

$\therefore$ area $= \frac{1}{4}\pi r^2$

$= \frac{1}{4} \times \pi \times 3^2 = \frac{9}{4}\pi \approx 7.07$ units^2 ✓

5 **a** If $\dfrac{d}{dx}F(x) = f(x)$ then $\dfrac{d}{dx}(-F(x)) = -f(x)$

$\therefore \int_a^b (-f(x))\, dx = -F(b) - (-F(a))$

$= -(F(b) - F(a))$

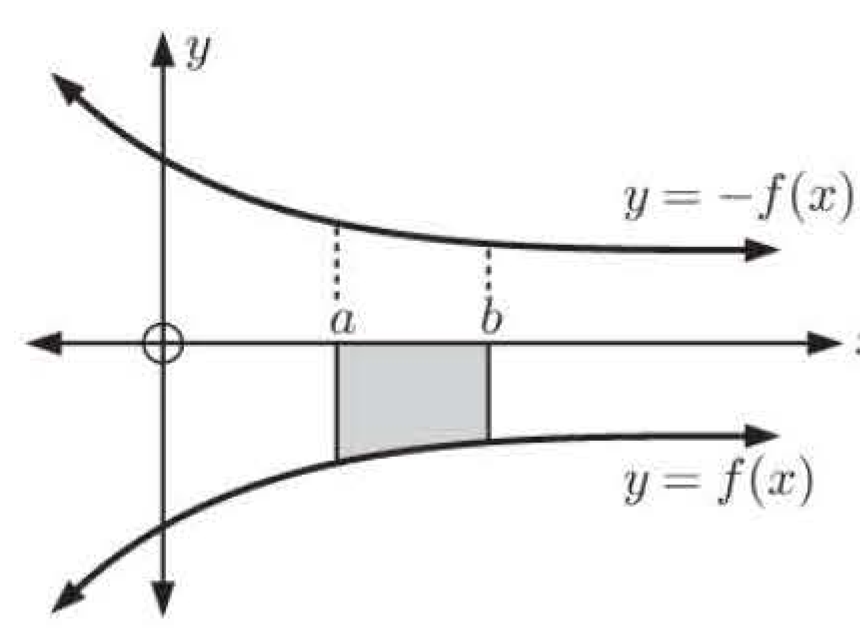
$= -\int_a^b f(x)\, dx$

b Since $y = -f(x)$ is a reflection of $y = f(x)$ in the x-axis, then

shaded area = area between the x-axis and $y = -f(x)$ from $x = a$ to $x = b$

$= \int_a^b (-f(x))\, dx$

$= -\int_a^b f(x)\, dx$ {using **a**}



c **i** $\int_0^1 (-x^2)\, dx = -\int_0^1 x^2\, dx$

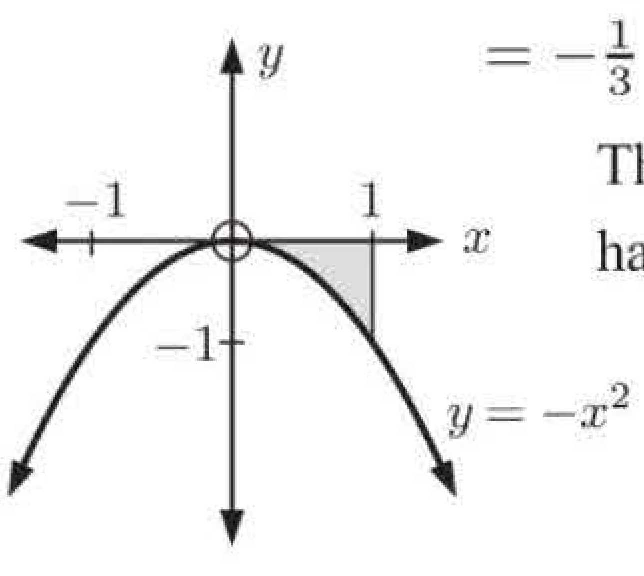
Now $f(x) = x^2$ has antiderivative

$F(x) = \frac{1}{3}x^3$

$\therefore \int_0^1 (-x^2)\, dx = -(F(1) - F(0))$

$= -\left(\frac{1}{3} - 0\right)$

$= -\frac{1}{3}$



The shaded region has area $\frac{1}{3}$ units^2.

ii $\int_0^1 (x^2 - x)\, dx = -\int_0^1 (x - x^2)\, dx$

$\{x^2 - x \leqslant 0$ for all $x \in [0, 1]\}$

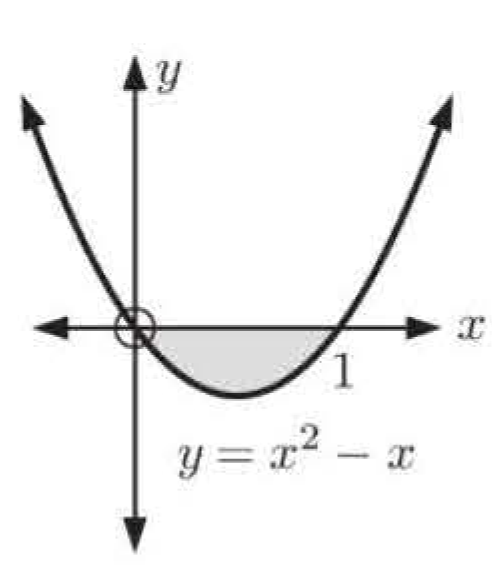
Now $f(x) = x - x^2$ has antiderivative

$F(x) = \frac{1}{2}x^2 - \frac{1}{3}x^3$

$\therefore \int_0^1 (x^2 - x)\, dx = -(F(1) - F(0))$

$= -\left(\frac{1}{2} - \frac{1}{3} - (0 - 0)\right)$

$= -\frac{1}{6}$

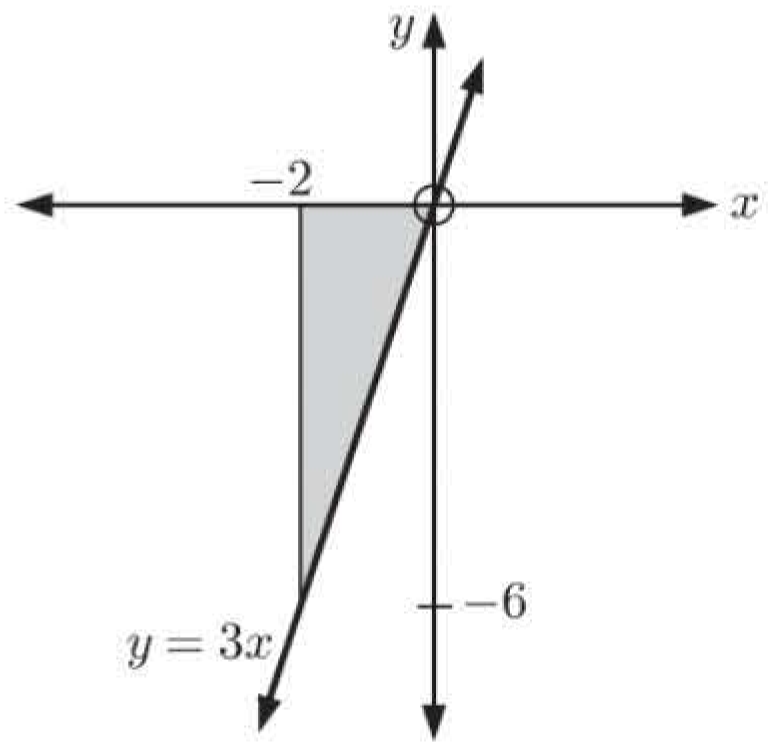


The shaded region has area $\frac{1}{6}$ units^2.

iii $\int_{-2}^{0} 3x\, dx = -\int_{-2}^{0} -3x\, dx$

Now $f(x) = -3x$ has antiderivative $F(x) = -\frac{3}{2}x^2$

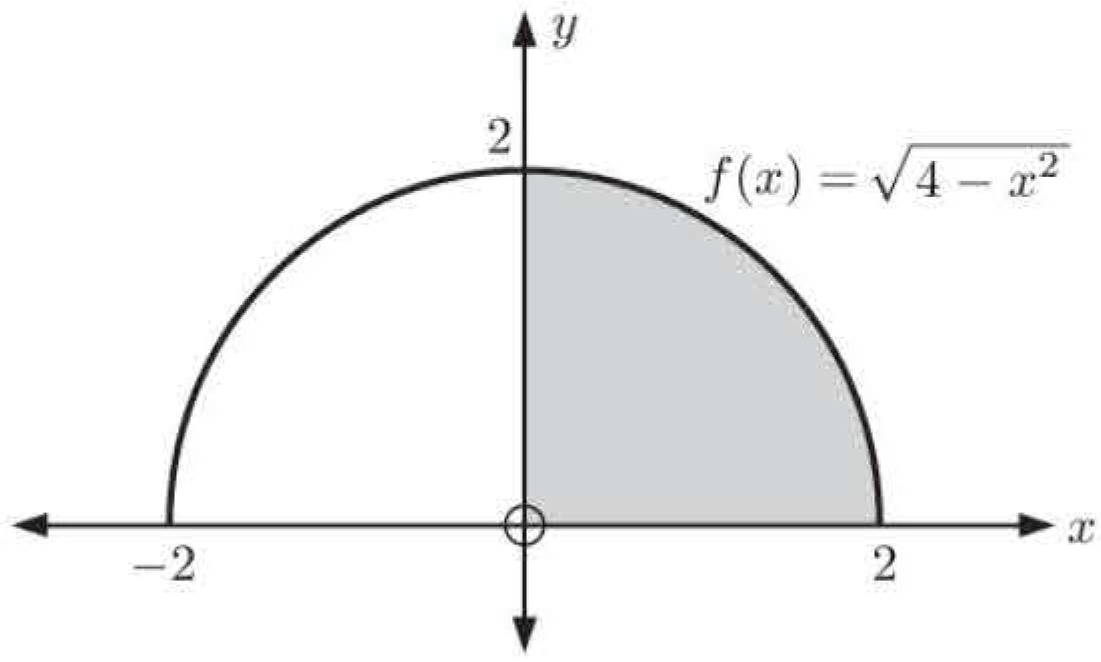
$\therefore \int_{-2}^{0} 3x\, dx = -(F(0) - F(-2))$
$= -(0 - (-6))$
$= -6$



The shaded region has area 6 units^2.

d $\int_0^2 \left(-\sqrt{4-x^2}\right) dx = -\int_0^2 \sqrt{4-x^2}\, dx$

Now $f(x) = \sqrt{4-x^2}$ is the top half of a circle with radius 2 units and centre (0, 0).



$\therefore \int_0^2 \left(-\sqrt{4-x^2}\right) dx = -\int_0^2 \sqrt{4-x^2}\, dx$
$= -$(shaded area)
$= -\frac{1}{4} \times \pi \times 2^2$
$= -\pi$

EXERCISE 21D

1 If $y = x^7$ then $\frac{dy}{dx} = 7x^6$

$\therefore \int 7x^6\, dx = x^7 + c$

$\therefore 7\int x^6\, dx = x^7 + c$

$\therefore \int x^6\, dx = \frac{1}{7}x^7 + c$

2 If $y = x^3 + x^2$ then $\frac{dy}{dx} = 3x^2 + 2x$

$\therefore \int (3x^2 + 2x)\, dx = x^3 + x^2 + c$

3 If $y = e^{2x+1}$ then $\frac{dy}{dx} = 2e^{2x+1}$

$\therefore \int 2e^{2x+1}\, dx = e^{2x+1} + c$

$\therefore 2\int e^{2x+1}\, dx = e^{2x+1} + c$

$\therefore \int e^{2x+1}\, dx = \frac{1}{2}e^{2x+1} + c$

4 If $y = (2x+1)^4$

then $\frac{dy}{dx} = 4(2x+1)^3 \times 2 = 8(2x+1)^3$ {chain rule}

$\therefore \int 8(2x+1)^3\, dx = (2x+1)^4 + c$

$\therefore 8\int (2x+1)^3\, dx = (2x+1)^4 + c$

$\therefore \int (2x+1)^3\, dx = \frac{1}{8}(2x+1)^4 + c$

5 If $y = x\sqrt{x} = x^{\frac{3}{2}}$

then $\frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}} = \frac{3}{2}\sqrt{x}$

$\therefore \int \frac{3}{2}\sqrt{x}\, dx = x\sqrt{x} + c$

$\therefore \frac{3}{2}\int \sqrt{x}\, dx = x\sqrt{x} + c$

$\therefore \int \sqrt{x}\, dx = \frac{2}{3}x\sqrt{x} + c$

6 If $y = \frac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$

then $\frac{dy}{dx} = -\frac{1}{2}x^{-\frac{3}{2}} = -\frac{1}{2x\sqrt{x}}$

$\therefore \int -\frac{1}{2}\left(\frac{1}{x\sqrt{x}}\right) dx = \frac{1}{\sqrt{x}} + c$

$\therefore -\frac{1}{2}\int \frac{1}{x\sqrt{x}}\, dx = \frac{1}{\sqrt{x}} + c$

$\therefore \int \frac{1}{x\sqrt{x}}\, dx = -\frac{2}{\sqrt{x}} + c$

7

$$\begin{aligned} \text{If} \quad y &= \cos 2x \\ \text{then} \quad \frac{dy}{dx} &= -2\sin 2x \\ \therefore \int -2\sin 2x\,dx &= \cos 2x + c \\ \therefore -2\int \sin 2x\,dx &= \cos 2x + c \\ \therefore \int \sin 2x\,dx &= -\tfrac{1}{2}\cos 2x + c \end{aligned}$$

8

$$\begin{aligned} \text{If} \quad y &= \sin(1-5x) \\ \text{then} \quad \frac{dy}{dx} &= -5\cos(1-5x) \\ \therefore \int -5\cos(1-5x)\,dx &= \sin(1-5x) + c \\ \therefore -5\int \cos(1-5x)\,dx &= \sin(1-5x) + c \\ \therefore \int \cos(1-5x)\,dx &= -\tfrac{1}{5}\sin(1-5x) + c \end{aligned}$$

9

$$\frac{d}{dx}\left[(x^2-x)^3\right] = 3(x^2-x)^2(2x-1) \qquad \{\text{chain rule}\}$$

$$\begin{aligned} \therefore \int 3(2x-1)(x^2-x)^2\,dx &= (x^2-x)^3 + c \\ \therefore 3\int (2x-1)(x^2-x)^2\,dx &= (x^2-x)^3 + c \\ \therefore \int (2x-1)(x^2-x)^2\,dx &= \tfrac{1}{3}(x^2-x)^3 + c \end{aligned}$$

10 Suppose $F(x)$ is the antiderivative of $f(x)$ and $G(x)$ is the antiderivative of $g(x)$.

$$\therefore \frac{d}{dx}(F(x)+G(x)) = f(x) + g(x)$$

$$\begin{aligned} \therefore \quad & \int [f(x)+g(x)]\,dx \\ &= F(x) + G(x) + c \\ &= (F(x)+c_1) + (G(x)+c_2) \\ &= \int f(x)\,dx + \int g(x)\,dx \end{aligned}$$

11

$$\begin{aligned} y &= \sqrt{1-4x} = (1-4x)^{\frac{1}{2}} \\ \therefore \frac{dy}{dx} &= \tfrac{1}{2}(1-4x)^{-\frac{1}{2}}(-4) \qquad \{\text{chain rule}\} \\ &= \frac{-2}{\sqrt{1-4x}} \\ \therefore \int \frac{-2}{\sqrt{1-4x}}\,dx &= \sqrt{1-4x} + c \\ \therefore -2\int \frac{1}{\sqrt{1-4x}}\,dx &= \sqrt{1-4x} + c \\ \therefore \int \frac{1}{\sqrt{1-4x}}\,dx &= -\tfrac{1}{2}\sqrt{1-4x} + c \end{aligned}$$

12

$$\frac{d}{dx}(\ln(5-3x+x^2)) = \frac{2x-3}{5-3x+x^2}$$

Now $5-3x+x^2 > 0$ for all x, as $a > 0$ and $\Delta = -11 < 0$

$$\begin{aligned} \therefore \int \frac{2x-3}{5-3x+x^2}\,dx &= \ln(5-3x+x^2) + c \\ \therefore \int \frac{4x-6}{5-3x+x^2}\,dx &= 2\ln(5-3x+x^2) + c \end{aligned}$$

13

$$\begin{aligned} \frac{d}{dx}(\csc x) &= -\csc x \cot x \\ &= -\frac{1}{\sin x} \times \frac{\cos x}{\sin x} \\ &= -\frac{\cos x}{\sin^2 x} \end{aligned}$$

$$\begin{aligned} \therefore \int -\frac{\cos x}{\sin^2 x}\,dx &= \csc x + c \\ \therefore -\int \frac{\cos x}{\sin^2 x}\,dx &= \csc x + c \\ \therefore \int \frac{\cos x}{\sin^2 x}\,dx &= -\csc x + c \end{aligned}$$

14

$$\frac{d}{dx}(\arctan x) = \frac{1}{x^2+1}, \quad x \in \mathbb{R}$$

$$\begin{aligned} \therefore \int \frac{1}{x^2+1}\,dx &= \arctan x + c \\ \therefore -3\int \frac{1}{x^2+1}\,dx &= -3\arctan x + c \\ \therefore \int \frac{-3}{x^2+1}\,dx &= -3\arctan x + c \end{aligned}$$

15
$$\frac{d}{dx}(2^x) = \frac{d}{dx}(e^{\ln 2})^x$$
$$= e^{(\ln 2)x} \times \ln 2$$
$$= 2^x \ln 2$$
$$\therefore \int 2^x \ln 2 \, dx = 2^x + c$$
$$\therefore \ln 2 \int 2^x \, dx = 2^x + c$$
$$\therefore \int 2^x \, dx = \frac{2^x}{\ln 2} + c$$

16
$$\frac{d}{dx}(x \ln x) = 1 \times \ln x + x \times \frac{1}{x}$$
{chain rule}
$$= \ln x + 1$$
$$\therefore \int (\ln x + 1) \, dx = x \ln x + c$$
$$\therefore \int \ln x \, dx + \int 1 \, dx = x \ln x + c$$
$$\therefore \int \ln x \, dx + x = x \ln x + c$$
$$\therefore \int \ln x \, dx = x \ln x - x + c$$

EXERCISE 21E.1

1 **a**
$$\int (x^4 - x^2 - x + 2) \, dx$$
$$= \frac{x^5}{5} - \frac{x^3}{3} - \frac{x^2}{2} + 2x + c$$

b
$$\int (\sqrt{x} + e^x) \, dx$$
$$= \int (x^{\frac{1}{2}} + e^x) \, dx$$
$$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + e^x + c$$
$$= \tfrac{2}{3}x^{\frac{3}{2}} + e^x + c$$

c
$$\int \left(3e^x - \frac{1}{x}\right) dx$$
$$= 3e^x - \ln|x| + c$$

d
$$\int \left(x\sqrt{x} - \frac{2}{x}\right) dx$$
$$= \int \left(x^{\frac{3}{2}} - \frac{2}{x}\right) dx$$
$$= \frac{x^{\frac{5}{2}}}{\frac{5}{2}} - 2\ln|x| + c$$
$$= \tfrac{2}{5}x^{\frac{5}{2}} - 2\ln|x| + c$$

e
$$\int \left(\frac{1}{x\sqrt{x}} + \frac{4}{x}\right) dx$$
$$= \int \left(x^{-\frac{3}{2}} + \frac{4}{x}\right) dx$$
$$= \frac{x^{-\frac{1}{2}}}{-\frac{1}{2}} + 4\ln|x| + c$$
$$= -2x^{-\frac{1}{2}} + 4\ln|x| + c$$

f
$$\int (\tfrac{1}{2}x^3 - x^4 + x^{\frac{1}{3}}) \, dx$$
$$= \tfrac{1}{2}\frac{x^4}{4} - \frac{x^5}{5} + \frac{x^{\frac{4}{3}}}{\frac{4}{3}} + c$$
$$= \tfrac{1}{8}x^4 - \tfrac{1}{5}x^5 + \tfrac{3}{4}x^{\frac{4}{3}} + c$$

g
$$\int \left(x^2 + \frac{3}{x}\right) dx$$
$$= \tfrac{1}{3}x^3 + 3\ln|x| + c$$

h
$$\int \left(\frac{1}{2x} + x^2 - e^x\right) dx$$
$$= \tfrac{1}{2}\ln|x| + \tfrac{1}{3}x^3 - e^x + c$$

i
$$\int \left(5e^x + \tfrac{1}{3}x^3 - \frac{4}{x}\right) dx$$
$$= 5e^x + \tfrac{1}{3}\frac{x^4}{4} - 4\ln|x| + c$$
$$= 5e^x + \tfrac{1}{12}x^4 - 4\ln|x| + c$$

2 **a**
$$\int (3\sin x - 2) \, dx$$
$$= -3\cos x - 2x + c$$

b
$$\int (4x - 2\cos x) \, dx$$
$$= 2x^2 - 2\sin x + c$$

c
$$\int (\sin x - 2\cos x + e^x) \, dx$$
$$= -\cos x - 2\sin x + e^x + c$$

d
$$\int (x^2\sqrt{x} - 10\sin x) \, dx$$
$$= \int (x^{\frac{5}{2}} - 10\sin x) \, dx$$
$$= \tfrac{2}{7}x^{\frac{7}{2}} + 10\cos x + c$$
$$= \tfrac{2}{7}x^3\sqrt{x} + 10\cos x + c$$

e
$$\int \left(\frac{x(x-1)}{3} + \cos x\right) dx$$
$$= \int \left(\tfrac{1}{3}x^2 - \tfrac{1}{3}x + \cos x\right) dx$$
$$= \tfrac{1}{9}x^3 - \tfrac{1}{6}x^2 + \sin x + c$$

f
$$\int (-\sin x + 2\sqrt{x}) \, dx$$
$$= \int (-\sin x + 2x^{\frac{1}{2}}) \, dx$$
$$= \cos x + \tfrac{4}{3}x^{\frac{3}{2}} + c$$
$$= \cos x + \tfrac{4}{3}x\sqrt{x} + c$$

3 **a**
$$\int (x^2 + 3x - 2) \, dx$$
$$= \tfrac{1}{3}x^3 + \tfrac{3}{2}x^2 - 2x + c$$

b
$$\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right) dx$$
$$= \int \left(x^{\frac{1}{2}} - x^{-\frac{1}{2}}\right) dx$$
$$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c$$
$$= \tfrac{2}{3}x^{\frac{3}{2}} - 2x^{\frac{1}{2}} + c$$

c
$$\int \left(2e^x - \frac{1}{x^2}\right) dx$$
$$= \int \left(2e^x - x^{-2}\right) dx$$
$$= 2e^x - \frac{x^{-1}}{-1} + c$$
$$= 2e^x + \frac{1}{x} + c$$

d $\int \left(\frac{1-4x}{x\sqrt{x}}\right) dx$
$= \int \left(\frac{1}{x\sqrt{x}} - \frac{4}{\sqrt{x}}\right) dx$
$= \int (x^{-\frac{3}{2}} - 4x^{-\frac{1}{2}})\, dx$
$= \frac{x^{-\frac{1}{2}}}{-\frac{1}{2}} - \frac{4x^{\frac{1}{2}}}{\frac{1}{2}} + c$
$= -2x^{-\frac{1}{2}} - 8x^{\frac{1}{2}} + c$

e $\int (2x+1)^2\, dx$
$= \int (4x^2+4x+1)\, dx$
$= \frac{4x^3}{3} + \frac{4x^2}{2} + x + c$
$= \frac{4}{3}x^3 + 2x^2 + x + c$

f $\int \frac{x^2+x-3}{x}\, dx$
$= \int \left(x + 1 - \frac{3}{x}\right) dx$
$= \frac{1}{2}x^2 + x - 3\ln|x| + c$

g $\int \frac{2x-1}{\sqrt{x}}\, dx$
$= \int \left(2x^{\frac{1}{2}} - x^{-\frac{1}{2}}\right) dx$
$= \frac{2x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c$
$= \frac{4}{3}x^{\frac{3}{2}} - 2x^{\frac{1}{2}} + c$

h $\int \frac{x^2-4x+10}{x^2\sqrt{x}}\, dx$
$= \int \left(\frac{x^2}{x^2\sqrt{x}} - \frac{4x}{x^2\sqrt{x}} + \frac{10}{x^2\sqrt{x}}\right) dx$
$= \int \left(x^{-\frac{1}{2}} - 4x^{-\frac{3}{2}} + 10x^{-\frac{5}{2}}\right) dx$
$= \frac{x^{\frac{1}{2}}}{\frac{1}{2}} - \frac{4x^{-\frac{1}{2}}}{-\frac{1}{2}} + \frac{10x^{-\frac{3}{2}}}{-\frac{3}{2}} + c$
$= 2x^{\frac{1}{2}} + 8x^{-\frac{1}{2}} - \frac{20}{3}x^{-\frac{3}{2}} + c$

i $\int (x+1)^3\, dx$
$= \int (x^3+3x^2+3x+1)\, dx$
$= \frac{1}{4}x^4 + x^3 + \frac{3}{2}x^2 + x + c$

4 **a** $\int (\sqrt{x} + \frac{1}{2}\cos x)\, dx$
$= \int (x^{\frac{1}{2}} + \frac{1}{2}\cos x)\, dx$
$= \frac{2}{3}x^{\frac{3}{2}} + \frac{1}{2}\sin x + c$

b $\int (2e^t - 4\sin t)\, dt$
$= 2e^t + 4\cos t + c$

c $\int \left(3\cos t - \frac{1}{t}\right) dt$
$= 3\sin t - \ln|t| + c$

d $\int (\sec^2 x + 2\sin x)\, dx$
$= \tan x - 2\cos x + c$

e $\int (\theta - \sin\theta)\, d\theta$
$= \frac{1}{2}\theta^2 - (-\cos\theta) + c$
$= \frac{1}{2}\theta^2 + \cos\theta + c$

f $\int \left(\frac{2}{\theta} - \sec^2\theta\right) d\theta$
$= 2\ln|\theta| - \tan\theta + c$

5 **a** $\frac{dy}{dx} = 6$
$\therefore\ y = \int 6\, dx$
$\therefore\ y = 6x + c$

b $\frac{dy}{dx} = 4x^2$
$\therefore\ y = \int 4x^2\, dx$
$\therefore\ y = \frac{4}{3}x^3 + c$

c $\frac{dy}{dx} = 5\sqrt{x} - x^2 = 5x^{\frac{1}{2}} - x^2$
$\therefore\ y = \int (5x^{\frac{1}{2}} - x^2)\, dx$
$\therefore\ y = \frac{10}{3}x^{\frac{3}{2}} - \frac{1}{3}x^3 + c$
$\therefore\ y = \frac{10}{3}x\sqrt{x} - \frac{1}{3}x^3 + c$

d $\frac{dy}{dx} = \frac{1}{x^2} = x^{-2}$
$\therefore\ y = \int x^{-2}\, dx$
$\therefore\ y = \frac{x^{-1}}{-1} + c$
$\therefore\ y = -\frac{1}{x} + c$

e $\frac{dy}{dx} = 2e^x - 5$
$\therefore\ y = \int (2e^x - 5)\, dx$
$\therefore\ y = 2e^x - 5x + c$

f $\frac{dy}{dx} = 4x^3 + 3x^2$
$\therefore\ y = \int (4x^3 + 3x^2)\, dx$
$= \frac{4x^4}{4} + \frac{3x^3}{3} + c$
$\therefore\ y = x^4 + x^3 + c$

6 **a** $\dfrac{dy}{dx} = (1-2x)^2$

$\therefore\ y = \int (1-2x)^2\, dx$

$= \int (1 - 4x + 4x^2)\, dx$

$= x - \dfrac{4x^2}{2} + \dfrac{4x^3}{3} + c$

$= x - 2x^2 + \frac{4}{3}x^3 + c$

b $\dfrac{dy}{dx} = \sqrt{x} - \dfrac{2}{\sqrt{x}}$

$= x^{\frac{1}{2}} - 2x^{-\frac{1}{2}}$

$\therefore\ y = \int (x^{\frac{1}{2}} - 2x^{-\frac{1}{2}})\, dx$

$= \dfrac{x^{\frac{3}{2}}}{\frac{3}{2}} - \dfrac{2x^{\frac{1}{2}}}{\frac{1}{2}} + c$

$= \frac{2}{3}x^{\frac{3}{2}} - 4\sqrt{x} + c$

c $\dfrac{dy}{dx} = \dfrac{x^2 + 2x - 5}{x^2}$

$= 1 + 2x^{-1} - 5x^{-2}$

$\therefore\ y = \int (1 + 2x^{-1} - 5x^{-2})\, dx$

$= x + 2\ln|x| - \dfrac{5x^{-1}}{-1} + c$

$= x + 2\ln|x| + \dfrac{5}{x} + c$

7 **a** $f'(x) = x^3 - 5\sqrt{x} + 3$

$= x^3 - 5x^{\frac{1}{2}} + 3$

$\therefore\ f(x) = \int (x^3 - 5x^{\frac{1}{2}} + 3)\, dx$

$= \frac{1}{4}x^4 - \frac{10}{3}x^{\frac{3}{2}} + 3x + c$

$= \frac{1}{4}x^4 - \frac{10}{3}x\sqrt{x} + 3x + c$

b $f'(x) = 2\sqrt{x}(1 - 3x)$

$= 2x^{\frac{1}{2}} - 6x^{\frac{3}{2}}$

$\therefore\ f(x) = \int (2x^{\frac{1}{2}} - 6x^{\frac{3}{2}})\, dx$

$= \dfrac{2x^{\frac{3}{2}}}{\frac{3}{2}} - \dfrac{6x^{\frac{5}{2}}}{\frac{5}{2}} + c$

$= \frac{4}{3}x^{\frac{3}{2}} - \frac{12}{5}x^{\frac{5}{2}} + c$

c $f'(x) = 3e^x - \dfrac{4}{x}$

$\therefore\ f(x) = \displaystyle\int \left(3e^x - \frac{4}{x}\right) dx$

$= 3e^x - 4\ln|x| + c$

8 $\dfrac{d}{dx}(e^x \sin x) = e^x \sin x + e^x \cos x$

$\therefore\ \int e^x(\sin x + \cos x)\, dx$

$= \int (e^x \sin x + e^x \cos x)\, dx$

$= e^x \sin x + c$

9 $\dfrac{d}{dx}(e^{-x} \sin x) = -e^{-x}\sin x + e^{-x}\cos x$

$= \dfrac{\cos x - \sin x}{e^x}$

$\therefore\ \displaystyle\int \frac{\cos x - \sin x}{e^x}\, dx = e^{-x}\sin x + c$

10 $\dfrac{d}{dx}(x \cos x) = \cos x + x(-\sin x)$

$= \cos x - x \sin x$

$\therefore\ \int (\cos x - x\sin x)\, dx = x\cos x + c$

$\therefore\ \int \cos x\, dx - \int x \sin x\, dx = x \cos x + c$

$\therefore\ \sin x - \int x\sin x\, dx = x\cos x + c$

$\therefore\ \int x \sin x\, dx = \sin x - x\cos x + c$

11 $\dfrac{d}{dx}(\sec x) = \sec x \tan x$

$\therefore\ \int \tan x \sec x\, dx = \sec x + c$

12 $\dfrac{d}{dx}(x - 3\arctan x) = 1 - 3\left(\dfrac{1}{x^2+1}\right),\quad x \in \mathbb{R}$

$= 1 - \dfrac{3}{x^2+1}$

$= \dfrac{1 + x^2 - 3}{x^2+1}$

$= \dfrac{x^2 - 2}{x^2 + 1}$

$\therefore\ \displaystyle\int \frac{x^2-2}{x^2+1}\, dx = x - 3\arctan x + c$

13 **a** $\dfrac{d}{dx}(\arccos x) = -\dfrac{1}{\sqrt{1-x^2}}, \quad x \in]-1, 1[$

$\dfrac{d}{dx}(\arcsin x) = \dfrac{1}{\sqrt{1-x^2}}, \quad x \in]-1, 1[$

b $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + c$

Also, $\displaystyle\int -\frac{1}{\sqrt{1-x^2}}\,dx = \arccos x + c$

$\therefore \displaystyle -\int \frac{1}{\sqrt{1-x^2}}\,dx = \arccos x + c$

$\therefore \displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx = -\arccos x + c$

c $\dfrac{d}{dx}(\arcsin x) = -\dfrac{d}{dx}(\arccos x)$

So, the solution can be written in terms of either.

EXERCISE 21E.2

1 **a** $f'(x) = 2x - 1$

$\therefore f(x) = \int (2x-1)\,dx$

$= \dfrac{2x^2}{2} - x + c$

$= x^2 - x + c$

But $f(0) = 3$, so $0 - 0 + c = 3$

$\therefore c = 3$

$\therefore f(x) = x^2 - x + 3$

b $f'(x) = 3x^2 + 2x$

$\therefore f(x) = \int (3x^2 + 2x)\,dx$

$= \dfrac{3x^3}{3} + \dfrac{2x^2}{2} + c$

$= x^3 + x^2 + c$

But $f(2) = 5$, so $8 + 4 + c = 5$

$\therefore c = -7$

$\therefore f(x) = x^3 + x^2 - 7$

c $f'(x) = e^x + \dfrac{1}{\sqrt{x}} = e^x + x^{-\frac{1}{2}}$

$\therefore f(x) = \int (e^x + x^{-\frac{1}{2}})\,dx$

$= e^x + 2x^{\frac{1}{2}} + c$

But $f(1) = 1$, so $e^1 + 2 + c = 1$

$\therefore c = -1 - e$

$\therefore f(x) = e^x + 2\sqrt{x} - 1 - e$

d $f'(x) = x - \dfrac{2}{\sqrt{x}} = x - 2x^{-\frac{1}{2}}$

$\therefore f(x) = \int (x - 2x^{-\frac{1}{2}})\,dx$

$= \dfrac{x^2}{2} - \dfrac{2x^{\frac{1}{2}}}{\frac{1}{2}} + c$

$= \frac{1}{2}x^2 - 4\sqrt{x} + c$

But $f(1) = 2$, so $\frac{1}{2} - 4 + c = 2$

$\therefore c = \frac{11}{2}$

$\therefore f(x) = \frac{1}{2}x^2 - 4\sqrt{x} + \frac{11}{2}$

2 **a** $f'(x) = x^2 - 4\cos x$

$\therefore f(x) = \int (x^2 - 4\cos x)\,dx$

$= \dfrac{x^3}{3} - 4\sin x + c$

But $f(0) = 3 \quad \therefore 0 - 4\sin(0) + c = 3$

$\therefore c = 3$

$\therefore f(x) = \dfrac{x^3}{3} - 4\sin x + 3$

b $f'(x) = 2\cos x - 3\sin x$

$\therefore f(x) = \int (2\cos x - 3\sin x)\,dx$

$= 2\sin x + 3\cos x + c$

But $f(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$

$\therefore 2\sin\frac{\pi}{4} + 3\cos\frac{\pi}{4} + c = \frac{1}{\sqrt{2}}$

$\therefore 2(\frac{1}{\sqrt{2}}) + 3(\frac{1}{\sqrt{2}}) + c = \frac{1}{\sqrt{2}}$

$\therefore c = -\frac{4}{\sqrt{2}}$

$\therefore c = -2\sqrt{2}$

$\therefore f(x) = 2\sin x + 3\cos x - 2\sqrt{2}$

c $f'(x) = \sqrt{x} - 2\sec^2 x$

$\therefore f(x) = \int (x^{\frac{1}{2}} - 2\sec^2 x)\,dx$

$= \frac{2}{3}x^{\frac{3}{2}} - 2\tan x + c$

But $f(\pi) = 0 \quad \therefore \frac{2}{3}\pi^{\frac{3}{2}} - 2\tan\pi + c = 0$

$\therefore c = -\frac{2}{3}\pi^{\frac{3}{2}}$

$\therefore f(x) = \frac{2}{3}x^{\frac{3}{2}} - 2\tan x - \frac{2}{3}\pi^{\frac{3}{2}}$

3 **a** Given:
$f''(x) = 2x + 1, \quad f'(1) = 3, \quad f(2) = 7$

$\therefore \; f'(x) = \int (2x+1)\,dx$
$= \dfrac{2x^2}{2} + x + c$
$= x^2 + x + c$

But $f'(1) = 3$ so $1 + 1 + c = 3$
$\therefore \; c = 1$

$\therefore \; f'(x) = x^2 + x + 1$

Then $f(x) = \int (x^2 + x + 1)\,dx$
$= \dfrac{x^3}{3} + \dfrac{x^2}{2} + x + k$

But $f(2) = 7$, so $\frac{8}{3} + 2 + 2 + k = 7$
$\therefore \; k = 7 - 4 - \frac{8}{3}$
$\therefore \; k = \frac{1}{3}$
$\therefore \; f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + \frac{1}{3}$

b Given: $f''(x) = 15\sqrt{x} + \dfrac{3}{\sqrt{x}}$,
$f'(1) = 12, \quad f(0) = 5$

Now $f''(x) = 15x^{\frac{1}{2}} + 3x^{-\frac{1}{2}}$

$\therefore \; f'(x) = \dfrac{15x^{\frac{3}{2}}}{\frac{3}{2}} + \dfrac{3x^{\frac{1}{2}}}{\frac{1}{2}} + c$
$= 10x^{\frac{3}{2}} + 6x^{\frac{1}{2}} + c$

But $f'(1) = 12$ so $10 + 6 + c = 12$
$\therefore \; c = -4$

$\therefore \; f'(x) = 10x^{\frac{3}{2}} + 6x^{\frac{1}{2}} - 4$
$\therefore \; f(x) = \int (10x^{\frac{3}{2}} + 6x^{\frac{1}{2}} - 4)\,dx$
$= \dfrac{10x^{\frac{5}{2}}}{\frac{5}{2}} + \dfrac{6x^{\frac{3}{2}}}{\frac{3}{2}} - 4x + k$
$= 4x^{\frac{5}{2}} + 4x^{\frac{3}{2}} - 4x + k$

But $f(0) = 5$, so $k = 5$

$\therefore \; f(x) = 4x^{\frac{5}{2}} + 4x^{\frac{3}{2}} - 4x + 5$

c Given: $f''(x) = \cos x, \quad f'(\frac{\pi}{2}) = 0$ and $f(0) = 3$

Now $f'(x) = \int \cos x\,dx = \sin x + c$

But $f'(\frac{\pi}{2}) = 0$ so $\sin(\frac{\pi}{2}) + c = 0$
$\therefore \; c = -1$

$\therefore \; f'(x) = \sin x - 1$

So, $f(x) = \int (\sin x - 1)\,dx$
$= -\cos x - x + k$

But $f(0) = 3$ so $-\cos 0 - 0 + k = 3$
$\therefore \; -1 + k = 3$
$\therefore \; k = 4$

So, $f(x) = -\cos x - x + 4$

d Given: $f''(x) = 2x$ and that $(1, 0)$ and $(0, 5)$ lie on the curve

Now $f'(x) = \int 2x\,dx = \dfrac{2x^2}{2} + c = x^2 + c$

$\therefore \; f(x) = \int (x^2 + c)\,dx = \dfrac{x^3}{3} + cx + k$

But $f(0) = 5$ so $0 + 0 + k = 5$ and so $k = 5$

and $f(1) = 0$ so $\frac{1}{3} + c + 5 = 0$ and so $c = -5\frac{1}{3}$

$\therefore \; f(x) = \frac{1}{3}x^3 - \frac{16}{3}x + 5$

EXERCISE 21F

1 **a** $\int (2x+5)^3\,dx$
$= \frac{1}{2} \times \dfrac{(2x+5)^4}{4} + c$
$= \frac{1}{8}(2x+5)^4 + c$

b $\displaystyle\int \frac{1}{(3-2x)^2}\,dx$
$= \int (3-2x)^{-2}\,dx$
$= \frac{1}{-2} \times \dfrac{(3-2x)^{-1}}{-1} + c$
$= \dfrac{1}{2(3-2x)} + c$

c $\displaystyle\int \frac{4}{(2x-1)^4}\,dx$
$= \int 4\,(2x-1)^{-4}\,dx$
$= 4(\frac{1}{2}) \times \dfrac{(2x-1)^{-3}}{-3} + c$
$= \dfrac{-2}{3(2x-1)^3} + c$

d $\int (4x-3)^7\, dx$
$= \frac{1}{4} \times \frac{(4x-3)^8}{8} + c$
$= \frac{1}{32}(4x-3)^8 + c$

e $\int \sqrt{3x-4}\, dx$
$= \int (3x-4)^{\frac{1}{2}}\, dx$
$= \frac{1}{3} \times \frac{(3x-4)^{\frac{3}{2}}}{\frac{3}{2}} + c$
$= \frac{2}{9}(3x-4)^{\frac{3}{2}} + c$

f $\int \frac{10}{\sqrt{1-5x}}\, dx$
$= \int 10(1-5x)^{-\frac{1}{2}}\, dx$
$= 10(\frac{1}{-5}) \times \frac{(1-5x)^{\frac{1}{2}}}{\frac{1}{2}} + c$
$= -4\sqrt{1-5x} + c$

g $\int 3(1-x)^4\, dx$
$= 3\int (1-x)^4\, dx$
$= 3(\frac{1}{-1}) \times \frac{(1-x)^5}{5} + c$
$= -\frac{3}{5}(1-x)^5 + c$

h $\int \frac{4}{\sqrt{3-4x}}\, dx$
$= \int 4(3-4x)^{-\frac{1}{2}}\, dx$
$= 4(\frac{1}{-4}) \times \frac{(3-4x)^{\frac{1}{2}}}{\frac{1}{2}} + c$
$= -2\sqrt{3-4x} + c$

2 **a** $\int \sin(3x)\, dx$
$= -\frac{1}{3}\cos(3x) + c$

b $\int (2\cos(-4x) + 1)\, dx$
$= 2 \times (\frac{1}{-4})\sin(-4x) + x + c$
$= -\frac{1}{2}\sin(-4x) + x + c$

c $\int \sec^2(2x)\, dx$
$= \frac{1}{2}\tan(2x) + c$

d $\int 3\cos\left(\frac{x}{2}\right)\, dx$
$= 6\sin\left(\frac{x}{2}\right) + c$

e $\int (3\sin(2x) - e^{-x})\, dx$
$= -\frac{3}{2}\cos(2x) + e^{-x} + c$

f $\int \left[e^{2x} - 2\sec^2\left(\frac{x}{2}\right)\right]\, dx$
$= \frac{1}{2}e^{2x} - 2 \times 2\tan\left(\frac{x}{2}\right) + c$
$= \frac{1}{2}e^{2x} - 4\tan\left(\frac{x}{2}\right) + c$

g $\int 2\sin(2x + \frac{\pi}{6})\, dx$
$= -\frac{2}{2}\cos(2x + \frac{\pi}{6}) + c$
$= -\cos(2x + \frac{\pi}{6}) + c$

h $\int -3\cos(\frac{\pi}{4} - x)\, dx$
$= -3 \times (-1)\sin(\frac{\pi}{4} - x) + c$
$= 3\sin(\frac{\pi}{4} - x) + c$

i $\int 4\sec^2\left(\frac{\pi}{3} - 2x\right)\, dx$
$= 4 \times (-\frac{1}{2})\tan\left(\frac{\pi}{3} - 2x\right) + c$
$= -2\tan\left(\frac{\pi}{3} - 2x\right) + c$

j $\int (\cos(2x) + \sin(2x))\, dx$
$= \frac{1}{2}\sin(2x) - \frac{1}{2}\cos(2x) + c$

k $\int (2\sin(3x) + 5\cos(4x))\, dx$
$= -\frac{2}{3}\cos(3x) + \frac{5}{4}\sin(4x) + c$

l $\int \left(\frac{1}{2}\cos(8x) - 3\sin x\right)\, dx$
$= \frac{1}{2}(\frac{1}{8})\sin(8x) + 3\cos x + c$
$= \frac{1}{16}\sin(8x) + 3\cos x + c$

3 **a** $\frac{dy}{dx} = \sqrt{2x-7} = (2x-7)^{\frac{1}{2}}$
$\therefore\ y = \frac{1}{2} \times \frac{(2x-7)^{\frac{3}{2}}}{\frac{3}{2}} + c$
$= \frac{1}{3}(2x-7)^{\frac{3}{2}} + c$

But $y = 11$ when $x = 8$
$\therefore\ \frac{1}{3}(16-7)^{\frac{3}{2}} + c = 11$
$\therefore\ \frac{1}{3}(27) + c = 11$
$\therefore\ 9 + c = 11$ and so $c = 2$
$\therefore\ y = \frac{1}{3}(2x-7)^{\frac{3}{2}} + 2$

b $f(x)$ has gradient function $f'(x) = \dfrac{4}{\sqrt{1-x}} = 4(1-x)^{-\frac{1}{2}}$

$\therefore \quad f(x) = 4(\frac{1}{-1}) \times \dfrac{(1-x)^{\frac{1}{2}}}{\frac{1}{2}} + c$
$= -8\sqrt{1-x} + c$

But $y = -11$ when $x = -3$
$\therefore \quad -8\sqrt{1-(-3)} + c = -11$
$\therefore \quad -8\sqrt{4} + c = -11$
$\therefore \quad -16 + c = -11$ and so $c = 5$
$\therefore \quad f(x) = 5 - 8\sqrt{1-x}$

Now $f(-8) = 5 - 8\sqrt{1-(-8)} = 5 - 8(3) = -19$, so the point is $(-8, -19)$.

4 **a** $\int \cos^2 x \, dx$
$= \int (\frac{1}{2} + \frac{1}{2}\cos(2x)) \, dx$
$= \frac{1}{2}x + \frac{1}{4}\sin(2x) + c$

b $\int \sin^2 x \, dx$
$= \int (\frac{1}{2} - \frac{1}{2}\cos(2x)) \, dx$
$= \frac{1}{2}x - \frac{1}{4}\sin(2x) + c$

c $\int (1 + \cos^2(2x)) \, dx$
$= \int (1 + \frac{1}{2} + \frac{1}{2}\cos(4x)) \, dx$
$= \int (\frac{3}{2} + \frac{1}{2}\cos(4x)) \, dx$
$= \frac{3}{2}x + \frac{1}{8}\sin(4x) + c$

d $\int (3 - \sin^2(3x)) \, dx$
$= \int (3 - (\frac{1}{2} - \frac{1}{2}\cos(6x))) \, dx$
$= \int (\frac{5}{2} + \frac{1}{2}\cos(6x)) \, dx$
$= \frac{5}{2}x + \frac{1}{12}\sin(6x) + c$

e $\int \frac{1}{2}\cos^2(4x) \, dx$
$= \int \frac{1}{2}(\frac{1}{2} + \frac{1}{2}\cos(8x)) \, dx$
$= \int (\frac{1}{4} + \frac{1}{4}\cos(8x)) \, dx$
$= \frac{1}{4}x + \frac{1}{32}\sin(8x) + c$

f $\int (1 + \cos x)^2 \, dx$
$= \int (1 + 2\cos x + \cos^2 x) \, dx$
$= \int (1 + 2\cos x + \frac{1}{2} + \frac{1}{2}\cos(2x)) \, dx$
$= \int \left(\frac{3}{2} + 2\cos x + \frac{1}{2}\cos(2x)\right) dx$
$= \frac{3}{2}x + 2\sin x + \frac{1}{4}\sin(2x) + c$

5 **a** $\int 3(2x-1)^2 \, dx$
$= 3\int (2x-1)^2 \, dx$
$= 3(\frac{1}{2})\,\dfrac{(2x-1)^3}{3} + c$
$= \frac{1}{2}(2x-1)^3 + c$

b $\int (x^2 - x)^2 \, dx$
$= \int \left(x^4 - 2x^3 + x^2\right) dx$
$= \dfrac{x^5}{5} - \dfrac{2x^4}{4} + \dfrac{x^3}{3} + c$
$= \frac{1}{5}x^5 - \frac{1}{2}x^4 + \frac{1}{3}x^3 + c$

c $\int (1-3x)^3 \, dx$
$= (\frac{1}{-3})\,\dfrac{(1-3x)^4}{4} + c$
$= -\frac{1}{12}(1-3x)^4 + c$

d $\int (1-x^2)^2 \, dx$
$= \int (1 - 2x^2 + x^4) \, dx$
$= x - \frac{2}{3}x^3 + \frac{1}{5}x^5 + c$

e $\int 4\sqrt{5-x} \, dx$
$= 4\int (5-x)^{\frac{1}{2}} \, dx$
$= 4(\frac{1}{-1})\,\dfrac{(5-x)^{\frac{3}{2}}}{\frac{3}{2}} + c$
$= -\frac{8}{3}(5-x)^{\frac{3}{2}} + c$

f $\int (x^2+1)^3 \, dx$
$= \int (x^6 + 3x^4 + 3x^2 + 1) \, dx$
$= \dfrac{x^7}{7} + \dfrac{3x^5}{5} + \dfrac{3x^3}{3} + x + c$
$= \frac{1}{7}x^7 + \frac{3}{5}x^5 + x^3 + x + c$

6 **a** $\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos(2\theta)$

$\therefore \quad \cos^4 x = (\frac{1}{2} + \frac{1}{2}\cos(2x))^2$
$= \frac{1}{4} + \frac{1}{2}\cos(2x) + \frac{1}{4}\cos^2(2x)$
$= \frac{1}{4} + \frac{1}{2}\cos(2x) + \frac{1}{4}(\frac{1}{2} + \frac{1}{2}\cos(4x))$
$= \frac{1}{4} + \frac{1}{2}\cos(2x) + \frac{1}{8} + \frac{1}{8}\cos(4x)$
$= \frac{1}{8}\cos(4x) + \frac{1}{2}\cos(2x) + \frac{3}{8}$ as required

b $\int \cos^4 x \, dx = \int (\frac{1}{8}\cos(4x) + \frac{1}{2}\cos(2x) + \frac{3}{8}) \, dx$
$= \frac{1}{32}\sin(4x) + \frac{1}{4}\sin(2x) + \frac{3}{8}x + c$

7 **a** $\int \left(2e^x + 5e^{2x}\right)\, dx$
$= 2e^x + 5(\frac{1}{2})e^{2x} + c$
$= 2e^x + \frac{5}{2}e^{2x} + c$

b $\int (3e^{5x-2})\, dx$
$= 3(\frac{1}{5})e^{5x-2} + c$
$= \frac{3}{5}e^{5x-2} + c$

c $\int (e^{7-3x})\, dx$
$= \frac{1}{-3}e^{7-3x} + c$
$= -\frac{1}{3}e^{7-3x} + c$

d $\int \frac{1}{2x-1}\, dx$
$= \frac{1}{2}\ln|2x-1| + c$

e $\int \frac{5}{1-3x}\, dx$
$= 5\int \frac{1}{1-3x}\, dx$
$= 5(\frac{1}{-3})\ln|1-3x| + c$
$= -\frac{5}{3}\ln|1-3x| + c$

f $\int \left(e^{-x} - \frac{4}{2x+1}\right) dx$
$= \frac{1}{-1}e^{-x} - 4(\frac{1}{2})\ln|2x+1| + c$
$= -e^{-x} - 2\ln|2x+1| + c$

g $\int (e^x + e^{-x})^2\, dx$
$= \int \left(e^{2x} + 2 + e^{-2x}\right)\, dx$
$= \frac{1}{2}e^{2x} + 2x + (\frac{1}{-2})e^{-2x} + c$
$= \frac{1}{2}e^{2x} + 2x - \frac{1}{2}e^{-2x} + c$

h $\int (e^{-x} + 2)^2\, dx$
$= \int \left(e^{-2x} + 4e^{-x} + 4\right) dx$
$= \frac{1}{-2}e^{-2x} + 4(\frac{1}{-1})e^{-x} + 4x + c$
$= -\frac{1}{2}e^{-2x} - 4e^{-x} + 4x + c$

i $\int \left(x - \frac{5}{1-x}\right) dx$
$= \frac{x^2}{2} - 5(\frac{1}{-1})\ln|1-x| + c$
$= \frac{1}{2}x^2 + 5\ln|1-x| + c$

8 **a** $\frac{dy}{dx} = (1-e^x)^2$
$= 1 - 2e^x + e^{2x}$
$\therefore\ y = x - 2e^x + \frac{1}{2}e^{2x} + c$

b $\frac{dy}{dx} = 1 - 2x + \frac{3}{x+2}$
$\therefore\ y = x - \frac{2x^2}{2} + 3\ln|x+2| + c$
$= x - x^2 + 3\ln|x+2| + c$

c $\frac{dy}{dx} = e^{-2x} + \frac{4}{2x-1}$
$\therefore\ y = \frac{1}{-2}e^{-2x} + 4(\frac{1}{2})\ln|2x-1| + c$
$= -\frac{1}{2}e^{-2x} + 2\ln|2x-1| + c$

9 Differentiating Tracy's answer gives
$\frac{d}{dx}\left(\frac{1}{4}\ln(4x) + c\right) = \frac{1}{4}\left(\frac{1}{4x}\right) \times 4 + 0,\ \ x > 0$
$= \frac{1}{4x},\ \ x > 0$

Differentiating Nadine's answer gives
$\frac{d}{dx}\left(\frac{1}{4}\ln(x) + c\right) = \frac{1}{4}\left(\frac{1}{x}\right) + 0,\ \ x > 0$
$= \frac{1}{4x},\ \ x > 0$

Both answers give the correct derivative and both are correct. This result occurs because $\ln(4x) = \ln 4 + \ln x$. Their answers differ by a constant which is accounted for by c.

10 Given: $f'(x) = p\sin(\frac{1}{2}x)$, $f(0) = 1$ and $f(2\pi) = 0$
$\therefore\ f(x) = -2p\cos(\frac{1}{2}x) + c$

But $f(0) = 1$,
so $-2p\cos(0) + c = 1$
$\therefore\ -2p + c = 1$
$\therefore\ c = 1 + 2p$ .... (1)

Also, $f(2\pi) = 0$, so $-2p\cos(\pi) + c = 0$
$\therefore\ 2p + c = 0$
$\therefore\ 2p + 1 + 2p = 0$ {using (1)}
$\therefore\ p = -\frac{1}{4}$
$\therefore\ c = \frac{1}{2}$ {from (1)}

$\therefore\ f(x) = \frac{1}{2}\cos(\frac{1}{2}x) + \frac{1}{2}$

11 $g''(x) = -\sin 2x$

Integrating both sides with respect to x, we get $g'(x) = \frac{1}{2}\cos 2x + c$, c some constant.

So, $g'(\pi) = \frac{1}{2}\cos(2\pi) + c = \frac{1}{2} + c$ and $g'(-\pi) = \frac{1}{2}\cos(-2\pi) + c = \frac{1}{2} + c = g'(\pi)$

$\therefore$ the gradients of the tangents to $y = g(x)$ at $x = \pi$ and $x = -\pi$ are equal.

12 **a**

$$\begin{aligned} f'(x) &= 2e^{-2x} \\ \therefore\ f(x) &= 2(\tfrac{1}{-2})e^{-2x} + c \\ &= -e^{-2x} + c \end{aligned}$$

But $f(0) = 3$ so $-e^0 + c = 3$

$\therefore\ c = 4$

$\therefore\ f(x) = -e^{-2x} + 4$

b

$$\begin{aligned} f'(x) &= 2x - \frac{2}{1-x} \\ \therefore\ f(x) &= \frac{2x^2}{2} - \frac{2}{-1}\ln|1-x| + c \\ &= x^2 + 2\ln|1-x| + c \end{aligned}$$

But $f(-1) = 3$ so $1 + 2\ln|2| + c = 3$

$\therefore\ c = 2 - 2\ln 2$

$\therefore\ f(x) = x^2 + 2\ln|1-x| + 2 - 2\ln 2$

c

$$\begin{aligned} f'(x) &= \sqrt{x} + \tfrac{1}{2}e^{-4x} \\ &= x^{\frac{1}{2}} + \tfrac{1}{2}e^{-4x} \\ \therefore\ f(x) &= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \tfrac{1}{2}(\tfrac{1}{-4})e^{-4x} + c \\ &= \tfrac{2}{3}x^{\frac{3}{2}} - \tfrac{1}{8}e^{-4x} + c \end{aligned}$$

But $f(1) = 0$

$\therefore\ \frac{2}{3} - \frac{1}{8}e^{-4} + c = 0$

$\therefore\ c = \frac{1}{8}e^{-4} - \frac{2}{3}$

$\therefore\ f(x) = \frac{2}{3}x^{\frac{3}{2}} - \frac{1}{8}e^{-4x} + \frac{1}{8}e^{-4} - \frac{2}{3}$

13

$$\begin{aligned} (\sin x + \cos x)^2 &= \sin^2 x + 2\sin x\cos x + \cos^2 x \\ &= 1 + \sin 2x \\ \therefore\ \int(\sin x + \cos x)^2\, dx &= \int(1 + \sin 2x)\, dx \\ &= x - \tfrac{1}{2}\cos(2x) + c \end{aligned}$$

14

$$\begin{aligned} (\cos x + 1)^2 &= \cos^2 x + 2\cos x + 1 \\ &= (\tfrac{1}{2} + \tfrac{1}{2}\cos 2x) + 2\cos x + 1 \\ &= \tfrac{1}{2}\cos 2x + 2\cos x + \tfrac{3}{2} \\ \therefore\ \int(\cos x + 1)^2\, dx &= \int(\tfrac{1}{2}\cos 2x + 2\cos x + \tfrac{3}{2})\, dx \\ &= \tfrac{1}{4}\sin 2x + 2\sin x + \tfrac{3}{2}x + c \end{aligned}$$

15

$$\begin{aligned} \frac{3}{x+2} - \frac{1}{x-2} &= \frac{3(x-2) - 1(x+2)}{(x+2)(x-2)} \\ &= \frac{3x - 6 - x - 2}{x^2 - 4} \\ &= \frac{2x - 8}{x^2 - 4} \\ \therefore\ \int \frac{2x-8}{x^2-4}\, dx &= \int\left(\frac{3}{x+2} - \frac{1}{x-2}\right) dx \\ &= 3\ln|x+2| - \ln|x-2| + c \end{aligned}$$

16

$$\begin{aligned} \frac{1}{2x-1} - \frac{1}{2x+1} &= \frac{1(2x+1) - 1(2x-1)}{(2x-1)(2x+1)} \\ &= \frac{2x + 1 - 2x + 1}{(2x-1)(2x+1)} \\ &= \frac{2}{4x^2 - 1} \end{aligned}$$

$$\begin{aligned} \therefore\ &\int \frac{2}{4x^2-1}\, dx \\ &= \int\left(\frac{1}{2x-1} - \frac{1}{2x+1}\right) dx \\ &= \tfrac{1}{2}\ln|2x-1| - \tfrac{1}{2}\ln|2x+1| + c \end{aligned}$$

EXERCISE 21G.1

1 **a** $u = x^3 + 1, \quad \dfrac{du}{dx} = 3x^2$

$$\begin{aligned}\therefore \int 3x^2(x^3+1)^4\,dx &= \int u^4 \frac{du}{dx}\,dx \\ &= \int u^4\,du \\ &= \tfrac{1}{5}u^5 + c \\ &= \tfrac{1}{5}(x^3+1)^5 + c\end{aligned}$$

b $u = x^3 + 1, \quad \dfrac{du}{dx} = 3x^2$

$$\begin{aligned}\therefore \int x^2 e^{x^3+1}\,dx &= \tfrac{1}{3}\int (3x^2)e^{x^3+1}\,dx \\ &= \tfrac{1}{3}\int e^u \frac{du}{dx}\,dx \\ &= \tfrac{1}{3}\int e^u\,du \\ &= \tfrac{1}{3}e^u + c \\ &= \tfrac{1}{3}e^{x^3+1} + c\end{aligned}$$

c $u = \sin x, \quad \dfrac{du}{dx} = \cos x$

$$\begin{aligned}\therefore \int \sin^4 x \cos x\,dx &= \int u^4 \frac{du}{dx}\,dx \\ &= \int u^4\,du \\ &= \frac{u^5}{5} + c \\ &= \tfrac{1}{5}\sin^5 x + c\end{aligned}$$

d $u = \dfrac{x-1}{x} = 1 - x^{-1}, \quad \dfrac{du}{dx} = \dfrac{1}{x^2}$

$$\begin{aligned}\therefore \int \frac{e^{\frac{x-1}{x}}}{x^2}\,dx &= \int e^u \frac{du}{dx}\,dx \\ &= \int e^u\,du \\ &= e^u + c \\ &= e^{\frac{x-1}{x}} + c\end{aligned}$$

2 **a**
$$\begin{aligned}&\int 4x^3(2+x^4)^3\,dx \\ &= \int u^3 \frac{du}{dx}\,dx \quad \left\{u = 2+x^4, \ \frac{du}{dx} = 4x^3\right\} \\ &= \int u^3\,du \\ &= \frac{u^4}{4} + c \\ &= \tfrac{1}{4}(2+x^4)^4 + c\end{aligned}$$

b
$$\begin{aligned}&\int \frac{2x}{\sqrt{x^2+3}}\,dx \\ &= \int ((x^2+3)^{-\frac{1}{2}} \times 2x)\,dx \\ &= \int u^{-\frac{1}{2}} \frac{du}{dx}\,dx \quad \left\{u = x^2+3, \ \frac{du}{dx} = 2x\right\} \\ &= \int u^{-\frac{1}{2}}\,du \\ &= \frac{u^{\frac{1}{2}}}{\frac{1}{2}} + c \\ &= 2\sqrt{u} + c \\ &= 2\sqrt{x^2+3} + c\end{aligned}$$

c
$$\begin{aligned}&\int \frac{x}{(1-x^2)^5}\,dx \\ &= -\tfrac{1}{2}\int (1-x^2)^{-5} \times (-2x)\,dx \\ &= -\tfrac{1}{2}\int u^{-5} \frac{du}{dx}\,dx \quad \{u = 1-x^2, \\ &= -\tfrac{1}{2}\int u^{-5}\,du \qquad\quad \frac{du}{dx} = -2x\} \\ &= -\tfrac{1}{2}\,\frac{u^{-4}}{-4} + c \\ &= \frac{1}{8(1-x^2)^4} + c\end{aligned}$$

d
$$\begin{aligned}&\int \sqrt{x^3+x}\,(3x^2+1)\,dx \\ &= \int \sqrt{u}\,\frac{du}{dx}\,dx \quad \{u = x^3+x, \\ &= \int u^{\frac{1}{2}}\,du \qquad\quad \frac{du}{dx} = 3x^2+1\} \\ &= \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c \\ &= \tfrac{2}{3}u^{\frac{3}{2}} + c \\ &= \tfrac{2}{3}(x^3+x)^{\frac{3}{2}} + c\end{aligned}$$

e
$$\begin{aligned}&\int (x^3+2x+1)^4(3x^2+2)\,dx \\ &= \int u^4 \frac{du}{dx}\,dx \quad \{u = x^3+2x+1, \\ &= \int u^4\,du \qquad\quad \frac{du}{dx} = 3x^2+2\} \\ &= \frac{u^5}{5} + c \\ &= \tfrac{1}{5}(x^3+2x+1)^5 + c\end{aligned}$$

f $\displaystyle\int \frac{x^2}{(3x^3-1)^4}\,dx$

$\displaystyle= \int (3x^3-1)^{-4} \times x^2\,dx$

$\displaystyle= \tfrac{1}{9}\int (3x^3-1)^{-4} \times 9x^2\,dx$

$\displaystyle= \tfrac{1}{9}\int u^{-4}\,\frac{du}{dx}\,dx$

$\displaystyle\{u = 3x^3-1,\ \ \frac{du}{dx} = 9x^2\}$

$\displaystyle= \tfrac{1}{9}\int u^{-4}\,du$

$\displaystyle= \tfrac{1}{9}\,\frac{u^{-3}}{-3} + c$

$\displaystyle= -\frac{1}{27(3x^3-1)^3} + c$

g $\displaystyle\int \frac{x+2}{(x^2+4x-3)^2}\,dx$

$= \frac{1}{2}\int (x^2+4x-3)^{-2}(2x+4)\,dx$

$= \frac{1}{2}\int u^{-2}\,\frac{du}{dx}\,dx \quad \{u = x^2+4x-3,$

$= \frac{1}{2}\int u^{-2}\,du \qquad \frac{du}{dx} = 2x+4\}$

$= \frac{1}{2}\,\frac{u^{-1}}{-1} + c$

$\displaystyle= \frac{-1}{2(x^2+4x-3)} + c$

h $\int x^4(x+1)^4(2x+1)\,dx$

$= \int (x^2+x)^4(2x+1)\,dx$

$= \int u^4\,\frac{du}{dx}\,dx \quad \{u = x^2+x,\ \ \frac{du}{dx} = 2x+1\}$

$= \int u^4\,du$

$= \frac{1}{5}u^5 + c$

$= \frac{1}{5}(x^2+x)^5 + c$

3 **a** $\int -2e^{1-2x}\,dx$

$= \int e^u\,\frac{du}{dx}\,dx \quad \{u = 1-2x,\ \ \frac{du}{dx} = -2\}$

$= \int e^u\,du$

$= e^u + c$

$= e^{1-2x} + c$

b $\int 2xe^{x^2}\,dx$

$= \int e^u\,\frac{du}{dx}\,dx \quad \{u = x^2,\ \ \frac{du}{dx} = 2x\}$

$= \int e^u\,du$

$= e^u + c$

$= e^{x^2} + c$

c $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2\int \frac{e^{\sqrt{x}}}{2\sqrt{x}}\,dx$

$= 2\int e^u\,\frac{du}{dx}\,dx \quad \{u = \sqrt{x},$

$= 2\int e^u\,du \qquad \frac{du}{dx} = \frac{1}{2\sqrt{x}}\}$

$= 2e^u + c$

$= 2e^{\sqrt{x}} + c$

d $\int (2x-1)e^{x-x^2}\,dx$

$= -\int (1-2x)e^{x-x^2}\,dx$

$= -\int e^u\,\frac{du}{dx}\,dx \quad \{u = x-x^2,$

$= -\int e^u\,du \qquad \frac{du}{dx} = 1-2x\}$

$= -e^u + c$

$= -e^{x-x^2} + c$

4 **a** Let $u = x^2+1,\ \ \dfrac{du}{dx} = 2x$

$\displaystyle\therefore\ \int \frac{2x}{x^2+1}\,dx = \int \frac{1}{x^2+1}\,(2x)\,dx$

$\displaystyle= \int \frac{1}{u}\,\frac{du}{dx}\,dx$

$\displaystyle= \int \frac{1}{u}\,du$

$= \ln|u| + c$

$= \ln\left|x^2+1\right| + c$

b Let $u = 2-x^2,\ \ \dfrac{du}{dx} = -2x$

$\displaystyle\therefore\ \int \frac{x}{2-x^2}\,dx = -\tfrac{1}{2}\int \frac{1}{2-x^2}\,(-2x)\,dx$

$\displaystyle= -\tfrac{1}{2}\int \frac{1}{u}\,\frac{du}{dx}\,dx$

$\displaystyle= -\tfrac{1}{2}\int \frac{1}{u}\,du$

$= -\frac{1}{2}\ln|u| + c$

$= -\frac{1}{2}\ln\left|2-x^2\right| + c$

c Let $u = x^2 - 3x$, $\dfrac{du}{dx} = 2x - 3$

$$\therefore \int \frac{2x-3}{x^2-3x}\,dx = \int \frac{1}{x^2-3x}(2x-3)\,dx = \int \frac{1}{u}\frac{du}{dx}\,dx = \int \frac{1}{u}\,du = \ln|u| + c = \ln\left|x^2-3x\right| + c$$

d Let $u = x^3 - x$, $\dfrac{du}{dx} = 3x^2 - 1$

$$\therefore \int \frac{6x^2-2}{x^3-x}\,dx = 2\int \frac{1}{x^3-3x}(3x^2-1)\,dx = 2\int \frac{1}{u}\frac{du}{dx}\,dx = 2\int \frac{1}{u}\,du = 2\ln|u| + c = 2\ln\left|x^3-x\right| + c$$

e Let $u = 5x - x^2$, $\dfrac{du}{dx} = 5 - 2x$

$$\therefore \int \frac{4x-10}{5x-x^2}\,dx = -2\int \frac{1}{5x-x^2}(5-2x)\,dx = -2\int \frac{1}{u}\frac{du}{dx}\,dx = -2\int \frac{1}{u}\,du = -2\ln|u| + c = -2\ln\left|5x-x^2\right| + c$$

f Let $u = x^3 - 3x$, $\dfrac{du}{dx} = 3x^2 - 3$

$$\therefore \int \frac{1-x^2}{x^3-3x}\,dx = -\tfrac{1}{3}\int \frac{1}{x^3-3x}(3x^2-3)\,dx = -\tfrac{1}{3}\int \frac{1}{u}\frac{du}{dx}\,dx = -\tfrac{1}{3}\int \frac{1}{u}\,du = -\tfrac{1}{3}\ln|u| + c = -\tfrac{1}{3}\ln\left|x^3-3x\right| + c$$

5 **a** Let $u = 3 - x^3$, $\dfrac{du}{dx} = -3x^2$

$$\therefore f(x) = \int x^2(3-x^3)^2\,dx = -\tfrac{1}{3}\int (-3x^2)(3-x^3)^2\,dx = -\tfrac{1}{3}\int u^2\frac{du}{dx}\,dx = -\tfrac{1}{3}\int u^2\,du = -\tfrac{1}{3}\times\frac{u^3}{3} + c = -\tfrac{1}{9}(3-x^3)^3 + c$$

b Let $u = \ln x$, $\dfrac{du}{dx} = \dfrac{1}{x}$

$$\therefore f(x) = \int \frac{4}{x\ln x}\,dx = 4\int \frac{1}{\ln x}\times\frac{1}{x}\,dx = 4\int u^{-1}\frac{du}{dx}\,dx = 4\int \frac{1}{u}\,du = 4\ln|u| + c = 4\ln|\ln x| + c$$

c Let $u = 1 - x^2$, $\dfrac{du}{dx} = -2x$

$$\therefore f(x) = \int x\sqrt{1-x^2}\,dx = -\tfrac{1}{2}\int (-2x)\sqrt{1-x^2}\,dx = -\tfrac{1}{2}\int \sqrt{u}\frac{du}{dx}\,dx = -\tfrac{1}{2}\int u^{\frac{1}{2}}\,du = -\tfrac{1}{2}\times\frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c = -\tfrac{1}{3}u^{\frac{3}{2}} + c = -\tfrac{1}{3}(1-x^2)^{\frac{3}{2}} + c$$

d Let $u = 1 - x^2$, $\dfrac{du}{dx} = -2x$

$$\therefore f(x) = \int xe^{1-x^2}\,dx = -\tfrac{1}{2}\int (-2x)e^{1-x^2}\,dx = -\tfrac{1}{2}\int e^u\frac{du}{dx}\,dx = -\tfrac{1}{2}\int e^u\,du = -\tfrac{1}{2}e^u + c = -\tfrac{1}{2}e^{1-x^2} + c$$

e Let $u = x^3 - x$, $\dfrac{du}{dx} = 3x^2 - 1$

$$\begin{aligned}\therefore\ f(x) &= \int \frac{1-3x^2}{x^3-x}\,dx \\ &= -\int \frac{3x^2-1}{x^3-x}\,dx \\ &= -\int \frac{1}{u}\frac{du}{dx}\,dx \\ &= -\int \frac{1}{u}\,du \\ &= -\ln|u| + c \\ &= -\ln\left|x^3-x\right| + c\end{aligned}$$

f Let $u = \ln x$, $\dfrac{du}{dx} = \dfrac{1}{x}$

$$\begin{aligned}\therefore\ f(x) &= \int \frac{(\ln x)^3}{x}\,dx \\ &= \int u^3 \frac{du}{dx}\,dx \\ &= \int u^3\,du \\ &= \frac{u^4}{4} + c \\ &= \tfrac{1}{4}(\ln x)^4 + c\end{aligned}$$

6 **a** Let $u = \sin x$, $\dfrac{du}{dx} = \cos x$

$$\begin{aligned}\therefore\ \int \sin^4 x\cos x\,dx &= \int u^4 \frac{du}{dx}\,dx \\ &= \int u^4\,du \\ &= \frac{u^5}{5} + c \\ &= \tfrac{1}{5}\sin^5 x + c\end{aligned}$$

b Let $u = \cos x$, $\dfrac{du}{dx} = -\sin x$

$$\begin{aligned}\therefore\ \int \frac{\sin x}{\sqrt{\cos x}}\,dx &= -\int \frac{-\sin x}{\sqrt{\cos x}}\,dx \\ &= -\int u^{-\frac{1}{2}} \frac{du}{dx}\,dx \\ &= -\int u^{-\frac{1}{2}}\,du \\ &= -\frac{u^{\frac{1}{2}}}{\frac{1}{2}} + c \\ &= -2\sqrt{\cos x} + c\end{aligned}$$

c Let $u = \cos x$, $\dfrac{du}{dx} = -\sin x$

$$\begin{aligned}\therefore\ \int \tan x\,dx &= \int \frac{\sin x}{\cos x}\,dx \\ &= -\int \frac{-\sin x}{\cos x}\,dx \\ &= -\int \frac{1}{u}\frac{du}{dx}\,dx \\ &= -\int \frac{1}{u}\,du \\ &= -\ln|u| + c \\ &= -\ln|\cos x| + c\end{aligned}$$

d Let $u = \sin x$, $\dfrac{du}{dx} = \cos x$

$$\begin{aligned}\therefore\ \int \sqrt{\sin x}\cos x\,dx &= \int u^{\frac{1}{2}} \frac{du}{dx}\,dx \\ &= \int u^{\frac{1}{2}}\,du \\ &= \tfrac{2}{3}u^{\frac{3}{2}} + c \\ &= \tfrac{2}{3}(\sin x)^{\frac{3}{2}} + c\end{aligned}$$

e Let $u = 2 + \sin x$, $\dfrac{du}{dx} = \cos x$

$$\begin{aligned}\therefore\ \int \frac{\cos x}{(2+\sin x)^2}\,dx &= \int u^{-2} \frac{du}{dx}\,dx \\ &= \int u^{-2}\,du \\ &= -u^{-1} + c \\ &= \frac{-1}{2+\sin x} + c\end{aligned}$$

f Let $u = \cos x$, $\dfrac{du}{dx} = -\sin x$

$$\begin{aligned}\therefore\ \int \frac{\sin x}{\cos^3 x}\,dx &= -\int \frac{-\sin x}{\cos^3 x}\,dx \\ &= -\int u^{-3} \frac{du}{dx}\,dx \\ &= -\int u^{-3}\,du \\ &= \frac{-u^{-2}}{-2} + c \\ &= \tfrac{1}{2}u^{-2} + c \\ &= \frac{1}{2\cos^2 x} + c\end{aligned}$$

g Let $u = 1 - \cos x$, $\dfrac{du}{dx} = \sin x$

$$\therefore \int \frac{\sin x}{1-\cos x}\,dx = \int \frac{1}{u}\frac{du}{dx}\,dx = \int \frac{1}{u}\,du = \ln|u| + c = \ln|1-\cos x| + c$$

h Let $u = \sin(2x) - 3$, $\dfrac{du}{dx} = 2\cos(2x)$

$$\therefore \int \frac{\cos(2x)}{\sin(2x)-3}\,dx = \tfrac{1}{2}\int \frac{2\cos(2x)}{\sin(2x)-3}\,dx = \tfrac{1}{2}\int \frac{1}{u}\frac{du}{dx}\,dx = \tfrac{1}{2}\int \frac{1}{u}\,du = \tfrac{1}{2}\ln|u| + c = \tfrac{1}{2}\ln|\sin(2x)-3| + c$$

i Let $u = x^2$, $\dfrac{du}{dx} = 2x$

$$\therefore \int x\sin(x^2)\,dx = \tfrac{1}{2}\int (2x)\sin(x^2)\,dx = \tfrac{1}{2}\int \sin u\,\frac{du}{dx}\,dx = \tfrac{1}{2}\int \sin u\,du = \tfrac{1}{2}(-\cos u) + c = -\tfrac{1}{2}\cos(x^2) + c$$

j Now $\displaystyle\int \frac{\sin^3 x}{\cos^5 x}\,dx = \int \tan^3 x\sec^2 x\,dx$

Let $u = \tan x$, $\dfrac{du}{dx} = \sec^2 x$

$$\therefore \int \frac{\sin^3 x}{\cos^5 x}\,dx = \int u^3\frac{du}{dx}\,dx = \int u^3\,du = \frac{u^4}{4} + c = \frac{\tan^4 x}{4} + c$$

k Let $u = \csc(2x)$,

$\dfrac{du}{dx} = -\csc(2x)\cot(2x) \times 2$

Now $\int \csc^3(2x)\cot(2x)\,dx$

$$= \int \csc^2(2x)\csc(2x)\cot(2x)\,dx = \int u^2\left(-\tfrac{1}{2}\frac{du}{dx}\right)dx = -\tfrac{1}{2}\int u^2\frac{du}{dx}\,dx = -\tfrac{1}{2}\int u^2\,du = -\tfrac{1}{2}\left(\frac{u^3}{3}\right) + c = -\tfrac{1}{6}\csc^3(2x) + c$$

l $\int \cos^3 x\,dx = \int \cos^2 x\cos x\,dx = \int (1-\sin^2 x)\cos x\,dx$

Let $u = \sin x$, $\dfrac{du}{dx} = \cos x$

$$\therefore \int \cos^3 x\,dx = \int (1-u^2)\frac{du}{dx}\,dx = \int (1-u^2)\,du = u - \frac{u^3}{3} + c = \sin x - \frac{\sin^3 x}{3} + c = \sin x - \tfrac{1}{3}\sin^3 x + c$$

7 **a** $\int \sin^5 x\,dx = \int \sin^4 x\sin x\,dx = \int (1-\cos^2 x)^2\sin x\,dx = \int (1 - 2\cos^2 x + \cos^4 x)\sin x\,dx$

Let $u = \cos x$, $\dfrac{du}{dx} = -\sin x$

$$\therefore \int \sin^5 x\,dx = -\int (1-2u^2+u^4)\frac{du}{dx}\,dx = -\int (1-2u^2+u^4)\,du = -u + \tfrac{2}{3}u^3 - \tfrac{1}{5}u^5 + c = -\cos x + \tfrac{2}{3}\cos^3 x - \tfrac{1}{5}\cos^5 x + c$$

b
$$\begin{aligned}
&\int \sin^4 x \cos^3 x \, dx \\
&= \int \sin^4 x \cos^2 x \cos x \, dx \\
&= \int \sin^4 x (1 - \sin^2 x) \cos x \, dx \\
&= \int (\sin^4 x - \sin^6 x) \cos x \, dx
\end{aligned}$$

Let $u = \sin x$, $\dfrac{du}{dx} = \cos x$

$$\begin{aligned}
\therefore \int \sin^4 x \cos^3 x \, dx &= \int (u^4 - u^6) \frac{du}{dx} \, dx \\
&= \int (u^4 - u^6) \, du \\
&= \tfrac{1}{5} u^5 - \tfrac{1}{7} u^7 + c \\
&= \tfrac{1}{5} \sin^5 x - \tfrac{1}{7} \sin^7 x + c
\end{aligned}$$

c
$$\begin{aligned}
&\int \sin^3(2x) \cos(2x) \, dx \\
&= \tfrac{1}{2} \int \sin^3(2x)(2\cos(2x)) \, dx
\end{aligned}$$

Let $u = \sin(2x)$, $\dfrac{du}{dx} = 2\cos(2x)$

$$\begin{aligned}
\therefore \quad &\int \sin^3(2x) \cos(2x) \, dx \\
&= \tfrac{1}{2} \int u^3 \frac{du}{dx} \, dx \\
&= \tfrac{1}{2} \int u^3 \, du \\
&= \tfrac{1}{2} \times \frac{u^4}{4} + c \\
&= \tfrac{1}{8} \sin^4(2x) + c
\end{aligned}$$

8 **a** Let $u = \cos x$, $\dfrac{du}{dx} = -\sin x$

$$\begin{aligned}
\therefore \quad f(x) &= \int \sin x e^{\cos x} \, dx \\
&= -\int e^{\cos x}(-\sin x) \, dx \\
&= -\int e^u \frac{du}{dx} \, dx \\
&= -\int e^u \, du \\
&= -e^u + c \\
&= -e^{\cos x} + c
\end{aligned}$$

b Let $u = \sin x - \cos x$,

$\dfrac{du}{dx} = \cos x + \sin x$

$$\begin{aligned}
\therefore \quad f(x) &= \int \frac{\sin x + \cos x}{\sin x - \cos x} \, dx \\
&= \int \frac{1}{u} \frac{du}{dx} \, dx \\
&= \int \frac{1}{u} \, du \\
&= \ln|u| + c \\
&= \ln|\sin x - \cos x| + c
\end{aligned}$$

c Let $u = \tan x$,

$\dfrac{du}{dx} = \sec^2 x = \dfrac{1}{\cos^2 x}$

$$\begin{aligned}
\therefore \quad \int \frac{e^{\tan x}}{\cos^2 x} \, dx &= \int e^u \frac{du}{dx} \, dx \\
&= \int e^u \, du \\
&= e^u + c \\
&= e^{\tan x} + c
\end{aligned}$$

9 **a** $\displaystyle\int \cot x \, dx = \int \frac{\cos x}{\sin x} \, dx$

Let $u = \sin x$, $\dfrac{du}{dx} = \cos x$

$$\begin{aligned}
\therefore \quad \int \cot x \, dx &= \int \frac{1}{u} \frac{du}{dx} \, dx \\
&= \int \frac{1}{u} \, du \\
&= \ln|u| + c \\
&= \ln|\sin x| + c
\end{aligned}$$

b $\displaystyle\int \cot(3x) \, dx = \int \frac{\cos(3x)}{\sin(3x)} \, dx$

Let $u = \sin(3x)$, $\dfrac{du}{dx} = 3\cos(3x)$

$$\begin{aligned}
\therefore \quad \int \cot(3x) \, dx &= \tfrac{1}{3} \int \frac{3\cos(3x)}{\sin(3x)} \, dx \\
&= \tfrac{1}{3} \int \frac{1}{u} \frac{du}{dx} \, dx \\
&= \tfrac{1}{3} \int \frac{1}{u} \, du \\
&= \tfrac{1}{3} \ln|u| + c \\
&= \tfrac{1}{3} \ln|\sin(3x)| + c
\end{aligned}$$

c Let $u = \cot x = \dfrac{\cos x}{\sin x}$,

$$\frac{du}{dx} = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x} = -\csc^2 x$$

$$\therefore \int \csc^2 x\, dx = -\int -\csc^2 x\, dx = -\int 1 \frac{du}{dx}\, dx = -\int 1\, du = -u + c = -\cot x + c$$

d $\displaystyle\int \sec x \tan x\, dx = \int \frac{1}{\cos x}\frac{\sin x}{\cos x}\, dx$

Let $u = \cos x$, $\dfrac{du}{dx} = -\sin x$

$$\therefore \int \sec x \tan x\, dx = -\int \frac{1}{u^2}\frac{du}{dx}\, dx = -\int u^{-2}\, du = -\frac{u^{-1}}{(-1)} + c = \frac{1}{\cos x} + c = \sec x + c$$

e $\displaystyle\int \csc x \cot x\, dx = \int \frac{1}{\sin x}\frac{\cos x}{\sin x}\, dx$

Let $u = \sin x$, $\dfrac{du}{dx} = \cos x$

$$\therefore \int \csc x \cot x\, dx = \int \frac{1}{u^2}\frac{du}{dx}\, dx = \int u^{-2}\, du = \frac{u^{-1}}{-1} + c = -\frac{1}{\sin x} + c = -\csc x + c$$

f $\displaystyle\int \tan(3x)\sec(3x)\, dx = \int \frac{\sin(3x)}{\cos(3x)}\frac{1}{\cos(3x)}\, dx$

Let $u = \cos(3x)$, $\dfrac{du}{dx} = -3\sin(3x)$

$$\therefore \int \tan(3x)\sec(3x)\, dx = -\tfrac{1}{3}\int \frac{1}{u^2}\frac{du}{dx}\, dx = -\tfrac{1}{3}\int u^{-2}\, du = -\tfrac{1}{3} \times \frac{u^{-1}}{-1} + c = \tfrac{1}{3} \times \frac{1}{\cos(3x)} + c = \tfrac{1}{3}\sec(3x) + c$$

g $\displaystyle\int \csc\left(\tfrac{x}{2}\right)\cot\left(\tfrac{x}{2}\right)\, dx = \int \frac{1}{\sin\left(\frac{x}{2}\right)}\frac{\cos\left(\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)}\, dx$

Let $u = \sin\left(\tfrac{x}{2}\right)$, $\dfrac{du}{dx} = \tfrac{1}{2}\cos\left(\tfrac{x}{2}\right)$

$$\therefore \int \csc\left(\tfrac{x}{2}\right)\cot\left(\tfrac{x}{2}\right)\, dx = 2\int \frac{1}{u^2}\frac{du}{dx}\, dx = 2\int u^{-2}\, du = 2 \times \frac{u^{-1}}{-1} + c = \frac{-2}{\sin\left(\frac{x}{2}\right)} + c = -2\csc\left(\tfrac{x}{2}\right) + c$$

h $\displaystyle\int \sec^3 x \sin x\, dx = \int (\cos x)^{-3}\sin x\, dx$

Let $u = \cos x$, $\dfrac{du}{dx} = -\sin x$

$$\therefore \int \sec^3 x \sin x\, dx = -\int \sec^3 x(-\sin x)\, dx = -\int u^{-3}\frac{du}{dx}\, dx = -\int u^{-3}\, du = \tfrac{1}{2}u^{-2} + c = \tfrac{1}{2}\cos^{-2} x + c = \tfrac{1}{2}\sec^2 x + c$$

i Let $u = \cot x$, $\dfrac{du}{dx} = -\csc^2 x$ {see **c**}

$$\therefore \int \frac{\csc^2 x}{\sqrt{\cot x}}\, dx = -\int \frac{-\csc^2 x}{\sqrt{\cot x}}\, dx = -\int \frac{1}{\sqrt{u}}\frac{du}{dx}\, dx = -\int u^{-\frac{1}{2}}\, du = -2u^{\frac{1}{2}} + c = -2\sqrt{\cot x} + c$$

10 **a**

$$u = \ln(x^2+7)$$
$$\therefore \frac{du}{dx} = \frac{2x}{x^2+7}$$
$$\therefore \int \frac{x\ln(x^2+7)}{x^2+7}\,dx$$
$$= \tfrac{1}{2}\int \frac{2x}{x^2+7}\ \ln(x^2+7)\,dx$$
$$= \tfrac{1}{2}\int u\,\frac{du}{dx}\,dx$$
$$= \tfrac{1}{2}\int u\,du$$
$$= \tfrac{1}{2}\times\frac{u^2}{2}+c$$
$$= \tfrac{1}{4}\left[\ln(x^2+7)\right]^2+c$$

b

$$u = x-16$$
$$\therefore \frac{du}{dx} = 1$$
Also, $x = u+16$
$$\therefore \int x^2\sqrt{x-16}\,dx$$
$$= \int (u+16)^2\sqrt{u}\,\frac{du}{dx}\,dx$$
$$= \int (u^2+32u+256)u^{\frac{1}{2}}\,du$$
$$= \int u^{\frac{5}{2}}+32u^{\frac{3}{2}}+256u^{\frac{1}{2}}\,du$$
$$= \frac{u^{\frac{7}{2}}}{\frac{7}{2}}+\frac{32u^{\frac{5}{2}}}{\frac{5}{2}}+\frac{256u^{\frac{3}{2}}}{\frac{3}{2}}+c$$
$$= \tfrac{2}{7}(x-16)^{\frac{7}{2}}+\tfrac{64}{5}(x-16)^{\frac{5}{2}}+\tfrac{512}{3}(x-16)^{\frac{3}{2}}+c$$

c

$$x = 3\tan\theta$$
$$\therefore \theta = \arctan\left(\tfrac{x}{3}\right)$$
Also, $\dfrac{d\theta}{dx} = \tfrac{1}{3}\left(\dfrac{1}{1+\left(\frac{x}{3}\right)^2}\right)$
$$\therefore 3\,\frac{d\theta}{dx} = \frac{1}{1+\frac{x^2}{9}}$$
$$\therefore \int \frac{1}{36+4x^2}\,dx = \tfrac{1}{36}\int \frac{1}{1+\frac{x^2}{9}}\,dx$$
$$= \tfrac{1}{36}\int 3\,\frac{d\theta}{dx}\,dx$$
$$= \tfrac{3}{36}\int 1\,d\theta$$
$$= \tfrac{1}{12}\theta+c$$
$$= \tfrac{1}{12}\arctan\left(\tfrac{x}{3}\right)+c$$

d Let $u = \sqrt{x-1}$

$$\therefore \frac{du}{dx} = \tfrac{1}{2}(x-1)^{-\frac{1}{2}}(1) = \frac{1}{2\sqrt{x-1}}$$
Also, $x = u^2+1$
$$\therefore \int \frac{\sqrt{x-1}}{x}\,dx = 2\int \frac{x-1}{x}\,\frac{1}{2\sqrt{x-1}}\,dx$$
$$= 2\int \frac{u^2}{u^2+1}\,\frac{du}{dx}\,dx$$
$$= 2\int \frac{u^2}{u^2+1}\,du$$
$$= 2\int \left(1-\frac{1}{u^2+1}\right)\,du$$
$$= 2(u-\arctan u)+c$$
$$= 2\sqrt{x-1}-2\arctan\sqrt{x-1}+c$$

e

$$x = \tfrac{1}{2}\sec\theta$$
$$\therefore x^2 = \tfrac{1}{4}\sec^2\theta$$
$$\therefore x^2 = \tfrac{1}{4}(1+\tan^2\theta)$$
$$\therefore 4x^2-1 = \tan^2\theta$$
$$\therefore \sqrt{4x^2-1} = \tan\theta$$
Also, $\dfrac{dx}{d\theta} = \tfrac{1}{2}\sec\theta\tan\theta$

Also, $2x = \sec\theta$
$$\therefore \cos\theta = \frac{1}{2x}$$
$$\therefore \theta = \arccos\left(\tfrac{1}{2x}\right)$$

$$\therefore \int \frac{\sqrt{4x^2-1}}{5x}\,dx = \tfrac{1}{5}\int \frac{\tan\theta}{\frac{1}{2}\sec\theta}\times\frac{dx}{d\theta}\times d\theta$$
$$= \tfrac{1}{5}\int \frac{\tan\theta}{\cancel{\frac{1}{2}\sec\theta}}\times\cancel{\tfrac{1}{2}\sec\theta}\tan\theta\,d\theta$$
$$= \tfrac{1}{5}\int \tan^2\theta\,d\theta$$
$$= \tfrac{1}{5}\int \sec^2\theta-1\,d\theta$$
$$= \tfrac{1}{5}(\tan\theta-\theta)+c$$
$$= \tfrac{1}{5}\sqrt{4x^2-1}-\tfrac{1}{5}\arccos\left(\tfrac{1}{2x}\right)+c$$

11 **a** $\dfrac{d}{dx}(\arcsin x) = \dfrac{1}{\sqrt{1-x^2}}, \quad x \in \,]-1, 1[$

$\therefore \displaystyle\int \frac{1}{\sqrt{1-x^2}}\, dx = \arcsin x + c$

b Let $\theta = \dfrac{x}{a}$

$\therefore \; x = a\theta$

$\therefore \; \dfrac{dx}{d\theta} = a$

$$\therefore \int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \int \frac{1}{\sqrt{a^2 - (a\theta)^2}} \frac{dx}{d\theta} \times d\theta$$
$$= \int \frac{1}{\sqrt{a^2(1-\theta^2)}} \times a \, d\theta$$
$$= \int \frac{1}{\sqrt{1-\theta^2}}\, d\theta$$
$$= \arcsin\theta + c$$
$$= \arcsin\left(\tfrac{x}{a}\right) + c$$

c Domain $= \{x \mid -a \leqslant x \leqslant a, \quad a \neq 0\}$

d **i**
$$\int \frac{4}{\sqrt{1-x^2}}\, dx$$
$$= 4\int \frac{1}{\sqrt{1-x^2}}\, dx$$
$$= 4\arcsin\left(\tfrac{x}{1}\right) + c$$
$$= 4\arcsin x + c$$

ii
$$\int \frac{3}{\sqrt{4-x^2}}\, dx$$
$$= 3\int \frac{1}{\sqrt{2^2-x^2}}\, dx$$
$$= 3\arcsin\left(\tfrac{x}{2}\right) + c$$

iii
$$\int \frac{1}{\sqrt{1-4x^2}}\, dx$$
$$= \tfrac{1}{2}\int \frac{1}{\sqrt{\frac{1}{4}-x^2}}\, dx$$
$$= \tfrac{1}{2}\arcsin\left(\frac{x}{\frac{1}{2}}\right) + c$$
$$= \tfrac{1}{2}\arcsin(2x) + c$$

iv
$$\int \frac{2}{\sqrt{4-9x^2}}\, dx$$
$$= 2\int \frac{1}{\sqrt{4-9x^2}}\, dx$$
$$= \tfrac{2}{3}\int \frac{1}{\sqrt{\frac{4}{9}-x^2}}\, dx$$
$$= \tfrac{2}{3}\arcsin\left(\tfrac{3x}{2}\right) + c$$

12 **a** $\dfrac{d}{dx}(\arctan x) = \dfrac{1}{x^2+1}, \quad x \in \mathbb{R}$

$\therefore \displaystyle\int \frac{1}{x^2+1}\, dx = \arctan x$

b Let $\theta = \dfrac{x}{a}$

$\therefore \; x = a\theta$

$\therefore \; \dfrac{dx}{d\theta} = a$

$\therefore \; dx = a\, d\theta$

$$\therefore \int \frac{1}{x^2+a^2}\, dx = \int \frac{1}{(a\theta)^2 + a^2} \times a\, d\theta$$
$$= \int \frac{a}{a^2(\theta^2+1)}\, d\theta$$
$$= \frac{1}{a}\int \frac{1}{\theta^2+1}\, d\theta$$
$$= \frac{1}{a}\arctan\theta + c$$
$$= \frac{1}{a}\arctan\left(\tfrac{x}{a}\right) + c, \quad a \neq 0$$

c Domain $= \{a, x \mid a, x \in \mathbb{R}, \quad a \neq 0\}$

d **i**
$$\int \frac{1}{x^2+16}\,dx$$
$$= \int \frac{1}{x^2+4^2}\,dx$$
$$= \tfrac{1}{4}\arctan\left(\tfrac{x}{4}\right) + c$$

ii
$$\int \frac{1}{4x^2+1}\,dx$$
$$= \tfrac{1}{4}\int \frac{1}{x^2+(\frac{1}{2})^2}\,dx$$
$$= \tfrac{1}{4}\left(\tfrac{1}{\frac{1}{2}}\right)\arctan\left(\tfrac{x}{\frac{1}{2}}\right) + c$$
$$= \tfrac{1}{2}\arctan(2x) + c$$

iii
$$\int \frac{1}{4+2x^2}\,dx$$
$$= \tfrac{1}{2}\int \frac{1}{2+x^2}\,dx$$
$$= \tfrac{1}{2}\left(\tfrac{1}{\sqrt{2}}\right)\arctan\left(\tfrac{x}{\sqrt{2}}\right) + c$$
$$= \tfrac{1}{2\sqrt{2}}\arctan\left(\tfrac{x}{\sqrt{2}}\right) + c$$

iv
$$\int \frac{5}{9+4x^2}\,dx$$
$$= \tfrac{5}{4}\int \frac{1}{\frac{9}{4}+x^2}\,dx$$
$$= \tfrac{5}{4}\left(\tfrac{1}{\frac{3}{2}}\right)\arctan\left(\tfrac{x}{\frac{3}{2}}\right) + c$$
$$= \tfrac{5}{6}\arctan\left(\tfrac{2x}{3}\right) + c$$

13 **a** Let $u = \sqrt{\dfrac{x-\xi}{x+\xi}}$

$$\therefore\ u^2 = \frac{x-\xi}{x+\xi}$$
$$= \frac{x+\xi-2\xi}{x+\xi}$$
$$= 1 - \frac{2\xi}{x+\xi}$$
$$= 1 - 2\xi(x+\xi)^{-1}$$
$$\therefore\ 2u\,\frac{du}{dx} = \frac{2\xi}{(x+\xi)^2}$$
$$\therefore\ \frac{u}{\xi}\,du = \frac{1}{(x+\xi)^2}\,dx$$
$$\therefore\ \int \frac{x-\xi}{(x+\xi)^3}\,dx = \int \frac{x-\xi}{x+\xi} \times \frac{1}{(x+\xi)^2}\,dx$$
$$= \int u^2 \times \frac{u}{\xi}\,du$$
$$= \frac{1}{\xi}\int u^3\,du$$
$$= \frac{1}{\xi}\left(\frac{u^4}{4} + c\right)$$
$$= \frac{1}{4\xi}\left(\frac{x-\xi}{x+\xi}\right)^2 + c$$
$$= \frac{x^2-2\xi x+\xi^2}{4\xi(x+\xi)^2} + c$$
$$= \frac{x^2-4\xi x+2\xi x+\xi^2}{4\xi(x+\xi)^2} + c$$
$$= \frac{(x+\xi)^2-4\xi x}{4\xi(x+\xi)^2} + c$$
$$= \frac{1}{4\xi} - \frac{x}{(x+\xi)^2} + c$$

b $$\begin{aligned}\int (x-\xi)^{\frac{1}{2}}(x+\xi)^{-\frac{5}{2}}\,dx &= \int \sqrt{\frac{x-\xi}{x+\xi}} \times \frac{1}{(x+\xi)^2}\,dx \\ &= \int u \times \frac{u}{\xi}\,du \quad \{\text{from part } \textbf{a},\quad \frac{u}{\xi}\,du = \frac{1}{(x+\xi)^2}\,dx\} \\ &= \frac{1}{\xi}\int u^2\,du \\ &= \frac{1}{\xi}\left(\frac{u^3}{3}+c\right) \\ &= \frac{1}{3\xi}\left(\frac{x-\xi}{x+\xi}\right)^{\frac{3}{2}} + c\end{aligned}$$

14 Let $u = x^{\frac{1}{6}}$

$\therefore\ x = u^6$

$\therefore\ \dfrac{dx}{du} = 6u^5$

$$\begin{aligned}\therefore\ \int \frac{1}{\sqrt{x}+\sqrt[3]{x}}\,dx &= \int \frac{1}{\sqrt{u^6}+\sqrt[3]{u^6}}\,\frac{dx}{du}\,du \\ &= \int \frac{1}{u^3+u^2} \times 6u^5\,du \\ &= 6\int \frac{u^5}{u^3+u^2}\,du \\ &= 6\int \frac{u^3}{u+1}\,du\end{aligned}$$

Let $v = u+1 \quad \therefore\ u = v-1$ and $\dfrac{dv}{du} = 1$

$$\begin{aligned}\therefore\ \int \frac{1}{\sqrt{x}+\sqrt[3]{x}}\,dx &= 6\int \frac{(v-1)^3}{v-1+1}\,\frac{dv}{du}\,du \\ &= 6\int \frac{v^3-3v^2+3v-1}{v}\,dv \\ &= 6\int v^2-3v+3-\frac{1}{v}\,dv \\ &= 6\left(\frac{v^3}{3}-\frac{3v^2}{2}+3v-\ln|v|+c\right) \\ &= 2v^3-9v^2+18v-6\ln|v|+c \\ &= 2(u+1)^3-9(u+1)^2+18(u+1)-6\ln|u+1|+c \quad \{v=u+1\} \\ &= 2(u^3+3u^2+3u+1)-9(u^2+2u+1)+18u+18-6\ln|u+1|+c \\ &= 2u^3+6u^2+6u+2-9u^2-\cancel{18u}-9+\cancel{18u}+18-6\ln|u+1|+c \\ &= 2u^3-3u^2+6u-6\ln|u+1|+c \\ &= 2(x^{\frac{1}{6}})^3-3(x^{\frac{1}{6}})^2+6(x^{\frac{1}{6}})-6\ln(x^{\frac{1}{6}}+1)+c \quad \{u=x^{\frac{1}{6}}\} \\ &= 2x^{\frac{1}{2}}-3x^{\frac{1}{3}}+6x^{\frac{1}{6}}-6\ln(x^{\frac{1}{6}}+1)+c\end{aligned}$$

EXERCISE 21G.2

1 **a** Let $u = x - 3, \quad \dfrac{du}{dx} = 1$

$\therefore \quad x = u + 3$

$$\therefore \quad \int x\sqrt{x-3}\,dx$$
$$= \int (u+3)\sqrt{u}\,\frac{du}{dx}\,dx$$
$$= \int \left(u^{\frac{3}{2}} + 3u^{\frac{1}{2}}\right)\,du$$
$$= \frac{u^{\frac{5}{2}}}{\frac{5}{2}} + \frac{3u^{\frac{3}{2}}}{\frac{3}{2}} + c$$
$$= \tfrac{2}{5}u^{\frac{5}{2}} + 2u^{\frac{3}{2}} + c$$
$$= \tfrac{2}{5}(x-3)^{\frac{5}{2}} + 2(x-3)^{\frac{3}{2}} + c$$

b Let $u = x + 1, \quad \dfrac{du}{dx} = 1$

$\therefore \quad x = u - 1$

$$\therefore \quad \int x^2\sqrt{x+1}\,dx$$
$$= \int (u-1)^2\sqrt{u}\,\frac{du}{dx}\,dx$$
$$= \int (u^2 - 2u + 1)u^{\frac{1}{2}}\,du$$
$$= \int \left(u^{\frac{5}{2}} - 2u^{\frac{3}{2}} + u^{\frac{1}{2}}\right)\,du$$
$$= \frac{u^{\frac{7}{2}}}{\frac{7}{2}} - \frac{2u^{\frac{5}{2}}}{\frac{5}{2}} + \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c$$
$$= \tfrac{2}{7}u^{\frac{7}{2}} - \tfrac{4}{5}u^{\frac{5}{2}} + \tfrac{2}{3}u^{\frac{3}{2}} + c$$
$$= \tfrac{2}{7}(x+1)^{\frac{7}{2}} - \tfrac{4}{5}(x+1)^{\frac{5}{2}} + \tfrac{2}{3}(x+1)^{\frac{3}{2}} + c$$

c Let $u = 3 - x^2, \quad \dfrac{du}{dx} = -2x$

$$\therefore \quad \int x^3\sqrt{3-x^2}\,dx$$
$$= -\tfrac{1}{2}\int x^2\sqrt{3-x^2}(-2x)\,dx$$
$$= -\tfrac{1}{2}\int x^2\sqrt{3-x^2}\,\frac{du}{dx}\,dx$$
$$= -\tfrac{1}{2}\int (3-u)\sqrt{u}\,du$$
$$= -\tfrac{1}{2}\int \left(3u^{\frac{1}{2}} - u^{\frac{3}{2}}\right)\,du$$
$$= -\tfrac{1}{2}\left[\frac{3u^{\frac{3}{2}}}{\frac{3}{2}} - \frac{u^{\frac{5}{2}}}{\frac{5}{2}}\right] + c$$
$$= -\tfrac{1}{2}\left[2u^{\frac{3}{2}} - \tfrac{2}{5}u^{\frac{5}{2}}\right] + c$$
$$= -u^{\frac{3}{2}} + \tfrac{1}{5}u^{\frac{5}{2}} + c$$
$$= -(3-x^2)^{\frac{3}{2}} + \tfrac{1}{5}(3-x^2)^{\frac{5}{2}} + c$$

d Let $u = t^2 + 2, \quad \dfrac{du}{dt} = 2t$

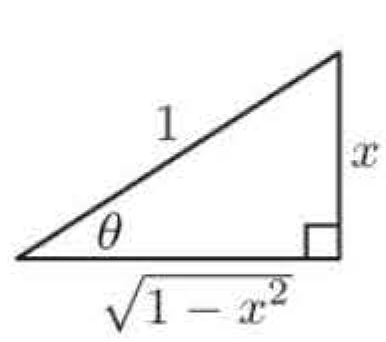
$$\therefore \quad \int t^3\sqrt{t^2+2}\,dt$$
$$= \tfrac{1}{2}\int t^2\sqrt{t^2+2}(2t)\,dt$$
$$= \tfrac{1}{2}\int t^2\sqrt{t^2+2}\,\frac{du}{dt}\,dt$$
$$= \tfrac{1}{2}\int (u-2)\sqrt{u}\,du$$
$$= \tfrac{1}{2}\int \left(u^{\frac{3}{2}} - 2u^{\frac{1}{2}}\right)\,du$$
$$= \tfrac{1}{2}\left[\frac{u^{\frac{5}{2}}}{\frac{5}{2}} - \frac{2u^{\frac{3}{2}}}{\frac{3}{2}}\right] + c$$
$$= \tfrac{1}{5}u^{\frac{5}{2}} - \tfrac{2}{3}u^{\frac{3}{2}} + c$$
$$= \tfrac{1}{5}(t^2+2)^{\frac{5}{2}} - \tfrac{2}{3}(t^2+2)^{\frac{3}{2}} + c$$

2 **a** Let $x = 3\tan\theta, \quad \dfrac{dx}{d\theta} = 3\sec^2\theta$

$$\therefore \quad \int \frac{x^2}{9+x^2}\,dx$$
$$= \int \frac{9\tan^2\theta}{9+9\tan^2\theta}\,3\sec^2\theta\,d\theta$$
$$= 3\int \frac{\tan^2\theta}{1+\tan^2\theta}\,\sec^2\theta\,d\theta$$
$$= 3\int \tan^2\theta\,d\theta \qquad \{\sec^2\theta = 1 + \tan^2\theta\}$$
$$= 3\int (\sec^2\theta - 1)\,d\theta$$
$$= 3\tan\theta - 3\theta + c$$
$$= x - 3\arctan\left(\tfrac{x}{3}\right) + c$$

b Let $x = \sin\theta, \quad \dfrac{dx}{d\theta} = \cos\theta$

$$\therefore \quad \int \frac{x^2}{\sqrt{1-x^2}}\,dx$$
$$= \int \frac{\sin^2\theta}{\sqrt{1-\sin^2\theta}}\,\cos\theta\,d\theta$$
$$= \int \frac{\sin^2\theta}{\cos\theta}\,\cos\theta\,d\theta$$



$$= \int \sin^2\theta\,d\theta$$
$$= \int \left(\tfrac{1}{2} - \tfrac{1}{2}\cos 2\theta\right)\,d\theta$$
$$= \tfrac{1}{2}\theta - \tfrac{1}{2}\left(\tfrac{1}{2}\right)\sin 2\theta + c$$
$$= \tfrac{1}{2}\arcsin x - \tfrac{1}{2}\sin\theta\cos\theta + c$$
$$= \tfrac{1}{2}\arcsin x - \tfrac{1}{2}x\sqrt{1-x^2} + c$$

$\{\text{since} \quad \cos\theta = \sqrt{1-\sin^2\theta}\}$

c $\displaystyle\int \frac{2x}{x^2+9}\,dx$ has the form $\displaystyle\int \frac{f'(x)}{f(x)}\,dx$

$$\therefore \int \frac{2x}{x^2+9}\,dx$$
$$= \ln\left|x^2+9\right| + c$$
$$= \ln(x^2+9) + c \quad \{x^2+9>0 \text{ for all } x\}$$

d Let $u = \ln x$, $\dfrac{du}{dx} = \dfrac{1}{x}$

$$\therefore \int \frac{4\ln x}{x\,(1+[\ln x]^2)}\,dx$$
$$= \int \frac{4u}{1+u^2}\,\frac{du}{dx}\,dx$$
$$= 2\int \frac{2u}{1+u^2}\,du$$

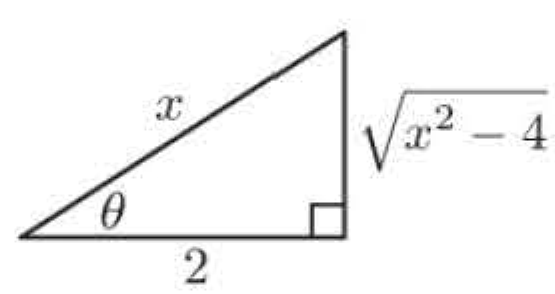
which has the form $\displaystyle\int \frac{f'(x)}{f(x)}\,dx$.

$\therefore$ the integral

$$= 2\ln\left|1+u^2\right| + c$$
$$= 2\ln(1+u^2) + c \quad \{1+u^2>0\}$$
$$= 2\ln\left(1+[\ln x]^2\right) + c$$

e Let $x = 2\sec\theta$, $\dfrac{dx}{d\theta} = 2\sec\theta\tan\theta$

$$\therefore \int \frac{\sqrt{x^2-4}}{x}\,dx$$
$$= \int \frac{\sqrt{4\sec^2\theta-4}}{2\sec\theta}\,2\sec\theta\tan\theta\,d\theta$$
$$= \tfrac{2}{2}\int \frac{\sqrt{\sec^2\theta-1}}{\cancel{\sec\theta}} \times 2\cancel{\sec\theta}\tan\theta\,d\theta$$
$$= 2\int \sqrt{\sec^2\theta-1}\,\tan\theta\,d\theta$$
$$= 2\int \tan\theta\tan\theta\,d\theta \quad \{\sec^2\theta-1=\tan^2\theta\}$$
$$= 2\int \tan^2\theta\,d\theta$$
$$= 2\int (\sec^2\theta-1)\,d\theta$$
$$= 2\tan\theta - 2\theta + c$$



$$= 2\,\frac{\sqrt{x^2-4}}{2} - 2\arccos\left(\tfrac{2}{x}\right) + c$$
$$= \sqrt{x^2-4} - 2\arccos\left(\tfrac{2}{x}\right) + c$$

f

$$\int \sin x\cos 2x\,dx$$
$$= \int \sin x(2\cos^2 x - 1)\,dx$$
$$= 2\int \cos^2 x\sin x\,dx - \int \sin x\,dx$$
$$= -2\int [\cos x]^2(-\sin x)\,dx - \int \sin x\,dx$$
$$= -2\frac{[\cos x]^3}{3} - (-\cos x) + c$$
$$= -\tfrac{2}{3}\cos^3 x + \cos x + c$$
$$= \cos x - \tfrac{2}{3}\cos^3 x + c$$

g $\displaystyle\int \frac{1}{\sqrt{9-4x^2}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{\frac{9}{4}-x^2}}\,dx$

Let $x = \frac{3}{2}\sin\theta$, so $\dfrac{dx}{d\theta} = \frac{3}{2}\cos\theta$

$\therefore$ the integral

$$= \tfrac{1}{2}\int \frac{1}{\sqrt{\frac{9}{4}-\frac{9}{4}\sin^2\theta}}\,\tfrac{3}{2}\cos\theta\,d\theta$$
$$= \tfrac{1}{2}\int \frac{1}{\frac{3}{2}\sqrt{1-\sin^2\theta}} \times \tfrac{3}{2}\cos\theta\,d\theta$$
$$= \tfrac{1}{2}\int \frac{\cos\theta}{\cos\theta}\,d\theta$$
$$= \tfrac{1}{2}\int 1\,d\theta$$
$$= \tfrac{1}{2}\theta + c$$
$$= \tfrac{1}{2}\arcsin\left(\tfrac{2x}{3}\right) + c \quad \{\text{since } \sin\theta = \tfrac{2x}{3}\}$$

h

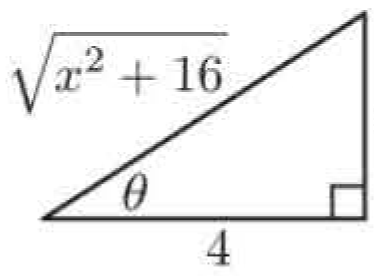
$$\int \frac{x^3}{1+x^2}\,dx$$
$$= \int \frac{x(1+x^2)-x}{1+x^2}\,dx$$
$$= \int \left(x - \frac{x}{1+x^2}\right)dx$$
$$= \int \left(x - \tfrac{1}{2}\left(\frac{2x}{1+x^2}\right)\right)dx$$
$$= \frac{x^2}{2} - \tfrac{1}{2}\ln\left|1+x^2\right| + c$$
$$= \frac{x^2}{2} - \tfrac{1}{2}\ln(1+x^2) + c \quad \{\text{as } 1+x^2>0\}$$

i Let $u = \ln x$, $\dfrac{du}{dx} = \dfrac{1}{x}$

$$\therefore \int \frac{1}{x\,(9+4[\ln x]^2)}\,dx$$
$$= \int \frac{1}{9+4u^2}\,\frac{du}{dx}\,dx$$
$$= \int \frac{1}{9+4u^2}\,du$$
$$= \tfrac{1}{4}\int \frac{1}{u^2+\frac{9}{4}}\,du$$
$$= \tfrac{1}{4}\left(\frac{1}{\frac{3}{2}}\right)\arctan\left(\frac{u}{\frac{3}{2}}\right)+c$$
$$= \tfrac{1}{6}\arctan\left(\frac{2u}{3}\right)+c$$
$$= \tfrac{1}{6}\arctan\left(\frac{2\ln x}{3}\right)+c$$

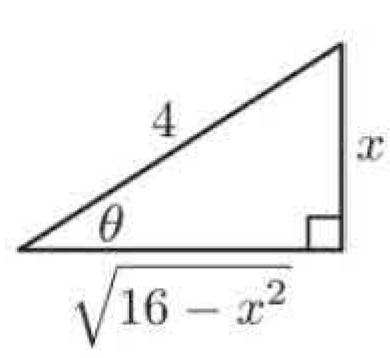
j Let $x = 4\tan\theta$, $\dfrac{dx}{d\theta} = 4\sec^2\theta$

$$\therefore \int \frac{1}{x(x^2+16)}\,dx$$
$$= \int \frac{1}{4\tan\theta(16\tan^2\theta+16)} \times 4\sec^2\theta\,d\theta$$
$$= \int \frac{1}{4\tan\theta \times 16\sec^2\theta} \times 4\sec^2\theta\,d\theta$$
$$= \tfrac{1}{16}\int \frac{1}{\tan\theta}\,d\theta$$
$$= \tfrac{1}{16}\int \frac{\cos\theta}{\sin\theta}\,d\theta$$
$$= \tfrac{1}{16}\ln|\sin\theta| + c \qquad \left\{\text{form } \frac{f'(\theta)}{f(\theta)}\right\}$$
$$= \tfrac{1}{16}\ln\left|\frac{x}{\sqrt{x^2+16}}\right| + c$$
$$= \tfrac{1}{16}\ln\left(\frac{|x|}{\sqrt{x^2+16}}\right) + c$$



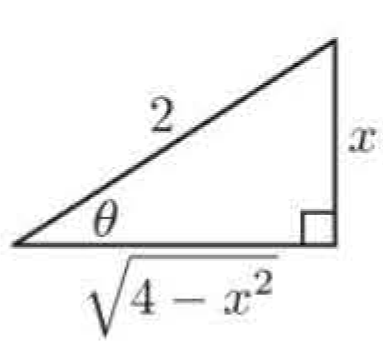
k Let $x = 4\sin\theta$, $\dfrac{dx}{d\theta} = 4\cos\theta$

$$\therefore \int \frac{1}{x^2\sqrt{16-x^2}}\,dx$$
$$= \int \frac{1}{16\sin^2\theta\sqrt{16-16\sin^2\theta}}\,4\cos\theta\,d\theta$$
$$= \int \frac{1}{16\sin^2\theta \times 4\cos\theta}\,4\cos\theta\,d\theta$$
$$= \tfrac{1}{16}\int \frac{1}{\sin^2\theta}\,d\theta$$
$$= \tfrac{1}{16}\int \csc^2\theta\,d\theta$$
$$= \tfrac{1}{16}(-\cot\theta)+c$$
$$= -\tfrac{1}{16}\cot\theta + c$$
$$= -\tfrac{1}{16}\,\frac{\sqrt{16-x^2}}{x}+c$$
$$= -\frac{\sqrt{16-x^2}}{16x}+c$$



l Let $x = 2\sin\theta$, $\dfrac{dx}{d\theta} = 2\cos\theta$

$$\therefore \int x^2\sqrt{4-x^2}\,dx$$
$$= \int 4\sin^2\theta\sqrt{4-4\sin^2\theta}\,2\cos\theta\,d\theta$$
$$= \int 4\sin^2\theta\,2\cos\theta\,2\cos\theta\,d\theta$$
$$= 4\int 4\sin^2\theta\cos^2\theta\,d\theta$$
$$= 4\int \sin^2(2\theta)\,d\theta$$
$$= 4\int \left(\tfrac{1}{2}-\tfrac{1}{2}\cos(4\theta)\right)\,d\theta$$
$$= 2\theta - 2(\tfrac{1}{4})\sin(4\theta)+c$$
$$= 2\theta - \tfrac{1}{2}\sin(4\theta)+c$$



Now $\sin\theta = \dfrac{x}{2}$, $\cos\theta = \dfrac{\sqrt{4-x^2}}{2}$

$$\therefore \sin 2\theta = 2\left(\frac{x}{2}\right)\frac{\sqrt{4-x^2}}{2} = \frac{x\sqrt{4-x^2}}{2}$$

and
$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$= \frac{4-x^2}{4} - \frac{x^2}{4}$$
$$= \frac{4-2x^2}{4}$$

$$\therefore \sin 4\theta = 2\left(\frac{x\sqrt{4-x^2}}{2}\right)\left(\frac{4-2x^2}{4}\right)$$
$$= \frac{x\sqrt{4-x^2}(2-x^2)}{2}$$

$$\therefore \int x^2\sqrt{4-x^2}\,dx$$
$$= 2\arcsin\left(\tfrac{x}{2}\right) - \tfrac{1}{4}x\sqrt{4-x^2}(2-x^2)+c$$

EXERCISE 21H

1 **a** We integrate by parts with

$$u = x \qquad v' = e^x$$
$$u' = 1 \qquad v = e^x$$

$$\therefore \quad \int xe^x\,dx = xe^x - \int 1e^x\,dx$$
$$= xe^x - e^x + c$$

b We integrate by parts with

$$u = x \qquad v' = \sin x$$
$$u' = 1 \qquad v = -\cos x$$

$$\therefore \quad \int x\sin x\,dx$$
$$= x(-\cos x) - \int 1(-\cos x)\,dx$$
$$= -x\cos x + \int \cos x\,dx$$
$$= -x\cos x + \sin x + c$$

c We integrate by parts with

$$u = \ln x \qquad v' = x^2$$
$$u' = \frac{1}{x} \qquad v = \frac{x^3}{3}$$

$$\therefore \quad \int x^2 \ln x\,dx$$
$$= \ln x\left(\frac{x^3}{3}\right) - \int \frac{1}{x}\,\frac{x^3}{3}\,dx$$
$$= \frac{x^3\ln x}{3} - \tfrac{1}{3}\int x^2\,dx$$
$$= \frac{x^3\ln x}{3} - \tfrac{1}{3}\,\frac{x^3}{3} + c$$
$$= \tfrac{1}{3}x^3\ln x - \tfrac{1}{9}x^3 + c$$

d We integrate by parts with

$$u = x \qquad v' = \sin 3x$$
$$u' = 1 \qquad v = -\tfrac{1}{3}\cos 3x$$

$$\therefore \quad \int x\sin 3x\,dx$$
$$= x(-\tfrac{1}{3}\cos 3x) - \int \left(-\tfrac{1}{3}\cos 3x\right)\,dx$$
$$= -\tfrac{1}{3}x\cos 3x + \tfrac{1}{3}(\tfrac{1}{3})\sin 3x + c$$
$$= -\tfrac{1}{3}x\cos 3x + \tfrac{1}{9}\sin 3x + c$$

e We integrate by parts with

$$u = x \qquad v' = \cos 2x$$
$$u' = 1 \qquad v = \tfrac{1}{2}\sin 2x$$

$$\therefore \quad \int x\cos 2x\,dx$$
$$= x(\tfrac{1}{2}\sin 2x) - \int \tfrac{1}{2}\sin 2x\,dx$$
$$= \tfrac{1}{2}x\sin 2x - \tfrac{1}{2}(-\tfrac{1}{2})\cos 2x + c$$
$$= \tfrac{1}{2}x\sin 2x + \tfrac{1}{4}\cos 2x + c$$

f We integrate by parts with

$$u = x \qquad v' = \sec^2 x$$
$$u' = 1 \qquad v = \tan x$$

$$\therefore \quad \int x\sec^2 x\,dx$$
$$= x\tan x - \int \tan x\,dx$$
$$= x\tan x + \int \frac{-\sin x}{\cos x}\,dx$$
$$= x\tan x + \ln|\cos x| + c$$

2 **a** $\int \ln x\,dx = \int 1 \times \ln x\,dx$

So, we integate by parts with

$$u = \ln x \qquad v' = 1$$
$$u' = \frac{1}{x} \qquad v = x$$

$$\therefore \quad \int \ln x\,dx = x\ln x - \int \left(\frac{1}{x}\right)x\,dx$$
$$= x\ln x - \int 1\,dx$$
$$= x\ln x - x + c$$

b $\int (\ln x)^2\,dx = \int (\ln x)(\ln x)\,dx$

So, we integrate by parts with

$$u = \ln x \qquad v' = \ln x$$
$$u' = \frac{1}{x} \qquad v = x\ln x - x \quad \{\text{using } \mathbf{g}\}$$

$$\therefore \quad \int (\ln x)^2\,dx$$
$$= \ln x(x\ln x - x) - \int \frac{1}{x}(x\ln x - x)\,dx$$
$$= x(\ln x)^2 - x\ln x - \int (\ln x - 1)\,dx$$
$$= x(\ln x)^2 - x\ln x - [x\ln x - x] + x + c$$
$$= x(\ln x)^2 - 2x\ln x + 2x + c$$

3 $\int \arctan x\,dx = \int 1\arctan x\,dx$

So, we integrate by parts with

$$u = \arctan x \qquad v' = 1$$
$$u' = \frac{1}{x^2+1} \qquad v = x$$

$$\therefore \quad \int \arctan x\,dx$$
$$= x\arctan x - \int \frac{x}{x^2+1}\,dx$$
$$= x\arctan x - \tfrac{1}{2}\int \frac{2x}{x^2+1}\,dx$$
$$= x\arctan x - \tfrac{1}{2}\ln\left|x^2+1\right| + c$$
$$= x\arctan x - \tfrac{1}{2}\ln(x^2+1) + c \quad \{\text{as } x^2+1 > 0\}$$

4 **a** We integrate by parts with $u = x^2$ $v' = e^{-x}$
$u' = 2x$ $v = -e^{-x}$

$$\therefore \int x^2 e^{-x}\,dx = -x^2 e^{-x} - \int 2x(-e^{-x})\,dx$$
$$= -x^2 e^{-x} + 2\int x e^{-x}\,dx$$

We integrate by parts again, this time with $u = x$ $v' = e^{-x}$
$u' = 1$ $v = -e^{-x}$

$$\therefore \int x^2 e^{-x}\,dx = -x^2 e^{-x} + 2\left[x(-e^{-x}) - \int -e^{-x}\,dx\right]$$
$$= -x^2 e^{-x} - 2xe^{-x} + 2\int e^{-x}\,dx$$
$$= -x^2 e^{-x} - 2xe^{-x} - 2e^{-x} + c$$

b We integrate by parts with $u = e^x$ $v' = \cos x$
$u' = e^x$ $v = \sin x$

$$\therefore \int e^x \cos x\,dx = e^x \sin x - \int e^x \sin x\,dx$$

We again integrate by parts with $u = e^x$ $v' = \sin x$
$u' = e^x$ $v = -\cos x$

$$\therefore \int e^x \cos x\,dx = e^x \sin x - \left[-e^x \cos x - \int e^x(-\cos x)\,dx\right] + c$$
$$= e^x \sin x + e^x \cos x - \int e^x \cos x\,dx + c$$
$$\therefore 2\int e^x \cos x\,dx = e^x(\sin x + \cos x) + c$$
$$\therefore \int e^x \cos x\,dx = \tfrac{1}{2}e^x(\sin x + \cos x) + c$$

c We integrate by parts with $u = e^{-x}$ $v' = \sin x$
$u' = -e^{-x}$ $v = -\cos x$

$$\therefore \int e^{-x} \sin x\,dx = -e^{-x}\cos x - \int -e^{-x}(-\cos x)\,dx$$
$$= -e^{-x}\cos x - \int e^{-x}\cos x\,dx$$

We integrate by parts again, this time with $u = e^{-x}$ $v' = \cos x$
$u' = -e^{-x}$ $v = \sin x$

$$\therefore \int e^{-x}\sin x\,dx = -e^{-x}\cos x - \left[e^{-x}\sin x - \int -e^{-x}\sin x\,dx\right] + c$$
$$= -e^{-x}\cos x - e^{-x}\sin x - \int e^{-x}\sin x\,dx + c$$
$$\therefore 2\int e^{-x}\sin x\,dx = -e^{-x}(\sin x + \cos x) + c$$
$$\therefore \int e^{-x}\sin x\,dx = -\tfrac{1}{2}e^{-x}(\sin x + \cos x) + c$$

d We integrate by parts with $u = x^2$ $v' = \sin x$
$u' = 2x$ $v = -\cos x$

$$\therefore \int x^2 \sin x\,dx = -x^2\cos x - \int -2x\cos x\,dx$$
$$= -x^2\cos x + \int 2x\cos x\,dx$$

We integrate by parts again, this time with $u = 2x$ $v' = \cos x$
$u' = 2$ $v = \sin x$

$$\therefore \int x^2\sin x\,dx = -x^2\cos x + \left[2x\sin x - \int 2\sin x\,dx\right]$$
$$= -x^2\cos x + 2x\sin x - 2\int \sin x\,dx$$
$$= -x^2\cos x + 2x\sin x - 2(-\cos x) + c$$
$$= -x^2\cos x + 2x\sin x + 2\cos x + c$$

5 **a** We integrate by parts with $a = u^2$ $b' = e^u$
$a' = 2u$ $b = e^u$

$$\therefore \int u^2 e^u\,du = u^2 e^u - \int 2ue^u\,du$$
$$= u^2 e^u - 2\int ue^u\,du$$

We integrate by parts again, this time with $a = u \quad b' = e^u$
$a' = 1 \quad b = e^u$

$$\therefore \int u^2 e^u \, du = u^2 e^u - 2\left[ue^u - \int e^u \, du\right]$$
$$= u^2 e^u - 2ue^u + 2e^u + c$$

b Let $u = \ln x$, $\dfrac{du}{dx} = \dfrac{1}{x} = \dfrac{1}{e^u}$ $\quad\therefore \int (\ln x)^2 \, dx = \int u^2 e^u \, du$

$$= u^2 e^u - 2ue^u + 2e^u + c \quad \{\text{using } \mathbf{a}\}$$
$$= (\ln x)^2 e^{\ln x} - 2 \ln x e^{\ln x} + 2e^{\ln x} + c$$
$$= x(\ln x)^2 - 2x \ln x + 2x + c$$

6 **a** We integrate by parts with $a = u \quad b' = \sin u$
$a' = 1 \quad b = -\cos u$

$$\therefore \int u \sin u \, du = -u \cos u - \int -\cos u \, du$$
$$= -u \cos u + \sin u + c$$

b Let $u^2 = 2x$, $2u \dfrac{du}{dx} = 2$

$$\therefore \frac{du}{dx} = \frac{1}{u}$$

$$\therefore \int \sin \sqrt{2x} \, dx = \int \sin u \, (u \, du)$$
$$= \int u \sin u \, du$$
$$= -u \cos u + \sin u + c$$
$$= -\sqrt{2x} \cos \sqrt{2x} + \sin \sqrt{2x} + c$$

7 Let $u^2 = 3x$, $2u \dfrac{du}{dx} = 3$

$$\therefore \frac{du}{dx} = \frac{3}{2u}$$

$$\therefore \int \cos \sqrt{3x} \, dx = \int \cos u \left(\frac{2u}{3}\right) du$$
$$= \tfrac{2}{3} \int u \cos u \, du$$

We integrate by parts with $a = u \quad b' = \cos u$
$a' = 1 \quad b = \sin u$

$$\therefore \int \cos \sqrt{3x} \, dx = \tfrac{2}{3}\left[u \sin u - \int \sin u \, du\right]$$
$$= \tfrac{2}{3} u \sin u - \tfrac{2}{3}(-\cos u) + c$$
$$= \tfrac{2}{3}\sqrt{3x} \, \sin \sqrt{3x} + \tfrac{2}{3} \cos \sqrt{3x} + c$$

EXERCISE 21I

1 **a** $\displaystyle\int \frac{e^x + e^{-x}}{e^x - e^{-x}} \, dx$ has the form $\displaystyle\int \frac{f'(x)}{f(x)} \, dx$

$$\therefore \int \frac{e^x + e^{-x}}{e^x - e^{-x}} \, dx = \ln\left|e^x - e^{-x}\right| + c$$

b $$\int 7^x \, dx = \frac{1}{\ln 7} \int 7^x \ln 7 \, dx$$
$$= \frac{7^x}{\ln 7} + c$$

c $$\int (3x+5)^5 \, dx = \tfrac{1}{3} \frac{(3x+5)^6}{6} + c$$
$$= \frac{(3x+5)^6}{18} + c$$

d $\displaystyle\int \frac{\sin x}{2 - \cos x} \, dx$ has the form $\displaystyle\int \frac{f'(x)}{f(x)} \, dx$

$$\therefore \int \frac{\sin x}{2 - \cos x} \, dx = \ln|2 - \cos x| + c$$
$$= \ln(2 - \cos x) + c$$
$\{\text{since } 2 - \cos x > 0\}$

e We integrate by parts with
$u = x \quad v' = \sec^2 x$
$u' = 1 \quad v = \tan x$

$$\therefore \int x \sec^2 x \, dx = x \tan x - \int \tan x \, dx$$
$$= x \tan x + \int \frac{-\sin x}{\cos x} \, dx$$
$\left\{\text{which has form } \displaystyle\int \frac{f'(x)}{f(x)} \, dx\right\}$
$$= x \tan x + \ln|\cos x| + c$$

f $$\int \cot 2x \, dx$$
$$= \int \frac{\cos 2x}{\sin 2x} \, dx$$
$$= \tfrac{1}{2} \int \frac{2 \cos 2x}{\sin 2x} \, dx$$
$\left\{\text{which has form } \displaystyle\int \frac{f'(x)}{f(x)} \, dx\right\}$
$$= \tfrac{1}{2} \ln|\sin 2x| + c$$

g Let $u = x + 3$, $\dfrac{du}{dx} = 1$

$\therefore \int x(x+3)^3 \, dx$
$= \int (u-3)u^3 \, du$
$= \int (u^4 - 3u^3) \, du$
$= \dfrac{u^5}{5} - \dfrac{3u^4}{4} + c$
$= \frac{1}{5}(x+3)^5 - \frac{3}{4}(x+3)^4 + c$

h $\displaystyle\int \frac{(x+1)^3}{x} \, dx$
$= \displaystyle\int \frac{x^3 + 3x^2 + 3x + 1}{x} \, dx$
$= \displaystyle\int \left(x^2 + 3x + 3 + \frac{1}{x}\right) dx$
$= \frac{1}{3}x^3 + \frac{3}{2}x^2 + 3x + \ln|x| + c$

2 **a** We integrate by parts with $u = x^2$ $v' = e^{-x}$
$u' = 2x$ $v = -e^{-x}$

$\therefore \int x^2 e^{-x} dx = x^2(-e^{-x}) - \int 2x(-e^{-x}) \, dx$
$= -x^2 e^{-x} + 2\int x e^{-x} \, dx$

We integrate by parts again, this time with $u = x$ $v' = e^{-x}$
$u' = 1$ $v = -e^{-x}$

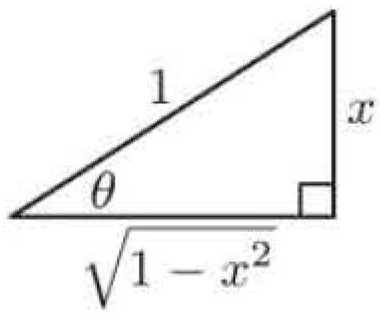
$\therefore \int x^2 e^{-x} dx = -x^2 e^{-x} + 2\left[x(-e^{-x}) - \int -e^{-x} \, dx\right]$
$= -x^2 e^{-x} - 2xe^{-x} + 2\int e^{-x} \, dx$
$= -x^2 e^{-x} - 2xe^{-x} - 2e^{-x} + c$

b Let $u = 1 - x$, $\dfrac{du}{dx} = -1$

$\therefore \int x\sqrt{1-x} \, dx$
$= \int (1-u)\sqrt{u} \, (-du)$
$= \int (u-1)u^{\frac{1}{2}} \, du$
$= \int (u^{\frac{3}{2}} - u^{\frac{1}{2}}) \, du$
$= \frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}} + c$
$= \frac{2}{5}(1-x)^{\frac{5}{2}} - \frac{2}{3}(1-x)^{\frac{3}{2}} + c$

c Let $x = \sin\theta$, $\dfrac{dx}{d\theta} = \cos\theta$

$\therefore \int x^2\sqrt{1-x^2} \, dx$
$= \int \sin^2\theta\sqrt{1-\sin^2\theta} \cos\theta \, d\theta$
$= \frac{1}{4}\int 4\sin^2\theta\cos^2\theta \, d\theta$
$= \frac{1}{4}\int \sin^2(2\theta) \, d\theta$
$= \frac{1}{4}\int \left(\frac{1}{2} - \frac{1}{2}\cos(4\theta)\right) d\theta$
$= \frac{1}{4}\left(\frac{1}{2}\theta - \frac{1}{8}\sin 4\theta\right) + c$
$= \frac{1}{8}\theta - \frac{1}{32}\sin 4\theta + c$



Now $\sin\theta = x$, $\cos\theta = \sqrt{1-x^2}$

$\therefore \sin 2\theta = 2x\sqrt{1-x^2}$

and $\cos 2\theta = 1 - 2\sin^2\theta = 1 - 2x^2$

$\therefore \sin 4\theta = 2\left(2x\sqrt{1-x^2}\right)\left(1-2x^2\right)$
$= 4x\sqrt{1-x^2}(1-2x^2)$

$\therefore \int x^2\sqrt{1-x^2} \, dx$
$= \frac{1}{8}\arcsin x - \frac{1}{32}(4x\sqrt{1-x^2}(1-2x^2)) + c$
$= \frac{1}{8}\arcsin x - \frac{1}{8}x\sqrt{1-x^2}(1-2x^2) + c$
$= \frac{1}{8}\arcsin x - \frac{1}{8}x\sqrt{1-x^2} + \frac{1}{4}x^3\sqrt{1-x^2} + c$

d Let $x = 2\sec\theta$, $\dfrac{dx}{d\theta} = 2\sec\theta\tan\theta$

$\therefore \displaystyle\int \frac{3}{x\sqrt{x^2-4}} \, dx$
$= \displaystyle\int \frac{3}{2\cancel{\sec\theta}\sqrt{4\sec^2\theta - 4}} \, 2\cancel{\sec\theta}\tan\theta \, d\theta$
$= \displaystyle\int \frac{3\tan\theta}{2\sqrt{\sec^2\theta - 1}} \, d\theta$
$= \int \frac{3}{2} \, d\theta$ $\{\sqrt{\sec^2\theta - 1} = \tan\theta\}$
$= \frac{3}{2}\theta + c$

Now $x = \dfrac{2}{\cos\theta}$ so $\cos\theta = \dfrac{2}{x}$

$\therefore \displaystyle\int \frac{3}{x\sqrt{x^2-4}} \, dx = \frac{3}{2}\arccos\left(\frac{2}{x}\right) + c$

e Let $u = x - 3,\ \dfrac{du}{dx} = 1$

$\therefore\ \int x^2\sqrt{x-3}\,dx$

$= \int (u+3)^2\sqrt{u}\,du$

$= \int (u^2+6u+9)\,u^{\frac{1}{2}}\,du$

$= \int (u^{\frac{5}{2}} + 6u^{\frac{3}{2}} + 9u^{\frac{1}{2}})\,du$

$= \frac{2}{7}u^{\frac{7}{2}} + \frac{12}{5}u^{\frac{5}{2}} + 6u^{\frac{3}{2}} + c$

$= \frac{2}{7}(x-3)^{\frac{7}{2}} + \frac{12}{5}(x-3)^{\frac{5}{2}} + 6(x-3)^{\frac{3}{2}} + c$

f Let $u = \cos x,\ \dfrac{du}{dx} = -\sin x$

$\therefore\ \int \tan^3 x\,dx$

$= \int \dfrac{\sin^3 x}{\cos^3 x}\,dx$

$= \int \dfrac{\sin x(1-\cos^2 x)}{\cos^3 x}\,dx$

$= \int \left(\dfrac{1}{\cos x} - \dfrac{1}{\cos^3 x}\right)(-\sin x)\,dx$

$= \int (u^{-1} - u^{-3})\,du$

$= \ln|u| + \dfrac{1}{2u^2} + c$

$= \ln|\cos x| + \dfrac{1}{2\cos^2 x} + c$

g Let $u = x + 2,\ \dfrac{du}{dx} = 1$

$\therefore\ \displaystyle\int \frac{\ln(x+2)}{(x+2)^2}\,dx = \int u^{-2}\ln u\,du$

We integrate by parts with

$a = \ln u \quad b' = u^{-2}$

$a' = \dfrac{1}{u} \quad b = -\dfrac{1}{u}$

$\therefore\ \displaystyle\int \frac{\ln(x+2)}{(x+2)^2}\,dx = -\frac{\ln u}{u} - \int -u^{-2}\,du$

$= -\dfrac{\ln u}{u} - \dfrac{1}{u} + c$

$= -\dfrac{\ln u + 1}{u} + c$

$= -\dfrac{\ln(x+2)+1}{x+2} + c$

h $\dfrac{1}{x^2+2x+3} = \dfrac{1}{(x+1)^2+2}$

$= \dfrac{1}{(x+1)^2+(\sqrt{2})^2}$

We let $x + 1 = \sqrt{2}\tan\theta,\ \dfrac{dx}{d\theta} = \sqrt{2}\sec^2\theta$

$\therefore\ \displaystyle\int \frac{1}{x^2+2x+3}\,dx$

$= \displaystyle\int \frac{1}{(x+1)^2+(\sqrt{2})^2}\,dx$

$= \displaystyle\int \frac{1}{2\tan^2\theta+2}\,(\sqrt{2}\sec^2\theta\,d\theta)$

$= \displaystyle\int \frac{\sqrt{2}\sec^2\theta}{2\sec^2\theta}\,d\theta = \int \tfrac{1}{\sqrt{2}}\,d\theta$

$= \frac{1}{\sqrt{2}}\theta + c$

Now $\tan\theta = \dfrac{x+1}{\sqrt{2}}$, so $\theta = \arctan\left(\dfrac{x+1}{\sqrt{2}}\right)$

$\therefore\ \displaystyle\int \frac{1}{x^2+2x+3}\,dx = \tfrac{1}{\sqrt{2}}\arctan\left(\frac{x+1}{\sqrt{2}}\right) + c$

3 **a** Let $x = 3\tan\theta$ so $\dfrac{dx}{d\theta} = 3\sec^2\theta$

$\therefore\ \displaystyle\int \frac{1}{x^2+9}\,dx$

$= \displaystyle\int \frac{1}{9\tan^2\theta+9} \times 3\sec^2\theta\,d\theta$

$= \displaystyle\int \frac{3\sec^2\theta}{9(\tan^2\theta+1)}\,d\theta$

$= \int \frac{1}{3}\,d\theta \quad \{\tan^2\theta + 1 = \sec^2\theta\}$

$= \frac{1}{3}\theta + c$

$= \frac{1}{3}\arctan\left(\frac{x}{3}\right) + c$

b Let $x = \sin^2\theta,\ \dfrac{dx}{d\theta} = 2\sin\theta\cos\theta$

$\therefore\ \displaystyle\int \frac{4}{\sqrt{x}\sqrt{1-x}}\,dx$

$= \displaystyle\int \frac{4}{\sqrt{\sin^2\theta}\sqrt{1-\sin^2\theta}} \times 2\sin\theta\cos\theta\,d\theta$

$= \displaystyle\int \frac{8\sin\theta\cos\theta}{\sin\theta\cos\theta}\,d\theta$

$= \int 8\,d\theta$

$= 8\theta + c$

$= 8\arcsin(\sqrt{x}) + c$

c Let $u = 2x$, $\dfrac{du}{dx} = 2$ $\quad\therefore$

$$\begin{aligned}\int \ln(2x)\,dx &= \tfrac{1}{2}\int \ln(2x)\times 2\,dx\\ &= \tfrac{1}{2}\int \ln u\,du\\ &= \tfrac{1}{2}(u\ln u - u) + c \quad \{\textbf{Ex. 21H Q 2 a}\}\\ &= \tfrac{1}{2}(2x)\ln(2x) - \tfrac{1}{2}(2x) + c\\ &= x\ln(2x) - x + c\end{aligned}$$

d We integrate by parts with $u = e^{-x}$, $v' = \cos x$
$u' = -e^{-x}$, $v = \sin x$

$$\therefore \int e^{-x}\cos x\,dx = e^{-x}\sin x + \int e^{-x}\sin x\,dx$$

We integrate by parts again, this time with $u = e^{-x}$, $v' = \sin x$
$u' = -e^{-x}$, $v = -\cos x$

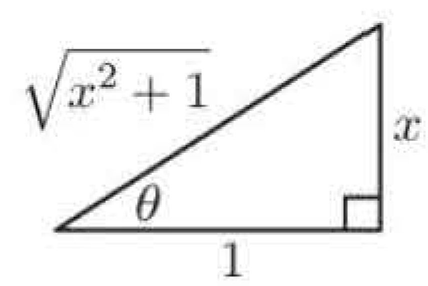
$$\therefore \int e^{-x}\cos x\,dx = e^{-x}\sin x - e^{-x}\cos x - \int e^{-x}\cos x\,dx + c$$

$$\therefore 2\int e^{-x}\cos x\,dx = e^{-x}(\sin x - \cos x) + c$$

$$\therefore \int e^{-x}\cos x\,dx = \tfrac{1}{2}e^{-x}(\sin x - \cos x) + c$$

e Let $x = \tan\theta$, $\dfrac{dx}{d\theta} = \sec^2\theta$

$$\begin{aligned}\therefore &\int \frac{1}{x(1+x^2)}\,dx\\ &= \int \frac{1}{\tan\theta(1+\tan^2\theta)}\times \sec^2\theta\,d\theta\\ &= \int \frac{1}{\tan\theta}\,d\theta \quad \{1+\tan^2\theta = \sec^2\theta\}\\ &= \int \frac{\cos\theta}{\sin\theta}\,d\theta\\ &= \ln|\sin\theta| + c\\ &= \ln\left|\frac{x}{\sqrt{x^2+1}}\right| + c\end{aligned}$$



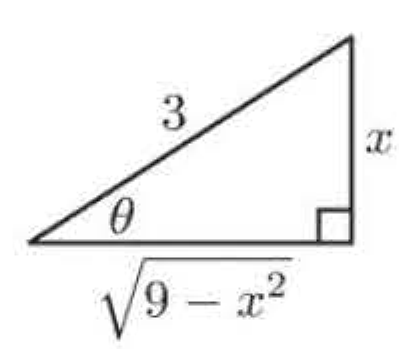
f Let $x = \tan\theta$, $\dfrac{dx}{d\theta} = \sec^2\theta$

$$\begin{aligned}\therefore &\int \frac{\arctan x}{1+x^2}\,dx\\ &= \int \frac{\arctan(\tan\theta)}{1+\tan^2\theta}\sec^2\theta\,d\theta\\ &= \int \frac{\theta\times\sec^2\theta}{\sec^2\theta}\,d\theta\\ &= \int \theta\,d\theta\\ &= \tfrac{1}{2}\theta^2 + c\\ &= \tfrac{1}{2}\arctan^2 x + c\end{aligned}$$

g Let $x = 3\sin\theta$, $\dfrac{dx}{d\theta} = 3\cos\theta$

$$\begin{aligned}\therefore \int \sqrt{9-x^2}\,dx &= \int \sqrt{9-9\sin^2\theta}\times 3\cos\theta\,d\theta\\ &= 9\int \cos^2\theta\,d\theta\\ &= 9\int \left(\tfrac{1}{2} + \tfrac{1}{2}\cos 2\theta\right)d\theta\\ &= 9\left(\tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta\right) + c\\ &= \tfrac{9}{2}\theta + \tfrac{9}{4}\sin 2\theta + c\end{aligned}$$

Now $\sin\theta = \dfrac{x}{3}$, so $\cos\theta = \dfrac{\sqrt{9-x^2}}{3}$



$$\therefore \sin 2\theta = 2\left(\frac{x}{3}\right)\frac{\sqrt{9-x^2}}{3} = \frac{2x\sqrt{9-x^2}}{9}$$

$$\begin{aligned}\therefore \int \sqrt{9-x^2}\,dx &= \tfrac{9}{2}\arcsin\left(\tfrac{x}{3}\right) + \tfrac{9}{4}\left(\frac{2x\sqrt{9-x^2}}{9}\right) + c\\ &= \tfrac{9}{2}\arcsin\left(\tfrac{x}{3}\right) + \frac{x\sqrt{9-x^2}}{2} + c\end{aligned}$$

h $\displaystyle\int \frac{(\ln x)^2}{x^2}\,dx = \int x^{-2}(\ln x)^2\,dx$

We integrate by parts with $u = (\ln x)^2 \quad v' = x^{-2}$

$$u' = \frac{2\ln x}{x} \qquad v = -\frac{1}{x}$$

$$\therefore \int \frac{(\ln x)^2}{x^2}\,dx = -\frac{(\ln x)^2}{x} - \int\left(\frac{2\ln x}{x}\right)\left(-\frac{1}{x}\right)dx$$

$$= -\frac{(\ln x)^2}{x} + 2\int x^{-2}\ln x\,dx$$

We integrate by parts again, this time with $u = \ln x \quad v' = x^{-2}$

$$u' = \frac{1}{x} \qquad v = -\frac{1}{x}$$

$$\therefore \int \frac{(\ln x)^2}{x^2}\,dx = \frac{-(\ln x)^2}{x} + 2\left(-\frac{\ln x}{x} - \int -x^{-2}\,dx\right)$$

$$= \frac{-(\ln x)^2}{x} - \frac{2\ln x}{x} - 2\left(\frac{1}{x}\right) + c$$

$$= -\frac{(\ln x)^2 + 2\ln x + 2}{x} + c$$

i Let $u = x - 3, \quad \dfrac{du}{dx} = 1 \qquad \therefore \displaystyle\int \frac{x}{\sqrt{x-3}}\,dx = \int \frac{u+3}{\sqrt{u}}\,du$

$$= \int (u^{\frac{1}{2}} + 3u^{-\frac{1}{2}})\,du$$

$$= \tfrac{2}{3}u^{\frac{3}{2}} + 6u^{\frac{1}{2}} + c$$

$$= \tfrac{2}{3}(x-3)^{\frac{3}{2}} + 6\sqrt{x-3} + c$$

j We integrate by parts with $u = \sin 4x \qquad v' = \cos x$

$$u' = 4\cos 4x \qquad v = \sin x$$

$$\therefore \int \sin 4x\cos x\,dx = \sin 4x\sin x - 4\int \cos 4x\sin x\,dx$$

We integrate by parts again, this time with $u = \cos 4x \qquad v' = \sin x$

$$u' = -4\sin 4x \qquad v = -\cos x$$

$$\therefore \int \sin 4x\cos x\,dx = \sin 4x\sin x - 4\left[-\cos 4x\cos x - \int 4\sin 4x\cos x\,dx\right] + c$$

$$= \sin 4x\sin x + 4\cos 4x\cos x + 16\int \sin 4x\cos x\,dx + c$$

$$\therefore -15\int \sin 4x\cos x\,dx = \sin 4x\sin x + 4\cos 4x\cos x + c$$

$$\therefore \int \sin 4x\cos x\,dx = -\tfrac{1}{15}(\sin 4x\sin x + 4\cos 4x\cos x) + c$$

k
$$\frac{2x+3}{x^2-2x+5} = \frac{2x-2}{x^2-2x+5} + \frac{5}{x^2-2x+5}$$

$$\therefore \int \frac{2x+3}{x^2-2x+5}\,dx = \int \frac{2x-2}{x^2-2x+5}\,dx + \int \frac{5}{x^2-2x+5}\,dx$$

Now $\displaystyle\int \frac{2x-2}{x^2-2x+5}\,dx$ has the form $\displaystyle\int \frac{f'(x)}{f(x)}\,dx,$

so $\displaystyle\int \frac{2x-2}{x^2-2x+5}\,dx = \ln\left|x^2-2x+5\right| + c$

$$= \ln(x^2-2x+5) + c \quad \text{since} \quad x^2-2x+5 > 0$$

For $\displaystyle\int \frac{5}{x^2-2x+5}\,dx$ we let $x - 1 = 2\tan\theta, \quad \dfrac{dx}{d\theta} = 2\sec^2\theta$

$$\therefore \quad \int \frac{5}{x^2-2x+5}\,dx = \int \frac{5}{(x-1)^2+4}\,dx$$
$$= \int \frac{5}{(2\tan\theta)^2+4} \times 2\sec^2\theta\,d\theta$$
$$= \int \frac{10\sec^2\theta}{4(\tan^2\theta+1)}\,d\theta$$
$$= \int \tfrac{5}{2}\,d\theta \qquad \{\tan^2\theta+1=\sec^2\theta\}$$
$$= \tfrac{5}{2}\theta + c$$
$$= \tfrac{5}{2}\arctan\left(\tfrac{x-1}{2}\right) + c$$

So, $\displaystyle\int \frac{2x+3}{x^2-2x+5}\,dx = \ln(x^2-2x+5) + \tfrac{5}{2}\arctan\left(\tfrac{x-1}{2}\right) + c,$ as $x^2-2x+5>0$

l Let $u = \sin x$, $\dfrac{du}{dx} = \cos x$ $\quad\therefore\quad \int \cos^3 x\,dx = \int \cos^2 x \cos x\,dx$

$$= \int (1-\sin^2 x)\cos x\,dx$$
$$= \int (1-u^2)\,du$$
$$= u - \frac{u^3}{3} + c$$
$$= \sin x - \tfrac{1}{3}\sin^3 x + c$$

m $\displaystyle\int \frac{x+4}{x^2+4}\,dx = \tfrac{1}{2}\int \frac{2x}{x^2+4}\,dx + \int \frac{4}{x^2+4}\,dx$

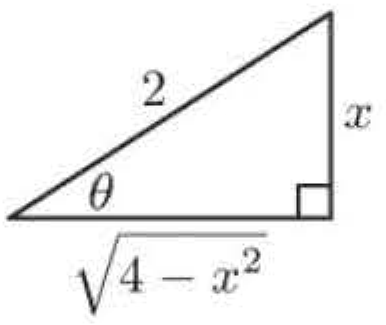
$$= \tfrac{1}{2}\ln|x^2+4| + \int \frac{4}{x^2+2^2}\,dx \quad \text{where} \quad x^2+4>0$$

Let $x = 2\tan\theta$, $\dfrac{dx}{d\theta} = 2\sec^2\theta$

$$\therefore \quad \int \frac{x+4}{x^2+4}\,dx = \tfrac{1}{2}\ln(x^2+4) + \int \frac{4}{4\tan^2\theta+4} \times 2\sec^2\theta\,d\theta$$
$$= \tfrac{1}{2}\ln(x^2+4) + \int \frac{8\sec^2\theta}{4(\tan^2\theta+1)}\,d\theta$$
$$= \tfrac{1}{2}\ln(x^2+4) + \int 2\,d\theta$$
$$= \tfrac{1}{2}\ln(x^2+4) + 2\theta + c$$
$$= \tfrac{1}{2}\ln(x^2+4) + 2\arctan\left(\tfrac{x}{2}\right) + c$$

n Let $x = 2\sin\theta$, $\dfrac{dx}{d\theta} = 2\cos\theta$

$$\therefore \quad \int \frac{1-2x}{\sqrt{4-x^2}}\,dx = \int \frac{1-4\sin\theta}{\sqrt{4-4\sin^2\theta}}(2\cos\theta)\,d\theta$$
$$= \int \frac{1-4\sin\theta}{2\sqrt{1-\sin^2\theta}}(2\cos\theta)\,d\theta$$
$$= \int \frac{1-4\sin\theta}{2\cos\theta}(2\cos\theta)\,d\theta$$
$$= \int (1-4\sin\theta)\,d\theta$$
$$= \theta + 4\cos\theta + c$$
$$= \arcsin\left(\tfrac{x}{2}\right) + 4\left(\frac{\sqrt{4-x^2}}{2}\right) + c$$
$$= \arcsin\left(\tfrac{x}{2}\right) + 2\sqrt{4-x^2} + c$$



o Let $u = 2 - x$, $\dfrac{du}{dx} = -1$

$$\begin{aligned}\therefore\ &\int \frac{x^3}{(2-x)^3}\,dx\\ &= \int \frac{(2-u)^3}{u^3}\,(-du)\\ &= \int \frac{(u-2)^3}{u^3}\,du\\ &= \int \frac{u^3-6u^2+12u-8}{u^3}\,du\\ &= \int \left(1 - \frac{6}{u} + 12u^{-2} - 8u^{-3}\right) du\\ &= u - 6\ln|u| - 12u^{-1} + 4u^{-2} + c\\ &= (2-x) - 6\ln|2-x| - \frac{12}{2-x} + \frac{4}{(2-x)^2} + c\end{aligned}$$

p Let $u = \sin x$, $\dfrac{du}{dx} = \cos x$

$$\begin{aligned}\therefore\ &\int \sin^5 x \cos^5 x\,dx\\ &= \int \sin^5 x\,(1-\sin^2 x)(1-\sin^2 x)\cos x\,dx\\ &= \int u^5(1-u^2)(1-u^2)\,du\\ &= \int u^5(1-2u^2+u^4)\,du\\ &= \int (u^5 - 2u^7 + u^9)\,du\\ &= \frac{u^6}{6} - \frac{u^8}{4} + \frac{u^{10}}{10} + c\\ &= \frac{\sin^6 x}{6} - \frac{\sin^8 x}{4} + \frac{\sin^{10} x}{10} + c\end{aligned}$$

EXERCISE 21J.1

1 **a** $$\begin{aligned}\int_0^1 x^3\,dx &= \left[\frac{x^4}{4}\right]_0^1\\ &= \tfrac{1}{4} - 0\\ &= \tfrac{1}{4}\end{aligned}$$

b $$\begin{aligned}\int_0^2 (x^2 - x)\,dx &= \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_0^2\\ &= (\tfrac{8}{3} - 2) - (0 - 0)\\ &= \tfrac{2}{3}\end{aligned}$$

c $$\begin{aligned}\int_0^1 e^x\,dx &= [e^x]_0^1\\ &= e^1 - e^0\\ &= e - 1\\ &\approx 1.72\end{aligned}$$

d $$\begin{aligned}&\int_1^4 \left(x - \frac{3}{\sqrt{x}}\right) dx\\ &= \int_1^4 (x - 3x^{-\frac{1}{2}})\,dx\\ &= \left[\frac{x^2}{2} - \frac{3x^{\frac{1}{2}}}{\frac{1}{2}}\right]_1^4\\ &= \left[\frac{x^2}{2} - 6\sqrt{x}\right]_1^4\\ &= \left[\tfrac{16}{2} - 12\right] - (\tfrac{1}{2} - 6)\\ &= 1\tfrac{1}{2}\end{aligned}$$

e $$\begin{aligned}&\int_4^9 \frac{x-3}{\sqrt{x}}\,dx\\ &= \int_4^9 (x^{\frac{1}{2}} - 3x^{-\frac{1}{2}})\,dx\\ &= \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{3x^{\frac{1}{2}}}{\frac{1}{2}}\right]_4^9\\ &= \left[\tfrac{2}{3}x^{\frac{3}{2}} - 6x^{\frac{1}{2}}\right]_4^9\\ &= \left[\tfrac{2}{3}(27) - 6(3)\right] - \left[\tfrac{2}{3}(8) - 6(2)\right]\\ &= (18 - 18) - (\tfrac{16}{3} - 12)\\ &= 6\tfrac{2}{3}\end{aligned}$$

f $$\begin{aligned}\int_1^3 \frac{1}{x}\,dx &= [\ln|x|]_1^3\\ &= \ln 3 - \ln 1\\ &= \ln 3 - 0\\ &= \ln 3\\ &\approx 1.10\end{aligned}$$

g $$\begin{aligned}&\int_1^2 (e^{-x} + 1)^2\,dx\\ &= \int_1^2 (e^{-2x} + 2e^{-x} + 1)\,dx\\ &= \left[(\tfrac{1}{-2})e^{-2x} + 2(\tfrac{1}{-1})e^{-x} + x\right]_1^2\\ &= \left[-\frac{e^{-2x}}{2} - 2e^{-x} + x\right]_1^2\\ &= \left(-\frac{e^{-4}}{2} - 2e^{-2} + 2\right) - \left(-\frac{e^{-2}}{2} - 2e^{-1} + 1\right)\\ &\approx 1.52\end{aligned}$$

h $$\begin{aligned}\int_2^6 \frac{1}{\sqrt{2x-3}}\,dx &= \int_2^6 (2x-3)^{-\frac{1}{2}}\,dx\\ &= \left[\tfrac{1}{2}\,\frac{(2x-3)^{\frac{1}{2}}}{\frac{1}{2}}\right]_2^6\\ &= \left[\sqrt{2x-3}\right]_2^6\\ &= \sqrt{9} - \sqrt{1}\\ &= 2\end{aligned}$$

i $\int_0^1 e^{1-x}\,dx = \left[(\frac{1}{-1})e^{1-x}\right]_0^1$
$= \left(\dfrac{e^0}{-1}\right) - \left(\dfrac{e^1}{-1}\right)$
$= -1 + e$
≈ 1.72

2 **a** $\int_0^{\frac{\pi}{6}} \cos x\,dx$
$= \left[\sin x\right]_0^{\frac{\pi}{6}}$
$= \sin\frac{\pi}{6} - \sin 0$
$= \frac{1}{2}$

b $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin x\,dx$
$= \left[-\cos x\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$
$= -\cos\frac{\pi}{2} + \cos\frac{\pi}{3}$
$= \frac{1}{2}$

c $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec^2 x\,dx$
$= \left[\tan x\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$
$= \tan\frac{\pi}{3} - \tan\frac{\pi}{4}$
$= \sqrt{3} - 1$

d $\int_0^{\frac{\pi}{6}} \sin(3x)\,dx$
$= \left[-\frac{1}{3}\cos(3x)\right]_0^{\frac{\pi}{6}}$
$= -\frac{1}{3}[\cos\frac{\pi}{2} - \cos 0]$
$= -\frac{1}{3}[0-1]$
$= \frac{1}{3}$

e $\int_0^{\frac{\pi}{4}} \cos^2 x\,dx$
$= \int_0^{\frac{\pi}{4}} (\frac{1}{2} + \frac{1}{2}\cos(2x))\,dx$
$= \left[\dfrac{x}{2} + \frac{1}{4}\sin(2x)\right]_0^{\frac{\pi}{4}}$
$= [\frac{\pi}{8} + \frac{1}{4}\sin\frac{\pi}{2}] - 0$
$= \frac{\pi}{8} + \frac{1}{4}$

f $\int_0^{\frac{\pi}{2}} \sin^2 x\,dx$
$= \int_0^{\frac{\pi}{2}} (\frac{1}{2} - \frac{1}{2}\cos(2x))\,dx$
$= \left[\dfrac{x}{2} - \frac{1}{4}\sin(2x)\right]_0^{\frac{\pi}{2}}$
$= [\frac{\pi}{4} - \frac{1}{4}\sin\pi] - 0$
$= \frac{\pi}{4}$

3 $\dfrac{4x+1}{x-1} = \dfrac{4x-4+1+4}{x-1}$
$= \dfrac{4(x-1)+5}{x-1}$
$= \dfrac{4\cancel{(x-1)}}{\cancel{x-1}} + \dfrac{5}{x-1}$
$= 4 + \dfrac{5}{x-1}$ as required

$\therefore \displaystyle\int_3^5 \frac{4x+1}{x-1}\,dx = \int_3^5 4 + \frac{5}{x-1}\,dx$
$= [4x + 5\ln|x-1|]_3^5$
$= 4(5) + 5\ln|5-1| - (4(3) + 5\ln|3-1|)$
$= 20 + 5\ln 4 - 12 - 5\ln 2$
$= 20 + 5\ln 2^2 - 12 - 5\ln 2$
$= 8 + 10\ln 2 - 5\ln 2$
$= 8 + 5\ln 2$

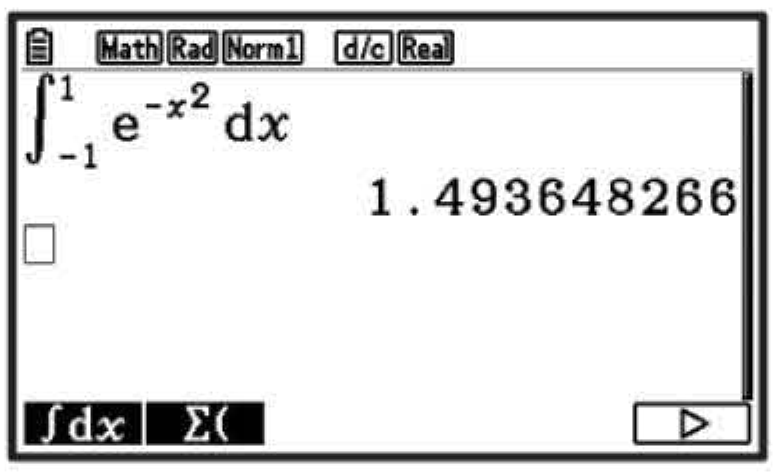
4 **a** $\displaystyle\int_m^{-2} \frac{1}{4-x}\,dx = \ln\frac{3}{2}$
$\therefore\ [-\ln|4-x|]_m^{-2} = \ln\frac{3}{2}$
$\therefore\ -\ln|4--2| + \ln|4-m| = \ln\frac{3}{2}$
$\therefore\ \ln|4-m| - \ln 6 = \ln\frac{3}{2}$
$\therefore\ \ln\left|\dfrac{4-m}{6}\right| = \ln\frac{3}{2}$
$\therefore\ \left|\dfrac{4-m}{6}\right| = \frac{3}{2}$
$\therefore\ \dfrac{4-m}{6} = \pm\frac{3}{2}$
$\therefore\ 4-m = \pm 9$
$\therefore\ m = 4 \pm 9$
$\therefore\ m = -5$ or 13

However, the solution $m = 13$ is invalid, since the vertical asymptote $x = 4$ lies between -2 and 13.

$\therefore\ m = -5$ is the only valid answer.

b $\displaystyle\int_m^{2m} (2x-1)\,dx = 4$
$\therefore\ \left[x^2 - x\right]_m^{2m} = 4$
$\therefore\ (2m)^2 - 2m - (m^2 - m) = 4$
$\therefore\ 4m^2 - 2m - m^2 + m = 4$
$\therefore\ 3m^2 - m - 4 = 0$
$\therefore\ (3m-4)(m+1) = 0$
$\therefore\ m = \frac{4}{3}$ or -1

5



$\therefore\ \int_{-1}^{1} e^{-x^2}\,dx \approx 1.49$

6 **a** $\int_0^1 -xe^{-x}\,dx$

Let $u = -x,\quad v' = e^{-x}$

$\therefore\ u' = -1,\quad v = -e^{-x}$

Using $\int uv'\,dx = uv - \int u'v\,dx$

$$\begin{aligned}\therefore\ & \int -xe^{-x}\,dx \\ &= -x(-e^{-x}) - \int(-1)(-e^{-x})\,dx \\ &= xe^{-x} - \int e^{-x}\,dx \\ &= xe^{-x} + e^{-x} + c \\ &= e^{-x}(x+1) + c\end{aligned}$$

$$\begin{aligned}\therefore\ & \int_0^1 -xe^{-x}\,dx \\ &= \left[e^{-x}(x+1)\right]_0^1 \\ &= e^{-1}(1+1) - e^0(0+1) \\ &= 2e^{-1} - 1 \\ &\approx -0.264\,241\,117\,7 \quad \{\text{using technology}\} \\ &\approx -0.264\end{aligned}$$

b $\int_0^{\frac{\pi}{2}} x\sin x\,dx$

Let $u = x,\quad v' = \sin x$

$\therefore\ u' = 1,\quad v = -\cos x$

Using $\int uv'\,dx = uv - \int u'v\,dx$

$$\begin{aligned}\therefore\ & \int x\sin x\,dx \\ &= x(-\cos x) - \int 1(-\cos x)\,dx \\ &= -x\cos x + \int \cos x\,dx \\ &= -x\cos x + \sin x + c\end{aligned}$$

$$\begin{aligned}\therefore\ & \int_0^{\frac{\pi}{2}} x\sin x\,dx \\ &= \Big[\sin x - x\cos x\Big]_0^{\frac{\pi}{2}} \\ &= (\sin(\tfrac{\pi}{2}) - \tfrac{\pi}{2}\cos(\tfrac{\pi}{2})) - (\sin 0 - 0\times\cos 0) \\ &= (1-0) - (0-0) \\ &= 1\end{aligned}$$

c $\int_1^3 \ln x\,dx$

Let $u = \ln x,\quad v' = 1$

$\therefore\ u' = \dfrac{1}{x},\quad v = x$

Using $\int uv'\,dx = uv - \int u'v\,dx$

$$\begin{aligned}\therefore\ \int \ln x\,dx &= \ln x \times x - \int \frac{1}{x}\times x\,dx \\ &= x\ln x - \int 1\,dx \\ &= x\ln x - x + c\end{aligned}$$

$$\begin{aligned}\therefore\ \int_1^3 \ln x\,dx &= [x\ln x - x]_1^3 \\ &= (3\ln 3 - 3) - (1\ln 1 - 1) \\ &= 3\ln 3 - \ln 1 - 3 + 1 \\ &= 3\ln 3 - 2 \\ &\approx 1.295\,836\,866 \quad \{\text{using technology}\} \\ &\approx 1.30\end{aligned}$$

EXERCISE 21J.2

1 **a** In $\displaystyle\int_1^2 \frac{x}{(x^2+2)^2}\,dx$

we let $u = x^2+2$, $\dfrac{du}{dx} = 2x$

when $x=1$, $u=3$
when $x=2$, $u=6$

$$\therefore \int_1^2 \frac{x}{(x^2+2)^2}\,dx = \int_1^2 u^{-2}\left(\tfrac{1}{2}\frac{du}{dx}\right)dx$$
$$= \tfrac{1}{2}\int_3^6 u^{-2}\,du$$
$$= \tfrac{1}{2}\left[\frac{u^{-1}}{-1}\right]_3^6 = \tfrac{1}{2}\left[-\tfrac{1}{6}-(-\tfrac{1}{3})\right]$$
$$= \tfrac{1}{12}$$

b In $\int_0^1 x^2 e^{x^3+1}\,dx$

we let $u = x^3+1$, $\dfrac{du}{dx} = 3x^2$

when $x=0$, $u=1$
when $x=1$, $u=2$

$$\therefore \int_0^1 x^2 e^{x^3+1}\,dx = \int_0^1 e^u\left(\tfrac{1}{3}\frac{du}{dx}\right)dx$$
$$= \tfrac{1}{3}\int_1^2 e^u\,du$$
$$= \tfrac{1}{3}\left[e^u\right]_1^2$$
$$= \tfrac{1}{3}(e^2-e)$$
$$\approx 1.56$$

c In $\int_0^3 x\sqrt{x^2+16}\,dx$

we let $u = x^2+16$, $\dfrac{du}{dx} = 2x$

when $x=0$, $u=16$
when $x=3$, $u=25$

$$\therefore \int_0^3 x\sqrt{x^2+16}\,dx$$
$$= \tfrac{1}{2}\int_0^3 2x\sqrt{x^2+16}\,dx$$
$$= \tfrac{1}{2}\int_0^3 u^{\frac{1}{2}}\frac{du}{dx}\,dx$$
$$= \tfrac{1}{2}\int_{16}^{25} u^{\frac{1}{2}}\,du$$
$$= \tfrac{1}{2}\left[\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right]_{16}^{25}$$
$$= \tfrac{1}{2}\times\tfrac{2}{3}\left[u^{\frac{3}{2}}\right]_{16}^{25}$$
$$= \tfrac{1}{3}(125-64)$$
$$= 20\tfrac{1}{3}$$

d In $\int_1^2 xe^{-2x^2}\,dx$

we let $u = -2x^2$, $\dfrac{du}{dx} = -4x$

when $x=1$, $u=-2$
when $x=2$, $u=-8$

$$\therefore \int_1^2 xe^{-2x^2}\,dx$$
$$= -\tfrac{1}{4}\int_1^2 -4xe^{-2x^2}\,dx$$
$$= -\tfrac{1}{4}\int_1^2 e^u\frac{du}{dx}\,dx$$
$$= -\tfrac{1}{4}\int_{-2}^{-8} e^u\,du$$
$$= -\tfrac{1}{4}\left[e^u\right]_{-2}^{-8}$$
$$= -\tfrac{1}{4}(e^{-8}-e^{-2})$$
$$\approx 0.0337$$

e In $\displaystyle\int_2^3 \frac{x}{2-x^2}\,dx$

we let $u = 2-x^2$, $\dfrac{du}{dx} = -2x$

when $x=2$, $u=-2$
when $x=3$, $u=-7$

$$\therefore \int_2^3 \frac{x}{2-x^2}\,dx = -\tfrac{1}{2}\int_2^3 \frac{1}{u}\frac{du}{dx}\,dx$$
$$= -\tfrac{1}{2}\int_{-2}^{-7}\frac{1}{u}\,du$$
$$= -\tfrac{1}{2}\left[\ln|u|\right]_{-2}^{-7}$$
$$= -\tfrac{1}{2}(\ln 7-\ln 2)$$
$$= -\tfrac{1}{2}\ln(\tfrac{7}{2})$$
$$\approx -0.626$$

f In $\displaystyle\int_1^2 \frac{\ln x}{x}\,dx$

we let $u = \ln x$, $\dfrac{du}{dx} = \dfrac{1}{x}$

when $x = 1$, $u = 0$

when $x = 2$, $u = \ln 2$

$$\therefore \int_1^2 \frac{\ln x}{x}\,dx = \int_1^2 u\,\frac{du}{dx}\,dx$$
$$= \int_0^{\ln 2} u\,du$$
$$= \left[\frac{u^2}{2}\right]_0^{\ln 2}$$
$$= \frac{(\ln 2)^2}{2} - 0$$
$$\approx 0.240$$

g In $\displaystyle\int_0^1 \frac{1-3x^2}{1-x^3+x}\,dx$

we let $u = 1 - x^3 + x$,

$\dfrac{du}{dx} = -3x^2 + 1$

when $x = 0$, $u = 1$

when $x = 1$, $u = 1$

$$\therefore \int_0^1 \frac{1-3x^2}{1-x^3+x}\,dx = \int_0^1 \frac{1}{u}\,\frac{du}{dx}\,dx$$
$$= \int_1^1 \frac{1}{u}\,du$$
$$= 0$$

h In $\displaystyle\int_2^4 \frac{6x^2-4x+4}{x^3-x^2+2x}\,dx$

we let $u = x^3 - x^2 + 2x$, $\dfrac{du}{dx} = 3x^2 - 2x + 2$

when $x = 2$, $u = 8$

when $x = 4$, $u = 56$

$$\therefore \int_2^4 \frac{6x^2-4x+4}{x^3-x^2+2x}\,dx = 2\int_2^4 \frac{3x^2-2x+2}{x^3-x^2+2x}\,dx$$
$$= 2\int_2^4 \frac{1}{u}\,\frac{du}{dx}\,dx$$
$$= 2\int_8^{56} \frac{1}{u}\,du$$
$$= 2\,[\ln|u|]_8^{56}$$
$$= 2(\ln 56 - \ln 8)$$
$$= 2\ln 7$$
$$\approx 3.89$$

2 **a** Let $u = \cos x$, $\dfrac{du}{dx} = -\sin x$

when $x = 0$, $u = \cos 0 = 1$

when $x = \frac{\pi}{3}$, $u = \cos\frac{\pi}{3} = \frac{1}{2}$

$$\therefore \int_0^{\frac{\pi}{3}} \frac{\sin x}{\sqrt{\cos x}}\,dx = -\int_0^{\frac{\pi}{3}} \frac{-\sin x}{\sqrt{\cos x}}\,dx$$
$$= -\int_0^{\frac{\pi}{3}} u^{-\frac{1}{2}}\,\frac{du}{dx}\,dx$$
$$= \int_{\frac{1}{2}}^{1} u^{-\frac{1}{2}}\,du = \left[2u^{\frac{1}{2}}\right]_{\frac{1}{2}}^{1}$$
$$= 2\sqrt{1} - 2\sqrt{\tfrac{1}{2}}$$
$$= 2 - \sqrt{2}$$

b Let $u = \sin x$, $\dfrac{du}{dx} = \cos x$

when $x = 0$, $u = \sin 0 = 0$

when $x = \frac{\pi}{6}$, $u = \sin\frac{\pi}{6} = \frac{1}{2}$

$$\therefore \int_0^{\frac{\pi}{6}} \sin^2 x\cos x\,dx = \int_0^{\frac{\pi}{6}} u^2\,\frac{du}{dx}\,dx$$
$$= \int_0^{\frac{1}{2}} u^2\,du$$
$$= \left[\frac{u^3}{3}\right]_0^{\frac{1}{2}}$$
$$= \tfrac{1}{3}(\tfrac{1}{2})^3$$
$$= \tfrac{1}{24}$$

c Let $u = \cos x$, $\dfrac{du}{dx} = -\sin x$

when $x = 0$, $u = \cos 0 = 1$

when $x = \frac{\pi}{4}$, $u = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$$\therefore \int_0^{\frac{\pi}{4}} \tan x\,dx = \int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos x}\,dx$$
$$= -\int_0^{\frac{\pi}{4}} \frac{1}{u}\frac{du}{dx}\,dx$$
$$= -\int_1^{\frac{1}{\sqrt{2}}} \frac{1}{u}\,du$$
$$= \int_{\frac{1}{\sqrt{2}}}^1 \frac{1}{u}\,du$$
$$= \Big[\ln|u|\Big]_{\frac{1}{\sqrt{2}}}^1$$
$$= \ln 1 - \ln\tfrac{1}{\sqrt{2}}$$
$$= \ln\sqrt{2} = \tfrac{1}{2}\ln 2$$

d Let $u = \sin x$, $\dfrac{du}{dx} = \cos x$

when $x = \frac{\pi}{6}$, $u = \sin\frac{\pi}{6} = \frac{1}{2}$

when $x = \frac{\pi}{2}$, $u = \sin\frac{\pi}{2} = 1$

$$\therefore \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cot x\,dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos x}{\sin x}\,dx$$
$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1}{u}\frac{du}{dx}\,dx$$
$$= \int_{\frac{1}{2}}^1 \frac{1}{u}\,du$$
$$= \Big[\ln|u|\Big]_{\frac{1}{2}}^1$$
$$= \ln 1 - \ln\tfrac{1}{2}$$
$$= \ln 2$$

e Let $u = 1 - \sin x$, $\dfrac{du}{dx} = -\cos x$

when $x = 0$, $u = 1 - \sin 0 = 1$

when $x = \frac{\pi}{6}$, $u = 1 - \sin\frac{\pi}{6} = \frac{1}{2}$

$$\therefore \int_0^{\frac{\pi}{6}} \frac{\cos x}{1-\sin x}\,dx = -\int_0^{\frac{\pi}{6}} \frac{-\cos x}{1-\sin x}\,dx$$
$$= -\int_0^{\frac{\pi}{6}} \frac{1}{u}\frac{du}{dx}\,dx$$
$$= -\int_1^{\frac{1}{2}} \frac{1}{u}\,du$$
$$= \int_{\frac{1}{2}}^1 \frac{1}{u}\,du$$
$$= [\ln|u|]_{\frac{1}{2}}^1$$
$$= \ln 1 - \ln\tfrac{1}{2}$$
$$= \ln 2$$

f Let $u = \tan x$, $\dfrac{du}{dx} = \sec^2 x$

when $x = 0$, $u = \tan 0 = 0$

when $x = \frac{\pi}{4}$, $u = \tan\frac{\pi}{4} = 1$

$$\therefore \int_0^{\frac{\pi}{4}} \sec^2 x \tan^3 x\,dx = \int_0^{\frac{\pi}{4}} u^3 \frac{du}{dx}\,dx$$
$$= \int_0^1 u^3\,du$$
$$= \left[\frac{u^4}{4}\right]_0^1$$
$$= \tfrac{1}{4}$$

3 In $\int_0^1 (x^2+2x)^n(x+1)\,dx$

we let $u = x^2 + 2x$, $\dfrac{du}{dx} = 2x + 2$

when $x = 0$, $u = 0$

when $x = 1$, $u = 3$

$$\therefore \int_0^1 (x^2+2x)^n(x+1)\,dx = \tfrac{1}{2}\int_0^1 (x^2+2x)^n(2x+2)\,dx$$
$$= \tfrac{1}{2}\int_0^1 u^n \frac{du}{dx}\,dx$$
$$= \tfrac{1}{2}\int_0^3 u^n\,du$$

If $n \neq -1$, the integral $= \frac{1}{2}\left[\dfrac{u^{n+1}}{n+1}\right]_0^3 = \frac{1}{2}\left(\dfrac{3^{n+1}}{n+1}\right)$

If $n = -1$, the integral $= \frac{1}{2}\displaystyle\int_0^3 \frac{1}{u}\,du = \frac{1}{2}[\ln|u|]_0^3$ which is undefined as $\ln 0$ is not defined.

4 **a** Let $u = x - 1$, $\dfrac{du}{dx} = 1$

when $x = 4$, $u = 3$

when $x = 3$, $u = 2$

$\therefore \int_3^4 x\sqrt{x-1}\,dx$

$= \int_2^3 (u+1)\sqrt{u}\,du$

$= \int_2^3 (u^{\frac{3}{2}} + u^{\frac{1}{2}})\,du$

$= \left[\dfrac{u^{\frac{5}{2}}}{\frac{5}{2}} + \dfrac{u^{\frac{3}{2}}}{\frac{3}{2}}\right]_2^3$

$= \left(\frac{2}{5}(3)^{\frac{5}{2}} + \frac{2}{3}(3)^{\frac{3}{2}}\right) - \left(\frac{2}{5}(2)^{\frac{5}{2}} + \frac{2}{3}(2)^{\frac{3}{2}}\right)$

$= \frac{2}{5}\left(9\sqrt{3}\right) + \frac{2}{3}\left(3\sqrt{3}\right) - \frac{2}{5}\left(4\sqrt{2}\right) - \frac{2}{3}\left(2\sqrt{2}\right)$

$= \left(\frac{18}{5} + 2\right)\sqrt{3} - \left(\frac{8}{5} + \frac{4}{3}\right)\sqrt{2}$

$= \frac{28\sqrt{3}}{5} - \frac{44\sqrt{2}}{15}$

b Let $u = x + 6$, $\dfrac{du}{dx} = 1$

when $x = 3$, $u = 9$

when $x = 0$, $u = 6$

$\therefore \int_0^3 x\sqrt{x+6}\,dx$

$= \int_6^9 (u-6)\sqrt{u}\,du$

$= \int_6^9 (u^{\frac{3}{2}} - 6u^{\frac{1}{2}})\,du$

$= \left[\dfrac{u^{\frac{5}{2}}}{\frac{5}{2}} - \dfrac{6u^{\frac{3}{2}}}{\frac{3}{2}}\right]_6^9$

$= \left(\frac{2}{5}(9)^{\frac{5}{2}} - 4(9)^{\frac{3}{2}}\right) - \left(\frac{2}{5}(6)^{\frac{5}{2}} - 4(6)^{\frac{3}{2}}\right)$

$= \frac{2}{5}\left(3^5\right) - 4\left(3^3\right) - \frac{2}{5} \times 36\sqrt{6} + 4 \times 6\sqrt{6}$

$= \frac{486}{5} - 108 - \frac{72}{5}\sqrt{6} + 24\sqrt{6}$

$= -\frac{54}{5} + \frac{48}{5}\sqrt{6}$

$= \frac{1}{5}(48\sqrt{6} - 54)$

c Let $u = x - 2$, $\dfrac{du}{dx} = 1$

when $x = 5$, $u = 3$

when $x = 2$, $u = 0$

$$\begin{aligned}\therefore \int_2^5 x^2\sqrt{x-2}\,dx &= \int_0^3 (u+2)^2\sqrt{u}\,du\\
&= \int_0^3 \left(u^2 + 4u + 4\right)\sqrt{u}\,du\\
&= \int_0^3 (u^{\frac{5}{2}} + 4u^{\frac{3}{2}} + 4u^{\frac{1}{2}})\,du\\
&= \left[\dfrac{u^{\frac{7}{2}}}{\frac{7}{2}} + \dfrac{4u^{\frac{5}{2}}}{\frac{5}{2}} + \dfrac{4u^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^3\\
&= \tfrac{2}{7}(3)^{\frac{7}{2}} + \tfrac{8}{5}(3)^{\frac{5}{2}} + \tfrac{8}{3}(3)^{\frac{3}{2}} - 0\\
&= \tfrac{2}{7}\left(27\sqrt{3}\right) + \tfrac{8}{5}\left(9\sqrt{3}\right) + \tfrac{8}{3}\left(3\sqrt{3}\right)\\
&= \tfrac{1054}{35}\sqrt{3}\end{aligned}$$

EXERCISE 21J.3

1 **a** $\int_1^4 \sqrt{x}\,dx = \int_1^4 x^{\frac{1}{2}}\,dx$

$= \left[\frac{2}{3}x^{\frac{3}{2}}\right]_1^4$

$= \frac{2}{3}(8) - \frac{2}{3}(1)$

$= \frac{14}{3} \approx 4.67$

$\int_1^4 (-\sqrt{x})\,dx = \int_1^4 \left(-x^{\frac{1}{2}}\right)\,dx$

$= \left[-\frac{2}{3}x^{\frac{3}{2}}\right]_1^4$

$= -\frac{2}{3}(8) - (-\frac{2}{3}(1))$

$= -\frac{14}{3} \approx -4.67$

b $\int_0^1 x^7\,dx = \left[\frac{1}{8}x^8\right]_0^1$

$= \frac{1}{8} - 0 = \frac{1}{8}$

$\int_0^1 (-x^7)\,dx = \left[-\frac{1}{8}x^8\right]_0^1$

$= -\frac{1}{8} - 0 = -\frac{1}{8}$

Property: $\int_a^b [-f(x)]\,dx = -\int_a^b f(x)\,dx$

2 **a** $\int_0^1 x^2\,dx$
$= \left[\frac{1}{3}x^3\right]_0^1$
$= \frac{1}{3} - 0$
$= \frac{1}{3}$

b $\int_1^2 x^2\,dx$
$= \left[\frac{1}{3}x^3\right]_1^2$
$= \frac{1}{3}(8) - \frac{1}{3}(1)$
$= \frac{7}{3}$

c $\int_0^2 x^2\,dx$
$= \left[\frac{1}{3}x^3\right]_0^2$
$= \frac{1}{3}(8) - 0$
$= \frac{8}{3}$

d $\int_0^1 3x^2\,dx$
$= \left[x^3\right]_0^1$
$= 1 - 0$
$= 1$

Properties: $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$
$\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx$, c a constant

3 **a** $\int_0^2 (x^3 - 4x)\,dx$
$= \left[\frac{1}{4}x^4 - 2x^2\right]_0^2$
$= \left[\frac{1}{4}(16) - 2(4)\right] - [0 - 0]$
$= -4$

b $\int_2^3 (x^3 - 4x)\,dx$
$= \left[\frac{1}{4}x^4 - 2x^2\right]_2^3$
$= \left[\frac{1}{4}(81) - 2(9)\right] - \left[\frac{1}{4}(16) - 2(4)\right]$
$= \frac{25}{4} = 6.25$

c $\int_0^3 (x^3 - 4x)\,dx = \left[\frac{1}{4}x^4 - 2x^2\right]_0^3$
$= \left[\frac{1}{4}(81) - 2(9)\right] - [0 - 0]$
$= \frac{9}{4} = 2.25$

Property: $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

4 **a** $\int_0^1 x^2\,dx = \left[\frac{1}{3}x^3\right]_0^1$
$= \frac{1}{3}(1) - 0$
$= \frac{1}{3}$

b $\int_0^1 \sqrt{x}\,dx = \int_0^1 x^{\frac{1}{2}}\,dx$
$= \left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^1$
$= \frac{2}{3}(1) - 0$
$= \frac{2}{3}$

c $\int_0^1 (x^2 + \sqrt{x})\,dx$
$= \int_0^1 (x^2 + x^{\frac{1}{2}})\,dx$
$= \left[\frac{1}{3}x^3 + \frac{2}{3}x^{\frac{3}{2}}\right]_0^1$
$= \left[\frac{1}{3}(1) + \frac{2}{3}(1)\right] - [0 + 0]$
$= 1$

Property: $\int_a^b f(x)\,dx + \int_a^b g(x)\,dx = \int_a^b [f(x) + g(x)]\,dx$

5 **a** $\int_0^3 f(x)\,dx =$ area between $f(x)$ and the x-axis from $x = 0$ to $x = 3$
$= 2 + 3 + 1.5 = 6.5$

b $\int_3^7 f(x)\,dx = -$(area between $f(x)$ and the x-axis from $x = 3$ to $x = 7$)
$= -\left(\frac{3}{2} + 3 + \frac{5}{2} + 2\right) = -9$

c $\int_2^4 f(x)\,dx =$ (area between $f(x)$ and the x-axis from $x = 2$ to $x = 3$)
$-$(area between $f(x)$ and the x-axis from $x = 3$ to $x = 4$)
$= 1.5 - 1.5 = 0$

d $\int_0^7 f(x)\,dx =$ (area between $f(x)$ and the x-axis from $x = 0$ to $x = 3$)
$-$(area between $f(x)$ and the x-axis from $x = 3$ to $x = 7$)
$= 6.5 - 9 = -2.5$

6 **a** $\int_0^4 f(x)\,dx$
$=$ area of semi-circle with radius 2
$= \frac{1}{2}\pi(2)^2 = 2\pi$

b $\int_4^6 f(x)\,dx$
$= -$(area of 2 by 2 rectangle)
$= -(2 \times 2) = -4$

c $\int_6^8 f(x)\,dx$
$= \text{area of semi-circle with radius } 1$
$= \frac{1}{2}\pi(1)^2 = \frac{\pi}{2}$

d $\int_0^8 f(x)\,dx$
$= \int_0^4 f(x)\,dx + \int_4^6 f(x)\,dx + \int_6^8 f(x)\,dx$
$= 2\pi + (-4) + \frac{\pi}{2} = \frac{5\pi}{2} - 4$

7 **a** $\int_2^4 f(x)\,dx + \int_4^7 f(x)\,dx$
$= \int_2^7 f(x)\,dx$

b $\int_1^3 g(x)\,dx + \int_3^8 g(x)\,dx + \int_8^9 g(x)\,dx$
$= \int_1^9 g(x)\,dx$

8 **a** $\int_1^3 f(x)\,dx + \int_3^6 f(x)\,dx = \int_1^6 f(x)\,dx$
$\therefore \int_3^6 f(x)\,dx = \int_1^6 f(x)\,dx - \int_1^3 f(x)\,dx$
$= (-3) - 2$
$= -5$

b $\int_0^2 f(x)\,dx + \int_2^4 f(x)\,dx + \int_4^6 f(x)\,dx = \int_0^6 f(x)\,dx$
$\therefore \int_2^4 f(x)\,dx = \int_0^6 f(x)\,dx - \int_4^6 f(x)\,dx - \int_0^2 f(x)\,dx$
$= (7) - (-2) - (5)$
$= 4$

9 **a** $\int_1^{-1} f(x)\,dx = -\int_{-1}^1 f(x)\,dx$
$= -(-4)$
$= 4$

b $\int_{-1}^1 (2 + f(x))\,dx = \int_{-1}^1 2\,dx + \int_{-1}^1 f(x)\,dx$
$= [2x]_{-1}^1 + (-4)$
$= (2 - (-2)) - 4$
$= 0$

c $\int_{-1}^1 2\,f(x)\,dx = 2\int_{-1}^1 f(x)\,dx$
$= 2(-4)$
$= -8$

d $\int_{-1}^1 k\,f(x)\,dx = 7$
$\therefore k\int_{-1}^1 f(x)\,dx = 7$
$\therefore k(-4) = 7$
$\therefore k = -\frac{7}{4}$

10 $\int_2^3 (g'(x) - 1)\,dx = \int_2^3 g'(x)\,dx + \int_2^3 -1\,dx$
$= [g(x)]_2^3 + [-x]_2^3$
$= (g(3) - g(2)) + (-3 - (-2))$
$= 5 - 4 - 1$
$= 0$

REVIEW SET 21A

1 **a** $\int_0^4 f(x)\,dx = \text{area of semi-circle with radius } 2$
$= \frac{1}{2} \times \pi \times 2^2$
$= 2\pi \text{ units}^2$

b $\int_4^6 f(x)\,dx = \text{area of square}$
$= 2 \times 2$
$= 4 \text{ units}^2$

2 **a** $\displaystyle\int \frac{4}{\sqrt{x}}\,dx = 4\int x^{-\frac{1}{2}}\,dx$
$\displaystyle = 4\,\frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c = 8\sqrt{x} + c$

b $\displaystyle\int \frac{3}{1-2x}\,dx = 3\int \frac{1}{1-2x}\,dx$
$= 3(\frac{1}{-2})\ln|1 - 2x| + c$
$= -\frac{3}{2}\ln|1 - 2x| + c$

c $\int \sin(4x - 5)\,dx = -\frac{1}{4}\cos(4x - 5) + c$

d $\int e^{4-3x}\,dx = \frac{1}{-3}e^{4-3x} + c$
$= -\frac{1}{3}e^{4-3x} + c$

3 **a** $\int_{-5}^{-1} \sqrt{1-3x}\, dx = \int_{-5}^{-1} (1-3x)^{\frac{1}{2}}\, dx$

$= \left[\frac{1}{-3} \times \frac{(1-3x)^{\frac{3}{2}}}{\frac{3}{2}}\right]_{-5}^{-1}$

$= -\frac{2}{9}\left[(1-3x)^{\frac{3}{2}}\right]_{-5}^{-1}$

$= -\frac{2}{9}\left(4^{\frac{3}{2}} - 16^{\frac{3}{2}}\right)$

$= -\frac{2}{9}(8-64) = 12\frac{4}{9}$

b $\int_0^{\frac{\pi}{2}} \cos\left(\frac{x}{2}\right)\, dx = \left[2\sin\left(\frac{x}{2}\right)\right]_0^{\frac{\pi}{2}}$

$= 2\sin(\frac{\pi}{4}) - 2\sin(0)$

$= 2\left(\frac{1}{\sqrt{2}}\right) - 2(0)$

$= \sqrt{2}$

c In $\displaystyle\int_0^1 \frac{4x^2}{(x^3+2)^3}\, dx$

we let $u = x^3 + 2$, $\frac{du}{dx} = 3x^2$

when $x = 0$, $u = 2$

when $x = 1$, $u = 3$

$\therefore \displaystyle\int_0^1 \frac{4x^2}{(x^3+2)^3}\, dx = \frac{4}{3}\int_0^1 \frac{3x^2}{(x^2+2)^3}\, dx$

$= \frac{4}{3}\displaystyle\int_0^1 \frac{1}{u^3}\frac{du}{dx}\, dx$

$= \frac{4}{3}\int_2^3 u^{-3}\, du$

$= \frac{4}{3}\left[\frac{u^{-2}}{-2}\right]_2^3$

$= -\frac{2}{3}\left[\frac{1}{u^2}\right]_2^3$

$= -\frac{2}{3}\left[\frac{1}{9} - \frac{1}{4}\right]$

$= \frac{5}{54}$

4 $y = \sqrt{x^2-4} = (x^2-4)^{\frac{1}{2}}$

$\therefore \frac{dy}{dx} = \frac{1}{2}\left(x^2-4\right)^{-\frac{1}{2}} \times 2x$

$= \frac{x}{(x^2-4)^{\frac{1}{2}}}$

$= \frac{x}{\sqrt{x^2-4}}$

$\therefore \displaystyle\int \frac{x}{\sqrt{x^2-4}}\, dx = \sqrt{x^2-4} + c$

5 $\int_0^b \cos x\, dx = \frac{1}{\sqrt{2}}$, $0 < b < \pi$

$\therefore [\sin x]_0^b = \frac{1}{\sqrt{2}}$

$\therefore \sin b - \sin 0 = \frac{1}{\sqrt{2}}$

$\therefore \sin b = \frac{1}{\sqrt{2}}$

$\therefore b = \frac{\pi}{4}, \frac{3\pi}{4}$ $\{0 < b < \pi\}$

6 **a** $\int (2-\cos x)^2\, dx$

$= \int (4 - 4\cos x + \cos^2 x)\, dx$

$= \int (4 - 4\cos x + \frac{1}{2} + \frac{1}{2}\cos 2x)\, dx$

$= \frac{9}{2}x - 4\sin x + \frac{1}{4}\sin 2x + c$

b $y = \sin(x^2)$

$\therefore \frac{dy}{dx} = \cos(x^2) \times 2x$

$\therefore \int 2x\cos(x^2)\, dx = \sin(x^2) + c$

$\therefore 2\int x\cos(x^2)\, dx = \sin(x^2) + c$

$\therefore \int x\cos(x^2)\, dx = \frac{1}{2}\sin(x^2) + c$

7 $\frac{d}{dx}(3x^2+x)^3 = 3(3x^2+x)^2(6x+1)$

$\therefore \int 3(3x^2+x)^2(6x+1)\, dx = (3x^2+x)^3 + c$

$\therefore 3\int (3x^2+x)^2(6x+1)\, dx = (3x^2+x)^3 + c$

$\therefore \int (3x^2+x)^2(6x+1)\, dx = \frac{1}{3}(3x^2+x)^3 + c$

8 **a** $\int_1^4 (f(x)+1)\,dx$
$= \int_1^4 f(x)\,dx + \int_1^4 1\,dx$
$= 3 + [x]_1^4$
$= 3 + (4-1) = 6$

b $\int_1^2 f(x)\,dx - \int_4^2 f(x)\,dx$
$= \int_1^2 f(x)\,dx + \int_2^4 f(x)\,dx$
$= \int_1^4 f(x)\,dx$
$= 3$

9 **a** Let $u = \sin x$, $\dfrac{du}{dx} = \cos x$

$\therefore \int \sin^7 x \cos x\,dx = \int u^7 \dfrac{du}{dx}\,dx$
$= \int u^7\,du$
$= \dfrac{u^8}{8} + c$
$= \dfrac{\sin^8 x}{8} + c$

b Let $u = \cos 2x$, $\dfrac{du}{dx} = -2\sin 2x$

$\therefore \int \tan 2x\,dx = \displaystyle\int \frac{\sin 2x}{\cos 2x}\,dx$
$= -\frac{1}{2}\displaystyle\int \frac{-2\sin 2x}{\cos 2x}\,dx$
$= -\frac{1}{2}\displaystyle\int \frac{1}{u}\frac{du}{dx}\,dx$
$= -\frac{1}{2}\displaystyle\int \frac{1}{u}\,du$
$= -\frac{1}{2}\ln|u| + c$
$= -\frac{1}{2}\ln|\cos(2x)| + c$

c Let $u = \sin x$, $\dfrac{du}{dx} = \cos x$

$\therefore \int e^{\sin x}\cos x\,dx = \int e^u \dfrac{du}{dx}\,dx$
$= \int e^u\,du$
$= e^u + c$
$= e^{\sin x} + c$

10 Given: $f''(x) = 2\sin(2x)$, $f'(\frac{\pi}{2}) = 0$, and $f(0) = 3$

Now $f'(x) = \int 2\sin(2x)\,dx$
$= -\cos(2x) + c$

But $f'(\frac{\pi}{2}) = 0$, so $-\cos(\pi) + c = 0$
$\therefore -(-1) + c = 0$
$\therefore c = -1$

$\therefore f'(x) = -\cos(2x) - 1$

$\therefore f(x) = \int(-\cos(2x) - 1)\,dx$
$= -\frac{1}{2}\sin(2x) - x + k$

But $f(0) = 3$, so $-\frac{1}{2}\sin(0) - 0 + k = 3$
$\therefore k = 3$

so $f(x) = -\frac{1}{2}\sin(2x) - x + 3$
$\therefore f(\frac{\pi}{2}) = -\frac{1}{2}\sin(\pi) - \frac{\pi}{2} + 3$
$= 3 - \frac{\pi}{2}$

11 **a** $\int_0^{\frac{\pi}{6}} 4\sin^2\left(\frac{x}{2}\right)\,dx$
$= 4\int_0^{\frac{\pi}{6}} \left(\frac{1}{2} - \frac{1}{2}\cos x\right)\,dx$
$= 4\left[\frac{1}{2}x - \frac{1}{2}\sin x\right]_0^{\frac{\pi}{6}}$
$= 4\left[\frac{\pi}{12} - \frac{1}{2}(\frac{1}{2}) - 0 + 0\right]$
$= 4(\frac{\pi}{12} - \frac{1}{4})$
$= \frac{\pi}{3} - 1$

b Let $u = \sin\theta$, $\dfrac{du}{d\theta} = \cos\theta$

when $\theta = \frac{\pi}{6}$, $\sin\theta = \sin\frac{\pi}{6} = \frac{1}{2}$
when $\theta = \frac{\pi}{2}$, $\sin\theta = \sin\frac{\pi}{2} = 1$

$\therefore \displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cot\theta\,d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos\theta}{\sin\theta}\,d\theta$
$= \displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1}{u}\frac{du}{d\theta}\,d\theta$
$= \displaystyle\int_{\frac{1}{2}}^{1} \frac{1}{u}\,du$
$= \Big[\ln|u|\Big]_{\frac{1}{2}}^{1}$
$= \ln 1 - \ln\frac{1}{2}$
$= \ln 2$

c Let $u = \tan x$, $\dfrac{du}{dx} = \sec^2 x$

when $x = \frac{\pi}{4}$, $u = \tan\frac{\pi}{4} = 1$

when $x = \frac{\pi}{3}$, $u = \tan\frac{\pi}{3} = \sqrt{3}$

$$\therefore \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec^2 x}{\tan x}\,dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{u}\frac{du}{dx}\,dx$$
$$= \int_1^{\sqrt{3}} \frac{1}{u}\,du$$
$$= \Big[\ln|u|\Big]_1^{\sqrt{3}}$$
$$= \ln\sqrt{3} - \ln 1$$
$$= \tfrac{1}{2}\ln 3$$

12 Let $u = 4 - x$, $\dfrac{du}{dx} = -1$

$$\therefore \int x^2\sqrt{4-x}\,dx$$
$$= \int (4-u)^2\sqrt{u}\,(-du)$$
$$= -\int (16 - 8u + u^2)u^{\frac{1}{2}}\,du$$
$$= -\int (16u^{\frac{1}{2}} - 8u^{\frac{3}{2}} + u^{\frac{5}{2}})\,du$$
$$= -\left(\frac{16u^{\frac{3}{2}}}{\frac{3}{2}} - \frac{8u^{\frac{5}{2}}}{\frac{5}{2}} + \frac{u^{\frac{7}{2}}}{\frac{7}{2}}\right) + c$$
$$= -\tfrac{32}{3}u^{\frac{3}{2}} + \tfrac{16}{5}u^{\frac{5}{2}} - \tfrac{2}{7}u^{\frac{7}{2}} + c$$
$$= -\tfrac{32}{3}(4-x)^{\frac{3}{2}} + \tfrac{16}{5}(4-x)^{\frac{5}{2}} - \tfrac{2}{7}(4-x)^{\frac{7}{2}} + c$$

13 $\int \arctan x\,dx = \int 1\arctan x\,dx$

so we integrate by parts with $u = \arctan x$ $\quad v' = 1$

$u' = \dfrac{1}{x^2+1}$ $\quad v = x$

$$\therefore \int \arctan x\,dx$$
$$= x\arctan x - \int \frac{1}{x^2+1}(x)\,dx \quad \left\{\int uv'\,dx = uv - \int u'v\,dx\right\}$$
$$= x\arctan x - \tfrac{1}{2}\int \frac{2x}{x^2+1}\,dx$$
$$= x\arctan x - \tfrac{1}{2}\ln\left|x^2+1\right| + c$$
$$= x\arctan x - \tfrac{1}{2}\ln(x^2+1) + c \quad \{x^2+1>0\}$$

Check: $\dfrac{d}{dx}\left(x\arctan x - \frac{1}{2}\ln(x^2+1) + c\right)$

$$= \arctan x + \frac{x}{x^2+1} - \frac{1}{2}\,\frac{2x}{x^2+1}$$
$$= \arctan x \quad \checkmark$$

14 **a** If $u(x) = x^2+1$, then $\dfrac{du}{dx} = 2x$

$$\therefore \int 2x(x^2+1)^3\,dx = \int \frac{du}{dx}\,u^3\,dx$$
$$= \int u^3\,du$$
$$= \frac{u^4}{4} + c$$
$$= \tfrac{1}{4}(x^2+1)^4 + c$$

b **i** $\int_0^1 2x(x^2+1)^3\,dx$

$$= \left[\frac{(x^2+1)^4}{4}\right]_0^1 \quad \{\text{using } \mathbf{a}\}$$
$$= \frac{(1^2+1)^4}{4} - \frac{(0+1)^4}{4}$$
$$= \tfrac{16}{4} - \tfrac{1}{4}$$
$$= \tfrac{15}{4}$$

ii $\int_{-1}^2 -x(1+x^2)^3\,dx$

$$= \int_{-1}^2 -\tfrac{1}{2} \times 2x(1+x^2)^3\,dx$$
$$= -\tfrac{1}{2}\int_{-1}^2 2x(x^2+1)^3\,dx$$
$$= -\tfrac{1}{2}\left[\frac{(x^2+1)^4}{4}\right]_{-1}^2$$
$$= -\tfrac{1}{2}\left[\frac{(4+1)^4}{4} - \frac{(1+1)^4}{4}\right]$$
$$= -\tfrac{1}{2}\left(\tfrac{625}{4} - \tfrac{16}{4}\right)$$
$$= -\tfrac{609}{8}$$

15 **a** $\dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{C}{x-1}$

$$= \frac{A(x^2-1)+Bx(x-1)+Cx(x+1)}{x(x+1)(x-1)}$$
$$= \frac{Ax^2 - A + Bx^2 - Bx + Cx^2 + Cx}{x(x^2-1)}$$
$$= \frac{x^2(A+B+C)+x(C-B)-A}{x(x^2-1)}$$
$$= \frac{x^2(-A-B-C)+x(B-C)+A}{x(1-x^2)}$$

So, if $\dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{C}{x-1} = \dfrac{4}{x(1-x^2)}$, then

$A = 4$, $B - C = 0$ and $-A-B-C = 0$

$\therefore\ B = C$

$\therefore\ -4-2B = 0$

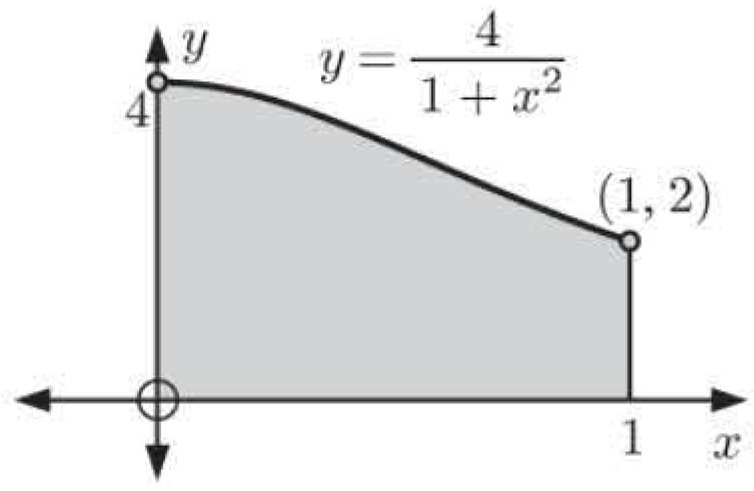
So, $A = 4$, $B = -2$, $C = -2$

b $$\int \frac{4}{x(1-x^2)}\,dx = \int \left(\frac{4}{x} - \frac{2}{x+1} - \frac{2}{x-1}\right)dx$$
$$= 4\ln|x| - 2\ln|x+1| - 2\ln|x-1| + c$$

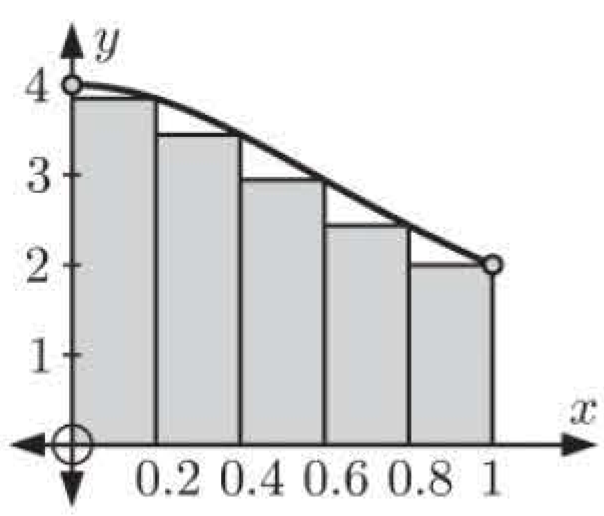
c $$\int_2^4 \frac{4}{x(1-x^2)}\,dx = [4\ln|x| - 2\ln|x+1| - 2\ln|x-1|]_2^4$$
$$= 4\ln 4 - 2\ln 5 - \cancel{2\ln 3} - 4\ln 2 + \cancel{2\ln 3} + 2\ln 1$$
$$= \ln\left(\frac{4^4}{5^2 \times 2^4}\right) = \ln\left(\tfrac{16}{25}\right)$$

REVIEW SET 21B

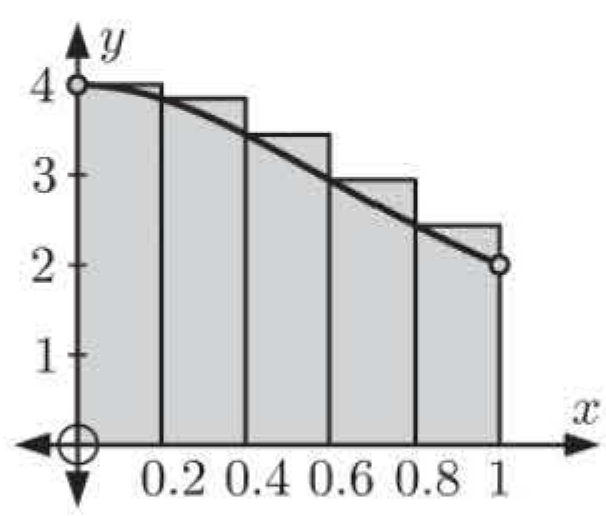
1 **a**



lower rectangles



upper rectangles



b

n	A_L	A_U
5	2.9349	3.3349
50	3.1215	3.1615
100	3.1316	3.1516
500	3.1396	3.1436

c $\displaystyle\int_0^1 \frac{4}{1+x^2}\,dx \approx 3.1416$

(the average of A_L and A_U for $n = 500$). This value agrees with π to 4 decimal places.

2 **a**
$$\frac{dy}{dx} = (x^2-1)^2$$
$$\begin{aligned}\therefore\ y &= \textstyle\int (x^2-1)^2\,dx\\ &= \textstyle\int (x^4-2x^2+1)\,dx\\ &= \tfrac{1}{5}x^5 - \tfrac{2}{3}x^3 + x + c\end{aligned}$$

b
$$\frac{dy}{dx} = 400 - 20e^{-\frac{x}{2}}$$
$$\begin{aligned}\therefore\ y &= \textstyle\int (400 - 20e^{-\frac{x}{2}})\,dx\\ &= 400x - \frac{20e^{-\frac{x}{2}}}{-\frac{1}{2}} + c\\ &= 400x + 40e^{-\frac{x}{2}} + c\end{aligned}$$

c
$$\begin{aligned}\frac{dy}{dx} &= x(x^2-1)^2\\ &= x^5 - 2x^3 + x\end{aligned}$$
$$\begin{aligned}\therefore\ y &= \textstyle\int (x^5 - 2x^3 + x)\,dx\\ &= \tfrac{1}{6}x^6 - \tfrac{2}{4}x^4 + \tfrac{1}{2}x^2 + c\\ &= \tfrac{1}{6}x^6 - \tfrac{1}{2}x^4 + \tfrac{1}{2}x^2 + c\end{aligned}$$

3 Using technology: **a** $\int_{-2}^{0} 4e^{-x^2}\,dx \approx 3.528$ **b** $\displaystyle\int_0^1 \frac{10x}{\sqrt{3x+1}}\,dx \approx 2.963$

4
$$\begin{aligned}\frac{d}{dx}(\ln x)^2 &= 2(\ln x)^1\left(\frac{1}{x}\right)\\ &= \frac{2\ln x}{x}\end{aligned}$$
$$\therefore\ \int \frac{2\ln x}{x}\,dx = (\ln x)^2 + c$$
$$\therefore\ \int \frac{\ln x}{x}\,dx = \tfrac{1}{2}(\ln x)^2 + c$$

5 Given: $f''(x) = 18x + 10,\quad f(0) = -1,\quad f(1) = 13$
$$\begin{aligned}f'(x) &= \textstyle\int (18x+10)\,dx\\ &= 9x^2 + 10x + c\end{aligned}$$
$\therefore\ f(x) = 3x^3 + 5x^2 + cx + d$

But $f(0) = -1$ so $d = -1$

$\therefore\ f(x) = 3x^3 + 5x^2 + cx - 1$

And $f(1) = 13$ so $3 + 5 + c - 1 = 13$

$\therefore\ c + 7 = 13$

$\therefore\ c = 6$

$\therefore\ f(x) = 3x^3 + 5x^2 + 6x - 1$

6
$$\begin{aligned}
\int_0^a e^{1-2x}\,dx &= \frac{e}{4}\\
\therefore\quad \left[\tfrac{1}{-2}e^{1-2x}\right]_0^a &= \frac{e}{4}\\
\therefore\quad (-\tfrac{1}{2}e^{1-2a}) - (-\tfrac{1}{2}e^{1}) &= \frac{e}{4}\\
\therefore\quad -\tfrac{1}{2}e^{1-2a} + \frac{e}{2} &= \frac{e}{4}\\
\therefore\quad \tfrac{1}{2}e^{1-2a} &= \frac{e}{4}\\
\therefore\quad e^{1-2a} &= \frac{e}{2}\\
\therefore\quad 1-2a &= \ln\left(\frac{e}{2}\right) = \ln e - \ln 2\\
\therefore\quad 1-2a &= 1-\ln 2\\
\therefore\quad 2a &= \ln 2\\
\therefore\quad a &= \tfrac{1}{2}\ln 2\\
\therefore\quad a &= \ln 2^{\frac{1}{2}}\\
\therefore\quad a &= \ln\sqrt{2}
\end{aligned}$$

7 **a** Let $y = \sqrt{2x+1} = (2x+1)^{\frac{1}{2}}$

$$\begin{aligned}
\therefore\quad \frac{dy}{dx} &= \cancel{\tfrac{1}{2}}(2x+1)^{-\frac{1}{2}} \times \cancel{2} = \frac{1}{\sqrt{2x+1}}\\
\therefore\quad \int \frac{1}{\sqrt{2x+1}}\,dx &= \sqrt{2x+1} + c\\
\therefore\quad \int_3^4 \frac{1}{\sqrt{2x+1}}\,dx &= \left[\sqrt{2x+1}\right]_3^4\\
&= \sqrt{9} - \sqrt{7}\\
&= 3 - \sqrt{7}\\
&\approx 0.354\,248\,688\,9
\end{aligned}$$

b Let $u = x^2,\quad v' = e^{x+1}$

$\therefore\ u' = 2x \qquad v = e^{x+1}$

$$\begin{aligned}
\therefore\quad \int x^2e^{x+1}\,dx &= x^2e^{x+1} - \int 2xe^{x+1}\,dx \quad \{\text{using } \int uv'\,dx = uv - \int u'v\,dx\}\\
&= x^2e^{x+1} - 2\int xe^{x+1}\,dx
\end{aligned}$$

Let $u = x,\quad v' = e^{x+1}$

$\therefore\ u' = 1 \qquad v = e^{x+1}$

$$\begin{aligned}
\therefore\quad \int xe^{x+1}\,dx &= xe^{x+1} - \int 1\times e^{x+1}\,dx\\
&= xe^{x+1} - e^{x+1} + c\\
\therefore\quad \int x^2e^{x+1}\,dx &= x^2e^{x+1} - 2(xe^{x+1} - e^{x+1} + c)\\
&= x^2e^{x+1} - 2xe^{x+1} + 2e^{x+1} + c\\
\therefore\quad \int_0^1 x^2e^{x+1}\,dx &= \left[x^2e^{x+1} - 2xe^{x+1} + 2e^{x+1}\right]_0^1\\
&= (e^2 - \cancel{2e^2} + \cancel{2e^2}) - (0e^1 - 0e^1 + 2e^1)\\
&= e^2 - 2e^1\\
&\approx 1.952\,492\,442
\end{aligned}$$

8 **a**
$$f''(x) = 3x^2 + 2x$$
$$\therefore\ f'(x) = \frac{3x^3}{3} + \frac{2x^2}{2} + c = x^3 + x^2 + c$$
$$\therefore\ f(x) = \frac{x^4}{4} + \frac{x^3}{3} + cx + d$$
But $f(0) = 3$ so $d = 3$
$$\therefore\ f(x) = \frac{x^4}{4} + \frac{x^3}{3} + cx + 3$$
Also, $f(2) = 3$ so $4 + \frac{8}{3} + 2c + 3 = 3$
$$\therefore\ \tfrac{20}{3} = -2c$$
$$\therefore\ c = -\tfrac{10}{3}$$
$$\therefore\ f(x) = \tfrac{1}{4}x^4 + \tfrac{1}{3}x^3 - \tfrac{10}{3}x + 3$$

b Now $f'(2) = 2^3 + 2^2 - \frac{10}{3} = 12 - \frac{10}{3} = \frac{26}{3}$

$\therefore$ the normal has gradient $-\frac{3}{26}$

$\therefore$ equation is $\dfrac{y-3}{x-2} = -\frac{3}{26}$
$$\therefore\ y - 3 = -\tfrac{3}{26}(x-2)$$
$$\therefore\ y = -\tfrac{3}{26}x + \tfrac{6}{26} + 3$$
or $3x + 26y = 84$

9 **a**
$$(e^x+2)^3 = (e^x)^3 + 3(e^x)^2(2) + 3(e^x)(2)^2 + (2)^3 = e^{3x} + 6e^{2x} + 12e^x + 8$$

b
$$\int_0^1 (e^x+2)^3\,dx = \left[\tfrac{1}{3}e^{3x} + 3e^{2x} + 12e^x + 8x\right]_0^1$$
$$= \left(\tfrac{1}{3}e^3 + 3e^2 + 12e + 8\right) - \left(\tfrac{1}{3} + 3 + 12\right)$$
$$= \tfrac{1}{3}e^3 + 3e^2 + 12e - 7\tfrac{1}{3}$$
$$\approx 54.148\,395\,88$$

10
$$\int \sin^5 x\cos x\,dx = \int (\sin x)^5 \cos x\,dx$$
$$= \int u^5 \frac{du}{dx}\,dx \qquad \left\{u = \sin x,\ \frac{du}{dx} = \cos x\right\}$$
$$= \int u^5\,du = \frac{u^6}{6} + c = \frac{\sin^6 x}{6} + c$$
$$\therefore\ \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sin^5 x\cos x\,dx = \left[\frac{\sin^6 x}{6}\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$
$$= \tfrac{1}{6}\left(\left(\tfrac{\sqrt{3}}{2}\right)^6 - \left(\tfrac{1}{\sqrt{2}}\right)^6\right)$$
$$= \tfrac{19}{384}$$
$$\approx 0.049\,479\,166\,67$$

11
$$f''(x) = 4x^2 - 3$$
$$\therefore\ f'(x) = \frac{4x^3}{3} - 3x + c$$
But $f'(0) = 6$ so $c = 6$
$$\therefore\ f'(x) = \frac{4x^3}{3} - 3x + 6$$
$$\therefore\ f(x) = \tfrac{4}{3}\frac{x^4}{4} - \frac{3x^2}{2} + 6x + d = \tfrac{1}{3}x^4 - \tfrac{3}{2}x^2 + 6x + d$$
But $f(2) = 3$, so $\frac{16}{3} - 6 + 12 + d = 3$
$$\therefore\ d = -3 - \tfrac{16}{3} = -\tfrac{25}{3}$$
$$\therefore\ f(x) = \tfrac{1}{3}x^4 - \tfrac{3}{2}x^2 + 6x - \tfrac{25}{3}$$
and $f(3) = 27 - \frac{27}{2} + 18 - \frac{25}{3} = 23\frac{1}{6}$

12 If $y = x\tan x$ then $\dfrac{dy}{dx} = \tan x + x\sec^2 x$
$$\therefore\ \int(\tan x + x\sec^2 x)\,dx = x\tan x + c$$
$$\therefore\ \int \tan x\,dx + \int x\sec^2 x\,dx = x\tan x + c$$
$$\therefore\ -\ln|\cos x| + \int x\sec^2 x\,dx = x\tan x + c \qquad \text{\{see Ex 21G.1, Q 6 c\}}$$
$$\therefore\ \int x\sec^2 x\,dx = x\tan x + \ln|\cos x| + c$$

13 **a** $\dfrac{4x-3}{2x+1} = \dfrac{2(2x+1)-5}{2x+1}$

$= 2 + \dfrac{-5}{2x+1}$

$\therefore \; A = 2, \; B = -5$

b $\displaystyle\int_0^2 \frac{4x-3}{2x+1}\,dx = \int_0^2 \left(2 - 5\left(\frac{1}{2x+1}\right)\right) dx$

$= \left[2x - 5\left(\tfrac{1}{2}\right)\ln|2x+1|\right]_0^2$

$= [4 - \tfrac{5}{2}\ln 5] - [0 - \tfrac{5}{2}\ln 1]$

$= 4 - \tfrac{5}{2}\ln 5$

≈ -0.0236

14 **a** We integrate by parts with $u = e^{-x}$ $\quad v' = \cos x$
$u' = -e^{-x}$ $\quad v = \sin x$

$\therefore \; \int e^{-x}\cos x\,dx = e^{-x}\sin x - \int -e^{-x}\sin x\,dx$

$= e^{-x}\sin x + \int e^{-x}\sin x\,dx$

We integrate by parts again, this time with $u = e^{-x}$ $\quad v' = \sin x$
$u' = -e^{-x}$ $\quad v = -\cos x$

$\therefore \; \int e^{-x}\cos x\,dx = e^{-x}\sin x + e^{-x}(-\cos x) - \int(-e^{-x})(-\cos x)\,dx + c$

$= e^{-x}\sin x - e^{-x}\cos x - \int e^{-x}\cos x\,dx + c$

$\therefore \; 2\int e^{-x}\cos x\,dx = e^{-x}(\sin x - \cos x) + c$

$\therefore \; \int e^{-x}\cos x\,dx = \tfrac{1}{2}e^{-x}(\sin x - \cos x) + c$

b We integrate by parts with $u = x^2$ $\quad v' = e^x$
$u' = 2x$ $\quad v = e^x$

$\therefore \; \int x^2e^x\,dx = x^2e^x - \int 2xe^x\,dx$

We integrate by parts again, this time with $u = 2x$ $\quad v' = e^x$
$u' = 2$ $\quad v = e^x$

$\therefore \; \int x^2e^x\,dx = x^2e^x - \left[2xe^x - \int 2e^x\,dx\right]$

$= x^2e^x - 2xe^x + 2\int e^x\,dx$

$= x^2e^x - 2xe^x + 2e^x + c$

$= e^x(x^2 - 2x + 2) + c$

c Let $u = 9 - x^2, \; \dfrac{du}{dx} = -2x$

$\therefore \; \displaystyle\int \frac{x^3}{\sqrt{9-x^2}}\,dx = -\tfrac{1}{2}\int \frac{x^2}{\sqrt{9-x^2}}(-2x)\,dx$

$= -\tfrac{1}{2}\displaystyle\int \frac{9-u}{\sqrt{u}}\,\frac{du}{dx}\,dx$

$= -\tfrac{1}{2}\displaystyle\int \frac{9-u}{\sqrt{u}}\,du$

$= -\tfrac{1}{2}\displaystyle\int \left(9u^{-\frac{1}{2}} - u^{\frac{1}{2}}\right) du$

$= -\tfrac{1}{2}\left[\dfrac{9u^{\frac{1}{2}}}{\frac{1}{2}} - \dfrac{u^{\frac{3}{2}}}{\frac{3}{2}}\right] + c$

$= -9u^{\frac{1}{2}} + \tfrac{1}{3}u^{\frac{3}{2}} + c$

$= -9\sqrt{9-x^2} + \tfrac{1}{3}\left(9-x^2\right)^{\frac{3}{2}} + c$

15 **a** $\displaystyle\int \frac{1}{x+2}\,dx - \int \frac{2}{x-1}\,dx = \ln|x+2| - 2\ln|x-1| + c = \ln\left(\frac{|x+2|}{(x-1)^2}\right) + c$

b $\displaystyle\frac{1}{x+2} - \frac{2}{x-1} = \frac{(x-1)-2(x+2)}{(x+2)(x-1)}$

$\displaystyle= \frac{-x-5}{(x+2)(x-1)}$

$\displaystyle= -\frac{x+5}{(x+2)(x-1)}$

$\displaystyle\therefore \int \frac{x+5}{(x+2)(x-1)}\,dx$

$\displaystyle= -\left[\int \frac{1}{x+2}\,dx - \int \frac{2}{x-1}\,dx\right]$

$\displaystyle= -\ln\left(\frac{|x+2|}{(x-1)^2}\right) + c$

$\displaystyle= \ln\left(\frac{(x-1)^2}{|x+2|}\right) + c$

REVIEW SET 21C

1 **a** $\displaystyle\int \left(2e^{-x} - \frac{1}{x} + 3\right) dx$

$= -2e^{-x} - \ln|x| + 3x + c$

b $\displaystyle\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 dx$

$\displaystyle= \int \left(x - 2 + \frac{1}{x}\right) dx$

$= \frac{1}{2}x^2 - 2x + \ln|x| + c$

c $\int (3 + e^{2x-1})^2\,dx$

$= \int (9 + 6e^{2x-1} + e^{4x-2})\,dx$

$= 9x + 3e^{2x-1} + \frac{1}{4}e^{4x-2} + c$

2 $f'(x) = x^2 - 3x + 2$

$\displaystyle\therefore\ f(x) = \frac{x^3}{3} - \frac{3x^2}{2} + 2x + c$

But $f(1) = 3$ so $\frac{1}{3} - \frac{3}{2} + 2 + c = 3$

$\therefore\ c = 1 - \frac{1}{3} + 1\frac{1}{2}$

$\therefore\ c = 2\frac{1}{6}$

$\therefore\ f(x) = \frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x + 2\frac{1}{6}$

3 **a** $\displaystyle\int_2^3 \frac{1}{\sqrt{3x-4}}\,dx$

$= \int_2^3 (3x-4)^{-\frac{1}{2}}\,dx$

$\displaystyle= \left[\frac{1}{3}\frac{(3x-4)^{\frac{1}{2}}}{\frac{1}{2}}\right]_2^3$

$= \left[\frac{2}{3}\sqrt{3x-4}\right]_2^3$

$= \frac{2}{3}\sqrt{5} - \frac{2}{3}\sqrt{2}$

$= \frac{2}{3}(\sqrt{5} - \sqrt{2})$

b $\int_0^{\frac{\pi}{3}} \cos^2\left(\frac{x}{2}\right)\,dx$

$= \int_0^{\frac{\pi}{3}} \left(\frac{1}{2} + \frac{1}{2}\cos x\right)\,dx$

$= \left[\frac{1}{2}x + \frac{1}{2}\sin x\right]_0^{\frac{\pi}{3}}$

$= \frac{\pi}{6} + \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right) - 0 - 0$

$= \frac{\pi}{6} + \frac{\sqrt{3}}{4}$

c Let $u = \cos x$, $\displaystyle\frac{du}{dx} = -\sin x$

when $x = 0$, $u = \cos 0 = 1$

when $x = \frac{\pi}{4}$, $u = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$\displaystyle\therefore\ \int_0^{\frac{\pi}{4}} \tan x\,dx = \int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos x}\,dx$

$\displaystyle= -\int_0^{\frac{\pi}{4}} \frac{-\sin x}{\cos x}\,dx$

$\displaystyle= -\int_0^{\frac{\pi}{4}} \frac{1}{u}\frac{du}{dx}\,dx$

$\displaystyle= -\int_1^{\frac{1}{\sqrt{2}}} \frac{1}{u}\,du$

$\displaystyle= \int_{\frac{1}{\sqrt{2}}}^1 \frac{1}{u}\,du$

$= [\ln|u|]_{\frac{1}{\sqrt{2}}}^1$

$= \ln 1 - \ln\frac{1}{\sqrt{2}}$

$= \frac{1}{2}\ln 2$

4 $\dfrac{d}{dx}(e^{-2x}\sin x) = -2e^{-2x}\sin x + e^{-2x}\cos x$ {product rule}
$= e^{-2x}(\cos x - 2\sin x)$

$\therefore \int_0^{\frac{\pi}{2}} e^{-2x}(\cos x - 2\sin x)\,dx = \left[e^{-2x}\sin x\right]_0^{\frac{\pi}{2}}$
$= e^{-\pi}(1) - e^0(0) = e^{-\pi}$

5 If $n \neq -1$, $\int (2x+3)^n\,dx = \frac{1}{2}\dfrac{(2x+3)^{n+1}}{n+1} + c = \dfrac{1}{2(n+1)}(2x+3)^{n+1} + c$

If $n = -1$, $\int (2x+3)^{-1}\,dx = \displaystyle\int \frac{1}{2x+3}\,dx = \frac{1}{2}\ln|2x+3| + c$

So, $\int (2x+3)^n\,dx = \begin{cases} \dfrac{1}{2(n+1)}(2x+3)^{n+1} + c & \text{if } n \neq -1 \\ \frac{1}{2}\ln|2x+3| + c & \text{if } n = -1 \end{cases}$

6 $f'(x) = 2\sqrt{x} + \dfrac{a}{\sqrt{x}}$
$= 2x^{\frac{1}{2}} + ax^{-\frac{1}{2}}$

$\therefore f(x) = \frac{4}{3}x^{\frac{3}{2}} + 2ax^{\frac{1}{2}} + c$
$= \dfrac{4x\sqrt{x}}{3} + 2a\sqrt{x} + c$

Now $f(0) = 2$ so $c = 2$

$\therefore f(x) = \dfrac{4x\sqrt{x}}{3} + 2a\sqrt{x} + 2$

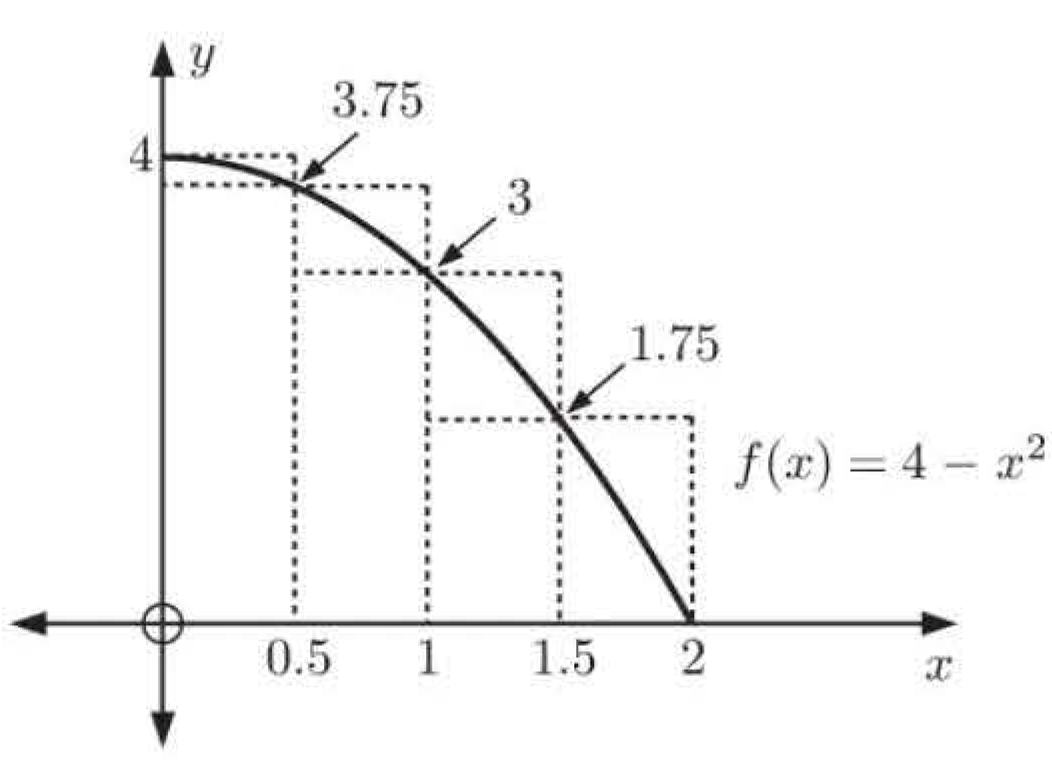
Also, $f(1) = 4$ so $\frac{4}{3} + 2a + 2 = 4$
$\therefore 2a = \frac{2}{3}$
$\therefore a = \frac{1}{3}$

$\therefore f'(x) = 2\sqrt{x} + \dfrac{1}{3\sqrt{x}} = \dfrac{6x+1}{3\sqrt{x}}$

Now $f(x)$ is only defined for $x > 0$,
so $f'(x) > 0$ for all x in the domain.
$\therefore$ the function has no stationary points.

7
$$\int_a^{2a} (x^2 + ax + 2)\,dx = \frac{73a}{2}$$
$$\therefore \left[\frac{x^3}{3} + \frac{ax^2}{2} + 2x\right]_a^{2a} = \frac{73a}{2}$$
$$\therefore \left(\frac{8a^3}{3} + \frac{a}{2}(4a^2) + 4a\right) - \left(\frac{a^3}{3} + \frac{a^3}{2} + 2a\right) = \frac{73a}{2}$$
$$\frac{8a^3}{3} + 2a^3 + 4a - \frac{a^3}{3} - \frac{a^3}{2} - 2a = \frac{73a}{2}$$
$$\therefore 16a^3 + 12a^3 + 24a - 2a^3 - 3a^3 - 12a = 219a$$
$$\therefore 23a^3 - 207a = 0$$
$$\therefore 23a(a^2 - 9) = 0$$
$$\therefore 23a(a+3)(a-3) = 0$$
$$\therefore a = 0 \text{ or } a = \pm 3$$

8 **a**



$A_U = 0.5\,[f(0) + f(0.5) + f(1) + f(1.5)]$
$= 0.5(4 + 3.75 + 3 + 1.75)$
$= 6.25$

$A_L = 0.5\,[f(0.5) + f(1) + f(1.5) + f(2)]$
$= 0.5(3.75 + 3 + 1.75 + 0)$
$= 4.25$

$\therefore 4.25 < \int_0^2 (4 - x^2)\,dx < 6.25$

$\therefore A = 4.25 = \frac{17}{4}$, $B = 6.25 = \frac{25}{4}$

b An estimate of $\int_0^2 (4-x^2)\,dx \approx \dfrac{A+B}{2} \approx \frac{42}{8} \approx \frac{21}{4}$.

9 **a**

$$\begin{aligned}
&\int \frac{2x}{\sqrt{x^2-5}}\,dx \\
&= \int 2x(x^2-5)^{-\frac{1}{2}}\,dx \\
&= \int u^{-\frac{1}{2}}\frac{du}{dx}\,dx \quad \left\{u = x^2-5,\ \frac{du}{dx} = 2x\right\} \\
&= \int u^{-\frac{1}{2}}\,du \\
&= \frac{u^{\frac{1}{2}}}{\frac{1}{2}} + c \\
&= 2\sqrt{u} + c \\
&= 2\sqrt{x^2-5} + c
\end{aligned}$$

b Let $u = \cos x$, $\dfrac{du}{dx} = -\sin x$

$$\begin{aligned}
\therefore\ \int \frac{\sin x}{\cos^4 x}\,dx &= -\int \frac{-\sin x}{\cos^4 x}\,dx \\
&= -\int u^{-4}\frac{du}{dx}\,dx \\
&= -\int u^{-4}\,du \\
&= \frac{u^{-3}}{3} + c = \frac{1}{3\cos^3 x} + c
\end{aligned}$$

c

$$\begin{aligned}
&\int 3x^2\sqrt{x^3-1}\,dx \\
&= \int 3x^2(x^3-1)^{\frac{1}{2}}\,dx \\
&= \int u^{\frac{1}{2}}\frac{du}{dx}\,dx \quad \left\{u = x^3-1,\ \frac{du}{dx} = 3x^2\right\} \\
&= \int u^{\frac{1}{2}}\,du \\
&= \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c \\
&= \tfrac{2}{3}u\sqrt{u} + c \\
&= \tfrac{2}{3}(x^3-1)\sqrt{x^3-1} + c
\end{aligned}$$

$$\begin{aligned}
\therefore\ &\int_1^2 3x^2\sqrt{x^3-1}\,dx \\
&= \tfrac{2}{3}\left[(x^3-1)\sqrt{x^3-1}\right]_1^2 \\
&= \tfrac{2}{3}\left((8-1)\sqrt{8-1} - 0\right) \\
&= \tfrac{2}{3} \times 7\sqrt{7} \\
&= \tfrac{14\sqrt{7}}{3}
\end{aligned}$$

10

$$\begin{aligned}
y &= \ln(\sec x), \quad \sec x > 0 \\
\therefore\ \frac{dy}{dx} &= \frac{\sec x \tan x}{\sec x} = \tan x \\
\therefore\ \int \tan x\,dx &= \ln(\sec x) + c, \quad \sec x > 0
\end{aligned}$$

11 **a**

$$\begin{aligned}
&\int \frac{5}{\sqrt{9-x^2}}\,dx \\
&= 5\int \frac{1}{\sqrt{3^2-x^2}}\,dx \\
&= 5\arcsin\left(\tfrac{x}{3}\right) + c
\end{aligned}$$

$\left\{\text{using } \displaystyle\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\left(\tfrac{x}{a}\right) + c,\ a \neq 0\right.$ from **Ex 21G.1**, Q **11 b**$\}$

b

$$\begin{aligned}
&\int \frac{1}{9+4x^2}\,dx \\
&= \tfrac{1}{4}\int \frac{1}{\frac{9}{4}+x^2}\,dx \\
&= \tfrac{1}{4}\left(\frac{1}{\frac{3}{2}}\right)\arctan\left(\frac{x}{\frac{3}{2}}\right) + c
\end{aligned}$$

$\left\{\text{using } \displaystyle\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\left(\tfrac{x}{a}\right) + c,\ a \neq 0\right.$ from **Ex 21G.1**, Q **12 b**$\}$

$$= \tfrac{1}{6}\arctan\left(\tfrac{2x}{3}\right) + c$$

c Let $u = x - 5$, $\dfrac{du}{dx} = 1$

when $x = 10$, $u = 5$, and when $x = 7$, $u = 2$

$$\begin{aligned}
\therefore\quad \int_7^{10} x\sqrt{x-5}\,dx &= \int_2^5 (u+5)\sqrt{u}\,du \\
&= \int_2^5 (u^{\frac{3}{2}} + 5u^{\frac{1}{2}})\,du \\
&= \left[\frac{u^{\frac{5}{2}}}{\frac{5}{2}} + \frac{5u^{\frac{3}{2}}}{\frac{3}{2}}\right]_2^5 \\
&= \tfrac{2}{5}\left(5^{\frac{5}{2}}\right) + \tfrac{10}{3}\left(5^{\frac{3}{2}}\right) - \left[\tfrac{2}{5}\left(2^{\frac{5}{2}}\right) + \tfrac{10}{3}\left(2^{\frac{3}{2}}\right)\right] \\
&= \tfrac{2}{5}\left(25\sqrt{5}\right) + \tfrac{10}{3}\left(5\sqrt{5}\right) - \tfrac{2}{5}\left(4\sqrt{2}\right) - \tfrac{10}{3}\left(2\sqrt{2}\right) \\
&= 10\sqrt{5} + \tfrac{50}{3}\sqrt{5} - \tfrac{8}{5}\sqrt{2} - \tfrac{20}{3}\sqrt{2} \\
&= \tfrac{80}{3}\sqrt{5} - \tfrac{124}{15}\sqrt{2}
\end{aligned}$$

12

$$\begin{array}{r|rrrr}
2 & 1 & 0 & -3 & 2 \\
 & 0 & 2 & 4 & 2 \\
\hline
 & 1 & 2 & 1 & \multicolumn{1}{|r}{4}
\end{array}$$

$$\therefore\quad \frac{x^3 - 3x + 2}{x - 2} = x^2 + 2x + 1 + \frac{4}{x-2}$$

$$\therefore\quad A = 1,\ B = 2,\ C = 1,\ D = 4$$

$$\begin{aligned}
\therefore\quad &\int \frac{x^3 - 3x + 2}{x-2}\,dx \\
&= \int \left(x^2 + 2x + 1 + \frac{4}{x-2}\right)\,dx \\
&= \frac{x^3}{3} + \frac{2x^2}{2} + x + 4\ln|x-2| + c \\
&= \tfrac{1}{3}x^3 + x^2 + x + 4\ln|x-2| + c
\end{aligned}$$

or

$$\begin{aligned}
&\int \frac{x^3 - 3x + 2}{x-2}\,dx \\
&= \int \left((x+1)^2 + \frac{4}{x-2}\right)\,dx \\
&= \frac{(x+1)^3}{3} + 4\ln|x-2| + c
\end{aligned}$$

13 **a** We integrate by parts with

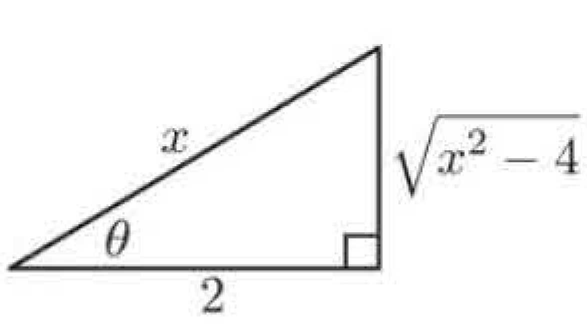
$$\begin{aligned}
u &= x & v' &= \cos x \\
u' &= 1 & v &= \sin x
\end{aligned}$$

Using $\int uv'\,dx = uv - \int u'v\,dx$,

$$\begin{aligned}
\therefore\quad &\int x\cos x\,dx \\
&= x\sin x - \int \sin x\,dx \\
&= x\sin x - (-\cos x) + c \\
&= x\sin x + \cos x + c
\end{aligned}$$

b Let $x = 2\sec\theta$, $\dfrac{dx}{d\theta} = 2\sec\theta\tan\theta$

$$\begin{aligned}
\therefore\quad &\int \frac{\sqrt{x^2-4}}{x}\,dx \\
&= \int \frac{\sqrt{4\sec^2\theta - 4}}{2\sec\theta} \times 2\sec\theta\tan\theta\,d\theta \\
&= \int \sqrt{4(\sec^2\theta - 1)}\ \tan\theta\,d\theta \\
&= \int 2\tan\theta\tan\theta\,d\theta \\
&= 2\int \tan^2\theta\,d\theta \\
&= 2\int (\sec^2\theta - 1)\,d\theta \\
&= 2[\tan\theta - \theta] + c \\
&= 2\tan\theta - 2\theta + c
\end{aligned}$$



$$\begin{aligned}
&= 2\left(\frac{\sqrt{x^2-4}}{2}\right) - 2\arccos\left(\tfrac{2}{x}\right) + c \\
&= \sqrt{x^2-4} - 2\arccos\left(\tfrac{2}{x}\right) + c
\end{aligned}$$

14 $$\frac{d}{dx}\left(\frac{e^{1-x}}{x^2}\right) = \frac{-e^{1-x}x^2 - e^{1-x}(2x)}{x^4} \quad \{\text{quotient rule}\}$$
$$= -\frac{e^{1-x}(x+2)}{x^3}$$
$$\therefore \quad \int_1^2 \frac{e^{1-x}(x+2)}{x^3}\,dx = -\int_1^2 -\frac{e^{1-x}(x+2)}{x^3}\,dx$$
$$= -\left[\frac{e^{1-x}}{x^2}\right]_1^2$$
$$= -\left(\frac{e^{-1}}{4} - \frac{e^0}{1}\right)$$
$$= 1 - \frac{1}{4e}$$

15 Let $u = \cos x$, $\frac{du}{dx} = -\sin x$

$$\therefore \quad \int \frac{\sin x}{\sqrt{\cos^n x}}\,dx = -\int \frac{-\sin x}{\sqrt{\cos^n x}}\,dx$$
$$= -\int \frac{1}{\sqrt{u^n}}\frac{du}{dx}\,dx$$
$$= -\int u^{-\frac{n}{2}}\,du$$
$$= -\frac{u^{1-\frac{n}{2}}}{1-\frac{n}{2}} + c \quad \text{provided } n \neq 2$$
$$= \frac{\cos^{1-\frac{n}{2}} x}{\frac{n}{2}-1} + c, \quad \text{for } n \neq 2$$

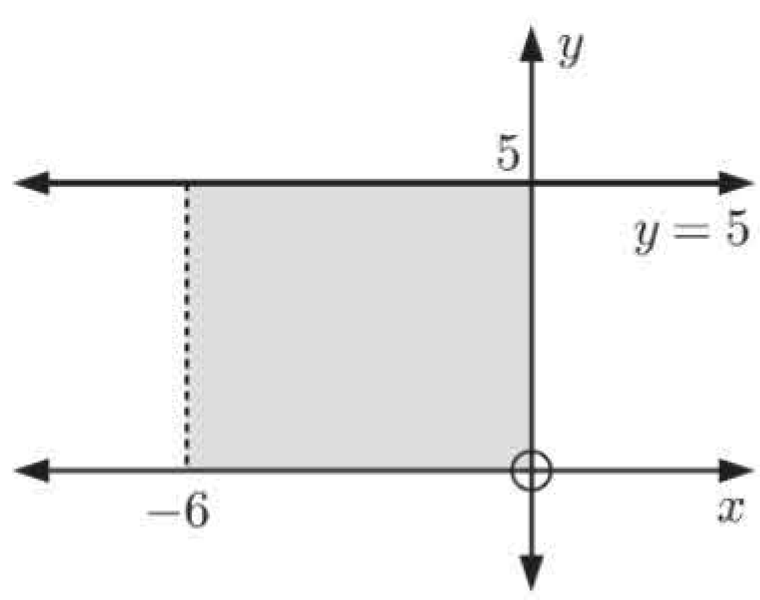
If $n = 2$, $$\int \frac{\sin x}{\sqrt{\cos^2 x}}\,dx = \int \frac{\sin x}{\cos x}\,dx$$
$$= \int \tan x\,dx$$
$$= -\ln|\cos x| + c, \quad \text{for } n = 2$$ {see **Ex 21G.1** Q **6 c**}

So, the integral is defined for all n.

Chapter 22
APPLICATIONS OF INTEGRATION

EXERCISE 22A

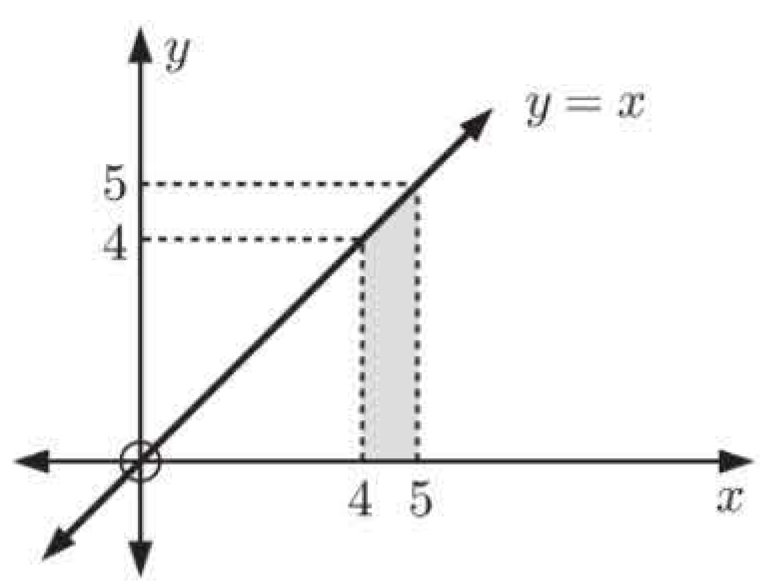
1 **a**



i $\text{Area} = 5 \times 6$
$= 30 \text{ units}^2$

ii $\text{Area} = \int_{-6}^{0} 5\, dx$
$= [5x]_{-6}^{0}$
$= 5(0) - 5(-6)$
$= 30 \text{ units}^2$

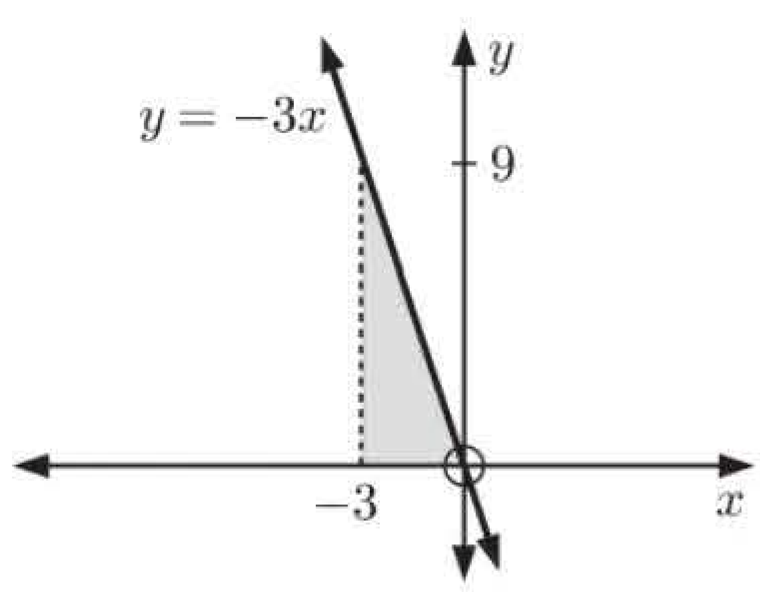
b



i $\text{Area} = \left(\dfrac{4+5}{2}\right) \times 1$
$= \frac{9}{2} \text{ units}^2$

ii $\text{Area} = \int_{4}^{5} x\, dx$
$= \left[\frac{1}{2}x^2\right]_4^5$
$= \frac{1}{2}(25) - \frac{1}{2}(16)$
$= \frac{9}{2} \text{ units}^2$

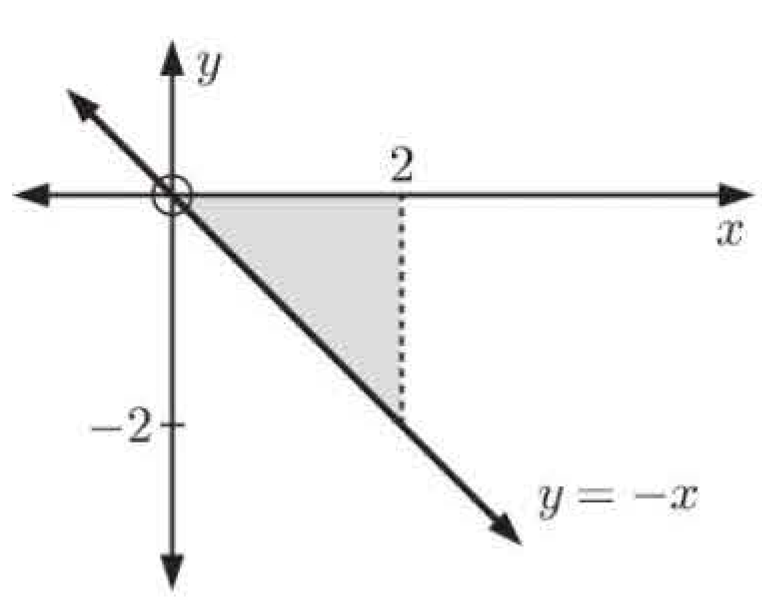
c



i $\text{Area} = \frac{1}{2} \times 3 \times 9$
$= \frac{27}{2} \text{ units}^2$

ii $\text{Area} = \int_{-3}^{0} (-3x)\, dx$
$= \left[-\frac{3}{2}x^2\right]_{-3}^{0}$
$= -\frac{3}{2}(0) - (-\frac{3}{2})(9)$
$= \frac{27}{2} \text{ units}^2$

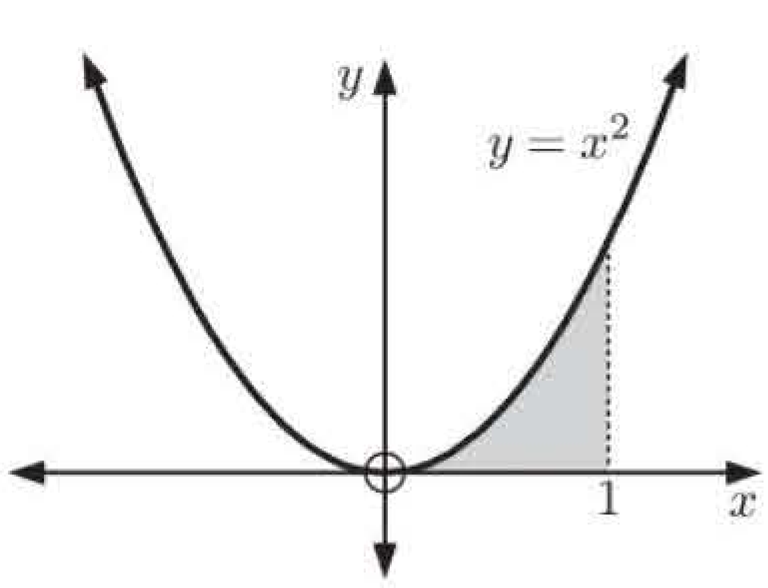
d



i $\text{Area} = \frac{1}{2} \times 2 \times 2$
$= 2 \text{ units}^2$

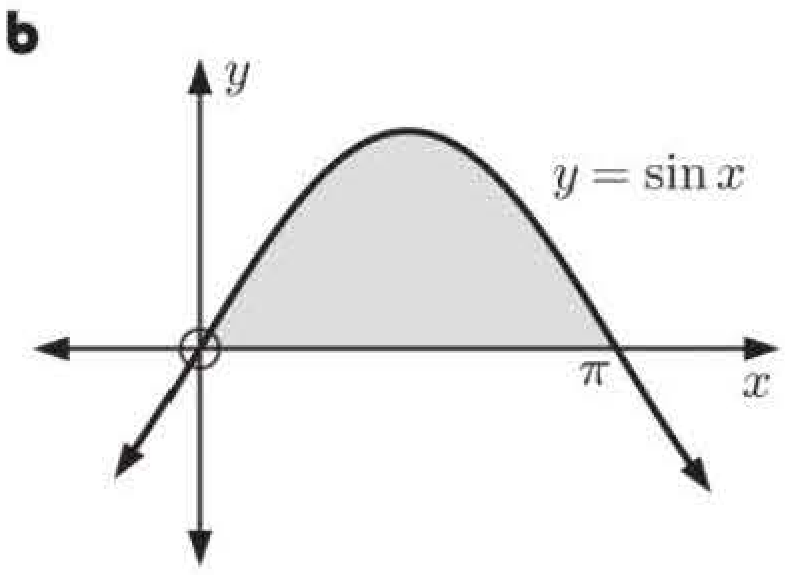
ii $\text{Area} = -\int_{0}^{2} -x\, dx$
$= -\left[-\frac{1}{2}x^2\right]_0^2$
$= -\left(-\frac{1}{2}(4) - (-\frac{1}{2})(0)\right)$
$= 2 \text{ units}^2$

2 **a**



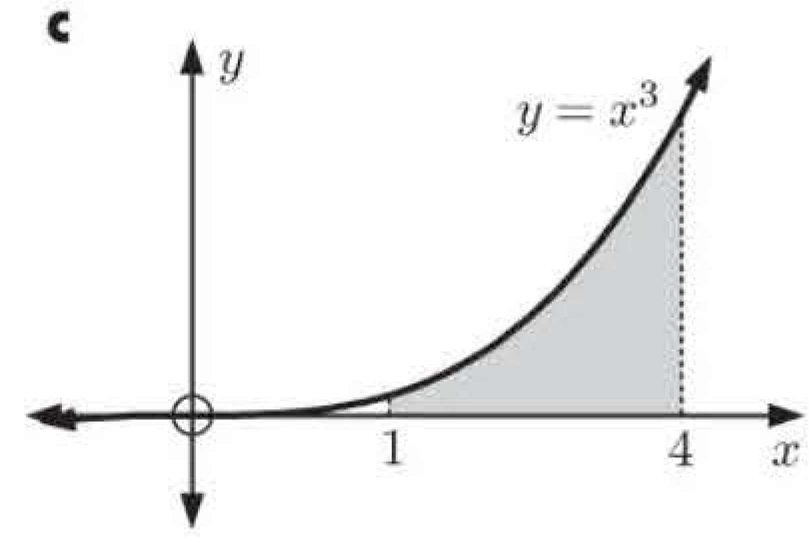
$\text{Area} = \int_0^1 x^2\, dx$
$= \left[\dfrac{x^3}{3}\right]_0^1$
$= \frac{1}{3} - 0$
$= \frac{1}{3} \text{ units}^2$

b



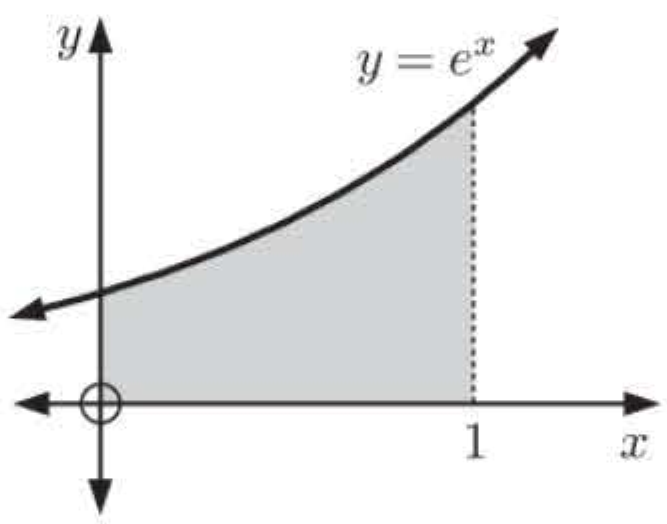
$\text{Area} = \int_0^{\pi} \sin x\, dx$
$= [-\cos x]_0^{\pi}$
$= -\cos\pi - (-\cos 0)$
$= 2 \text{ units}^2$

c



$\text{Area} = \int_1^4 x^3\, dx$
$= \left[\dfrac{x^4}{4}\right]_1^4$
$= \frac{256}{4} - \frac{1}{4}$
$= 63\frac{3}{4} \text{ units}^2$

d



$$\text{Area} = \int_0^1 e^x\, dx$$
$$= [e^x]_0^1$$
$$= (e-1) \text{ units}^2$$

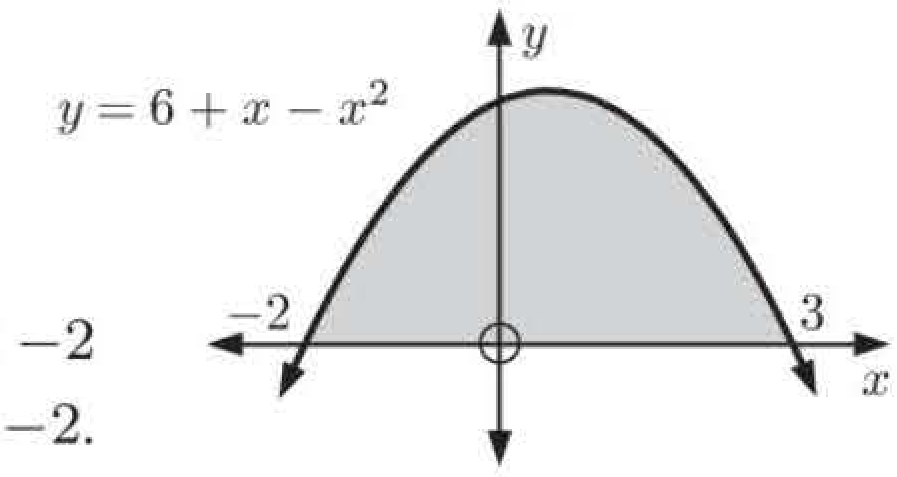
e The graph cuts the x-axis at $y = 0$.

$\therefore \quad 6 + x - x^2 = 0$

$\therefore \quad (3-x)(2+x) = 0$

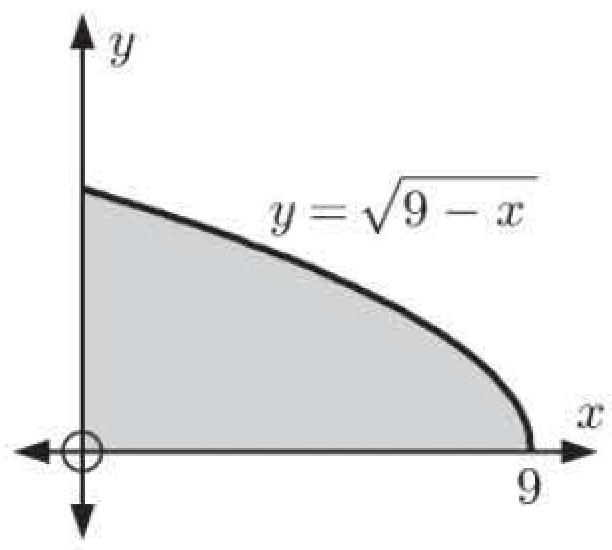
$\therefore \quad x = 3$ or -2

The x-intercepts are 3 and -2.



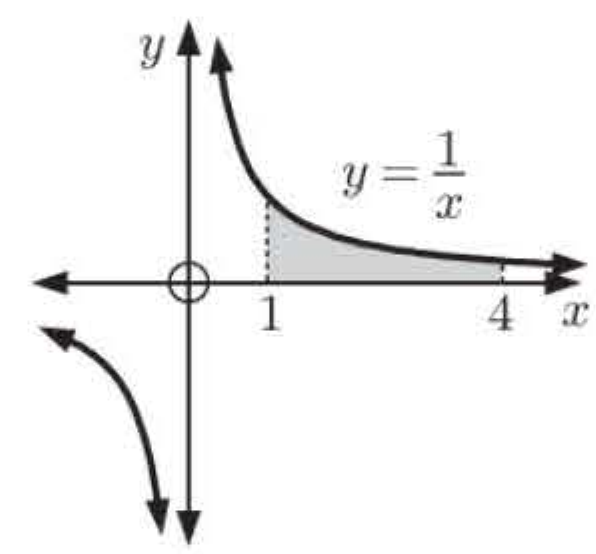
$$\text{Area} = \int_{-2}^{3} (6 + x - x^2)\, dx$$
$$= \left[6x + \frac{x^2}{2} - \frac{x^3}{3}\right]_{-2}^{3}$$
$$= (18 + \tfrac{9}{2} - 9) - (-12 + 2 + \tfrac{8}{3})$$
$$= 20\tfrac{5}{6} \text{ units}^2$$

f



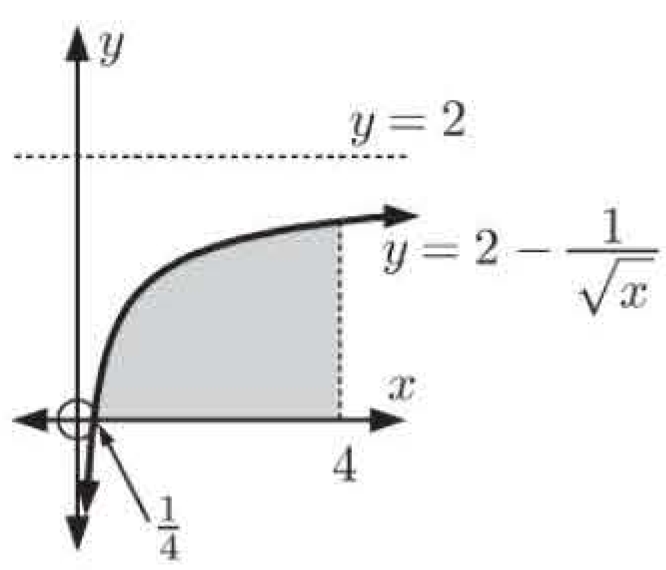
$$\text{Area} = \int_0^9 (9-x)^{\frac{1}{2}}\, dx$$
$$= \left[\left(\tfrac{1}{-1}\right) \frac{(9-x)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^9$$
$$= -\tfrac{2}{3}\left[(9-x)^{\frac{3}{2}}\right]_0^9$$
$$= -\tfrac{2}{3}(0 - 27)$$
$$= 18 \text{ units}^2$$

g



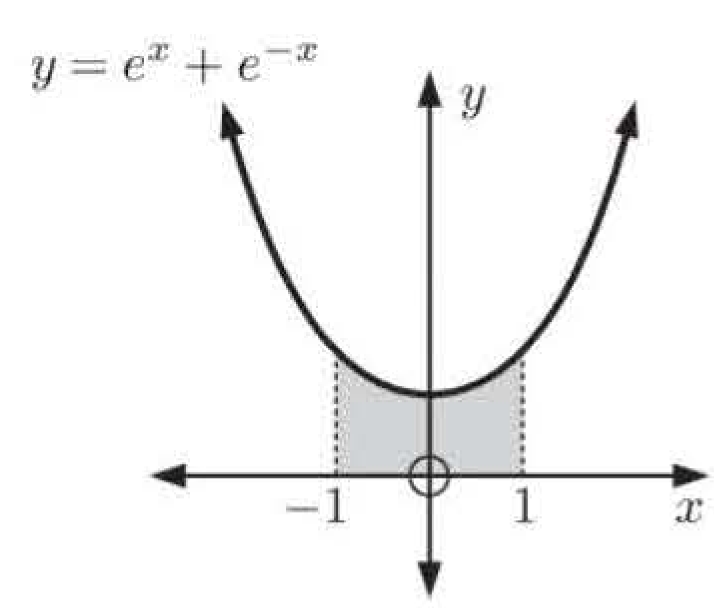
$$\text{Area} = \int_1^4 \frac{1}{x}\, dx$$
$$= [\ln x]_1^4 \quad \{x > 0\}$$
$$= \ln 4 - \ln 1$$
$$= \ln 4 - 0$$
$$= \ln 4 \text{ units}^2$$

h



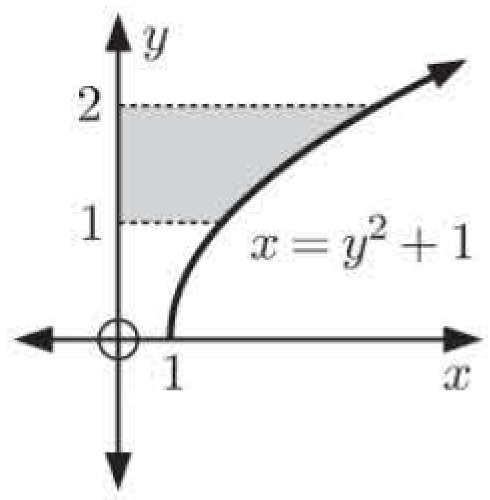
$$\text{Area} = \int_{\frac{1}{4}}^{4} \left(2 - \tfrac{1}{\sqrt{x}}\right) dx$$
$$= \int_{\frac{1}{4}}^{4} (2 - x^{-\frac{1}{2}})\, dx$$
$$= \left[2x - \frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right]_{\frac{1}{4}}^{4}$$
$$= \left[2x - 2\sqrt{x}\right]_{\frac{1}{4}}^{4}$$
$$= (8-4) - (\tfrac{1}{2} - 1)$$
$$= 4\tfrac{1}{2} \text{ units}^2$$

i



$$\text{Area} = \int_{-1}^{1} (e^x + e^{-x})\, dx$$
$$= \left[e^x - e^{-x}\right]_{-1}^{1}$$
$$= (e - e^{-1}) - (e^{-1} - e)$$
$$= \left(2e - \frac{2}{e}\right) \text{ units}^2$$

3 **a**



Area

$= \int_1^2 x\,dy$

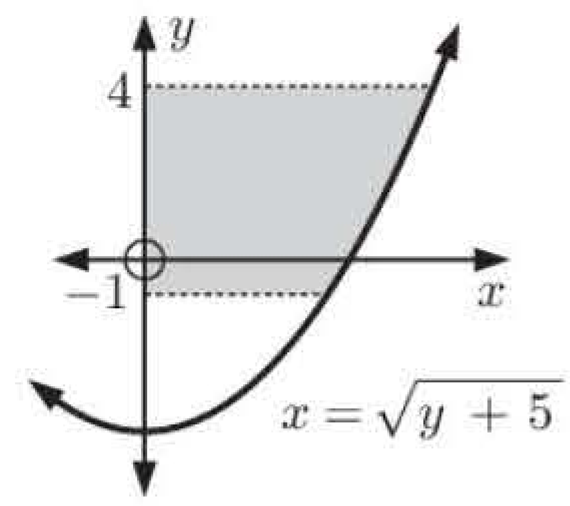
$= \int_1^2 (y^2+1)\,dy$

$= \left[\dfrac{y^3}{3} + y\right]_1^2$

$= \left(\frac{8}{3} + 2\right) - \left(\frac{1}{3} + 1\right)$

$= 3\frac{1}{3}$ units^2

b



Area

$= \int_{-1}^4 x\,dy$

$= \int_{-1}^4 (y+5)^{\frac{1}{2}}\,dy$

$= \left[\dfrac{(y+5)^{\frac{3}{2}}}{\frac{3}{2}}\right]_{-1}^4$

$= \left[\frac{2}{3}(y+5)^{\frac{3}{2}}\right]_{-1}^4$

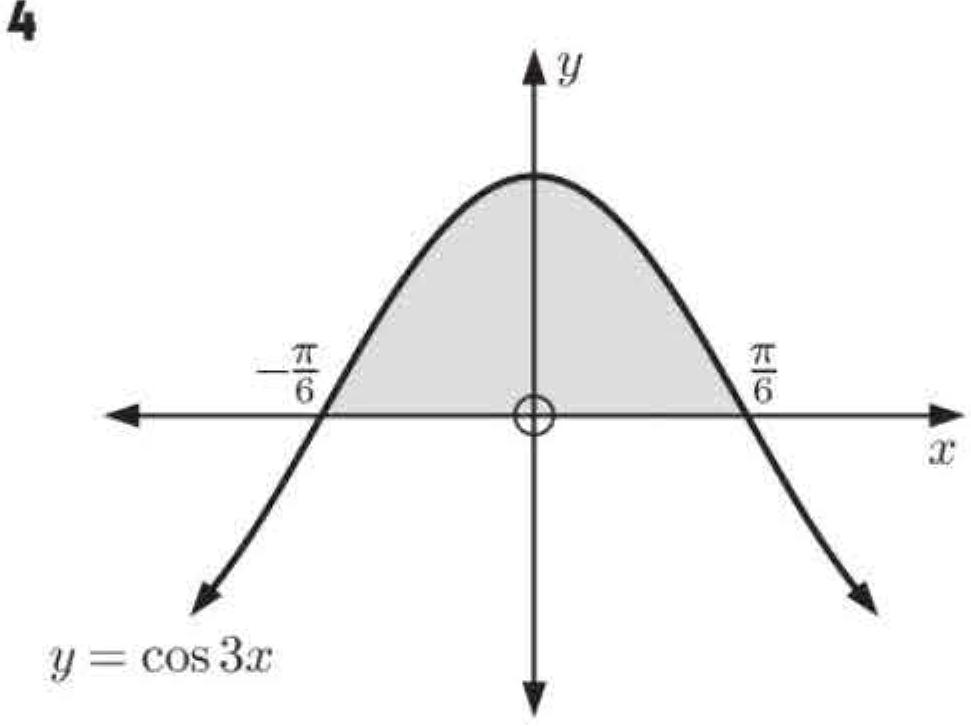
$= \frac{2}{3}(9)^{\frac{3}{2}} - \frac{2}{3}(4)^{\frac{3}{2}}$

$= \frac{2}{3}(27-8)$

$= \frac{2}{3} \times 19$

$= 12\frac{2}{3}$ units^2

4



$y = \cos 3x$ has zeros at $\begin{cases} \frac{\pi}{6} \\ -\frac{\pi}{6} \end{cases} + \frac{2k\pi}{3}$, k an integer

$\therefore$ area $= \displaystyle\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \cos 3x\,dx$

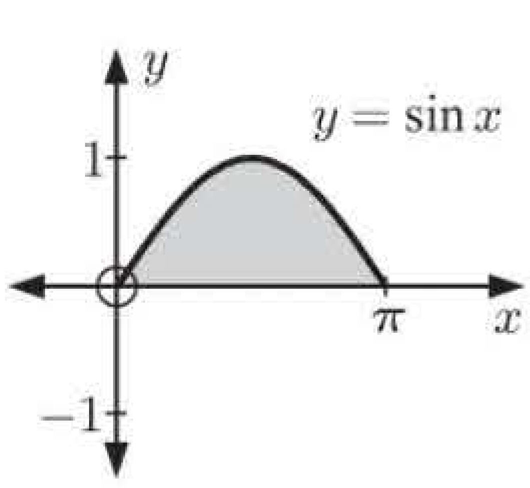
$= \left[\frac{1}{3}\sin 3x\right]_{-\frac{\pi}{6}}^{\frac{\pi}{6}}$

$= \frac{1}{3}\left(\sin(\frac{\pi}{2}) - \sin(-\frac{\pi}{2})\right)$

$= \frac{1}{3}(1 - (-1))$

$= \frac{2}{3}$ units^2

5 **a**



Area $= \int_0^\pi \sin x\,dx$

$= [-\cos x]_0^\pi$

$= [-\cos\pi + \cos 0]$

$= -(-1) + 1$

$= 2$ units^2

b Since $\sin^2 x \geqslant 0$ always, the function never drops below the x-axis.

$\therefore$ area $= \int_0^\pi \sin^2 x\,dx$

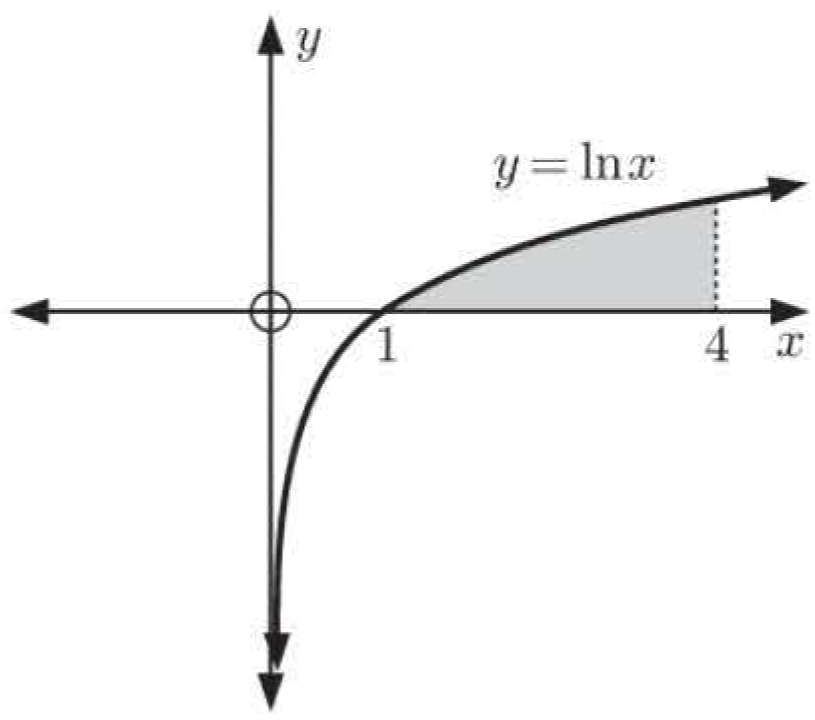
$= \int_0^\pi (\frac{1}{2} - \frac{1}{2}\cos(2x))\,dx$

$= \left[\dfrac{x}{2} - \frac{1}{4}\sin(2x)\right]_0^\pi$

$= \left[\frac{\pi}{2} - \frac{1}{4}\sin(2\pi)\right] - \left[0 - \frac{1}{4}\sin 0\right]$

$= \frac{\pi}{2}$ units^2

6



$$\begin{aligned}\text{Area} &= \int_1^4 \ln x \, dx \\ &= [x \ln x - x]_1^4 \\ &= 4\ln 4 - 4 - \ln 1 + 1 \\ &= (4\ln 4 - 3) \text{ units}^2\end{aligned}$$

7 **a**

$$\begin{aligned} u &= x \qquad & v' &= \sin x \\ \therefore\ u' &= 1 & v &= -\cos x \end{aligned}$$

$$\begin{aligned}\therefore\ \int x \sin x \, dx &= -x\cos x - \int -\cos x \, dx \\ &= -x \cos x + \sin x + c\end{aligned}$$

b

$$\begin{aligned}\text{Area} &= \int_1^{\frac{\pi}{2}} x \sin x \, dx \\ &= \Big[-x\cos x + \sin x\Big]_1^{\frac{\pi}{2}} \\ &= -\tfrac{\pi}{2}\cos(\tfrac{\pi}{2}) + \sin(\tfrac{\pi}{2}) + \cos 1 - \sin 1 \\ &= (1 + \cos 1 - \sin 1) \text{ units}^2\end{aligned}$$

EXERCISE 22B

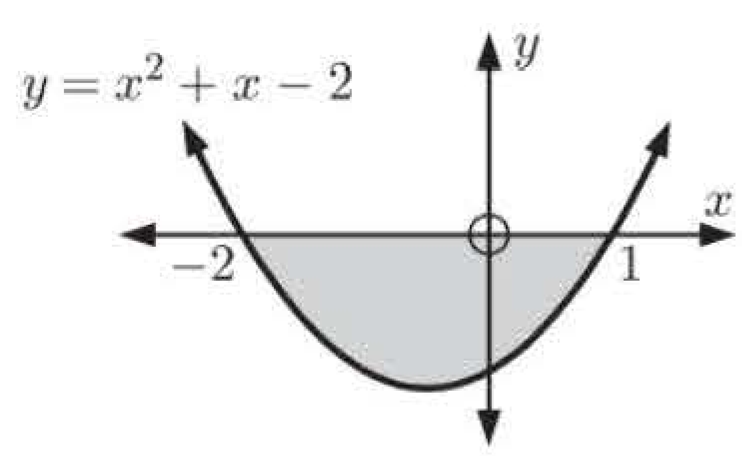
1 **a** The curve cuts the x-axis when $y = 0$.

$\therefore\ x^2 + x - 2 = 0$

$\therefore\ (x+2)(x-1) = 0$

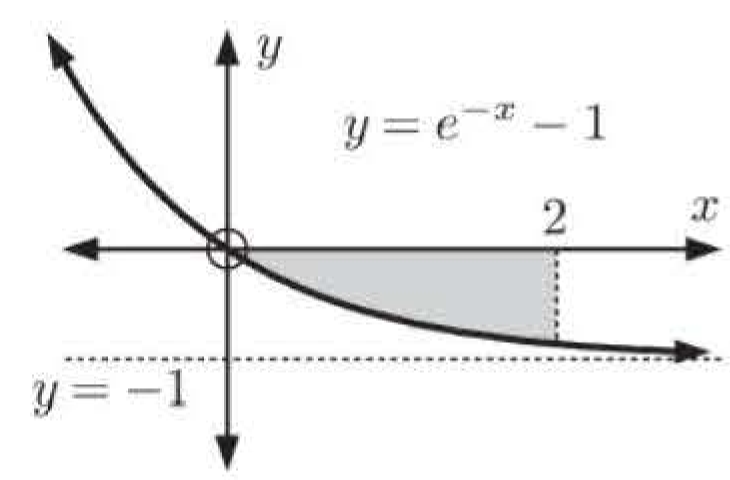
$\therefore\ x = -2$ or 1

$\therefore$ the x-intercepts are -2 and 1



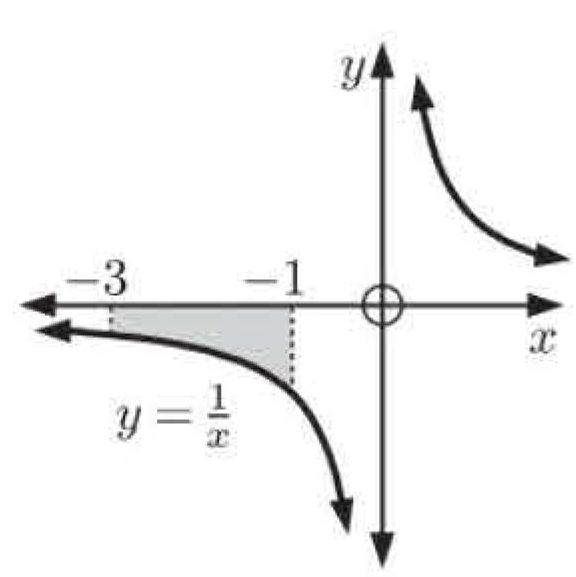
$$\begin{aligned}\text{Area} &= \int_{-2}^{1} [0 - (x^2 + x - 2)] \, dx \\ &= \int_{-2}^{1} (-x^2 - x + 2) \, dx \\ &= \left[-\frac{x^3}{3} - \frac{x^2}{2} + 2x\right]_{-2}^{1} \\ &= (-\tfrac{1}{3} - \tfrac{1}{2} + 2) - (\tfrac{8}{3} - 2 - 4) \\ &= 4\tfrac{1}{2} \text{ units}^2\end{aligned}$$

b The curve cuts the x-axis at $(0, 0)$.



$$\begin{aligned}\text{Area} &= \int_0^2 [0 - (e^{-x} - 1)] \, dx \\ &= \int_0^2 (1 - e^{-x}) \, dx \\ &= \left[x + e^{-x}\right]_0^2 \\ &= \left(2 + e^{-2}\right) - \left(0 + e^0\right) \\ &= (1 + e^{-2}) \text{ units}^2\end{aligned}$$

c



Area

$$\begin{aligned} &= \int_{-3}^{-1} \left(0 - \frac{1}{x}\right) dx \\ &= -\int_{-3}^{-1} \left(\frac{1}{x}\right) dx \\ &= -\,[\ln |x|]_{-3}^{-1} \\ &= -(\ln 1 - \ln 3) \\ &= \ln 3 \text{ units}^2\end{aligned}$$

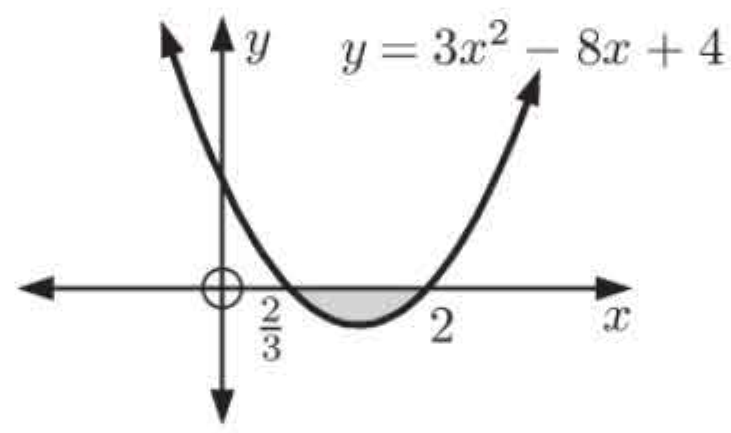
d The curve cuts the x-axis when $y = 0$.

$\therefore\ 3x^2 - 8x + 4 = 0$

$\therefore\ (3x-2)(x-2) = 0$

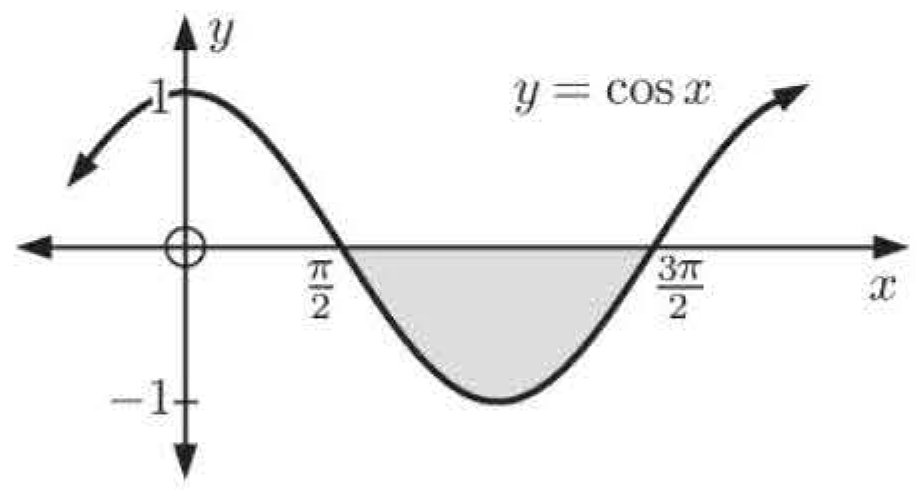
$\therefore\ x = 2$ or $\frac{2}{3}$

$\therefore$ the x-intercepts are 2 and $\frac{2}{3}$.



$$\text{Area} = \int_{\frac{2}{3}}^{2} [0 - (3x^2 - 8x + 4)]\, dx$$
$$= \int_{\frac{2}{3}}^{2} (-3x^2 + 8x - 4)\, dx$$
$$= \Big[-x^3 + 4x^2 - 4x\Big]_{\frac{2}{3}}^{2}$$
$$= (-8 + 16 - 8) - (-\tfrac{8}{27} + \tfrac{16}{9} - \tfrac{8}{3})$$
$$= 1\tfrac{5}{27} \text{ units}^2$$

e



$$\text{Area} = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} [0 - \cos x]\, dx$$
$$= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} -\cos x\, dx$$
$$= \Big[-\sin x\Big]_{\frac{\pi}{2}}^{\frac{3\pi}{2}}$$
$$= -\sin(\tfrac{3\pi}{2}) - \big(-\sin(\tfrac{\pi}{2})\big)$$
$$= -(-1) - (-1)$$
$$= 2 \text{ units}^2$$

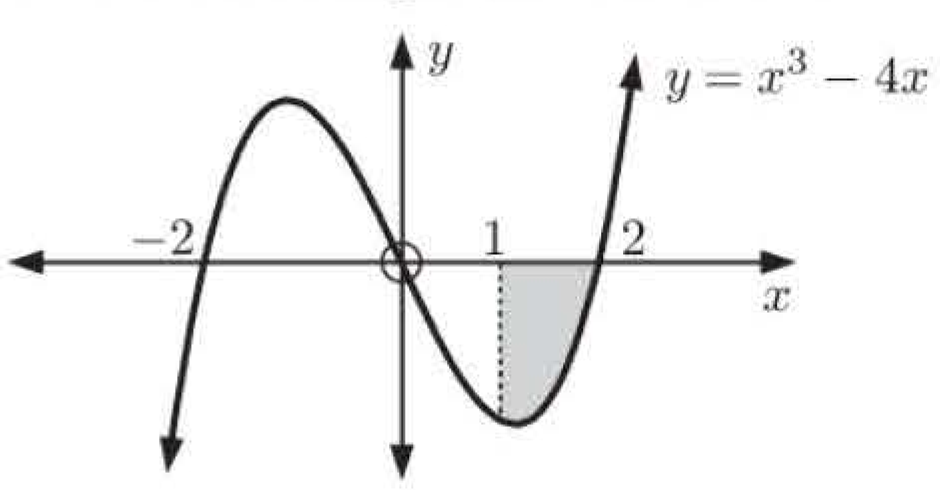
f The curve cuts the x-axis when $y = 0$.

$\therefore\ x^3 - 4x = 0$

$\therefore\ x(x^2 - 4) = 0$

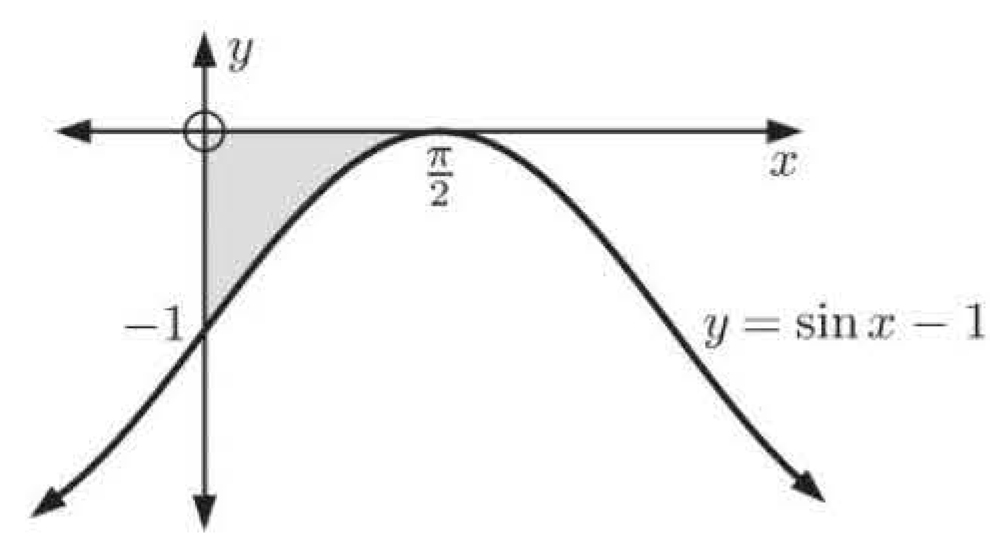
$\therefore\ x(x+2)(x-2) = 0$

$\therefore$ the x-intercepts are 0 and ± 2.



$$\text{Area} = \int_1^2 [0 - (x^3 - 4x)]\, dx$$
$$= \int_1^2 (-x^3 + 4x)\, dx$$
$$= \left[-\frac{x^4}{4} + 2x^2\right]_1^2$$
$$= (-4 + 8) - (-\tfrac{1}{4} + 2)$$
$$= 2\tfrac{1}{4} \text{ units}^2$$

g $y = \sin x - 1$ is the graph of $\sin x$ translated vertically -1 unit downwards.



$$\text{Area} = \int_0^{\frac{\pi}{2}} [0 - (\sin x - 1)]\, dx$$
$$= \int_0^{\frac{\pi}{2}} (1 - \sin x)\, dx$$
$$= \Big[x + \cos x\Big]_0^{\frac{\pi}{2}}$$
$$= (\tfrac{\pi}{2} + \cos\tfrac{\pi}{2}) - (0 + \cos 0)$$
$$= (\tfrac{\pi}{2} - 1) \text{ units}^2$$

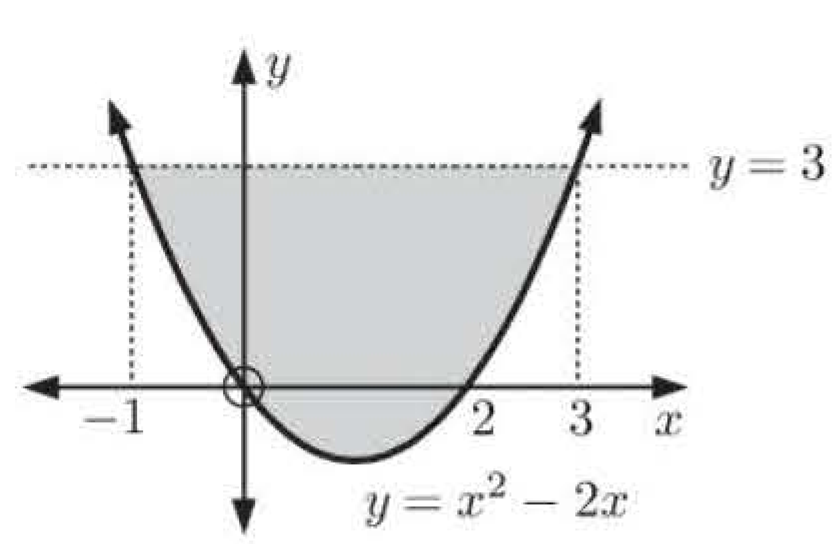
2 $y = x^2 - 2x$ meets $y = 3$

when $x^2 - 2x = 3$

$\therefore\ x^2 - 2x - 3 = 0$

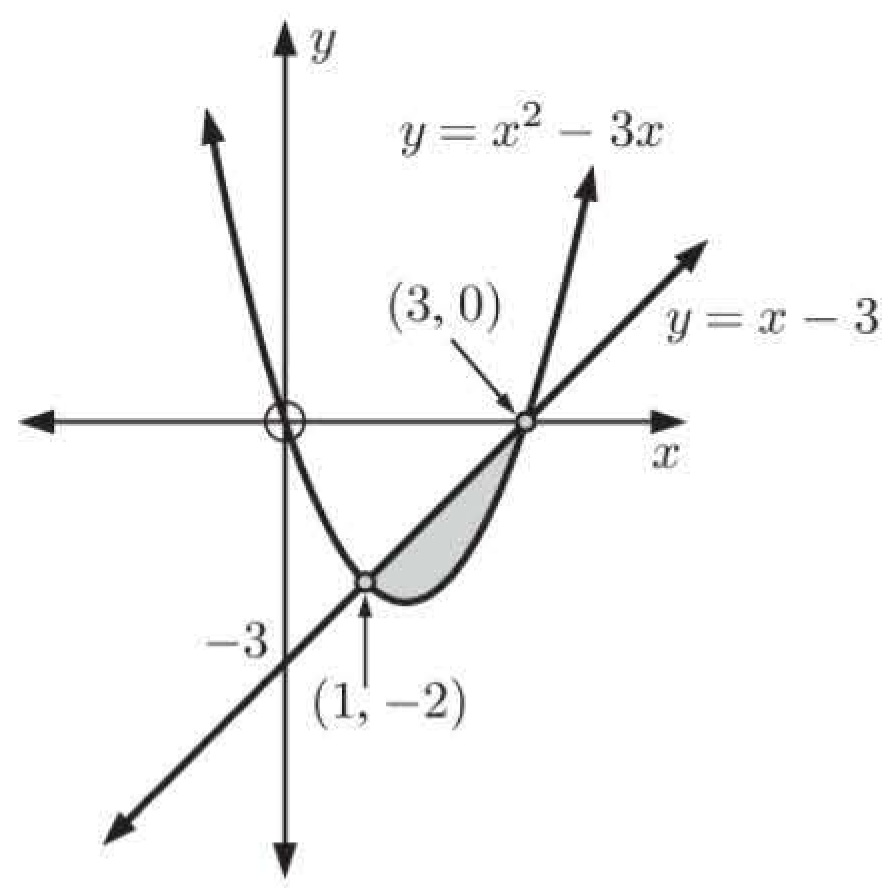
$\therefore\ (x-3)(x+1) = 0$

$\therefore\ x = 3$ or -1



$$A = \int_{-1}^{3} [3 - (x^2 - 2x)]\, dx$$
$$= \int_{-1}^{3} (3 + 2x - x^2)\, dx$$
$$= \left[3x + x^2 - \frac{x^3}{3}\right]_{-1}^{3}$$
$$= (9 + 9 - 9) - (-3 + 1 + \tfrac{1}{3})$$
$$= 10\tfrac{2}{3} \text{ units}^2$$

3 **a**



b The graphs meet where $x - 3 = x^2 - 3x$

$$\therefore \quad x^2 - 3x - x + 3 = 0$$
$$\therefore \quad x^2 - 4x + 3 = 0$$
$$\therefore \quad (x-1)(x-3) = 0$$
$$\therefore \quad x = 1 \text{ or } 3$$

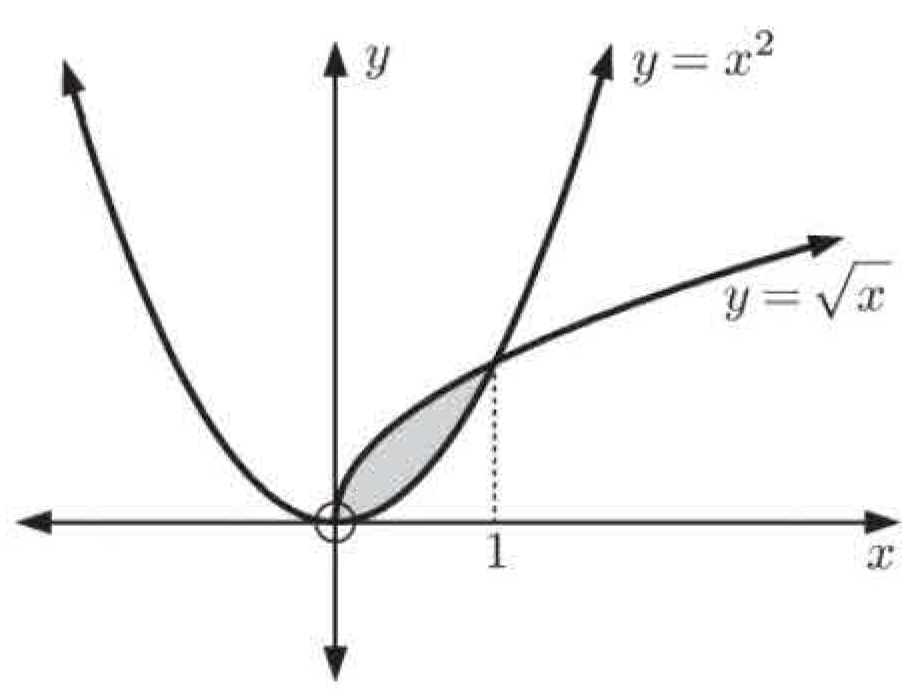
$\therefore$ the graphs meet at $(1, -2)$ and $(3, 0)$.

c $$\text{Area} = \int_1^3 [(x-3) - (x^2 - 3x)]\, dx$$
$$= \int_1^3 (-3 + 4x - x^2)\, dx$$
$$= \left[-3x + 2x^2 - \frac{x^3}{3}\right]_1^3$$
$$= (-9 + 18 - 9) - (-3 + 2 - \tfrac{1}{3})$$
$$= 1\tfrac{1}{3} \text{ units}^2$$

4 $y = \sqrt{x}$ meets $y = x^2$ where $\sqrt{x} = x^2$

$$\therefore \quad x = x^4$$
$$\therefore \quad x^4 - x = 0$$
$$\therefore \quad x(x^3 - 1) = 0$$
$$\therefore \quad x(x-1)(x^2 + x + 1) = 0$$
$$\therefore \quad x = 0 \text{ or } 1$$

The factor $(x^2 + x + 1)$ has no real root since $\Delta = -3$ which is < 0.



$$\text{Area} = \int_0^1 \left(\sqrt{x} - x^2\right)\, dx$$
$$= \int_0^1 (x^{\frac{1}{2}} - x^2)\, dx$$
$$= \left[\tfrac{2}{3}x^{\frac{3}{2}} - \frac{x^3}{3}\right]_0^1$$
$$= \tfrac{2}{3} - \tfrac{1}{3}$$
$$= \tfrac{1}{3} \text{ unit}^2$$

5 **a** $y = e^x - 1$ has no vertical asymptotes.

As $x \to \infty$, $e^x - 1 \to \infty$

As $x \to -\infty$, $e^x \to 0$

so $e^x - 1 \to -1^+$

$\therefore$ $y = -1$ is a horizontal asymptote.

$y = 0$ when $e^x - 1 = 0$

$$\therefore \quad e^x = 1$$
$$\therefore \quad x = 0$$

$\therefore$ x-intercept is $(0, 0)$.

This is also the y-intercept.

$y = 2 - 2e^{-x}$ has no vertical asymptotes.

As $x \to \infty$, $e^{-x} \to 0$

so $2 - 2e^{-x} \to 2^-$

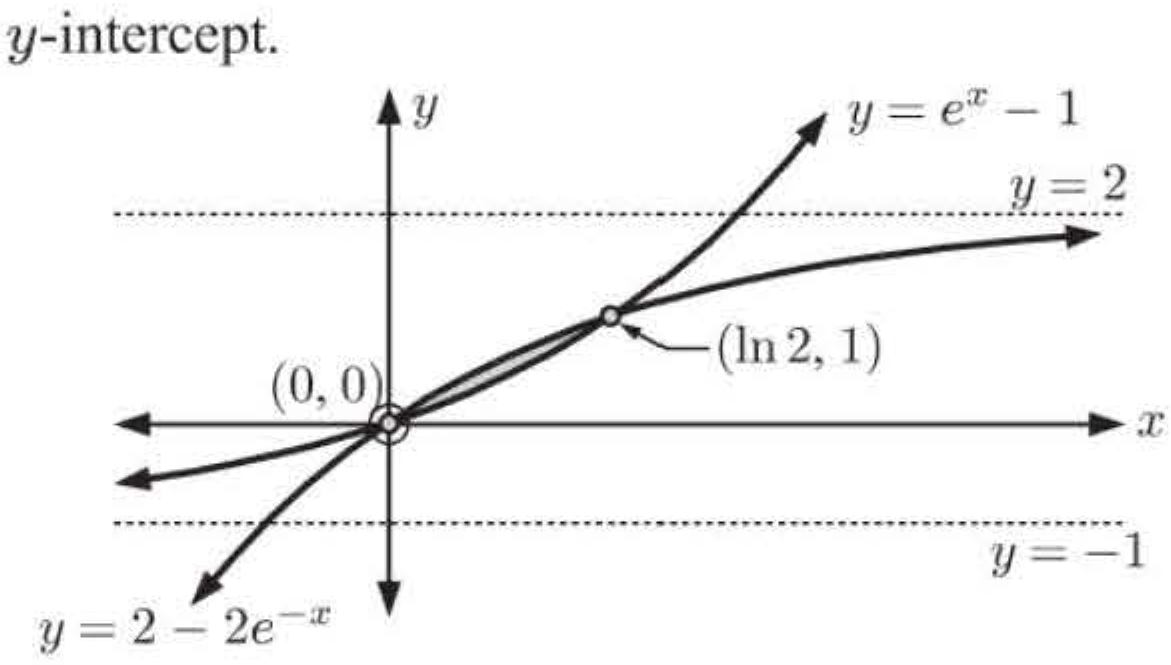
$\therefore$ $y = 2$ is a horizontal asymptote.

$y = 0$ when $2 - 2e^{-x} = 0$

$$\therefore \quad e^{-x} = 1$$
$$\therefore \quad x = 0$$

$\therefore$ x-intercept is $(0, 0)$.

This is also the y-intercept.



b $y = e^x - 1$ meets $y = 2 - 2e^{-x}$

where $e^x - 1 = 2 - 2e^{-x}$

$\therefore \quad e^{2x} - e^x = 2e^x - 2 \quad \{\times\ e^x\}$

$\therefore \quad e^{2x} - 3e^x + 2 = 0$

$\therefore \quad (e^x - 1)(e^x - 2) = 0$

$\therefore \quad e^x = 1 \text{ or } 2$

$\therefore \quad x = 0 \text{ or } \ln 2$

$\therefore$ the graphs meet at $(0, 0)$ and $(\ln 2, 1)$.

c $\text{Area} = \int_0^{\ln 2} [(2 - 2e^{-x}) - (e^x - 1)]\, dx$

$= \int_0^{\ln 2} (3 - e^x - 2e^{-x})\, dx$

$= \left[3x - e^x + 2e^{-x}\right]_0^{\ln 2}$

$= (3\ln 2 - 2 + 1) - (0 - 1 + 2)$

$= 3\ln 2 - 2$

$\approx 0.0794 \text{ units}^2$

6 $y = 2e^x$ meets $y = e^{2x}$ where

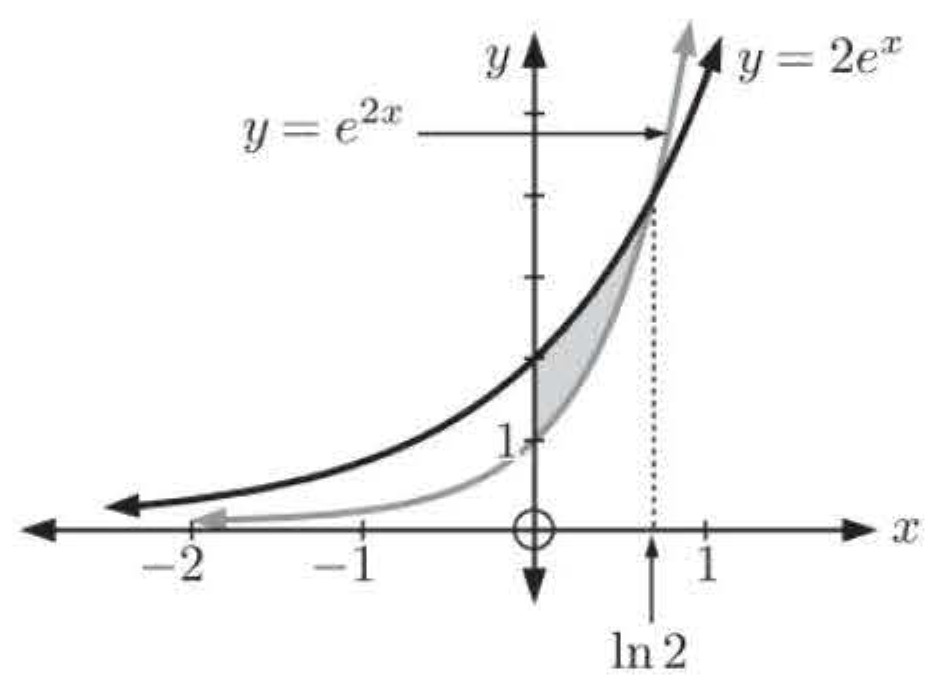
$2e^x = e^{2x}$

$\therefore \quad e^{2x} - 2e^x = 0$

$\therefore \quad e^x(e^x - 2) = 0$

$\therefore \quad e^x = 2 \qquad \{e^x > 0 \text{ for all } x\}$

$\therefore \quad x = \ln 2$



$\text{Area} = \int_0^{\ln 2} (2e^x - e^{2x})\, dx$

$= \left[2e^x - \frac{1}{2}e^{2x}\right]_0^{\ln 2}$

$= (4 - 2) - (2 - \frac{1}{2})$

$= \frac{1}{2} \text{ unit}^2$

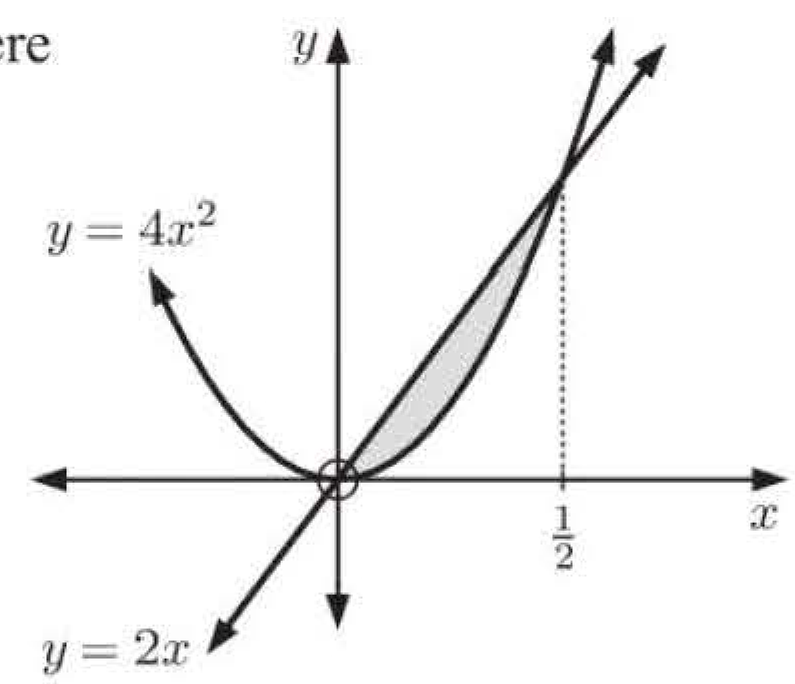
7 **a** $y = 2x$ meets $y = 4x^2$ where

$2x = 4x^2$

$\therefore \quad 4x^2 - 2x = 0$

$\therefore \quad 2x(2x - 1) = 0$

$\therefore \quad x = 0 \text{ or } \frac{1}{2}$



Area

$= \int_0^{\frac{1}{2}} (2x - 4x^2)\, dx$

$= \left[x^2 - \frac{4}{3}x^3\right]_0^{\frac{1}{2}}$

$= \left(\frac{1}{4} - \frac{4}{3}\left(\frac{1}{8}\right)\right) - (0 - 0)$

$= \frac{1}{12} \text{ unit}^2$

b $y = 2x$ meets $y^2 = 4x$ where

$(2x)^2 = 4x$

$\therefore \quad 4x^2 = 4x$

$\therefore \quad 4x^2 - 4x = 0$

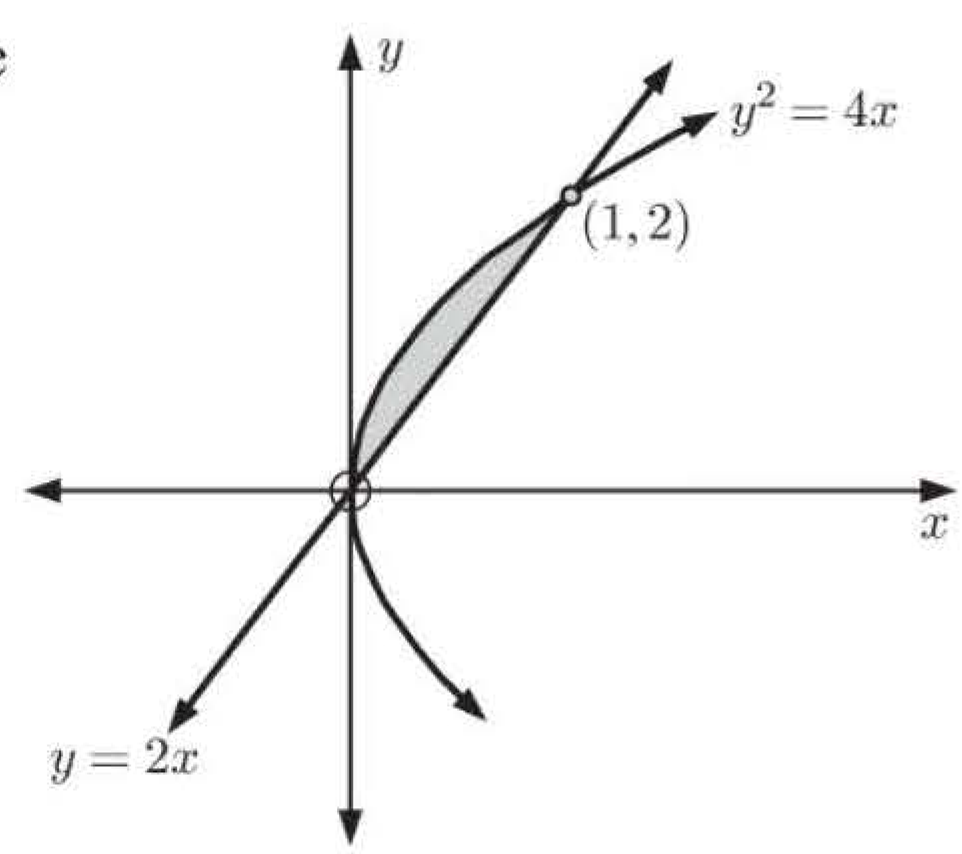
$\therefore \quad 4x(x - 1) = 0$

$\therefore \quad x = 0 \text{ or } 1$

The upper part of $y^2 = 4x$

is $y = \sqrt{4x}$

or $y = 2\sqrt{x}$



Area

$= \int_0^1 (2\sqrt{x} - 2x)\, dx$

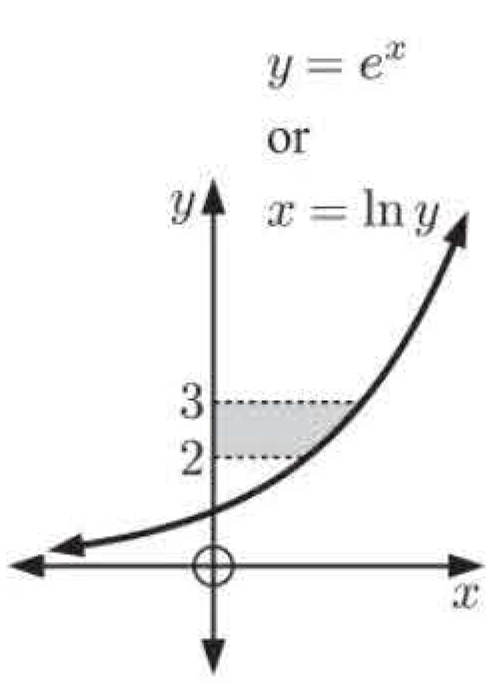
$= \int_0^1 (2x^{\frac{1}{2}} - 2x)\, dx$

$= \left[\frac{4}{3}x^{\frac{3}{2}} - x^2\right]_0^1$

$= \frac{4}{3} - 1$

$= \frac{1}{3} \text{ unit}^2$

8



$\text{Area} = \int_2^3 (x - 0)\ dy$

$= \int_2^3 \ln y\, dy$

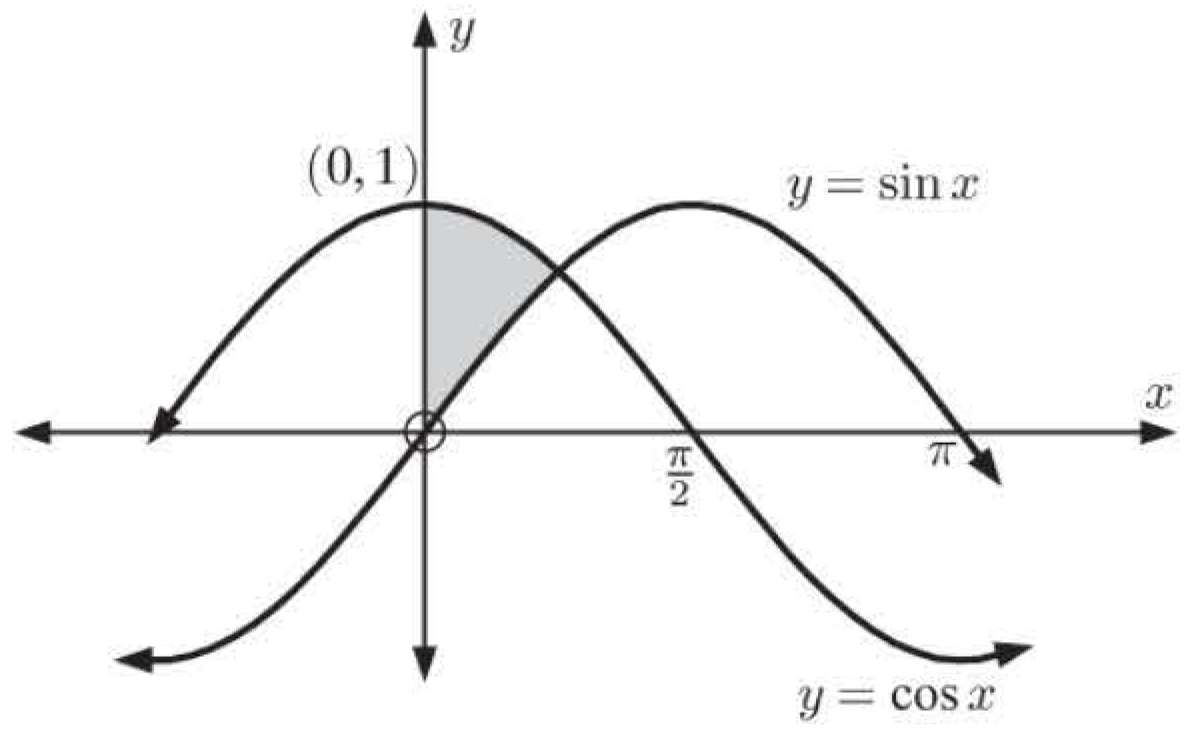
$= [y\ln y - y]_2^3$

$= (3\ln 3 - 3) - (2\ln 2 - 2)$

$= \ln 27 - \ln 4 - 1$

$= \left(\ln\left(\frac{27}{4}\right) - 1\right) \text{ units}^2$

9



The curves $y = \cos x$ and $y = \sin x$ meet when $x = \frac{\pi}{4}$.

$$\therefore \quad A = \int_0^{\frac{\pi}{4}} (\cos x - \sin x)\, dx$$
$$= [\sin x + \cos x]_0^{\frac{\pi}{4}}$$
$$= (\sin \tfrac{\pi}{4} + \cos \tfrac{\pi}{4}) - (\sin 0 + \cos 0)$$
$$= \left(\tfrac{1}{\sqrt{2}} + \tfrac{1}{\sqrt{2}}\right) - (0+1)$$
$$= (\sqrt{2} - 1) \text{ units}^2$$

10 **a** Point A has y-coordinate 1 and lies on the graph of $y = \tan x$ on the interval $[0, \frac{\pi}{2}[$.

At this point, $\tan x = 1$ $\quad \therefore \; x = \frac{\pi}{4}$ $\quad \therefore$ A is at $(\frac{\pi}{4}, 1)$.

b Consider $\tan x = \dfrac{\sin x}{\cos x}$

Let $u = \cos x$, $\dfrac{du}{dx} = -\sin x$

When $x = 0$, $u = \cos 0 = 1$

When $x = \frac{\pi}{4}$, $u = \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}$

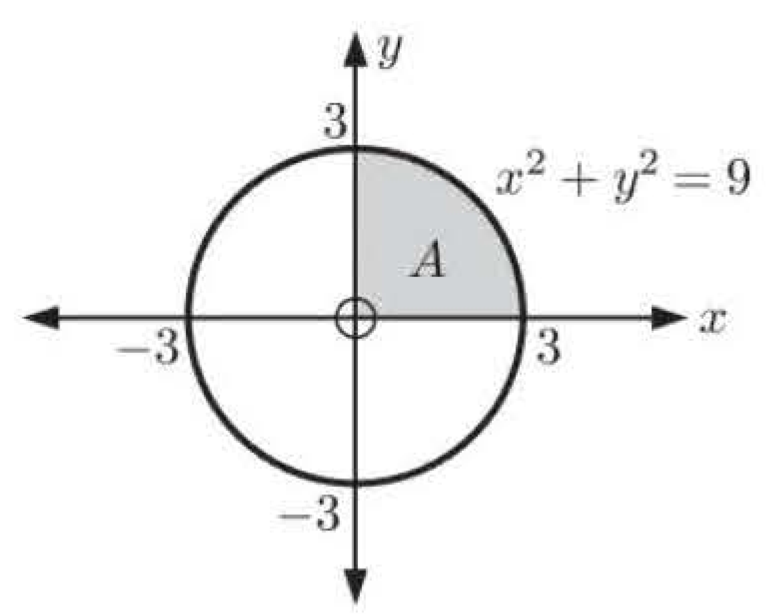
$$\therefore \text{ area} = \int_0^{\frac{\pi}{4}} \tan x \, dx$$
$$= -\int_0^{\frac{\pi}{4}} \frac{-\sin x}{\cos x}\, dx$$
$$= -\int_0^{\frac{\pi}{4}} \frac{1}{u}\frac{du}{dx}\, dx$$
$$= -\int_1^{\frac{1}{\sqrt{2}}} \frac{1}{u}\, du$$
$$= \int_{\frac{1}{\sqrt{2}}}^1 \frac{1}{u}\, du$$
$$= [\ln|u|]_{\frac{1}{\sqrt{2}}}^1$$
$$= \ln 1 - \ln \tfrac{1}{\sqrt{2}}$$
$$= \ln \sqrt{2} \text{ units}^2$$

11 **a** Now $x^2 + y^2 = 9 \quad \therefore \; y^2 = 9 - x^2$

$\therefore \; y = \pm\sqrt{9 - x^2}$

In the upper half of the circle all y-values are $\geqslant 0$

$\therefore \; y = +\sqrt{9 - x^2}$ is the required equation.



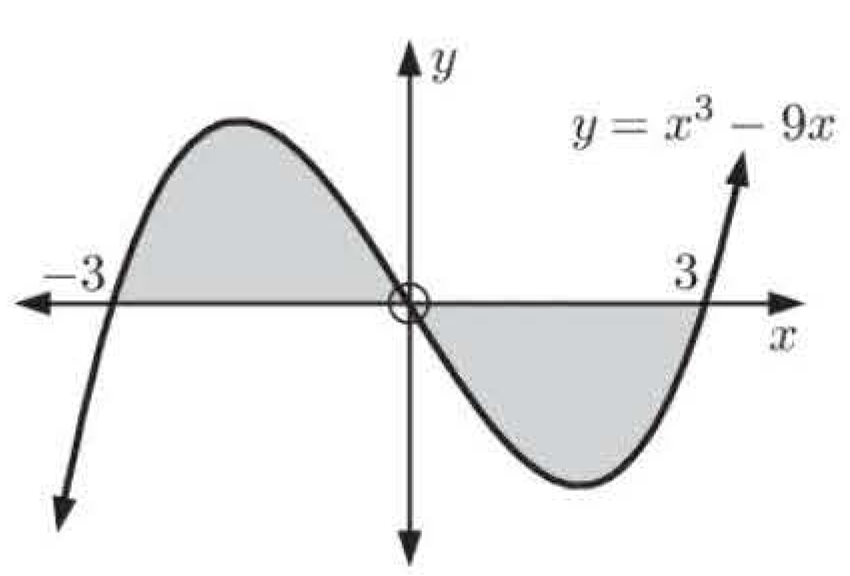
b The shaded area is A where $A = \int_0^3 \sqrt{9 - x^2}\, dx$

This is a quarter of the area of a circle with radius 3 units.

$\therefore \; A = \frac{1}{4}(\pi \times 3^2) = \frac{9\pi}{4} \approx 7.07 \text{ units}^2$

12 **a**

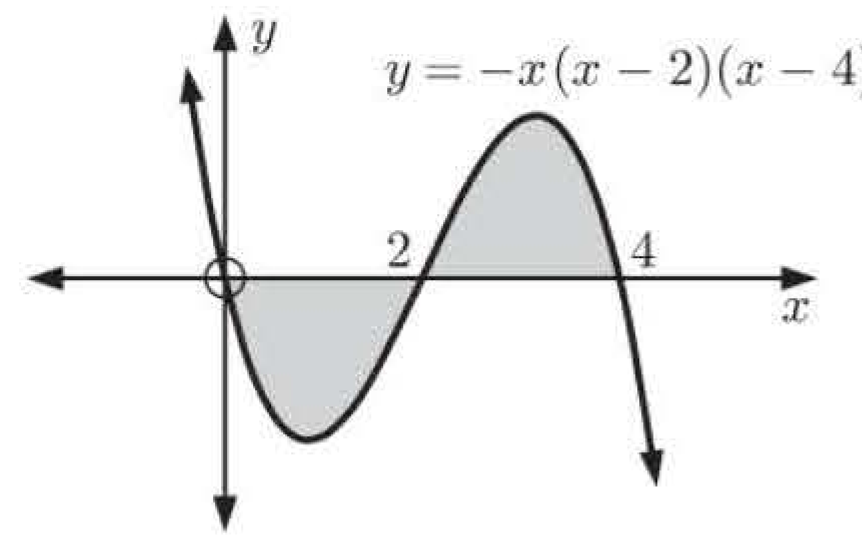
$$f(x) = x^3 - 9x$$
$$= x(x^2 - 9)$$
$$= x(x+3)(x-3)$$



$\therefore \; y = f(x)$ cuts the x-axis at $0, \pm 3$

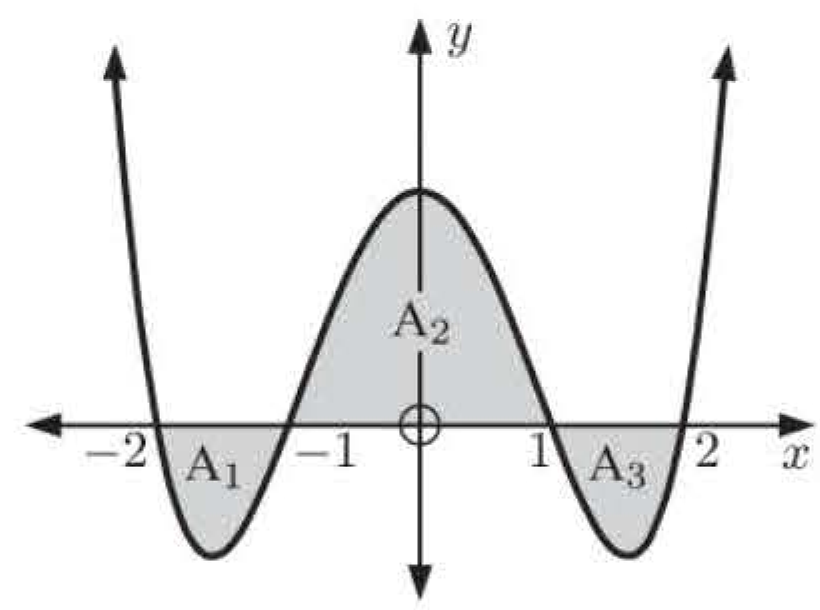
$$\text{Area} = \int_{-3}^0 (x^3 - 9x)\, dx + \int_0^3 [0 - (x^3 - 9x)]\, dx$$
$$= \left[\frac{x^4}{4} - \frac{9x^2}{2}\right]_{-3}^0 + \left[-\frac{x^4}{4} + \frac{9x^2}{2}\right]_0^3$$
$$= 0 - (\tfrac{81}{4} - \tfrac{81}{2}) + (-\tfrac{81}{4} + \tfrac{81}{2}) - 0$$
$$= 40\tfrac{1}{2} \text{ units}^2$$

b $f(x) = -x(x-2)(x-4)$
$= -x^3 + 6x^2 - 8x$
$\therefore \quad y = f(x)$ cuts the x-axis at 0, 2, and 4



$$\begin{aligned}\text{Area} &= \int_0^2 [0 - (-x^3 + 6x^2 - 8x)]\,dx \\ &\qquad + \int_2^4 (-x^3 + 6x^2 - 8x)\,dx \\ &= \int_0^2 (x^3 - 6x^2 + 8x)\,dx + \int_2^4 (-x^3 + 6x^2 - 8x)\,dx \\ &= \left[\frac{x^4}{4} - 2x^3 + 4x^2\right]_0^2 + \left[-\frac{x^4}{4} + 2x^3 - 4x^2\right]_2^4 \\ &= ([4 - 16 + 16] - 0) + ([-64 + 128 - 64] - [-4 + 16 - 16]) \\ &= 8 \text{ units}^2\end{aligned}$$

c $f(x) = x^4 - 5x^2 + 4$
$= (x^2 - 1)(x^2 - 4)$
$= (x+1)(x-1)(x+2)(x-2)$
$\therefore \quad y = f(x)$ cuts the x-axis at $\pm 1, \pm 2$



$$\begin{aligned}A_1 &= \int_{-2}^{-1} [0 - (x^4 - 5x^2 + 4)]\,dx \\ &= \int_{-2}^{-1} (-x^4 + 5x^2 - 4)\,dx \\ &= \left[-\frac{x^5}{5} + \frac{5x^3}{3} - 4x\right]_{-2}^{-1} \\ &= (\tfrac{1}{5} - \tfrac{5}{3} + 4) - (\tfrac{32}{5} - \tfrac{40}{3} + 8) \\ &= \tfrac{22}{15} \text{ units}^2\end{aligned}$$

$$\begin{aligned}A_2 &= \int_{-1}^{1} (x^4 - 5x^2 + 4)\,dx \\ &= \left[\frac{x^5}{5} - \frac{5x^3}{3} + 4x\right]_{-1}^{1} \\ &= (\tfrac{1}{5} - \tfrac{5}{3} + 4) - (-\tfrac{1}{5} + \tfrac{5}{3} - 4) \\ &= \tfrac{76}{15} \text{ units}^2\end{aligned}$$

By symmetry, $A_3 = A_1$
$\therefore$ area $= \frac{22}{15} + \frac{76}{15} + \frac{22}{15} = \frac{120}{15} = 8$ units^2

13 **a** $y = \sin(2x)$ is the curve C_1 and $y = \sin x$ is the curve C_2.

b The curves meet when $\sin(2x) = \sin x$
$\therefore \quad 2\sin x\cos x - \sin x = 0$
$\therefore \quad \sin x(2\cos x - 1) = 0$
$\therefore \quad \sin x = 0$ or $\cos x = \frac{1}{2}$

$\therefore \quad x = 0 + k\pi$ or $x = \begin{cases}\frac{\pi}{3} \\ \frac{5\pi}{3}\end{cases} + 2k\pi$, k an integer
$\therefore$ the x-coordinate of $A = \frac{\pi}{3}$
{smallest positive solution}
$\therefore$ A is at $(\frac{\pi}{3}, \frac{\sqrt{3}}{2})$

c
$$\begin{aligned}\text{Area} &= \int_0^{\frac{\pi}{3}} (\sin(2x) - \sin x)\,dx + \int_{\frac{\pi}{3}}^{\pi} (\sin x - \sin(2x))\,dx \\ &= \left[-\tfrac{1}{2}\cos(2x) + \cos x\right]_0^{\frac{\pi}{3}} + \left[-\cos x + \tfrac{1}{2}\cos(2x)\right]_{\frac{\pi}{3}}^{\pi} \\ &= \left(-\tfrac{1}{2}\cos\tfrac{2\pi}{3} + \cos\tfrac{\pi}{3}\right) - \left(-\tfrac{1}{2}\cos 0 + \cos 0\right) + \left(-\cos\pi + \tfrac{1}{2}\cos 2\pi\right) \\ &\qquad - \left(-\cos\tfrac{\pi}{3} + \tfrac{1}{2}\cos\tfrac{2\pi}{3}\right) \\ &= (\tfrac{1}{4} + \tfrac{1}{2}) - (-\tfrac{1}{2} + 1) + (1 + \tfrac{1}{2}) - (-\tfrac{1}{2} - \tfrac{1}{4}) \\ &= 2\tfrac{1}{2} \text{ units}^2\end{aligned}$$

14 **a** **i** The graphs meet where $x^3 - 4x = 3x + 6$

$$\therefore \quad x^3 - 7x - 6 = 0$$
$$\therefore \quad (x+2)(x^2 - 2x - 3) = 0 \quad \{\text{diagram shows intersection at } -2\}$$
$$\therefore \quad (x+2)(x+1)(x-3) = 0$$
$$\therefore \quad x = -2, -1 \text{ or } 3$$

$$\therefore \quad \text{area} = \int_{-2}^{-1} \left([x^3 - 4x] - [3x+6]\right)\, dx + \int_{-1}^{3} \left([3x+6] - [x^3 - 4x]\right)\, dx$$
$$= \int_{-2}^{-1} \left(x^3 - 7x - 6\right)\, dx + \int_{-1}^{3} \left(-x^3 + 7x + 6\right)\, dx$$

ii Area $= \int_{-2}^{3} \left|x^3 - 7x - 6\right|\, dx$

b Using technology, area $= 32\frac{3}{4}$ units^2

15 **a** The graphs meet where

$$x^3 - 5x = 2x^2 - 6$$
$$\therefore \quad x^3 - 2x^2 - 5x + 6 = 0$$
$$\therefore \quad (x-1)(x^2 - x - 6) = 0$$
$$\therefore \quad (x-1)(x-3)(x+2) = 0$$
$$\therefore \quad x = -2,\ 1, \text{ or } 3$$

So, area $= \int_{-2}^{3} \left|x^3 - 2x^2 - 5x + 6\right|\, dx$

$= 21\frac{1}{12}$ units^2 {technology}

b The graphs meet where

$$-x^3 + 3x^2 + 6x - 8 = 5x - 5$$
$$\therefore \quad x^3 - 3x^2 - x + 3 = 0$$
$$\therefore \quad (x-1)(x^2 - 2x - 3) = 0$$
$$\therefore \quad (x-1)(x-3)(x+1) = 0$$
$$\therefore \quad x = -1,\ 1, \text{ or } 3$$

So, area $= \int_{-1}^{3} \left|x^3 - 3x^2 - x + 3\right|\, dx$

$= 8$ units^2 {technology}

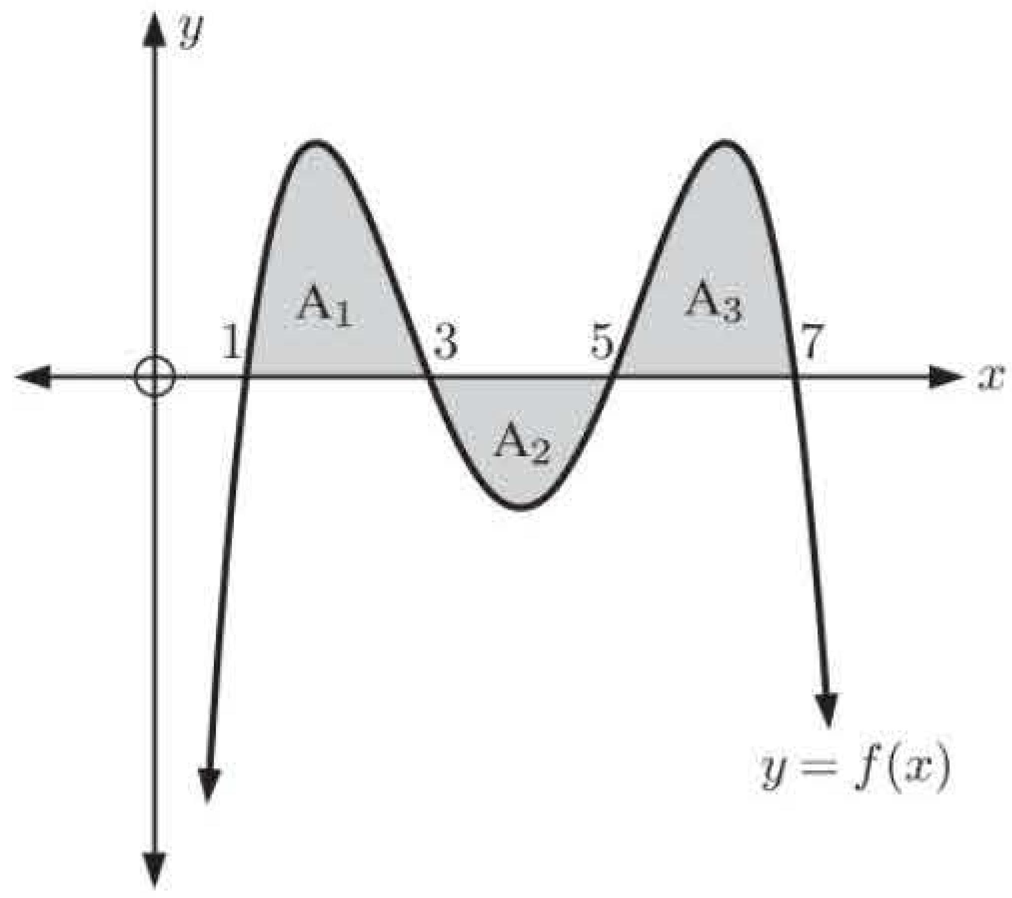
c The graphs meet where

$$2x^3 - 3x^2 + 18 = x^3 + 10x - 6$$
$$\therefore \quad x^3 - 3x^2 - 10x + 24 = 0$$
$$\therefore \quad (x-2)(x^2 - x - 12) = 0$$
$$\therefore \quad (x-2)(x-4)(x+3) = 0$$
$$\therefore \quad x = -3,\ 2, \text{ or } 4$$

So, area $= \int_{-3}^{4} \left|x^3 - 3x^2 - 10x + 24\right|\, dx$

$= 101\frac{3}{4}$ units^2 {technology}

16



a $\int_1^7 f(x)\, dx$ only gives us the correct area provided that $f(x)$ is positive on the interval $1 \leqslant x \leqslant 7$.
But $f(x)$ is not positive for $3 \leqslant x \leqslant 5$,
so $\int_1^7 f(x)\, dx = A_1 - A_2 + A_3$
which is *not* the shaded area.

b Shaded area

$$= \int_1^3 f(x)\, dx + \int_3^5 [0 - f(x)]\, dx + \int_5^7 f(x)\, dx$$
$$= \int_1^3 f(x)\, dx - \int_3^5 f(x)\, dx + \int_5^7 f(x)\, dx$$

17 **a** $y = \cos(2x)$ is the curve C_2 and $y = \cos^2 x$ is the curve C_1.

b Point A lies on $y = \cos(2x)$. When $x = 0$, $y = \cos 0 = 1$. $\therefore$ A is at $(0, 1)$.
Point B lies on $y = \cos(2x)$. When $x = \frac{\pi}{4}$, $y = \cos\frac{\pi}{2} = 0$. $\therefore$ B is at $(\frac{\pi}{4}, 0)$.
Point C lies on $y = \cos^2 x$. When $x = \frac{\pi}{2}$, $y = \cos^2\frac{\pi}{2} = 0$. $\therefore$ C is at $(\frac{\pi}{2}, 0)$.
Point D lies on $y = \cos(2x)$. When $x = \frac{3\pi}{4}$, $y = \cos\frac{3\pi}{2} = 0$. $\therefore$ D is at $(\frac{3\pi}{4}, 0)$.
Point E lies where the curves meet. Now $\cos(2\pi) = \cos^2\pi = 1$. $\therefore$ E is at $(\pi, 1)$.

c Area $= \int_0^{\pi} (\cos^2 x - \cos(2x))\, dx$

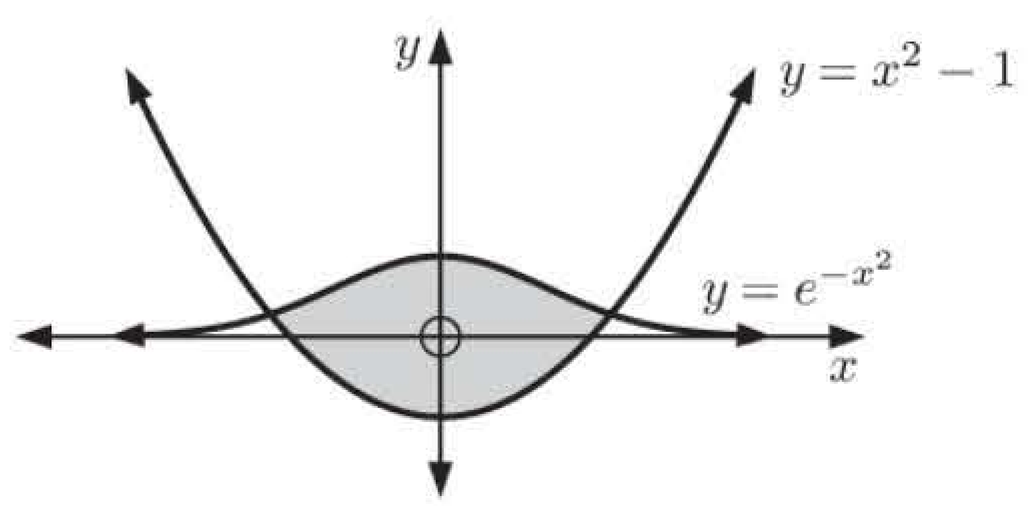
$= \int_0^{\pi} \left(\frac{1}{2} + \frac{1}{2}\cos(2x) - \cos(2x)\right) dx$

$= \int_0^{\pi} \left(\frac{1}{2} - \frac{1}{2}\cos(2x)\right) dx$

$= \left[\frac{x}{2} - \frac{1}{4}\sin(2x)\right]_0^{\pi} = (\frac{\pi}{2} - 0) - (0 - 0) = \frac{\pi}{2}$ units^2

18 **a** The graphs meet when $e^{-x^2} = x^2 - 1$

$\therefore\ x = \pm 1.1307$ {technology}

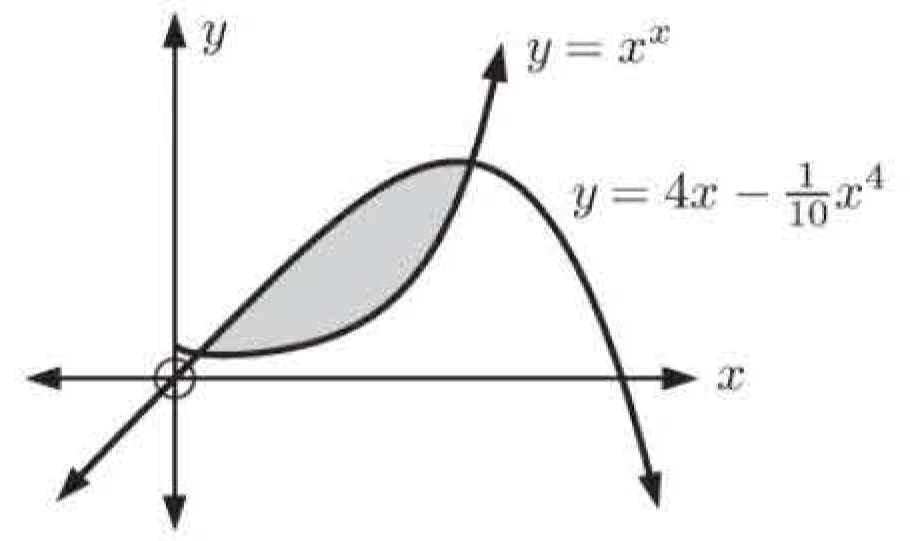


$\therefore$ area $= \int_{-1.1307}^{1.1307} [e^{-x^2} - (x^2 - 1)]\, dx$

≈ 2.88 units^2 {technology}

b The graphs meet when $x^x = 4x - \frac{1}{10}x^4$

$\therefore\ x \approx 0.1832$ or 2.2696 {technology}



$\therefore$ area $= \int_{0.1832}^{2.2696} (4x - \frac{1}{10}x^4 - x^x)\, dx$

≈ 4.97 units^2 {technology}

19 **a** Area $= \displaystyle\int_1^k \frac{1}{1+2x}\, dx = 0.2$ units^2

$\therefore\ \left[\frac{1}{2}\ln(1+2x)\right]_1^k = 0.2,\ 1 + 2x > 0$

$\therefore\ [\ln(1+2x)]_1^k = 0.4$

$\therefore\ \ln(1+2k) - \ln 3 = 0.4$

{since $k \geqslant 1$, $1 + 2x > 0$ for all x in the shaded region}

$\therefore\ \ln\left(\dfrac{1+2k}{3}\right) = 0.4$

$\therefore\ \dfrac{1+2k}{3} = e^{0.4}$

$\therefore\ 1 + 2k = 3e^{0.4}$

$\therefore\ k = \dfrac{3e^{0.4} - 1}{2} \approx 1.7377$

b Area $= \int_0^b \sqrt{x}\, dx$

$\therefore\ \int_0^b x^{\frac{1}{2}}\, dx = 1$

$\therefore\ \left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^b = 1$

$\therefore\ \frac{2}{3}b\sqrt{b} - 0 = 1$

$\therefore\ b\sqrt{b} = \frac{3}{2}$

$\therefore\ b^{\frac{3}{2}} = 1.5$

$\therefore\ b = (1.5)^{\frac{2}{3}}$

≈ 1.3104

20 **a** $y = x^2$ meets $y = k$ where $x^2 = k$

$\therefore\ x = \pm\sqrt{k}$

Now, the area $= \int_0^{\sqrt{k}} (k - x^2)\, dx$

$\therefore\ \int_0^{\sqrt{k}} (k - x^2)\, dx = 2.4$

$\therefore\ \left[kx - \dfrac{x^3}{3}\right]_0^{\sqrt{k}} = 2.4$

$\therefore\ k\sqrt{k} - \dfrac{k\sqrt{k}}{3} - 0 = 2.4$

$\therefore\ \dfrac{2k\sqrt{k}}{3} = 2.4$

$\therefore\ k^{\frac{3}{2}} = 3.6$

$\therefore\ k = (3.6)^{\frac{2}{3}}$

≈ 2.3489

b By symmetry, the area bounded by $x = 0$ and $x = a$ is $\frac{1}{2}(6a)$ units^2.

$\therefore\ \int_0^a (x^2 + 2)\, dx = 3a$

$\therefore\ \left[\dfrac{x^3}{3} + 2x\right]_0^a = 3a$

$\therefore\ \dfrac{a^3}{3} + 2a - 0 = 3a$

$\therefore\ a^3 + 6a = 9a$

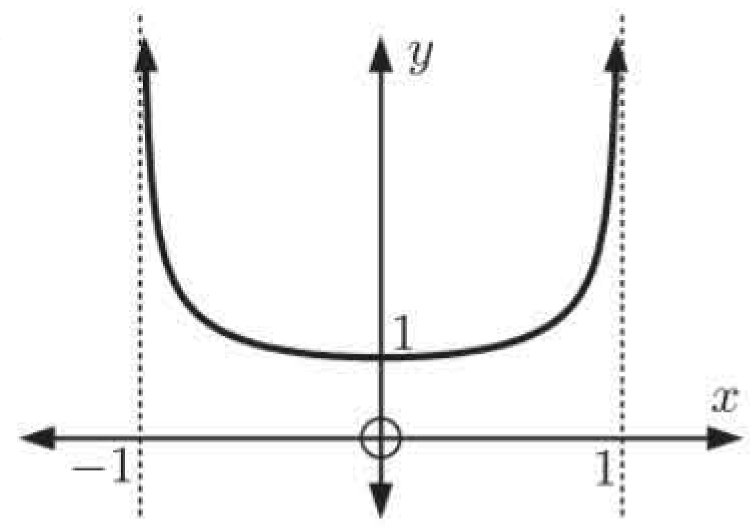
$\therefore\ a^3 - 3a = 0$

$\therefore\ a(a^2 - 3) = 0$

$\therefore\ a = 0$ or $\pm\sqrt{3}$

$\therefore\ a = \sqrt{3}$ {as $a > 0$}

21 **a**



b **i** If $f(x) = \dfrac{1}{\sqrt{1-x^2}}$ then $f(-x) = \dfrac{1}{\sqrt{1-(-x)^2}}$

$$= \frac{1}{\sqrt{1-x^2}}$$

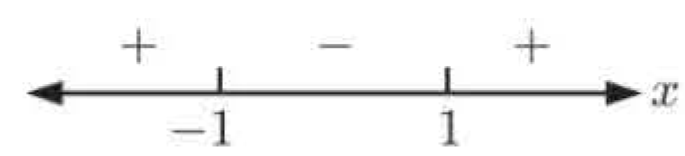
$$= f(x) \text{ for all } x$$

$\therefore$ $f(x)$ is an even function, and so it is symmetric about the y-axis.

ii $f(x)$ is defined when $1 - x^2 > 0$

$\therefore \quad x^2 - 1 < 0$

$\therefore \quad (x+1)(x-1) < 0$



$\therefore \quad x \in \,]-1, 1[$

c Area $= \displaystyle\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}}\,dx$

$$= \Big[\arcsin(x)\Big]_0^{\frac{1}{2}}$$

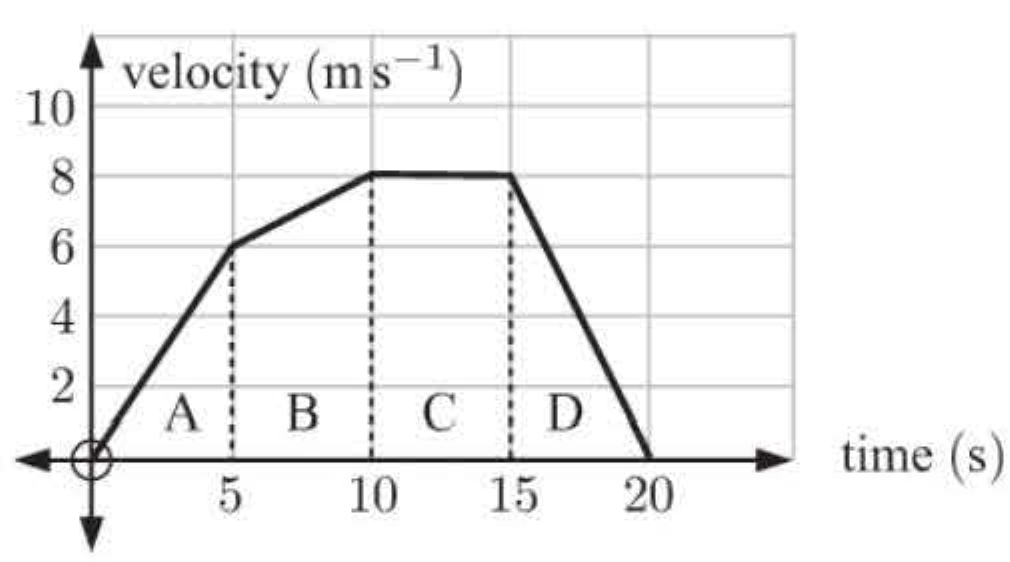
$$= \arcsin\left(\tfrac{1}{2}\right) - \arcsin(0)$$

$$= \tfrac{\pi}{6} - 0$$

$$= \tfrac{\pi}{6} \text{ units}^2$$

EXERCISE 22C.1

1



Total distance travelled

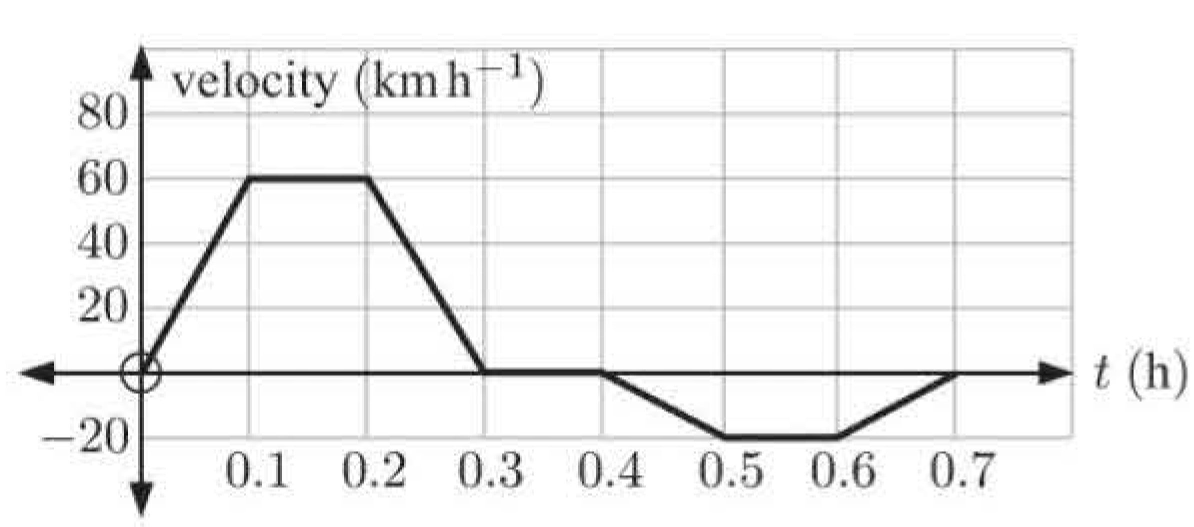
$= \text{area A} + \text{area B} + \text{area C} + \text{area D}$

$= \frac{1}{2}(5 \times 6) + \left(\dfrac{6+8}{2}\right) 5 + 5 \times 8 + \frac{1}{2}(5 \times 8)$

$= 15 + 35 + 40 + 20$

$= 110$ m

2



a **i** The graph above the t-axis indicates that the velocity is positive and the car is travelling forwards.

ii The graph below the t-axis indicates that the velocity is negative and the car is travelling backwards (opposite direction).

b Total distance travelled $=$ area above the t-axis $+$ area below the t-axis

$$= \left(\tfrac{0.1}{2} + 0.1 + \tfrac{0.1}{2}\right) 60 + \left(\tfrac{0.1}{2} + 0.1 + \tfrac{0.1}{2}\right) 20$$

$$= 12 + 4$$

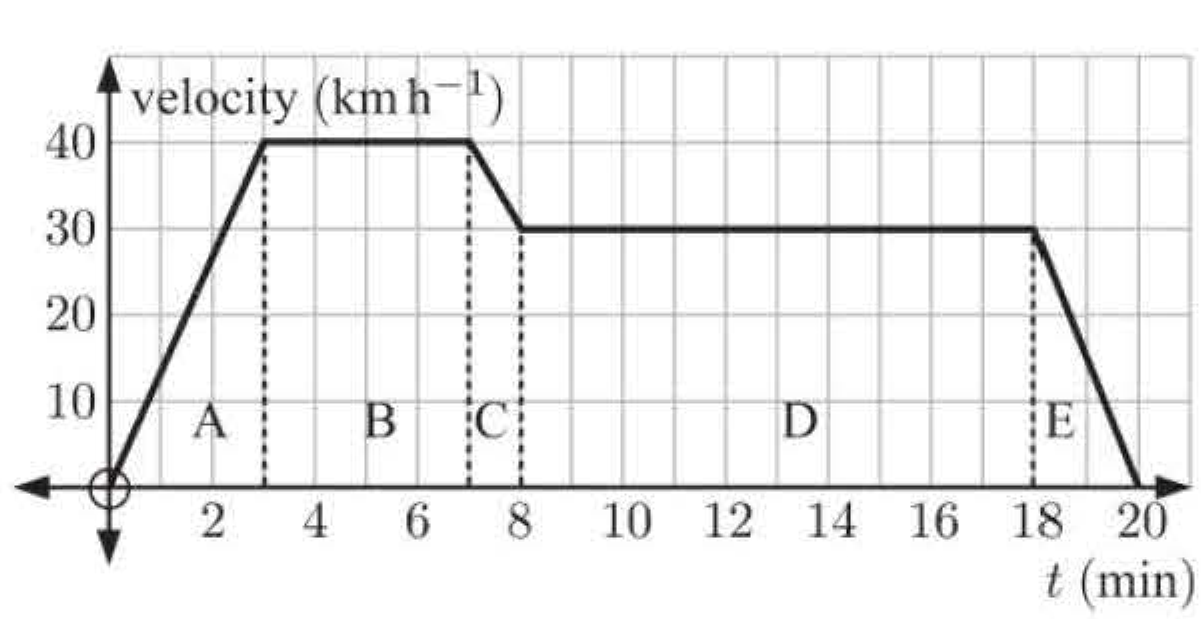
$$= 16 \text{ km}$$

c Final displacement $=$ area above the t-axis $-$ area below the t-axis

$$= 12 - 4$$

$= 8$ km from the starting point (on the positive side)

3 **a**



b Total distance travelled

$= \text{area A} + \text{area B} + \text{area C} + \text{area D} + \text{area E}$

$= \frac{1}{60}\left[\frac{1}{2}(3\times 40) + (40\times 4) + \left(\frac{40+30}{2}\right)1 + (10\times 30) + \frac{1}{2}(2\times 30)\right]$

{the factor $\frac{1}{60}$ accounts for the fact that the times are in minutes while the speeds are in km h^{-1}}

$= \frac{1}{60}(60+160+35+300+30)$

$= 9.75$ km

EXERCISE 22C.2

1 **a**
$$\begin{aligned} s(t) &= \int (1-2t)\,dt \\ &= t - 2\left(\frac{t^2}{2}\right) + c \\ &= t - t^2 + c \end{aligned}$$

But $s(0) = 2$

$\therefore\ 0 - 0^2 + c = 2$

$\therefore\ c = 2$

$\therefore\ s(t) = t - t^2 + 2$ cm

b The particle changes direction when

$v(t) = 0$

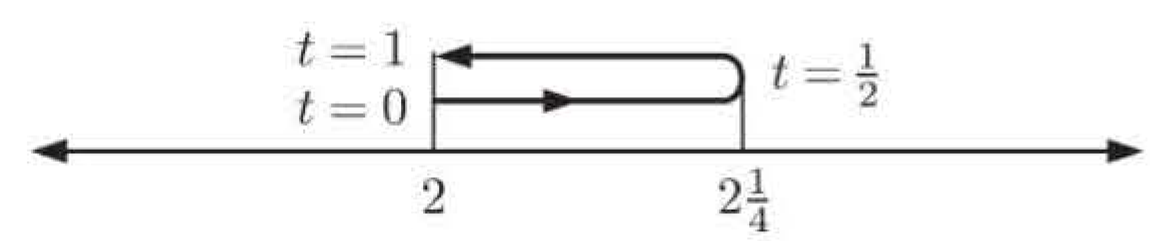
$\therefore\ 1 - 2t = 0$

$\therefore\ t = \frac{1}{2}$ s

Now $s(\frac{1}{2}) = \frac{1}{2} - (\frac{1}{2})^2 + 2 = 2\frac{1}{4}$ cm

and $s(1) = 1 - 1 + 2 = 2$ cm

$\therefore$ motion diagram is:



$\therefore$ total distance travelled $= \frac{1}{4} + \frac{1}{4}$

$= \frac{1}{2}$ cm

c Displacement $= s(1) - s(0)$

$= 2 - 2$

$= 0$ cm

2 **a**
$$\begin{aligned} s(t) &= \int (t^2 - t - 2)\,dt \\ &= \tfrac{1}{3}t^3 - \tfrac{1}{2}t^2 - 2t + c \end{aligned}$$

But $s(0) = 0$, $\therefore\ c = 0$

$\therefore\ s(t) = \frac{1}{3}t^3 - \frac{1}{2}t^2 - 2t$ cm

b P changes direction when $v(t) = 0$

$\therefore\ t^2 - t - 2 = 0$

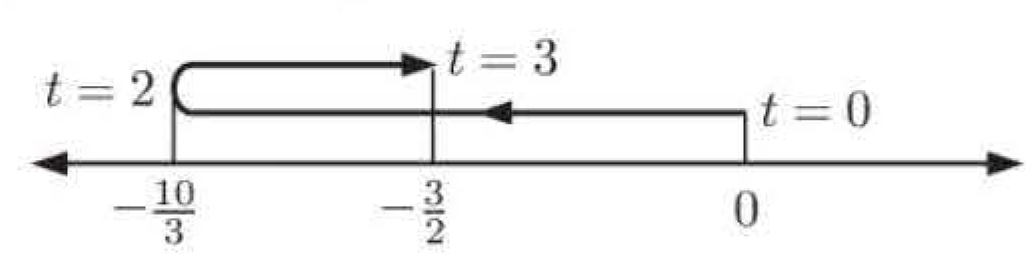
$\therefore\ (t-2)(t+1) = 0$

$\therefore$ P changes direction when $t = 2$ (since $t \geqslant 0$)

Now $s(2) = \dfrac{2^3}{3} - \dfrac{2^2}{2} - 2(2) = -\frac{10}{3}$

and $s(3) = \dfrac{3^3}{3} - \dfrac{3^2}{2} - 2(3) = -\frac{3}{2}$

$\therefore$ motion diagram is:



$\therefore$ total distance travelled $= \frac{10}{3} + (\frac{10}{3} - \frac{3}{2})$

$= 5\frac{1}{6}$ cm

c Displacement

$= s(3) - s(0)$

$= -\frac{3}{2} - 0$

$= -\frac{3}{2}$ cm ($1\frac{1}{2}$ cm left of its starting point)

3 **a**
$$\begin{aligned} s(t) &= \int (32+4t)\,dt \\ &= 32t + 4\left(\frac{t^2}{2}\right) + c \end{aligned}$$

$\therefore\ s(t) = 32t + 2t^2 + c$

But $s(0) = 16$

$\therefore\ 0 + 0 + c = 16$

$\therefore\ c = 16$

$\therefore\ s(t) = 32t + 2t^2 + 16$ m

b We check for any changes in direction.

These occur when $v(t) = 0$

$$\therefore \ 32 + 4t = 0$$
$$\therefore \ 4t = -32$$
$$\therefore \ t = -8$$

But $0 \leqslant t \leqslant t_1$, so the object does not change direction on the interval.

$$\therefore \ \text{displacement} = s(t_1) - s(0) = \int_0^{t_1} (32 + 4t)\, dt$$

c $a(t) = v'(t) = 4$

$\therefore$ the object is travelling with constant acceleration of $4\ \text{m}\,\text{s}^{-2}$.

4 $s(t) = \int (\cos(2t))\, dt = \frac{1}{2}\sin(2t) + c$

But $s(\frac{\pi}{4}) = 1$ $\therefore$ $\frac{1}{2}\sin(\frac{\pi}{2}) + c = 1$

$$\therefore \ c + \tfrac{1}{2} = 1$$
$$\therefore \ c = \tfrac{1}{2}$$
$$\therefore \ s(t) = \tfrac{1}{2}\sin(2t) + \tfrac{1}{2}$$

$$\therefore \ s(\tfrac{\pi}{3}) = \tfrac{1}{2}\sin(\tfrac{2\pi}{3}) + \tfrac{1}{2} = \tfrac{1}{2} \times \tfrac{\sqrt{3}}{2} + \tfrac{1}{2} = \tfrac{\sqrt{3}}{4} + \tfrac{1}{2} = \tfrac{\sqrt{3}+2}{4}\ \text{m}$$

5 $x'(t) = 16t - 4t^3$ units s^{-1}, $t \geqslant 0$

$= 4t(4 - t^2)$

$= 4t(2 + t)(2 - t)$ which has sign diagram:

$\therefore$ a direction reversal occurs at $t = 2$.

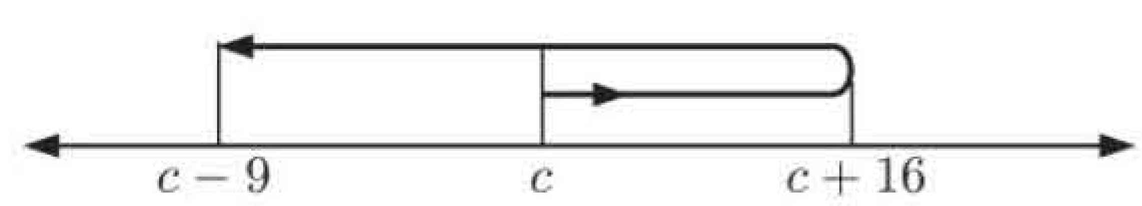
Now $x(t) = \int (16t - 4t^3)\, dt = 8t^2 - t^4 + c$

a $x(0) = c$

$x(2) = 32 - 16 + c = c + 16$

$x(3) = 72 - 81 + c = c - 9$

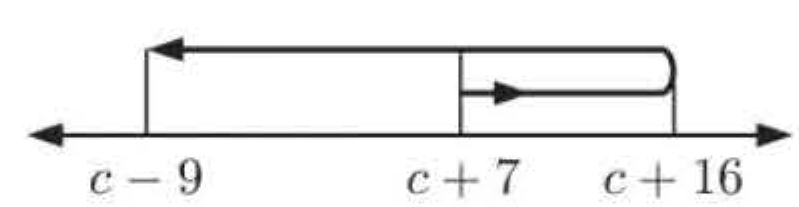
$\therefore$ motion diagram for $0 \leqslant t \leqslant 3$ is:



$\therefore$ total distance travelled $= (c + 16 - c) + (c + 16 - [c - 9])$
$= 41$ units

b $x(1) = 7 + c = c + 7$

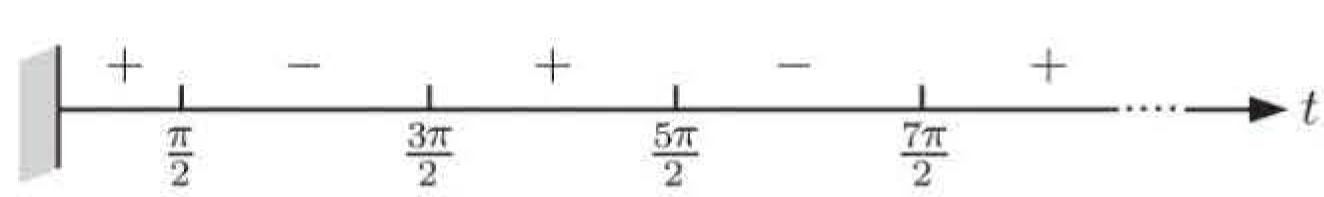
$\therefore$ motion diagram for $1 \leqslant t \leqslant 3$ is:



$\therefore$ total distance travelled $= (c + 16 - [c + 7]) + (c + 16 - [c - 9])$
$= 34$ units

6 **a** $v(t) = \cos t\ \text{m}\,\text{s}^{-1}$, $t \geqslant 0$

$\therefore$ $v(t)$ has sign diagram:



$\therefore$ a direction reversal occurs at $t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \ldots$

$s(t) = \int \cos t\, dt = \sin t + c$

$\therefore$ $s(0) = c$

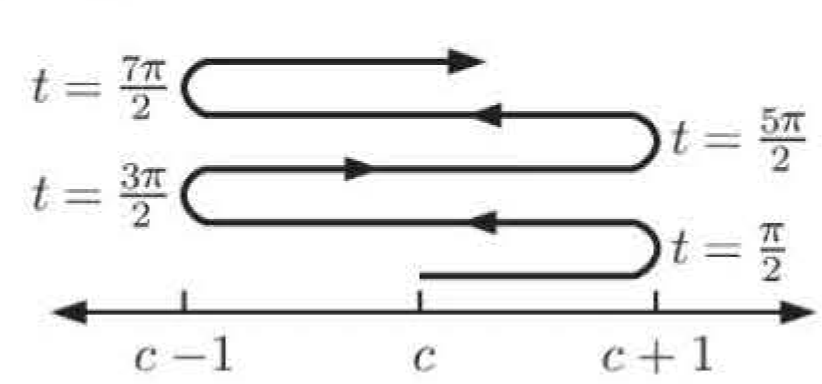
$s\left(\frac{\pi}{2}\right) = c + 1$

$s\left(\frac{3\pi}{2}\right) = c - 1$

$s\left(\frac{5\pi}{2}\right) = c + 1$

$s\left(\frac{7\pi}{2}\right) = c - 1$

The motion diagram is:



$\therefore$ the particle oscillates between the points $(c - 1)$ and $(c + 1)$.

b distance $= (c+1)-(c-1)$
$= 2$ m

7 $v(t) = 50 - 10e^{-0.5t}$ m s^{-1}, $t \geqslant 0$

a $v(0) = 50 - \dfrac{10}{e^0} = 50 - 10 = 40$ m s^{-1}

b $v(3) = 50 - \dfrac{10}{e^{1.5}} \approx 47.8$ m s^{-1}

c The velocity reaches 45 m s^{-1}

when $45 = 50 - 10e^{-0.5t}$

$\therefore\ 10e^{-\frac{t}{2}} = 5$

$\therefore\ e^{\frac{t}{2}} = 2$

$\therefore\ \dfrac{t}{2} = \ln 2$

$\therefore\ t = 2\ln 2 \approx 1.39$ seconds

d $v(t) = 50 - \dfrac{10}{e^{\frac{t}{2}}}$

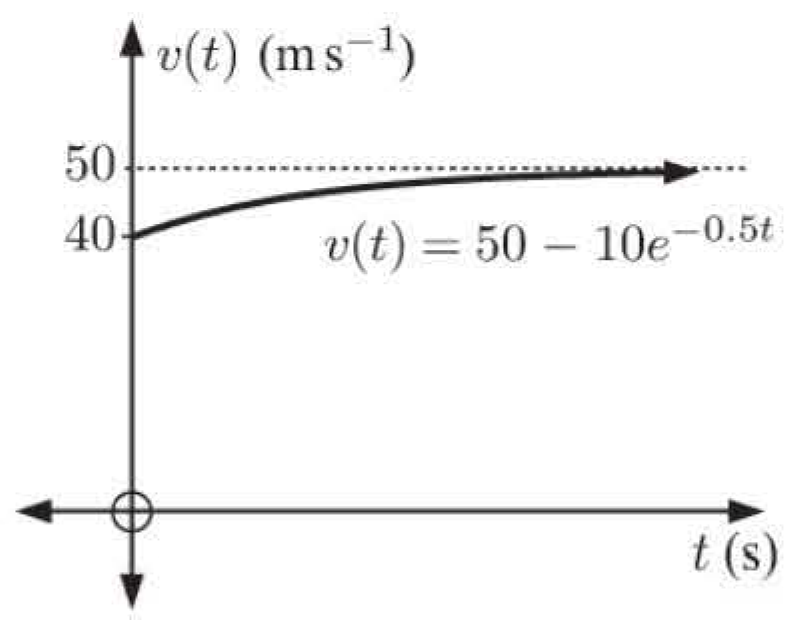
As $t \to \infty$, $\dfrac{10}{e^{\frac{t}{2}}} \to 0^+$

$\therefore\ v(t) \to 50^-$

e $a(t) = v'(t)$
$= -10e^{-0.5t}(-0.5)$
$= 5e^{-0.5t}$ m s^{-2}

$\therefore\ a(t) > 0$ for all t $\{e^x > 0$ for all $x\}$
$\therefore$ the acceleration is always positive

f



g total distance travelled
$= \int_0^3 (50 - 10e^{-0.5t})\,dt$
$= \left[50t + 20e^{-0.5t}\right]_0^3$
$= 150 + 20e^{-1.5} - 20$
≈ 134.5 m

8 **a** $v(t) = \displaystyle\int \frac{-1}{(t+1)^2}\,dt$
$= \int -(t+1)^{-2}\,dt$
$= (t+1)^{-1} + c$

But $v(0) = 0$

$\therefore\ \dfrac{1}{0+1} + c = 0$

$\therefore\ c + 1 = 0$

$\therefore\ c = -1$

$\therefore\ v(t) = \dfrac{1}{t+1} - 1$ m s^{-1}

b $s(t) = \displaystyle\int \left(\frac{1}{t+1} - 1\right) dt$
$= \ln|t+1| - t + c$

But $s(0) = 0$

$\therefore\ \ln 1 - 0 + c = 0$

$\therefore\ c = 0$

$\therefore\ s(t) = \ln|t+1| - t$ m

c $s(2) = \ln 3 - 2$ m
≈ -0.901 m

$v(2) = \dfrac{1}{2+1} - 1$
$= -\frac{2}{3}$ m s^{-1}

$a(2) = \dfrac{-1}{(2+1)^2}$
$= -\frac{1}{9}$ m s^{-2}

The object is approximately 0.901 m to the left of the origin, travelling left at $\frac{2}{3}$ m s^{-1}, with acceleration $-\frac{1}{9}$ m s^{-2}.

9 **a** $v(t) = \displaystyle\int \left(\frac{t}{10} - 3\right) dt$
$= \frac{1}{10}\left(\dfrac{t^2}{2}\right) - 3t + c$
$= \frac{1}{20}t^2 - 3t + c$ m s^{-1}

But $v(0) = 45$

$\therefore\ \frac{1}{20}(0)^2 - 3(0) + c = 45$

$\therefore\ c = 45$

$\therefore\ v(t) = \frac{1}{20}t^2 - 3t + 45$ m s^{-1}

b $\int_0^{60}(\frac{1}{20}t^2 - 3t + 45)\,dt = \left[\frac{1}{60}t^3 - \frac{3}{2}t^2 + 45t\right]_0^{60}$

$= \frac{1}{60}(60)^3 - \frac{3}{2}(60)^2 + 45(60)$

$= 900$

The train travels a total of 900 m in the first 60 seconds.

10 $a(t) = 4e^{-\frac{t}{20}}$ m s^{-2}

$\therefore\ v(t) = \int 4e^{-\frac{t}{20}}\,dt$

$= 4 \times \dfrac{1}{-\frac{1}{20}} e^{-\frac{t}{20}} + c$

$= -80e^{-\frac{t}{20}} + c$

Now $v(0) = 20$ m s^{-1}

$\therefore\ c = 100$

$\therefore\ v(t) = 100 - 80e^{-\frac{t}{20}}$ m s^{-1}

a As $t \to \infty,\ e^{-\frac{t}{20}} \to 0^+ \quad \therefore\ v(t) \to 100^-$ m s^{-1}

$\therefore$ the object approaches a limiting velocity of 100 m s^{-1}.

b The total distance travelled $= \int_0^{10} (100 - 80e^{-\frac{t}{20}})\,dt \qquad \{v(t) > 0$ for $0 \leqslant t \leqslant 10\}$

$= \left[100t + 1600e^{-\frac{t}{20}}\right]_0^{10}$

$= \left(1000 + 1600e^{-\frac{1}{2}}\right) - (0 + 1600)$

≈ 370.4 m

EXERCISE 22D

1 The marginal cost is $C'(x)$ and $C'(x) = 3.15 + 0.004x$ € per gadget

$\therefore\ C(x) = \int(3.15 + 0.004x)\,dx$

$= 3.15x + 0.002x^2 + c$

But $C(0) = 450$ so $c = 450$

$\therefore\ C(x) = 3.15x + 0.002x^2 + 450$ euros

$\therefore\ C(800) = 3.15(800) + 0.002(800)^2 + 450$

$=$ €4250

So, the total cost is €4250.

2 **a** The marginal profit is $P'(x)$ and $P'(x) = 15 - 0.03x$ dollars per plate

$\therefore\ P(x) = \int(15 - 0.03x)\,dx$

$= 15x - 0.015x^2 + c$

But $P(0) = -650$ so $c = -650$

$\therefore\ P(x) = 15x - 0.015x^2 - 650$ dollars

b The maximum profit occurs when $P'(x) = 0$, which is when $15 - 0.03x = 0$

$\therefore\ 0.03x = 15$

$\therefore\ x = \dfrac{15}{0.03}$

$\therefore\ x = 500$

Now $P''(x) = -0.03 < 0 \quad \therefore$ the profit is at a maximum when $x = 500$ plates.

The maximum profit $= P(500) = 15(500) - 0.015(500)^2 - 650$

$= \$3100$

c In order for a profit to be made, $P(x)$ must be greater than 0

$\therefore\ 15x - 0.015x^2 - 650 > 0$

Using technology, the x-intercepts of $P(x)$ are $x_1 = 45.39$ and $x_2 = 954.6$

Since we cannot produce part plates, a profit is made for $46 \leqslant x \leqslant 954$.

3 $E'(t) = 350(80+0.15t)^{0.8} - 120(80+0.15t)$ calories per day

Total energy needs over the first week $= \int_0^7 E'(t)\,dt$

$$= \int_0^7 [350(80+0.15t)^{0.8} - 120(80+0.15t)]\,dt$$

$$= \left[\tfrac{1}{0.15} \times \frac{350(80+0.15t)^{1.8}}{1.8} - 9600t - 9t^2\right]_0^7$$

$$\approx 14\,400 \text{ calories}$$

4 $\dfrac{dT}{dx} = \dfrac{-20}{x^{0.63}} = -20x^{-0.63}$ $\quad\therefore\quad T = \int -20x^{-0.63}\,dx = \dfrac{-20x^{0.37}}{0.37} + c$

Now when $x = 3$, $T = 100$

$\therefore\quad \dfrac{-20(3^{0.37})}{0.37} + c = 100$

$\therefore\quad c = 100 + \dfrac{20(3^{0.37})}{0.37} \approx 181.1639$

$\therefore\quad T \approx \dfrac{-20x^{0.37}}{0.37} + 181.1639$

So, when $x = 6$, $T \approx -104.8925 + 181.1639 \approx 76.27$

$\therefore$ the outer surface temperature is about $76.3°\text{C}$.

5 **a** When $x = 0$, deflection $= 0$ $\therefore$ $y = 0$.

And when $x = 0$, the tangent is horizontal $\therefore$ $\dfrac{dy}{dx} = 0$.

b $\dfrac{d^2y}{dx^2} = -\tfrac{1}{10}(1-x)^2$

$\therefore\quad \dfrac{dy}{dx} = \int -\tfrac{1}{10}(1-x)^2\,dx$

$= -\tfrac{1}{10}(\tfrac{1}{-1}) \times \dfrac{(1-x)^3}{3} + c$

$= \tfrac{1}{30}(1-x)^3 + c$

From **a**, when $x = 0$, $\dfrac{dy}{dx} = 0$

$\therefore\quad \tfrac{1}{30}(1-0)^3 + c = 0$

$\therefore\quad c = -\tfrac{1}{30}$

$\therefore\quad \dfrac{dy}{dx} = \tfrac{1}{30}(1-x)^3 - \tfrac{1}{30}$

$\therefore\quad y = \int [\tfrac{1}{30}(1-x)^3 - \tfrac{1}{30}]\,dx$

$= \tfrac{1}{30}(\tfrac{1}{-1})\dfrac{(1-x)^4}{4} - \tfrac{1}{30}x + d$

$= -\dfrac{(1-x)^4}{120} - \dfrac{x}{30} + d$

Also from **a**, when $x = 0$, $y = 0$

$\therefore\quad -\tfrac{1}{120} - 0 + d = 0$

$\therefore\quad d = \tfrac{1}{120}$

$\therefore\quad y = \dfrac{1}{120} - \dfrac{(1-x)^4}{120} - \dfrac{x}{30}$

c Maximum deflection occurs at the right hand end where $x \approx 1$

and at $x \approx 1$, $y \approx \tfrac{1}{120} - 0 - \tfrac{1}{30} \approx -0.025$ m

$\therefore$ the maximum deflection is about 2.5 cm.

6 **a** $\dfrac{d^2y}{dx^2} = 0.01\left(2x - \dfrac{x^2}{2}\right) = 0.02x - 0.005x^2$

$\therefore\quad \dfrac{dy}{dx} = \int (0.02x - 0.005x^2)\,dx = 0.01x^2 - \tfrac{0.005}{3}x^3 + c$

The sag, $y = \int (0.01x^2 - \tfrac{0.005}{3}x^3 + c)\,dx$

$\therefore\quad y = \tfrac{0.01}{3}x^3 - \tfrac{0.005}{12}x^4 + cx + d$

Now when $x = 0,\ y = 0$ $\quad \therefore\ 0 - 0 + 0 + d = 0$

$\therefore\ d = 0$

$\therefore\ y = \frac{0.01}{3}x^3 - \frac{0.005}{12}x^4 + cx$

Also, when $x = 4,\ y = 0$ $\quad \therefore\ \frac{0.01}{3}\left(4^3\right) - \frac{0.005}{12}\left(4^4\right) + 4c = 0$

$\therefore\ 4c = \frac{0.005}{12}\left(4^4\right) - \frac{0.01}{3}\left(4^3\right)$

$\therefore\ c = \frac{0.005}{12}\left(4^3\right) - \frac{0.01}{3}\left(4^2\right)$

$\therefore\ c = -\frac{0.08}{3}$

$\therefore\ y = \left(\frac{0.01}{3}x^3 - \frac{0.005}{12}x^4 - \frac{0.08}{3}x\right)$ m

b The maximum sag occurs when $\dfrac{dy}{dx} = 0 \quad \therefore\ 0.01x^2 - \frac{0.005}{3}x^3 - \frac{0.08}{3} = 0$

$\therefore\ 6x^2 - x^3 - 16 = 0$

Using technology, the three solutions are $x = -1.464,\ 2,$ and 5.464

But the maximum lies between 0 and 4, so it must occur when $x = 2$.

When $x = 2$, $\quad y = \frac{0.01}{3}\left(2^3\right) - \frac{0.005}{12}\left(2^4\right) - \frac{0.08}{3}(2)$

$\approx -0.033\,33$ m

≈ -3.333 cm $\quad \therefore$ the maximum sag is ≈ 3.33 cm

c At the point 1 m from P, $\quad x = 3$ m, so $\quad y = \frac{0.01}{3}\left(3^3\right) - \frac{0.005}{12}\left(3^4\right) - \frac{0.08}{3}(3)$

$= -0.023\,75$ m

$= -2.375$ cm $\quad \therefore$ the sag is 2.375 cm

d At the point 1 m from P, $\quad x = 3$ m, so $\quad \dfrac{dy}{dx} = 0.01\left(3^2\right) - \frac{0.005}{3}\left(3^3\right) - \frac{0.08}{3} \approx 0.0183$

$\therefore$ the angle θ that the plank makes with the horizontal is such that $\tan\theta \approx 0.0183$

$\therefore\ \theta \approx \tan^{-1}(0.0183) \approx 1.05°$

7 The cost per unit volume, $\dfrac{dC}{dV} = \frac{1}{2}x^2 + 4$ dollars per m^3 (at depth x).

Since the volume of a well x m deep is $V = \pi r^2 x$, $\dfrac{dV}{dx} = \pi r^2$

Now $\dfrac{dC}{dx} = \dfrac{dC}{dV}\dfrac{dV}{dx}$ $\quad$ {chain rule}

$$\therefore\ \frac{dC}{dx} = \left(\tfrac{1}{2}x^2 + 4\right)\pi r^2$$

$$\therefore\ C = \int \frac{dC}{dx}\,dx$$

$$= \int [\pi r^2(\tfrac{1}{2}x^2 + 4)]\,dx$$

$$= \pi r^2\left(\frac{x^3}{6} + 4x\right) + c$$

So, the cost of digging a well h metres deep

$$= \pi r^2\left(\frac{h^3}{6} + 4h\right) + c$$

Now if the initial cost $= C_0$ when $h = 0$,

$$\pi r^2(\tfrac{0}{6} + 0) + c = C_0$$

$$\therefore\ c = C_0$$

$$\therefore\ C(h) = \pi r^2\left(\frac{h^3 + 24h}{6}\right) + C_0$$

8 $\quad y = \sin x,\ 0 \leqslant x \leqslant \pi$

$$\therefore\ \frac{dy}{dx} = \cos x$$

$\therefore\ L = \int_0^{\pi} \sqrt{1 + \cos^2 x}\,dx \approx 3.820\,20$ units $\quad$ {technology}

9 **a** The yield Y per unit area A is proportional to $\dfrac{1}{\sqrt{x+4}}$.

$\therefore \quad \dfrac{dY}{dA} \propto \dfrac{1}{\sqrt{x+4}}$

$\therefore \quad \dfrac{dY}{dA} = \dfrac{k}{\sqrt{x+4}}$ for some constant k.

b The shaded area $A = \text{length} \times \text{width}$

$\therefore \quad A = (4-2p)x$

$\therefore \quad \dfrac{dA}{dx} = 4 - 2p$

Now $\dfrac{dY}{dx} = \dfrac{dY}{dA}\,\dfrac{dA}{dx}$ {chain rule}

$\therefore \quad \dfrac{dY}{dx} = \dfrac{k}{\sqrt{x+4}} \times (4-2p)$

$\therefore \quad \dfrac{dY}{dx} = \dfrac{k(4-2p)}{\sqrt{x+4}}$

c $\dfrac{dY}{dx}$ is the instantaneous rate of change of the yield with respect to the distance x from the canal.

$\therefore$ total yield $= Y = \displaystyle\int_0^p \frac{dY}{dx}\,dx = \int_0^p \frac{k(4-2p)}{\sqrt{x+4}}\,dx$

d Using **c**,

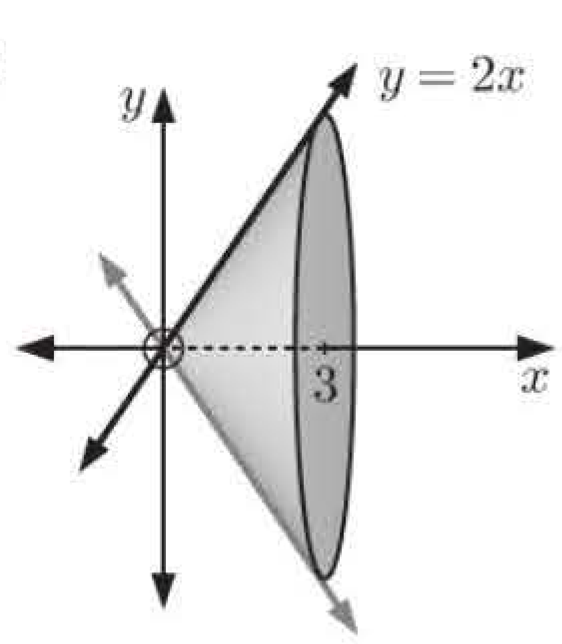
$$\begin{aligned} Y &= k(4-2p)\textstyle\int_0^p (x+4)^{-\frac{1}{2}}\,dx \\ &= k(4-2p) \times \left[\dfrac{(x+4)^{\frac{1}{2}}}{\frac{1}{2}}\right]_0^p \\ &= 2k(4-2p)\left[\sqrt{x+4}\right]_0^p \\ &= 4k(2-p)\left[\sqrt{p+4} - \sqrt{4}\right] \\ \therefore \quad Y &= 4k(2-p)\left(\sqrt{p+4} - 2\right) \end{aligned}$$

e Using technology to graph Y and find its maximum, we find that the maximum occurs when $p \approx 0.9735$ km

$\therefore$ the orchard is $0.974 \text{ km} \times 2.05 \text{ km}$.

EXERCISE 22E.1

1 **a**



Volume $= \pi \int_0^3 (2x)^2\,dx$

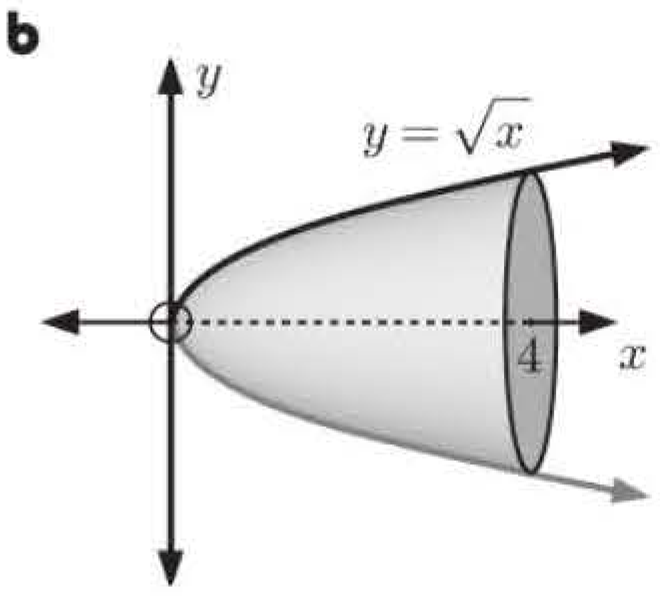
$= 4\pi \int_0^3 x^2\,dx$

$= 4\pi \left[\frac{1}{3}x^3\right]_0^3$

$= 4\pi(9-0)$

$= 36\pi$ units^3

b



Volume $= \pi \int_0^4 (\sqrt{x})^2\,dx$

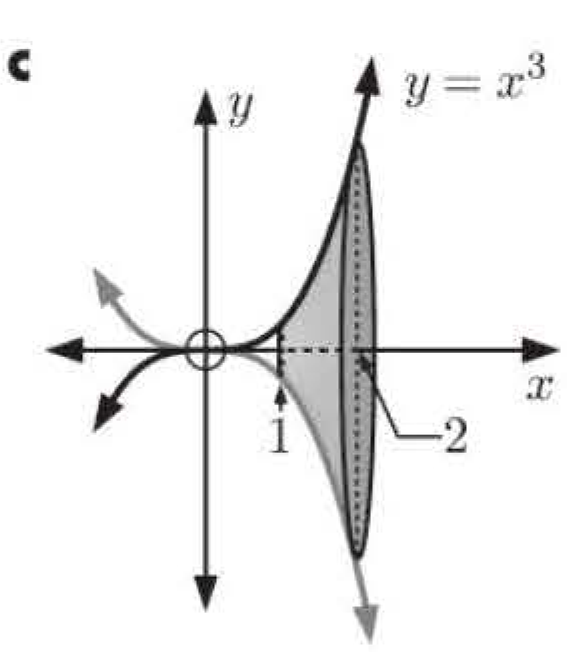
$= \pi \int_0^4 x\,dx$

$= \pi \left[\frac{1}{2}x^2\right]_0^4$

$= \pi(8-0)$

$= 8\pi$ units^3

c



Volume $= \pi \int_1^2 (x^3)^2\,dx$

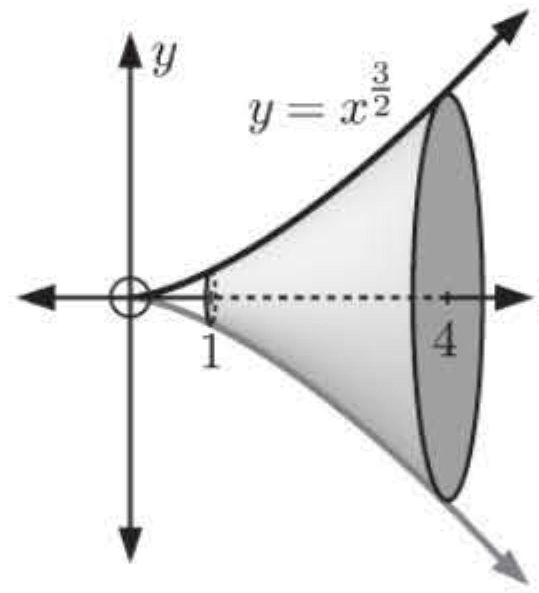
$= \pi \int_1^2 x^6\,dx$

$= \pi \left[\frac{1}{7}x^7\right]_1^2$

$= \pi\left(\frac{128}{7} - \frac{1}{7}\right)$

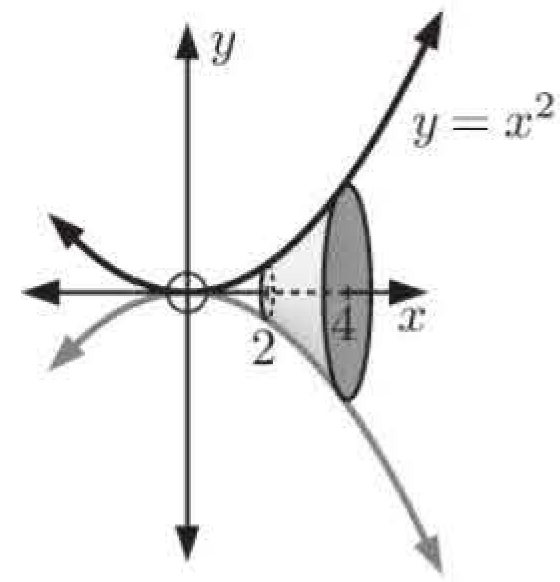
$= \frac{127\pi}{7}$ units^3

d



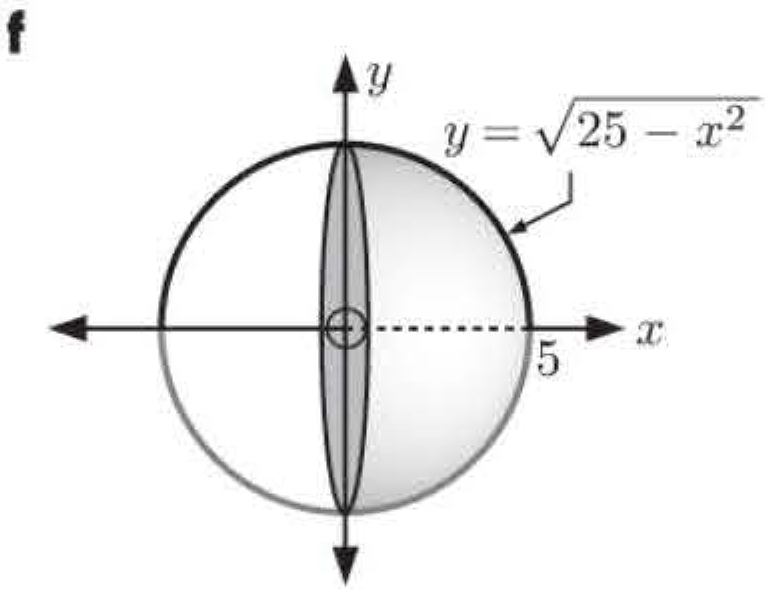
$$\begin{aligned}\text{Volume} &= \pi \int_1^4 (x^{\frac{3}{2}})^2\, dx \\ &= \pi \int_1^4 x^3\, dx \\ &= \pi \left[\tfrac{1}{4}x^4\right]_1^4 \\ &= \pi\left(\tfrac{256}{4} - \tfrac{1}{4}\right) \\ &= \tfrac{255\pi}{4} \text{ units}^3\end{aligned}$$

e



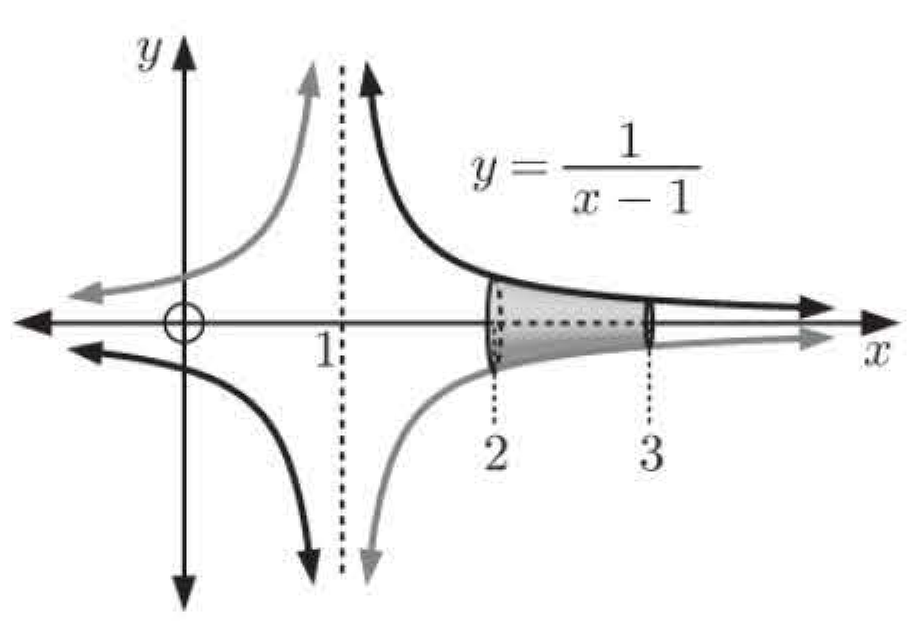
$$\begin{aligned}\text{Volume} &= \pi \int_2^4 (x^2)^2\, dx \\ &= \pi \int_2^4 x^4\, dx \\ &= \pi \left[\tfrac{1}{5}x^5\right]_2^4 \\ &= \pi\left(\tfrac{1024}{5} - \tfrac{32}{5}\right) \\ &= \tfrac{992\pi}{5} \text{ units}^3\end{aligned}$$

f



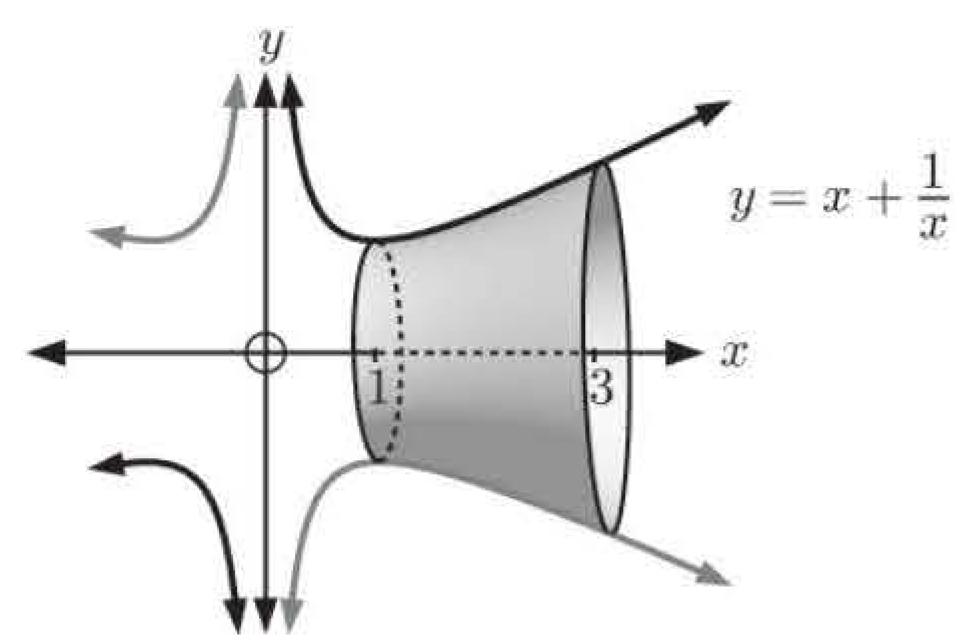
$$\begin{aligned}\text{Volume} &= \pi \int_0^5 (25 - x^2)\, dx \\ &= \pi \left[25x - \frac{x^3}{3}\right]_0^5 \\ &= \pi\left(125 - \tfrac{125}{3}\right) \\ &= \pi\left(\tfrac{2}{3}\right)125 \\ &= \tfrac{250\pi}{3} \text{ units}^3\end{aligned}$$

g



$$\begin{aligned}\text{Volume} &= \pi \int_2^3 \left(\frac{1}{x-1}\right)^2 dx \\ &= \pi \int_2^3 (x-1)^{-2}\, dx \\ &= \pi \left[-\frac{1}{x-1}\right]_2^3 \\ &= \pi\left(-\tfrac{1}{2} + 1\right) \\ &= \tfrac{\pi}{2} \text{ units}^3\end{aligned}$$

h



$$\begin{aligned}\text{Volume} &= \pi \int_1^3 \left(x + \frac{1}{x}\right)^2 dx \\ &= \pi \int_1^3 \left(x^2 + 2 + x^{-2}\right)\, dx \\ &= \pi \left[\frac{x^3}{3} + 2x - \frac{1}{x}\right]_1^3 \\ &= \pi\left[9 + 6 - \tfrac{1}{3} - \left(\tfrac{1}{3} + 2 - 1\right)\right] \\ &= \tfrac{40\pi}{3} \text{ units}^3\end{aligned}$$

2 **a**

$$\begin{aligned}\text{Volume} &= \pi \int_1^3 \left(\frac{x^3}{x^2+1}\right)^2 dx \\ &\approx 5.926\pi \quad \{\text{using technology}\} \\ &\approx 18.6 \text{ units}^3\end{aligned}$$

b

$$\begin{aligned}\text{Volume} &= \pi \int_0^2 \left(e^{\sin x}\right)^2 dx \\ &\approx 9.613\pi \quad \{\text{using technology}\} \\ &\approx 30.2 \text{ units}^3\end{aligned}$$

3 **a**

$$\begin{aligned}V &= \pi \int_0^6 \left(\frac{x}{2} + 4\right)^2 dx \\ &= \pi \int_0^6 \left(\tfrac{1}{4}x^2 + 4x + 16\right)\, dx \\ &= \pi \left[\frac{x^3}{12} + \frac{4x^2}{2} + 16x\right]_0^6 \\ &= \pi(18 + 72 + 96) - 0 \\ &= 186\pi \text{ units}^3\end{aligned}$$

b

$$\begin{aligned}V &= \pi \int_1^2 (x^2+3)^2\, dx \\ &= \pi \int_1^2 \left(x^4 + 6x^2 + 9\right)\, dx \\ &= \pi \left[\frac{x^5}{5} + \frac{6x^3}{3} + 9x\right]_1^2 \\ &= \pi\left[\left(\tfrac{32}{5} + 16 + 18\right) - \left(\tfrac{1}{5} + 2 + 9\right)\right] \\ &= \pi\left(\tfrac{146}{5}\right) \\ &= \tfrac{146\pi}{5} \text{ units}^3\end{aligned}$$

c $V = \pi \int_0^4 (e^x)^2 \, dx$
$= \pi \int_0^4 e^{2x} \, dx$
$= \pi \left[\frac{1}{2}e^{2x}\right]_0^4$
$= \pi \left(\frac{1}{2}e^8 - \frac{1}{2}\right)$
$= \frac{\pi}{2}(e^8 - 1)$ units^3

4 **a** If we take a vertical slice of the bowl, we get a circle.

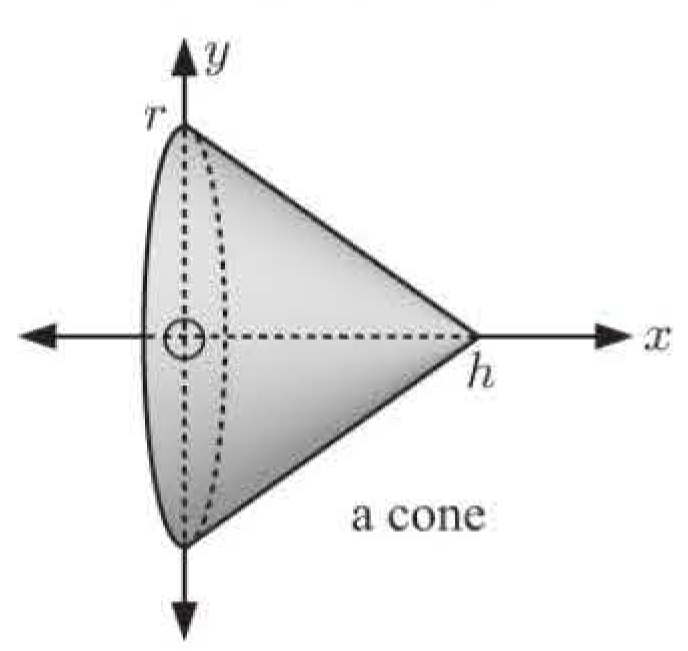
b Volume of revolution
$= \pi \int_a^b y^2 \, dx$
$= \pi \int_0^4 \left(4\sqrt{x}\right)^2 \, dx$
$= \int_0^4 \pi \left(4\sqrt{x}\right)^2 \, dx$

c Capacity $= \int_0^4 \pi \times 16x \, dx$
$= \int_0^4 16\pi x \, dx$
$= \left[8\pi x^2\right]_0^4$
$= 8\pi \times 16$
$= 128\pi$ units^3
≈ 402 units^3

5 **a** Volume $= \pi \int_5^8 y^2 \, dx$
$= \pi \int_5^8 (64 - x^2) \, dx$
$= \pi \left[64x - \dfrac{x^3}{3}\right]_5^8$
$= \pi \left[\left(512 - \frac{512}{3}\right) - \left(320 - \frac{125}{3}\right)\right]$
$= 63\pi$ units^3

b 63π cm$^3 \approx 198$ cm^3

6 **a** a cone of base radius r and height h



b [AB] has gradient $= \dfrac{r-0}{0-h} = -\dfrac{r}{h}$

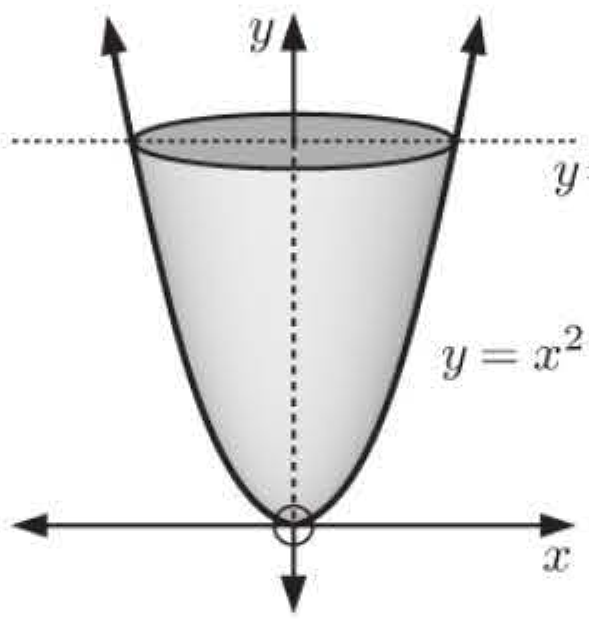
$\therefore$ its equation is $y = -\left(\dfrac{r}{h}\right)x + r$

c $V = \pi \displaystyle\int_0^h \left(\frac{-r}{h}x + r\right)^2 dx$
$= \pi r^2 \displaystyle\int_0^h \left(-\frac{x}{h} + 1\right)^2 dx$
$= \pi r^2 \displaystyle\int_0^h \left(\frac{x^2}{h^2} - \frac{2x}{h} + 1\right) dx$
$= \pi r^2 \left[\dfrac{x^3}{3h^2} - \dfrac{2x^2}{2h} + x\right]_0^h$
$= \pi r^2 \left[\left(\dfrac{h}{3} - h + h\right) - 0\right]$
$= \frac{1}{3}\pi r^2 h$ units^3

7 **a** a sphere of radius r

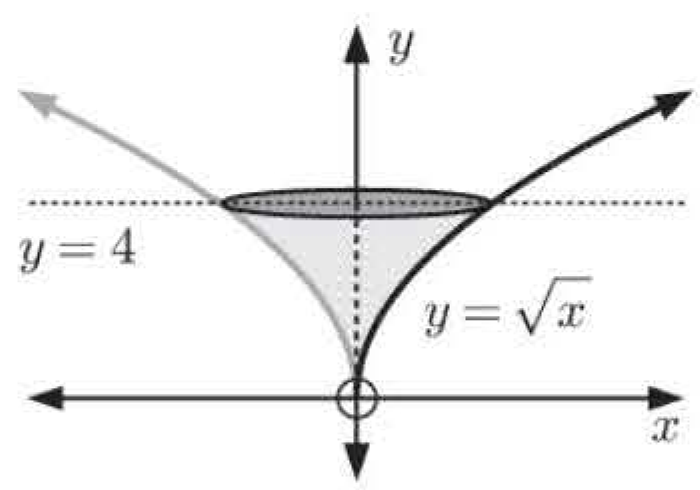
b $V = \pi \int_{-r}^{r} y^2 \, dx = 2\pi \int_0^r (r^2 - x^2) \, dx$
$= 2\pi \left[r^2 x - \dfrac{x^3}{3}\right]_0^r$
$= 2\pi(r^3 - \dfrac{r^3}{3} - 0)$
$= 2\pi \times \frac{2}{3}r^3$
$= \frac{4}{3}\pi r^3$ units^3

8 **a**



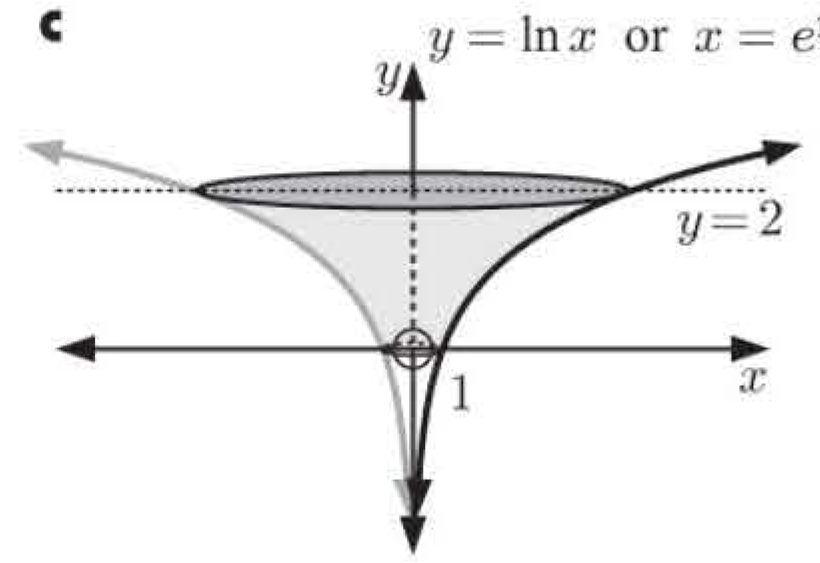
$$\begin{aligned}\text{Volume} &= \pi\int_0^4 x^2\,dy\\ &= \pi\int_0^4 y\,dy\\ &= \pi\left[\frac{y^2}{2}\right]_0^4\\ &= \pi(8-0)\\ &= 8\pi \text{ units}^3\end{aligned}$$

b



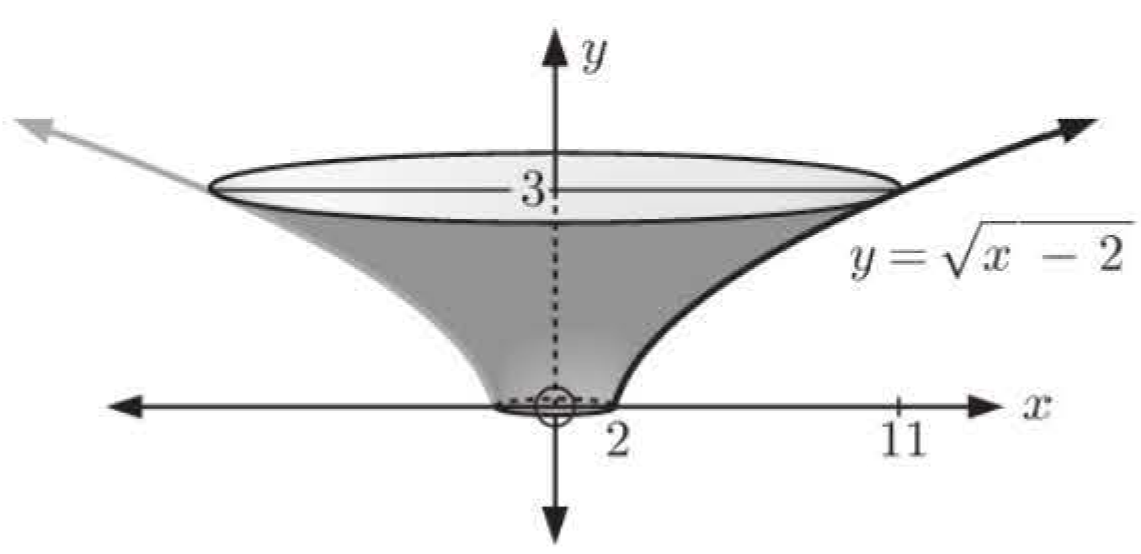
$$\begin{aligned}\text{Volume} &= \pi\int_1^4 x^2\,dy\\ &= \pi\int_1^4 y^4\,dy\\ &= \pi\left[\frac{y^5}{5}\right]_1^4\\ &= \pi\left(\tfrac{4^5}{5}-\tfrac{1}{5}\right)\\ &= \tfrac{1023}{5}\pi \text{ units}^3\end{aligned}$$

c



$$\begin{aligned}\text{Volume} &= \pi\int_0^2 x^2\,dy\\ &= \pi\int_0^2 (e^y)^2\,dy\\ &= \pi\int_0^2 e^{2y}\,dy\\ &= \pi\left[\tfrac{1}{2}e^{2y}\right]_0^2\\ &= \pi\left(\tfrac{1}{2}e^4-\tfrac{1}{2}\right)\\ &= \tfrac{\pi}{2}(e^4-1) \text{ units}^3\end{aligned}$$

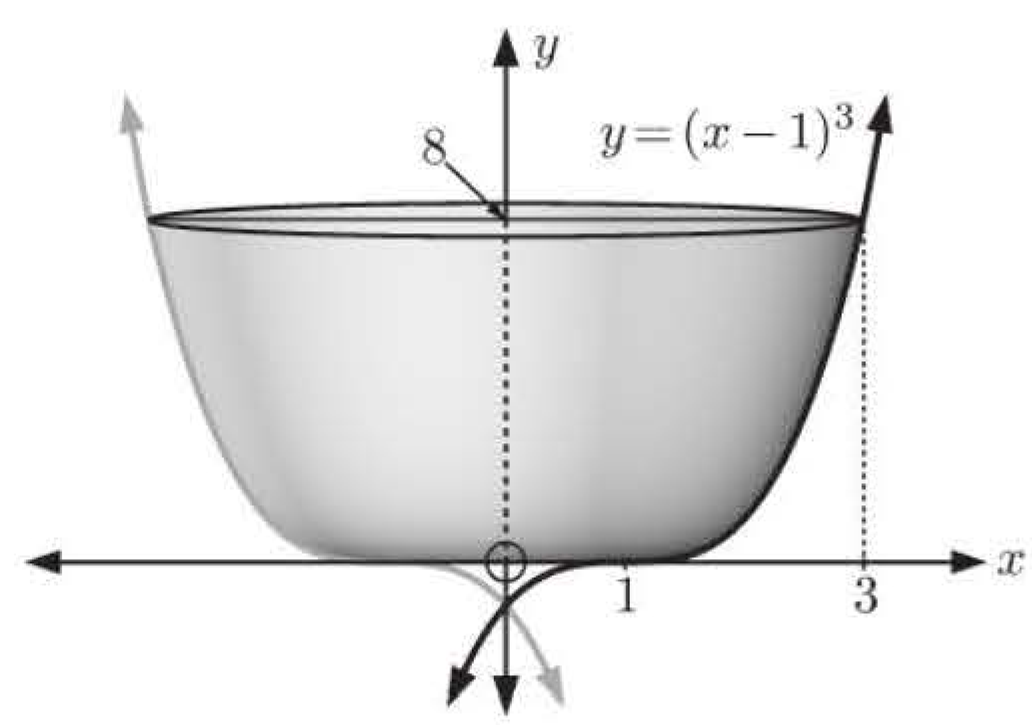
d



When $x=2$, $y=0$
When $x=11$, $y=3$
Now $y=\sqrt{x-2}$
$\therefore\ y^2=x-2$
$\therefore\ x=y^2+2$

$$\begin{aligned}\therefore\ \text{volume} &= \pi\int_0^3 x^2\,dy\\ &= \pi\int_0^3 (y^2+2)^2\,dy\\ &= \pi\int_0^3 (y^4+4y^2+4)\,dy\\ &= \pi\left[\frac{y^5}{5}+\frac{4}{3}y^3+4y\right]_0^3\\ &= \pi\left(\tfrac{3^5}{5}+36+12-0\right)\\ &= \tfrac{483}{5}\pi \text{ units}^3\end{aligned}$$

e

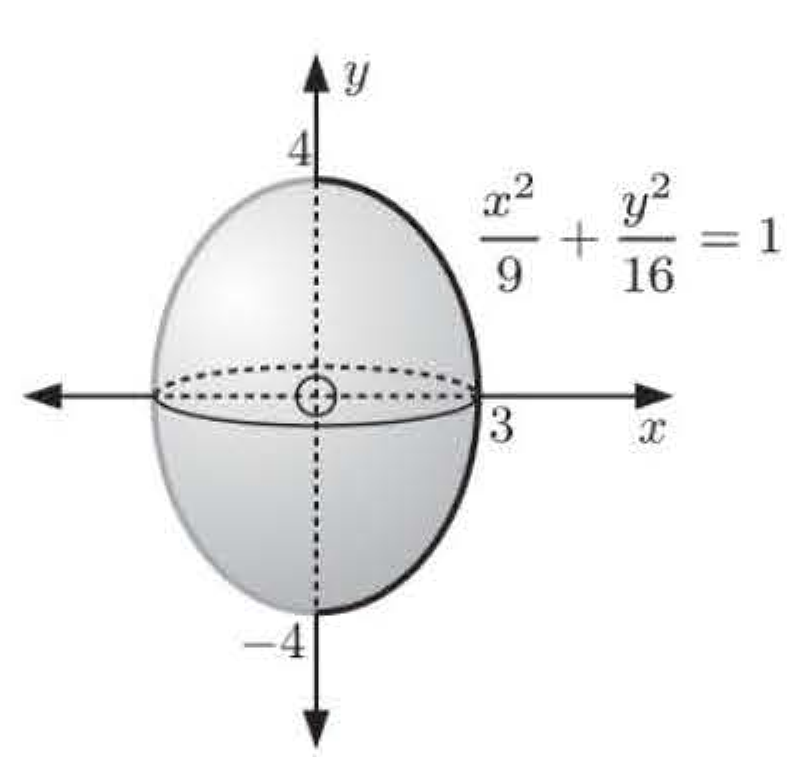


When $x=1$, $y=0$
When $x=3$, $y=8$
Now $y=(x-1)^3$
$\therefore\ x-1=y^{\frac{1}{3}}$
$\therefore\ x=y^{\frac{1}{3}}+1$

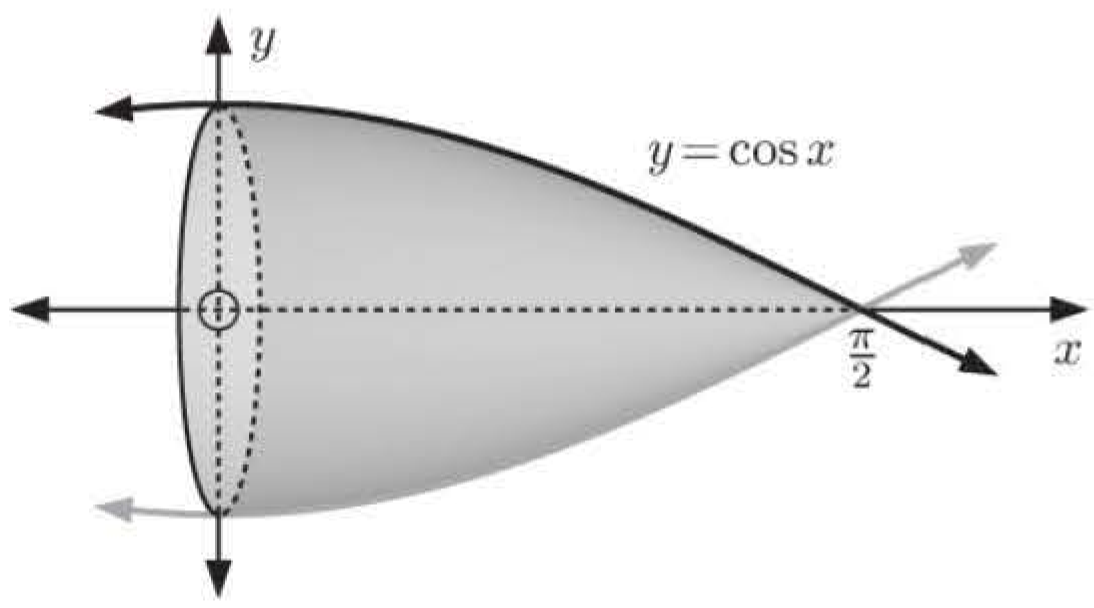
$$\begin{aligned}\therefore\ \text{volume} &= \pi\int_0^8 x^2\,dy\\ &= \pi\int_0^8 (y^{\frac{1}{3}}+1)^2\,dy\\ &= \pi\int_0^8 \left(y^{\frac{2}{3}}+2y^{\frac{1}{3}}+1\right)dy\\ &= \left[\tfrac{3}{5}y^{\frac{5}{3}}+\tfrac{3}{2}y^{\frac{4}{3}}+y\right]_0^8\\ &= \pi\left(\tfrac{3}{5}\times 32+\tfrac{3}{2}\times 16+8-0\right)\\ &= \tfrac{256}{5}\pi \text{ units}^3\end{aligned}$$

9 $\dfrac{x^2}{9}+\dfrac{y^2}{16}=1,\ x\geqslant 0 \qquad \therefore\ x^2=9\left(1-\dfrac{y^2}{16}\right)$

$$\begin{aligned}\therefore\ \text{volume} &= \pi\int_{-4}^4 x^2\,dy\\ &= \pi\int_{-4}^4 \left(9-\tfrac{9}{16}y^2\right)dy\\ &= \pi\left[9y-\tfrac{3}{16}y^3\right]_{-4}^4\\ &= \pi[(36-12)-(-36+12)]\\ &= 48\pi \text{ units}^3\end{aligned}$$

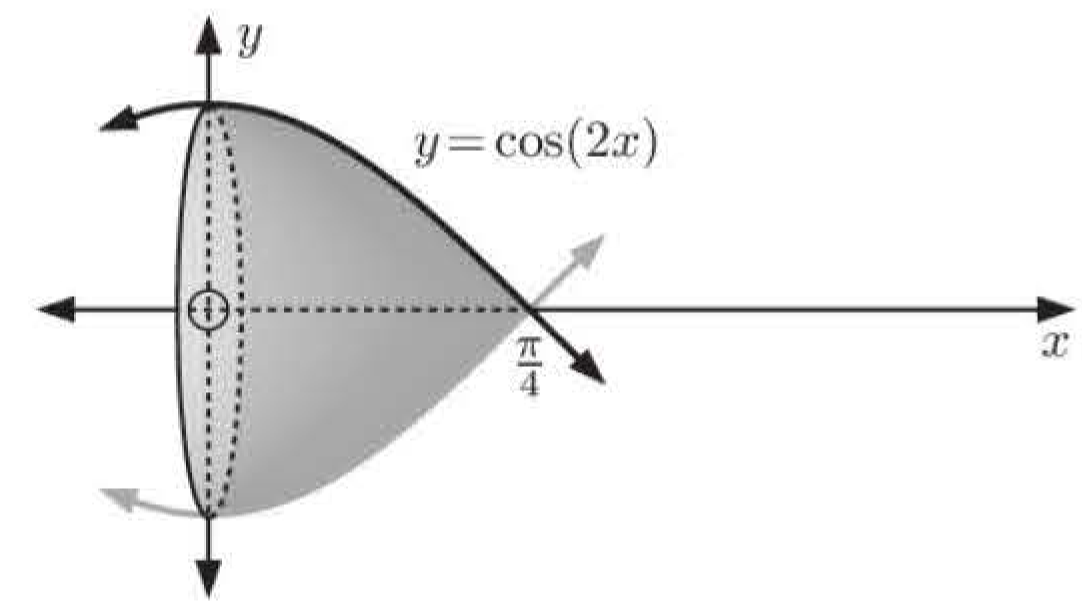


10 **a**



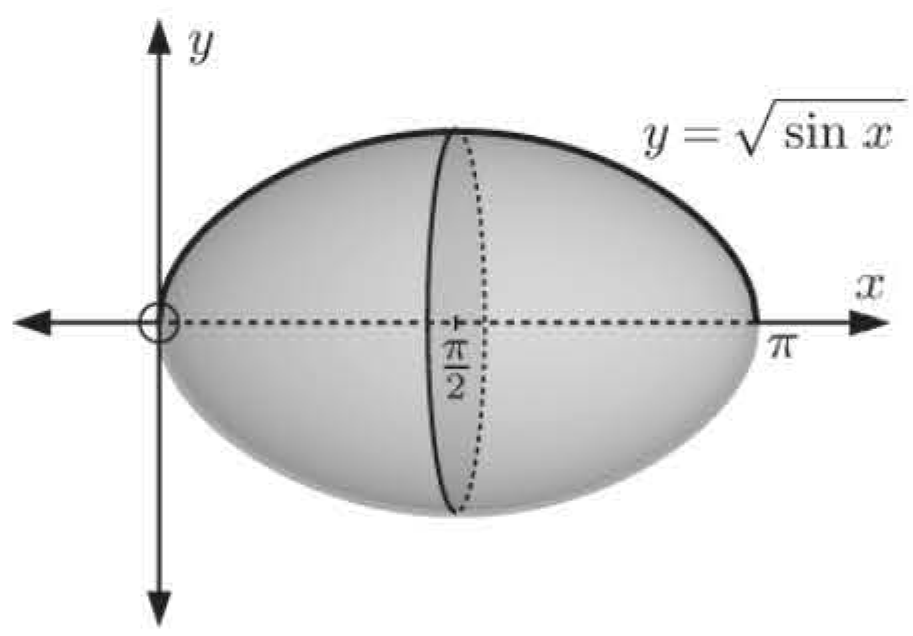
$$\begin{aligned}\text{Volume} &= \pi \int_0^{\frac{\pi}{2}} (\cos x)^2 \, dx \\ &= \pi \int_0^{\frac{\pi}{2}} \cos^2 x \, dx \\ &= \pi \int_0^{\frac{\pi}{2}} \left(\tfrac{1}{2} + \tfrac{1}{2}\cos(2x)\right) dx \\ &= \pi \left[\tfrac{1}{2}x + \tfrac{1}{2}\left(\tfrac{1}{2}\right)\sin(2x)\right]_0^{\frac{\pi}{2}} \\ &= \pi \left[\tfrac{\pi}{4} + \tfrac{1}{4}\sin\pi - 0\right] \\ &= \tfrac{\pi^2}{4} \text{ units}^3\end{aligned}$$

b



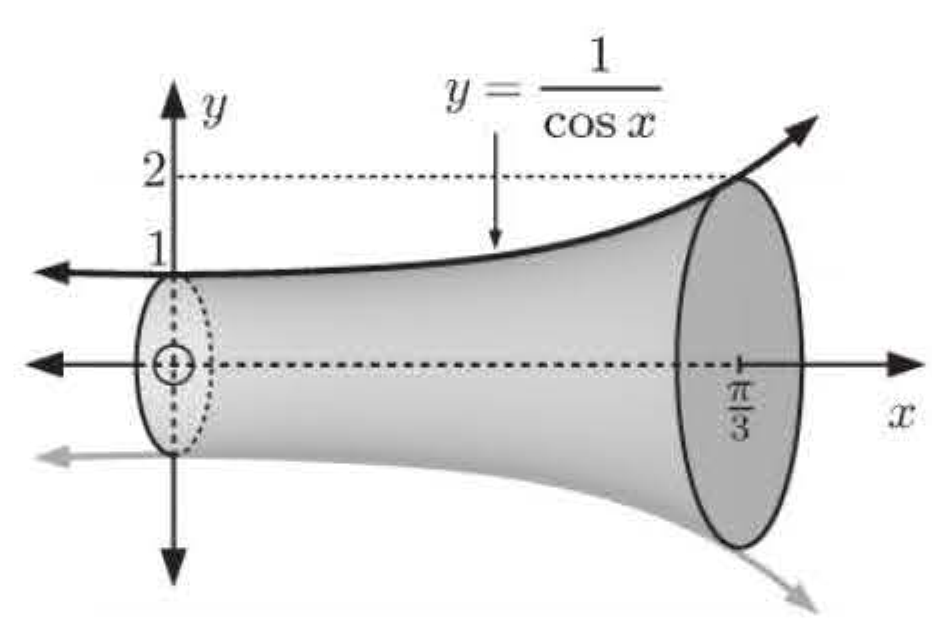
$$\begin{aligned}\text{Volume} &= \pi \int_0^{\frac{\pi}{4}} \cos^2(2x) \, dx \\ &= \pi \int_0^{\frac{\pi}{4}} \left(\tfrac{1}{2} + \tfrac{1}{2}\cos(4x)\right) dx \\ &= \pi \left[\tfrac{1}{2}x + \tfrac{1}{2}\left(\tfrac{1}{4}\right)\sin(4x)\right]_0^{\frac{\pi}{4}} \\ &= \pi \left[\tfrac{\pi}{8} + \tfrac{1}{8}\sin\pi - 0\right] \\ &= \tfrac{\pi^2}{8} \text{ units}^3\end{aligned}$$

c



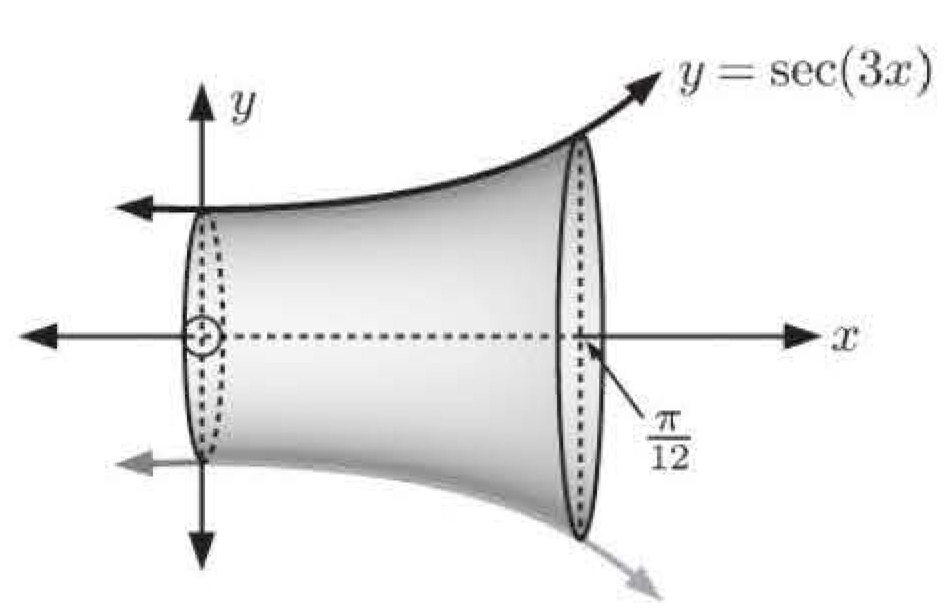
$$\begin{aligned}\text{Volume} &= \pi \int_0^{\pi} \sin x \, dx \\ &= \pi \left[-\cos x\right]_0^{\pi} \\ &= \pi \left[-\cos\pi - -\cos 0\right] \\ &= \pi(2) \\ &= 2\pi \text{ units}^3\end{aligned}$$

d



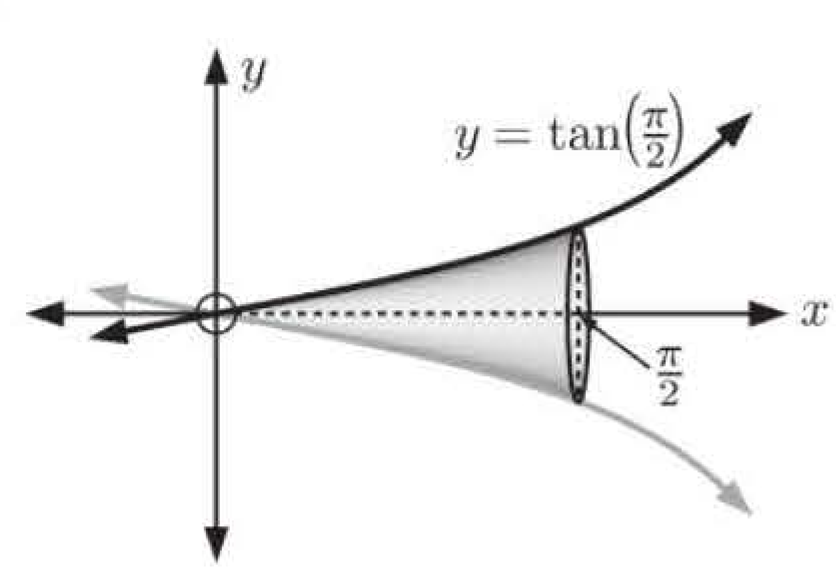
$$\begin{aligned}\text{Volume} &= \pi \int_0^{\frac{\pi}{3}} \frac{1}{\cos^2 x} \, dx \\ &= \pi \Big[\tan x\Big]_0^{\frac{\pi}{3}} \\ &= \pi \left(\tan\tfrac{\pi}{3} - \tan 0\right) \\ &= \pi(\sqrt{3} - 0) \\ &= \pi\sqrt{3} \text{ units}^3\end{aligned}$$

e



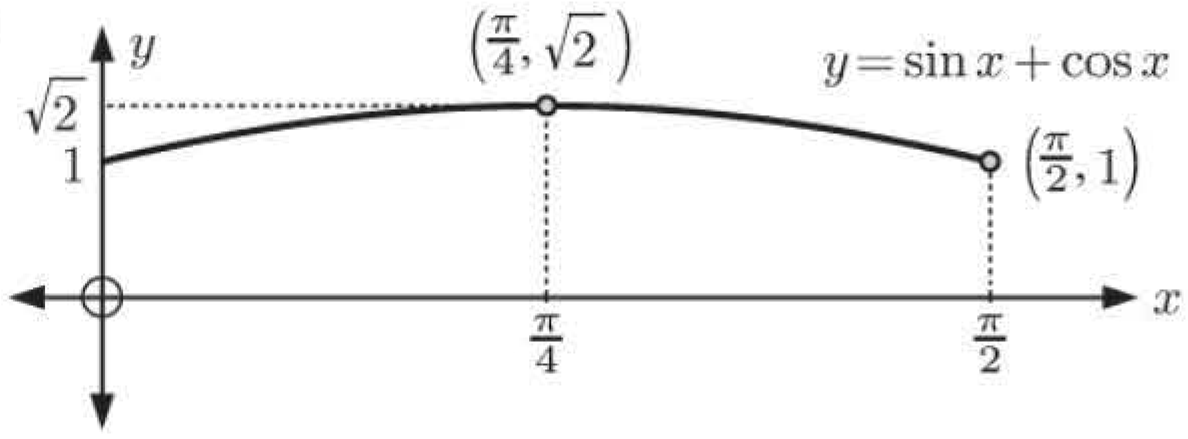
$$\begin{aligned}\text{Volume} &= \pi \int_0^{\pi\frac{\pi}{12}} \sec^2(3x) \, dx \\ &= \pi \left[\tfrac{1}{3}\tan(3x)\right]_0^{\frac{\pi}{12}} \\ &= \tfrac{\pi}{3}\left(\tan\left(\tfrac{\pi}{4}\right) - 0\right) \\ &= \tfrac{\pi}{3} \text{ units}^3\end{aligned}$$

f



$$\begin{aligned}\text{Volume} &= \pi \int_0^{\frac{\pi}{2}} \tan^2\left(\tfrac{x}{2}\right) \, dx \\ &= \pi \int_0^{\frac{\pi}{2}} \left(\sec^2\left(\tfrac{x}{2}\right) - 1\right) dx \\ &= \pi \left[2\tan\left(\tfrac{x}{2}\right) - x\right]_0^{\frac{\pi}{2}} \\ &= \pi \left(2\tan\tfrac{\pi}{4} - \tfrac{\pi}{2} - 0\right) \\ &= \pi\left(2 - \tfrac{\pi}{2}\right) \text{ units}^3\end{aligned}$$

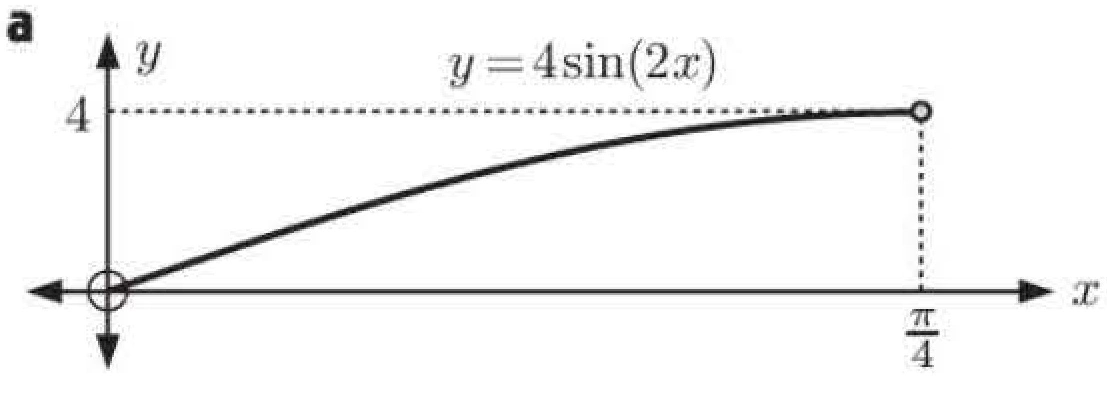
11 **a**



b Volume

$$\begin{aligned}
&= \pi \int_0^{\frac{\pi}{4}} (\sin x + \cos x)^2 \, dx \\
&= \pi \int_0^{\frac{\pi}{4}} \left(\sin^2 x + 2\sin x \cos x + \cos^2 x\right) dx \\
&= \pi \int_0^{\frac{\pi}{4}} (1 + \sin(2x)) \, dx \\
&= \pi \left[x - \tfrac{1}{2}\cos(2x)\right]_0^{\frac{\pi}{4}} \\
&= \pi \left[\left(\tfrac{\pi}{4} - \tfrac{1}{2}\cos\left(\tfrac{\pi}{2}\right)\right) - \left(0 - \tfrac{1}{2}\cos 0\right)\right] \\
&= \pi \left(\tfrac{\pi}{4} + \tfrac{1}{2}\right) \text{ units}^3
\end{aligned}$$

12 **a**



b Volume

$$\begin{aligned}
&= \pi \int_0^{\frac{\pi}{4}} (4\sin(2x))^2 \, dx \\
&= 16\pi \int_0^{\frac{\pi}{4}} \sin^2(2x) \, dx \\
&= 16\pi \int_0^{\frac{\pi}{4}} \left(\tfrac{1}{2} - \tfrac{1}{2}\cos(4x)\right) dx \\
&= 16\pi \left[\frac{x}{2} - \tfrac{1}{2}\left(\tfrac{1}{4}\right)\sin(4x)\right]_0^{\frac{\pi}{4}} \\
&= 16\pi \left[\left(\tfrac{\pi}{8} - \tfrac{1}{8}\sin\pi\right) - \left(0 - \tfrac{1}{8}\sin 0\right)\right] \\
&= 2\pi^2 \text{ units}^3
\end{aligned}$$

EXERCISE 22E.2

1 **a** The graphs meet where $4 - x^2 = 3$

$$\therefore \quad x^2 = 1$$
$$\therefore \quad x = \pm 1$$

$\therefore$ A is at $(-1, 3)$ and B is at $(1, 3)$.

b
$$\begin{aligned}
V &= \pi \int_{-1}^{1} \left((4 - x^2)^2 - 3^2\right) dx \\
&= \pi \int_{-1}^{1} \left(16 - 8x^2 + x^4 - 9\right) dx \\
&= \pi \int_{-1}^{1} \left(x^4 - 8x^2 + 7\right) dx \\
&= \pi \left[\frac{x^5}{5} - \frac{8x^3}{3} + 7x\right]_{-1}^{1} \\
&= \pi \left(\tfrac{1}{5} - \tfrac{8}{3} + 7 - \left(\tfrac{-1}{5} - \tfrac{-8}{3} - 7\right)\right) \\
&= \tfrac{136\pi}{15} \text{ units}^3
\end{aligned}$$

2 **a** The graphs meet where $e^{\frac{x}{2}} = e$

$$\therefore \quad e^{\frac{x}{2}} = e^1$$
$$\therefore \quad \frac{x}{2} = 1$$
$$\therefore \quad x = 2$$

$\therefore$ A is at $(2, e)$.

b
$$\begin{aligned}
V &= \pi \int_0^2 \left(e^2 - \left(e^{\frac{x}{2}}\right)^2\right) dx \\
&= \pi \int_0^2 \left(e^2 - e^x\right) dx \\
&= \pi \left[e^2 x - e^x\right]_0^2 \\
&= \pi \left[(2e^2 - e^2) - (0 - 1)\right] \\
&= \pi(e^2 + 1) \text{ units}^3
\end{aligned}$$

3 **a** The graphs meet where $x = \dfrac{1}{x}$

$$\therefore \quad x^2 = 1$$
$$\therefore \quad x = \pm 1$$
$$\therefore \quad x = 1 \quad \{\text{as } x > 0\}$$

$\therefore$ A is at $(1, 1)$.

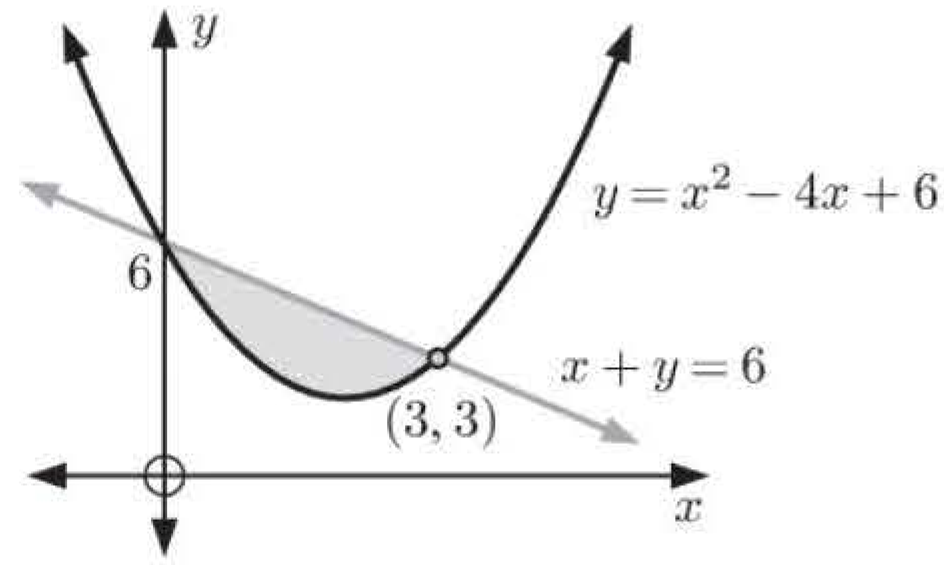
b
$$\begin{aligned}
V &= \pi \int_1^2 \left(x^2 - \left(\frac{1}{x}\right)^2\right) dx \\
&= \pi \int_1^2 \left(x^2 - x^{-2}\right) dx \\
&= \pi \left[\frac{x^3}{3} - \frac{x^{-1}}{-1}\right]_1^2 \\
&= \pi \left[\left(\tfrac{8}{3} + \tfrac{1}{2}\right) - \left(\tfrac{1}{3} + 1\right)\right] \\
&= \tfrac{11\pi}{6} \text{ units}^3
\end{aligned}$$

4 **a** The graphs meet where $x^2 - 4x + 6 = 6 - x$

$\therefore\ x^2 - 3x = 0$

$\therefore\ x(x-3) = 0$

$\therefore\ x = 0$ or 3



b $V = \pi \int_0^3 \left[(6-x)^2 - (x^2 - 4x + 6)^2\right] dx$

$= \pi \int_0^3 \left[(36 - 12x + x^2) - (x^4 - 4x^3 + 6x^2 - 4x^3 + 16x^2 - 24x + 6x^2 - 24x + 36)\right] dx$

$= \pi \int_0^3 \left(-x^4 + 8x^3 - 27x^2 + 36x\right) dx$

$= \pi \left[-\dfrac{x^5}{5} + 2x^4 - 9x^3 + 18x^2\right]_0^3$

$= \pi \left(-\frac{3^5}{5} + 2(3^4) - 9(27) + 18(9) - 0\right)$

$= \frac{162}{5}\pi$ units^3

5 **a** The curves meet where $\sqrt{x-4} = 1$

$\therefore\ x - 4 = 1$

$\therefore\ x = 5$

$\therefore$ A is at $(5, 1)$.

b $V = \pi \int_5^8 \left(\left(\sqrt{x-4}\right)^2 - 1^2\right) dx$

$= \pi \int_5^8 (x - 4 - 1)\, dx$

$= \pi \int_5^8 (x - 5)\, dx$

$= \pi \left[\dfrac{x^2}{2} - 5x\right]_5^8$

$= \pi \left[(32 - 40) - (\frac{25}{2} - 25)\right]$

$= \frac{9\pi}{2}$ units^3

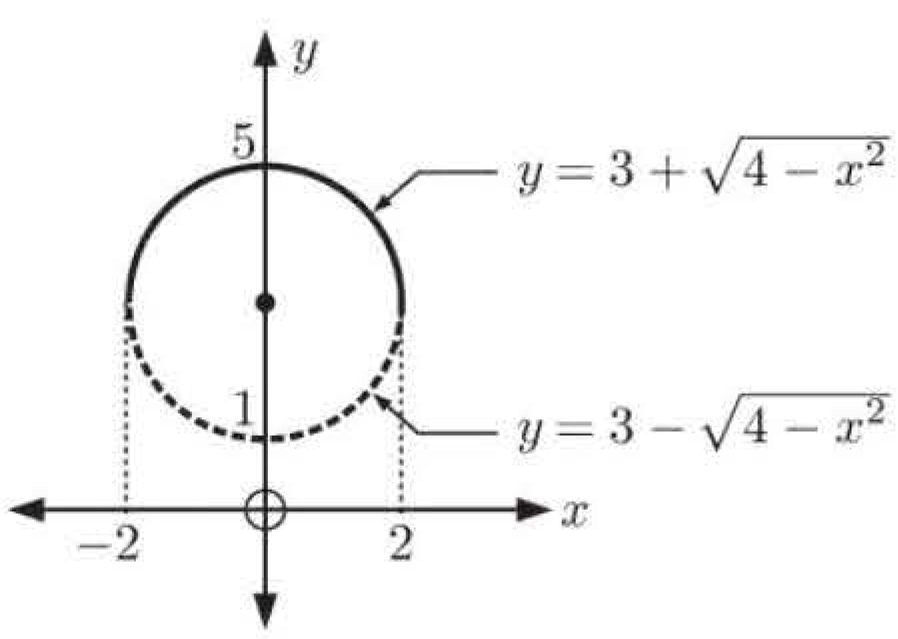
6 **a** $x^2 + (y-3)^2 = 4$

$\therefore\ (y-3)^2 = 4 - x^2$

$\therefore\ y - 3 = \pm\sqrt{4 - x^2}$

$\therefore\ y = 3 \pm \sqrt{4 - x^2}$

b



c $V = \pi \int_{-2}^{2} \left[\left(3 + \sqrt{4 - x^2}\right)^2 - \left(3 - \sqrt{4 - x^2}\right)^2\right] dx$

$= 2\pi \int_0^2 \left[\left(3 + \sqrt{4 - x^2}\right)^2 - \left(3 - \sqrt{4 - x^2}\right)^2\right] dx$

$= 2\pi \int_0^2 \left[\left(9 + 6\sqrt{4 - x^2} + 4 - x^2\right) - \left(9 - 6\sqrt{4 - x^2} + 4 - x^2\right)\right] dx$

$= 2\pi \int_0^2 12\sqrt{4 - x^2}\, dx$

$= 24\pi \int_0^2 \sqrt{4 - x^2}\, dx$

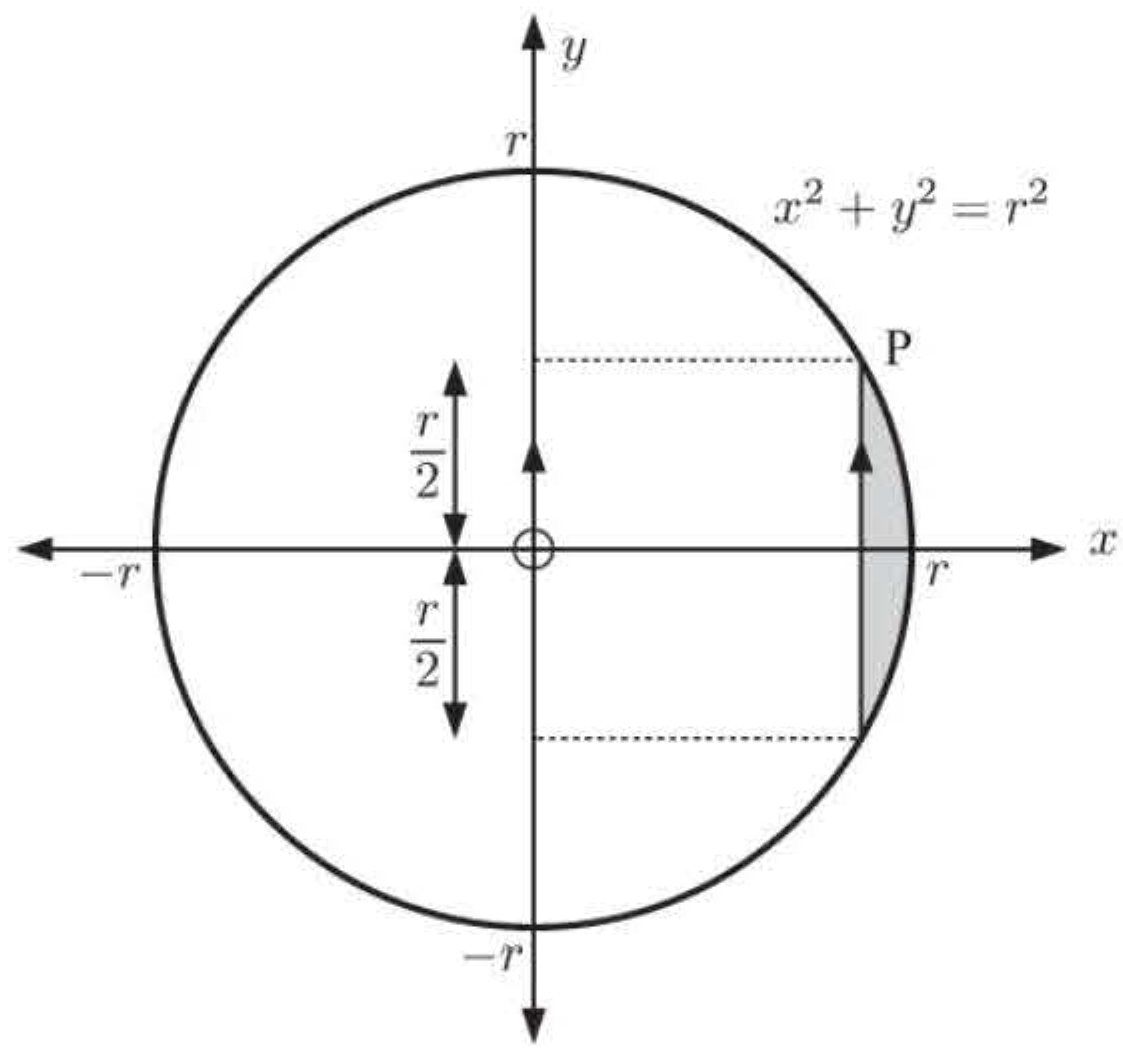
Let $x = 2\sin u$, $\dfrac{dx}{du} = 2\cos u$

when $x = 0$, $u = 0$

when $x = 2$, $u = \frac{\pi}{2}$

$$\begin{aligned}
\therefore\quad V &= 24\pi \int_0^2 \sqrt{4-(2\sin u)^2}\, dx \\
&= 24\pi \int_0^{\frac{\pi}{2}} \sqrt{4-4\sin^2 u}\, \frac{dx}{du}\, du \\
&= 48\pi \int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2 u}\, (2\cos u)\, du \\
&= 48\pi \int_0^{\frac{\pi}{2}} 2\cos^2 u\, du \qquad \{\sqrt{1-\sin^2 u} = \cos u\} \\
&= 48\pi \int_0^{\frac{\pi}{2}} (1+\cos 2u)\, du \\
&= 48\pi \left[u + \tfrac{1}{2}\sin 2u\right]_0^{\frac{\pi}{2}} \\
&= 48\pi \left(\tfrac{\pi}{2} + \tfrac{1}{2}(0) - 0\right) \\
&= 24\pi^2 \text{ units}^3 \quad (\approx 237 \text{ units}^3)
\end{aligned}$$

7 Since the chord is parallel to the y-axis, the y-coordinate of P is $\dfrac{r}{2}$.



When $\quad y = \dfrac{r}{2},\quad x^2 + \left(\dfrac{r}{2}\right)^2 = r^2$

$$\therefore\quad x^2 = \tfrac{3}{4}r^2$$

$$\therefore\quad x = \pm\tfrac{\sqrt{3}}{2}r$$

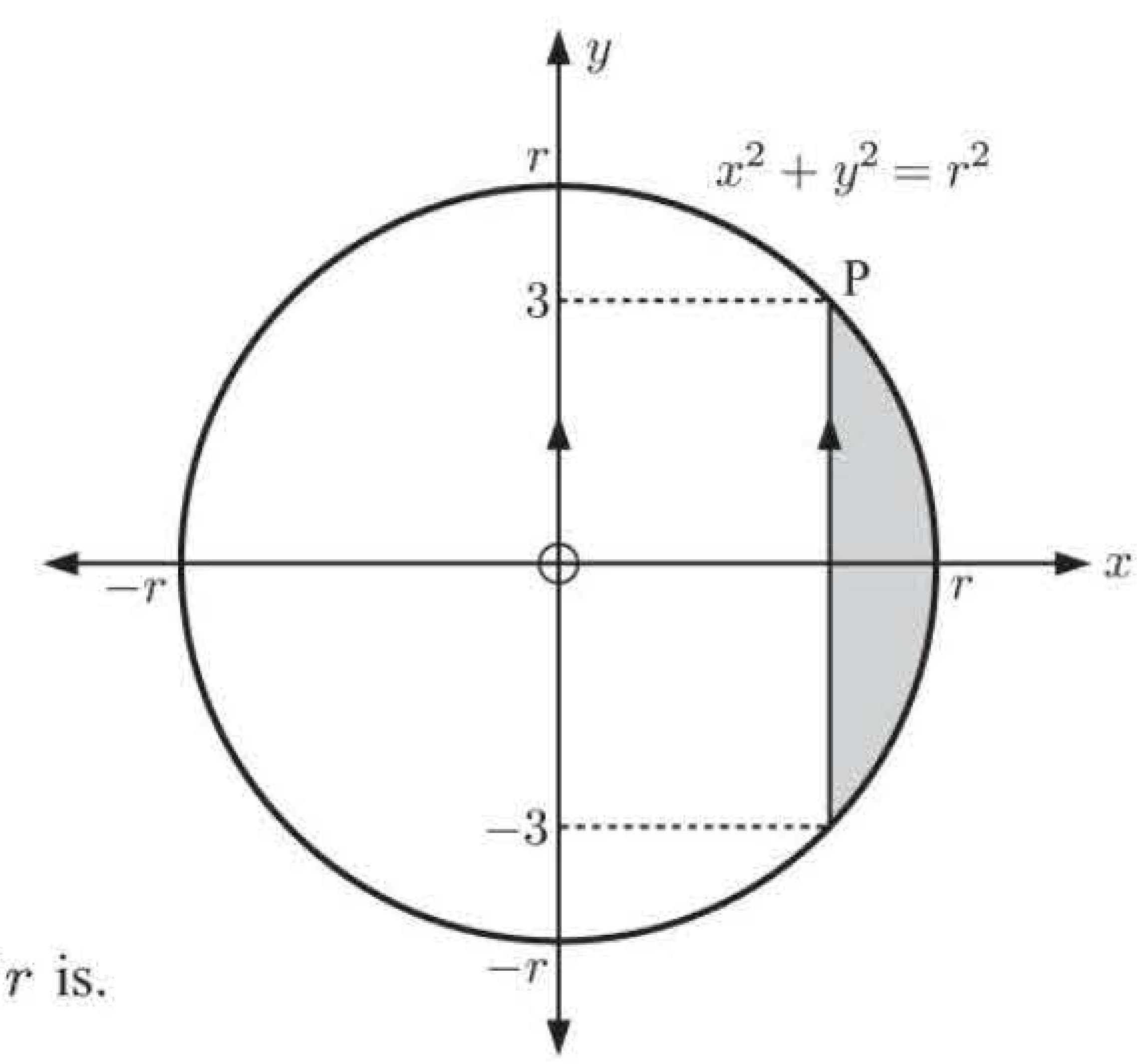
$\therefore$ the coordinates of P are $\left(\frac{\sqrt{3}}{2}r,\ \dfrac{r}{2}\right)$.

$$\begin{aligned}
\therefore\quad V &= 2\pi \int_0^{\frac{r}{2}} \left[(r^2-y^2) - \tfrac{3}{4}r^2\right] dy \\
&= 2\pi \left[r^2 y - \frac{y^3}{3} - \tfrac{3}{4}r^2 y\right]_0^{\frac{r}{2}} \\
&= 2\pi \left(\frac{r^3}{2} - \frac{r^3}{24} - \frac{3r^3}{8} - 0\right) \\
&= 2\pi \left(\frac{r^3}{12}\right) \\
&= \frac{\pi r^3}{6} \text{ units}^3
\end{aligned}$$

8 When $\quad y = 3,\quad x^2 + 3^2 = r^2$

$$\therefore\quad x^2 = r^2 - 9$$

$$\therefore\quad x = \pm\sqrt{r^2-9}$$



$\therefore$ the coordinates of P are $(\sqrt{r^2-9},\ 3)$.

$$\begin{aligned}
\therefore\quad V &= 2\pi \int_0^3 \left[(r^2-y^2) - (r^2-9)\right] dy \\
&= 2\pi \int_0^3 (9-y^2)\, dy \\
&= 2\pi \left[9y - \frac{y^3}{3}\right]_0^3 \\
&= 2\pi(27-9-0) \\
&= 36\pi \text{ units}^3, \text{ no matter what the value of } r \text{ is.}
\end{aligned}$$

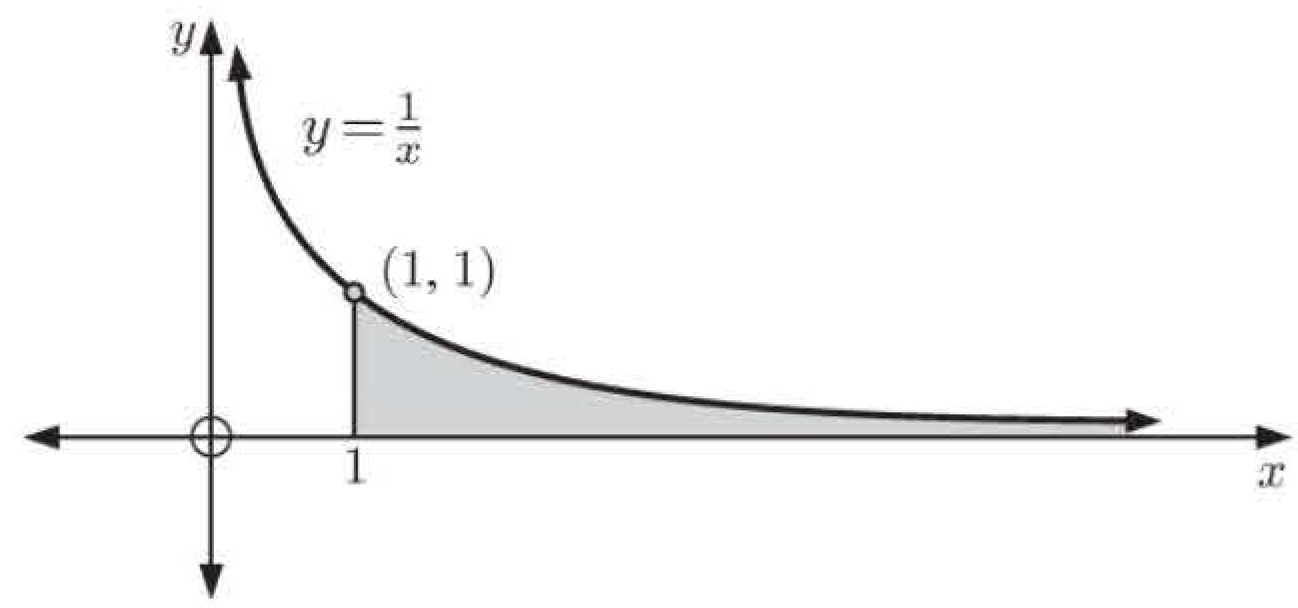
$\therefore$ the volume is independent of r.

9 The shaded area $= \int_1^\infty \frac{1}{x}\, dx$

$= \lim_{t\to\infty} \int_1^t \frac{1}{x}\, dx$

$= \lim_{t\to\infty} [\ln(x)]_1^t,\ x > 0$

$= \lim_{t\to\infty} \ln t,$ which is infinite



The volume of revolution $= \pi \int_1^\infty \left(\frac{1}{x}\right)^2 dx$

$= \pi \lim_{t\to\infty} \int_1^t x^{-2}\, dx$

$= \pi \lim_{t\to\infty} \left[-\frac{1}{x}\right]_1^t$

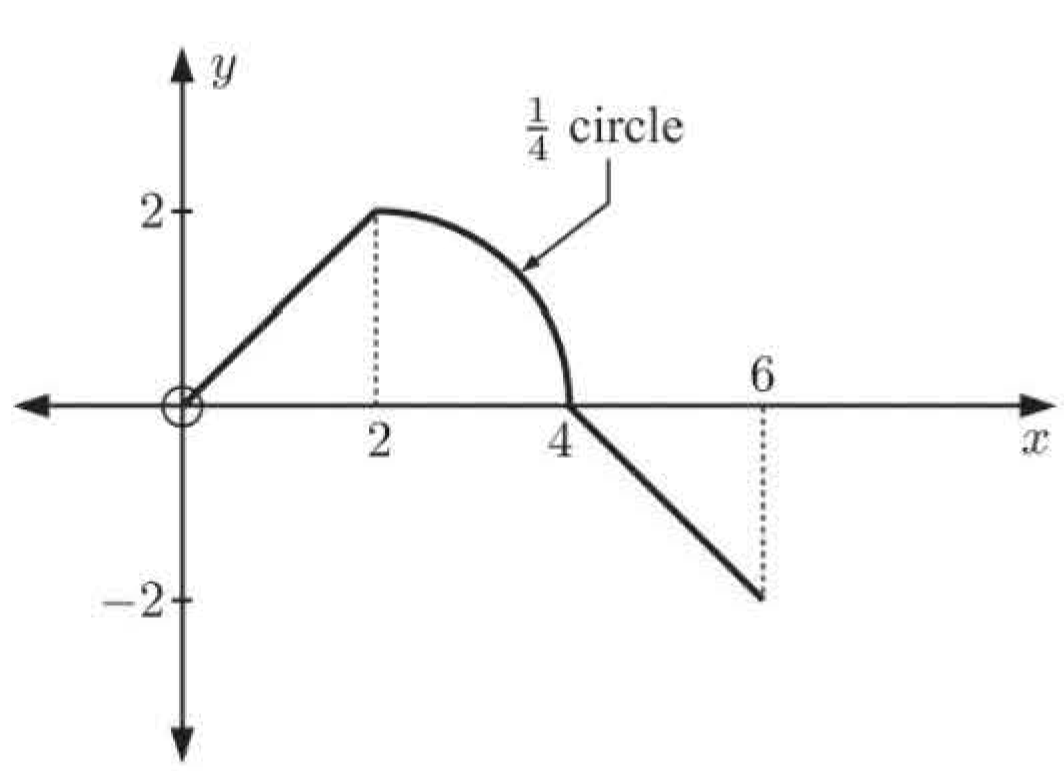
$= \pi \lim_{t\to\infty} \left(-\frac{1}{t} + 1\right)$

$= \pi,$ which is finite

REVIEW SET 22A

1 shaded area $= \int_a^b [f(x) - g(x)]\ dx + \int_b^c [g(x) - f(x)]\ dx + \int_c^d [f(x) - g(x)]\ dx$

2



a $\int_0^4 f(x)\ dx =$ area of triangle + area of $\frac{1}{4}$ circle

$= \frac{1}{2}(2\times 2) + \frac{1}{4}\pi(2)^2$

$= (2+\pi)$ units^2

b $\int_4^6 f(x)\ dx = -$area of triangle below x-axis

$= -\frac{1}{2}(2\times 2)$

$= -2$ units^2

c $\int_0^6 f(x)\ dx = \int_0^4 f(x)\ dx + \int_4^6 f(x)\ dx$

$= (2+\pi) + (-2)$

$= \pi$ units^2

3 $\int_{-1}^3 f(x)\ dx$ gives us the correct area only if $f(x)$ is non-negative on the interval $-1 \leqslant x \leqslant 3$.

In this case $f(x)$ is negative for $1 < x < 3$, so $\int_{-1}^3 f(x)\ dx$ does not provide the correct answer.

(The shaded area which is below the x-axis is given by $\int_1^3 [0 - f(x)]\ dx = -\int_1^3 f(x)\ dx$.

$\therefore$ total area shaded $= \int_{-1}^1 f(x)\ dx - \int_1^3 f(x)\ dx$)

4 $y = k$ meets $y = x^2$ where $x^2 = k$ $\quad\therefore\ x = \pm\sqrt{k}$

By symmetry, $\int_0^{\sqrt{k}} (k - x^2)\ dx = \frac{1}{2}\times 5\frac{1}{3} = \frac{1}{2}\times\frac{16}{3}$

$\therefore \left[kx - \frac{x^3}{3}\right]_0^{\sqrt{k}} = \frac{8}{3}$

$\therefore k\sqrt{k} - \frac{k\sqrt{k}}{3} = \frac{8}{3}$

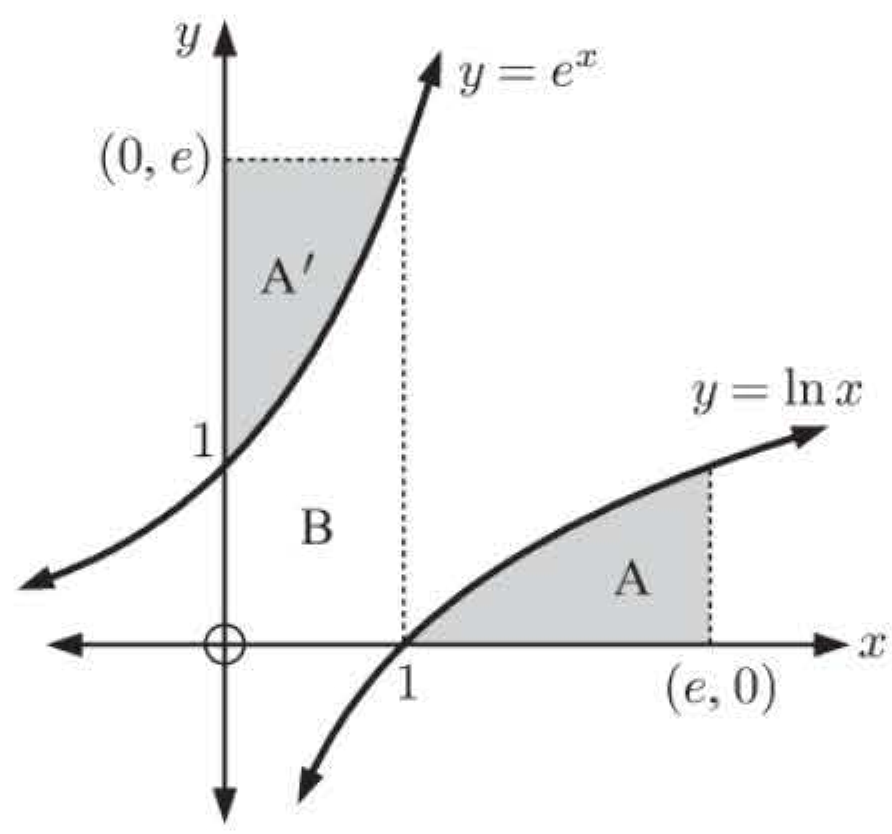
$\therefore \frac{2}{3}k\sqrt{k} = \frac{8}{3}$

$\therefore k\sqrt{k} = 4$

$\therefore k^{\frac{3}{2}} = 4$

$\therefore k = 4^{\frac{2}{3}} = \sqrt[3]{16}$

5



$y = e^x$ and $y = \ln x$ are inverse functions, so they are symmetrical about $y = x$

$\therefore$ area A = area A′

But area A′ + area B = area of rectangle

$\therefore$ area A + area B $= e \times 1 = e$

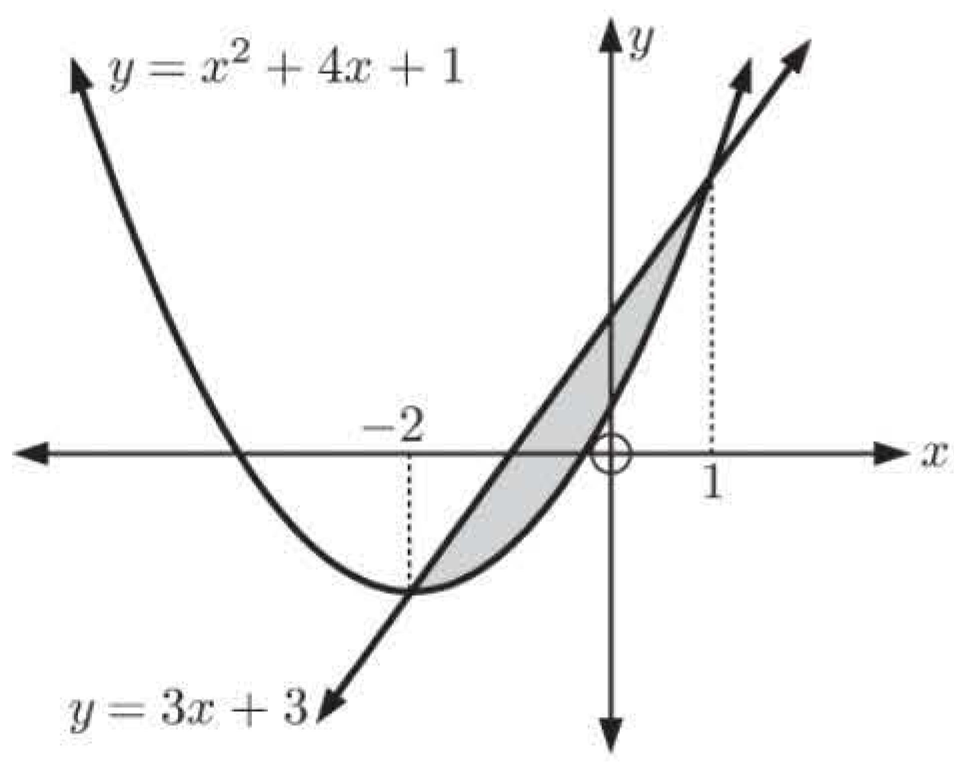
Since area A $= \int_1^e \ln x \, dx$

and area B $= \int_0^1 e^x \, dx$,

$$\int_0^1 e^x \, dx + \int_1^e \ln x \, dx = e$$

6 $y = x^2 + 4x + 1$ meets $y = 3x + 3$ where

$$x^2 + 4x + 1 = 3x + 3$$
$$\therefore \quad x^2 + x - 2 = 0$$
$$\therefore \quad (x+2)(x-1) = 0$$
$$\therefore \quad x = -2 \text{ or } 1$$



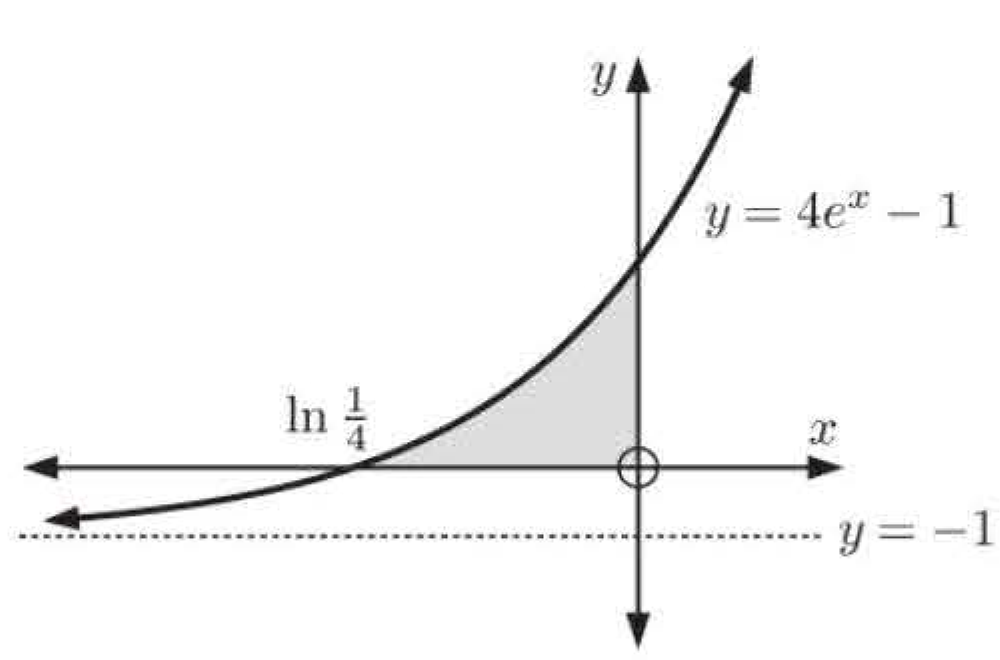
$$\begin{aligned}
\therefore \quad \text{area} &= \int_{-2}^{1} [(3x+3) - (x^2+4x+1)] \, dx \\
&= \int_{-2}^{1} (-x^2 - x + 2) \, dx \\
&= \left[-\frac{x^3}{3} - \frac{x^2}{2} + 2x \right]_{-2}^{1} \\
&= \left(-\tfrac{1}{3} - \tfrac{1}{2} + 2\right) - \left(\tfrac{8}{3} - 2 - 4\right) \\
&= -\tfrac{1}{3} - \tfrac{1}{2} + 2 - \tfrac{8}{3} + 2 + 4 \\
&= 4\tfrac{1}{2} \text{ units}^2
\end{aligned}$$

7 Consider $y = 4e^x - 1$.

The x-intercept occurs when $y = 0$

$$\therefore \quad 4e^x - 1 = 0$$
$$\therefore \quad e^x = \tfrac{1}{4}$$
$$\therefore \quad x = \ln \tfrac{1}{4} < 0$$

$y = 4e^x - 1$ is the graph of $y = e^x$ with a vertical stretch of factor 4 and a vertical translation of -1.

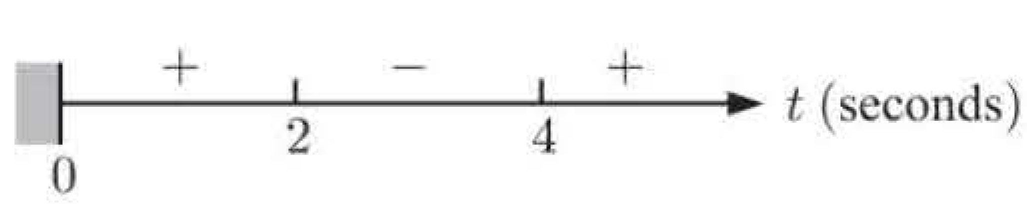


$$\begin{aligned}
\text{Area} &= \int_{\ln \frac{1}{4}}^{0} (4e^x - 1) \, dx \\
&= [4e^x - x]_{\ln \frac{1}{4}}^{0} \\
&= \left(4e^0 - 0\right) - \left(4e^{\ln \frac{1}{4}} - \ln \tfrac{1}{4}\right) \\
&= 4 - 0 - 4\left(\tfrac{1}{4}\right) + \ln \tfrac{1}{4} \\
&= 3 + \ln \tfrac{1}{4} \text{ units}^2 \\
&= (3 - \ln 4) \text{ units}^2
\end{aligned}$$

8 **a** $v(t) = t^2 - 6t + 8 \text{ m s}^{-1}, \quad t \geqslant 0$

$= (t-4)(t-2)$

which has sign diagram:



b Now $s(t) = \int (t^2 - 6t + 8)\, dt$

$$= \frac{t^3}{3} - 3t^2 + 8t + c$$

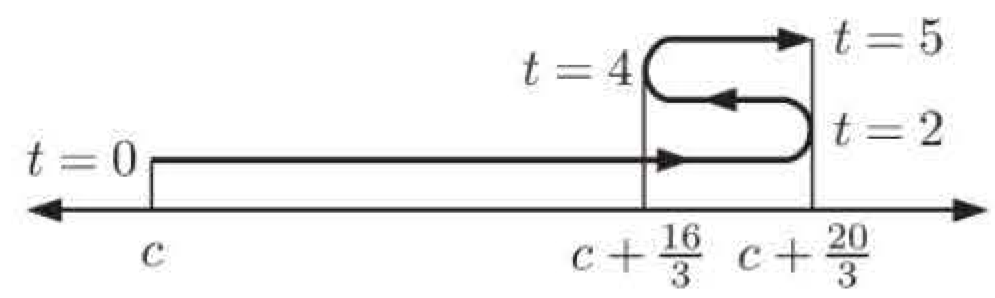
$\therefore\ s(0) = c$

$s(2) = c + 6\frac{2}{3}$

$s(4) = c + 5\frac{1}{3}$

$s(5) = c + 6\frac{2}{3}$

the motion diagram is:



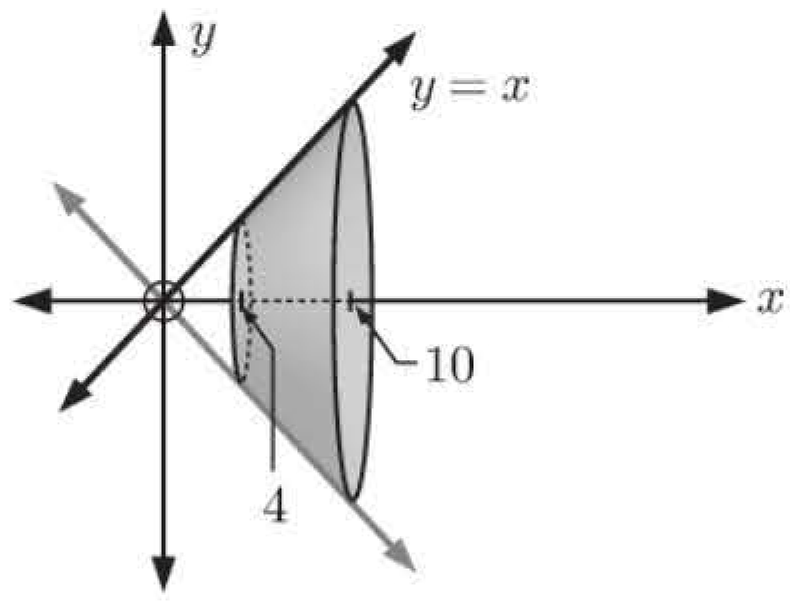
The particle moves in the positive direction initially, then at $t = 2$, $6\frac{2}{3}$ m from its starting point, it changes direction. It changes direction again at $t = 4$, $5\frac{1}{3}$ m from its starting point. When $t = 5$ it is $6\frac{2}{3}$ m from its starting point.

c After 5 seconds, the particle is $6\frac{2}{3}$ m to the right of its starting point.

d The total distance travelled $= (c + \frac{20}{3} - c) + [(c + \frac{20}{3}) - (c + \frac{16}{3})] + [(c + \frac{20}{3}) - (c + \frac{16}{3})]$

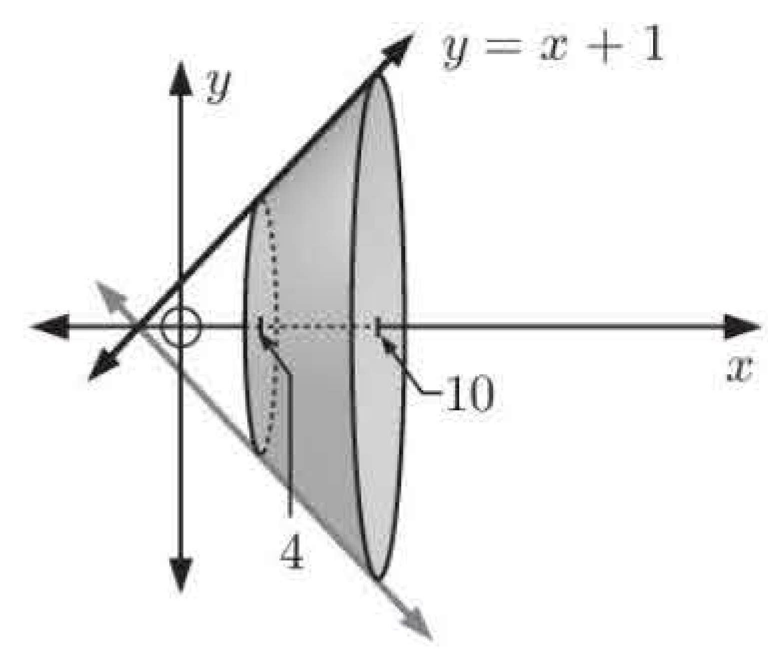
$$= 9\tfrac{1}{3} \text{ m}$$

9 **a**



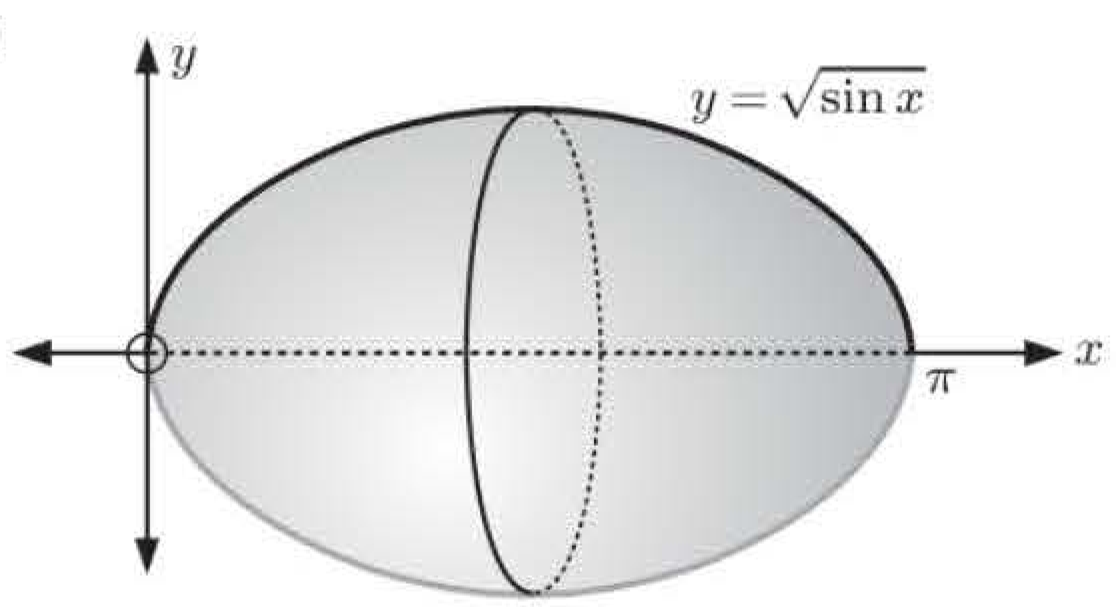
$$V = \pi \int_4^{10} x^2\, dx$$
$$= \pi \left[\frac{x^3}{3}\right]_4^{10}$$
$$= \pi \left(\frac{1000}{3} - \frac{64}{3}\right)$$
$$= \frac{936\pi}{3}$$
$$= 312\pi \text{ units}^3$$

b



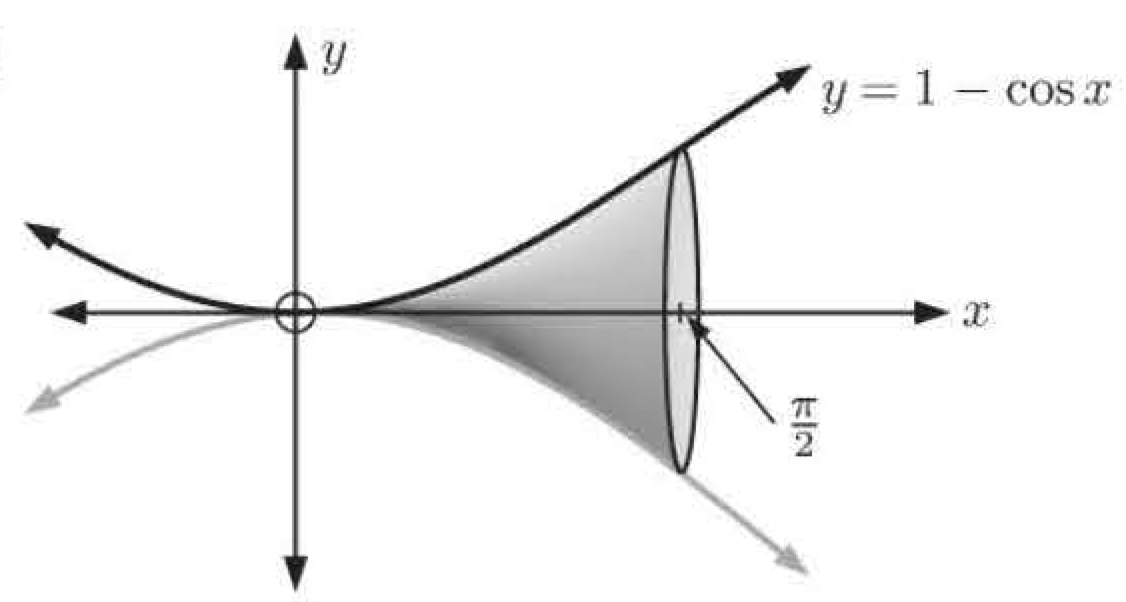
$$V = \pi \int_4^{10} (x+1)^2\, dx$$
$$= \pi \left[\frac{(x+1)^3}{3}\right]_4^{10}$$
$$= \pi \left(\frac{11^3}{3} - \frac{5^3}{3}\right)$$
$$= \frac{1206\pi}{3} = 402\pi \text{ units}^3$$

c



$$V = \pi \int_0^{\pi} \left(\sqrt{\sin x}\right)^2\, dx$$
$$= \pi \int_0^{\pi} \sin x\, dx$$
$$= \pi \left[-\cos x\right]_0^{\pi}$$
$$= \pi (1 - (-1))$$
$$= 2\pi \text{ units}^3$$

d



$$V = \pi \int_0^{\frac{\pi}{2}} (1 - \cos x)^2\, dx$$
$$= \pi \int_0^{\frac{\pi}{2}} (1 - 2\cos x + \cos^2 x)\, dx$$
$$= \pi \int_0^{\frac{\pi}{2}} (1 - 2\cos x + \tfrac{1}{2} + \tfrac{1}{2}\cos 2x)\, dx$$
$$= \pi \left[\tfrac{3}{2}x - 2\sin x + \tfrac{1}{4}\sin 2x\right]_0^{\frac{\pi}{2}}$$
$$= \pi \left[\left(\tfrac{3}{2}(\tfrac{\pi}{2}) - 2\sin(\tfrac{\pi}{2}) + \tfrac{1}{4}\sin\pi\right) - \left(\tfrac{3}{2}(0) - 2\sin 0 + \tfrac{1}{4}\sin 0\right)\right]$$
$$= \pi \left(\tfrac{3\pi}{4} - 2\right) \text{ units}^3$$
$$= \pi \left(\tfrac{3\pi - 8}{4}\right) \text{ units}^3$$

10 $\dfrac{d^2y}{dx^2} = k(L-x)^2$

$\therefore \quad \dfrac{dy}{dx} = \displaystyle\int k(L-x)^2\,dx = \dfrac{-k(L-x)^3}{3} + c$

But when $x=0$ the tangent is horizontal and so $\dfrac{dy}{dx} = 0.$

$\therefore \quad \dfrac{-kL^3}{3} + c = 0$ and so $c = \dfrac{kL^3}{3}$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{-k(L-x)^3}{3} + \dfrac{kL^3}{3}$

$$\therefore \quad y = \int\left(\frac{-k(L-x)^3}{3} + \frac{kL^3}{3}\right)dx$$

$$= \frac{k(L-x)^4}{12} + \frac{kL^3}{3}x + d$$

But when $x=0,\ y=0 \quad \therefore \quad \dfrac{kL^4}{12} + d = 0$

$$\therefore \quad d = -\frac{kL^4}{12}$$

$$\therefore \quad y = \frac{k(L-x)^4}{12} + \frac{kL^3x}{3} - \frac{kL^4}{12}$$

The greatest deflection occurs when $x \approx L$

$\therefore \quad y \approx \dfrac{k(0)^4}{12} + \dfrac{kL^4}{3} - \dfrac{kL^4}{12} = \dfrac{kL^4}{4}$

$\therefore$ the greatest deflection is about $\dfrac{kL^4}{4}$ metres.

11 **a** The shaded area $= \int_0^2 ax(x-2)\,dx$

$= 4$ units^2

$\therefore \quad \int_0^2 (ax^2 - 2ax)\,dx = 4$

$\therefore \quad \left[\dfrac{ax^3}{3} - ax^2\right]_0^2 = 4$

$\therefore \quad \left(\dfrac{8a}{3} - 4a\right) - 0 = 4$

$\therefore \quad \dfrac{8a}{3} - \dfrac{12a}{3} = 4$

$\therefore \quad -\dfrac{4a}{3} = 4$

$\therefore \quad a = -3$

$\therefore \quad y = -3x(x-2)$

b Suppose A has coordinates $(k, -3k(k-2))$.

$\therefore$ gradient of [OA] $= \dfrac{-3k(k-2) - 0}{k - 0}$

$= -3(k-2)$

$\therefore$ equation of [OA] is $y = -3(k-2)x$

If [OA] divides the shaded region into equal areas,

$\int_0^k [-3x(x-2) - (-3(k-2)x)]\,dx = 2$

$\therefore \quad \int_0^k (-3x^2 + 6x + 3kx - 6x)\,dx = 2$

$\therefore \quad \int_0^k (-3x^2 + 3kx)\,dx = 2$

$\therefore \quad \left[-x^3 + \dfrac{3kx^2}{2}\right]_0^k = 2$

$\therefore \quad -k^3 + \dfrac{3k^3}{2} = 2$

$\therefore \quad \dfrac{k^3}{2} = 2$

$\therefore \quad k^3 = 4$

$\therefore \quad k = \sqrt[3]{4}$

$\therefore$ the x-coordinate of A is $\sqrt[3]{4}$.

12 **a** A is the *upper half* of a circle centre $(2, 0)$ and radius 2.

$$\therefore \quad (x-2)^2 + (y-0)^2 = 2^2$$
$$(x-2)^2 + y^2 = 4$$
$$y^2 = 4 - (x-2)^2$$
$$y^2 = 4 - x^2 + 4x - 4$$
$$y^2 = 4x - x^2$$
$$\therefore \quad y = \pm\sqrt{4x - x^2}$$

So, $y = \sqrt{4x - x^2}$ 

or $y = -\sqrt{4x - x^2}$ 

Since A is the upper half of the circle, $y_A = \sqrt{4x - x^2}$

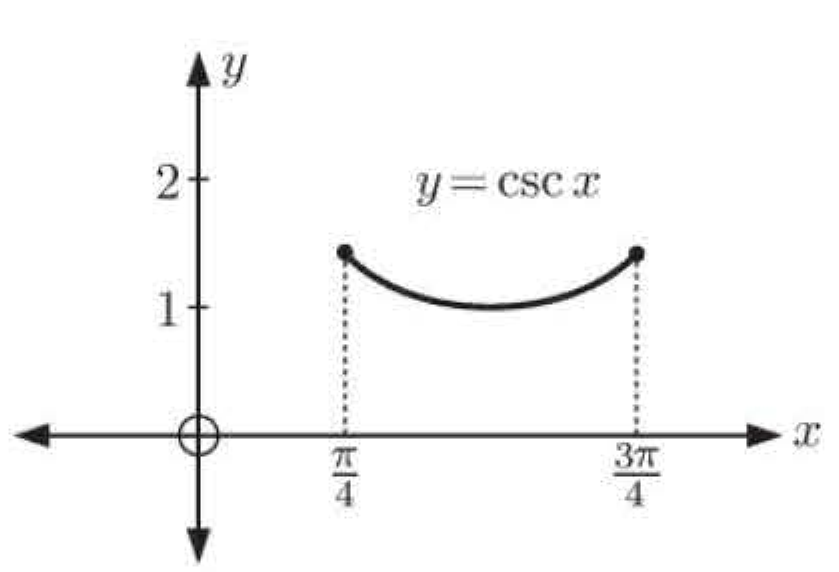
b Now B is the *lower half* of a circle centre $(5, 0)$ and radius 1.

$$\therefore \quad (x-5)^2 + (y-0)^2 = 1^2$$
$$(x-5)^2 + y^2 = 1$$
$$y^2 = 1 - (x-5)^2$$
$$y^2 = 1 - x^2 + 10x - 25$$
$$\therefore \quad y = \pm\sqrt{10x - x^2 - 24}$$
$$\therefore \quad y_B = -\sqrt{10x - x^2 - 24}$$

c
$$\int_0^4 y_A\, dx$$
$$= \tfrac{1}{2}\pi r^2 \text{ where } r = 2$$
$$= \tfrac{1}{2}\pi (2)^2$$
$$= 2\pi$$

$$\int_4^6 y_B\, dx$$
$$= -\tfrac{1}{2}\pi r^2 \text{ where } r = 1$$
$$= -\tfrac{1}{2}\pi (1)^2$$
$$= -\tfrac{\pi}{2}$$

d
$$\int_0^6 f(x)\, dx$$
$$= \int_0^4 y_A\, dx + \int_4^6 y_B\, dx$$
$$= 2\pi + (-\tfrac{\pi}{2})$$
$$= \tfrac{3\pi}{2}$$

13 

$$V = \pi \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \csc^2 x\, dx$$
$$= \pi \left[-\cot x\right]_{\frac{\pi}{4}}^{\frac{3\pi}{4}}$$
$$= \pi \left(-\cot\left(\tfrac{3\pi}{4}\right) - -\cot\left(\tfrac{\pi}{4}\right)\right)$$
$$= \pi\left(-(-1) + 1\right)$$
$$= 2\pi \text{ units}^3$$

14
$$\frac{dT}{dx} = \frac{k}{x} = kx^{-1}$$
$$\therefore \quad T = k\ln x + c \quad \{x > 0\}$$

When $x = r_1,\ T = T_0$

$$\therefore \quad k\ln r_1 + c = T_0$$
$$\therefore \quad c = T_0 - k\ln r_1$$
$$\therefore \quad T = k\ln x + T_0 - k\ln r_1$$
$$= T_0 + k\ln\left(\frac{x}{r_1}\right)$$

So, when $x = r_2$,

$$T = T_0 + k\ln\left(\frac{r_2}{r_1}\right)$$

$\therefore$ the outer surface has temperature $T_0 + k\ln\left(\dfrac{r_2}{r_1}\right)$.

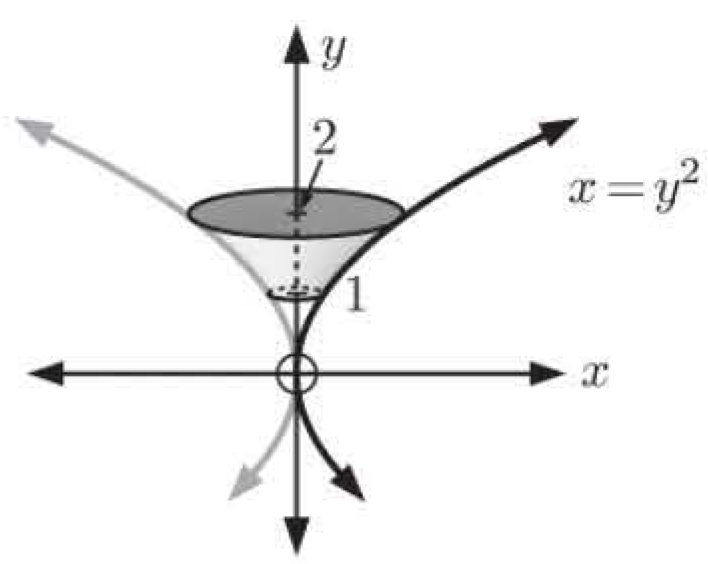
15 The gradient of the straight line is $\dfrac{0-8}{4-0} = \frac{-8}{4} = -2$

$\therefore$ the straight line has equation $y = -2x + 8$

$\therefore$ the volume of revolution

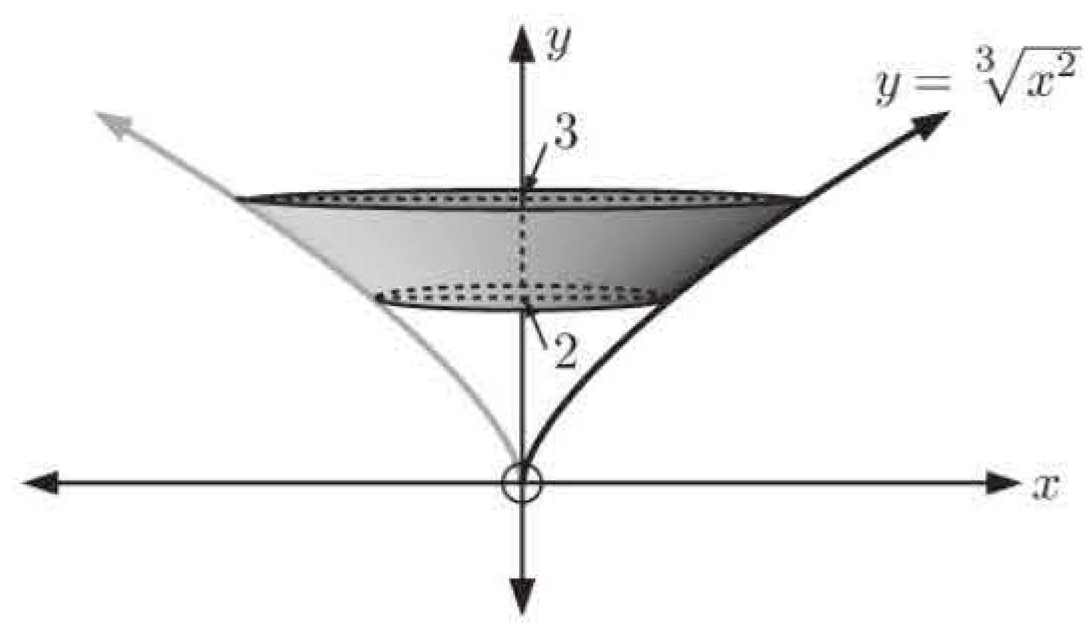
$$\begin{aligned}
&= \pi \int_0^2 \left(x^2\right)^2 \, dx + \pi \int_2^4 (-2x+8)^2 \, dx \\
&= \pi \int_0^2 x^4 \, dx + \pi \int_2^4 (4x^2 - 32x + 64) \, dx \\
&= \pi \left[\tfrac{1}{5}x^5\right]_0^2 + \pi \left[\tfrac{4}{3}x^3 - 16x^2 + 64x\right]_2^4 \\
&= \pi \left(\tfrac{1}{5}(2)^5 - 0\right) + \pi \left[\tfrac{4}{3}(4)^3 - 16(4)^2 + 64(4) - \left(\tfrac{4}{3}(2)^3 - 16(2)^2 + 64(2)\right)\right] \\
&= \pi \times \tfrac{32}{5} + \pi \left(\tfrac{256}{3} - \tfrac{224}{3}\right) \\
&= \tfrac{32\pi}{5} + \tfrac{32\pi}{3} \\
&= \tfrac{256\pi}{15} \quad \text{as required}
\end{aligned}$$

16 **a**



$$\begin{aligned}
V &= \pi \int_1^2 x^2 \, dy \\
&= \pi \int_1^2 y^4 \, dy \\
&= \pi \left[\frac{y^5}{5}\right]_1^2 \\
&= \pi \left(\tfrac{32}{5} - \tfrac{1}{5}\right) \\
&= \tfrac{31\pi}{5} \text{ units}^3
\end{aligned}$$

b



$y = \sqrt[3]{x^2} \quad \therefore \; x^2 = y^3$

$$\begin{aligned}
V &= \pi \int_2^3 x^2 \, dy \\
&= \pi \int_2^3 y^3 \, dy \\
&= \pi \left[\frac{y^4}{4}\right]_2^3 \\
&= \pi \left(\tfrac{81}{4} - \tfrac{16}{4}\right) \\
&= \tfrac{65\pi}{4} \text{ units}^3
\end{aligned}$$

REVIEW SET 22B

1 **a** $a(t) = v'(t)$

$\therefore \; a(t) = 2 - 6t \;\; \text{m s}^{-2}$

b $s(t) = \int (2t - 3t^2) \, dt$

$\therefore \; s(t) = t^2 - t^3 + c \;\; \text{m}$

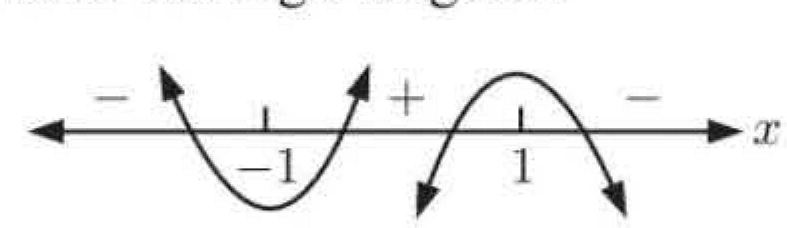
c Change in displacement after two seconds

$$\begin{aligned}
&= s(2) - s(0) \\
&= 2^2 - 2^3 + c - (0^2 - 0^3 + c) \\
&= 4 - 8 + c - c \\
&= -4 \text{ m} \quad (4 \text{ m to the left})
\end{aligned}$$

2 **a** $f(x) = \dfrac{x}{1+x^2} \quad \therefore \; f'(x) = \dfrac{1(1+x^2) - x(2x)}{(1+x^2)^2}$ {quotient rule}

$$\begin{aligned}
&= \frac{1 + x^2 - 2x^2}{(1+x^2)^2} \\
&= \frac{1 - x^2}{(1+x^2)^2} \\
&= \frac{(1+x)(1-x)}{(1+x^2)^2}
\end{aligned}$$

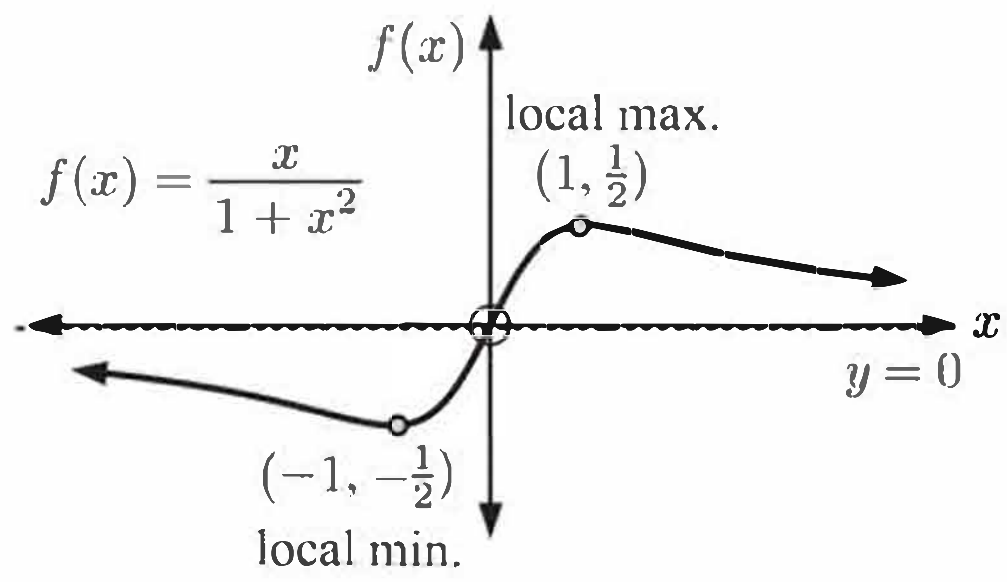
which has sign diagram:



$\therefore$ there is a local minimum at $\left(-1, -\frac{1}{2}\right)$ and a local maximum at $\left(1, \frac{1}{2}\right)$.

b As $x \to \infty$, $f(x) \to 0^+$.
As $x \to -\infty$, $f(x) \to 0^-$.

c



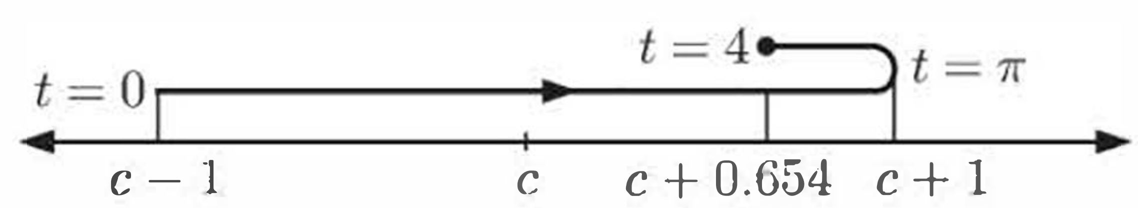
d
$$\text{Area} = \int_{-2}^{0} \left[0 - \frac{x}{1+x^2}\right] dx$$
$$= \int_{-2}^{0} \frac{-x}{1+x^2}\, dx$$
$$\approx 0.805 \text{ units}^2$$

3 $v(t) = \sin t$ which has sign diagram:

Now $s(t) = \int \sin t\, dt$
$= -\cos t + c$ metres

$\therefore$ $s(0) = -1 + c$
$s(\pi) = 1 + c$
$s(4) = -\cos 4 + c \approx c + 0.654$

motion diagram:



$\therefore$ total distance travelled $= [(c+1) - (c-1)] + [(c+1) - (c+0.654)]$
≈ 2.35 m

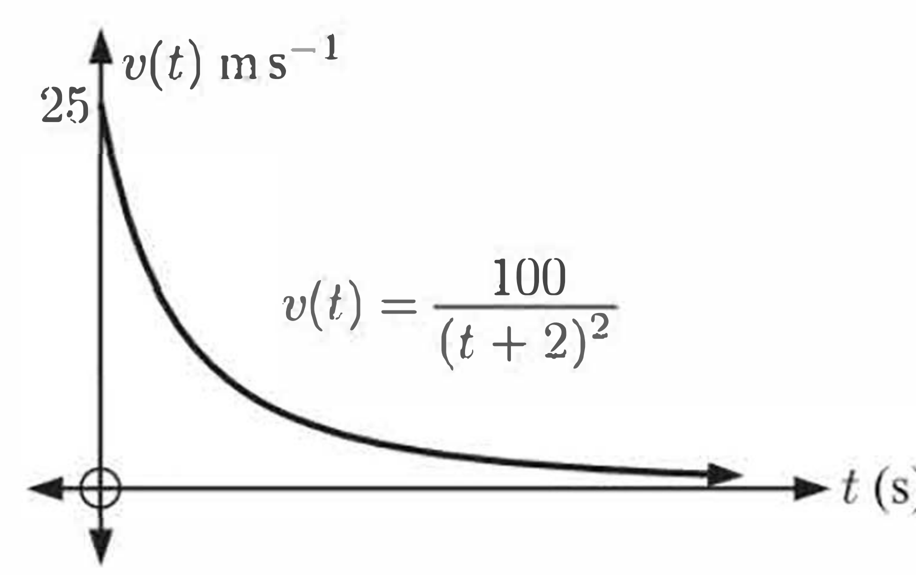
4 $v(t) = \dfrac{100}{(t+2)^2} = 100(t+2)^{-2}\ \text{m s}^{-1}$

a At $t = 0$, $v(0) = \dfrac{100}{2^2} = 25\ \text{m s}^{-1}$

At $t = 3$, $v(3) = \dfrac{100}{5^2} = 4\ \text{m s}^{-1}$

b As $t \to \infty$, $v(t) \to 0^+$

c



d As $v(t)$ is always positive, the boat is always travelling forwards.

$$s(t) = \int v(t)\, dt$$
$$= \int 100(t+2)^{-2}\, dt$$
$$= -100(t+2)^{-1} + c$$
$$= \frac{-100}{t+2} + c$$

$\therefore$ $s(0) = c - 50$ m

$\therefore$ when the boat has travelled 30 m,
$s(t) = c - 20$ m

$\therefore$ $c - 20 = \dfrac{-100}{t+2} + c$

$\therefore$ $\dfrac{-100}{t+2} = -20$

$\therefore$ $t + 2 = 5$

$\therefore$ $t = 3$ seconds

e
$$a(t) = v'(t)$$
$$= -200(t+2)^{-3}$$
$$= \frac{-200}{(t+2)^3}\ \text{m s}^{-2}, \quad t \geqslant 0$$

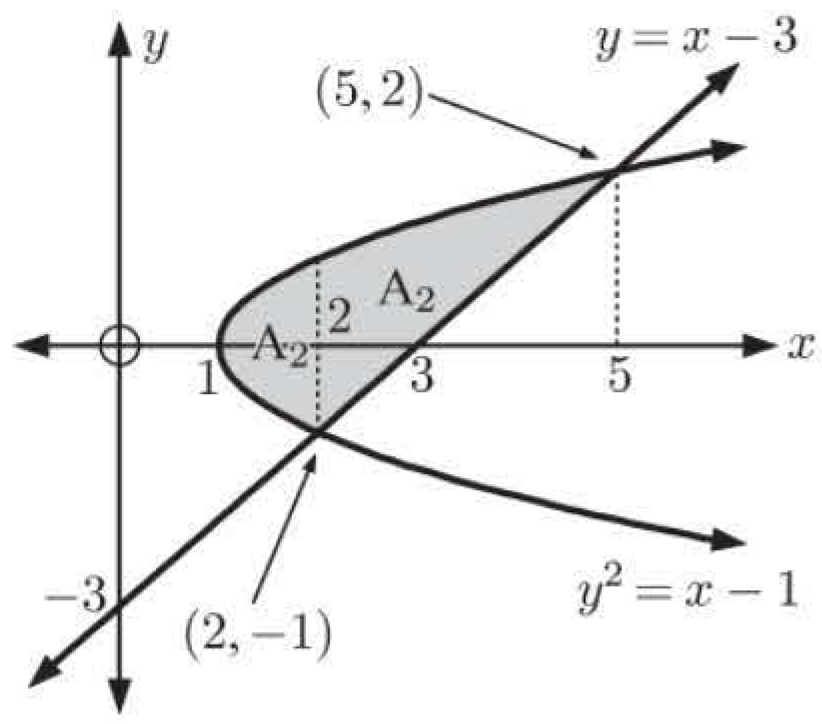
f
$$\frac{dv}{dt} = \frac{-200}{(t+2)^3} = -\tfrac{1}{5}\,\frac{1000}{(t+2)^3}$$
$$= -\tfrac{1}{5}\left(\frac{100}{(t+2)^2}\right)^{\frac{3}{2}}$$
$$= -\tfrac{1}{5}\, v^{\frac{3}{2}}$$

$\therefore$ $\dfrac{dv}{dt} = -kv^{\frac{3}{2}}$ where $k = \frac{1}{5}$

5 **a** The graphs meet when $\cos 2x = e^{3x}$
Using technology, $x = 0$ and
$x \approx -0.7292$

b Shaded area $\approx \int_{-0.7292}^{0} (\cos 2x - e^{3x})\, dx$
≈ 0.2009 units^2 {using technology}

6 **a**



b $y^2 = x - 1$ meets $y = x - 3$ where

$$x - 1 = (x-3)^2$$
$$\therefore\ x - 1 = x^2 - 6x + 9$$
$$\therefore\ x^2 - 7x + 10 = 0$$
$$\therefore\ (x-5)(x-2) = 0$$
$$\therefore\ x = 2 \text{ or } x = 5$$

$\therefore$ the graphs meet at $(5, 2)$ and $(2, -1)$.

c Area $= A_1 + A_2$

$$= 2\int_1^2 (x-1)^{\frac{1}{2}}\,dx + \int_2^5 [(x-1)^{\frac{1}{2}} - (x-3)]\,dx$$
$$= 2\left[\tfrac{2}{3}(x-1)^{\frac{3}{2}}\right]_1^2 + \left[\tfrac{2}{3}(x-1)^{\frac{3}{2}} - \frac{x^2}{2} + 3x\right]_2^5$$
$$= 2\left[\tfrac{2}{3} - 0\right] + \left[\left(\tfrac{2}{3}(8) - \tfrac{25}{2} + 15\right) - \left(\tfrac{2}{3} - 2 + 6\right)\right] = 4\tfrac{1}{2} \text{ units}^2$$

7

$$\int_0^m \sin x\,dx = \tfrac{1}{2}$$
$$\therefore\ [-\cos x]_0^m = \tfrac{1}{2}$$
$$\therefore\ -\cos m + \cos 0 = \tfrac{1}{2}$$
$$\therefore\ \cos m = \tfrac{1}{2}$$
$$\therefore\ m = \tfrac{\pi}{3} \quad \{0 < m < \tfrac{\pi}{2}\}$$

8 **a** The graphs meet where

$$x^2 = \sin x$$
$$\therefore\ x = 0 \text{ or } \approx 0.8767 \quad \{\text{using technology}\}$$
$$\therefore\ a \approx 0.8767$$

b area $\approx \int_0^{0.8767} (\sin x - x^2)\,dx$

≈ 0.1357 units^2 {using technology}

9 **a** $y = \cos(2x)$ meets the x-axis where $2x = \frac{\pi}{2}$, or $x = \frac{\pi}{4}$.

$$\therefore\ V = \pi\int_{\frac{\pi}{16}}^{\frac{\pi}{4}} \cos^2(2x)\,dx = \pi\int_{\frac{\pi}{16}}^{\frac{\pi}{4}} \left(\tfrac{1}{2} + \tfrac{1}{2}\cos(4x)\right)dx$$
$$= \pi\left[\tfrac{1}{2}x + \tfrac{1}{8}\sin(4x)\right]_{\frac{\pi}{16}}^{\frac{\pi}{4}}$$
$$= \pi\left[\left(\tfrac{\pi}{8} + \tfrac{1}{8}\sin\pi\right) - \left(\tfrac{\pi}{32} + \tfrac{1}{8}\sin\left(\tfrac{\pi}{4}\right)\right)\right]$$
$$= \pi\left(\tfrac{\pi}{8} - \tfrac{\pi}{32} - \tfrac{1}{8}\left(\tfrac{1}{\sqrt{2}}\right)\right)$$
$$= \pi\left(\tfrac{3\pi}{32} - \tfrac{1}{8\sqrt{2}}\right) \text{ units}^3$$

b $V = \pi\int_0^2 (e^{-x} + 4)^2\,dx$

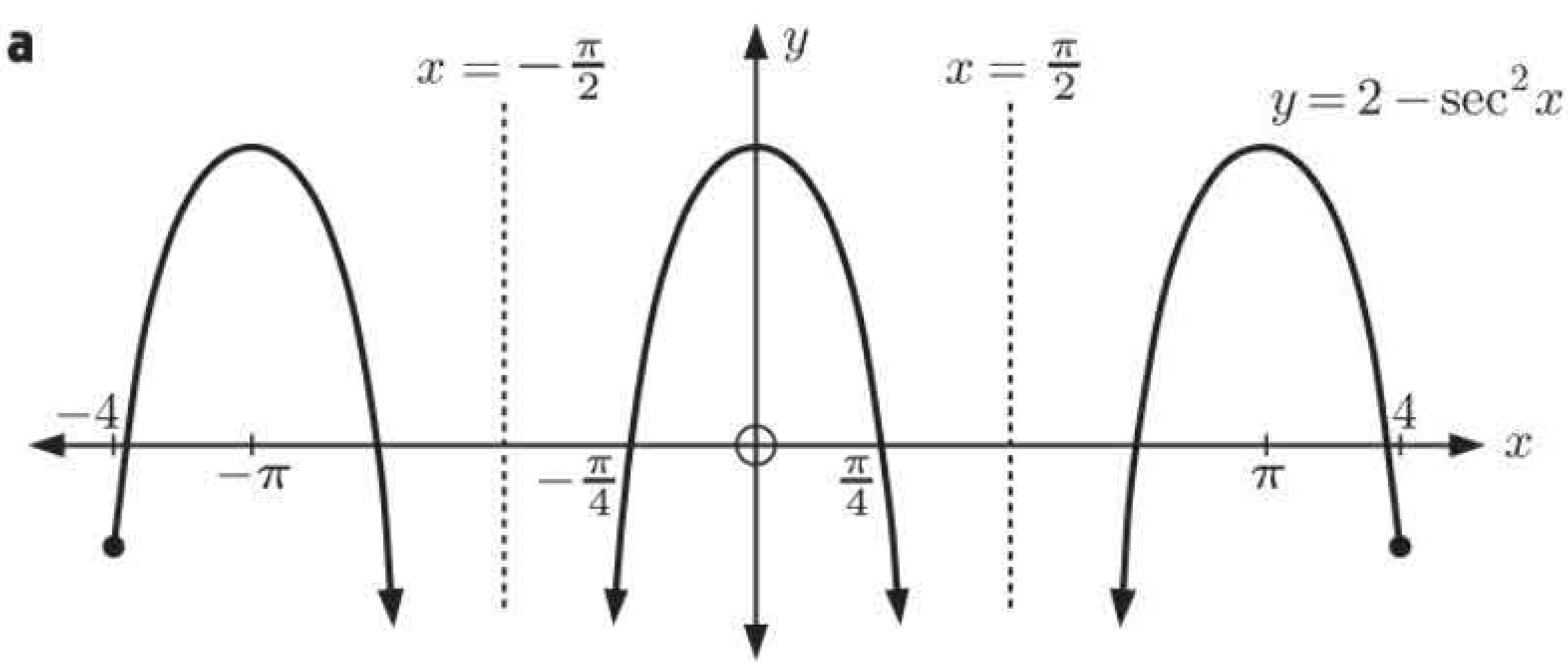
$$= \pi\int_0^2 \left(e^{-2x} + 8e^{-x} + 16\right)dx$$
$$= \pi\left[\tfrac{1}{-2}e^{-2x} + \tfrac{8}{-1}e^{-x} + 16x\right]_0^2$$
$$= \pi\left[\left(-\tfrac{1}{2}e^{-4} - 8e^{-2} + 32\right) - \left(-\tfrac{1}{2} - 8\right)\right]$$
$$= \pi\left(\frac{81}{2} - \frac{1}{2e^4} - \frac{8}{e^2}\right) \text{ units}^3$$
$$\approx 124 \text{ units}^3$$

10 $y = 2x^3 - 9x$ meets $y = 3x^2 - 10$ when $2x^3 - 9x = 3x^2 - 10$

$$\therefore \quad 2x^3 - 3x^2 - 9x + 10 = 0$$
$$\therefore \quad (x-1)(2x^2 - x - 10) = 0$$
$$\therefore \quad (x-1)(2x-5)(x+2) = 0$$
$$\therefore \quad x = -2,\ 1, \text{ or } \tfrac{5}{2}$$

$$\therefore \text{ total area} = \int_{-2}^{\frac{5}{2}} \left|2x^3 - 3x^2 - 9x + 10\right| dx$$
$$\approx 31.2 \text{ units}^2$$

11 **a**



b $f(x) = 2 - \sec^2 x$ is undefined when $\cos x = 0$.
On the domain $x \in [-4, 4]$ this is when $x = -\frac{\pi}{2}, \frac{\pi}{2}$.
$\therefore$ the vertical asymptotes are $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$

c When $y = 0$, $2 - \sec^2 x = 0$

$$\therefore \quad \cos^2 x = \tfrac{1}{2}$$
$$\therefore \quad \cos x = \pm\tfrac{1}{\sqrt{2}}$$

$\therefore \quad x = -\frac{5\pi}{4}, -\frac{3\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$ $\{$for $x \in [-4, 4]\}$

$\therefore$ the x-intercepts are $-\frac{5\pi}{4}, -\frac{3\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$

When $x = 0$, $y = 2 - \sec^2(0) = 2 - 1^2 = 1$

$\therefore$ the y-intercept is 1.

d area

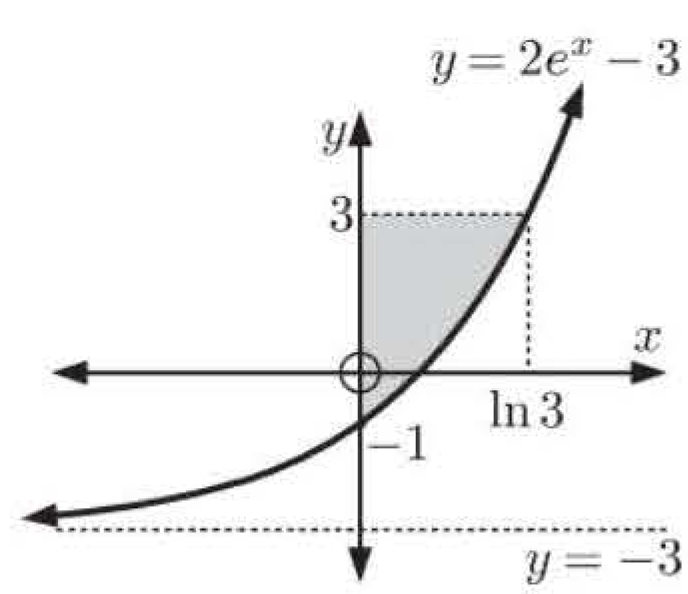
$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (2 - \sec^2 x)\, dx$$
$$= [2x - \tan x]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$
$$= \left(\tfrac{\pi}{2} - 1\right) - \left(-\tfrac{\pi}{2} - (-1)\right)$$
$$= (\pi - 2) \text{ units}^2$$

12 $x = \ln\left(\dfrac{y+3}{2}\right)$

$$\therefore \quad \frac{y+3}{2} = e^x$$
$$\therefore \quad y = 2e^x - 3$$

When $y = 3$, $x = \ln\left(\dfrac{3+3}{2}\right)$

$$\therefore \quad x = \ln 3$$

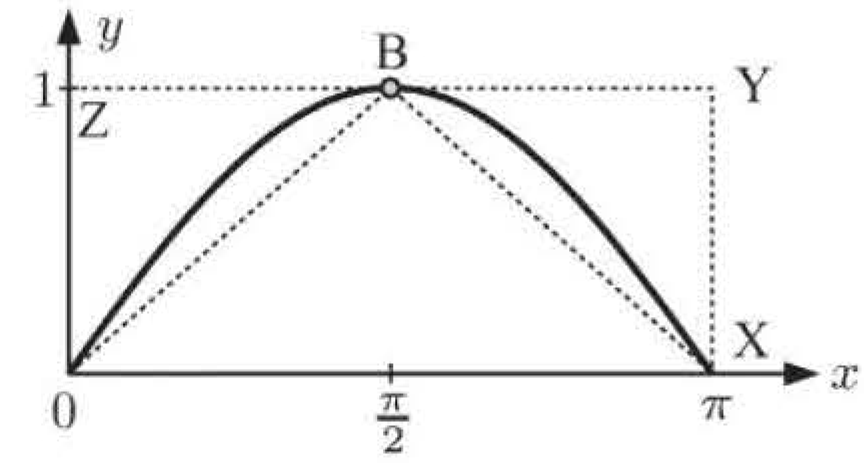


$$\text{Area} = \int_0^{\ln 3} (3 - [2e^x - 3])\, dx$$
$$= \int_0^{\ln 3} (6 - 2e^x)\, dx$$
$$= [6x - 2e^x]_0^{\ln 3}$$
$$= \left(6\ln 3 - 2e^{\ln 3}\right) - (0 - 2)$$
$$= (6\ln 3 - 2 \times 3 + 2)$$
$$= 6\ln 3 - 4 \text{ units}^2$$
$$\approx 2.59 \text{ units}^2$$

13 **a** From the graph,

area $\triangle$OBX $<$ area under the curve $<$ area OXYZ

$$\therefore \quad \tfrac{1}{2}\pi(1) < \int_0^{\pi} \sin x\, dx < \pi(1)$$
$$\therefore \quad \tfrac{\pi}{2} < \int_0^{\pi} \sin x\, dx < \pi$$



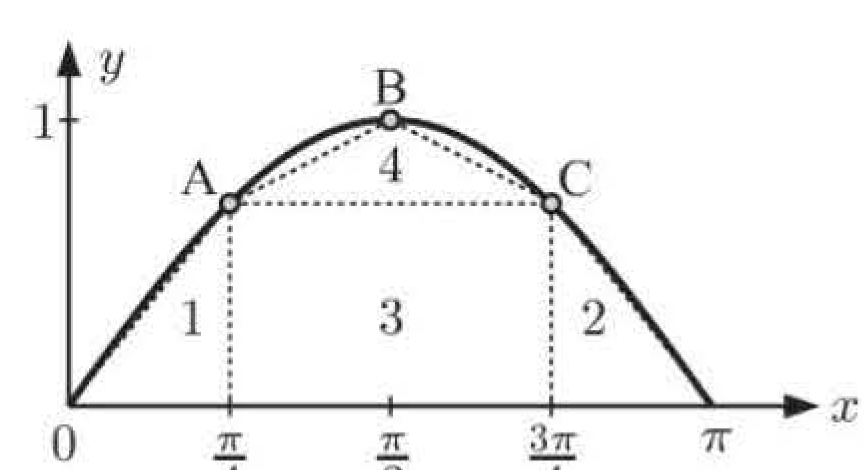
b If we partition the diagram as shown:

Area $1 = \frac{1}{2}\left(\frac{\pi}{4}\right)\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi\sqrt{2}}{16}$

Area $2 = \frac{\pi\sqrt{2}}{16}$

Area $3 = \frac{\pi}{2} \times \frac{\sqrt{2}}{2} = \frac{\pi\sqrt{2}}{4}$

Area $4 = \frac{1}{2}\left(\frac{\pi}{2}\right)\left(1 - \frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}\left(1 - \frac{\sqrt{2}}{2}\right)$

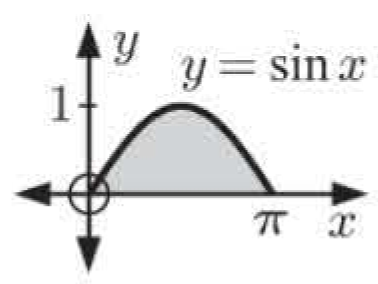


The area under the arch is greater than the total area of the 4 sections.

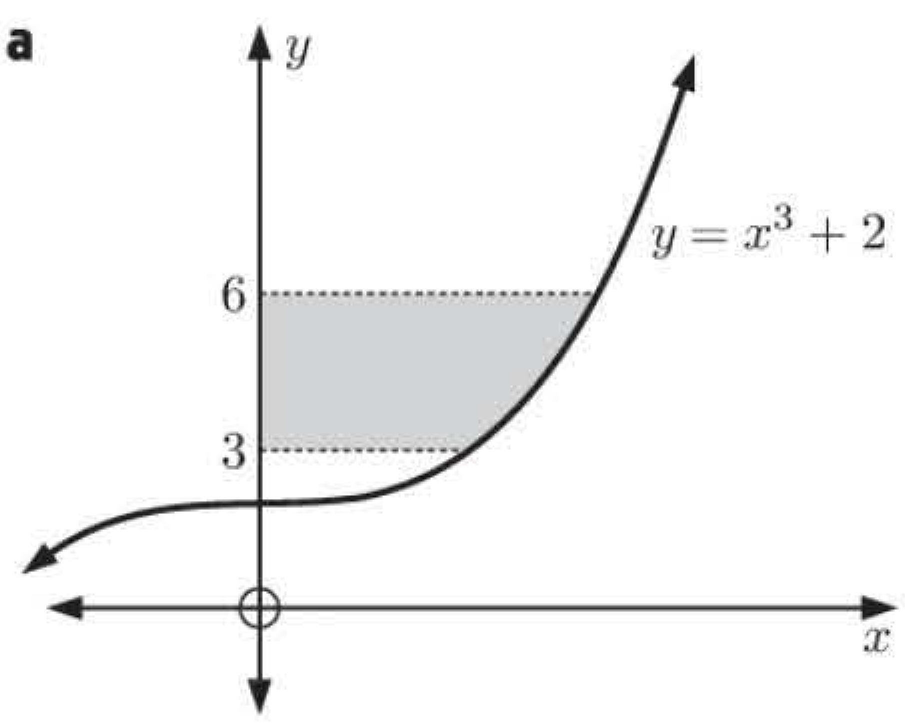
$$\text{Total area of the sections} = \tfrac{\pi\sqrt{2}}{16} + \tfrac{\pi\sqrt{2}}{16} + \tfrac{\pi\sqrt{2}}{4} + \tfrac{\pi}{4}\left(1 - \tfrac{\sqrt{2}}{2}\right)$$
$$= \frac{\pi\sqrt{2} + \pi\sqrt{2} + 4\pi\sqrt{2} + 4\pi - 2\pi\sqrt{2}}{16}$$
$$= \frac{4\pi + 4\pi\sqrt{2}}{16}$$
$$= \tfrac{\pi + \pi\sqrt{2}}{4}$$
$$= \tfrac{\pi}{4}(1 + \sqrt{2}) \text{ units}^2$$

$\therefore$ the area under the arch is greater than $\frac{\pi}{4}(1+\sqrt{2})$ units^2.

c $A = \int_0^\pi \sin x \, dx$
$= [-\cos x]_0^\pi$
$= [-\cos \pi + \cos 0]$
$= -(-1) + 1$
$= 2$ units^2



14 **a**



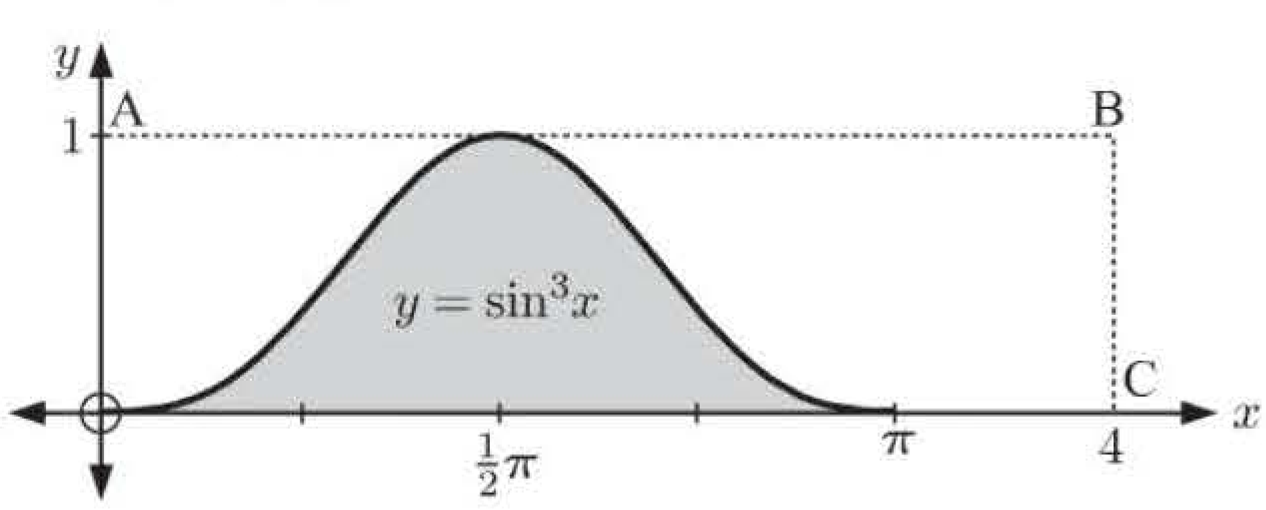
b $y = x^3 + 2$
$\therefore \; x^3 = y - 2$
$\therefore \; x = (y-2)^{\frac{1}{3}}$
$\therefore \; x = f(y)$
$= \sqrt[3]{y-2}$

c $\text{Area} = \int_3^6 x \, dy$
$= \int_3^6 (y-2)^{\frac{1}{3}} \, dy$
$= \left[\dfrac{(y-2)^{\frac{4}{3}}}{\frac{4}{3}}\right]_3^6$
$= \frac{3}{4}\left(4^{\frac{4}{3}} - 1^{\frac{4}{3}}\right)$
$= \frac{3}{4}\left(4\sqrt[3]{4} - 1\right)$ units^2
≈ 4.01 units^2

15 The curves meet when
$x^3 + x^2 + 2x + 6 = 7x^2 - x - 4$
$\therefore \; x^3 - 6x^2 + 3x + 10 = 0$
$\therefore \; (x+1)(x^2 - 7x + 10) = 0$
$\therefore \; (x+1)(x-2)(x-5) = 0$
$\therefore \; x = -1, 2, \text{ or } 5$

$\therefore$ area enclosed
$= \int_{-1}^5 \left|x^3 - 6x^2 + 3x + 10\right| \, dx$
$= 40\frac{1}{2}$ units^2

16 Consider the graph of $y = \sin^3 x$, $0 \leqslant x \leqslant \pi$



Now $\int_0^\pi \sin^3 x \, dx =$ shaded area

But the shaded area $<$ area of rectangle ABCO

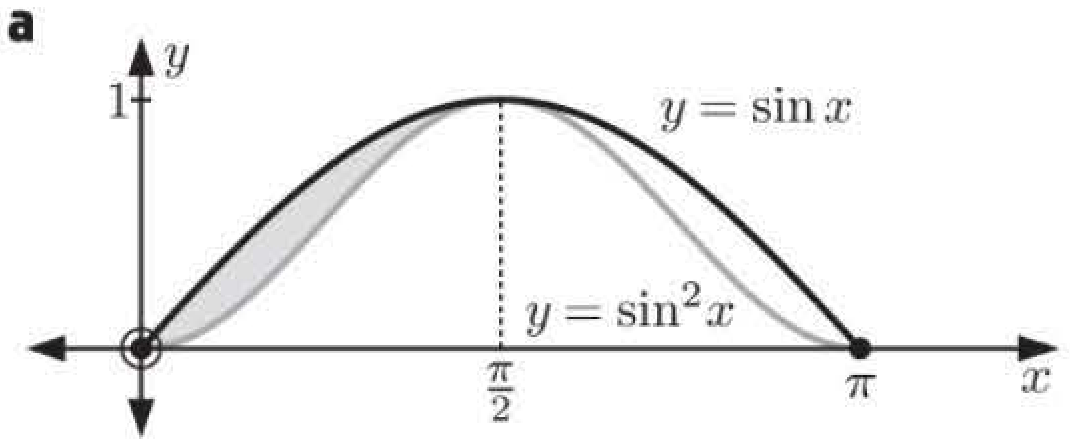
$\therefore \; \int_0^\pi \sin^3 x \, dx < 4$

REVIEW SET 22C

1 **a** $a(t) = 6t - 30 \text{ cm s}^{-2}$
$v(t) = \int (6t - 30) \, dt$
$= 3t^2 - 30t + c$
But $v(0) = 27$
$\therefore \; 0 - 0 + c = 27$
$\therefore \; c = 27$
$\therefore \; v(t) = 3t^2 - 30t + 27 \text{ cm s}^{-1}$

b Displacement after 6 seconds
$= \int_0^6 (3t^2 - 30t + 27) \, dt$
$= \left[t^3 - 15t^2 + 27t\right]_0^6$
$= 6^3 - 15(6)^2 + 27(6) - 0$
$= -162$ cm
(162 cm to the left of the origin)

2 **a**



b Area $= \int_0^{\frac{\pi}{2}} (\sin x - \sin^2 x)\, dx$

$$= \int_0^{\frac{\pi}{2}} (\sin x - (\tfrac{1}{2} - \tfrac{1}{2}\cos 2x))\, dx$$
$$= \int_0^{\frac{\pi}{2}} (\sin x + \tfrac{1}{2}\cos 2x - \tfrac{1}{2})\, dx$$
$$= \left[-\cos x + \tfrac{1}{4}\sin 2x - \tfrac{1}{2}x\right]_0^{\frac{\pi}{2}}$$
$$= \left(0 + \tfrac{1}{4}(0) - \tfrac{\pi}{4}\right) - (-1 + 0 - 0)$$
$$= \left(1 - \tfrac{\pi}{4}\right) \text{ units}^2$$

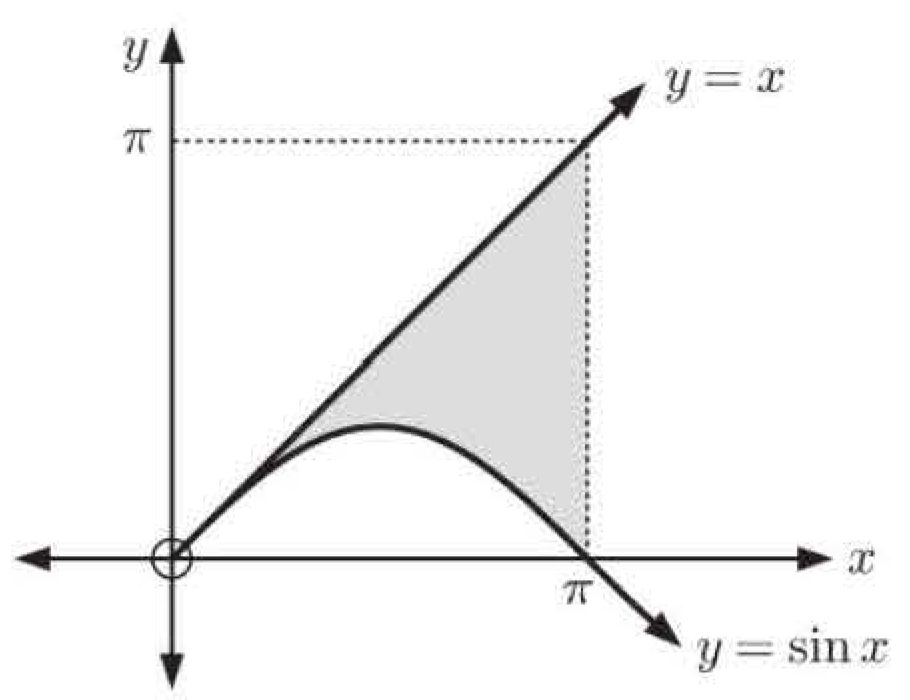
3 The area between $x = 0$ and $x = a$ is 2 units^2.

$$\therefore \int_0^a e^x\, dx = 2$$
$$\therefore [e^x]_0^a = 2$$
$$\therefore e^a - e^0 = 2$$
$$\therefore e^a = 3$$
$$\therefore a = \ln 3$$

The area between $x = a = \ln 3$ and $x = b$ is 2 units^2.

$$\therefore \int_{\ln 3}^b e^x\, dx = 2$$
$$\therefore [e^x]_{\ln 3}^b = 2$$
$$\therefore e^b - e^{\ln 3} = 2$$
$$\therefore e^b - 3 = 2$$
$$\therefore e^b = 5$$
$$\therefore b = \ln 5$$

4



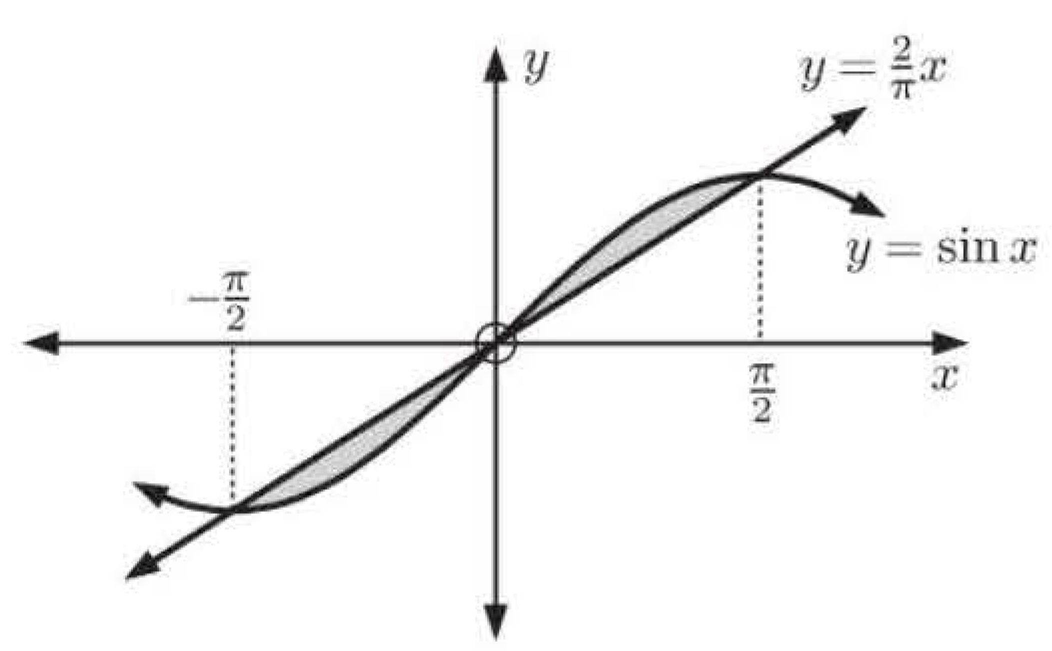
Required area = area of △ − area under sine curve

$$= \tfrac{1}{2}\pi \times \pi - \int_0^{\pi} \sin x\, dx$$
$$= \tfrac{\pi^2}{2} - [-\cos x]_0^{\pi}$$
$$= \tfrac{\pi^2}{2} - [-\cos\pi + \cos 0]$$
$$= \left(\tfrac{\pi^2}{2} - 2\right) \text{ units}^2$$

5 The graphs meet when $\frac{2}{\pi}x = \sin x$

$\therefore\ x = -\frac{\pi}{2}, 0, \frac{\pi}{2}$ {using technology}

$$\therefore \text{ area} = \int_{-\frac{\pi}{2}}^{0} (\tfrac{2}{\pi}x - \sin x)\, dx + \int_0^{\frac{\pi}{2}} (\sin x - \tfrac{2}{\pi}x)\, dx$$
$$= \left[\frac{x^2}{\pi} + \cos x\right]_{-\frac{\pi}{2}}^{0} + \left[-\cos x - \frac{x^2}{\pi}\right]_0^{\frac{\pi}{2}}$$
$$= (0 + 1) - (\tfrac{\pi}{4} + 0) + (0 - \tfrac{\pi}{4}) - (-1 - 0)$$
$$= (2 - \tfrac{\pi}{2}) \text{ units}^2$$



6 The coordinates of B are $(2, 4 + k)$

$\therefore$ area rectangle OABC $= 2 \times (4 + k)$
$= 8 + 2k$

$\therefore$ since the two shaded regions are equal in area, each area is $4 + k$ units^2.

$$\therefore \int_0^2 (x^2 + k)\, dx = 4 + k$$
$$\therefore \left[\frac{x^3}{3} + kx\right]_0^2 = 4 + k$$
$$\therefore \tfrac{8}{3} + 2k = 4 + k$$
$$\therefore k = 4 - \tfrac{8}{3}$$
$$\therefore k = 1\tfrac{1}{3}$$

7 $y=\sqrt{4-x^2}$ is a semi-circle above the x-axis with centre O and radius 2.

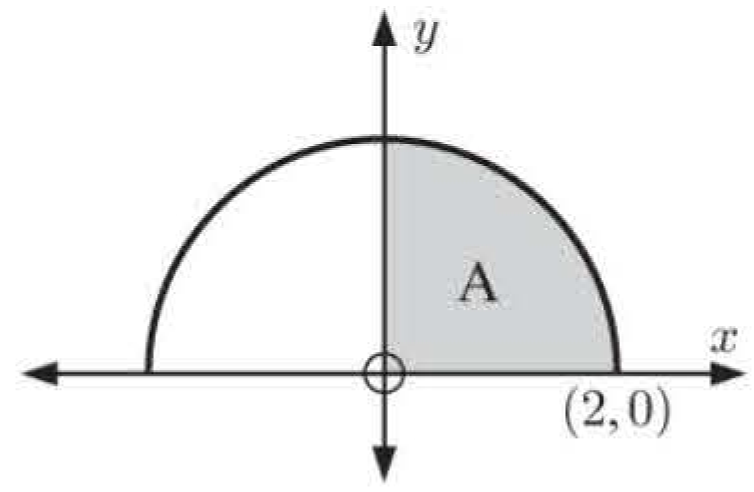
Now $\int_0^2 \sqrt{4-x^2}\,dx$

$= $ shaded area

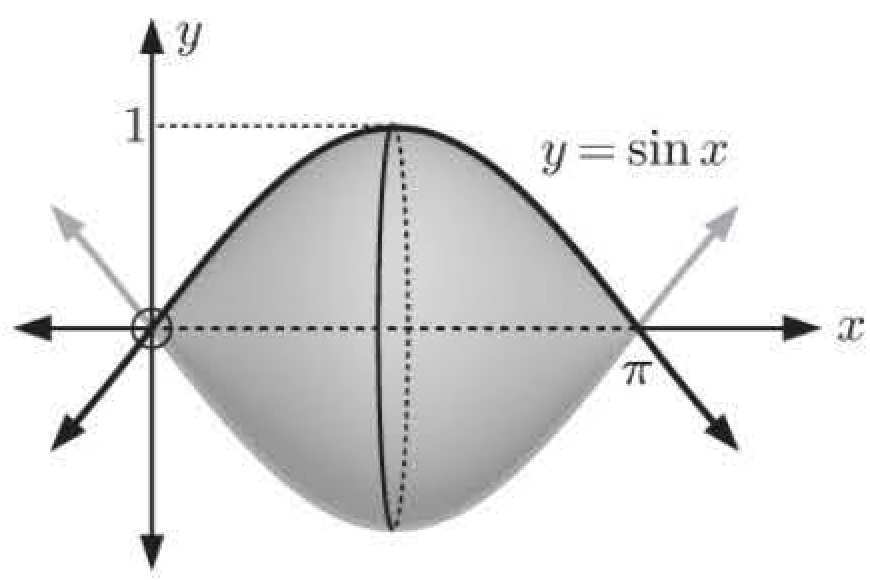
$= \frac{1}{4}$ of the area of a circle of radius 2 units

$= \frac{1}{4}\pi(2^2)$

$= \pi$ units^2

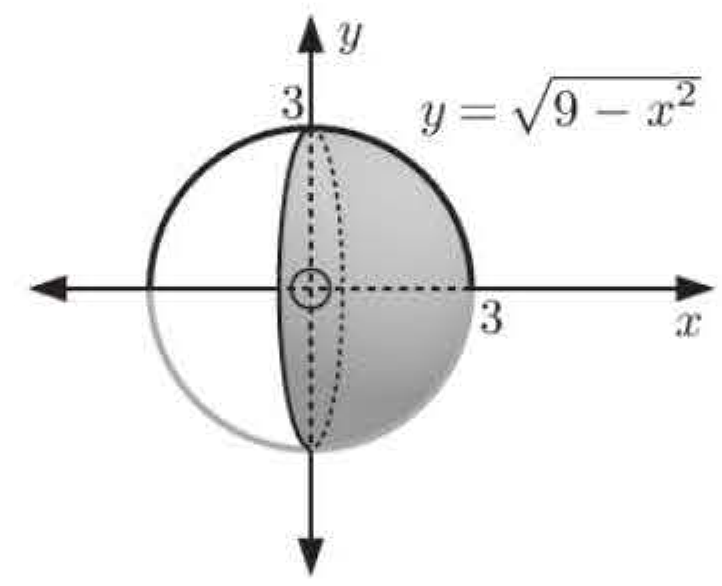


8 **a**



$$V=\pi\int_0^{\pi}\sin^2 x\,dx$$
$$=\pi\int_0^{\pi}\left(\tfrac{1}{2}-\tfrac{1}{2}\cos(2x)\right)dx$$
$$=\pi\left[\tfrac{1}{2}x-\tfrac{1}{2}\left(\tfrac{1}{2}\right)\sin(2x)\right]_0^{\pi}$$
$$=\pi\left[\tfrac{1}{2}\pi-\tfrac{1}{4}\sin 2\pi-0\right]$$
$$=\tfrac{\pi^2}{2}\text{ units}^3$$

b



$$V=\pi\int_0^{3}(9-x^2)\,dx$$
$$=\pi\left[9x-\frac{x^3}{3}\right]_0^3$$
$$=\pi\left[27-\tfrac{27}{3}-0\right]$$
$$=18\pi\text{ units}^3$$

9 $y=x^3$ meets $y=7x^2-10x$

when $x^3=7x^2-10x$

$\therefore\ x^3-7x^2+10x=0$

$\therefore\ x(x^2-7x+10)=0$

$\therefore\ x(x-2)(x-5)=0$

$\therefore\ x=0$, 2, or 5

$\therefore$ total area $=\int_0^5 \left|x^3-7x^2+10x\right|\,dx$

$= 21\frac{1}{12}$ units^2

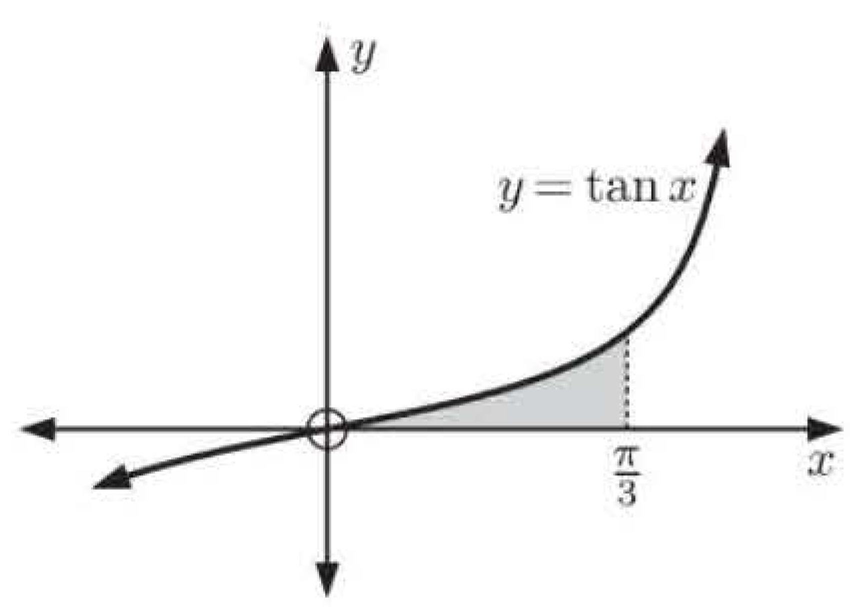
10 area $=\int_0^{\frac{\pi}{3}}\tan x\,dx$

$= \left[-\ln|\cos x|\right]_0^{\frac{\pi}{3}}$ {see **Exercise 21G.1** Q **6 c**}

$= -\ln\cos\frac{\pi}{3}+\ln\cos 0$

$= -\ln(\frac{1}{2})+\ln 1$

$= \ln 2$ units^2



11 **a**

$$\frac{dI}{dt}=-\frac{100}{t^2},\quad t\geqslant 0.2\text{ seconds}$$

$$\therefore\ I(t)=\int\frac{dI}{dt}\,dt=\int -100t^{-2}\,dt=100t^{-1}+c$$

Now $I(2)=150$ milliamps, so $\dfrac{100}{2}+c=150$ and so $c=100$

$$\therefore\ I(t)=\left(\frac{100}{t}+100\right)\text{ milliamps}$$

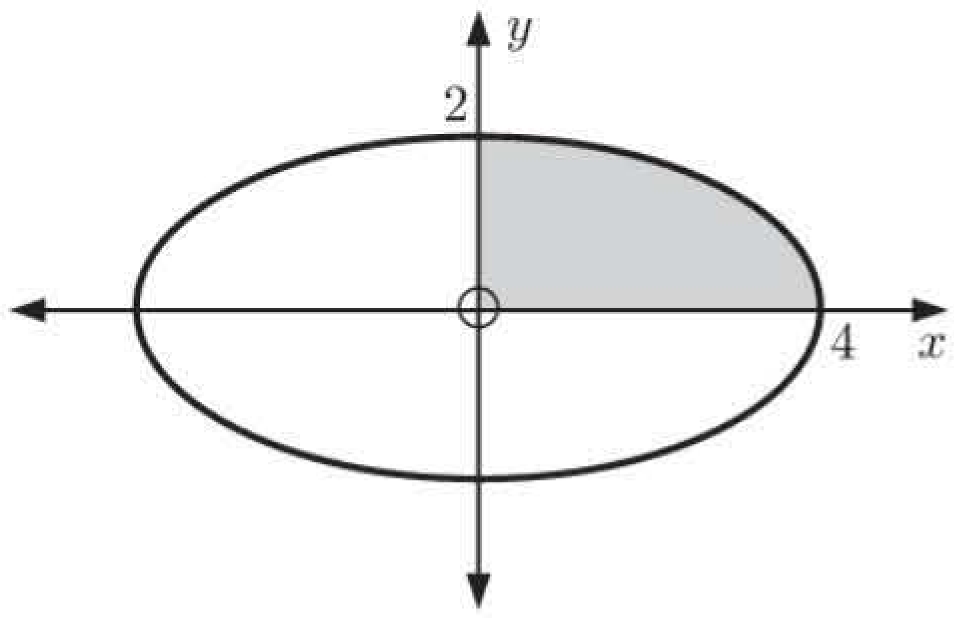
b **i** $I(20)=\frac{100}{20}+100$

$= 105$ milliamps

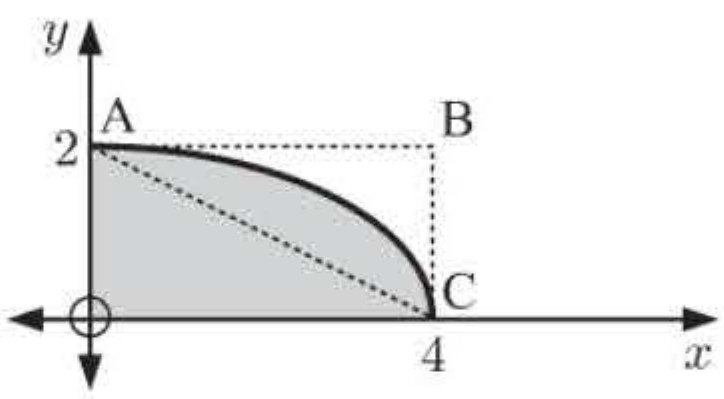
ii As $t\to\infty$,

$I(t)\to 100$ milliamps (above)

12 **a**



b



Now area $\triangle$AOC < shaded area < area ABCO

$\therefore \quad \frac{1}{2}(2 \times 4) < \int_0^4 \frac{1}{2}\sqrt{16-x^2}\,dx < 2 \times 4$

$\therefore \quad 4 < \int_0^4 \frac{1}{2}\sqrt{16-x^2}\,dx < 8$

$\therefore \quad 8 < \int_0^4 \sqrt{16-x^2}\,dx < 16$

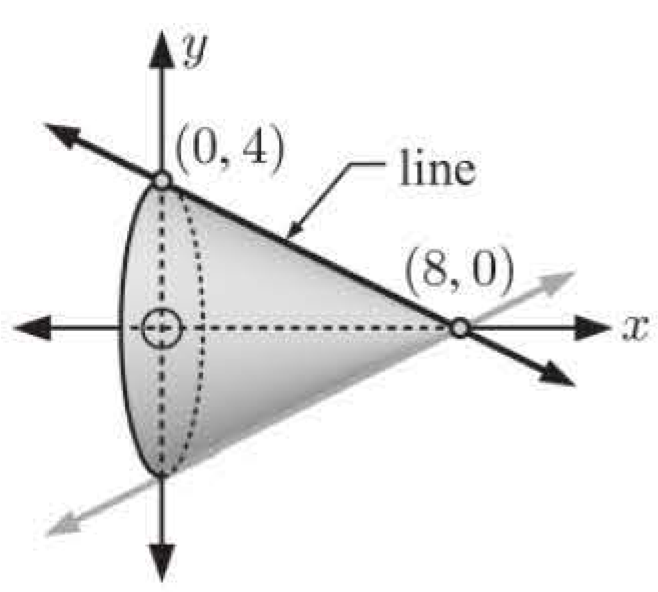
13 **a** $V = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi \times 4^2 \times 8$

$= \frac{1}{3}\pi \times 128$

$= \frac{128\pi}{3}$ units^3

b



gradient $= \frac{0-4}{8-0} = -\frac{1}{2}$

$\therefore$ the line has equation $y = -\frac{1}{2}x + 4$

$\therefore \quad V = \pi \int_0^8 \left(-\frac{1}{2}x+4\right)^2 dx$

$$= \pi \int_0^8 \left(\frac{x^2}{4} - 4x + 16\right) dx$$

$$= \pi \left[\frac{x^3}{12} - \frac{4x^2}{2} + 16x\right]_0^8$$

$= \pi\left(\frac{128}{3} - 128 + 128 - 0\right)$

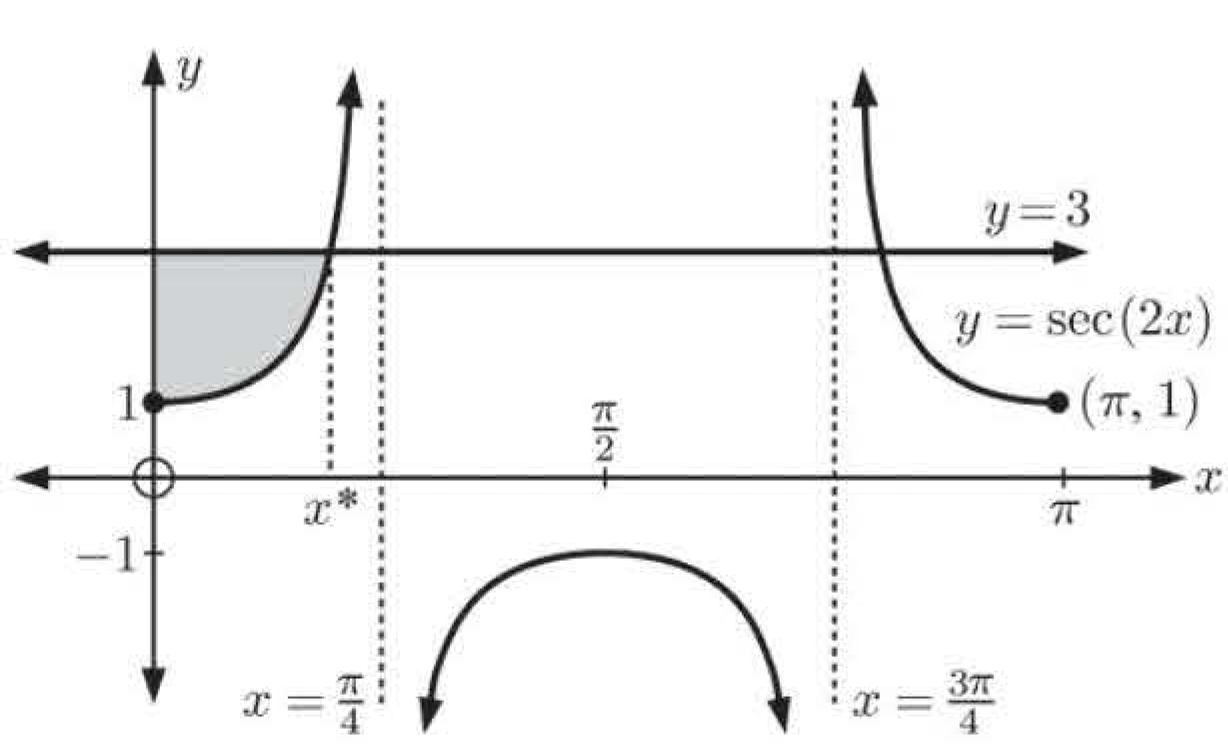
$= \frac{128\pi}{3}$ units^3 ✓

14 **a** $$\frac{d}{dx}\left[\ln(\tan x + \sec x)\right] = \frac{\sec^2 x + \sec x \tan x}{\tan x + \sec x}$$

$$= \sec x$$

$\therefore \quad \int \sec x\,dx = \ln|\tan x + \sec x| + c$

b **i**



ii $y = \sec(2x)$ and $y = 3$ meet when $\sec(2x) = 3$

$\therefore \quad \cos(2x) = \frac{1}{3}$

$\therefore \quad x^* \approx 0.615$ {see the graph}

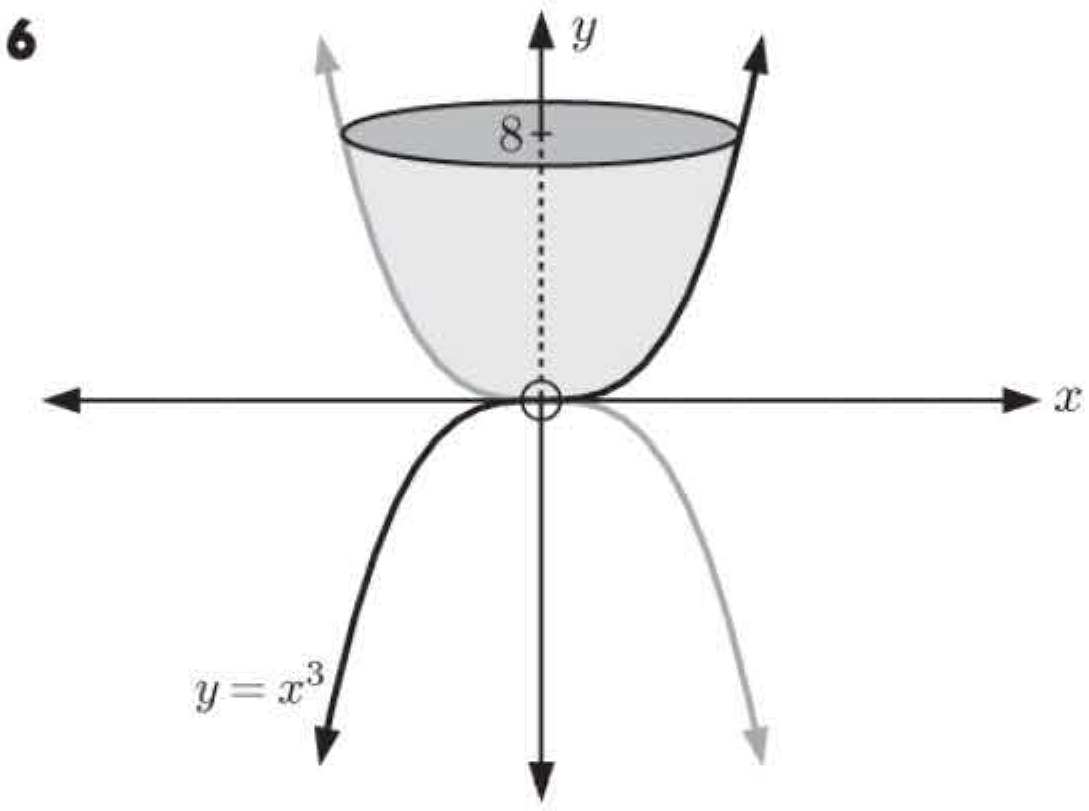
$$\therefore \quad \text{shaded area} \approx (\text{rectangle area}) - \int_0^{0.615} \sec(2x)\, dx$$
$$\approx 0.615 \times 3 - \left[\tfrac{1}{2} \ln|\tan(2x) + \sec(2x)|\right]_0^{0.615}$$
$$\approx 1.846 - \left[\tfrac{1}{2} \ln|\tan(1.23) + \sec(1.23)| - \tfrac{1}{2} \ln|0+1|\right]$$
$$\approx 0.965 \text{ units}^2$$

15 $y = \sin x$ and $y = \cos x$ meet where $\sin x = \cos x$

$$\therefore \quad \frac{\sin x}{\cos x} = 1$$
$$\therefore \quad \tan x = 1$$
$$\therefore \quad x = \tfrac{\pi}{4}$$

Hence
$$V = \pi \int_0^{\frac{\pi}{4}} (\cos^2 x - \sin^2 x)\, dx$$
$$= \pi \int_0^{\frac{\pi}{4}} \cos(2x)\, dx$$
$$= \pi \left[\tfrac{1}{2} \sin(2x)\right]_0^{\frac{\pi}{4}}$$
$$= \pi \left(\tfrac{1}{2} \sin\left(\tfrac{\pi}{2}\right) - \tfrac{1}{2} \sin 0\right)$$
$$= \pi \left(\tfrac{1}{2}(1) - 0\right)$$
$$= \tfrac{\pi}{2} \text{ units}^3$$

16



Now $x^3 = y$

$$\therefore \quad (x^3)^{\frac{2}{3}} = y^{\frac{2}{3}}$$
$$\therefore \quad x^2 = y^{\frac{2}{3}}$$

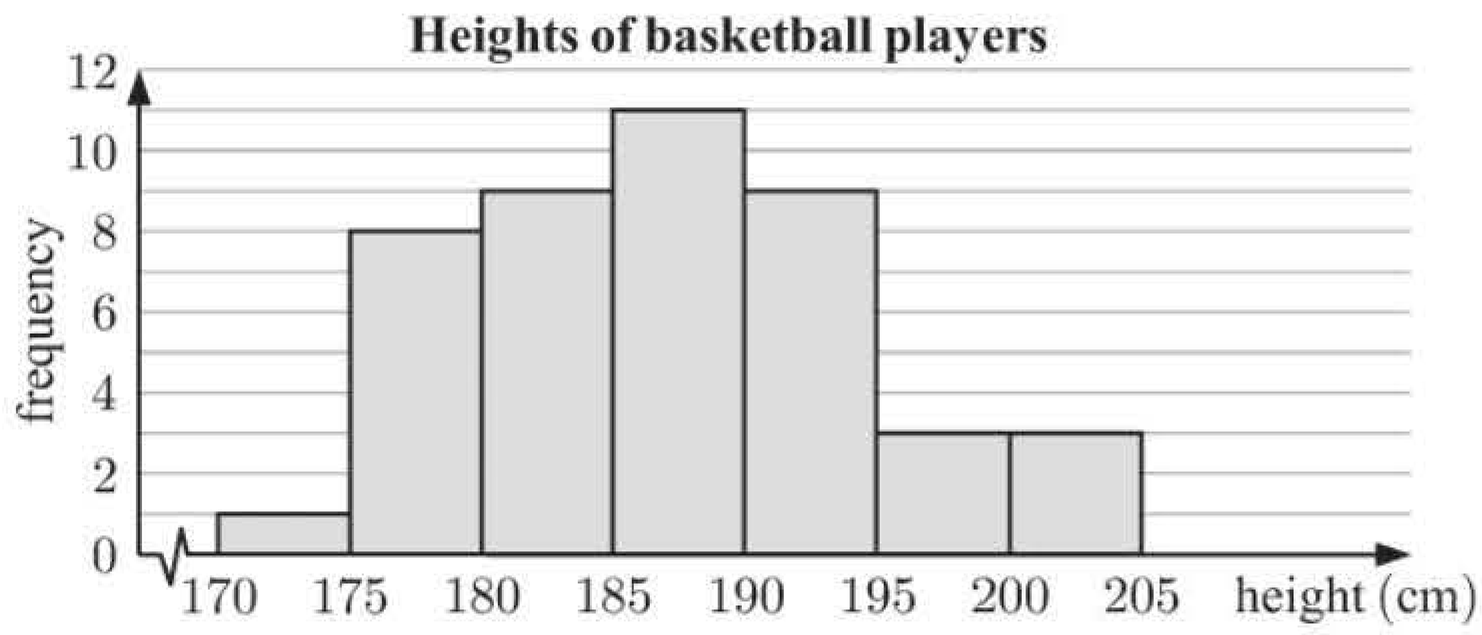
Volume
$$V = \pi \int_0^8 x^2\, dy$$
$$= \pi \int_0^8 y^{\frac{2}{3}}\, dy$$
$$= \pi \left[\frac{y^{\frac{5}{3}}}{\frac{5}{3}}\right]_0^8$$
$$= \tfrac{3\pi}{5}\left(8^{\frac{5}{3}} - 0^{\frac{5}{3}}\right)$$
$$= \tfrac{3\pi}{5} \times (2^3)^{\frac{5}{3}}$$
$$= \tfrac{3\pi}{5} \times 2^5$$
$$= \tfrac{96\pi}{5} \text{ units}^3$$

Chapter 23
DESCRIPTIVE STATISTICS

EXERCISE 23A

1 **a** Heights can take any value from 170 cm to 205 cm, including decimal values such as 181.37 cm. The 'height' variable can take any real number between 170 and 205.

b



c The modal class is the class occurring most often. This is $185 \leqslant H < 190$ cm.

d The distribution is slightly positively skewed, as there is more of a 'tail' to the right.

2 **a** The data is continuous numerical. Actual time is continuous and could be measured to the nearest second, millisecond, and so on. After it has been rounded to the nearest minute, it becomes discrete numerical data.

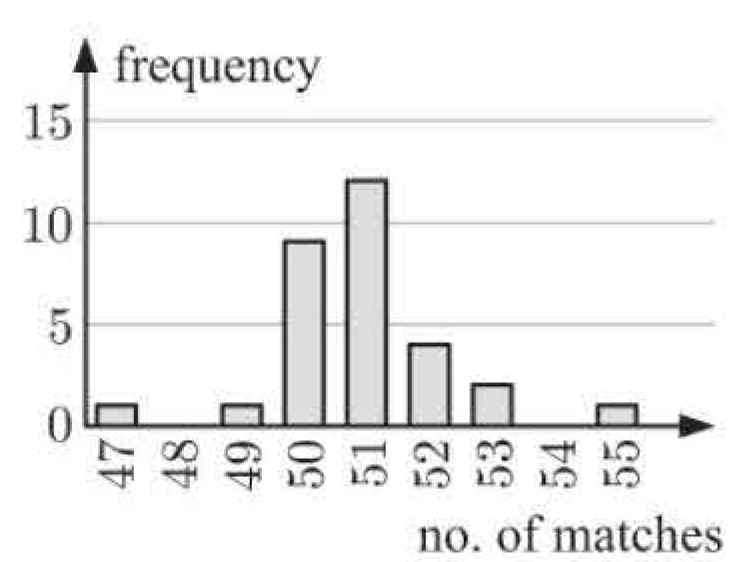
b

Time (min)	*Tally*	*Frequency*
0 - 9	卌 |	6
10 - 19	卌 卌 卌 卌 卌 |	26
20 - 29	卌 卌 |||	13
30 - 39	卌 ||||	9
40 - 49	卌 |	6

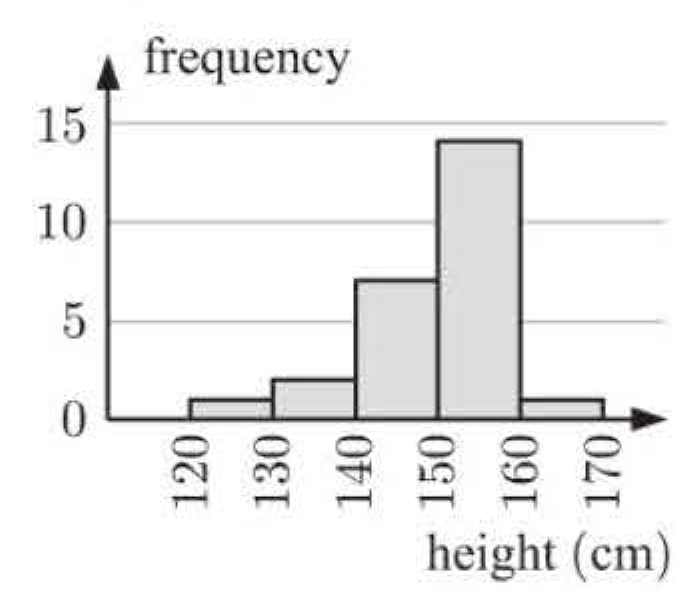
c The distribution is positively skewed, or skewed to the high end.

d The travelling time modal class is 10 - 19 min, if considering classes. (The mode is actually 10, as 10 occurs most frequently.)

3 **a** The data is discrete numerical, so a column graph should be used.



b The data is continuous, so a frequency histogram should be used.



4 **a** Number which are $\geqslant 400$ mm is $14 + 6 = 20$ seedlings.

b $12 + 18 + 42 + 28 + 14 + 6 = 120$ seedlings have been sampled.

$$\therefore \ \% \text{ between 349 and 400} = \frac{42+28}{120} \times 100\%$$
$$= \frac{70}{120} \times 100\%$$
$$\approx 58.3\%$$

c **i** Number less than 400 mm

$$= \frac{12+18+42+28}{120} \times 1462$$
$$= \frac{100}{120} \times 1462$$
$$\approx 1218 \text{ seedlings}$$

ii Number between 374 mm and 425 mm

$$= \frac{28+14}{120} \times 1462$$
$$= \frac{42}{120} \times 1462$$
$$\approx 512 \text{ seedlings}$$

EXERCISE 23B.1

1 **a** **i** $\text{mean} = \dfrac{2+3+3+3+4+ .... +9+9}{23} = \frac{129}{23} \approx 5.61$

ii median $=$ 12th score (when in order) $= 6$

iii mode $= 6$ (6 occurs most often)

b **i** $\text{mean} = \dfrac{10+12+12+15+ .... +20+21}{15} = \frac{245}{15} \approx 16.3$

ii median $=$ 8th score (when in order) $= 17$

iii mode $= 18$

c **i** $\text{mean} = \dfrac{22.4+24.6+21.8+ .... +23.5}{11} = \frac{273}{11} \approx 24.8$

ii median $=$ 6th score (when in order) $= 24.9$

iii mode $= 23.5$

2 **a** $\text{mean of set A} = \dfrac{3+4+4+5+ .... +10}{13} \approx 6.46$ $\quad$ $\text{mean of set B} = \dfrac{3+4+4+5+ .... +15}{13} \approx 6.85$

b median of set A $=$ 7th score $= 7$ $\quad$ median of set B $=$ 7th score $= 7$

c The data sets are the same except for the last value, and the last value of set A is less than that of set B. So, the mean of set A is less than that of set B.

d The middle value of both data sets is the same, so the median is the same.

3 **a** $\text{mean} = \dfrac{23\,000 + 46\,000 + 23\,000 + ... + 32\,000}{10} = \$29\,300$

median $=$ middle score when in order of size $= \dfrac{\$23\,000 + \$24\,000}{2} = \$23\,500$

mode $= \$23\,000$

b The mode is unsatisfactory because it is the lowest salary. It does not take the higher values into account.

c The median is too close to the lower end of the distribution since the data is positively skewed. So the median is not a satisfactory measure of the middle.

4 **a** $\text{mean} = \dfrac{3+1+0+0+ .... +1+0+0}{31} = \frac{99}{31} \approx 3.19$

median $=$ 16th score (when in order) $= 0$

mode $= 0$ (most frequently occurring score)

b The median is not in the centre, as the data is very positively skewed.

c The mode is the lowest value. It does not take the higher values into account.

d Yes, 21 and 42. $\quad$ **e** No, as this would ignore actual valid data.

5 **a** $\text{mean} = \dfrac{43+55+41+37}{4} = \frac{176}{4} = 44$ points $\quad$ **b** another 44 points

c **i** $\text{new mean} = \dfrac{176+25}{5} = 40.2$ points

ii It will increase the new mean to 40.3 points as 41 points is greater than the old mean of 40.2 points.

$\left\{ \dfrac{5 \times 40.2 + 41}{6} \approx 40.3 \right\}$

6 $\text{mean} = \dfrac{\text{total}}{12}$ $\quad \therefore \; \$15\,467 = \dfrac{\text{total}}{12}$ $\quad \therefore \;$ total $= \$15\,467 \times 12 = \$185\,604$

7 mean $= \dfrac{\text{total}}{12}$ $\quad\therefore\quad 262 = \dfrac{\text{total}}{12}$ $\quad\therefore\quad$ total $= 262 \times 12 = 3144$ km

8 **a** mean birth mass $= \dfrac{75 + 70 + 80 + ... + 83}{8} = \dfrac{567}{8} \approx 70.9$ grams

b mean after 2 weeks $= \dfrac{210 + 200 + 200 + ... + 230}{8} = \dfrac{1681}{8} \approx 210$ grams

c mean increase $\approx (210.13 - 70.88)$ grams ≈ 139 grams

9 $\mu = \dfrac{\sum\limits_{i=1}^{n} x_i}{n} \quad\therefore\quad 11.6 = \dfrac{\sum\limits_{i=1}^{10} x_i}{10} \quad\therefore\quad \sum\limits_{i=1}^{10} x_i = 11.6 \times 10 = 116$

10 Total for first 14 matches $= 14 \times 16.5$ goals $= 231$ goals

$\therefore$ new average $= \dfrac{231 + 21 + 24}{16} = \dfrac{276}{16} = 17.25$ goals per game

11 **a** mean selling price $= \dfrac{146\,400 + 127\,600 + 211\,000 + ... + 162\,500}{10} = \$163\,770$

median selling price $= \dfrac{5\text{th} + 6\text{th}}{2} = \dfrac{146\,400 + 148\,000}{2} = \$147\,200$

These figures differ by \$16 570. There are more selling prices at the lower end of the market.

b **i** Use the mean as it tends to inflate the average house value of that district.

ii Use the median as you want to buy at the lowest price possible.

12
$$\begin{aligned} \frac{5 + 9 + 11 + 12 + 13 + 14 + 17 + x}{8} &= 12 \\ \therefore\ \frac{81 + x}{8} &= 12 \\ \therefore\ 81 + x &= 96 \\ \therefore\ x &= 15 \end{aligned}$$

13
$$\begin{aligned} \frac{3 + 0 + a + a + 4 + a + 6 + a + 3}{9} &= 4 \\ \therefore\ \frac{4a + 16}{9} &= 4 \\ \therefore\ 4a + 16 &= 36 \\ \therefore\ 4a &= 20 \\ \therefore\ a &= 5 \end{aligned}$$

14
$$\begin{aligned} \frac{29 + 36 + 32 + 38 + 35 + 34 + 39 + x}{8} &= 35 \\ \therefore\ \frac{243 + x}{8} &= 35 \\ \therefore\ 243 + x &= 280 \\ \therefore\ x &= 37 \end{aligned}$$

So, her 8th result was 37.

15 Total for first 10 measurements $= 10 \times 15.7$
$= 157$

Total for next 20 measurements $= 20 \times 14.3$
$= 286$

$\therefore$ mean $= \dfrac{157 + 286}{30} \approx 14.8$

16 If there are 9 measurements with a median of 12, then 12 must be one of the unknown measurements. So, the measurements are 7, 9, 11, 12, 13, 14, 17, 19, and a.

mean $= \dfrac{7 + 9 + 11 + 12 + 13 + 14 + 17 + 19 + a}{9} = \dfrac{102 + a}{9}$

$$\begin{aligned} \therefore\ \frac{102 + a}{9} &= 12 \\ \therefore\ 102 + a &= 108 \\ \therefore\ a &= 6 \end{aligned}$$

So, the other measurements are 6 and 12.

17 Scores were 5 7 9 9 10 a b where $a \leqslant b$ say.

$$\text{mean} = \frac{5+7+9+9+10+a+b}{7} = 8$$
$$\therefore \frac{40+a+b}{7} = 8$$
$$\therefore 40+a+b = 56$$
$$\therefore a+b = 16 \qquad \{a \leqslant 12,\ b \leqslant 12\}$$

Possibilities are:

a	5	6	7	8
b	11	10	9	8
	✗	✗	✓	✗

reject as modes are 8 and 9
reject as modes are 9 and 10
reject as modes are 5 and 9

So, the missing results are 7 and 9.

EXERCISE 23B.2

1 **a** The mode is 1 head, as this is the result which occurs most often.

b The median is the average of the 15th and 16th scores $= \frac{1+1}{2} = 1$ head

c

x	f	fx
0	4	0
1	12	12
2	11	22
3	3	9
$\sum$	30	43

$$\text{mean} = \frac{\sum fx}{\sum f} = \frac{43}{30} \approx 1.43 \text{ heads}$$

2 **a** **i**

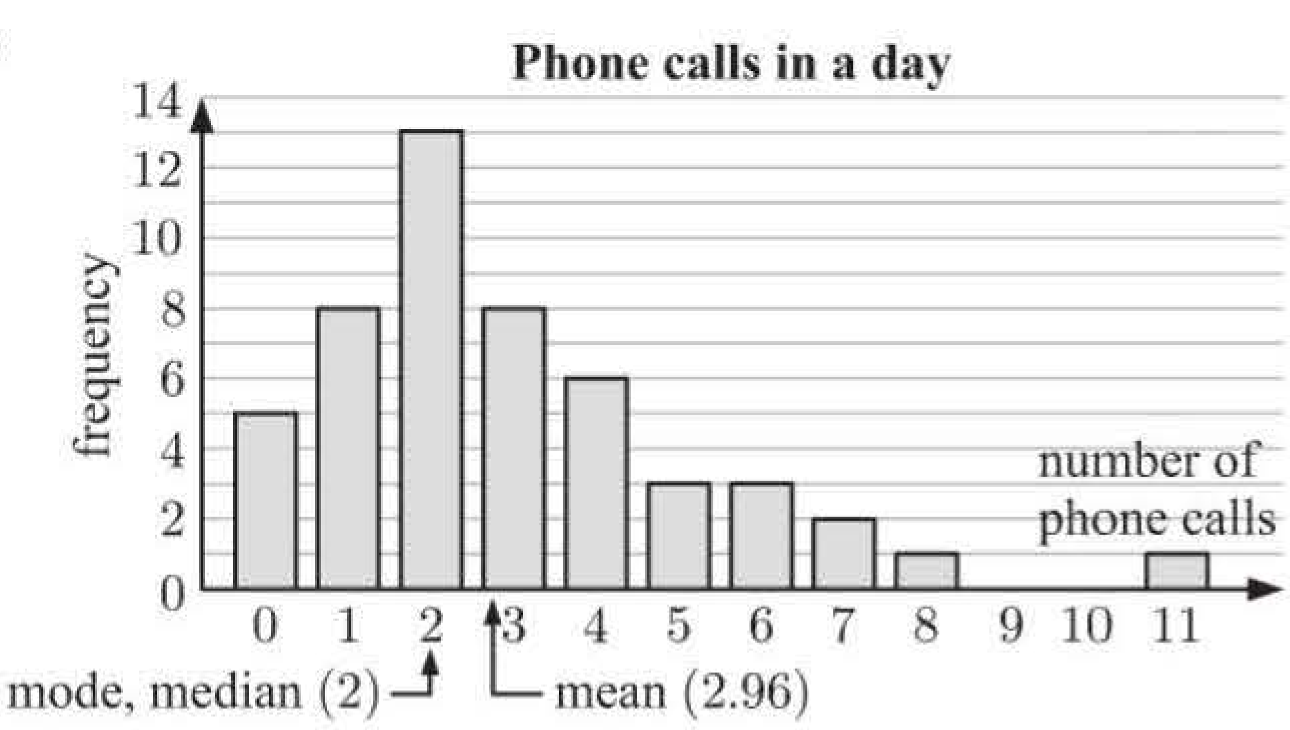
x	f	fx
0	5	0
1	8	8
2	13	26
3	8	24
4	6	24
5	3	15
6	3	18
7	2	14
8	1	8
9	0	0
10	0	0
11	1	11
$\sum$	50	148

$$\text{mean} = \frac{\sum fx}{\sum f} = \frac{148}{50} = 2.96 \text{ phone calls}$$

ii median
= average of 25th and 26th scores (when in order)
$$= \frac{2+2}{2} \quad \left\{\begin{array}{l}\text{13 scores are 1 or 0}\\ \text{26 scores are 2, 1, or 0}\end{array}\right\}$$
= 2 phone calls

iii mode = 2 phone calls {occurs most often}

b



c The distribution is positively skewed. 11 is an outlier.

d The mean takes into account the larger numbers of phone calls.

e The mean, as it best represents all the data.

3 **a** **i** mode = 49 matches {occurs most often}

ii median = average of 15th and 16th values (when in order)
$$= \frac{49+49}{2} = 49 \text{ matches} \quad \{9 \text{ are 47 or 48 and the next 11 are 49}\}$$

iii

x	f	fx
47	5	235
48	4	192
49	11	539
50	6	300
51	3	153
52	1	52
$\sum$	30	1471

$$\text{mean} = \frac{\sum fx}{\sum f} = \frac{1471}{30} \approx 49.0 \text{ matches}$$

b No, as they claim the average is 50 matches per box.

c The sample of only 30 is not large enough. The company could have won its case by arguing that a larger sample would have found an average of 50 matches per box.

4 **a** **i**

x	f	fx
1	5	5
2	28	56
3	15	45
4	8	32
5	2	10
6	1	6
$\sum$	59	154

$$\text{mean} = \frac{\sum fx}{\sum f} = \frac{154}{59} \approx 2.61 \text{ children}$$

ii mode $= 2$ children {occurs most often}

iii median $=$ 30th score $= 2$ children

b This school has more children per family (2.61) than the average Australian family (2.2).

c Positive as the higher values are more spread out.

d The mean is higher than the mode and median.

5 **a**

Donation (x)	*Frequency* (f)	fx
1	7	7
2	9	18
3	2	6
4	4	16
5	8	40
Total	30	87

b Total number of donations $= 7+9+2+4+8$
$= 30$

c **i** $$\text{mean} = \frac{\sum fx}{\sum f} = \frac{87}{30} = 2.9$$

$\therefore$ the mean donation is \$2.90.

ii mode $= \$2$ {occurs most often}

iii median

$=$ average of the 15th and 16th values (when in order)

$$= \frac{2+2}{2} = 2$$

$\therefore$ the median donation is \$2.

d The mode can be found easily using the graph only, as it is the value with the tallest column.

6 **a**

$$\text{mean} = \frac{\sum fx}{\sum f}$$

$$\therefore\ 4.45 = \frac{1\times 0 + 2\times 2 + 3\times 3 + 4\times 5 + 5\times x + 6\times 4 + 7\times 1}{0+2+3+5+x+4+1}$$

$$\therefore\ 4.45 = \frac{64+5x}{15+x}$$

$$\therefore\ 4.45(15+x) = 64+5x$$

$$\therefore\ 2.75 = 0.55x$$

$$\therefore\ x = 5$$

b From a total of $2+3+5+5+4+1 = 20$ students, $5+5+4+1 = 15$ students scored 4 or more.

$\therefore\ \frac{15}{20} = 75\%$ of the students passed.

7 **a** **Without fertiliser**

2	||
3	卌 卌 |
4	卌 卌 卌 ||||
5	卌 卌 卌 卌 卌 ||||
6	卌 卌 卌 卌 卌 卌 卌 卌 卌 卌 |
7	卌 卌 卌 卌 卌
8	卌 卌 ||
9	|

x	f	fx	cf
2	2	4	2
3	11	33	13
4	19	76	32
5	29	145	61
6	51	306	112
7	25	175	137
8	12	96	149
9	1	9	150

i mean $= \dfrac{\sum fx}{\sum f} = \dfrac{844}{150} \approx 5.63$ peas/pod **ii** mode $= 6$ peas/pod {occurs most often}

iii median = average of 75th and 76th scores $= \dfrac{6+6}{2} = 6$ peas/pod

b **With fertiliser**

3	||||
4	卌 卌 |||
5	卌 卌 |
6	卌 卌 卌 卌 卌 |||
7	卌 卌 卌 卌 卌 卌 卌 卌 卌 ||
8	卌 卌 卌 卌 卌 ||
9	卌 卌 ||||
10	||||
11	|
13	|

x	f	fx	cf
3	4	12	4
4	13	52	17
5	11	55	28
6	28	168	56
7	47	329	103
8	27	216	130
9	14	126	144
10	4	40	148
11	1	11	149
13	1	13	150

i mean $= \dfrac{\sum fx}{\sum f} = \dfrac{1022}{150} \approx 6.81$ peas/pod **ii** mode $= 7$ peas/pod {occurs most often}

iii median = average of 75th and 76th scores $= \dfrac{7+7}{2} = 7$ peas/pod

c The mean best represents the centre for this data.

d Yes, as a mean of 6.81 peas per pod is significantly greater than a mean of 5.63 peas per pod.

Note: The total yield of the crop may not have improved as, for example, the number of pods per plant may have decreased when using the fertiliser.

8 The 31 scores in order are: {15 scores below 10}, 10.1, 10.4, 10.7, 10.9, {12 scores above 11}
Median = 16th score (when in order) = 10.1 cm

9 **a** **Brand A**

x	f	fx
46	1	46
47	2	94
48	3	144
49	7	343
50	10	500
51	20	1020
52	15	780
53	3	159
55	1	55
$\sum$	62	3141

$$\text{mean} = \frac{\sum fx}{\sum f} = \frac{3141}{62} \approx 50.7$$

Brand B

x	f	fx
48	3	144
49	17	833
50	30	1500
51	7	357
52	2	104
53	1	53
54	1	54
$\sum$	61	3045

$$\text{mean} = \frac{\sum fx}{\sum f} = \frac{3045}{61} \approx 49.9$$

b Based on average contents, the C.P.S. should not prosecute either manufacturer. To the nearest toothpick, the average contents for A is 51 and for B is 50.

10 **a** **i** median salary

$= \dfrac{10\text{th} + 11\text{th}}{2}$ (when in order)

$= \dfrac{35\,000 + 28\,000}{2}$

$= €31\,500$

ii modal salary $= €28\,000$ {occurs most often}

iii

x	f	fx
50 000	1	50 000
42 000	3	126 000
35 000	6	210 000
28 000	10	280 000
$\sum$	20	666 000

mean $= \dfrac{\sum fx}{\sum f}$

$= \dfrac{666\,000}{20}$

$= €33\,300$

b The mode, as it is the most commonly occurring value.

EXERCISE 23B.3

1

midpoint (x)	f	fx
4.5	2	9
14.5	5	72.5
24.5	7	171.5
34.5	27	931.5
44.5	9	400.5
$\sum$	50	1585

$\therefore$ mean result $\approx \dfrac{1585}{50}$

≈ 31.7

2

midpoint (x)	f	fx
2500	4	10 000
3500	4	14 000
4500	9	40 500
5500	14	77 000
6500	23	149 500
7500	16	120 000
$\sum$	70	411 000

a 70

b $\approx 411\,000$ litres

≈ 411 kL

c mean

$\approx \dfrac{\sum fx}{\sum f}$

$\approx \dfrac{411\,000}{70}$

≈ 5870 litres

3 **a** $5 + 10 + 25 + 40 + 10 + 15 + 10 + 10 = 125$ people

b

midpoint (x)	*frequency* (f)	fx
85	5	425
95	10	950
105	25	2625
115	40	4600
125	10	1250
135	15	2025
145	10	1450
155	10	1550
$\sum$	125	14 875

mean

$\approx \dfrac{\sum fx}{\sum f}$

$\approx \dfrac{14\,875}{125}$

≈ 119 marks

c $\frac{15}{125} = \frac{3}{25}$ scored < 100

d There are $15 + 10 + 10 = 35$ people who scored more than 130 for the test.

$\therefore$ % who scored more than $130 = \frac{35}{125} \times 100\% = 28\%$

EXERCISE 23C.1

1 **a** Looking at the graphs, Sample A appears to have the wider spread.

b *Sample A*:

x	f	fx
4	1	4
5	2	10
6	3	18
7	4	28
8	5	40
9	4	36
10	3	30
11	2	22
12	1	12
$\sum$	25	200

$\therefore$ mean $= \frac{200}{25} = 8$

Sample B:

x	f	fx
6	2	12
7	6	42
8	9	72
9	6	54
10	2	20
$\sum$	25	200

$\therefore$ mean $= \frac{200}{25} = 8$

c *Sample A*:

x	$x-\mu$	$(x-\mu)^2$	f	$f(x-\mu)^2$
4	-4	16	1	16
5	-3	9	2	18
6	-2	4	3	12
7	-1	1	4	4
8	0	0	5	0
9	1	1	4	4
10	2	4	3	12
11	3	9	2	18
12	4	16	1	16
			$\sum$	100

$$\therefore \quad \sigma = \sqrt{\frac{\sum(x-\mu)^2}{n}} = \sqrt{\tfrac{100}{25}} = 2$$

Sample B:

x	$x-\mu$	$(x-\mu)^2$	f	$f(x-\mu)^2$
6	-2	4	2	8
7	-1	1	6	6
8	0	0	9	0
9	1	1	6	6
10	2	4	2	8
			$\sum$	28

$$\therefore \quad \sigma = \sqrt{\frac{\sum(x-\mu)^2}{n}} = \sqrt{\tfrac{28}{25}} \approx 1.06$$

The standard deviation is higher for Sample A.

$\therefore$ Sample A has a greater spread.

d The standard deviation is calculated using all data values, not just two. (Range only uses maximum and minimum; IQR only uses the upper and lower quartiles.)

2 **a** *Andrew*: $\mu = \dfrac{23+17+....+28+32}{8} = 25$

x	$x-\mu$	$(x-\mu)^2$
23	-2	4
17	-8	64
31	6	36
25	0	0
25	0	0
19	-6	36
28	3	9
32	7	49
	$\sum$	198

$$\therefore \quad \sigma = \sqrt{\frac{\sum(x-\mu)^2}{n}} = \sqrt{\tfrac{198}{8}} \approx 4.97$$

Brad: $\mu = \dfrac{9+29+....+38+43}{8} = 30.5$

x	$x-\mu$	$(x-\mu)^2$
9	-21.5	462.25
29	-1.5	2.25
41	10.5	110.25
26	-4.5	20.25
14	-16.5	272.25
44	13.5	182.25
38	7.5	56.25
43	12.5	156.25
	$\sum$	1262

$$\therefore \quad \sigma = \sqrt{\frac{\sum(x-\mu)^2}{n}} = \sqrt{\tfrac{1262}{8}} \approx 12.6$$

b Andrew, as he has the smaller standard deviation.

3 **a** Rockets have mean $= \dfrac{0+10+1+9+11+0+8+5+6+7}{10} = \dfrac{57}{10} = 5.7$ runs

Bullets have mean $= \dfrac{4+3+4+1+4+11+7+6+12+5}{10} = \dfrac{57}{10} = 5.7$ runs

Rockets' range $= 11 - 0 = 11$ runs Bullets' range $= 12 - 1 = 11$ runs

b We suspect the Rockets, as they have two zeros.

c *Rockets*

x	$(x-\mu)^2$
0	$(-5.7)^2$
10	$(4.3)^2$
1	$(-4.7)^2$
9	$(3.3)^2$
11	$(5.3)^2$
0	$(-5.7)^2$
8	$(2.3)^2$
5	$(-0.7)^2$
6	$(0.3)^2$
7	$(1.3)^2$
	152.1

$\sigma = \sqrt{\dfrac{\sum(x-\mu)^2}{n}}$

$= \sqrt{\dfrac{152.1}{10}}$

$= 3.9$ runs

greater variability

Bullets

x	$(x-\mu)^2$
4	$(-1.7)^2$
3	$(-2.7)^2$
4	$(-1.7)^2$
1	$(-4.7)^2$
4	$(-1.7)^2$
11	$(5.3)^2$
7	$(1.3)^2$
6	$(0.3)^2$
12	$(6.3)^2$
5	$(-0.7)^2$
	108.1

$\sigma = \sqrt{\dfrac{\sum(x-\mu)^2}{n}}$

$= \sqrt{\dfrac{108.1}{10}}$

≈ 3.29 runs

d The standard deviation, as it takes all values into account, not just the lowest and highest.

4 **a** We suspect variability in standard deviation since the factors may change every day.

b **i** mean **ii** standard deviation

c A low standard deviation would mean less variability in the volume of soft drink per can.

5 **a**

x	$(x-\mu)^2$
79	10^2
64	$(-5)^2$
59	$(-10)^2$
71	2^2
68	$(-1)^2$
68	$(-1)^2$
74	5^2
483	256

$\mu = \dfrac{\sum x}{n}$

$= \dfrac{483}{7} = 69$ kg

$\sigma = \sqrt{\dfrac{\sum(x-\mu)^2}{n}}$

$= \sqrt{\dfrac{256}{7}}$

≈ 6.05 kg

b

x	$(x-\mu)^2$
89	10^2
74	$(-5)^2$
69	$(-10)^2$
81	2^2
78	$(-1)^2$
78	$(-1)^2$
84	5^2
553	256

$\mu = \dfrac{\sum x}{n}$

$= \dfrac{553}{7}$

$= 79$ kg

$\sigma \approx 6.05$ kg

c The distribution has simply shifted by 10 kg. The mean increases by 10 kg and the standard deviation remains the same. In general, changing the values by a constant does not affect standard deviation.

6 **a**

x	$(x-\mu)^2$
0.8	$(-0.21)^2$
1.1	$(0.09)^2$
1.2	$(0.19)^2$
0.9	$(-0.11)^2$
1.2	$(0.19)^2$
1.2	$(0.19)^2$
0.9	$(-0.11)^2$
0.7	$(-0.31)^2$
1.0	$(-0.01)^2$
1.1	$(0.09)^2$
10.1	0.289

$\mu = \dfrac{\sum x}{n}$

$= \dfrac{10.1}{10}$

$= 1.01$ kg

$\sigma = \sqrt{\dfrac{\sum(x-\mu)^2}{n}}$

$= \sqrt{\dfrac{0.289}{10}}$

$= 0.17$ kg

b

x	$(x-\mu)^2$
1.6	$(-0.42)^2$
2.2	$(0.18)^2$
2.4	$(0.38)^2$
1.8	$(-0.22)^2$
2.4	$(0.38)^2$
2.4	$(0.38)^2$
1.8	$(-0.22)^2$
1.4	$(-0.62)^2$
2.0	$(-0.02)^2$
2.2	$(0.18)^2$
20.2	1.156

$\mu = \dfrac{\sum x}{n}$

$= \dfrac{20.2}{10}$

$= 2.02$ kg

$\sigma = \sqrt{\dfrac{\sum(x-\mu)^2}{n}}$

$= \sqrt{\dfrac{1.156}{10}}$

$= 0.34$ kg

c Doubling the values doubles the mean and the standard deviation.

7 **a** $\mu = \dfrac{0.8+0.6+0.7+0.8+0.4+2.8}{6}$

≈ 1.017

x	$(x-\mu)^2$
0.8	$(-0.217)^2$
0.6	$(-0.417)^2$
0.7	$(-0.317)^2$
0.8	$(-0.217)^2$
0.4	$(-0.617)^2$
2.8	$(1.783)^2$
$\sum$	3.928

$\therefore \quad \sigma = \sqrt{\dfrac{\sum(x-\mu)^2}{n}} \approx \sqrt{\dfrac{3.928}{6}}$

≈ 0.809

b $\mu = \dfrac{0.8+0.6+0.7+0.8+0.4}{5}$

$= 0.66$

x	$(x-\mu)^2$
0.8	$(0.14)^2$
0.6	$(-0.06)^2$
0.7	$(0.04)^2$
0.8	$(0.14)^2$
0.4	$(-0.26)^2$
$\sum$	0.112

$\therefore \quad \sigma = \sqrt{\dfrac{\sum(x-\mu)^2}{n}} = \sqrt{\dfrac{0.112}{5}}$

≈ 0.150

c The extreme value greatly increases the standard deviation.

8 $\mu = \dfrac{1+3+5+7+4+5+p+q}{8} = 5$

$\therefore \quad 25+p+q = 40$

$\therefore \quad p+q = 15$

$\therefore \quad q = 15-p$

and $\sigma = \sqrt{\dfrac{(-4)^2+(-2)^2+0^2+2^2+(-1)^2+0^2+(p-5)^2+(q-5)^2}{8}} = \sqrt{5.25}$

$\therefore \quad \dfrac{16+4+4+1+(p-5)^2+(15-p-5)^2}{8} = 5.25$

$\therefore \quad 25+p^2-10p+25+100-20p+p^2 = 42$

$\therefore \quad 2p^2-30p+108 = 0$

$\therefore \quad p^2-15p+54 = 0$

$\therefore \quad (p-6)(p-9) = 0$

$\therefore \quad p = 6$ or 9 and $q = 9$ or 6

But $p < q$ $\quad \therefore \quad p = 6, \; q = 9$

9 $\mu = \dfrac{3+9+5+5+6+4+a+6+b+8}{10} = 6$

$\therefore \quad \dfrac{46+a+b}{10} = 6$

$\therefore \quad 46+a+b = 60$

$\therefore \quad a+b = 14$

$\therefore \quad b = 14-a$

and $\sigma = \sqrt{\dfrac{(-3)^2+3^2+(-1)^2+(-1)^2+(-2)^2+(a-6)^2+(b-6)^2+2^2}{10}} = \sqrt{3.2}$

$\therefore \quad 9+9+1+1+4+4+(a-6)^2+(14-a-6)^2 = 32$

$\therefore \quad 28+a^2-12a+36+64-16a+a^2 = 32$

$\therefore \quad 2a^2-28a+96 = 0$

$\therefore \quad a^2-14a+48 = 0$

$\therefore \quad (a-6)(a-8) = 0$

$\therefore \quad a = 6$ or 8 and $b = 8$ or 6

But $a > b$ $\quad \therefore \quad a = 8, \; b = 6$

10 **a** $\sum_{i=1}^{n}(x_i-\mu)^2=\sum_{i=1}^{n}(x_i^{\,2})-2\mu\sum_{i=1}^{n}x_i+\sum_{i=1}^{n}\mu^2$

$=\sum_{i=1}^{n}(x_i^{\,2})-2\mu(x_1+x_2+....+x_n)+n\mu^2$

$=\sum_{i=1}^{n}(x_i^{\,2})-2\mu(n\mu)+n\mu^2 \qquad \{\mu=\frac{x_1+x_2+....+x_n}{n}\}$

$=\sum_{i=1}^{n}(x_i^{\,2})-n\mu^2$

b $\sigma=\sqrt{\frac{\sum_{i=1}^{25}(x_i-\mu)^2}{25}}=5.2$

$\therefore \frac{\sum_{i=1}^{25}(x_i-\mu)^2}{25}=27.04$

$\therefore \sum_{i=1}^{25}(x_i-\mu)^2=676$

$\therefore \sum_{i=1}^{25}(x_i^{\,2})-25\mu^2=676$ {using **a**}

$\therefore 2568.25-25\mu^2=676$

$\therefore 1892.25=25\mu^2$

$\therefore \mu^2=75.69$

$\therefore \mu=8.7$ {assuming positive data}

EXERCISE 23C.2

1

x	f	fx	$x-\mu$	$f(x-\mu)^2$
0	14	0	-1.7241	41.62
1	18	18	-0.7241	9.44
2	13	26	0.2759	0.99
3	5	15	1.2759	8.14
4	3	12	2.2759	15.54
5	2	10	3.2759	21.46
6	2	12	4.2759	36.57
7	1	7	5.2759	27.83
$\sum$	58	100		161.59

$\mu=\frac{\sum fx}{\sum f}=\frac{100}{58}\approx 1.72$ children

$\sigma=\sqrt{\frac{\sum f(x-\mu)^2}{\sum f}}\approx\sqrt{\frac{161.59}{58}}\approx 1.67$ children

2

x	f	fx	$x-\mu$	$f(x-\mu)^2$
11	2	22	-3.48	24.22
12	1	12	-2.48	6.150
13	4	52	-1.48	8.762
14	5	70	-0.48	1.152
15	6	90	0.52	1.622
16	4	64	1.52	9.242
17	2	34	2.52	12.70
18	1	18	3.52	12.39
$\sum$	25	362		76.24

$\mu=\frac{\sum fx}{\sum f}=\frac{362}{25}\approx 14.5$ years

$\sigma=\sqrt{\frac{\sum f(x-\mu)^2}{\sum f}}=\sqrt{\frac{76.24}{25}}\approx 1.75$ years

3

x	f	fx	$x-\mu$	$f(x-\mu)^2$
33	1	33	-4.2708	18.24
35	5	175	-2.2708	25.78
36	7	252	-1.2708	11.30
37	13	481	-0.2708	0.95
38	12	456	0.7292	6.38
39	8	312	1.7292	23.92
40	2	80	2.7292	14.90
$\sum$	48	1789		101.47

$\mu=\frac{\sum fx}{\sum f}=\frac{1789}{48}\approx 37.3$ toothpicks

$\sigma=\sqrt{\frac{\sum f(x-\mu)^2}{\sum f}}=\sqrt{\frac{101.47}{48}}\approx 1.45$ toothpicks

4

Midpoint (x)	f	fx	$f(x-\mu)^2$
41	1	41	52.80
43	1	43	27.74
45	3	135	32.01
47	7	329	11.23
49	11	539	5.91
51	5	255	37.35
53	2	106	44.80
$\sum$	30	1448	211.87

$$\mu = \frac{\sum fx}{\sum f} = \frac{1448}{30} \approx 48.3 \text{ cm}$$

$$\sigma = \sqrt{\frac{\sum f(x-\mu)^2}{\sum f}} \approx \sqrt{\frac{211.87}{30}} \approx 2.66 \text{ cm}$$

5

Midpoint (x)	f	fx	$f(x-\mu)^2$
364.995	17	6204.9	10 881.53
374.995	38	14 249.8	8895.42
384.995	47	18 094.8	1320.23
394.995	57	22 514.7	1259.13
404.995	18	7289.9	3889.62
414.995	10	4150.0	6100.9
424.995	10	4250.0	12 040.9
434.995	3	1305.0	5994.27
$\sum$	200	78 059	50 382

$$\mu = \frac{\sum fx}{\sum f} = \frac{78\,059}{200} \approx \$390.30$$

$$\sigma = \sqrt{\frac{\sum f(x-\mu)^2}{\sum f}} = \sqrt{\frac{50\,382}{200}} \approx \$15.87$$

REVIEW SET 23A

1 **a** **i** There are 30 colonies.

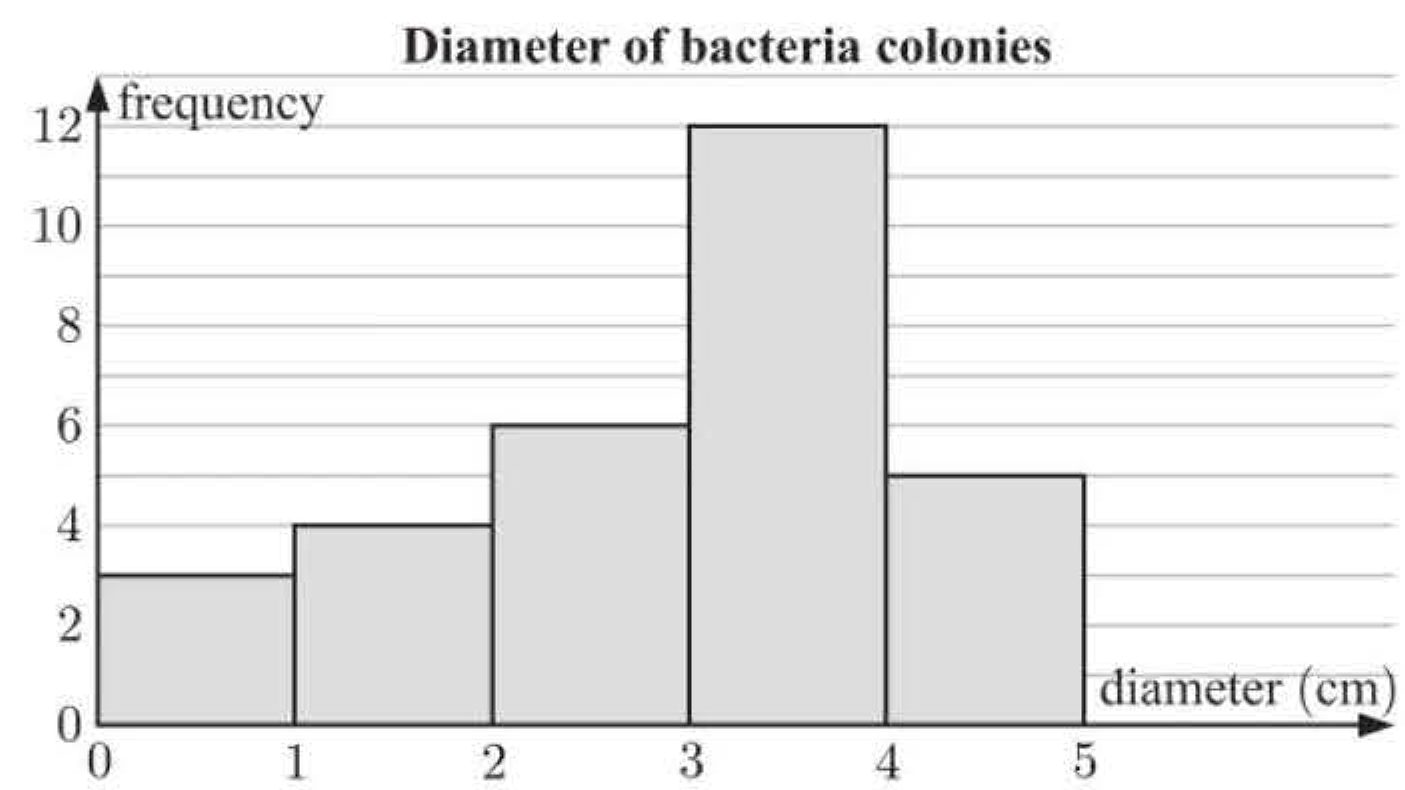
$\therefore$ median = average of 15th and 16th colonies

$$= \frac{3.1+3.2}{2} = 3.15 \text{ cm}$$

ii range $= 4.9 - 0.4 = 4.5$ cm

b

Diameter	*Tally*	*Frequency*
0.0 - 0.9	|||	3
1.0 - 1.9	||||	4
2.0 - 2.9	~~||||~~ |	6
3.0 - 3.9	~~||||~~ ~~||||~~ ||	12
4.0 - 4.9	~~||||~~	5



c The distribution is slightly negatively skewed.

2 If the mode is 6 then one of the unknown numbers must be 6.
Suppose the other unknown number is x.

$$\therefore \frac{4+6+9+6+3+x}{6} = 6$$

$$\therefore 28 + x = 36$$

$$\therefore x = 8$$

Since $a > b$, $a = 8$ and $b = 6$.

3 **a** We first organise the data into tables:

Girls:

Time x (s)	f	fx
33	1	33
34	3	102
35	5	175
36	4	144
37	4	148
38	1	38
39	1	39
40	0	0
41	1	41
Total	20	720

Boys:

Time x (s)	f	fx
32	1	32
33	4	132
34	5	170
35	6	210
36	3	108
37	1	37
Total	20	689

Both boys and girls have 20 member squads, so the median is the average of the 10th and 11th swimmer.

Girls:
$$\text{median} = \frac{36+36}{2} = 36 \text{ s}$$
$$\text{mean} = \frac{\sum fx}{\sum f} = \frac{720}{20} = 36 \text{ s}$$

The tallest column on the *Girls* histogram is the '35' column.

$\therefore$ the modal class is 34.5 - 35.5 s.

Boys:
$$\text{median} = \frac{34+35}{2} = 34.5 \text{ s}$$
$$\text{mean} = \frac{\sum fx}{\sum f} = \frac{689}{20} = 34.45 \text{ s}$$

The tallest column on the *Boys* histogram is the '35' column.

$\therefore$ the modal class is 34.5 - 35.5 s.

So,

Distribution	*Girls*	*Boys*
shape	positively skewed	approximately symmetrical
median	36 s	34.5 s
mean	36 s	34.45 s
modal class	34.5 - 35.5 s	34.5 - 35.5 s

b The girls' distribution is positively skewed and the boys' distribution is approximately symmetrical. The median and mean swim times for boys are both about 1.5 seconds lower than for girls. Despite this, the distributions have the same modal class because of the skewness in the girls' distribution. The analysis supports the conjecture that boys generally swim faster than girls with less spread of times.

4 $$\text{mean} = \frac{2+5+k+k+3+k+7+4}{8} = \frac{21+3k}{8}$$
$$\therefore \frac{21+3k}{8} = 6$$
$$\therefore 21+3k = 48$$
$$\therefore 3k = 27$$
$$\therefore k = 9$$

5 **a** **i** mode $= 4$ {occurs most often}

ii total $= 2+6+12+8+6+5+3+2 = 44$

median = average of 22nd and 23rd data values

$$= \frac{5+5}{2} = 5$$

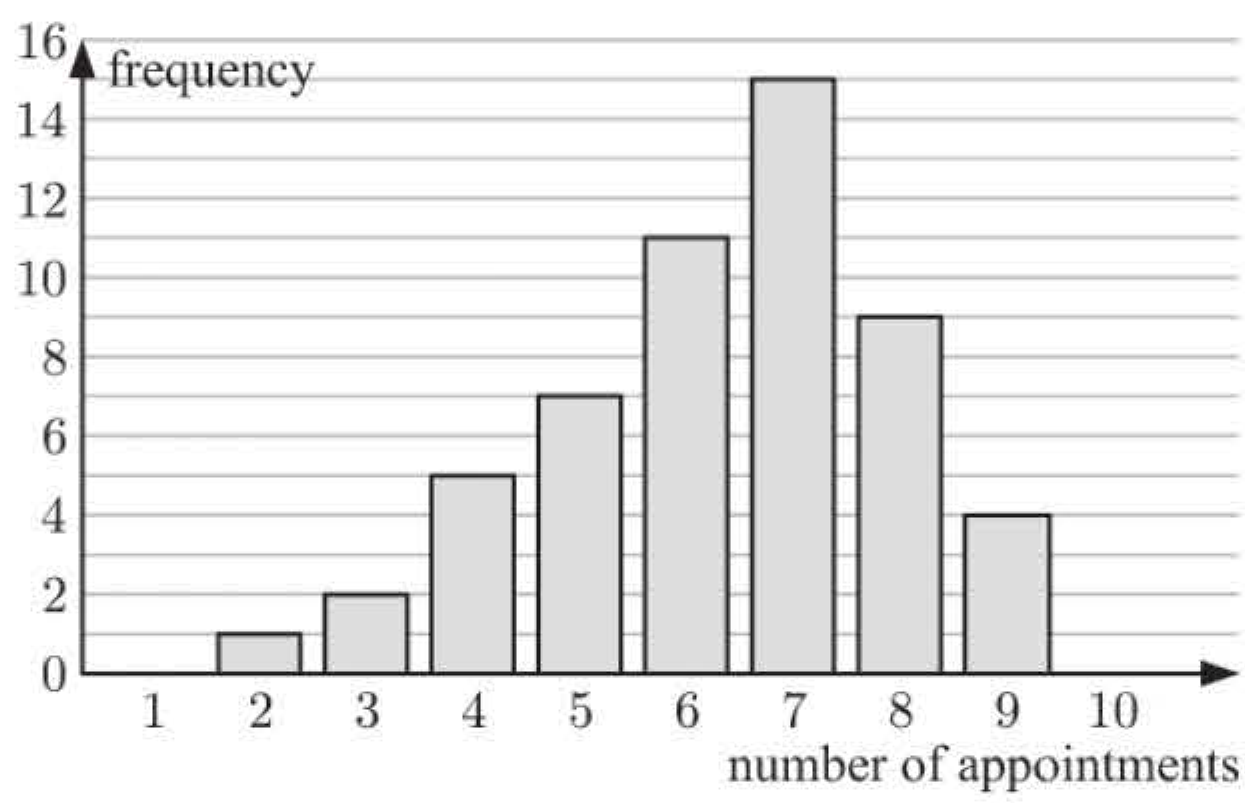
iii

x	f	fx
2	2	4
3	6	18
4	12	48
5	8	40
6	6	36
7	5	35
8	3	24
9	2	18
$\sum$	44	223

$$\text{mean} = \frac{\sum fx}{\sum f} = \frac{223}{44} \approx 5.07$$

b The data is positively skewed.

6 **a**



b The data is negatively skewed.

c **i** mode = 7 appointments {occurs most often}

ii

Number of appointments x	*Frequency f*	fx
2	1	2
3	2	6
4	5	20
5	7	35
6	11	66
7	15	105
8	9	72
9	4	36
$\sum$	54	342

$$\text{mean} = \frac{\sum fx}{\sum f} = \frac{342}{54} \approx 6.33 \text{ appointments}$$

7 $$\frac{a+b+c+d+e}{5} = 8$$

$\therefore\ a+b+c+d+e = 40 \quad .... (1)$

$$\mu = \frac{(10-a)+(10-b)+(20-c)+(20-d)+(50-e)}{5} = \frac{110-(a+b+c+d+e)}{5}$$

$$= \frac{110-40}{5} \quad \{\text{using (1)}\}$$

$$= \frac{70}{5}$$

$$= 14$$

8 $$\mu = \frac{12+13+8+10+14+7+a+b}{8} = 10$$

$\therefore\ 64+a+b = 80$

$\therefore\ a+b = 16$

$\therefore\ b = 16-a$

and $\sigma = \sqrt{\dfrac{2^2+3^2+(-2)^2+0^2+4^2+(-3)^2+(a-10)^2+(b-10)^2}{8}} = \sqrt{8.5}$

$$\therefore \frac{4+9+4+16+9+(a-10)^2+(16-a-10)^2}{8} = 8.5$$
$$\therefore 42+a^2-20a+100+36-12a+a^2 = 68$$
$$\therefore 2a^2-32a+110=0$$
$$\therefore a^2-16a+55=0$$
$$\therefore (a-5)(a-11)=0$$
$$\therefore a=5 \text{ or } 11 \quad \text{and} \quad b=11 \text{ or } 5$$
But $a<b \quad \therefore a=5,\ b=11$

9 **a** $\mu = \dfrac{8+11+12+9+a}{5}$
$= \dfrac{40+a}{5}$
$= 8+\frac{1}{5}a$

b $$\sigma^2 = \frac{\sum {x_i}^2}{n} - \mu^2 = 6$$
$$\therefore \frac{8^2+11^2+12^2+9^2+a^2}{5} - \mu^2 = 6$$
$$\therefore \frac{410+a^2}{5} - \left(\frac{40+a}{5}\right)^2 = 6$$
$$\therefore \frac{5(410+a^2)-(1600+80a+a^2)}{25} = 6$$
$$\therefore 2050+5a^2-1600-80a-a^2 = 150$$
$$\therefore 4a^2-80a+300=0$$
$$\therefore a^2-20a+75=0$$
$$\therefore (a-5)(a-15)=0$$
$$\therefore a=5 \text{ or } 15$$

10 Let the five consecutive integers be $(x-2)$, $(x-1)$, x, $(x+1)$, and $(x+2)$.

$$\mu = \frac{(x-2)+(x-1)+x+(x+1)+(x+2)}{5} = \frac{5x}{5} = x$$

$$\sigma^2 = \frac{\sum(x_i-\mu)^2}{n} = \frac{(x-2-x)^2+(x-1-x)^2+(x-x)^2+(x+1-x)^2+(x+2-x)^2}{5}$$
$$= \frac{(-2)^2+(-1)^2+0^2+1^2+2^2}{5}$$
$$= \frac{4+1+1+4}{5}$$
$$= \frac{10}{5}$$
$$= 2$$

REVIEW SET 23B

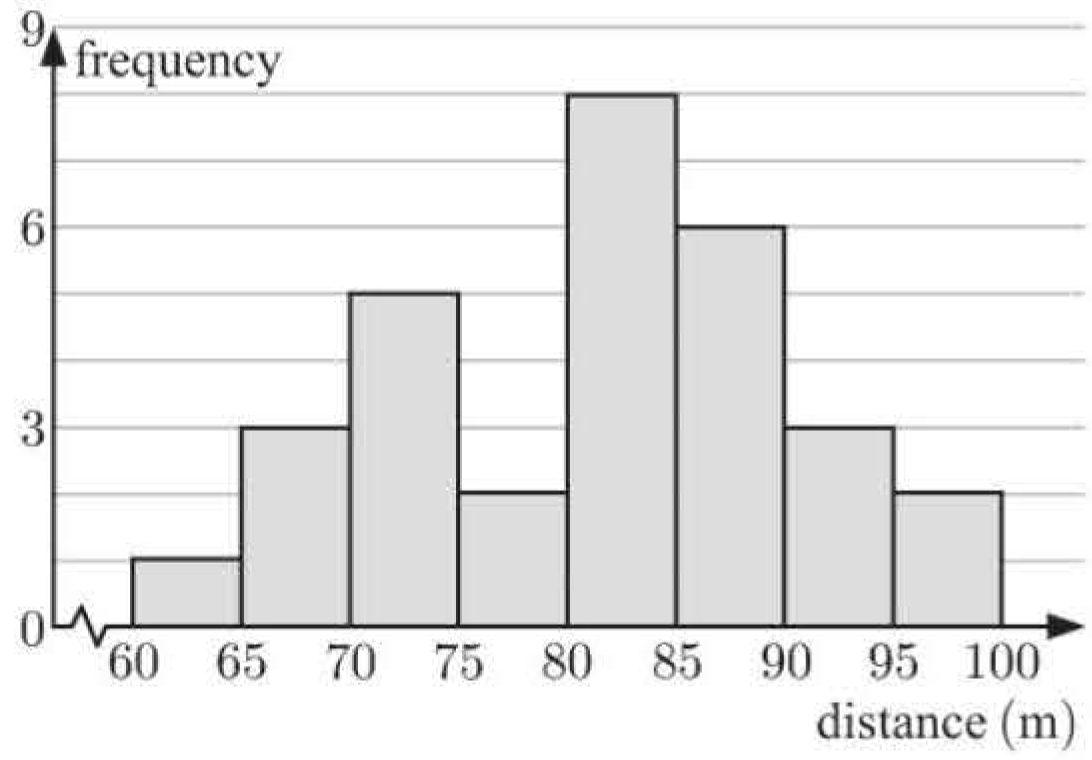
1 **a** highest $= 97.5$ m, lowest $= 64.6$ m

b The range $= 97.5 - 64.6 = 32.9$
So, if intervals of length 5 are used we need about 7 of them.
We choose $60 \leqslant d < 65$, $65 \leqslant d < 70$, $70 \leqslant d < 75$, and so on.

c **Distances thrown by Thabiso**

Distance (m)	*Tally*	*Frequency* (f)
$60 \leqslant d < 65$	$\vert$	1
$65 \leqslant d < 70$	$\vert\vert\vert$	3
$70 \leqslant d < 75$	$\cancel{\vert\vert\vert\vert}$	5
$75 \leqslant d < 80$	$\vert\vert$	2
$80 \leqslant d < 85$	$\cancel{\vert\vert\vert\vert}\ \vert\vert\vert$	8
$85 \leqslant d < 90$	$\cancel{\vert\vert\vert\vert}\ \vert$	6
$90 \leqslant d < 95$	$\vert\vert\vert$	3
$95 \leqslant d < 100$	$\vert\vert$	2
	Total	30

d **Frequency histogram displaying the distance Thabiso throws a baseball**



e Using technology:

i $\mu \approx 81.1$ m **ii** median ≈ 83.1 m

2 **a** $$\text{mean} = \frac{(k-2)+k+(k+3)+(k+3)}{4} = \frac{4k+4}{4} = \frac{\cancel{4}(k+1)}{\cancel{4}} = k+1 \quad \text{as required}$$

b The members are now

$k,\ k+2,\ k+5,$ and $k+5$.

$\therefore$ new mean

$$= \frac{k+(k+2)+(k+5)+(k+5)}{4} = \frac{4k+12}{4} = \frac{\cancel{4}(k+3)}{\cancel{4}} = k+3$$

3

Scores	f	midpt x	fx	$x-\mu$	$f(x-\mu)^2$
$0 \leqslant x < 10$	1	5	5	-21	441
$10 \leqslant x < 20$	13	15	195	-11	1573
$20 \leqslant x < 30$	27	25	675	-1	27
$30 \leqslant x < 40$	17	35	595	9	1377
$40 \leqslant x < 50$	2	45	90	19	722
$\sum$	60		1560		4140

$$\mu = \frac{\sum fx}{\sum f} = \frac{1560}{60} = 26$$

$$\sigma = \sqrt{\frac{\sum f(x-\mu)^2}{\sum f}} = \sqrt{\frac{4140}{60}} \approx 8.31$$

4

Litres (x)	f	fx	$f(x-\mu)^2$
17.5	5	87.5	1299.38
22.5	13	292.5	1607.71
27.5	17	467.5	636.87
32.5	29	942.5	36.42
37.5	27	1012.5	406.32
42.5	18	765	1419.16
47.5	7	332.5	1348.45
$\sum$	116	3900	6754.31

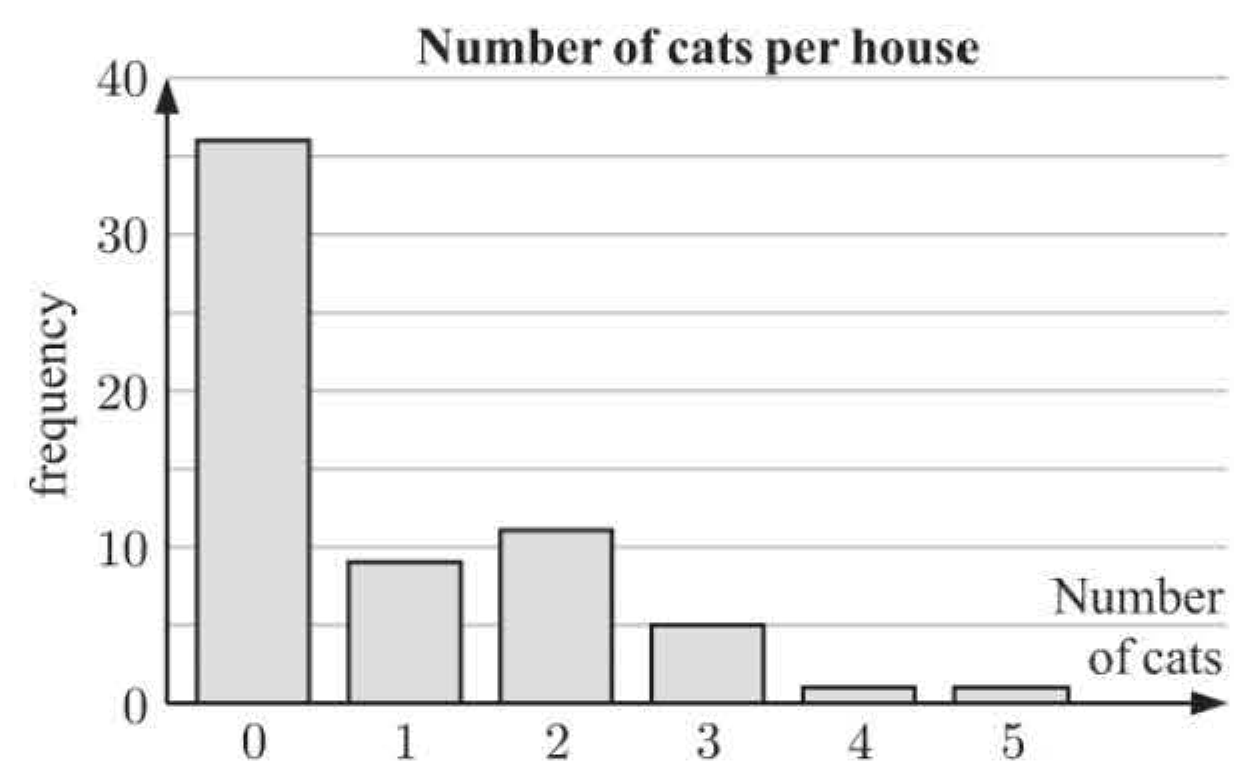
$$\mu = \frac{\sum fx}{\sum f} = \frac{3900}{116} \approx 33.6 \text{ litres}$$

$$\sigma = \sqrt{\frac{\sum f(x-\mu)^2}{\sum f}} \approx \sqrt{\frac{6754.31}{116}} \approx 7.63 \text{ litres}$$

5 **a** Using technology, $\mu \approx 49.6$, $\sigma \approx 1.60$.

b This does not justify the claim. A larger sample is needed.

6 **a**



b The distribution is positively skewed.

c **i** The mode is 0 cats.

ii

Number of cats	*Frequency*	fx
0	36	0
1	9	9
2	11	22
3	5	15
4	1	4
5	1	5
Total	63	55

$$\mu = \frac{\sum fx}{\sum f} = \frac{55}{63} \approx 0.873$$

iii There are 63 values, so the median is the $\frac{63+1}{2} = 32$nd value.

$\therefore$ median $= 0$ cats.

d The mean, as it suggests that some people have cats. (The mode and median are both 0.)

7 **a** total cars $= 32 + 85 + 123 + 97 + 62 + 27 = 426$ cars

b

Midpoint (x)	f	fx	$f(x-\mu)^2$
0.5	32	16	178.100
1.5	85	127.5	157.021
2.5	123	307.5	15.866
3.5	97	339.5	39.836
4.5	62	279	166.927
5.5	27	148.5	188.300
$\sum$	426	1218	746.05

$$\mu = \frac{\sum fx}{\sum f} = \frac{1218}{426} \approx 2.86 \text{ cars}$$

$$\sigma = \sqrt{\frac{\sum f(x-\mu)^2}{\sum f}} = \sqrt{\frac{746.05}{426}} \approx 1.32 \text{ cars}$$

8 **a** $\mu = \dfrac{42+58+74+62+51+45+73+54+66+84}{10}$

$= 60.9$

b median = average of 5th and 6th values (when in order)

$= \dfrac{58+62}{2} = 60$

c

x	$(x-\mu)^2$
42	357.21
58	8.41
74	171.61
62	1.21
51	98.01
45	252.81
73	146.41
54	47.61
66	26.01
84	533.61
$\sum$	1642.9

$$\sigma = \sqrt{\frac{\sum(x-\mu)^2}{n}} = \sqrt{\frac{1642.9}{10}} \approx 12.8$$

9 **a** Kevin: $\mu = 41.2$, Felicity: $\mu = 39.5$ {using technology}

b Kevin: $\sigma \approx 7.61$, Felicity: $\sigma \approx 9.22$ {using technology}

c Felicity has a lower mean, so she generally solved the puzzles faster.

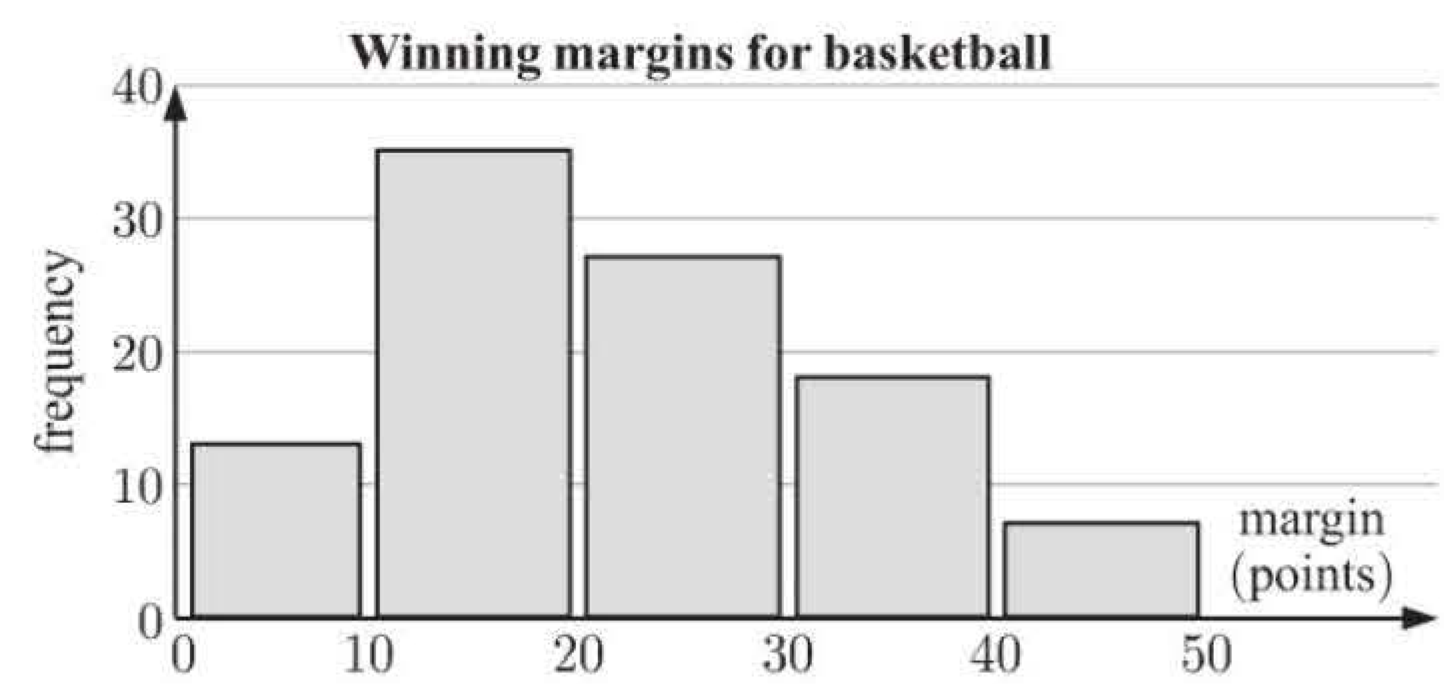
d Kevin has a lower standard deviation, so he was more consistent.

10
$$\begin{aligned}\sigma &= \frac{\sum x_i^2}{n} - \mu^2 \\ &= \frac{\sum_{i=1}^{20} x_i^2}{20} - \mu^2 \\ &= \frac{2872}{20} - 11^2 \\ &= 22.6\end{aligned}$$

REVIEW SET 23C

1 **a** The data is discrete.

b



c No, as we do not know each individual data value, only the intervals they fall in.

2 **a**

Score	f	*Product*
2	3	6
5	2	10
x	4	$4x$
$x+6$	1	$x+6$
Total	10	$5x+22$

$$\begin{aligned}\text{mean} &= 5.7 \\ \therefore \frac{5x+22}{10} &= 5.7 \\ \therefore 5x + 22 &= 57 \\ \therefore 5x &= 35 \\ \therefore x &= 7\end{aligned}$$

b The data set is:

Score x	f	$x-\mu$	$(x-\mu)^2$	$f(x-\mu)^2$
2	3	-3.7	13.69	41.07
5	2	-0.7	0.49	0.98
7	4	1.3	1.69	6.76
13	1	7.3	53.29	53.29
$\sum f$	10		$\sum f(x-\mu)^2$	102.1

$$\begin{aligned}\text{Variance} &= \frac{\sum f(x-\mu)^2}{\sum f} \\ &= \frac{102.1}{10} \\ &= 10.21 \\ \therefore \sigma^2 &\approx 10.2\end{aligned}$$

3 Use technology or

Midpoint (x)	f	fx
274.5	14	3843
324.5	34	11 033
374.5	68	25 466
424.5	72	30 564
474.5	54	25 623
524.5	23	12 063.5
574.5	7	4021.5
$\sum$	272	112 614

$$\begin{aligned}\mu &= \frac{\sum fx}{\sum f} \\ &= \frac{112\,614}{272} \\ &\approx 414 \text{ customers}\end{aligned}$$

4 The mode is 36, so at least one of m and n must be 36.

$$\mu = \frac{33+18+25+40+36+41+36+x}{8} = 32$$

$$\therefore \quad 229 + x = 256$$

$$\therefore \quad x = 27 \text{ is the other number.}$$

Now $m < n$, $\quad \therefore \quad m = 27, \; n = 36$

5 **a** In order, the data set is $\{m-3, \; m-2, \; m, \; m+1, \; m+4, \; m+6\}$

median = average of 3rd and 4th values

$$= \frac{m+(m+1)}{2}$$

$$= m + \tfrac{1}{2}$$

$$\text{mean} = \frac{(m-3)+(m-2)+m+(m+1)+(m+4)+(m+6)}{6}$$

$$= \frac{6m+6}{6}$$

$$= m+1$$

b $$\sigma^2 = \frac{\sum (x_i - \mu)^2}{n} = \frac{(m-3-(m+1))^2 + (m-2-(m+1))^2 + ... + (m+6-(m+1))^2}{6}$$

$$= \frac{(-4)^2 + (-3)^2 + (-1)^2 + 0^2 + 3^2 + 5^2}{6}$$

$$= \frac{60}{6}$$

$$= 10$$

6 Using technology with x values 74.995, 84.995, 94.995, and so on, $\mu \approx$ €103.51 and $\sigma \approx$ €19.40

7 **a** No, it will not be the same. Extreme values have less effect on the standard deviation of a larger population.

b **i** The mean would be used. **ii** The standard deviation would be used.

c A low standard deviation means that the weight of biscuits in each packet is, on average, close to 250 g.

8

Midpoint (x)	f	fx	$f(x-\mu)^2$
6.5	5	32.5	21.126
7.5	19	142.5	21.170
8.5	38	323	0.117
9.5	22	209	19.623
10.5	6	63	22.685
$\sum$	90	770	84.72

$$\mu = \frac{\sum fx}{\sum f} = \frac{770}{90} \approx 8.56 \text{ hours}$$

$$\sigma = \sqrt{\frac{\sum f(x-\mu)^2}{\sum f}} \approx \sqrt{\frac{84.72}{90}} \approx 0.970 \text{ hours}$$

9 **a** Roger: $\mu \approx 84.1, \; \sigma \approx 6.60$

Clinton: $\mu \approx 76.8, \; \sigma \approx 3.83$ {using technology}

b Clinton has a lower mean, so he generally has the lower score.

c Roger has a higher standard variation, so he has greater variation in his scores.

Chapter 24
PROBABILITY

EXERCISE 24A

1 **a** $P(\text{inside a square}) = \dfrac{113}{145} \approx 0.779$ **b** $P(\text{on a line}) = \dfrac{32}{145} \approx 0.221$

2 Total frequency $= 17 + 38 + 19 + 4 = 78$

a $P(20 \text{ to } 39 \text{ seconds}) = \frac{38}{78} \approx 0.487$ **b** $P(> 60 \text{ seconds}) = \frac{4}{78} \approx 0.051$

c $P(\text{between 20 and 59 seconds inclusive}) = \dfrac{38+19}{78} \approx 0.731$

3

Calls/day	*No. of days*
0	2
1	7
2	11
3	8
4	7
5	4
6	3
7	0
8	1

a Survey lasted $2+7+11+8+7+4+3+0+1$
$= 43$ days

b **i** $P(0 \text{ calls}) \approx \dfrac{2}{43} \approx 0.0465$ **ii** $P(\geqslant 5 \text{ calls}) \approx \dfrac{4+3+0+1}{43} \approx 0.186$ **iii** $P(< 3 \text{ calls}) \approx \dfrac{2+7+11}{43} \approx 0.465$

4 Total frequency
$= 37 + 81 + 48 + 17 + 6 + 1$
$= 190$

a $P(4 \text{ days gap}) \approx \dfrac{17}{190} \approx 0.0895$ **b** $P(\text{at least 4 days gap}) \approx \dfrac{17+6+1}{190} \approx 0.126$

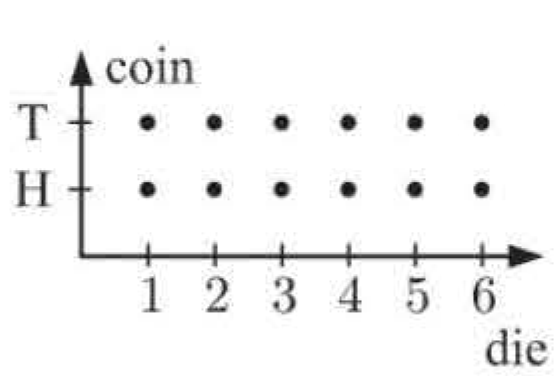
EXERCISE 24B

1 **a** {A, B, C, D} **b** {BB, BG, GB, GG}

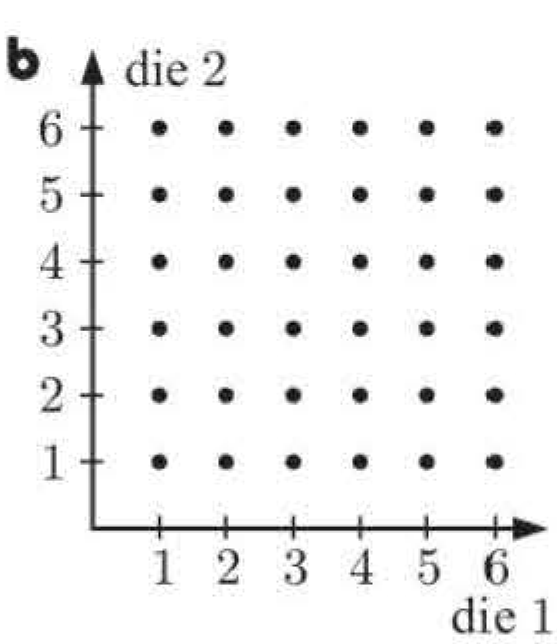
c {ABCD, ABDC, ACBD, ACDB, ADBC, ADCB, BACD, BADC, BCAD, BCDA, BDAC, BDCA, CABD, CADB, CBAD, CBDA, CDAB, CDBA, DABC, DACB, DBAC, DBCA, DCAB, DCBA}

d {GGG, GGB, GBG, BGG, GBB, BGB, BBG, BBB}

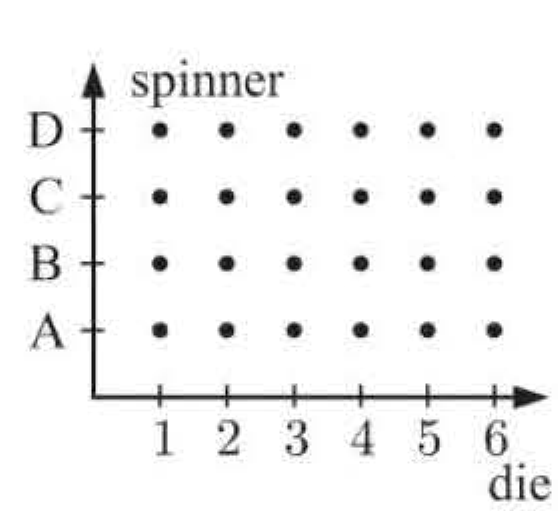
2 **a**



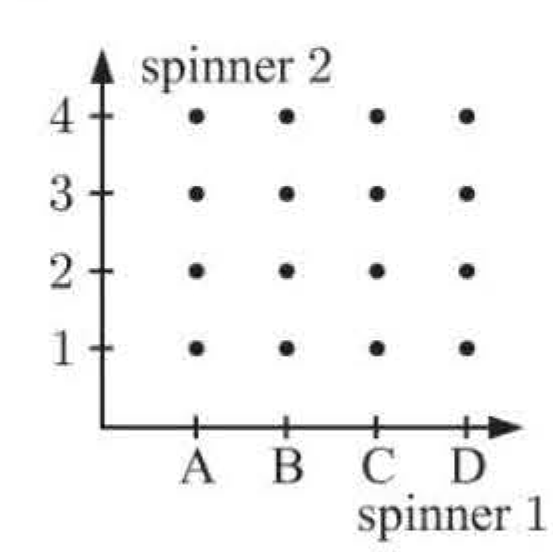
b



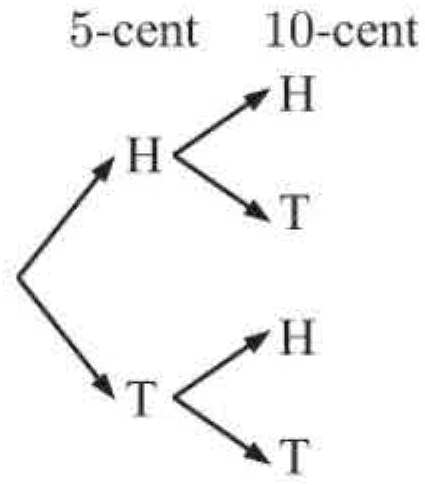
c



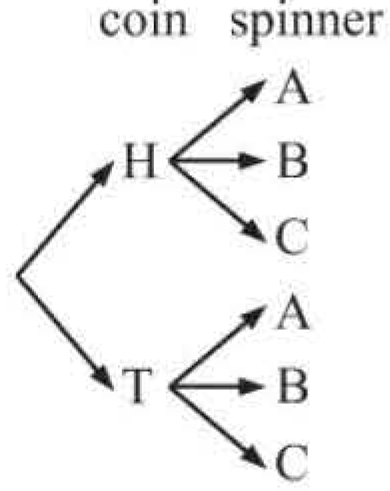
d



3 **a**



b



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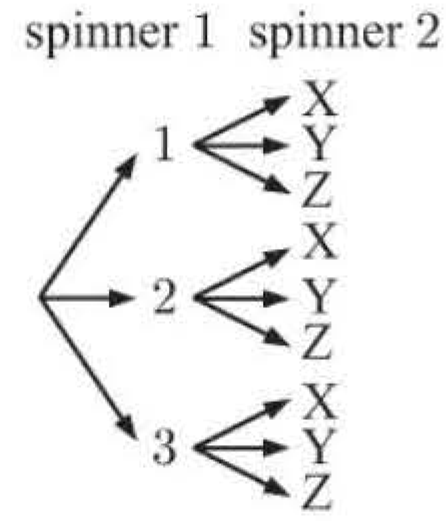
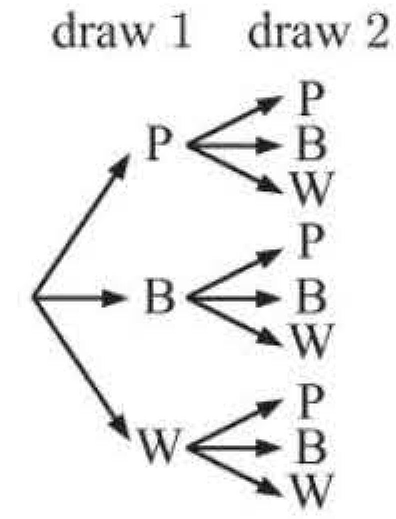


d



EXERCISE 24C.1

1 Total number of marbles $= 5 + 3 + 7 = 15$

a $\text{P(red)} = \frac{3}{15} = \frac{1}{5}$

b $\text{P(green)} = \frac{5}{15} = \frac{1}{3}$

c $\text{P(blue)} = \frac{7}{15}$

d $\text{P(not red)} = \dfrac{5+7}{15} = \frac{12}{15}$ or $\frac{4}{5}$

e $\text{P(neither green nor blue)} = \text{P(red)} = \frac{1}{5}$

f $\text{P(green or red)} = \dfrac{5+3}{15} = \frac{8}{15}$

2 **a** 8 are brown and so 4 are white.

b **i** $\text{P(brown)} = \frac{8}{12} = \frac{2}{3}$

ii $\text{P(white)} = \frac{4}{12} = \frac{1}{3}$

3 **a** P(multiple of 4)
$= \text{P}(4, 8, 12, 16, 20, 24, 28, 32, 36)$
$= \frac{9}{36}$
$= \frac{1}{4}$

b $\text{P(between 6 and 9 inclusive)}$
$= \text{P}(6, 7, 8, \text{ or } 9)$
$= \frac{4}{36}$
$= \frac{1}{9}$

c $\text{P}(> 20)$
$= \text{P}(21, 22, 23, 24, ..., 35, 36)$
$= \dfrac{36-20}{36}$
$= \frac{16}{36}$
$= \frac{4}{9}$

d $\text{P}(9) = \frac{1}{36}$

e P(multiple of 13)
$= \text{P}(13 \text{ or } 26)$
$= \frac{2}{36}$
$= \frac{1}{18}$

f $\text{P(odd multiple of 3)}$
$= \text{P}(3, 9, 15, 21, 27, \text{ or } 33)$
$= \frac{6}{36}$
$= \frac{1}{6}$

g $\text{P(multiple of 4 and 6)}$
$= \text{P(multiple of 12)}$
$= \text{P}(12, 24, 36)$
$= \frac{3}{36}$
$= \frac{1}{12}$

h $\text{P(multiple of 4 or 6)}$
$= \text{P}(4, 6, 8, 12, 16, 18, 20, 24, 28, 30, 32, 36)$
$= \frac{12}{36}$
$= \frac{1}{3}$

4 **a** P(on Tuesday)
$= \frac{1}{7}$

b P(on a weekend)
$= \frac{2}{7}$

c P(in July)
$= \dfrac{4 \times 31}{365 \times 3 + 366}$ {over a 4 year period}
$= \frac{124}{1461}$

d $\text{P(in January or February)}$
$= \dfrac{4 \times 31 + 3 \times 28 + 1 \times 29}{3 \times 365 + 1 \times 366}$ {over a 4 year period}
$= \frac{237}{1461} \quad (= \frac{79}{487})$

5 **a** Let A denote Antti, K denote Kai, and N denote Neda.
Possible orders are: {AKN, ANK, KAN, KNA, NAK, NKA}

b **i** $\text{P(A in middle)} = \frac{2}{6}$
$= \frac{1}{3}$

ii $\text{P(A at left end)} = \frac{2}{6}$
$= \frac{1}{3}$

iii P(A does not sit at right end)
$= 1 - \text{P(A at right end)}$
$= 1 - \frac{2}{6}$
$= \frac{4}{6} = \frac{2}{3}$

iv P(K and N are together) $= \frac{4}{6}$
$= \frac{2}{3}$

6 Let G denote 'a girl' and B denote 'a boy'.

a Possible orders are: {GGG, GGB, GBG, BGG, GBB, BGB, BBG, BBB}

b **i** P(all boys) = P(BBB) $= \frac{1}{8}$

ii P(all girls) = P(GGG) $= \frac{1}{8}$

iii P(boy, then girl, then girl)
= P(BGG)
$= \frac{1}{8}$

iv P(2 girls and a boy)
= P(GGB or GBG or BGG)
$= \frac{3}{8}$

v P(girl is eldest)
= P(GGG or GBG or GBB or GGB)
$= \frac{4}{8} = \frac{1}{2}$

vi P(at least one boy)
$= \frac{7}{8}$ {all except GGG}

7 **a** {ABCD, ABDC, ACBD, ACDB, ADBC, ADCB, BACD, BADC, BCAD, BCDA, BDAC, BDCA, CABD, CADB, CBAD, CBDA, CDAB, CDBA, DABC, DACB, DBAC, DBCA, DCAB, DCBA}

b **i** P(A sits on one end) $= \frac{12}{24} = \frac{1}{2}$

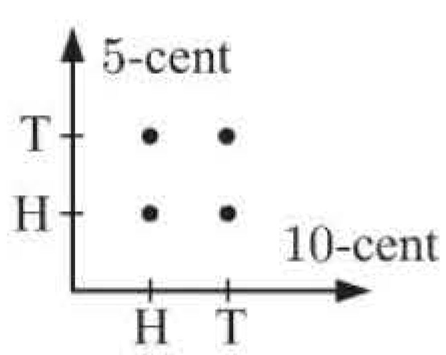
ii P(B sits on one of the two middle seats) $= \frac{12}{24} = \frac{1}{2}$

iii P(A and B are together) $= \frac{12}{24} = \frac{1}{2}$

iv P(A, B, and C are together) $= \frac{12}{24} = \frac{1}{2}$

EXERCISE 24C.2

1



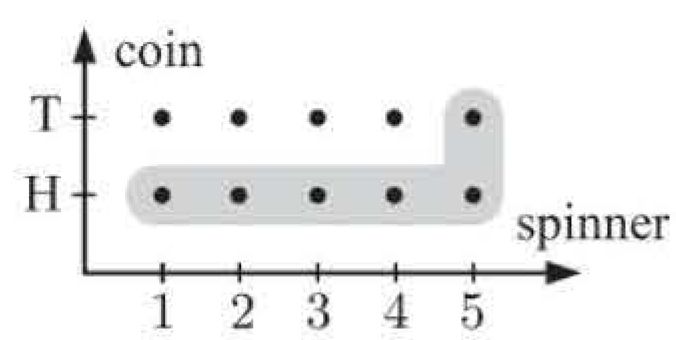
a P(2 heads) $= \frac{1}{4}$

b P(2 tails) $= \frac{1}{4}$

c P(exactly 1 head)
= P(HT or TH)
$= \frac{2}{4} = \frac{1}{2}$

d P(at least one H)
= P(HT or TH or HH)
$= \frac{3}{4}$

2 **a**



b There are $2 \times 5 = 10$ possible outcomes.

c **i** P(T and 3)
$= \frac{1}{10}$

ii P(H and even)
= P(H2 or H4)
$= \frac{2}{10} = \frac{1}{5}$

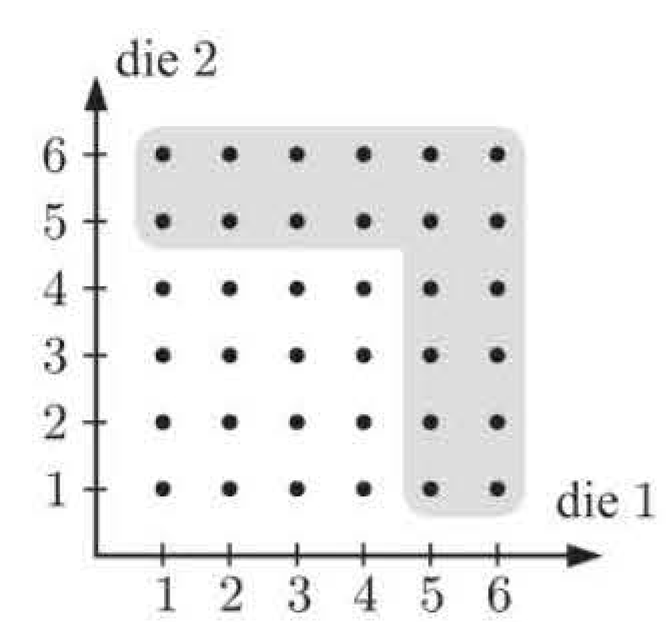
iii P(an odd)
= P(H1, T1, H3, T3, H5, T5)
$= \frac{6}{10} = \frac{3}{5}$

iv P(H or 5)
$= \frac{6}{10}$
$= \frac{3}{5}$ {shaded}

3 **a** P(two 3s)
= P((3, 3))
$= \frac{1}{36}$

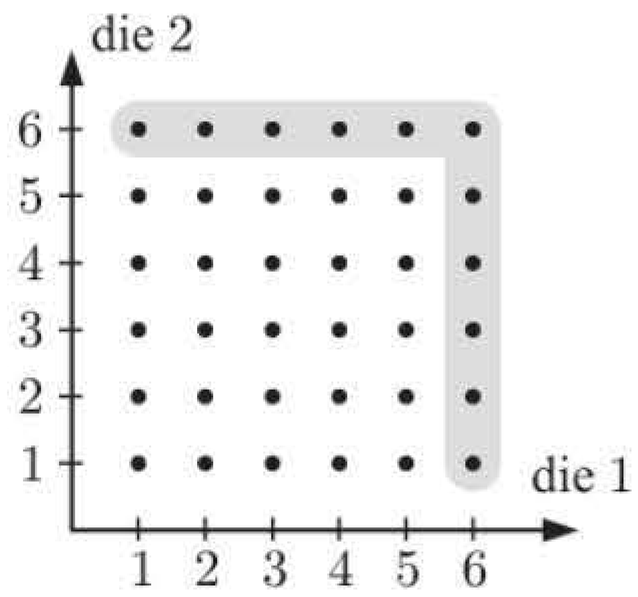
b P(5 and a 6)
= P((5, 6), (6, 5))
$= \frac{2}{36}$
$= \frac{1}{18}$

c P(5 or a 6)
$= \frac{20}{36}$
$= \frac{5}{9}$



d P(at least one 6)

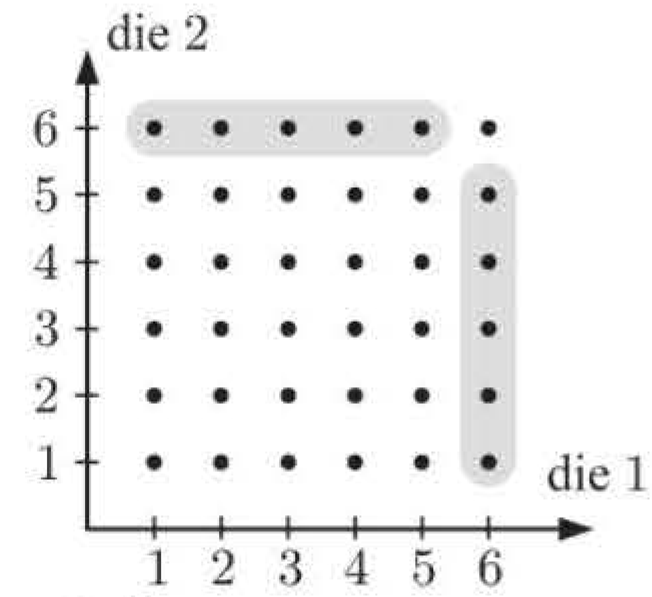
$= \frac{11}{36}$



e P(exactly one 6)

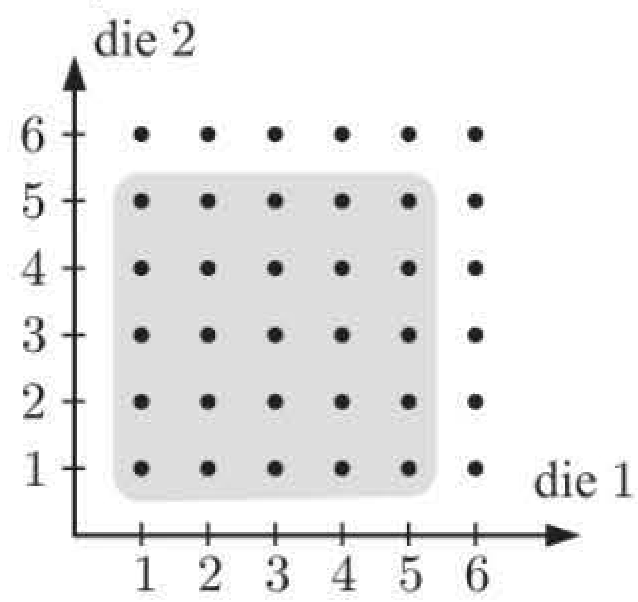
$= \frac{10}{36}$

$= \frac{5}{18}$



f P(no sixes)

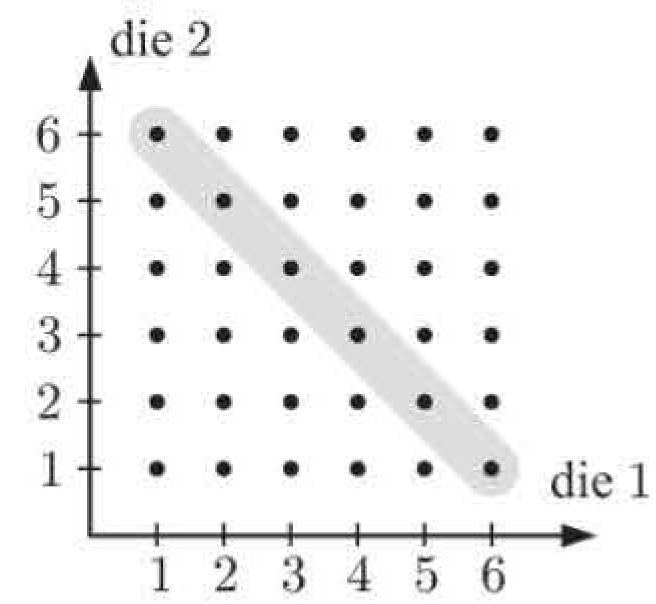
$= \frac{25}{36}$



g P(sum of 7)

$= \frac{6}{36}$

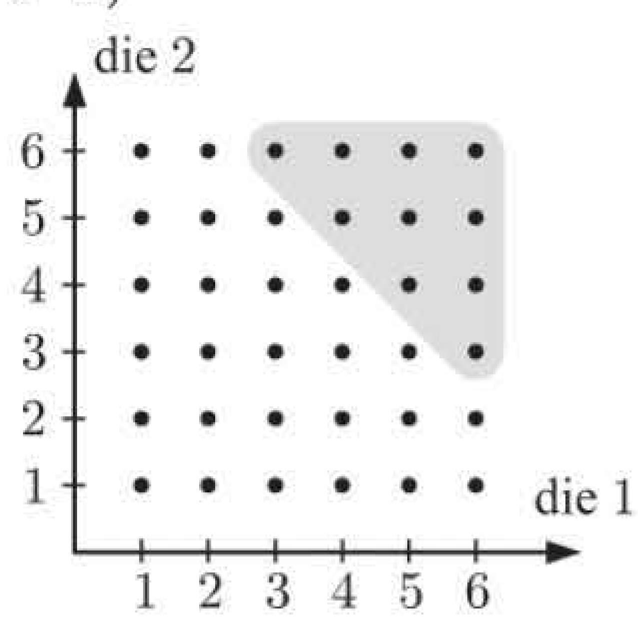
$= \frac{1}{6}$



h P(sum > 8)

$= \frac{10}{36}$

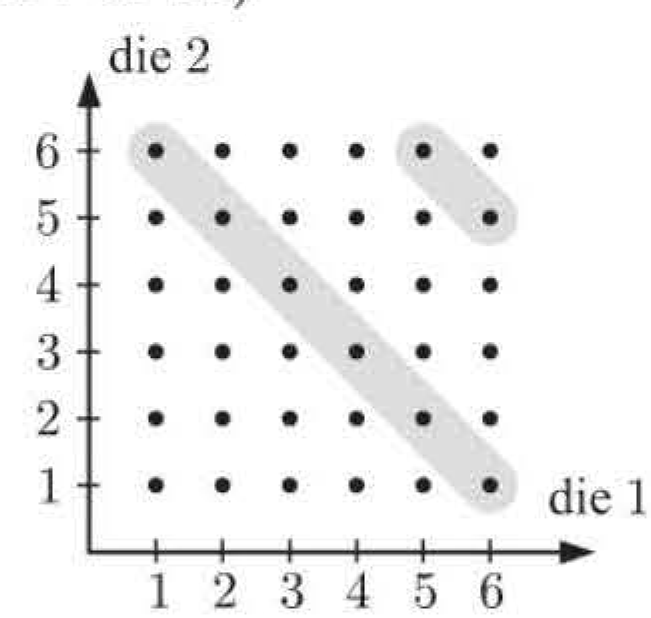
$= \frac{5}{18}$



i P(sum of 7 or 11)

$= \dfrac{6+2}{36}$

$= \frac{2}{9}$

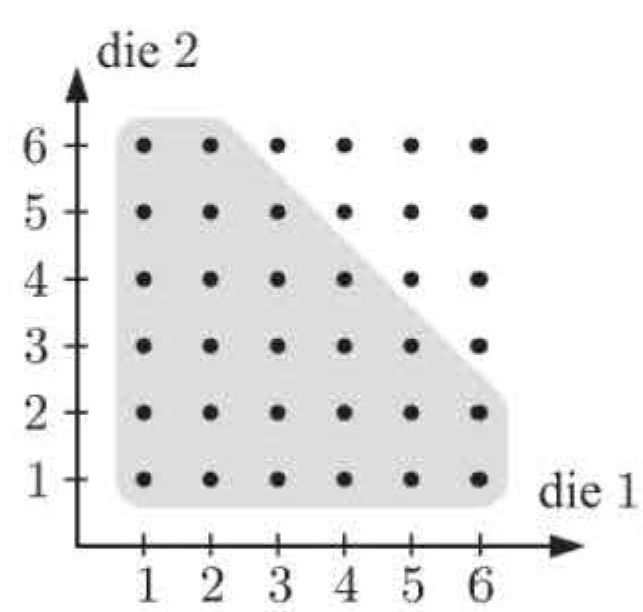


j P(sum no more than 8)

$=$ P(sum $\leqslant 8$)

$= \frac{26}{36}$

$= \frac{13}{18}$



EXERCISE 24D

1 We extend the table to include the totals:

	Employed	Unemployed	Total
Attended university	225	34	259
Did not attend university	197	81	278
Total	422	115	537

a 259 out of the 537 adults surveyed attended university.

$\therefore$ P(attended university) $\approx \frac{259}{537} \approx 0.482$

b 197 out of the 537 adults surveyed did not attend university and are currently employed.

$\therefore$ P(did not attend university and is currently employed) $\approx \frac{197}{537} \approx 0.367$

c 115 out of the 537 adults surveyed were unemployed.

$\therefore$ P(unemployed) $\approx \frac{115}{537} \approx 0.214$

d Of the 259 adults who attended university, 225 are currently employed.

$\therefore$ P(employed given that they attended university) $\approx \frac{225}{259} \approx 0.869$

e Of the 115 unemployed adults, 34 attended university.

$\therefore$ P(attended university given that they are currently unemployed) $\approx \frac{34}{115} \approx 0.296$

2 We extend the table to include the totals:

	Adult	Child	Total
Season ticket holder	1824	779	2603
Not a season ticket holder	3247	1660	4907
Total	5071	2439	7510

a Total match attendance was 7510.

b **i** P(child) $= \frac{2439}{7510} \approx 0.325$ **ii** P(not a season ticket holder) $= \frac{4907}{7510} \approx 0.653$

iii P(adult season ticket holder) $= \frac{1824}{7510} \approx 0.243$

3 We extend the table to include the totals:

	Single	Double	Family	Total
Peak season	125	220	98	443
Off-peak season	248	192	152	592
Total	373	412	250	1035

a P(peak season) $= \frac{443}{1035} \approx 0.428$

b P(single room in the off-peak season) $= \frac{248}{1035} \approx 0.240$

c P(single room or double room) $= \frac{373+412}{1035} = \frac{785}{1035} \approx 0.758$

d Of the 592 off-peak season bookings, 152 were for family rooms.

$\therefore$ P(family room given it was in the off-peak season) $= \frac{152}{592} \approx 0.257$

e $412 + 250 = 662$ bookings were not for a single room. Of these, $220 + 98 = 318$ were in the peak season.

$\therefore$ P(peak season given it was not a single room) $= \frac{318}{662} \approx 0.480$

EXERCISE 24E.1

1 **a** P(rains on any one day)

$= \frac{6}{7}$

b P(rains on 2 successive days)

$= \text{P(R and R)}$

$= \frac{6}{7} \times \frac{6}{7}$

$= \frac{36}{49}$

c P(rains on 3 successive days)

$= \text{P(R and R and R)}$

$= \frac{6}{7} \times \frac{6}{7} \times \frac{6}{7} = \frac{216}{343}$

2 **a** P(H, then H, then H)

$= \text{P(H and H and H)}$

$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$

$= \frac{1}{8}$

b P(T, then H, then T)

$= \text{P(T and H and T)}$

$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$

$= \frac{1}{8}$

3 Let A be the event of photocopier A malfunctioning and B be the event of photocopier B malfunctioning.

a P(both malfunction)

$= \text{P}(A \text{ and } B)$

$= 0.08 \times 0.12$

$= 0.0096$

b P(both work)

$= \text{P}(A' \text{ and } B')$

$= 0.92 \times 0.88$

$= 0.8096$

4 **a** P(they will be happy)
$= \text{P(B, then G, then B, then G)}$
$= \text{P(B and G and B and G)}$
$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$
$= \frac{1}{16}$

b P(they will be unhappy)
$= 1 - \text{P(they will be happy)}$
$= 1 - \frac{1}{16}$
$= \frac{15}{16}$

5 Let J be the event of Jiri hitting the target and B be the event of Benita hitting the target.

a P(both hit)
$= \text{P}(JB)$
$= 0.7 \times 0.8$
$= 0.56$

b P(both miss)
$= \text{P}(J'B')$
$= 0.3 \times 0.2$
$= 0.06$

c P(J hits and B misses)
$= \text{P}(JB')$
$= 0.7 \times 0.2$
$= 0.14$

d P(B hits and J misses)
$= \text{P}(BJ')$
$= 0.8 \times 0.3$
$= 0.24$

6 Let H be the event the archer hits the bullseye. $\therefore\ \text{P}(H) = \frac{2}{5},\ \text{P}(H') = \frac{3}{5}$

a P(3 hits)
$= \text{P}(HHH)$
$= \frac{2}{5} \times \frac{2}{5} \times \frac{2}{5}$
$= \frac{8}{125}$

b P(2 hits then a miss)
$= \text{P}(HHH')$
$= \frac{2}{5} \times \frac{2}{5} \times \frac{3}{5}$
$= \frac{12}{125}$

c P(all misses)
$= \text{P}(H'H'H')$
$= \frac{3}{5} \times \frac{3}{5} \times \frac{3}{5}$
$= \frac{27}{125}$

EXERCISE 24E.2

1 **a** P(all strawberry creams)
$= \text{P(SSS)}$
$= \frac{8}{12} \times \frac{7}{11} \times \frac{6}{10}$
$= \frac{14}{55}$

b P(none is a strawberry cream)
$= \text{P(S}'\text{S}'\text{S}')$
$= \frac{4}{12} \times \frac{3}{11} \times \frac{2}{10}$
$= \frac{1}{55}$

2 **a** P(both red)
$= \text{P(RR)}$
$= \frac{7}{10} \times \frac{6}{9}$
$= \frac{7}{15}$

b P(GR)
$= \frac{3}{10} \times \frac{7}{9}$
$= \frac{7}{30}$

c P(a green and a red)
$= \text{P(GR or RG)}$
$= \frac{3}{10} \times \frac{7}{9} + \frac{7}{10} \times \frac{3}{9}$
$= \frac{7}{15}$

3 **a** P(wins first prize) $= \frac{3}{100}$

b P(wins 1st and 2nd)
$= \text{P(WW)}$
$= \frac{3}{100} \times \frac{2}{99}$
$\approx 0.000\,606$

c P(wins all 3)
$= \text{P(WWW)}$
$= \frac{3}{100} \times \frac{2}{99} \times \frac{1}{98}$
$\approx 0.000\,006\,18$

d P(wins none of them)
$= \text{P(W}'\text{W}'\text{W}')$
$= \frac{97}{100} \times \frac{96}{99} \times \frac{95}{98}$
≈ 0.912

4 **a** P(does not contain captain)
$= \text{P(C}'\text{C}'\text{C}')$
$= \frac{6}{7} \times \frac{5}{6} \times \frac{4}{5}$
$= \frac{4}{7}$

b P(does not contain captain or vice captain)
$= \text{P(OOO)}$ $\quad \{\text{O} \equiv \text{other}\}$
$= \frac{5}{7} \times \frac{4}{6} \times \frac{3}{5}$
$= \frac{2}{7}$

5 **a** P(two boys) = P(first selected is a boy *and* second selected is a boy)
= P(first selected is a boy) × P(second selected is a boy)
$= \frac{5}{7} \times \frac{4}{6}$
$= \frac{20}{42} = \frac{10}{21}$

b P(eldest two students) $=$ P(either of the two eldest students *and* the remaining student)

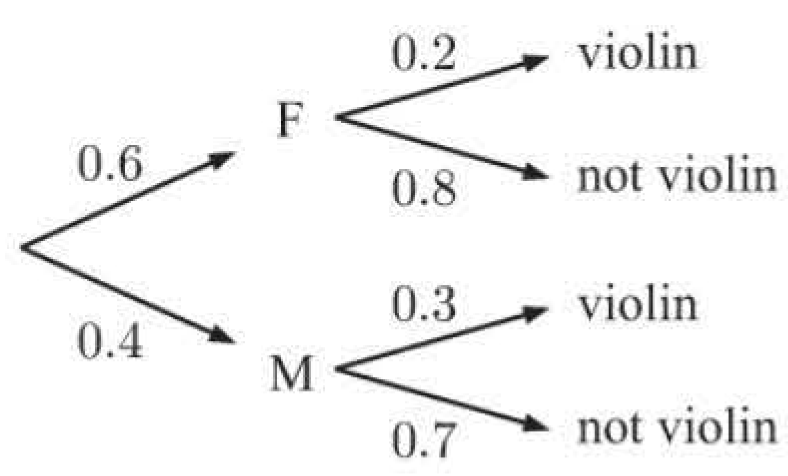
$= \text{P(either of the two eldest students)} \times \text{P(the remaining student)}$

$= \frac{2}{7} \times \frac{1}{6}$

$= \frac{2}{42} = \frac{1}{21}$

EXERCISE 24F

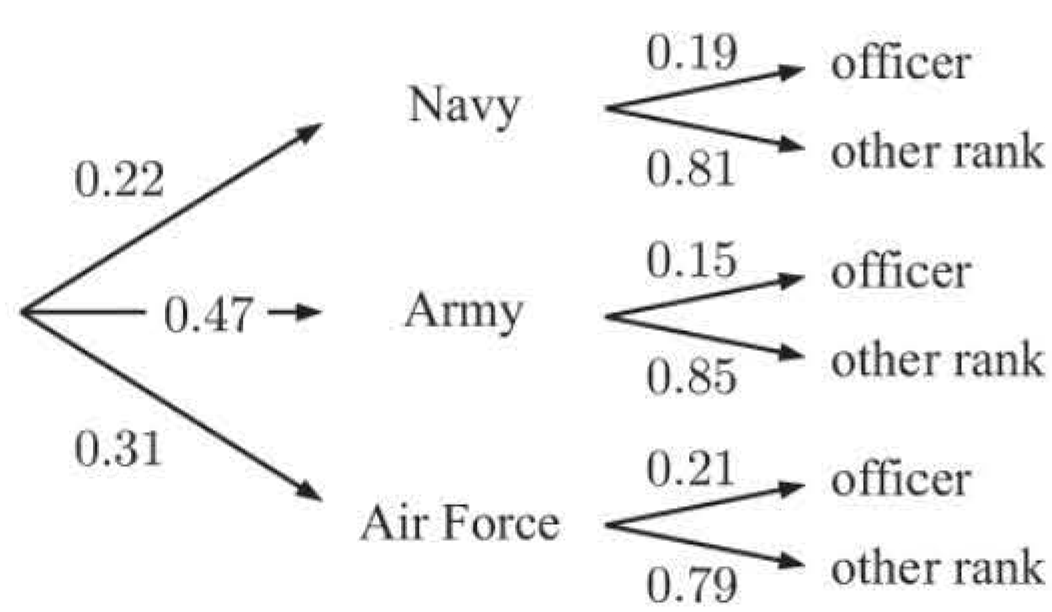
1 **a**



b **i** $\text{P(male and not violin)} = 0.4 \times 0.7$
$= 0.28$

ii P(plays the violin)
$= \text{P(F and V)} + \text{P(M and V)}$
$= 0.6 \times 0.2 + 0.4 \times 0.3$
$= 0.24$

2 **a**

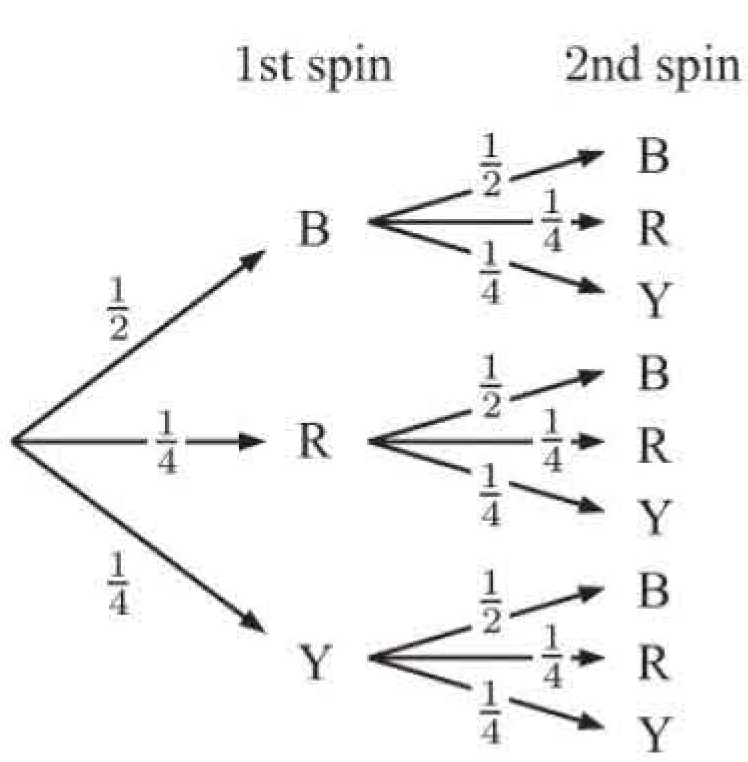


b **i** $\text{P(officer)} = \text{P(N and O)} + \text{P(A and O)} + \text{P(AF and O)}$ {where O represents officer}
$= 0.22 \times 0.19 + 0.47 \times 0.15 + 0.31 \times 0.21$
$= 0.1774 \approx 0.177$

ii P(not an officer in the navy)
$= \text{P((N and O)}')$
$= 1 - \text{P(N and O)}$
$= 1 - 0.22 \times 0.19$
$= 0.9582 \approx 0.958$

iii P(not an army or air force officer)
$= 1 - \text{(P(army or air force officer))}$
$= 1 - \text{(P(A and O)} + \text{P(AF and O))}$
$= 1 - (0.47 \times 0.15 + 0.31 \times 0.21)$
$= 0.8644 \approx 0.864$

3 **a**



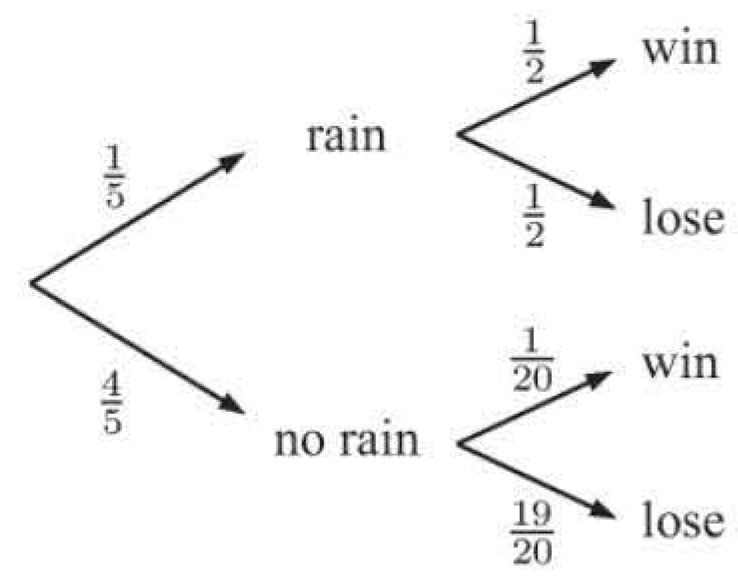
b P(both black)
$= \text{P(BB)}$
$= \frac{1}{2} \times \frac{1}{2}$
$= \frac{1}{4}$

c P(both yellow)
$= \text{P(YY)}$
$= \frac{1}{4} \times \frac{1}{4}$
$= \frac{1}{16}$

d P(both different)
$= \text{P(BR or BY or RB or RY or YB or YR)}$
$= \frac{1}{2} \times \frac{1}{4} + \frac{1}{2} \times \frac{1}{4} + \frac{1}{4} \times \frac{1}{2} + \frac{1}{4} \times \frac{1}{4}$
$+ \frac{1}{4} \times \frac{1}{2} + \frac{1}{4} \times \frac{1}{4}$
$= \frac{4}{8} + \frac{2}{16}$
$= \frac{5}{8}$

e P(B appears on either spin)
$= \text{P(BB or BR or BY or RB or YB)}$
$= \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{4} + \frac{1}{2} \times \frac{1}{4} + \frac{1}{4} \times \frac{1}{2}$
$+ \frac{1}{4} \times \frac{1}{2}$
$= 4(\frac{1}{8}) + \frac{1}{4}$
$= \frac{3}{4}$

4



P(Mudlark wins)

$= $ P(rain and win or no rain and win)

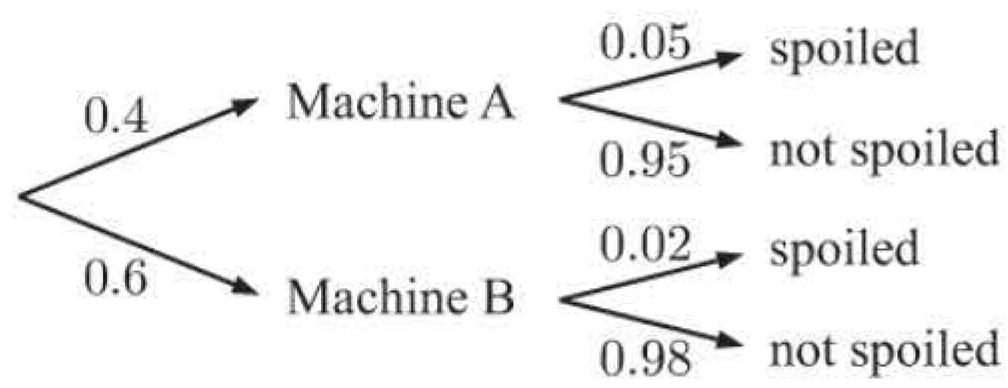
$= \frac{1}{5} \times \frac{1}{2} + \frac{4}{5} \times \frac{1}{20}$

$= \frac{1}{10} + \frac{4}{100}$

$= \frac{14}{100}$

$= \frac{7}{50}$

5



P(next is spoiled)

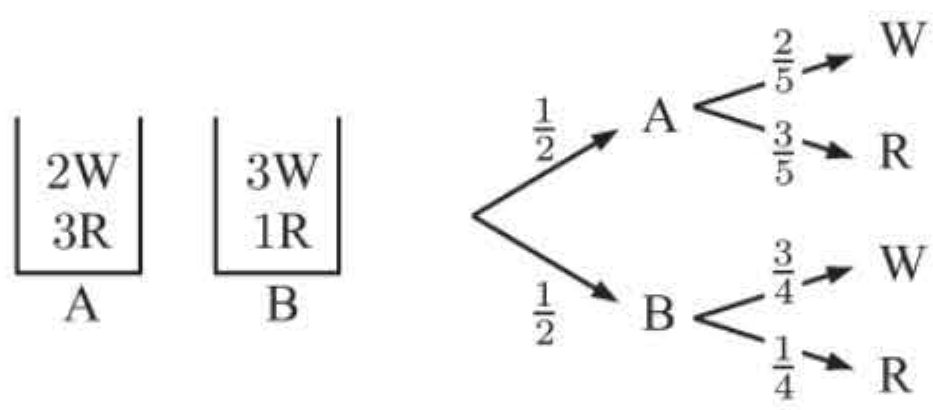
$= $ P(from A and spoiled or from B and spoiled)

$= 0.4 \times 0.05 + 0.6 \times 0.02$

$= 0.020 + 0.012$

$= 0.032 \qquad (3.2\%)$

6



P(red)

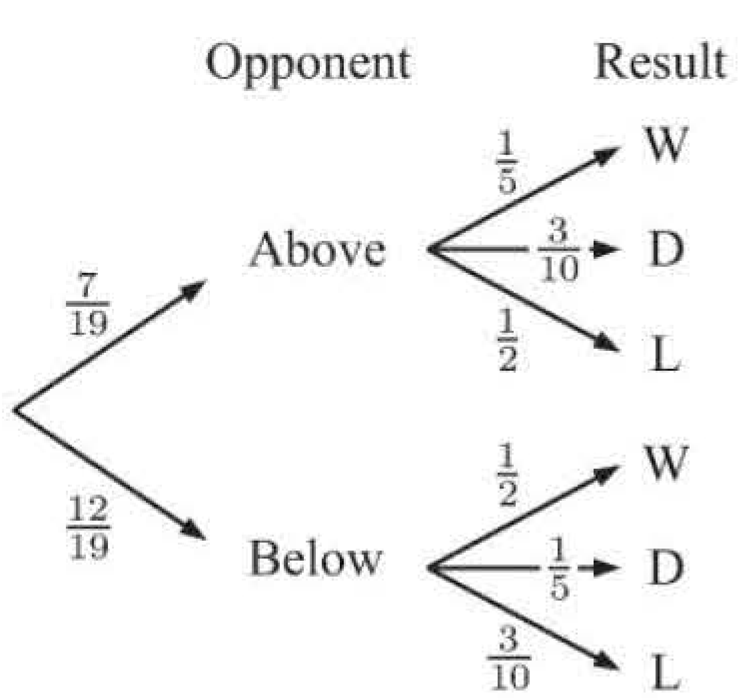
$= $ P(A and red or B and red)

$= \frac{1}{2} \times \frac{3}{5} + \frac{1}{2} \times \frac{1}{4}$

$= \frac{3}{10} + \frac{1}{8}$

$= \frac{17}{40}$

7 There are 7 teams above Tottenham and 12 teams below Tottenham.



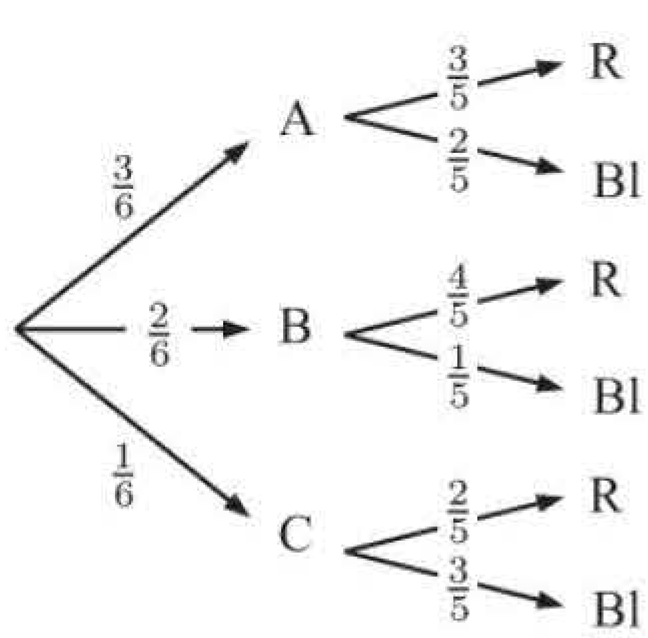
$\therefore$ P(Draw)

$= \frac{7}{19} \times \frac{3}{10} + \frac{12}{19} \times \frac{1}{5}$

$= \frac{21}{190} + \frac{24}{190}$

$= \frac{9}{38}$

8



a P(blue) $= $ P(A and Bl or B and Bl or C and Bl)

$= \frac{3}{6} \times \frac{2}{5} + \frac{2}{6} \times \frac{1}{5} + \frac{1}{6} \times \frac{3}{5}$

$= \frac{11}{30}$

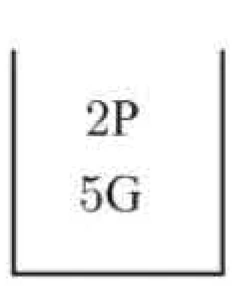
b P(red) $= 1 - $ P(blue)

$= 1 - \frac{11}{30}$

$= \frac{19}{30}$

EXERCISE 24G

1



a P(different colours)

$= $ P(PG or GP)

$= \frac{2}{7} \times \frac{5}{7} + \frac{5}{7} \times \frac{2}{7}$

$= \frac{20}{49}$

b P(different colours)

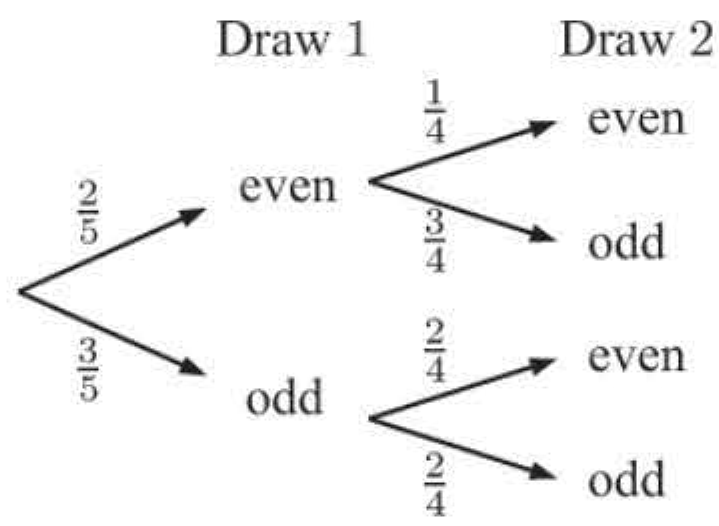
$= $ P(PG or GP)

$= \frac{2}{7} \times \frac{5}{6} + \frac{5}{7} \times \frac{2}{6}$

$= \frac{20}{42}$

$= \frac{10}{21}$

2 **a**

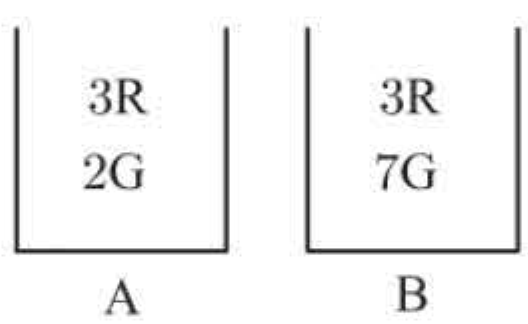


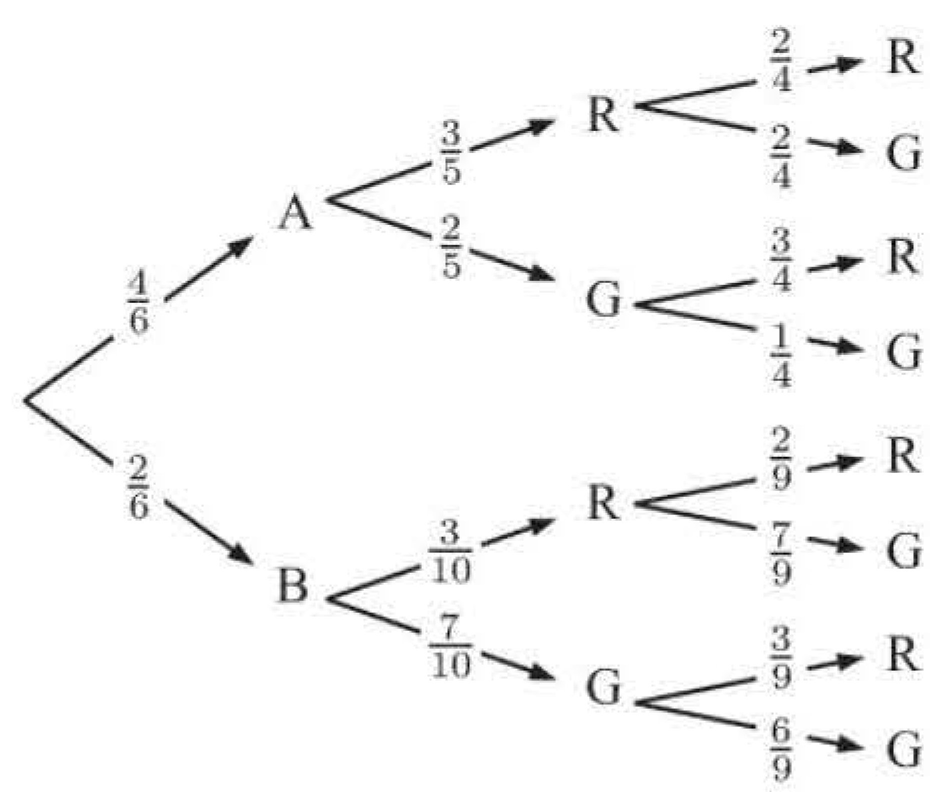
b **i** P(both odd)
$= \text{P(odd and odd)}$
$= \frac{3}{5} \times \frac{2}{4}$
$= \frac{3}{10}$

ii P(both even)
$= \text{P(even and even)}$
$= \frac{2}{5} \times \frac{1}{4}$
$= \frac{1}{10}$

iii P(one odd and other even)
$= 1 - \text{P(both odd)} - \text{P(both even)}$
$= 1 - \frac{3}{10} - \frac{1}{10}$
$= \frac{6}{10}$
$= \frac{3}{5}$

3

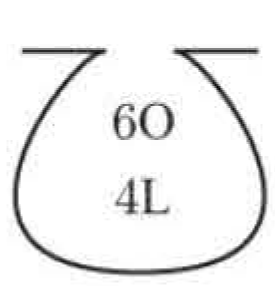


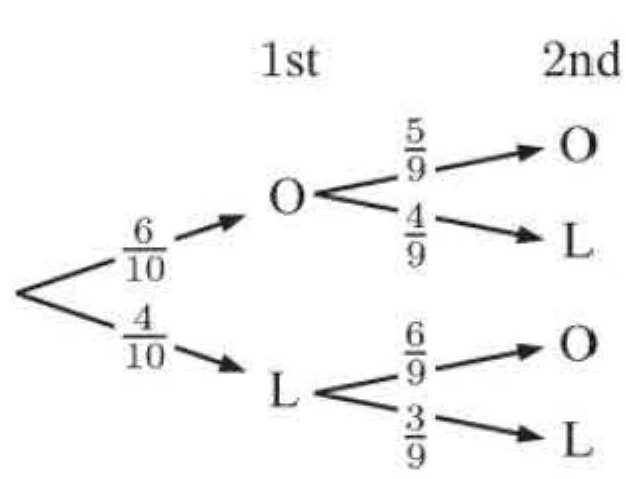


a P(both green)
$= \text{P(AGG or BGG)}$
$= \frac{4}{6} \times \frac{2}{5} \times \frac{1}{4} + \frac{2}{6} \times \frac{7}{10} \times \frac{6}{9}$
$= \frac{1}{15} + \frac{7}{45}$
$= \frac{10}{45}$
$= \frac{2}{9}$

b P(different in colour)
$= 1 - \text{P(both green)} - \text{P(both red)}$
$= 1 - \frac{2}{9} - \text{P(ARR or BRR)}$
$= \frac{7}{9} - (\frac{4}{6} \times \frac{3}{5} \times \frac{2}{4} + \frac{2}{6} \times \frac{3}{10} \times \frac{2}{9})$
$= \frac{7}{9} - (\frac{1}{5} + \frac{1}{45})$
$= \frac{5}{9}$

4





a **i** P(both O)
$= \frac{6}{10} \times \frac{5}{9}$
$= \frac{1}{3}$

ii P(both L)
$= \frac{4}{10} \times \frac{3}{9}$
$= \frac{2}{15}$

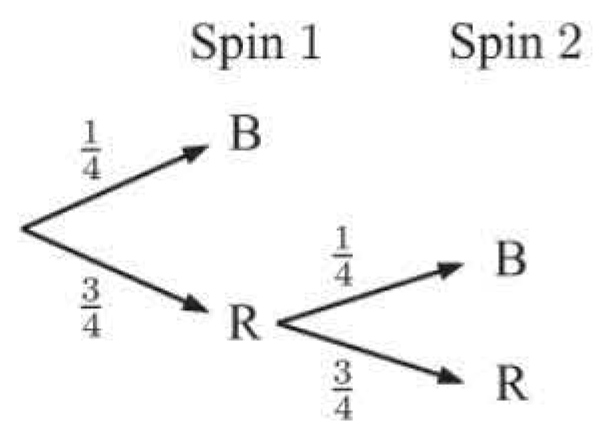
iii P(OL)
$= \frac{6}{10} \times \frac{4}{9}$
$= \frac{4}{15}$

iv P(LO)
$= \frac{4}{10} \times \frac{6}{9}$
$= \frac{4}{15}$

b $\frac{1}{3} + \frac{2}{15} + \frac{4}{15} + \frac{4}{15}$
$= \frac{5}{15} + \frac{2}{15} + \frac{4}{15} + \frac{4}{15}$
$= \frac{15}{15}$ which is 1

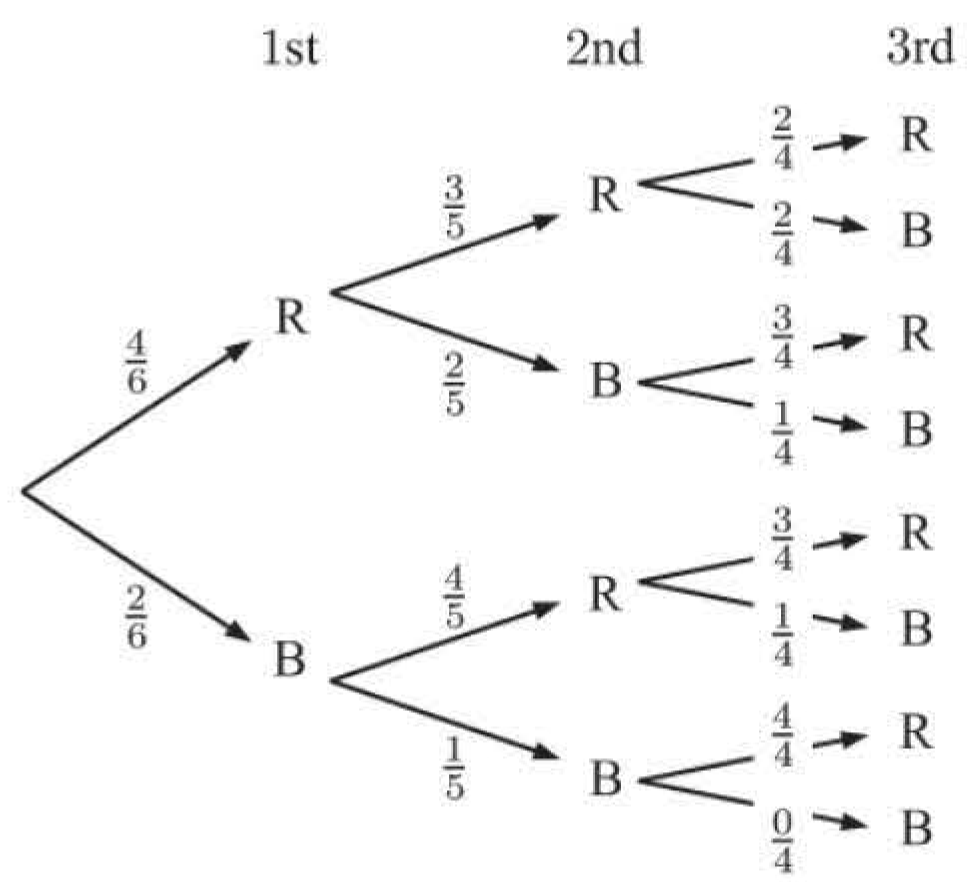
The answer must be 1 as the four categories **i**, **ii**, **iii**, **iv** are all the possibilities that could occur.

5 **a**



b P(blue)
$= \text{P}(\text{B}) + \text{P}(\text{RB})$
$= \frac{1}{4} + \frac{3}{4} \times \frac{1}{4}$
$= \frac{7}{16}$

6

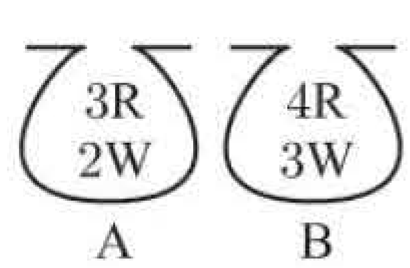


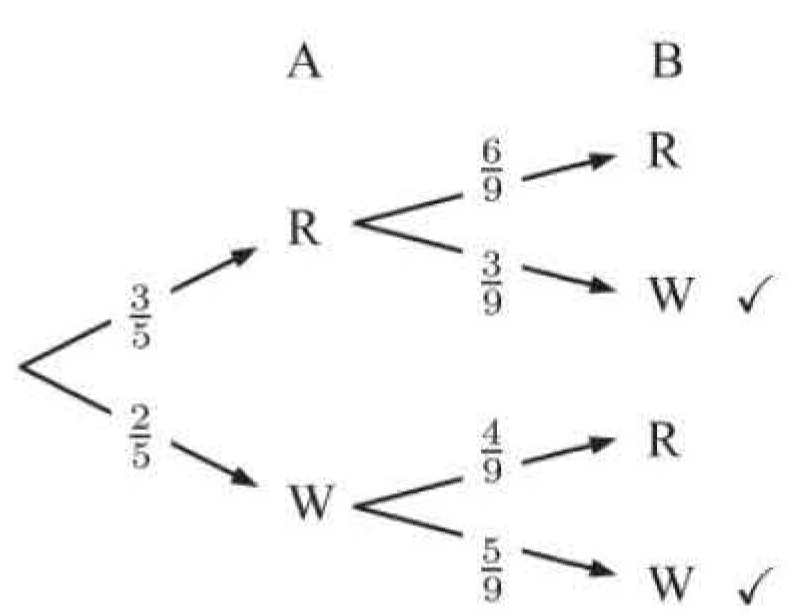
a P(all red)
$= \text{P}(\text{RRR})$
$= \frac{4}{6} \times \frac{3}{5} \times \frac{2}{4}$
$= \frac{1}{5}$

b P(only two are red)
$= \text{P}(\text{RRB or RBR or BRR})$
$= \frac{4}{6} \times \frac{3}{5} \times \frac{2}{4} + \frac{4}{6} \times \frac{2}{5} \times \frac{3}{4} + \frac{2}{6} \times \frac{4}{5} \times \frac{3}{4}$
$= 3 \times \left(\frac{24}{6 \times 5 \times 4}\right)$
$= \frac{3}{5}$

c P(at least two are red)
$= \text{P}(\text{all red or only two are red})$
$= \frac{1}{5} + \frac{3}{5}$ {from **a** and **b**}
$= \frac{4}{5}$

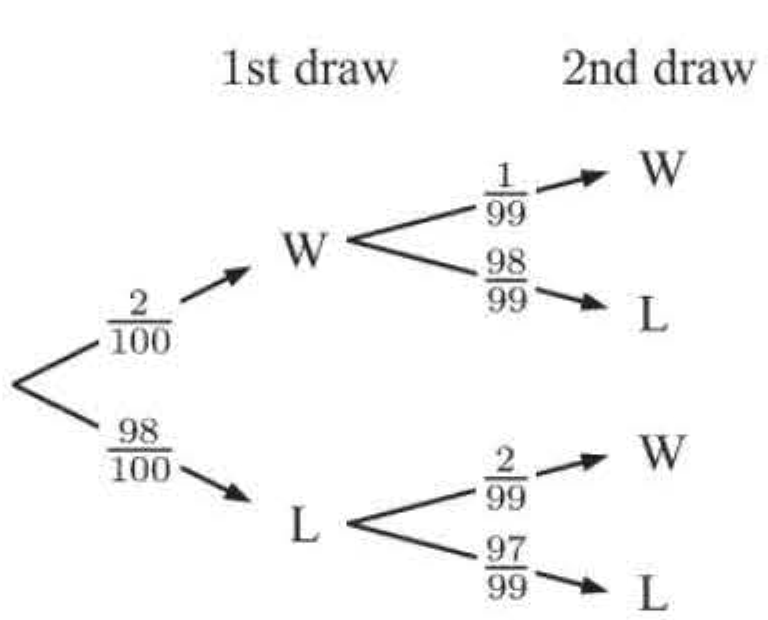
7





P(marble from B is W)
$= \text{P}(\text{RW or WW})$ {paths ticked}
$= \frac{3}{5} \times \frac{3}{9} + \frac{2}{5} \times \frac{5}{9}$
$= \frac{19}{45}$

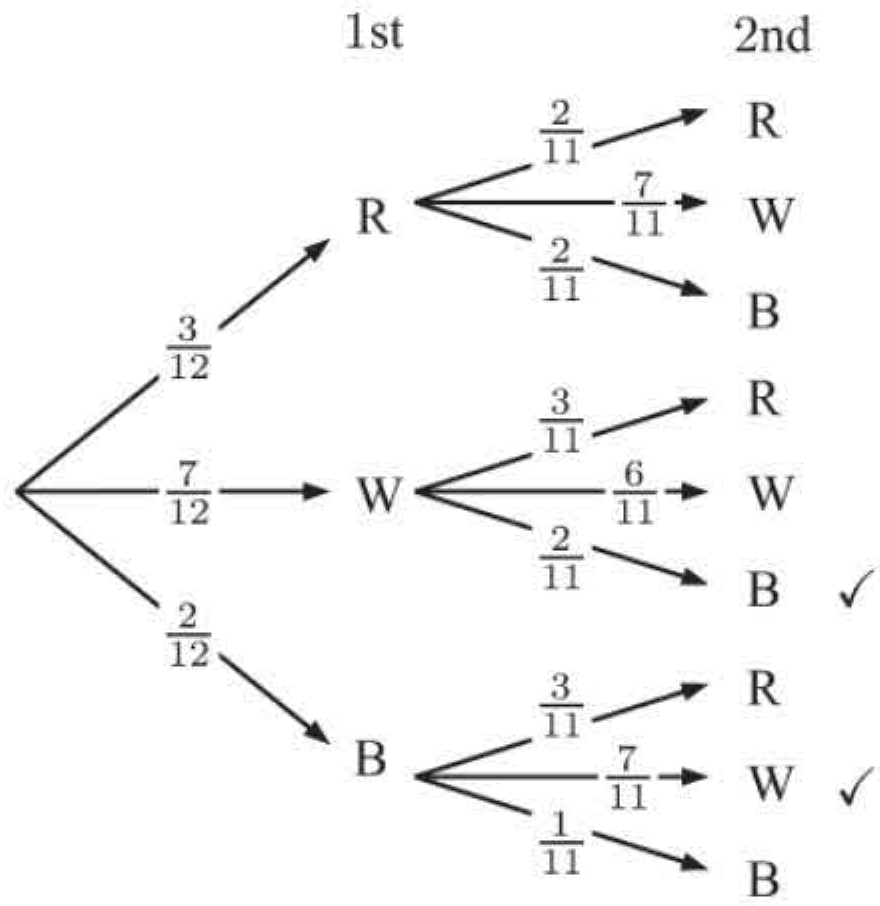
8



a P(wins both)
$= \text{P}(\text{WW})$
$= \frac{2}{100} \times \frac{1}{99}$
$\approx 0.000\,202$

b P(wins neither)
$= \text{P}(\text{LL})$
$= \frac{98}{100} \times \frac{97}{99}$
≈ 0.960

c P(wins at least one prize)
$= 1 - \text{P}(\text{wins neither})$
$= 1 - \frac{98}{100} \times \frac{97}{99}$
≈ 0.0398

9 

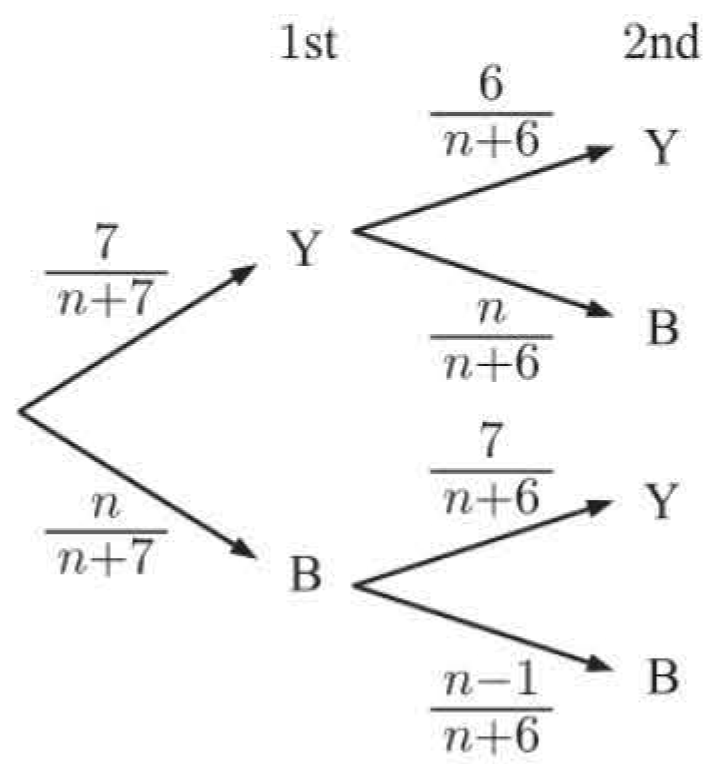
P(one white and one black)

$= \text{P(WB or BW)}$ {paths ticked}

$= \frac{7}{12} \times \frac{2}{11} + \frac{2}{12} \times \frac{7}{11}$

$= \frac{7}{33}$

10 There are $(n+7)$ markers in total.



$$\text{P(YY)} = \frac{3}{13}$$

$$\therefore \quad \frac{7}{n+7} \times \frac{6}{n+6} = \frac{3}{13}$$

$$\therefore \quad \frac{42}{n^2+13n+42} = \frac{3}{13}$$

$$\therefore \quad 546 = 3n^2 + 39n + 126$$

$$\therefore \quad 3(n^2+13n-140) = 0$$

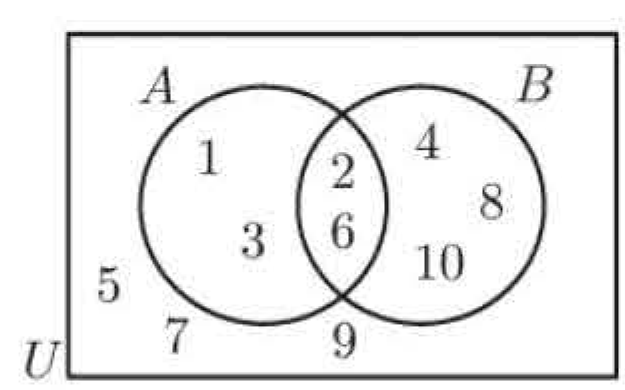
$$\therefore \quad 3(n-7)(n+20) = 0$$

$$\therefore \quad n = 7 \qquad \{n \geqslant 0\}$$

$\therefore$ there are 7 blue markers in the bag to start with.

EXERCISE 24H.1

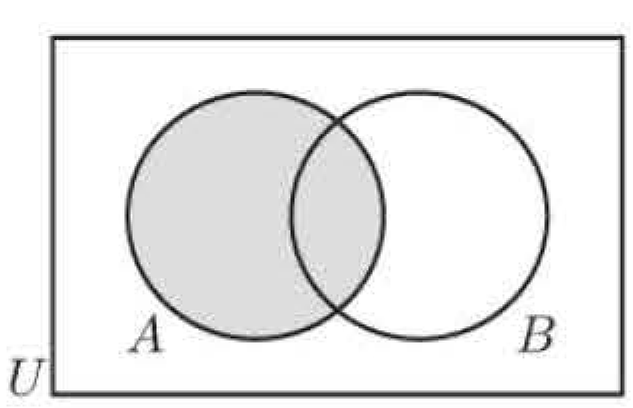
1 **a** $A = \{1, 2, 3, 6\}$, $B = \{2, 4, 6, 8, 10\}$

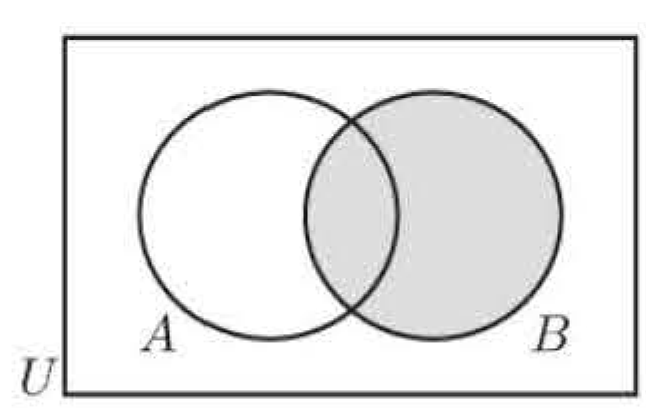
b 

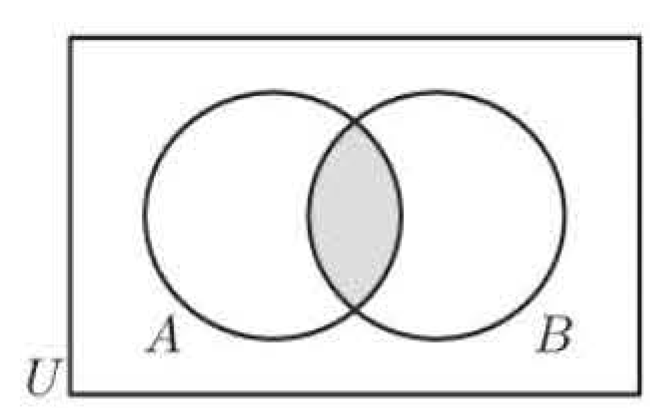
c **i** $n(A) = 4$

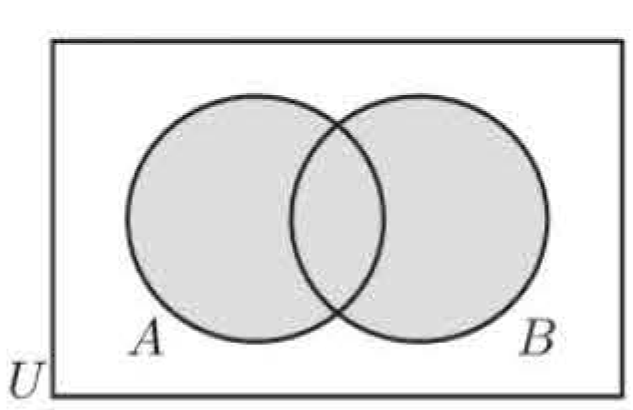
ii $A \cup B = \{1, 2, 3, 4, 6, 8, 10\}$

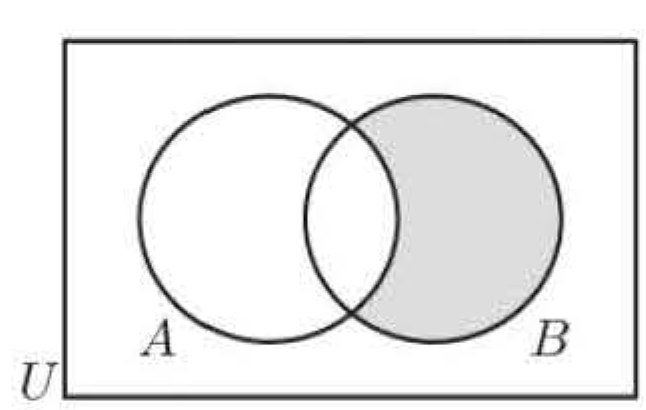
iii $A \cap B = \{2, 6\}$

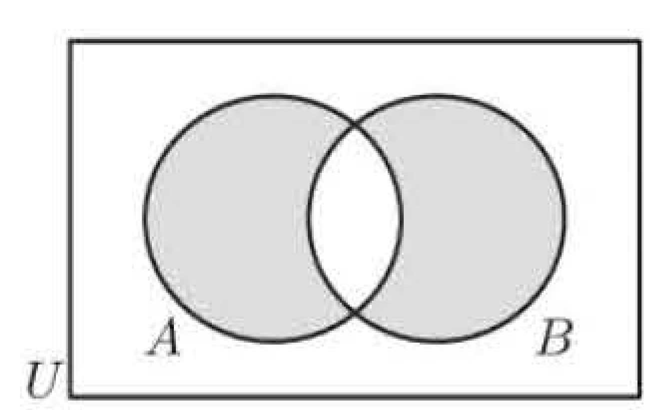
2 **a** 

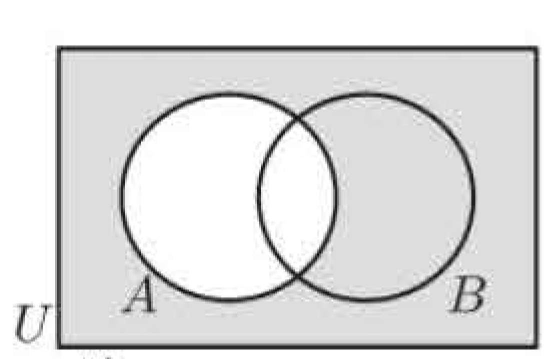
b 

c 

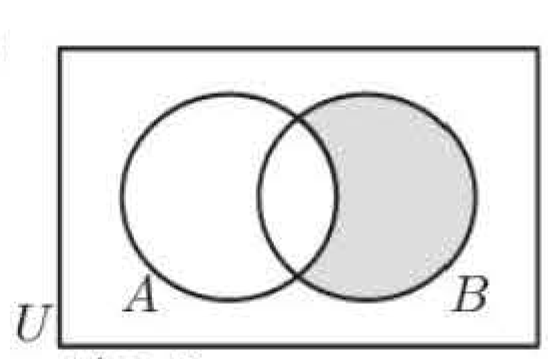
d 

e 

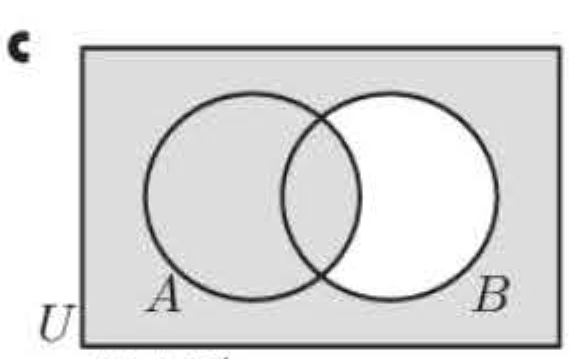
f 

3 **a** 

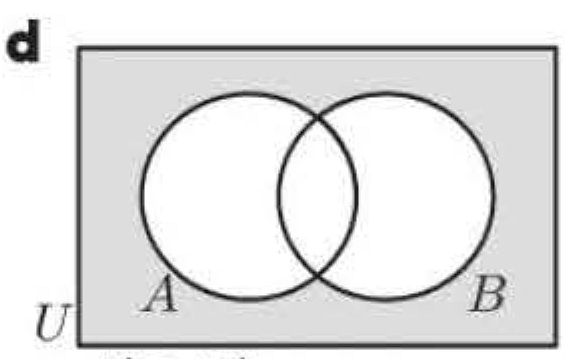
A'

b 

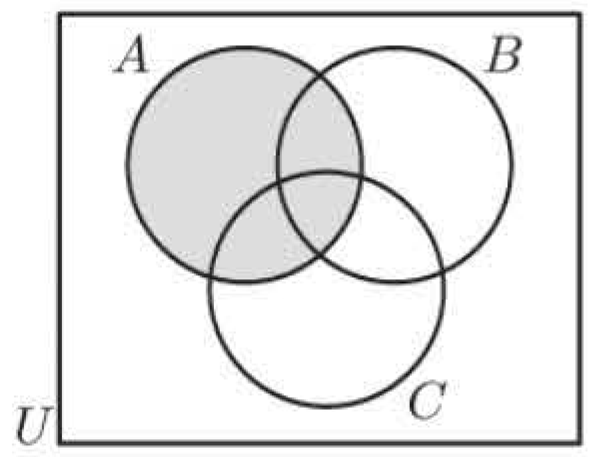
$A' \cap B$

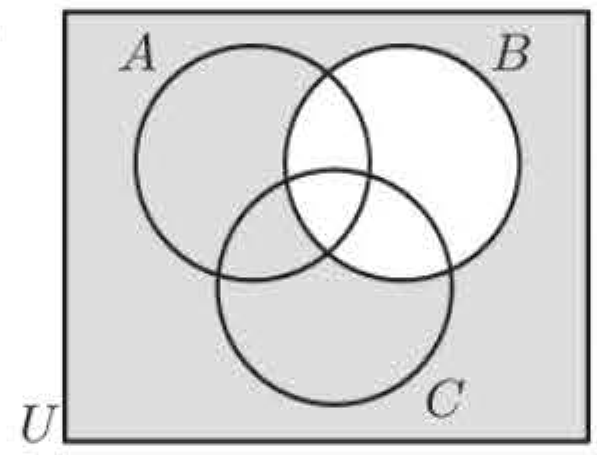
c 

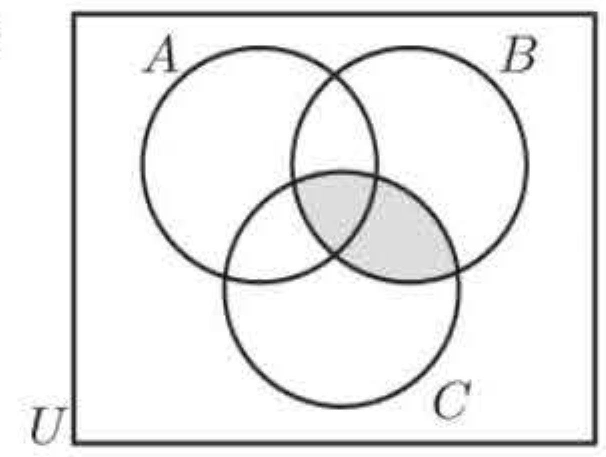
$A \cup B'$

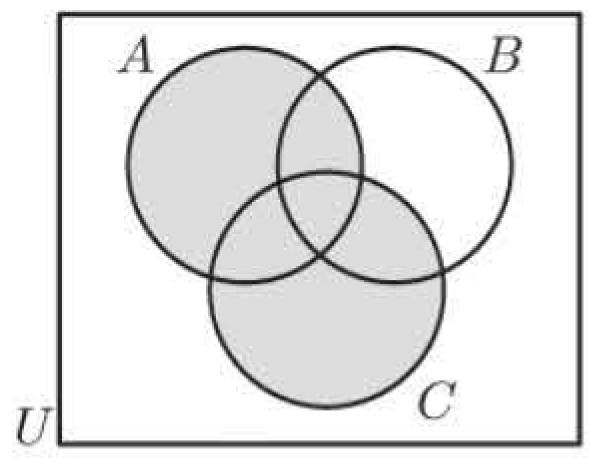
d 

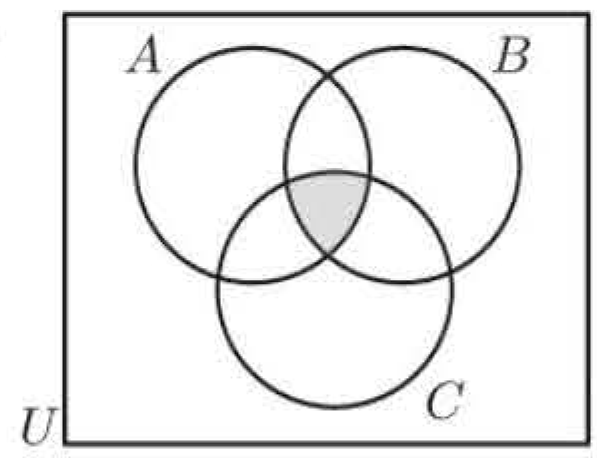
$A' \cap B'$

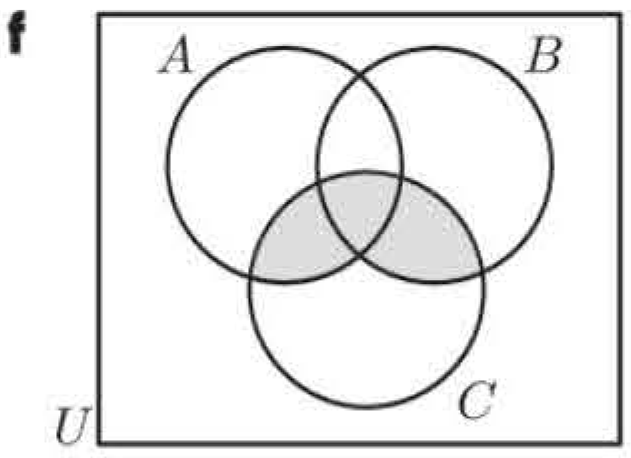
4 **a** 

b 

c 

d 

e 

f 

5 **a** Total number in the class $= 3 + 5 + 17 + 4 = 29$

b Number who study both $= 17$ {the intersection}

c Number who study at least one $= 5 + 17 + 4 = 26$ {the union}

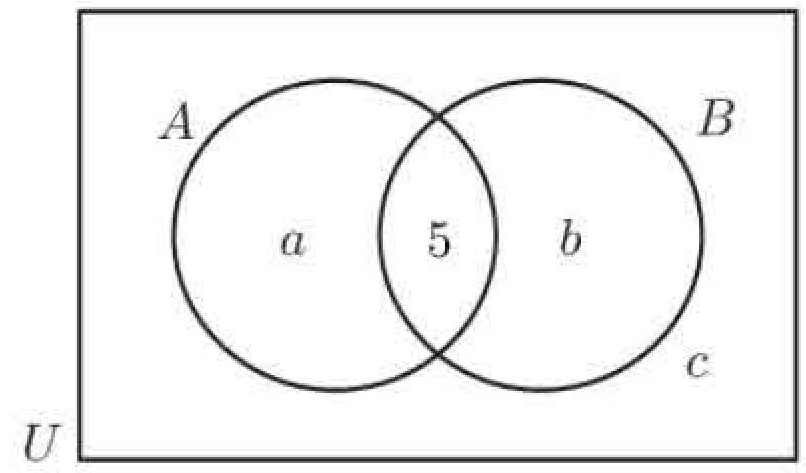
d Number who study only Chemistry $= 5$

6 **a** Total number in the survey $= 37 + 9 + 15 + 4 = 65$

b Number who liked both $= 9$ {the intersection}

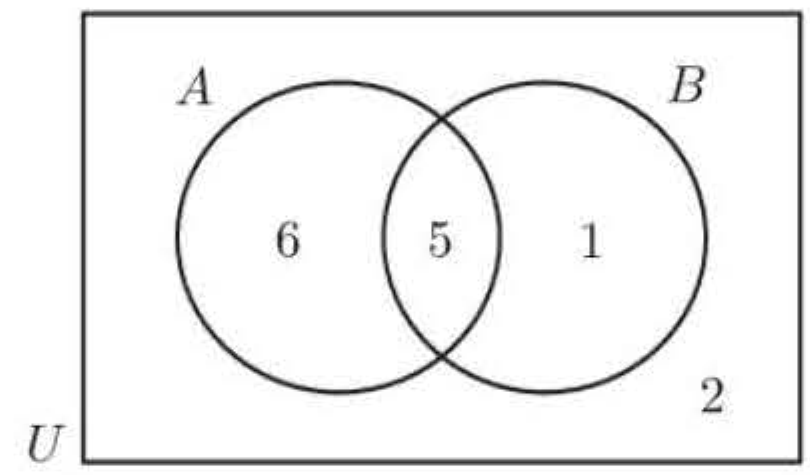
c Number who liked neither $= 4$

d Number who liked exactly one $= 37 + 15 = 52$

7 **a** 

$$\begin{aligned} a + 5 &= 11 \quad \{\text{since } n(A) = 11\} \\ \therefore\ a &= 6 \\ 6 + 5 + b &= 12 \quad \{\text{since } n(A \cup B) = 12\} \\ \therefore\ b &= 1 \\ 6 + c &= 8 \quad \{\text{since } n(B') = 8\} \\ \therefore\ c &= 2 \end{aligned}$$

$\therefore$ the completed Venn diagram is:

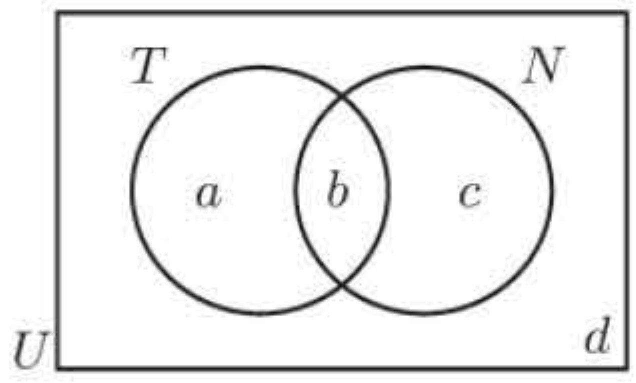


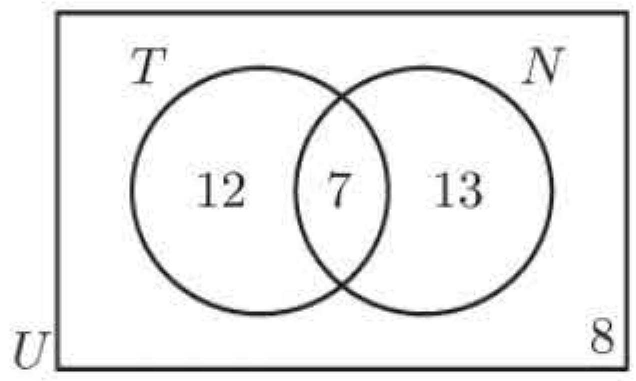
b **i** $n(U) = 6 + 5 + 1 + 2 = 14$

$$\begin{aligned} \therefore\ P(A \cup B) &= \frac{n(A \cup B)}{n(U)} \\ &= \frac{6+5+1}{14} \\ &= \tfrac{12}{14} \\ &= \tfrac{6}{7} \end{aligned}$$

ii $$\begin{aligned} P(A') &= \frac{n(A')}{n(U)} \\ &= \frac{1+2}{14} \\ &= \tfrac{3}{14} \end{aligned}$$

8





T represents those playing tennis
N represents those playing netball

$$\therefore \begin{cases} a+b+c+d=40 \\ a+b=19 \\ b+c=20 \\ d=8 \end{cases}$$

So, $a+b+c=32$
$\therefore$ $19+c=32$ and $a+20=32$
$\therefore$ $c=13$ and $a=12$

Hence, $12+b=19$
$\therefore$ $b=7$

a P(plays tennis)
$= \dfrac{12+7}{40}$
$= \frac{19}{40}$

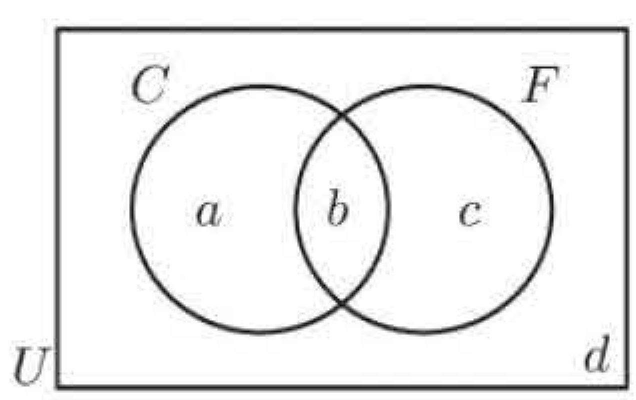
b P(does not play netball)
$= \dfrac{12+8}{40}$
$= \frac{1}{2}$

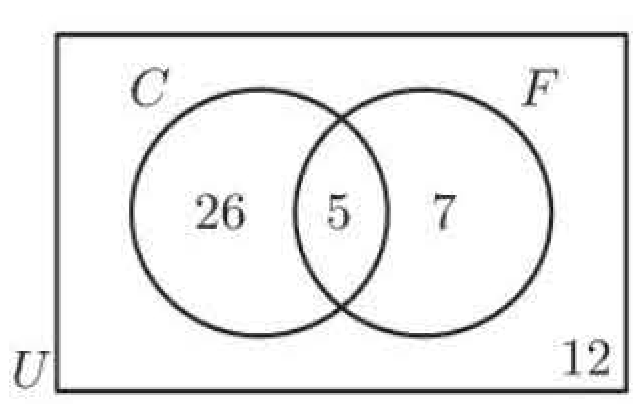
c P(plays at least one)
$= \dfrac{12+7+13}{40}$
$= \frac{32}{40}$
$= \frac{4}{5}$

d P(plays one and only one)
$= \dfrac{12+13}{40}$
$= \frac{25}{40}$
$= \frac{5}{8}$

e P(plays netball, but not tennis) $= \frac{13}{40}$

9





C represents men who gave chocolates.
F represents men who gave flowers.

$$\therefore \begin{cases} a+b+c+d=50 \\ a+b=31 \\ b+c=12 \\ b=5 \end{cases}$$

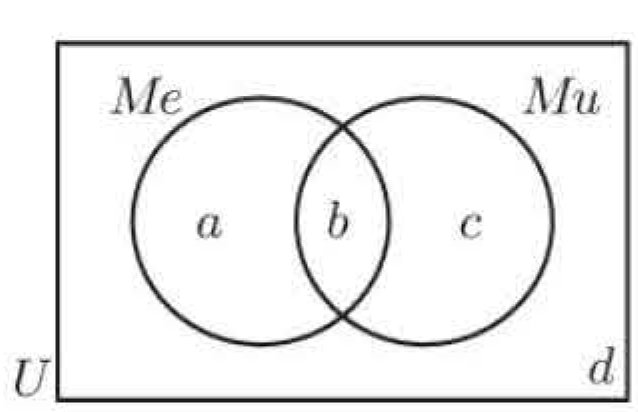
Thus $c=7$, $a=26$ and $26+5+7+d=50$ $\therefore$ $d=12$

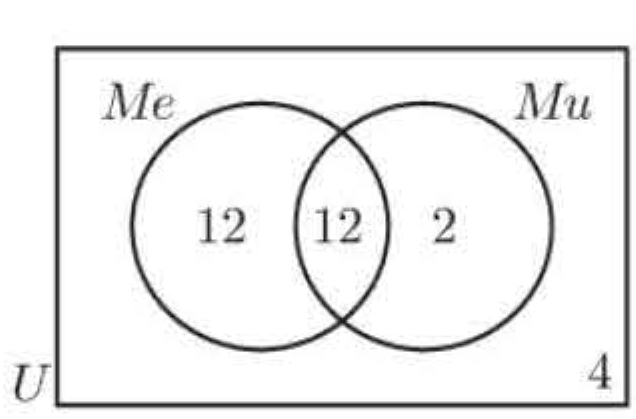
a P(C or F)
$= \dfrac{26+5+7}{50}$
$= \frac{38}{50} = \frac{19}{25}$

b P(C but not F)
$= \frac{26}{50}$
$= \frac{13}{25}$

c P(neither C nor F)
$= \frac{12}{50}$
$= \frac{6}{25}$

10





Me represents children who had measles.
Mu represents children who had mumps.

$$\therefore \begin{cases} a+b+c+d=30 \\ a+b=24 \\ b=12 \\ a+b+c=26 \end{cases}$$

$\therefore$ $26+d=30$ $\therefore$ $d=4$
$24+c=26$ $\therefore$ $c=2$
and $a+12=24$ $\therefore$ $a=12$

a $\text{P}(Mu)$
$= \frac{14}{30}$
$= \frac{7}{15}$

b $\text{P}(Mu, \text{ but not } Me)$
$= \frac{2}{30}$
$= \frac{1}{15}$

c $\text{P}(\text{neither } Mu \text{ nor } Me)$
$= \frac{4}{30}$
$= \frac{2}{15}$

11 **a** $4+2+1+a = 10$ $\{10 \text{ watched a movie}\}$
$\therefore\ a = 3$

$4+2+1+3+6+12+9+b = 40$ $\{40 \text{ individuals in total}\}$
$\therefore\ 37+b = 40$
$\therefore\ b = 3$

b **i** $\text{P(sport)} = \dfrac{6+2+1+3}{40} = \frac{12}{40} = \frac{3}{10}$

ii $\text{P(drama and sport)} = \dfrac{3+1}{40} = \frac{4}{40} = \frac{1}{10}$

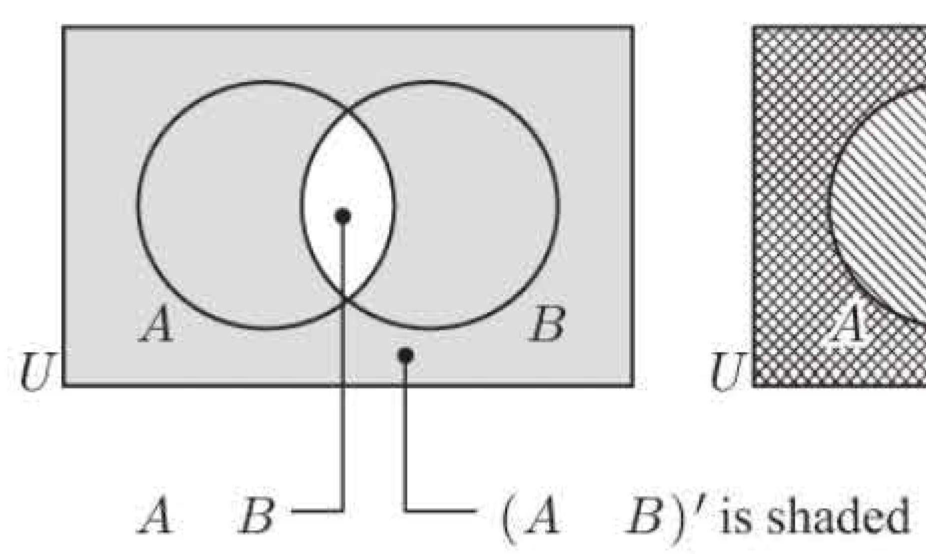
iii $\text{P(movie but not sport)} = \dfrac{4+3}{40} = \frac{7}{40}$

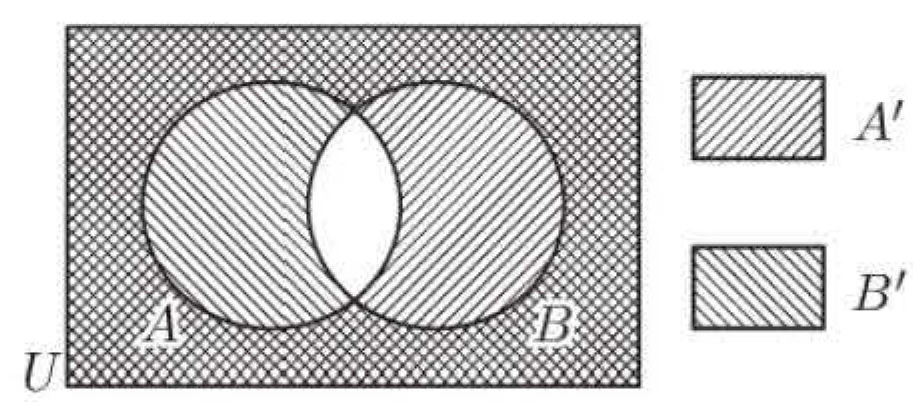
iv $\text{P(drama but not movie)} = \dfrac{12+3}{40}$
$= \frac{15}{40} = \frac{3}{8}$

v $\text{P(drama or a movie)} = \dfrac{12+3+3+1+4+2}{40} = \frac{25}{40} = \frac{5}{8}$

EXERCISE 24H.2

1 **a**

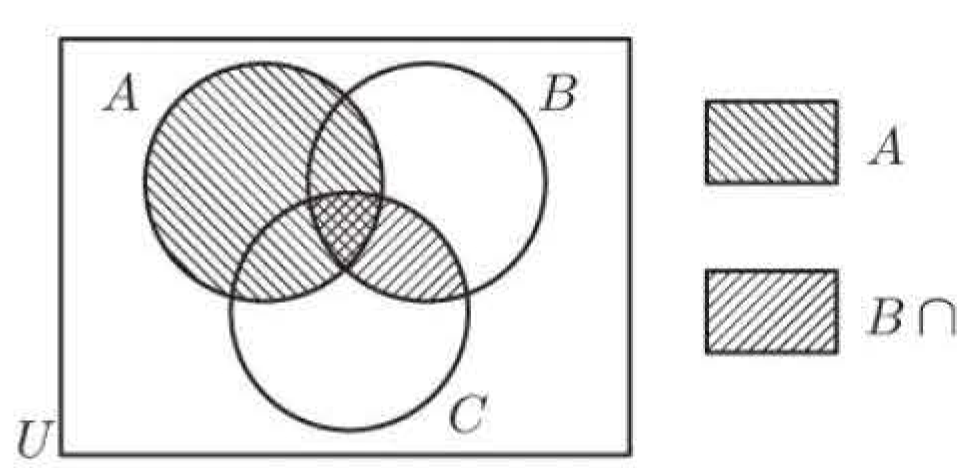


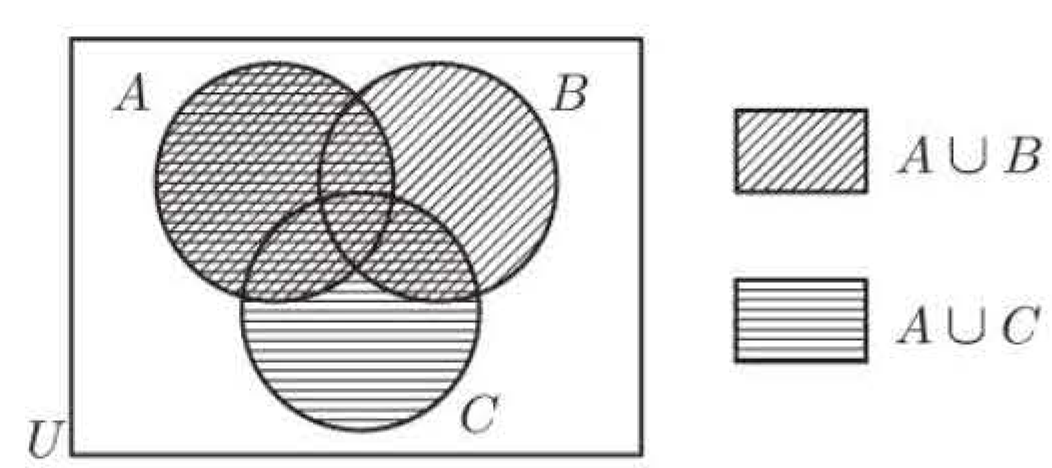


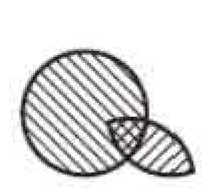
So $A' \cup B'$ is the region containing either type of shading.

Thus, as the regions are the same, $(A \cap B)' = A' \cup B'$ is verified.

b



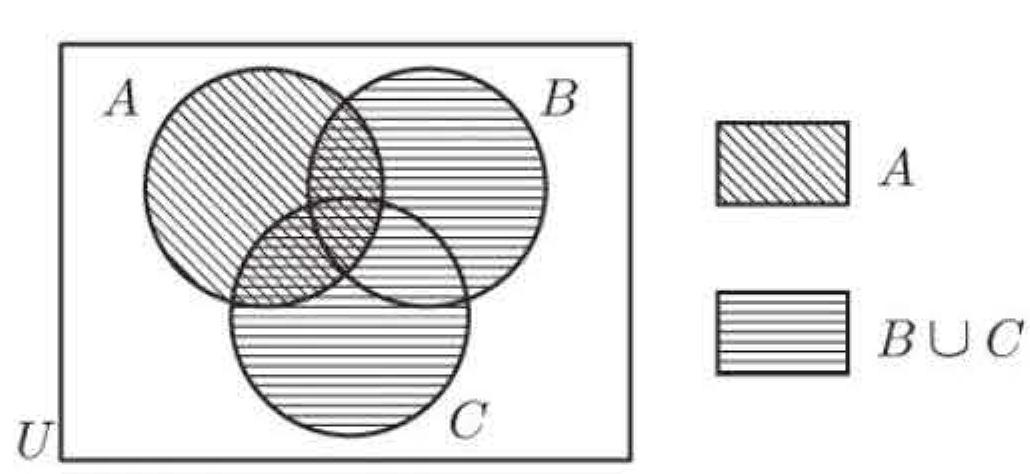


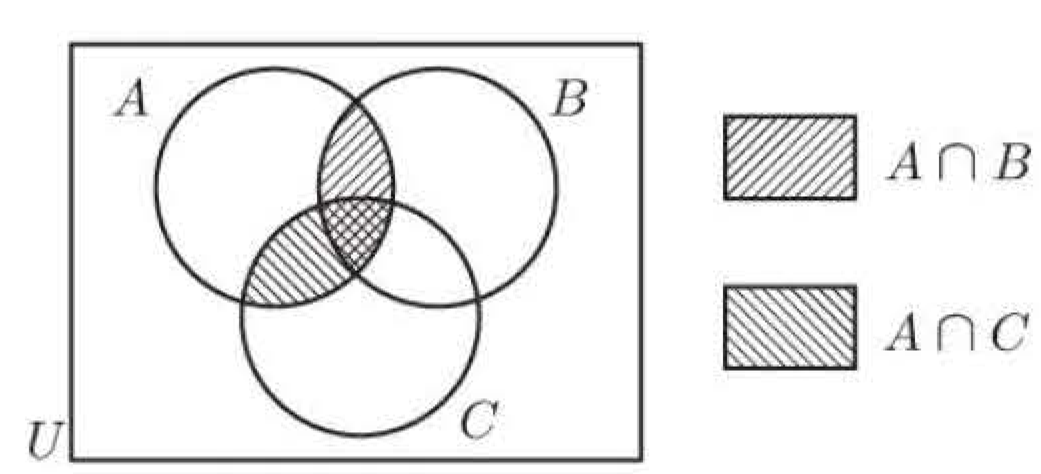
$A \cup (B \cap C)$ consists of the shaded region 

$(A \cup B) \cap (A \cup C)$ consists of the 'double shaded' region.

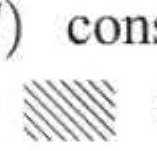
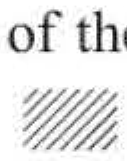
As the two regions are identical
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ is verified.

c





$A \cap (B \cup C)$ consists of the double shaded region 

$(A \cap B) \cup (A \cap C)$ consists of the region shaded. (all forms  and )

As the regions are identical, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ is verified.

2 **a** $A = \{7, 14, 21, 28, 35, ..., 98\}$
$B = \{5, 10, 15, 20, 25, ..., 95\}$

i as $98 = 7 \times 14$, $n(A) = 14$ **ii** as $95 = 5 \times 19$, $n(B) = 19$

iii $A \cap B = \{35, 70\}$ $\therefore$ $n(A \cap B) = 2$

iv $A \cup B = \{5, 7, 10, 14, 15, 20, 21, 25, 28, 30, 35, 40, 42, 45, 49, 50, 55, 56, 60, 63, 65, 70, 75, 77, 80, 84, 85, 90, 91, 95, 98\}$ $\therefore$ $n(A \cup B) = 31$

b $n(A) + n(B) - n(A \cap B)$
$= 14 + 19 - 2$
$= 31$
$= n(A \cup B)$ ✓

c From the diagram, $n(A) + n(B) - n(A \cap B)$
$= (a + b) + (b + c) - b$
$= a + b + c$
$= n(A \cup B)$

3 **a** **i** $\text{P}(B) = \dfrac{n(B)}{n(U)} = \dfrac{b + c}{a + b + c + d}$

ii $\text{P}(A \cap B) = \dfrac{n(A \cap B)}{n(U)} = \dfrac{b}{a + b + c + d}$

iii $\text{P}(A \cup B) = \dfrac{n(A \cup B)}{n(U)} = \dfrac{a + b + c}{a + b + c + d}$

iv $\text{P}(A) + \text{P}(B) - \text{P}(A \cap B) = \dfrac{a + b + b + c - b}{a + b + c + d}$
$= \dfrac{a + b + c}{a + b + c + d}$

b $\text{P}(A \cup B) = \text{P}(A) + \text{P}(B) - \text{P}(A \cap B)$ {using **iii** and **iv**}

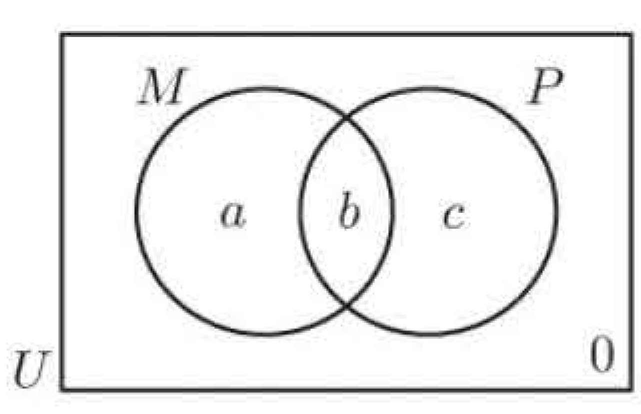
EXERCISE 24I

1 $\text{P}(A \cup B) = \text{P}(A) + \text{P}(B) - \text{P}(A \cap B)$
$\therefore$ $0.9 = 0.4 + \text{P}(B) - 0.1$
$\therefore$ $\text{P}(B) = 0.6$

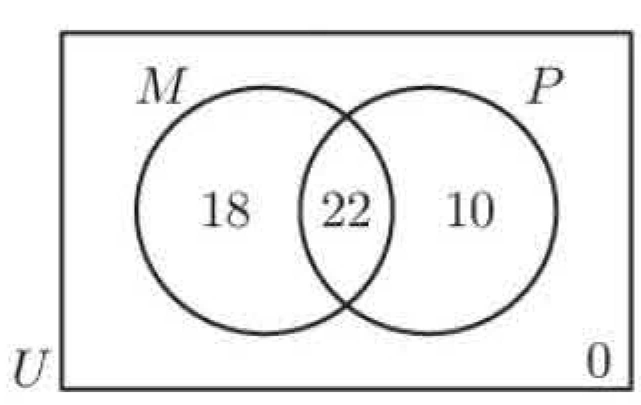
2 $\text{P}(X \cup Y) = \text{P}(X) + \text{P}(Y) - \text{P}(X \cap Y)$
$\therefore$ $0.9 = 0.6 + 0.5 - \text{P}(X \cap Y)$
$\therefore$ $\text{P}(X \cap Y) = 0.2$

3 $\text{P}(A \cup B) = \text{P}(A) + \text{P}(B)$ {A and B are mutually exclusive}
$\therefore$ $0.8 = \text{P}(A) + 0.45$
$\therefore$ $\text{P}(A) = 0.35$

4 **a**



$a + b + c = 50$ $\therefore$ $a + 32 = 50$, $\therefore$ $a = 18$
$a + b = 40$ $\therefore$ $18 + b = 40$, $\therefore$ $b = 22$
$b + c = 32$ $\therefore$ $22 + c = 32$, $\therefore$ $c = 18$

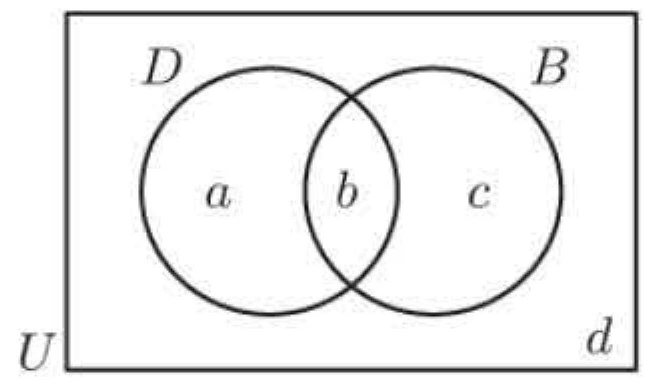


So 22 study both.

b **i** $\text{P}(M \text{ but not } P) = \frac{18}{50} = \frac{9}{25}$

ii $\text{P}(P \text{ given } M) = \dfrac{22}{18 + 22} = \frac{22}{40} = \frac{11}{20}$

5



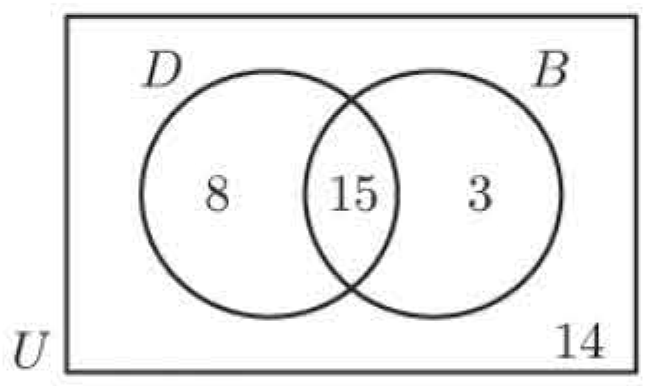
$$\begin{aligned} a+b+c+d &= 40 \quad .... (1) \\ a+b &= 23 \quad .... (2) \\ b+c &= 18 \quad .... (3) \\ a+b+c &= 26 \quad .... (4) \end{aligned}$$

$\therefore\ d = 14$ {using (1) and (4)}

$23 + c = 26$ and $a + 18 = 26$

$\therefore\ c = 3$ and $a = 8$

Thus $b = 18 - c = 15$



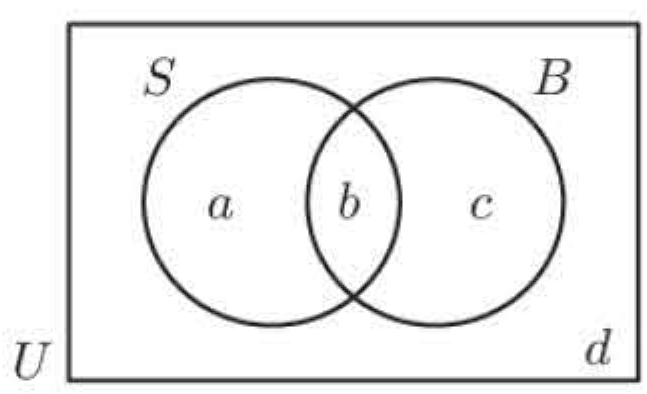
a P(D and B)
$= \frac{15}{40}$
$= \frac{3}{8}$

b P(neither D nor B)
$= \frac{14}{40}$
$= \frac{7}{20}$

c P(D, but not B)
$= \frac{8}{40}$
$= \frac{1}{5}$

d P(B given D)
$= \frac{15}{23}$

6

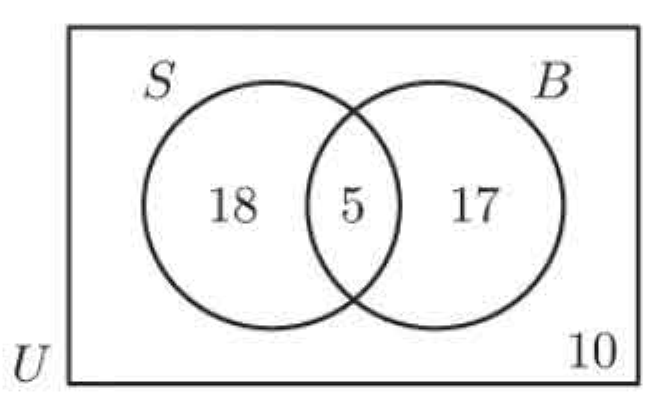


$$\begin{aligned} a+b+c+d &= 50 \\ a+b &= 23 \\ b+c &= 22 \\ b &= 5 \end{aligned}$$

$\therefore\ c = 17,\ a = 18$

and $18 + 5 + 17 + d = 50$

$\therefore\ d = 10$



a P(not B)
$= \text{P}(B')$
$= \frac{28}{50}$
$= \frac{14}{25}$

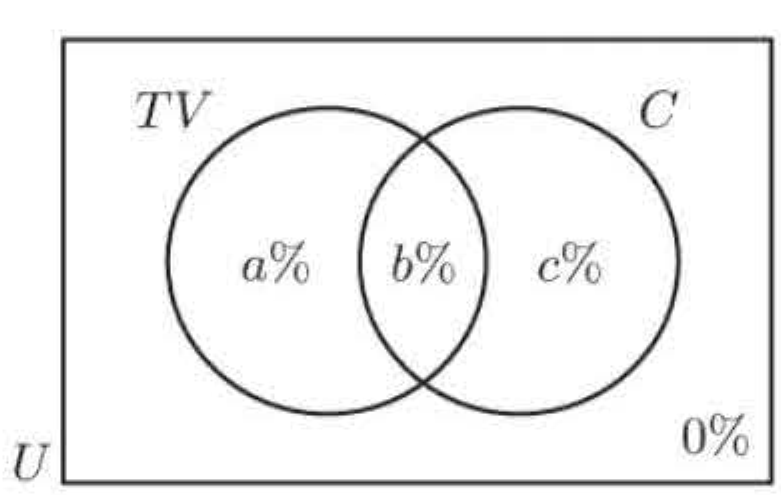
b P(B or S)
$= \dfrac{18+5+17}{50}$
$= \frac{40}{50}$
$= \frac{4}{5}$

c P(neither B nor S)
$= \frac{10}{50}$
$= \frac{1}{5}$

d P(B, given S)
$= \dfrac{5}{18+5}$
$= \frac{5}{23}$

e P(S, given B')
$= \dfrac{18}{18+10}$
$= \frac{18}{28}$
$= \frac{9}{14}$

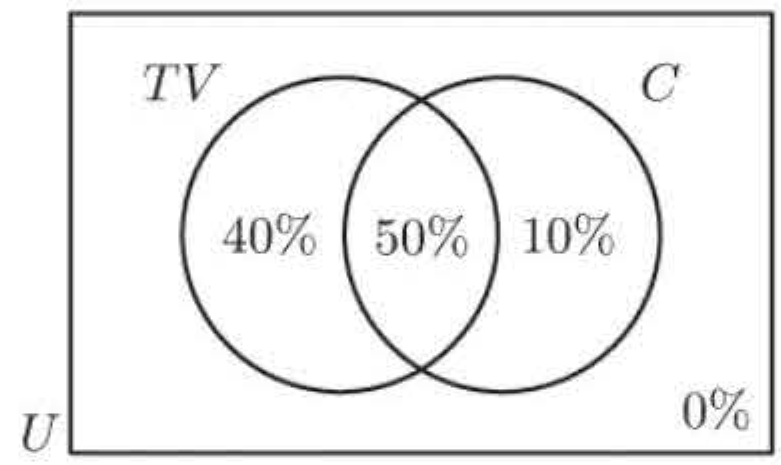
7



$$\begin{cases} a+b+c = 100 \\ a+b = 90 \\ b+c = 60 \end{cases}$$

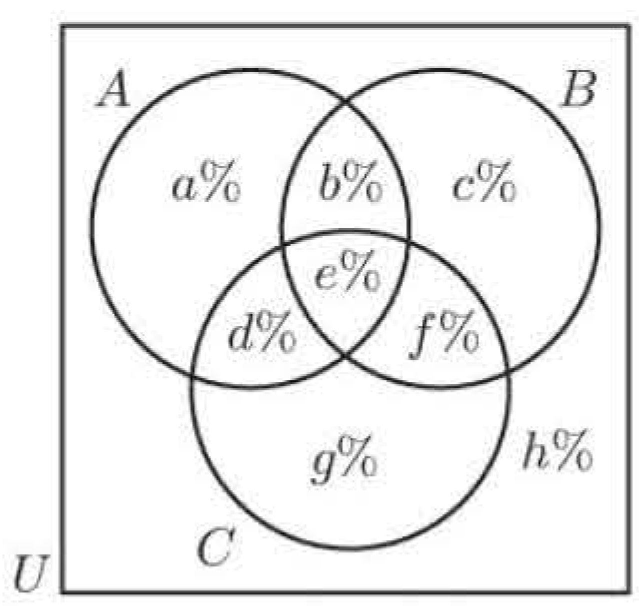
$\therefore\ c = 10$ and $a = 40$

$\therefore\ b = 50$

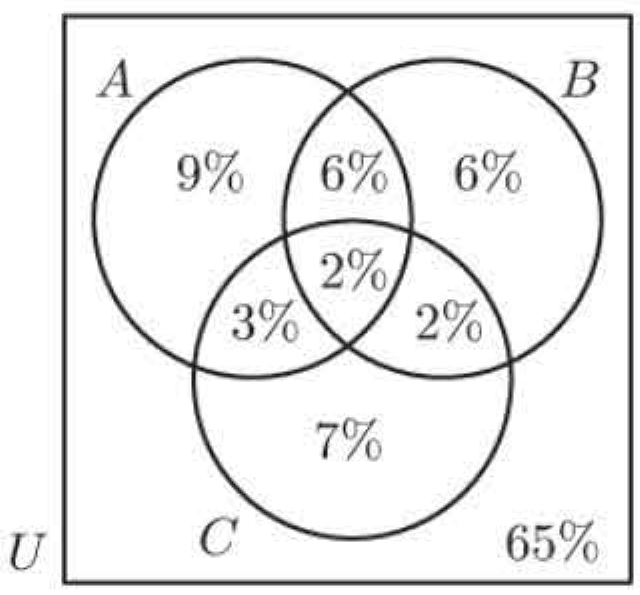


$$\text{P}(TV, \text{ given } C) = \frac{50}{50+10} = \frac{5}{6}$$

8



$$\begin{aligned} a+b+c+d+e+f+g+h &= 100 \\ a+b+d+e &= 20 \\ b+c+e+f &= 16 \\ d+e+f+g &= 14 \\ b+e &= 8 \\ d+e &= 5 \\ e+f &= 4 \\ e &= 2 \end{aligned}$$



$\therefore \; e = 2, \; f = 2, \; d = 3, \; b = 6, \quad \begin{cases} a + 6 + 3 + 2 = 20 \\ 6 + c + 2 + 2 = 16 \\ 3 + 2 + 2 + g = 14 \end{cases} \quad \therefore \quad \begin{cases} a = 9 \\ c = 6 \\ g = 7 \end{cases}$

a P(none)
$= \frac{65}{100}$
$= \frac{13}{20}$

b P(at least one)
$= 1 - \text{P(none)}$
$= 1 - \frac{13}{20}$
$= \frac{7}{20}$

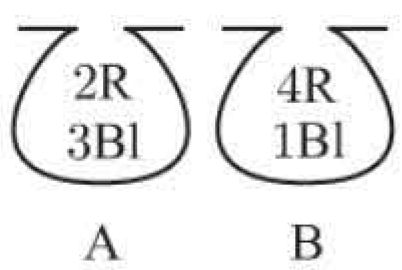
c P(exactly one)
$= \dfrac{9+6+7}{100}$
$= \frac{22}{100}$
$= \frac{11}{50}$

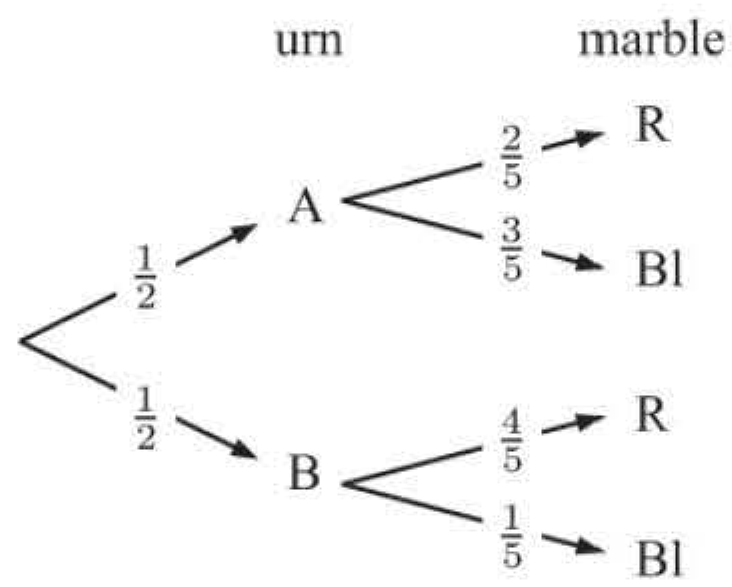
d P(A or B)
$= \frac{9+6+6+3+2+2}{100}$
$= \frac{28}{100}$
$= \frac{7}{25}$

e P(A, given at least one)
$= \frac{9+6+2+3}{35}$
$= \frac{20}{35}$
$= \frac{4}{7}$

f P(C, given A or B or both)
$= \frac{3+2+2}{9+6+6+3+2+2}$
$= \frac{7}{28}$
$= \frac{1}{4}$

9

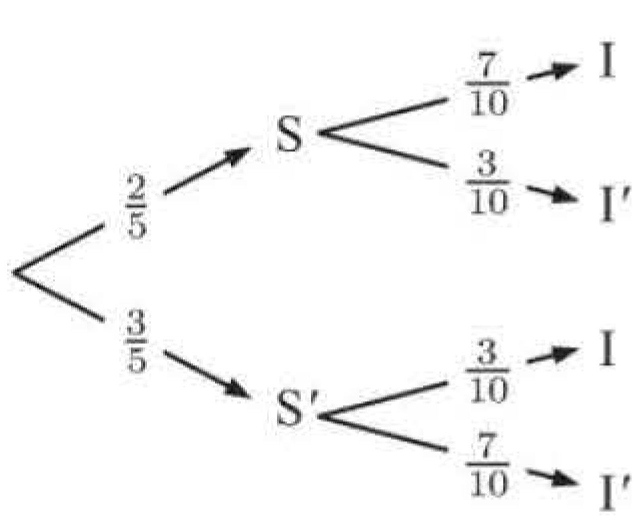




a $\text{P(R)} = \frac{1}{2} \times \frac{2}{5} + \frac{1}{2} \times \frac{4}{5}$
$= \frac{3}{5}$

b $\text{P(B | R)} = \dfrac{\text{P(B} \cap \text{R)}}{\text{P(R)}}$
$= \dfrac{\frac{1}{2} \times \frac{4}{5}}{\frac{3}{5}}$
$= \frac{2}{3}$

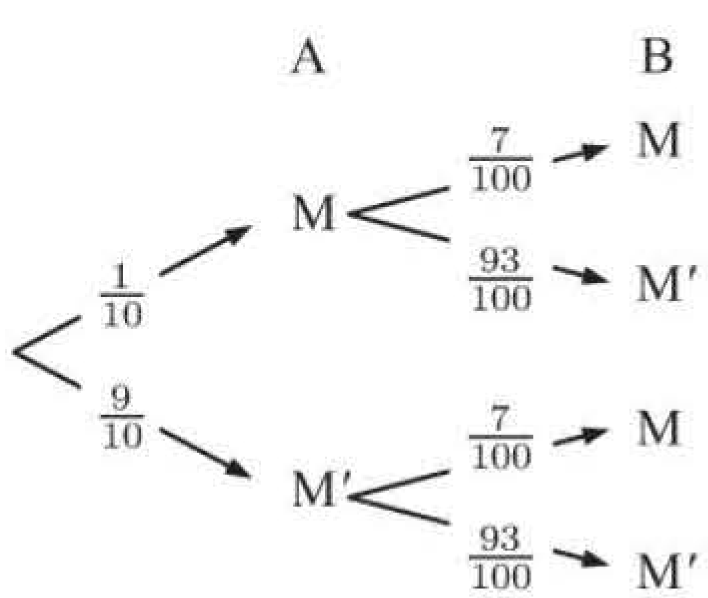
10



a $\text{P(I)} = \frac{2}{5} \times \frac{7}{10} + \frac{3}{5} \times \frac{3}{10}$
$= \frac{23}{50}$ (or 0.46)

b $\text{P(S | I)} = \dfrac{\text{P(S} \cap \text{I)}}{\text{P(I)}}$
$= \dfrac{\frac{2}{5} \times \frac{7}{10}}{\frac{23}{50}}$
$= \frac{14}{23}$

11



P(B | at least one malfunctions)
$= \dfrac{\text{P(B} \cap \text{at least one malfunctions)}}{\text{P(at least one malfunctions)}}$
$= \dfrac{\frac{1}{10} \times \frac{7}{100} + \frac{9}{10} \times \frac{7}{100}}{\frac{1}{10} \times \frac{7}{100} + \frac{1}{10} \times \frac{93}{100} + \frac{9}{10} \times \frac{7}{100}}$
$= \frac{7+63}{7+93+63}$
$= \frac{70}{163}$

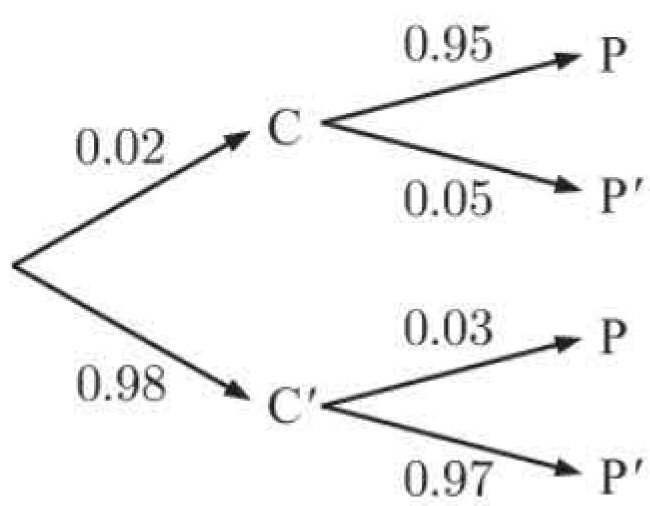
12 $\text{P}(B) = 0.5, \quad \text{P}(G) = 0.6, \quad \text{P}(G \mid B) = 0.9,$ where B is "the boy eats his lunch" and G is "the girl eats her lunch"

a P(both eat lunch)
$= \text{P}(B \cap G)$
$= \text{P}(G \mid B) \times \text{P}(B) \qquad \left\{ \text{as } \; \text{P}(G \mid B) = \dfrac{\text{P}(G \cap B)}{\text{P}(B)} \right\}$
$= 0.9 \times 0.5$
$= 0.45$

b $\text{P}(B \mid G)$
$= \dfrac{\text{P}(B \cap G)}{\text{P}(G)}$
$= \dfrac{0.45}{0.6}$
$= 0.75$

c $\quad$ P(at least one eats lunch)
$= \text{P}(B \cup G)$
$= \text{P}(B) + \text{P}(G) - \text{P}(B \cap G)$
$= 0.5 + 0.6 - 0.45$
$= 0.65$

13



a $\quad$ P(P)
$= 0.02 \times 0.95 + 0.98 \times 0.03$
$= 0.0484$

b $\quad$ P(C | P)
$= \dfrac{\text{P}(\text{C} \cap \text{P})}{\text{P}(\text{P})}$
$= \dfrac{0.02 \times 0.95}{0.0484}$
≈ 0.393

14 The coins are H, H $\quad$ T, T $\quad$ and H, T.

Any one of these 6 faces could be seen uppermost, $\therefore$ P(falls H) $= \frac{3}{6} = \frac{1}{2}$

Now $\quad \text{P(HH coin | falls H)} = \dfrac{\text{P(HH coin} \cap \text{falls H)}}{\text{P(falls H)}}$

$$= \frac{\text{P(HH)}}{\text{P(falls H)}} = \frac{\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}$$

EXERCISE 24J

1 $\quad \text{P}(R \cap S)$
$= \text{P}(R) + \text{P}(S) - \text{P}(R \cup S)$
$= 0.4 + 0.5 - 0.7$
$= 0.2$

Also, $\quad \text{P}(R) \times \text{P}(S)$
$= 0.4 \times 0.5$
$= 0.2$

So, $\text{P}(R \cap S) = \text{P}(R) \times \text{P}(S)$ and hence R and S are independent events.

2 **a** $\quad \text{P}(A \cap B)$
$= \text{P}(A) + \text{P}(B) - \text{P}(A \cup B)$
$= \frac{2}{5} + \frac{1}{3} - \frac{1}{2}$
$= \frac{7}{30}$

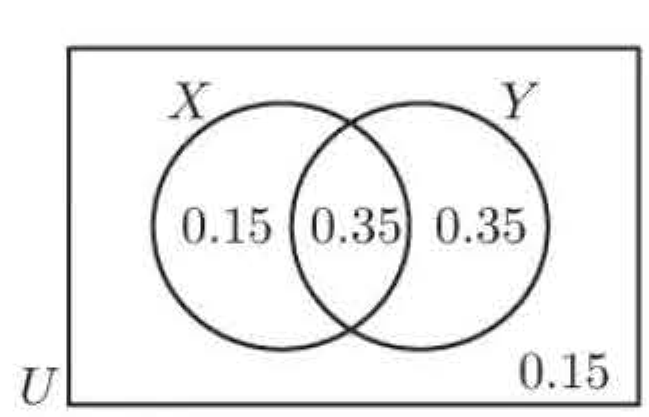
b $\quad \text{P}(B \mid A)$
$= \dfrac{\text{P}(B \cap A)}{\text{P}(A)}$
$= \dfrac{\frac{7}{30}}{\frac{2}{5}}$
$= \frac{7}{12}$

c $\quad \text{P}(A \mid B)$
$= \dfrac{\text{P}(A \cap B)}{\text{P}(B)}$
$= \dfrac{\frac{7}{30}}{\frac{1}{3}}$
$= \frac{7}{10}$

A and B are not independent as $\text{P}(A \mid B) \neq \text{P}(A)$.

3 **a** As X and Y are independent
$\text{P}(X \cap Y) = \text{P}(X) \times \text{P}(Y)$
$= 0.5 \times 0.7$
$= 0.35$
$\therefore$ P(both X and Y) $= 0.35$

b $\quad$ P(X or Y)
$= \text{P}(X \cup Y)$
$= \text{P}(X) + \text{P}(Y) - \text{P}(X \cap Y)$
$= 0.5 + 0.7 - 0.35$
$= 0.85$



c $\quad$ P(neither X nor Y)
$= 0.15$

d $\quad$ P(X but not Y)
$= 0.15$

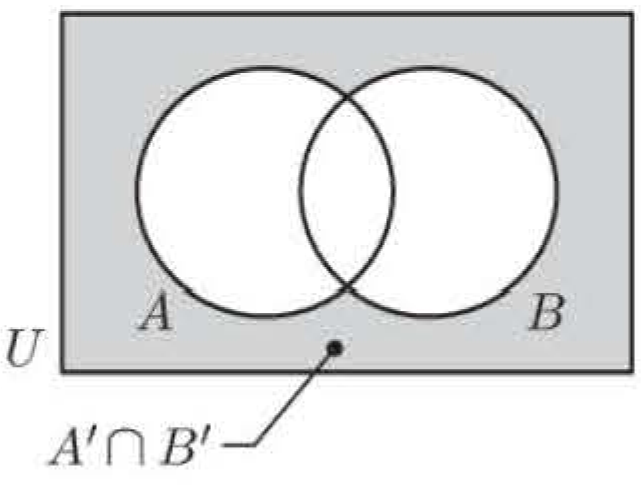
e $\quad \text{P}(X \mid Y) = \dfrac{\text{P}(X \cap Y)}{\text{P}(Y)} = \dfrac{0.35}{0.70} = 0.5$

4 $\quad$ P(at least one solves it)
$= 1 - \text{P(no-one solves it)}$
$= 1 - \text{P}(\text{A}' \text{ and } \text{B}' \text{ and } \text{C}')$
$= 1 - \frac{2}{5} \times \frac{1}{3} \times \frac{1}{2}$ {each student's ability to solve the problem is independent}
$= 1 - \frac{1}{15}$
$= \frac{14}{15}$

5 **a** $\quad$ P(at least one 6)
$= 1 - \text{P(no 6s)}$
$= 1 - \text{P}(6' \text{ and } 6' \text{ and } 6')$
$= 1 - \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6}$
$= 1 - \frac{125}{216}$
$= \frac{91}{216}$

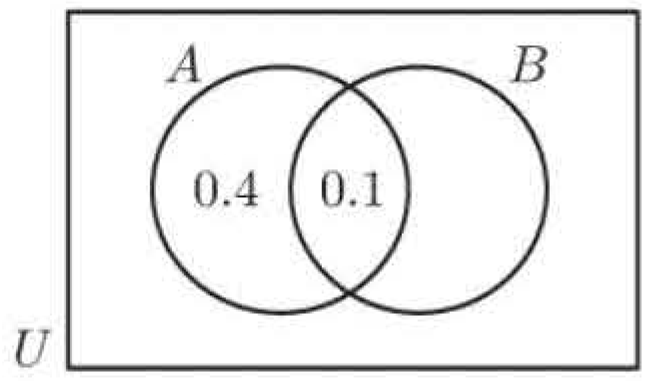
b $\quad$ P(at least one 6 in n throws)
$= 1 - (\frac{5}{6})^n$
So we want $1 - (\frac{5}{6})^n > 0.99$
$\therefore \; -(\frac{5}{6})^n > -0.01$
$\therefore \; (\frac{5}{6})^n < 0.01$
$\therefore \; n \log(\frac{5}{6}) < \log(0.01)$
$\therefore \; n > \dfrac{\log(0.01)}{\log(\frac{5}{6})}$ $\quad$ {as $\log(\frac{5}{6}) < 0$}
$\therefore \; n > 25.2585$
$\therefore \; n = 26$

6 A and B are independent, so $\text{P}(A \cap B) = \text{P}(A)\,\text{P}(B)$ .... (1)



Now $\quad \text{P}(A' \cap B')$
$= 1 - \text{P}(A \cup B)$
$= 1 - [\text{P}(A) + \text{P}(B) - \text{P}(A \cap B)]$
$= 1 - \text{P}(A) - \text{P}(B) + \text{P}(A \cap B)$
$= 1 - \text{P}(A) - \text{P}(B) + \text{P}(A)\,\text{P}(B)$ $\quad$ {using (1)}
$= [1 - \text{P}(A)]\,[1 - \text{P}(B)]$
$= \text{P}(A')\,\text{P}(B')$ $\qquad \therefore \; A'$ and B' are also independent.

7



$\therefore \; \text{P}(A) = 0.5$
and $\quad \text{P}(A \cap B) = \text{P}(A) \times \text{P}(B)$ $\quad$ {A and B are independent}
$\therefore \; 0.1 = 0.5 \times \text{P}(B)$
$\therefore \; \text{P}(B) = 0.2$
Now $\quad \text{P}(A \cup B') = \text{P}(A) + \text{P}(B') - \text{P}(A \cap B')$
$= 0.5 + 0.8 - 0.4$
$= 0.9$

8 **a** **i** $\text{P}(C \mid D) = \dfrac{\text{P}(C \cap D)}{\text{P}(D)}$, so $\text{P}(C \cap D) = \text{P}(C \mid D)\,\text{P}(D)$
Similarly, $\text{P}(C \cap D') = \text{P}(C \mid D')\,\text{P}(D')$
Now $\quad \text{P}(C \cap D) + \text{P}(C \cap D') = \text{P}(C)$
$\therefore \; \text{P}(C \mid D)\,\text{P}(D) + \text{P}(C \mid D')\,\text{P}(D') = \text{P}(C)$
$\therefore \; \frac{6}{13}\,\text{P}(D) + \frac{3}{7}\,[1 - \text{P}(D)] = \frac{9}{20}$
$\therefore \; \frac{6}{13}\,\text{P}(D) + \frac{3}{7} - \frac{3}{7}\,\text{P}(D) = \frac{9}{20}$
$\therefore \; \frac{3}{91}\,\text{P}(D) = \frac{3}{140}$
$\therefore \; \text{P}(D) = \frac{91}{140} = \frac{13}{20}$

ii $P(C \cap D) = P(C \mid D)\, P(D) = \frac{6}{13} \times \frac{13}{20} = \frac{3}{10}$

Now $P(C' \cup D') = 1 - P(C \cap D) = 1 - \frac{3}{10} = \frac{7}{10}$

b $P(C \mid D) = \frac{6}{13}$ and $P(C) = \frac{9}{20}$

$\therefore$ C and D are not independent as $P(C \mid D) \neq P(C)$

or

$P(C \cap D) = \frac{3}{10}$ and $P(C)\, P(D) = \frac{9}{20} \times \frac{13}{20} = \frac{117}{400}$

$\therefore$ C and D are not independent as $P(C \cap D) \neq P(C)\, P(D)$

9 **a** **i** P(Ruba wins on her third turn) = P(4 non-aces, then an ace)

$$= \left(\tfrac{12}{13}\right)^4 \times \left(\tfrac{1}{13}\right) \approx 0.0558$$

ii P(Ruba wins on her nth turn) = P($2(n-1)$ non-aces, then an ace)

$$= \left(\tfrac{12}{13}\right)^{2(n-1)} \times \tfrac{1}{13}$$

$\therefore$ P(Ruba wins prior to her $(n+1)$th turn)

= P(Ruba wins on her 1st or 2nd or 3rd or .... or nth turn)

$$= \left(\tfrac{12}{13}\right)^0 \times \tfrac{1}{13} + \left(\tfrac{12}{13}\right)^2 \times \tfrac{1}{13} + \left(\tfrac{12}{13}\right)^4 \times \tfrac{1}{13} + ... + \left(\tfrac{12}{13}\right)^{2(n-1)} \times \tfrac{1}{13}$$

$$= \tfrac{1}{13}\Bigg(\underbrace{1 + \left(\tfrac{12}{13}\right)^2 + \left(\tfrac{12}{13}\right)^4 + ... + \left(\tfrac{12}{13}\right)^{2(n-1)}}_{\text{geometric series with } u_1 = 1,\ r = \left(\frac{12}{13}\right)^2,\ \text{"}n\text{"} = n}\Bigg)$$

$$= \tfrac{1}{13}\left(\frac{1 - \left(\frac{12}{13}\right)^{2n}}{1 - \left(\frac{12}{13}\right)^2}\right) = \tfrac{1}{13}\left(\frac{1 - \left(\frac{12}{13}\right)^{2n}}{\frac{25}{169}}\right)$$

$$= \frac{169}{13 \times 25}\left(1 - \left(\tfrac{12}{13}\right)^{2n}\right)$$

$$= \tfrac{13}{25}\left(1 - \left(\tfrac{12}{13}\right)^{2n}\right)$$

iii As $n \to \infty$, $1 - \left(\frac{12}{13}\right)^{2n} \to 1$

$\therefore$ $\frac{13}{25}\left(1 - \left(\frac{12}{13}\right)^{2n}\right) \to \frac{13}{25}$

$\therefore$ P(Ruba wins the game) $= \frac{13}{25}$

b Let X be the number of times Ruba wins the game.

Then X is binomial with $n = 7$ trials of probability $p = \frac{13}{25}$.

$\therefore$ P(Ruba will win more games than Hania) $= P(X \geqslant 4)$

$$= 1 - P(X \leqslant 3)$$
$$\approx 1 - 0.456$$
$$\approx 0.544$$

10 The man will step over the cliff in his first four steps if either:

(1) he steps towards the cliff on his first step

(2) he steps away from the cliff on his first step, but towards the cliff on his next two steps.

P(case (1)) $= \frac{2}{5}$

P(case (2)) $= \frac{3}{5} \times \left(\frac{2}{5}\right)^2 = \frac{12}{125}$

$\therefore$ P(the man steps over the cliff in his first four steps) $= \frac{2}{5} + \frac{12}{125} = \frac{62}{125}$

$\therefore$ P(the man does **not** step over the cliff in his first four steps) $= 1 - \frac{62}{125}$

$$= \frac{63}{125}$$

EXERCISE 24K

1 The total number of different committees is $\binom{11}{4}$.

The number of ways of both sisters being on a committee with any 2 others is $\binom{2}{2} \times \binom{9}{2}$.

$\therefore$ P(both sisters are on the committee) $= \dfrac{\binom{2}{2} \times \binom{9}{2}}{\binom{11}{4}} = \frac{6}{55}$

2 AIDS and SAID are 2 of the 4! different orderings. $\therefore$ P(AIDS or SAID) $= \dfrac{2}{4!} = \frac{1}{12}$

3 There are $\binom{12}{7}$ different teams that can be selected.

$\therefore$ P(captain and vice captain are chosen) $= \dfrac{\binom{2}{2} \times \binom{10}{5}}{\binom{12}{7}} \approx 0.318$

4 P(none of the golfers was killed) $= \dfrac{\binom{3}{0} \times \binom{19}{4}}{\binom{22}{4}} \approx 0.530$

5

5	4	3	2	1

$\therefore$ there are 5! different possible seating arrangements.

a

2	3	2	1	1

There are $2 \times 3!$ seating arrangements if K and J sit at the ends

$\therefore$ P(K and J sit at the ends) $= \dfrac{2 \times 3!}{5!} = \frac{1}{10}$

b K and J can sit together in 2! ways. They as a pair plus the other three people can then be ordered in 4! ways.

$\therefore$ P(sit together) $= \dfrac{2! \times 4!}{5!} = \frac{2}{5}$

6 There are $\binom{16}{5}$ different committees possible.

a P(all men)

$= \dfrac{\binom{9}{5} \times \binom{7}{0}}{\binom{16}{5}}$

$= 0.0288$

b P(at least 3 men)

$=$ P(3 men or 4 men or 5 men)

$= \dfrac{\binom{9}{3}\binom{7}{2} + \binom{9}{4}\binom{7}{1} + \binom{9}{5}\binom{7}{0}}{\binom{16}{5}}$

≈ 0.635

c P(at least one of each sex)

$= 1 -$ P(no men or no women)

$= 1 - \dfrac{\binom{9}{0}\binom{7}{5} + \binom{9}{5}\binom{7}{0}}{\binom{16}{5}}$

≈ 0.966

7 If there are no restrictions there are 6! different orderings possible. A, B, and C can be ordered in 3! ways. This triple together with the 3 others can be ordered in 4! ways.

$\therefore$ P(A, B, C together) $= \dfrac{3! \times 4!}{6!} = \frac{1}{5}$

8 There are $\binom{14}{7}$ different committees possible.

a P(only senior students) $= \dfrac{\binom{11}{7}\binom{3}{0}}{\binom{14}{7}}$

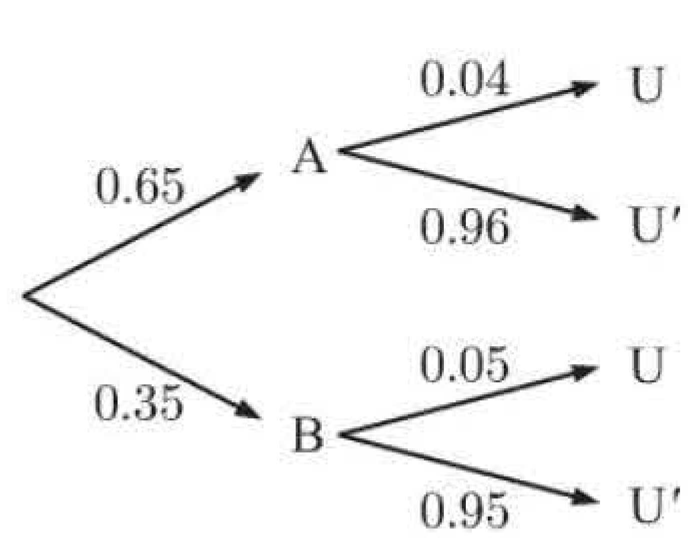
≈ 0.0962

b P(all three junior students chosen) $= \dfrac{\binom{11}{4}\binom{3}{3}}{\binom{14}{7}}$

≈ 0.0962

EXERCISE 24L

1



a P(underfilled)

$=$ P(A and U or B and U)

$= 0.65 \times 0.04 + 0.35 \times 0.05$

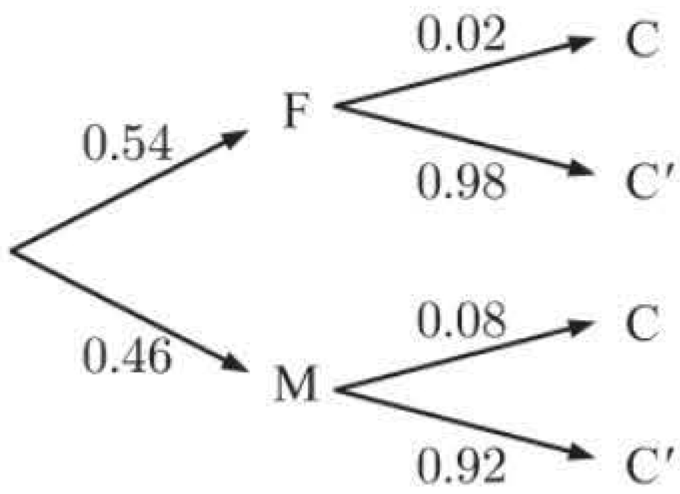
$= 0.0435$

b $\text{P}(A \mid U) = \dfrac{\text{P}(U \mid A)\,\text{P}(A)}{\text{P}(U)}$

$= \dfrac{0.04 \times 0.65}{0.0435}$

≈ 0.598

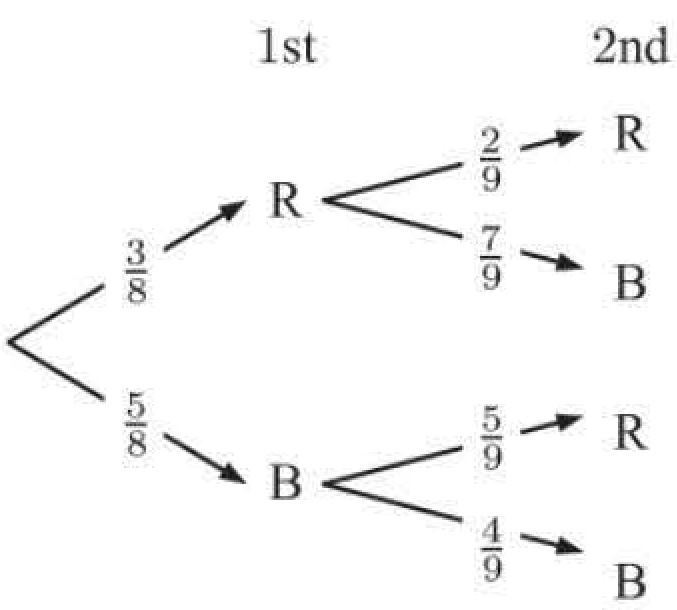
2



a $$\begin{aligned}P(M \mid C) &= \frac{P(C \mid M) \times P(M)}{P(C \mid M) \times P(M) + P(C \mid F) \times P(F)} \\ &= \frac{0.08 \times 0.46}{0.08 \times 0.46 + 0.02 \times 0.54} \\ &\approx 0.773\end{aligned}$$

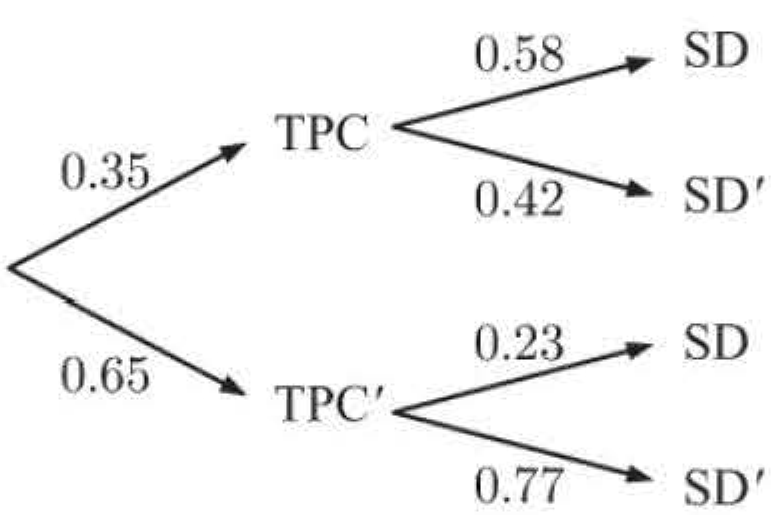
b $$\begin{aligned}P(F \mid C') &= \frac{P(C' \mid F) \times P(F)}{P(C' \mid F) \times P(F) + P(C' \mid M) \times P(M)} \\ &= \frac{0.98 \times 0.54}{0.98 \times 0.54 + 0.92 \times 0.46} \\ &\approx 0.556\end{aligned}$$

3



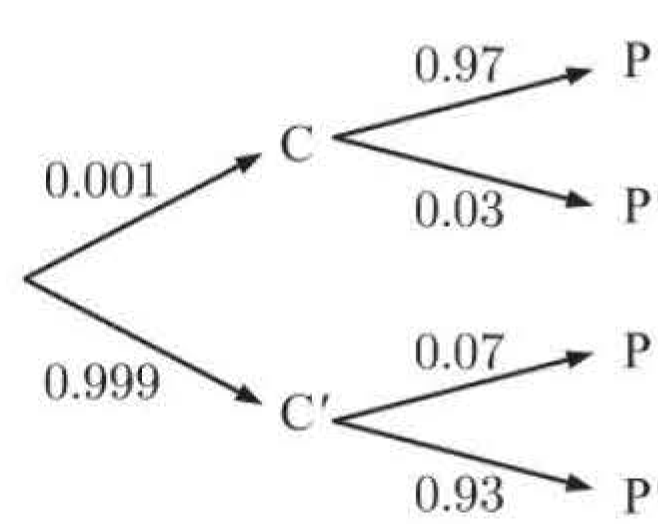
$$\begin{aligned}P(BB \mid RR \text{ or } BB) &= \frac{P(BB \cap (RR \text{ or } BB))}{P(RR \text{ or } BB)} \\ &= \frac{P(BB)}{P(RR \text{ or } BB)} \\ &= \frac{\frac{5}{8} \times \frac{4}{9}}{\frac{3}{8} \times \frac{2}{9} + \frac{5}{8} \times \frac{4}{9}} \\ &= \tfrac{20}{26} \\ &= \tfrac{10}{13}\end{aligned}$$

4

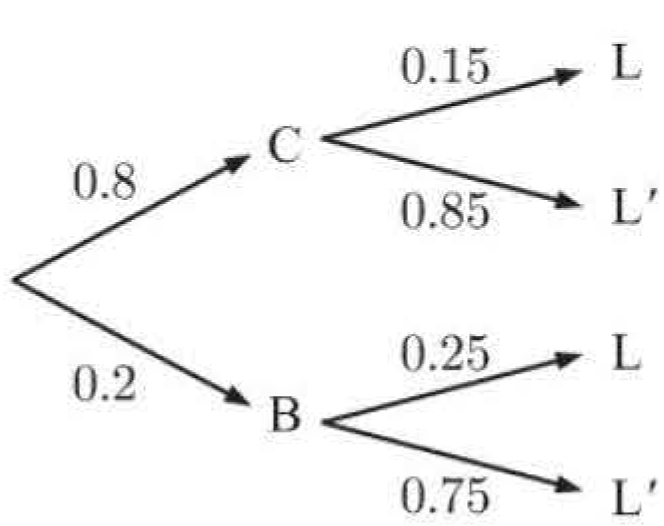


$$\begin{aligned}&P(TPC' \mid SD) \\ &= \frac{P(TPC' \cap SD)}{P(SD)} \\ &= \frac{0.65 \times 0.23}{0.35 \times 0.58 + 0.65 \times 0.23} \\ &\approx 0.424\end{aligned}$$

5 $$\begin{aligned}&P(C \mid P) \\ &= \frac{P(P \mid C) \times P(C)}{P(P \mid C) \times P(C) + P(P \mid C') \times P(C')} \\ &= \frac{0.97 \times 0.001}{0.97 \times 0.001 + 0.07 \times 0.999} \\ &\approx 0.0137\end{aligned}$$

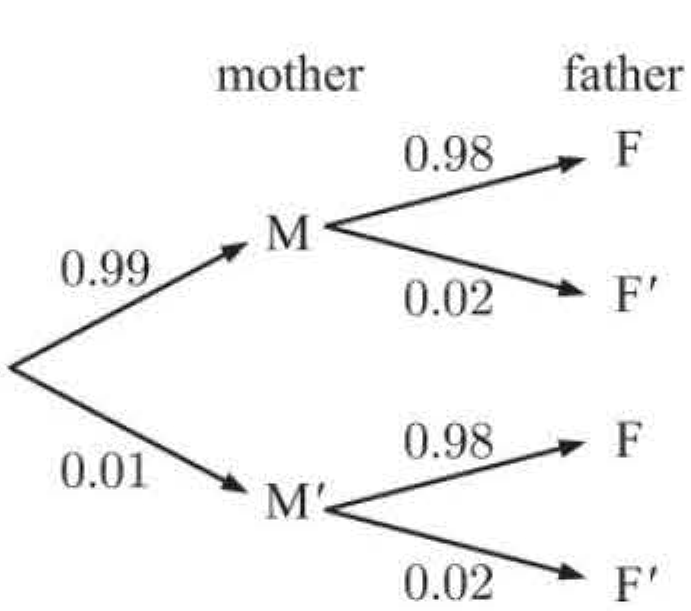


6



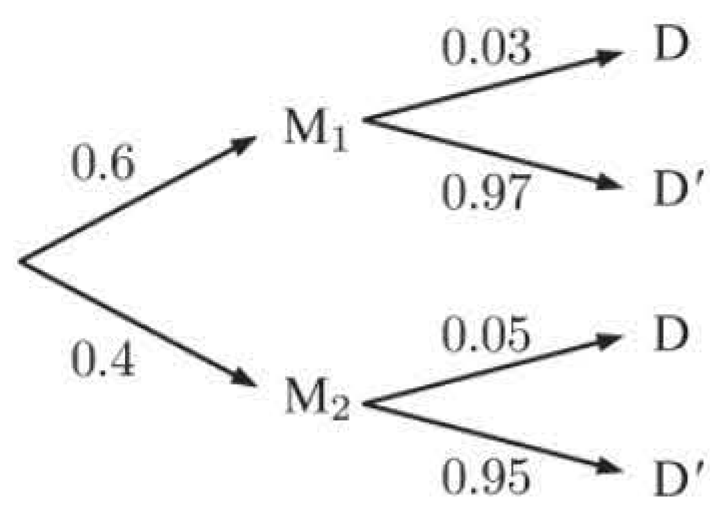
$$\begin{aligned}&P(B \mid L') \\ &= \frac{P(L' \mid B) \times P(B)}{P(L' \mid B) \times P(B) + P(L' \mid C) \times P(C)} \\ &= \frac{0.75 \times 0.2}{0.75 \times 0.2 + 0.85 \times 0.8} \\ &= \tfrac{15}{83} \approx 0.181\end{aligned}$$

7



$$\begin{aligned}P(M \mid \text{only 1 alive}) &= \frac{P(M \cap (\text{only 1 alive}))}{P(\text{only 1 alive})} \\ &= \frac{P(MF')}{P(MF' \text{ or } M'F)} \\ &= \frac{0.99 \times 0.02}{0.99 \times 0.02 + 0.01 \times 0.98} \\ &= \tfrac{99}{148} \approx 0.669\end{aligned}$$

8



a $$\text{P}(\text{M}_1 \mid \text{D}) = \frac{\text{P}(\text{D} \mid \text{M}_1) \times \text{P}(\text{M}_1)}{\text{P}(\text{D} \mid \text{M}_1) \times \text{P}(\text{M}_1) + \text{P}(\text{D} \mid \text{M}_2) \times \text{P}(\text{M}_2)}$$
$$= \frac{0.03 \times 0.6}{0.03 \times 0.6 + 0.05 \times 0.4}$$
$$= \tfrac{9}{19} \approx 0.474$$

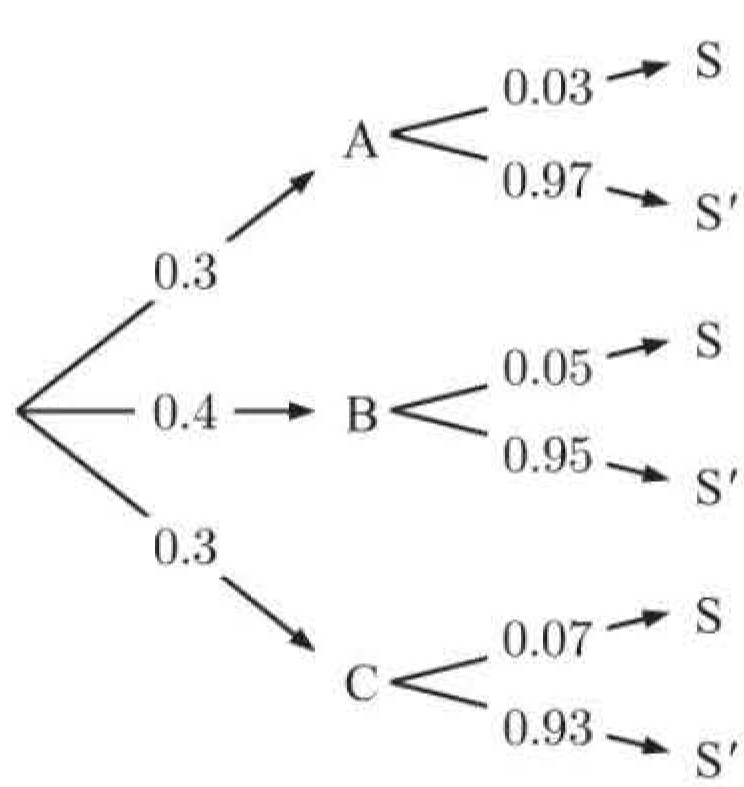
b $$\text{P}(\text{M}_2 \mid \text{D}) = 1 - \tfrac{9}{19}$$
$$= \tfrac{10}{19}$$

9 **a** $$\begin{aligned} \text{P}(B) &= \text{P}(B \text{ and in } A_1 \text{ or } B \text{ and in } A_2 \text{ or } B \text{ and in } A_3) \\ &= \text{P}((B \cap A_1) \cup (B \cap A_2) \cup (B \cap A_3)) \\ &= \text{P}(B \cap A_1) + \text{P}(B \cap A_2) + \text{P}(B \cap A_3) \\ &\qquad \text{as } B \cap A_1,\ B \cap A_2,\ B \cap A_3 \text{ are disjoint} \\ &= \text{P}(B \mid A_1)\,\text{P}(A_1) + \text{P}(B \mid A_2)\,\text{P}(A_2) + \text{P}(B \mid A_3)\,\text{P}(A_3) \\ &\qquad \left\{\text{since } \text{P}(X \mid Y) = \frac{\text{P}(X \cap Y)}{\text{P}(Y)} \Rightarrow \text{P}(X \cap Y) = \text{P}(X \mid Y)\,\text{P}(Y)\right\} \end{aligned}$$

b $$\text{P}(A_i \mid B) = \frac{\text{P}(A_i \cap B)}{\text{P}(B)} = \frac{\text{P}(B \mid A_i)\,\text{P}(A_i)}{\text{P}(B)}$$

where $\text{P}(B) = \text{P}(B \mid A_1)\,\text{P}(A_1) + \text{P}(B \mid A_2)\,\text{P}(A_2) + \text{P}(B \mid A_3)\,\text{P}(A_3)$
$$= \sum_{j=1}^{3} \text{P}(B \mid A_j)\,\text{P}(A_j)$$

10

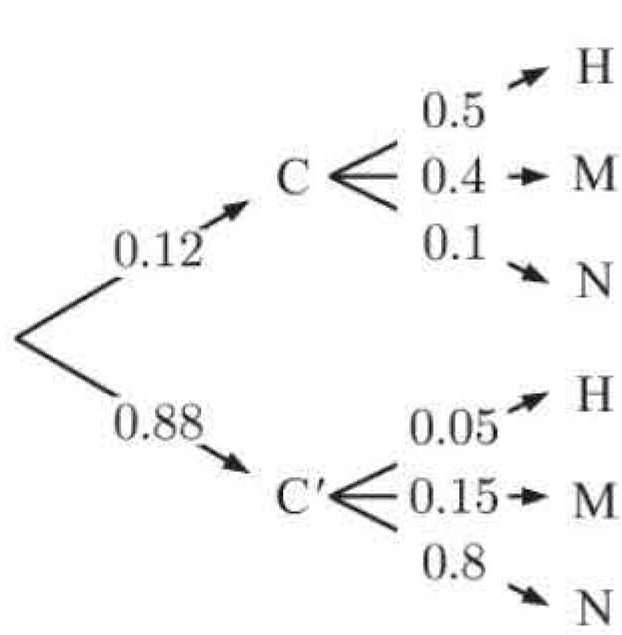


a $$\begin{aligned} \text{P}(\text{S}') &= \text{P}(\text{A} \cap \text{S}' \text{ or } \text{B} \cap \text{S}' \text{ or } \text{C} \cap \text{S}') \\ &= 0.3 \times 0.97 + 0.4 \times 0.95 + 0.3 \times 0.93 \\ &= 0.95 \end{aligned}$$

b $$\text{P}(\text{A} \mid \text{S}') = \frac{\text{P}(\text{A} \cap \text{S}')}{\text{P}(\text{S}')} = \frac{0.3 \times 0.97}{0.95} \approx 0.306$$

c $$\begin{aligned} \text{P}(\text{A or C} \mid \text{S}) &= \frac{\text{P}((\text{A} \cup \text{C}) \cap \text{S})}{\text{P}(\text{S})} \\ &= \frac{\text{P}((\text{A} \cap \text{S}) \cup (\text{C} \cap \text{S}))}{1 - \text{P}(\text{S}')} \\ &= \frac{0.3 \times 0.03 + 0.3 \times 0.07}{0.05} \\ &= 0.6 \end{aligned}$$

11



a $$\begin{aligned} &\text{P}(\text{H}) \\ &= \text{P}(\text{C} \cap \text{H} \text{ or } \text{C}' \cap \text{H}) \\ &= 0.12 \times 0.5 + 0.88 \times 0.05 \\ &= 0.104 \end{aligned}$$

b $$\begin{aligned} \text{P}(\text{C} \mid \text{M}) &= \frac{\text{P}(\text{C} \cap \text{M})}{\text{P}(\text{M})} \\ &= \frac{0.12 \times 0.4}{0.12 \times 0.4 + 0.88 \times 0.15} \\ &\approx 0.267 \end{aligned}$$

c $$\begin{aligned} \text{P}(\text{C} \mid \text{N}) &= \frac{\text{P}(\text{C} \cap \text{N})}{\text{P}(\text{N})} \\ &= \frac{0.12 \times 0.1}{0.12 \times 0.1 + 0.88 \times 0.8} \\ &\approx 0.0168 \end{aligned}$$

REVIEW SET 24A

1 ABCD, ABDC, ACBD, ACDB, ADBC, ADCB, BACD, BADC, BCAD, BCDA, BDAC, BDCA, CABD, CADB, CBAD, CBDA, CDAB, CDBA, DABC, DACB, DBAC, DBCA, DCAB, DCBA

a There are 24 possible orderings.
$\therefore$ P(A is next to C)
$= \frac{12}{24}$ {12 have A next to C}
$= \frac{1}{2}$

b P(exactly one person between A and C)
$= \frac{8}{24}$ {8 have one person between A and C}
$= \frac{1}{3}$

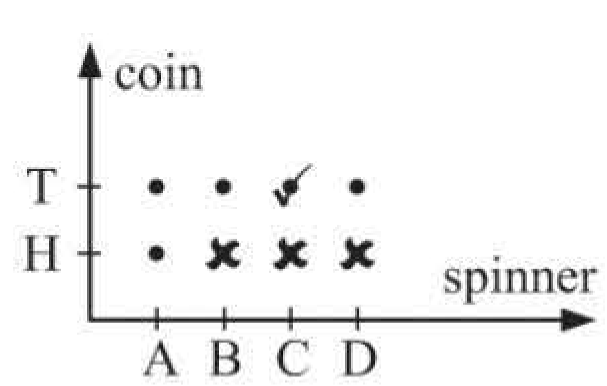
2 **a** $\text{P}(A') = 1 - \text{P}(A)$
$= 1 - m$

b m can be any value between 0 and 1 inclusive.
$\therefore\ 0 \leqslant m \leqslant 1$

c **i** P(A exactly once)
$= \text{P}(AA') + \text{P}(A'A)$
$= m(1-m) + (1-m)m$
$= 2m(1-m)$

ii P(A at least once)
$= 1 - \text{P}(A'A')$
$= 1 - (1-m)(1-m)$
$= 1 - (1 - 2m + m^2)$
$= 1 - 1 + 2m - m^2$
$= 2m - m^2$

3



a Consonants are B, C and D
$\therefore$ P(H and a consonant)
$= \frac{3}{8}$ {those with a ✗}

b P(T and C)
$= \frac{1}{8}$ {those with a ✓}

c P(T or vowel)
= P(T or A)
= P(T) + P(A) − P(T and A)
$= \frac{4}{8} + \frac{2}{8} - \frac{1}{8}$
$= \frac{5}{8}$

4 $\text{P}(M) = \frac{3}{5}$, $\text{P}(W) = \frac{2}{3}$, where M is the event "the man is alive in 25 years", and W is the event "the woman is alive in 25 years".

a P(M and W)
$= \frac{3}{5} \times \frac{2}{3}$
{assuming independence}
$= \frac{2}{5}$

b P(at least one)
= P(M or W)
= P(M) + P(W) − P(M and W)
$= \frac{3}{5} + \frac{2}{3} - \frac{2}{5}$
$= \frac{13}{15}$

c P(M' and W)
$= (1 - \frac{3}{5}) \times \frac{2}{3}$
$= \frac{2}{5} \times \frac{2}{3}$
$= \frac{4}{15}$

5 **a** $\text{P}(X \cap Y) = 0$ {X and Y are mutually exclusive events}

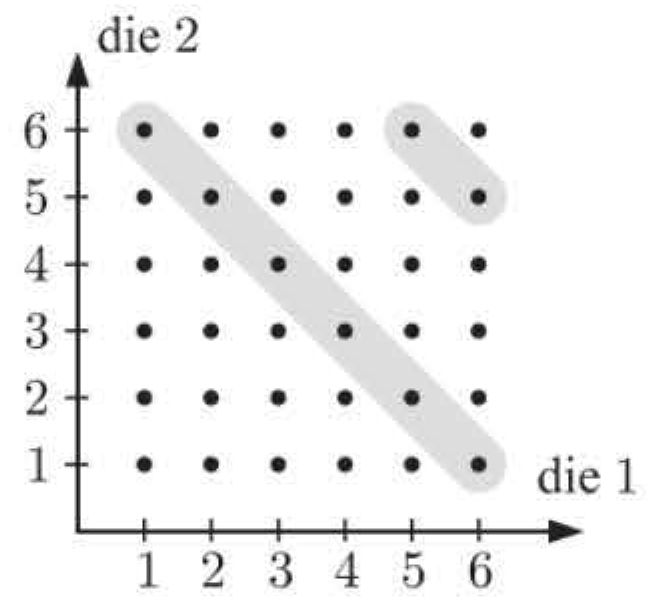
b $\text{P}(X \cup Y) = \text{P}(X) + \text{P}(Y) - \text{P}(X \cap Y)$
$\therefore\ 0.8 = \text{P}(X) + 0.35 - 0$
$\therefore\ \text{P}(X) = 0.45$

c P(X or Y but not both) = P(X or Y) {X and Y mutually exclusive}
$= \text{P}(X \cup Y)$
$= 0.8$

6 **a** Two events are independent if the occurrence of either event does not affect the probability that the other occurs. For A and B independent, $\text{P}(A) \times \text{P}(B) = \text{P}(A \text{ and } B)$.

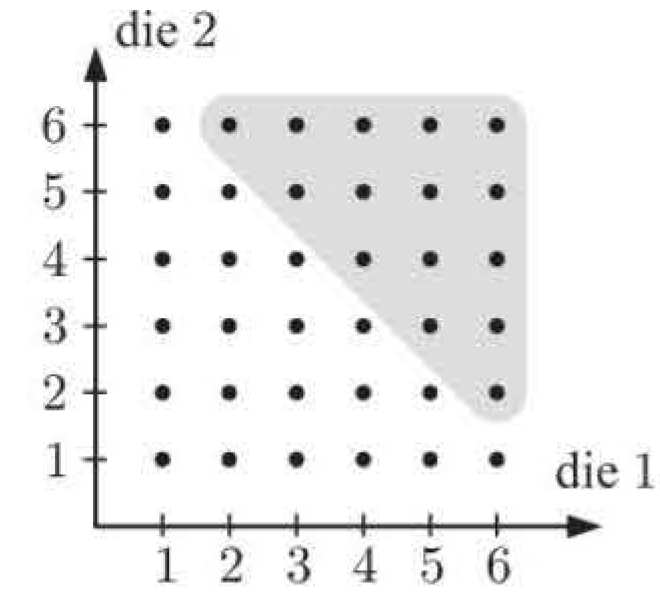
b Two events A and B are mutually exclusive if they have no common outcomes.
$\therefore$ P(A and B) = 0 and so P(A or B) = P(A) + P(B).

7 **a**



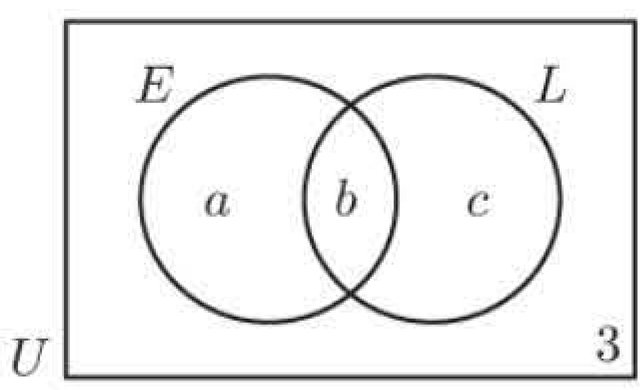
P(sum of 7 or 11)
$= \frac{8}{36}$
$= \frac{2}{9}$

b



P(sum of at least 8)
$= \frac{15}{36}$
$= \frac{5}{12}$

8

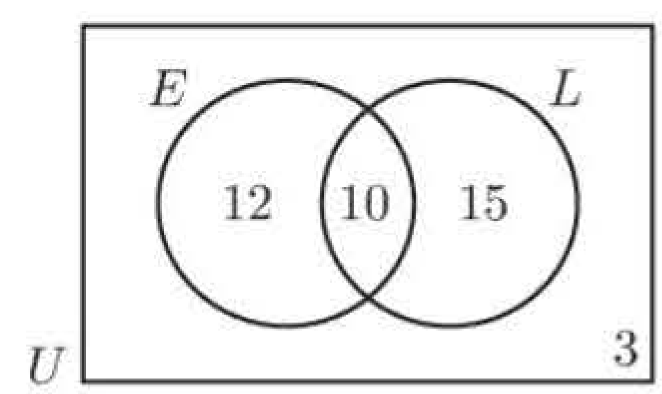


$a+b+c=37$
$a+b=22$
$b+c=25$

$\therefore\ 22+c=37$ and $a+25=37$
$\therefore\ c=15$ and $a=12$
Hence, $b=22-a=10$

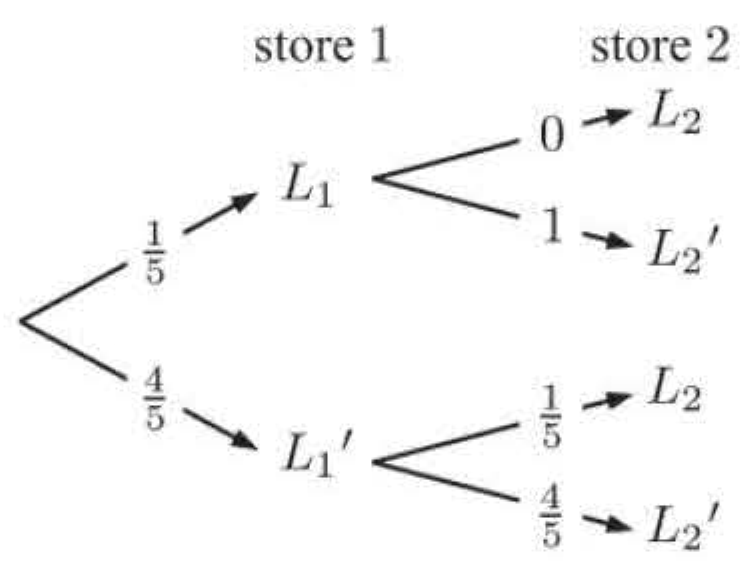
a P(E and L)
$= \frac{10}{40}$
$= \frac{1}{4}$

b P(at least one)
$= \frac{12+10+15}{40}$
$= \frac{37}{40}$



c $\mathrm{P}(E \mid L) = \frac{10}{15+10} = \frac{10}{25} = \frac{2}{5}$

9 Let L_i be the event that the salesman leaves his sunglasses in store i.



$$\begin{aligned}
\mathrm{P}(L_1 \mid L_1 \text{ or } L_2) &= \frac{\mathrm{P}(L_1 \cap (L_1 \text{ or } L_2))}{\mathrm{P}(L_1 \text{ or } L_2)} \\
&= \frac{\mathrm{P}(L_1)}{\mathrm{P}(L_1 L_2' \text{ or } L_1' L_2)} \\
&= \frac{\frac{1}{5}}{\frac{1}{5} + \frac{4}{5} \times \frac{1}{5}} \\
&= \frac{5}{9}
\end{aligned}$$

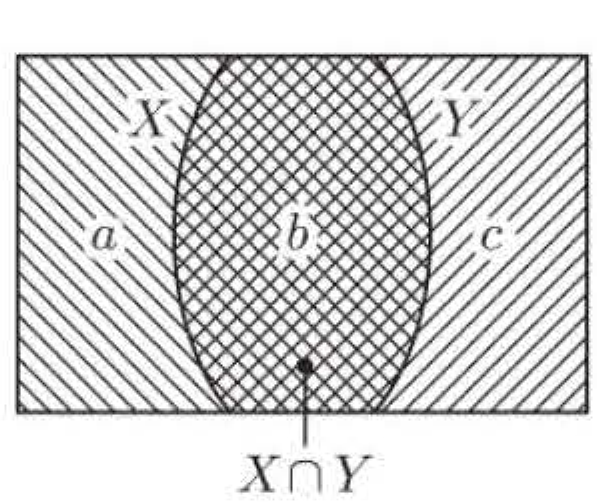
10 $\mathrm{P(M)} = \frac{4}{5}$, $\mathrm{P(M')} = \frac{1}{5}$

a There are $\binom{5}{3} = 10$ ways in which Mae wins 3 games and Ravi wins 2 games.

P(M wins 3 games)
$= 10(\frac{4}{5})^3(\frac{1}{5})^2$
≈ 0.205

b P(M wins 4 or 5 games)
$= \binom{5}{4}(\frac{4}{5})^4(\frac{1}{5})^1 + \binom{5}{5}(\frac{4}{5})^5$
≈ 0.737

11 Since $X' \cap Y' = \varnothing$, every element in the universal set is in either X or Y or both.
$\therefore$ we can construct the Venn diagram alongside:



Now $\quad P(X' \mid Y) = \frac{2}{3}$

$$\therefore \quad \frac{P(X' \cap Y)}{P(Y)} = \frac{2}{3}$$

$$\therefore \quad P(X' \cap Y) = \tfrac{2}{3} \times P(Y)$$
$$= \tfrac{2}{3} \times \tfrac{5}{6}$$
$$= \tfrac{5}{9}$$

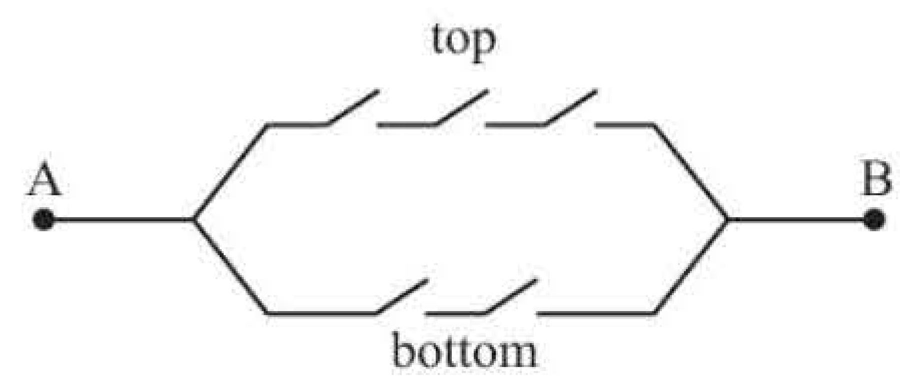
So, $\quad c = \frac{5}{9}$

and $\quad P(Y) = b + c$

$$\therefore \quad b = \tfrac{5}{6} - \tfrac{5}{9} = \tfrac{5}{18}$$
$$\therefore \quad a = 1 - b - c$$
$$= 1 - \tfrac{5}{18} - \tfrac{5}{9}$$
$$= \tfrac{1}{6}$$
$$\therefore \quad P(X) = a + b$$
$$= \tfrac{1}{6} + \tfrac{5}{18} = \tfrac{4}{9}$$

12 P(any switch is closed) $= \frac{2}{3}$

Current will flow through a wire if all switches along that wire are closed.



$\therefore$ P(current flows through top wire) $= \left(\frac{2}{3}\right)^3$

P(current flows through bottom wire) $= \left(\frac{2}{3}\right)^2$

P(current flows through both wires) = P(all five switches closed)

$$= \left(\tfrac{2}{3}\right)^5$$

$\therefore$ P(current flows from A to B) = P(top) + P(bottom) − P(top and bottom)

$$= \left(\tfrac{2}{3}\right)^3 + \left(\tfrac{2}{3}\right)^2 - \left(\tfrac{2}{3}\right)^5$$
$$= \tfrac{148}{243}$$

13 **a** E and S are the only letters common to all three names.

$\therefore$ P(all letters are the same) = P(all Es) + P(all Ss)

$$= \tfrac{1}{5} \times \tfrac{2}{6} \times \tfrac{1}{5} + \tfrac{1}{5} \times \tfrac{1}{6} \times \tfrac{1}{5}$$
$$= \tfrac{1}{50}$$

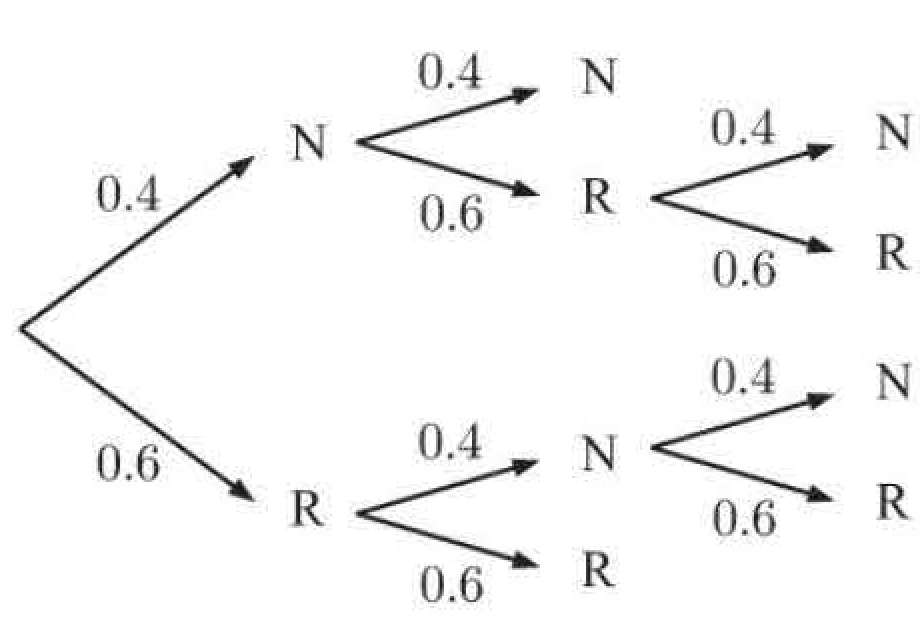
b Exactly two letters will be the same if:

N is selected from JONES and EVANS	$\frac{1}{5} \times \frac{1}{5} = \frac{1}{25}$
E is selected from JONES and PETERS (but not EVANS)	$\frac{1}{5} \times \frac{2}{6} \times \frac{4}{5} = \frac{8}{150}$
E is selected from JONES and EVANS (but not PETERS)	$\frac{1}{5} \times \frac{1}{5} \times \frac{4}{6} = \frac{4}{150}$
E is selected from PETERS and EVANS (but not JONES)	$\frac{2}{6} \times \frac{1}{5} \times \frac{4}{5} = \frac{8}{150}$
S is selected from JONES and PETERS (but not EVANS)	$\frac{1}{5} \times \frac{1}{6} \times \frac{4}{5} = \frac{4}{150}$
S is selected from JONES and EVANS (but not PETERS)	$\frac{1}{5} \times \frac{1}{5} \times \frac{5}{6} = \frac{5}{150}$
S is selected from PETERS and EVANS (but not JONES)	$\frac{1}{6} \times \frac{1}{5} \times \frac{4}{5} = \frac{4}{150}$
	Total $= \frac{13}{50}$

$\therefore$ P(only two of the letters are the same) $= \frac{13}{50}$

REVIEW SET 24B

1



P(Niklas wins)

$$= (0.4)(0.4) + (0.4)(0.6)(0.4) + (0.6)(0.4)(0.4)$$
$$= \tfrac{44}{125}$$
$$= 0.352$$

2 **a** P(win first 3 prizes)

$= \text{P(WWW)}$

$= \frac{4}{500} \times \frac{3}{499} \times \frac{2}{498}$

$\approx 0.000\,000\,193$

b P(win at least one of the 3 prizes)

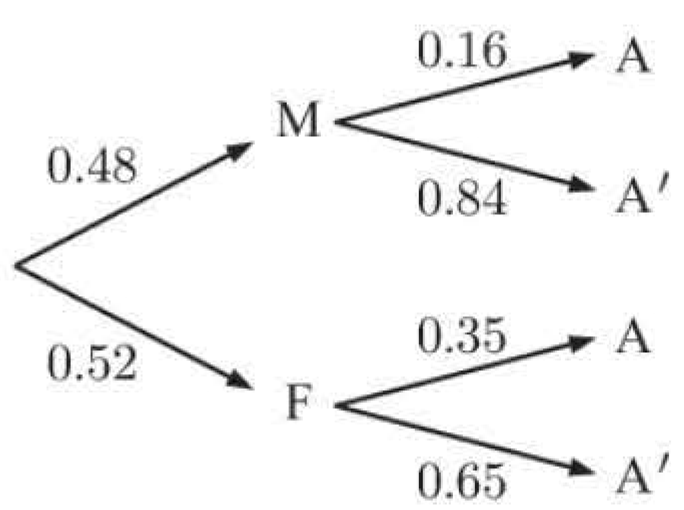
$= 1 - \text{P(wins none of them)}$

$= 1 - \text{P}(\text{W}'\text{W}'\text{W}')$

$= 1 - \frac{496}{500} \times \frac{495}{499} \times \frac{494}{498}$

≈ 0.0239

3



a $\text{P(A)} = \text{P}(\text{M} \cap \text{A} \text{ or } \text{F} \cap \text{A})$

$= 0.48 \times 0.16 + 0.52 \times 0.35$

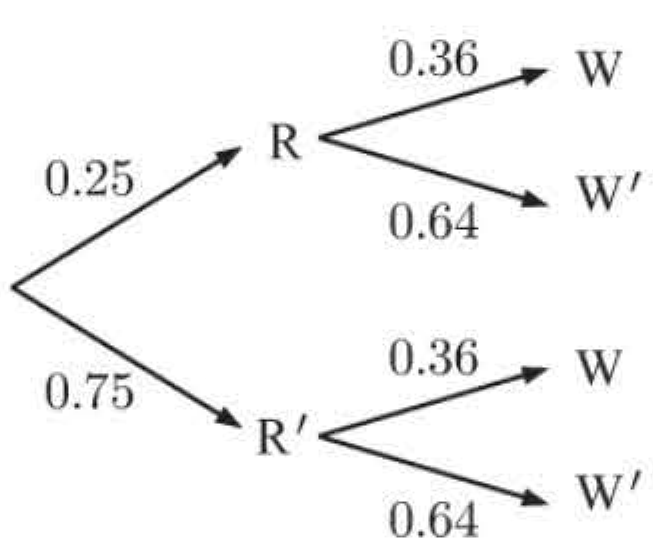
$= 0.2588 \approx 0.259$

b $\text{P}(\text{F} \mid \text{A}) = \dfrac{\text{P}(\text{F} \cap \text{A})}{\text{P(A)}}$

$= \dfrac{0.52 \times 0.35}{0.2588}$

≈ 0.703

4 **a**



b **i** P(R and W)

$= 0.25 \times 0.36$

$= 0.09$

ii P(R or W)

$= \text{P(R)} + \text{P(W)} - \text{P(R and W)}$

$= 0.25 + 0.36 - 0.09$

$= 0.52$

or $\text{P(R or W)} = 1 - \text{P}(\text{R}'\text{W}')$

$= 1 - 0.75 \times 0.64$

$= 0.52$

c We have assumed that the two events (rain and wind) are independent.

5 $\text{P(A)} = 0.1$, $\text{P(B)} = 0.2$, $\text{P(C)} = 0.3$

$\therefore$ P(group solves it) $=$ P(at least one solves it)

$= 1 - \text{P(no-one solves it)}$

$= 1 - \text{P}(\text{A}' \text{ and } \text{B}' \text{ and } \text{C}')$

$= 1 - (0.9 \times 0.8 \times 0.7)$

$= 0.496$

6 **a** **i** $\text{P}(A') = 1 - \text{P}(A)$

$= 1 - 0.11$

$= 0.89$

ii $\text{P}(B) = 0.7$

$\therefore \dfrac{14}{n(U)} = 0.7$

$\therefore n(U) = \dfrac{14}{0.7}$

$= 20$

b **i** $\text{P}(A \cap B) = \text{P}(A) \times \text{P}(B)$

$= 0.11 \times 0.7$

$= 0.077$

ii $\text{P}(A \mid B) = \text{P}(A)$

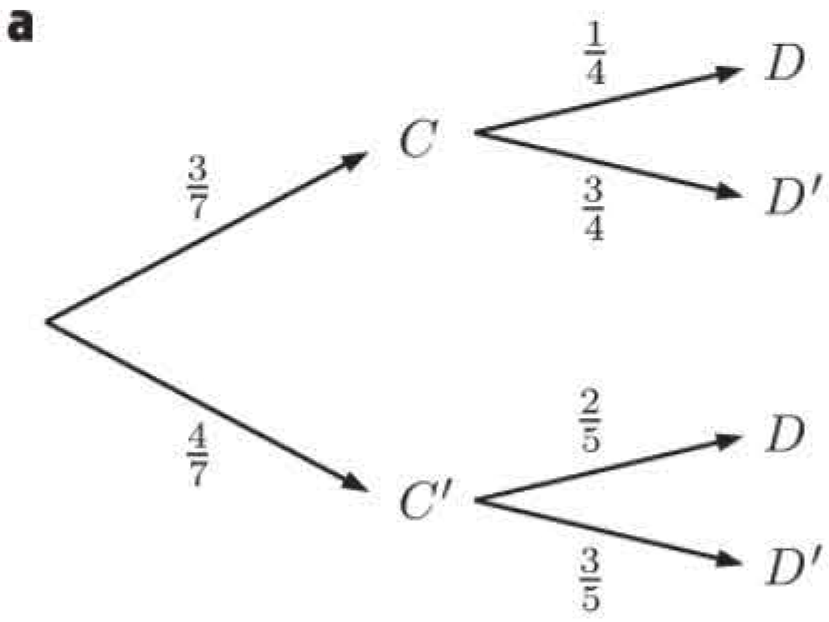
$= 0.11$

c $\text{P}(A \cup B) = \text{P}(A) + \text{P}(B)$

$= 0.11 + 0.7$

$= 0.81$

7 **a**



b **i** $\text{P}(CD) = \frac{3}{7} \times \frac{1}{4}$
$= \frac{3}{28}$

ii $\text{P(at least one pet)} = 1 - \text{P}(C'D')$
$= 1 - \frac{4}{7} \times \frac{3}{5}$
$= \frac{23}{35}$

8 **a**

	Female	*Male*	*Total*
Smoker	$60 - 40 = 20$	40	60
Non-smoker	$90 - 20 = 70$	$110 - 40 = 70$	$200 - 60 = 140$
Total	90	$200 - 90 = 110$	200

b **i** $\text{P(female non-smoker)} = \frac{70}{200} = \frac{7}{20}$

ii $\text{P(male given non-smoker)} = \frac{70}{140} = \frac{1}{2}$

c **i** P(two non-smoking females)
$= \frac{70}{200} \times \frac{69}{199}$
≈ 0.121

ii P(one is a smoker and the other is not)
$= \text{P}(\text{SS}') + \text{P}(\text{S}'\text{S})$
$= \frac{60}{200} \times \frac{140}{199} + \frac{140}{200} \times \frac{60}{199}$
$= \frac{42}{199} + \frac{42}{199}$
$= \frac{84}{199}$
≈ 0.422

9 **a** $\text{P}(M) \times \text{P}(C) = 0.91 \times 0.88$
≈ 0.801
and $\text{P}(M \cap C) = 0.85$
$\therefore\ \text{P}(M \cap C) \neq \text{P}(M)\ \text{P}(C)$,
so M and C are not independent.

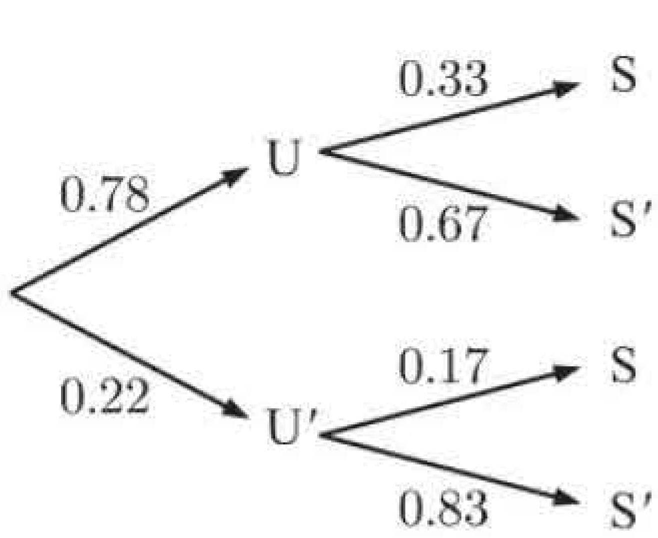
b $\text{P}(M' \mid C) = \dfrac{\text{P}(M' \cap C)}{\text{P}(C)}$

Now $\text{P}(M' \cap C) + \text{P}(M \cap C) = \text{P}(C)$
$\therefore\ \text{P}(M' \cap C) = \text{P}(C) - \text{P}(M \cap C)$
$= 0.88 - 0.85$
$= 0.03$
$\therefore\ \text{P}(M' \mid C) = \frac{0.03}{0.88} = \frac{3}{88}$

10 There are $\binom{10}{5}$ different ways to choose the group.

$\therefore\ \text{P(2 Year 12, 2 Year 11)} = \dfrac{\binom{3}{2}\binom{4}{2}\binom{3}{1}}{\binom{10}{5}}$
$= \frac{3}{14}$

11

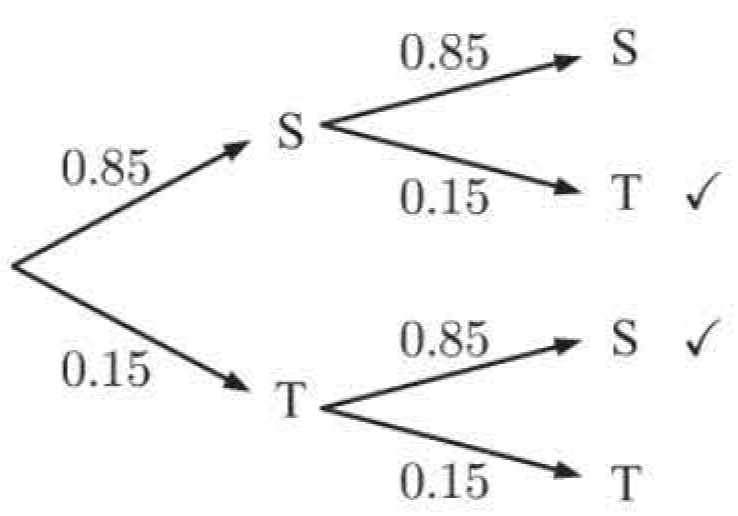


$\text{P}(\text{U}' \mid \text{S}) = \dfrac{\text{P}(\text{U}' \cap \text{S})}{\text{P}(\text{S})}$
$= \dfrac{0.22 \times 0.17}{0.22 \times 0.17 + 0.78 \times 0.33}$
≈ 0.127

12 **a** $\text{P(all doctors)} = \dfrac{\binom{6}{5} \times \binom{4}{0}}{\binom{10}{5}} \approx 0.0238$

b $\text{P(at least 2 doctors)} = 1 - \text{P(1 doctor)} = 1 - \dfrac{\binom{6}{1} \times \binom{4}{4}}{\binom{10}{5}} \approx 0.976$

13



$$\text{P(twins first} \mid \text{3 children)} = \frac{\text{P(twins first} \cap \text{3 children)}}{\text{P(3 children)}}$$
$$= \frac{0.15 \times 0.85}{0.85 \times 0.15 + 0.15 \times 0.85}$$
$$= \tfrac{1}{2}$$

14 There are $\binom{10}{4} = 210$ ways to select the four numbers.

a 2 cannot be the second largest number, as there is only 1 number smaller than 2.

$\therefore \; \mathrm{P}(X = 2) = 0$

b There are $\binom{6}{2}\binom{1}{1}\binom{3}{1} = 45$ ways to choose the numbers so that $X = 7$.

$\binom{6}{2}$: 2 numbers below 7, $\binom{1}{1}$: 7, $\binom{3}{1}$: 1 number above 7

$\therefore \; \mathrm{P}(X = 7) = \frac{45}{210} = \frac{3}{14}$

c There are $\binom{8}{2}\binom{1}{1}\binom{1}{1} = 28$ ways to choose the numbers so that $X = 9$.

$\binom{8}{2}$: 2 numbers below 9, $\binom{1}{1}$: 9, $\binom{1}{1}$: 10

$\therefore \; \mathrm{P}(X = 9) = \frac{28}{210} = \frac{2}{15}$

REVIEW SET 24C

1 BBBB, BBBG, BBGB, BGBB, GBBB, BBGG, BGBG, BGGB, GBBG, GBGB, GGBB, BGGG, GBGG, GGBG, GGGB, GGGG

P(2B and 2G)
$= \frac{6}{16}$ ← 6 have 2B and 2G
$= \frac{3}{8}$

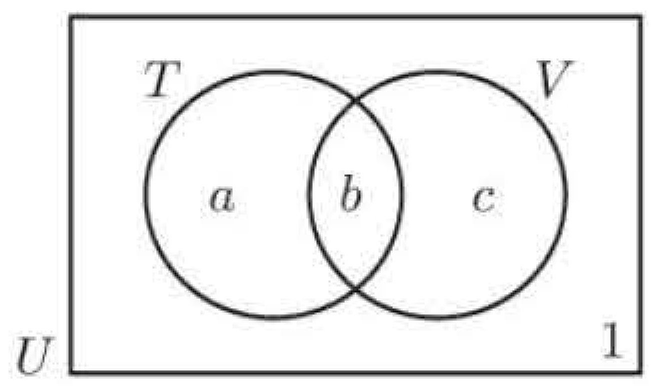
2 **a** P(both blue)
$= \mathrm{P(BB)}$
$= \frac{5}{12} \times \frac{4}{11}$
$= \frac{5}{33}$

b P(both same colour)
$= \mathrm{P(BB \text{ or } RR \text{ or } YY)}$
$= \frac{5}{12} \times \frac{4}{11} + \frac{3}{12} \times \frac{2}{11} + \frac{4}{12} \times \frac{3}{11}$
$= \frac{19}{66}$

c P(at least one R)
$= 1 - \mathrm{P(no\ reds)}$
$= 1 - \mathrm{P(R'R')}$
$= 1 - \frac{9}{12} \times \frac{8}{11}$
$= 1 - \frac{6}{11}$
$= \frac{5}{11}$

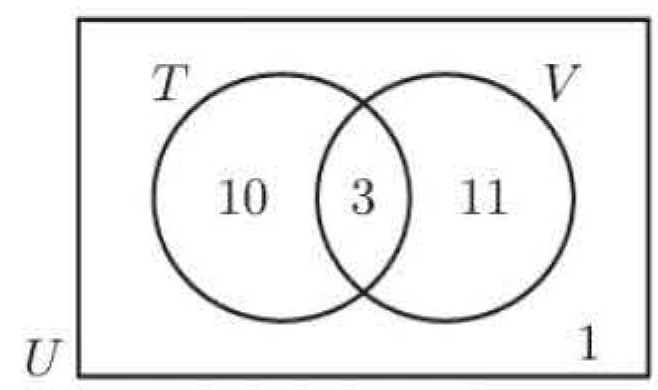
d P(exactly one Y)
$= \mathrm{P(YY' \text{ or } Y'Y)}$
$= \frac{4}{12} \times \frac{8}{11} + \frac{8}{12} \times \frac{4}{11}$
$= \frac{16}{33}$

3 **a**



$a + b + c = 24$
$a + b = 13$
$b + c = 14$
Also $b = 13 - a$
$= 3$

$\therefore \; 13 + c = 24$ and $a + 14 = 24$
$\therefore \; c = 11$ and $a = 10$



i $\mathrm{P}(T \text{ and } V)$
$= \frac{3}{25}$

ii P(at least one)
$= 1 - \mathrm{P(neither)}$
$= 1 - \frac{1}{25}$
$= \frac{24}{25}$

iii $\mathrm{P}(V \mid T')$
$= \dfrac{11}{11 + 1}$
$= \frac{11}{12}$

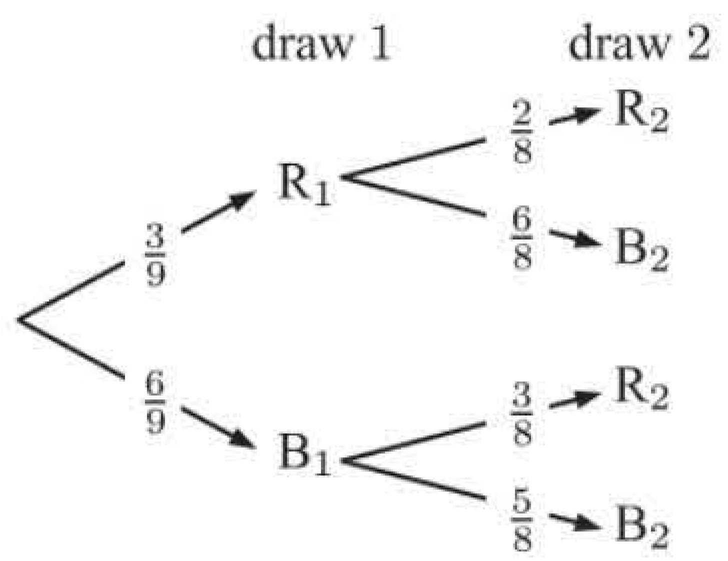
b **i** $\mathrm{P}(T'T'T')$
$= \frac{12}{25} \times \frac{11}{24} \times \frac{10}{23}$
$= \frac{11}{115}$

ii P(at least one plays tennis)
$= 1 - \mathrm{P}(\text{none play tennis})$
$= 1 - \frac{11}{115}$
$= \frac{104}{115}$

4 **a** There are now 3 red and 5 blue balls remaining.

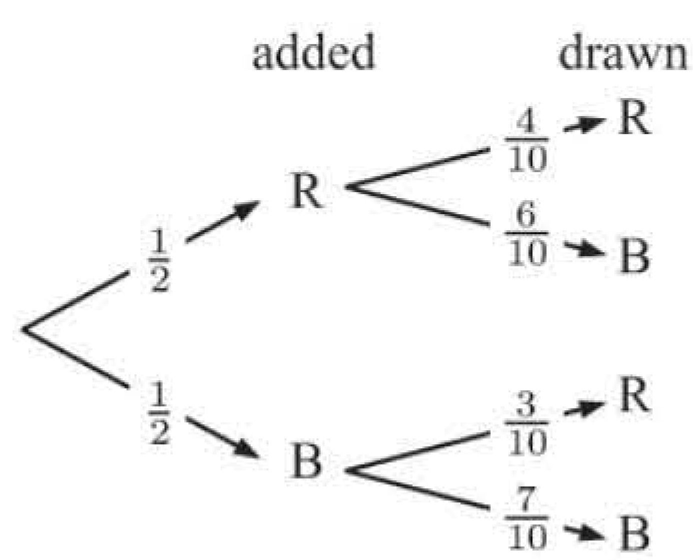
$\therefore\ \mathrm{P}(\text{blue}) = \frac{5}{8}$

b



$$\mathrm{P}(\mathrm{R}_1 \mid \mathrm{R}_2) = \frac{\mathrm{P}(\mathrm{R}_1 \cap \mathrm{R}_2)}{\mathrm{P}(\mathrm{R}_2)} = \frac{\frac{3}{9} \times \frac{2}{8}}{\frac{3}{9} \times \frac{2}{8} + \frac{6}{9} \times \frac{3}{8}} = \frac{1}{4}$$

c



$$\mathrm{P}(\text{red added} \mid \text{blue drawn}) = \frac{\mathrm{P}(\text{red added} \cap \text{blue drawn})}{\mathrm{P}(\text{blue drawn})} = \frac{\frac{1}{2} \times \frac{6}{10}}{\frac{1}{2} \times \frac{6}{10} + \frac{1}{2} \times \frac{7}{10}} = \frac{6}{13}$$

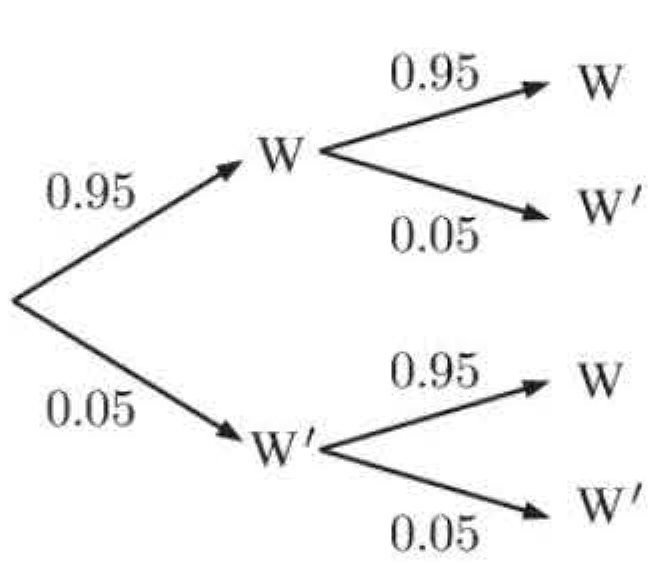
5 **a** $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$
$= \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A)\,\mathrm{P}(B)$ $\{A$ and B are independent$\}$
$= 0.8 + 0.65 - 0.8 \times 0.65$
$= 0.93$

b $\mathrm{P}(A \mid B) = \mathrm{P}(A)$ $\{A$ and B are independent$\}$
$= 0.8$

c $$\mathrm{P}(A' \mid B') = \frac{\mathrm{P}(A' \cap B')}{\mathrm{P}(B')} = \frac{1 - \mathrm{P}(A \cup B)}{1 - \mathrm{P}(B)} = \frac{1 - 0.93}{1 - 0.65} = 0.2$$

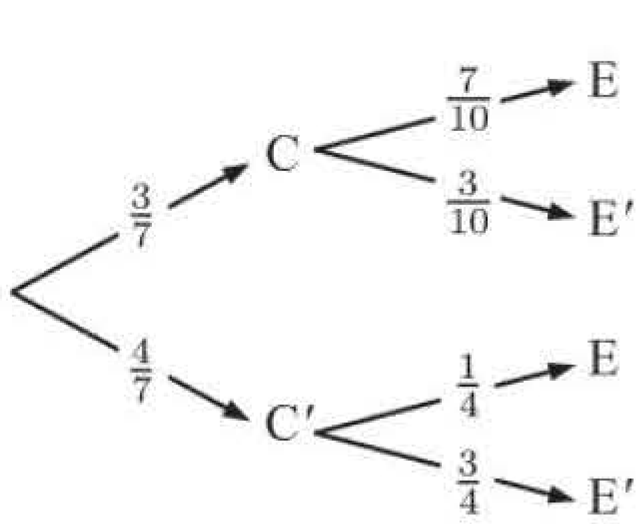
d $\mathrm{P}(B \mid A) = \mathrm{P}(B)$
$= 0.65$

6



P(works on at least one day)
$= 0.95 \times 0.95 + 0.95 \times 0.05 + 0.05 \times 0.95$
$= 0.9975$

7



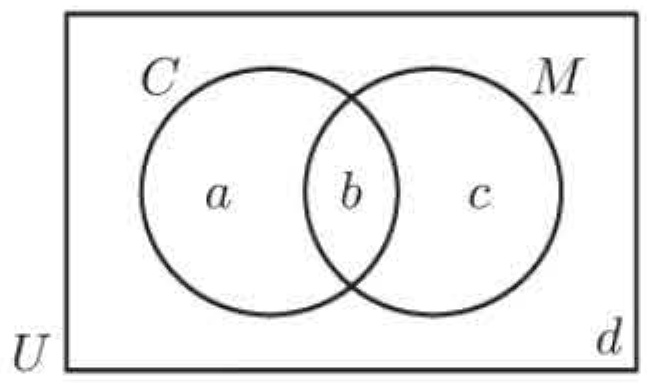
a $\mathrm{P}(\mathrm{E}) = \frac{3}{7} \times \frac{7}{10} + \frac{4}{7} \times \frac{1}{4}$
$= \frac{3}{10} + \frac{1}{7}$
$= \frac{31}{70}$

b $$\mathrm{P}(\mathrm{C} \mid \mathrm{E}) = \frac{\mathrm{P}(\mathrm{C \text{ and } E})}{\mathrm{P}(\mathrm{E})} = \frac{\frac{3}{7} \times \frac{7}{10}}{\frac{31}{70}} = \frac{21}{31}$$

8 **a**

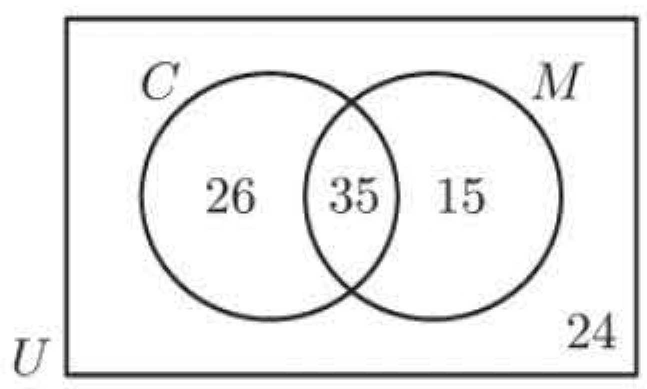
	Men	*Women*	*Total*
Tea	15	24	$15+24=39$
Coffee	$50-15=35$	$50-24=26$	$35+26=61$
Total	50	50	100

b



$$\begin{aligned} b &= 35 && \{35 \text{ men prefer coffee}\} \\ a+35 &= 61 && \{61 \text{ people prefer coffee}\} \\ \therefore\ a &= 26 && \\ 35+c &= 50 && \{50 \text{ men were surveyed}\} \\ \therefore\ c &= 15 && \\ 26+35+15+d &= 100 && \{100 \text{ people were surveyed}\} \\ \therefore\ d &= 24 && \end{aligned}$$

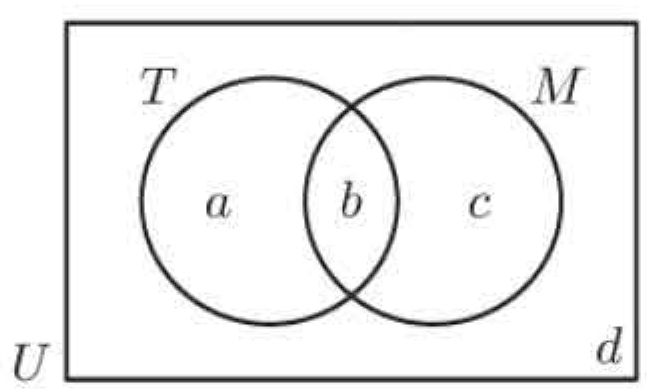
$\therefore$ the Venn diagram is:



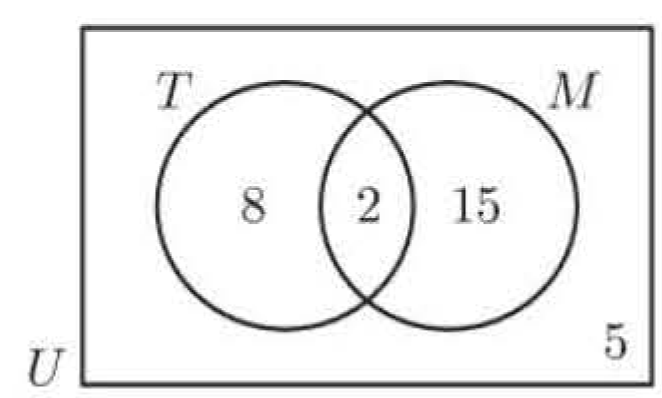
c **i** $P(C') = \dfrac{15+24}{100} = \frac{39}{100} = 0.39$

ii $P(M \mid C) = \dfrac{35}{26+35} = \frac{35}{61} \approx 0.574$

9 **a**



$$\begin{aligned} a+b &= 10 && \{n(T)=10\} \\ b+c &= 17 && \{n(M)=17\} \\ d &= 5 && \{n((T\cup M)')=5\} \\ \therefore\ a+b+c &= 30-5 && \{n(U)=30\} \\ \therefore\ a+17 &= 25 && \\ \therefore\ a &= 8 && \\ \text{and}\quad 8+b &= 10 && \\ \therefore\ b &= 2 && \\ \text{and}\quad 2+c &= 17 && \\ \therefore\ c &= 15 && \end{aligned}$$



b **i** $P(T\cap M) = \frac{2}{30} = \frac{1}{15}$

ii $P((T\cap M) \mid M) = \frac{2}{17}$

10 **b** Of 100 000 females born, 98 956 are still alive at age 15.
Of 100 000 males born, 98 555 are still alive at age 15.

$$\therefore\ \text{P(reaching the age of 15)} = \frac{98\,956 + 98\,555}{200\,000} = \frac{197\,511}{200\,000} = 0.987\,555 \approx 0.9876$$

c **i** There are 98 555 boys alive at age 15, and 53 942 still alive at 75.

$$\therefore\ \text{probability} = \frac{53\,942}{98\,555} \approx 0.547$$

ii There are 98 956 females alive at age 15, and 72 656 alive at age 75.

$\therefore$ P(15 year old girl does *not* reach 75)

$$= 1 - \frac{72\,656}{98\,956} = \frac{26\,300}{98\,956} \approx 0.266$$

d A 20 year old of either gender is expected to live for longer than 30 years, so it is unlikely the insurance company will have to pay out the policy.

11 **a** P(at least one component needs replacing) $= 1 - \text{P(no components need replacing)}$

$$= 1 - \tfrac{19}{20} \times \tfrac{49}{50} \times \tfrac{99}{100}$$
$$= 0.078\,31$$

b P(exactly one component needs replacing) $= \tfrac{1}{20} \times \tfrac{49}{50} \times \tfrac{99}{100} + \tfrac{19}{20} \times \tfrac{1}{50} \times \tfrac{99}{100} + \tfrac{19}{20} \times \tfrac{49}{50} \times \tfrac{1}{100}$

$$= 0.076\,63$$

12 **A** Peter will win at least two consecutive games out of 3 serving first if

(1) he wins the second game (served by John), **and**

(2) he wins at least one of the other two games (served by Peter).

$$\begin{aligned} \text{P(event 1)} &= 1 - q \quad \{\text{John loses his serve}\} \\ \text{P(event 2)} &= 1 - \text{P(Peter loses both)} \\ &= 1 - (1-p)^2 \\ &= 1 - (1 - 2p + p^2) \\ &= p(2-p) \\ \therefore\ \text{P}(\mathbf{A}) &= p(1-q)(2-p) \end{aligned}$$

B Peter will win at least two consecutive games out of 3 when John serves first if

(1) he wins the second game (served by Peter), **and**

(2) he wins at least one of the other two games (served by John).

$$\begin{aligned} \text{P(event 1)} &= p \quad \{\text{Peter wins his serve}\} \\ \text{P(event 2)} &= 1 - \text{P(Peter loses both)} \\ &= 1 - q^2 \\ &= (1-q)(1+q) \\ \therefore\ \text{P}(\mathbf{B}) &= p(1-q)(1+q) \end{aligned}$$

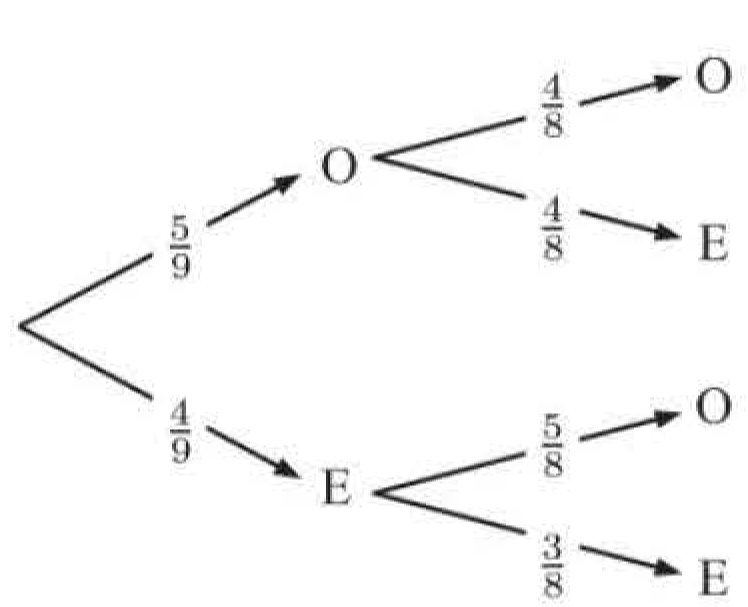
Now $p + q > 1$

$\therefore\ q > 1 - p$

$\therefore\ 1 + q > 2 - p$

$\therefore\ \text{P}(\mathbf{B}) > \text{P}(\mathbf{A})$, and so **B** is more likely than **A**.

13



If the sum of the numbers is even, then the numbers are either both even or both odd.

$$\begin{aligned} \therefore\ \text{P(both odd | sum even)} &= \frac{\text{P(both odd} \cap \text{sum even)}}{\text{P(sum even)}} \\ &= \frac{\text{P(OO)}}{\text{P(OO or EE)}} \\ &= \frac{\frac{5}{9} \times \frac{4}{8}}{\frac{5}{9} \times \frac{4}{8} + \frac{4}{9} \times \frac{3}{8}} \\ &= \tfrac{5}{8} \end{aligned}$$

14 **a** There are $\binom{52}{5}$ possible poker hands.

4 of these are a royal flush {10, J, Q, K, A of hearts, clubs, diamonds, or spades}

$\therefore$ P(royal flush in any order) $= \dfrac{4}{\binom{52}{5}} \approx 1.54 \times 10^{-6}$

b When order is important, there are $52 \times 51 \times 50 \times 49 \times 48$ possible poker hands.

4 of these are a royal flush in the order 10, J, Q, K, A.

$\therefore$ P(royal flush in the order 10, J, Q, K, A) $= \dfrac{4}{52 \times 51 \times 50 \times 49 \times 48} \approx 1.28 \times 10^{-8}$

Chapter 25

DISCRETE RANDOM VARIABLES

EXERCISE 25A

1 **a** The quantity of fat in a sausage is a continuous random variable.

b The mark out of 50 for a geography test is a discrete random variable.

c The weight of a seventeen year old student is a continuous random variable.

d The volume of water in a cup of coffee is a continuous random variable.

e The number of trout in a lake is a discrete random variable.

f The number of the hairs on a cat is a discrete random variable.

g The length of hairs on a horse is a continuous random variable.

h The height of a sky-scraper is a continuous random variable.

2 **a** **i** The random variable X is the height of water in the rain gauge.

ii $0 \leqslant X \leqslant 400$ mm **iii** The variable is a continuous random variable.

b **i** The random variable X is the stopping distance.

ii $0 \leqslant X \leqslant 50$ m **iii** The variable is a continuous random variable.

c **i** The random variable X is the number of times that the switch is turned on or off before it fails.

ii X can be any integer $\geqslant 1$ **iii** The variable is a discrete random variable.

3 **a** Since X is the number of weighing devices that are accurate, $X = 0, 1, 2, 3,$ or 4.

b

		YYNN		
		YNYN		
	YYYN	YNNY	NNNY	
	YYNY	NYYN	NNYN	
	YNYY	NYNY	NYNN	
YYYY	NYYY	NNYY	YNNN	NNNN
$(X=4)$	$(X=3)$	$(X=2)$	$(X=1)$	$(X=0)$

c **i** If two are accurate then $X = 2$.

ii If at least two are accurate then 2, 3, or 4 are accurate $\therefore$ $X = 2, 3,$ or 4.

4 **a** If 3 coins are tossed then the number of heads X can be $0, 1, 2,$ or 3.

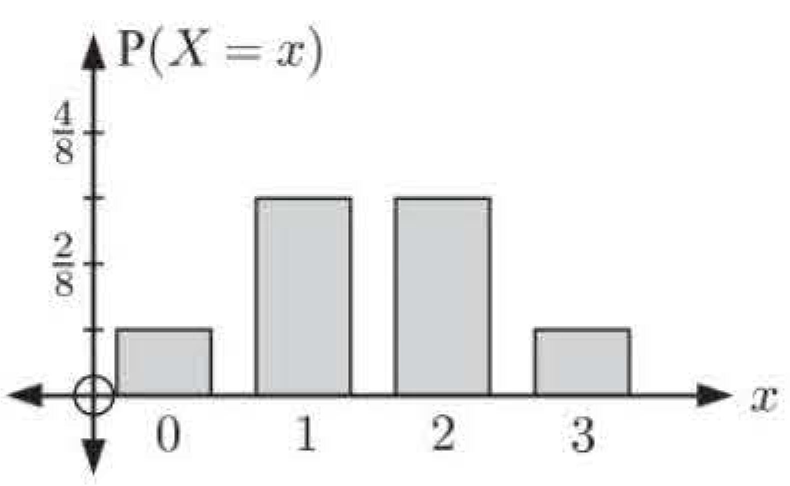
b Suppose H represents heads, T represents tails.

	HHT	TTH	
	HTH	THT	
HHH	THH	HTT	TTT
$(X=3)$	$(X=2)$	$(X=1)$	$(X=0)$

c $P(X=0) = \frac{1}{8}$ $P(X=1) = \frac{3}{8}$

$P(X=2) = \frac{3}{8}$ $P(X=3) = \frac{1}{8}$

d



EXERCISE 25B

1 **a** $$\sum_{x=0}^{2} P(X=x) = 1$$

$$\therefore\ 0.3 + k + 0.5 = 1$$

$$\therefore\ k = 0.2$$

b $$\sum_{x=0}^{3} P(X=x) = 1$$

$$\therefore\ k + 2k + 3k + k = 1$$

$$\therefore\ 7k = 1$$

$$\therefore\ k = \tfrac{1}{7}$$

2 **a** $P(2) = 0.1088$ (from table)

b Since this is a probability distribution, $\sum P(x_i) = 1$

$\therefore\ a + 0.3333 + 0.1088 + 0.0084 + 0.0007 + 0.0000 = 1$

$\therefore\ a + 0.4512 = 1$

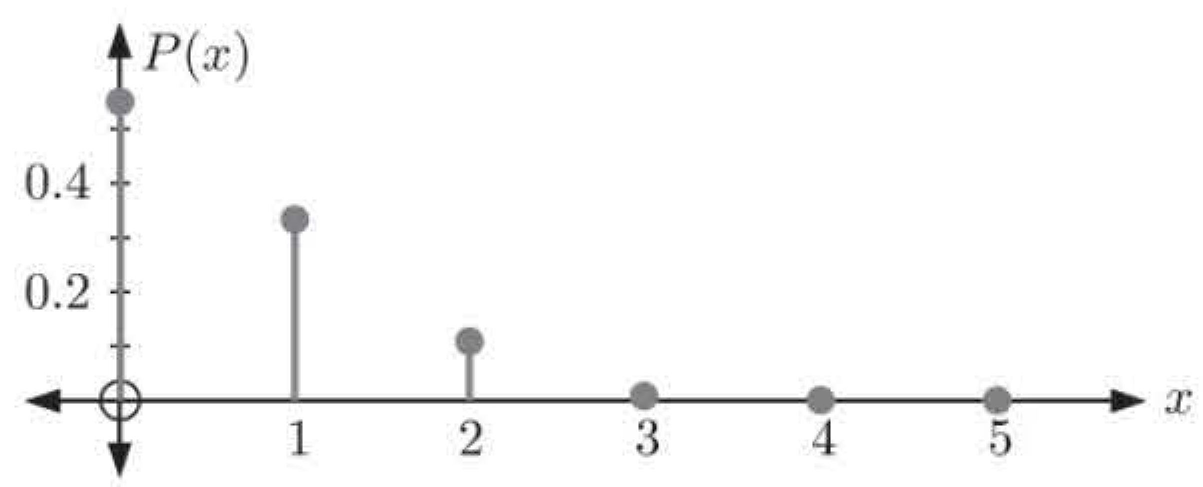
$\therefore\ a = 0.5488$

This is the probability that Jason does not hit a home run in a game.

c $P(1) + P(2) + P(3) + P(4) + P(5) = 0.3333 + 0.1088 + 0.0084 + 0.0007 + 0.0000$
$= 0.4512$

This represents the probability that Jason hits at least one home run in a game.

d



e Jason is most likely to score 0 home runs, so this is the mode of the distribution.
$p_0 = 0.5488$ Since $p_0 \geqslant 0.5$, the median is 0 home runs.

3 **a** Sum of probabilities $\sum P(x_i) = 0.2 + 0.3 + 0.4 + 0.2 = 1.1$
Since this sum $\neq 1$, this is not a valid probability distribution.

b $P(5) = -0.2$, so not all of the probabilities lie in $0 \leqslant P(x_i) \leqslant 1$.
$\therefore$ this is not a valid probability distribution.

4 **a** The random variable represents the number of hits that Sally has in each game.
$X = 0, 1, 2, 3, 4,$ or 5.

b **i** $0.07 + 0.14 + k + 0.46 + 0.08 + 0.02 = 1$ $\left\{\text{since } \sum_{x=0}^{5} \mathrm{P}(X = x) = 1\right\}$

$\therefore\ k + 0.77 = 1$

$\therefore\ k = 0.23$

ii $\mathrm{P}(X \geqslant 2)$
$= \mathrm{P}(X = 2 \text{ or } X = 3 \text{ or } X = 4 \text{ or } X = 5)$
$= P(2) + P(3) + P(4) + P(5)$
$= 0.23 + 0.46 + 0.08 + 0.02$
$= 0.79$

iii $\mathrm{P}(1 \leqslant X \leqslant 3)$
$= P(1) + P(2) + P(3)$
$= 0.14 + 0.23 + 0.46$
$= 0.83$

c Sally is most likely to have 3 hits, so this is the mode of the distribution.

$$\begin{aligned} p &= 0.07 \\ p_0 + p_1 &= 0.07 + 0.14 = 0.21 \\ p_0 + p_1 + p_2 &= 0.21 + 0.23 = 0.44 \\ p_0 + p_1 + p_2 + p_3 &= 0.44 + 0.46 = 0.90 \end{aligned}$$

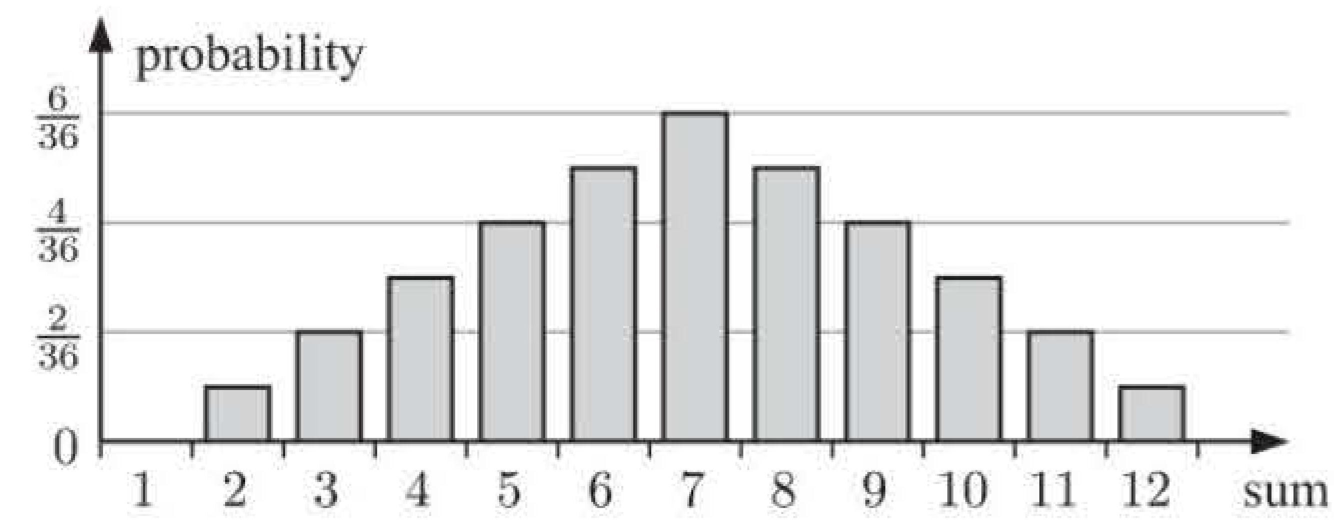
Since $p_0 + p_1 + p_2 + p_3 \geqslant 0.5$, the median is 3 hits.

5 **a** When rolling a die twice, the sample space is:

	6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)
	5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
roll 1	4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
	3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
	2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
	1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
		1	2	3	4	5	6
				roll 2			

b **i** $P(0) = 0$ $\quad P(1) = 0$
$P(2) = \frac{1}{36}$ $\quad P(3) = \frac{2}{36}$
$P(4) = \frac{3}{36}$ $\quad P(5) = \frac{4}{36}$
$P(6) = \frac{5}{36}$ $\quad P(7) = \frac{6}{36}$
$P(8) = \frac{5}{36}$ $\quad P(9) = \frac{4}{36}$
$P(10) = \frac{3}{36}$ $\quad P(11) = \frac{2}{36}$
$P(12) = \frac{1}{36}$

ii



iii The sum of the results for the two rolls is most likely to be 7, so this is the mode.

$$p_2 = \tfrac{1}{36}$$
$$p_2 + p_3 = \tfrac{1}{36} + \tfrac{2}{36} = \tfrac{3}{36}$$
$$p_2 + p_3 + p_4 = \tfrac{3}{36} + \tfrac{3}{36} = \tfrac{6}{36}$$
$$p_2 + p_3 + p_4 + p_5 = \tfrac{6}{36} + \tfrac{4}{36} = \tfrac{10}{36}$$
$$p_2 + p_3 + p_4 + p_5 + p_6 = \tfrac{10}{36} + \tfrac{5}{36} = \tfrac{15}{36}$$
$$p_2 + p_3 + p_4 + p_5 + p_6 + p_7 = \tfrac{15}{36} + \tfrac{6}{36} = \tfrac{21}{36} \approx 0.583$$

Since $p_2 + p_3 + p_4 + p_5 + p_6 + p_7 \approx 0.583 \geqslant 0.5$, the median is 7.

6 **a** $P(x) = k(x+2), \quad x = 1, 2, 3$

$\therefore \; P(1) = 3k, \quad P(2) = 4k, \quad P(3) = 5k$

Since this is a probability distribution, $3k + 4k + 5k = 1$

$\therefore \; 12k = 1$

$\therefore \; k = \frac{1}{12}$

b $P(x) = \dfrac{k}{x+1}, \quad x = 0, 1, 2, 3$

$\therefore \; P(0) = k, \quad P(1) = \dfrac{k}{2},$

$P(2) = \dfrac{k}{3}, \quad P(3) = \dfrac{k}{4}$

Since $\sum P(x_i) = 1, \quad k + \dfrac{k}{2} + \dfrac{k}{3} + \dfrac{k}{4} = 1$

$\therefore \; \dfrac{12k + 6k + 4k + 3k}{12} = 1$

$\therefore \; \dfrac{25k}{12} = 1$

$\therefore \; k = \frac{12}{25}$

7 **a** $\mathrm{P}(X = x) = k\left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{4-x}, \qquad x = 0, 1, 2, 3, 4$

$\mathrm{P}(X = 0) = k\left(\frac{1}{3}\right)^0 \left(\frac{2}{3}\right)^4 = \dfrac{16k}{81} \approx 0.1975k$ $\qquad \mathrm{P}(X = 1) = k\left(\frac{1}{3}\right)^1 \left(\frac{2}{3}\right)^3 = \dfrac{8k}{81} \approx 0.0988k$

$\mathrm{P}(X = 2) = k\left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^2 = \dfrac{4k}{81} \approx 0.0494k$ $\qquad \mathrm{P}(X = 3) = k\left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^1 = \dfrac{2k}{81} \approx 0.0247k$

$\mathrm{P}(X = 4) = k\left(\frac{1}{3}\right)^4 \left(\frac{2}{3}\right)^0 = \dfrac{k}{81} \approx 0.0123k$

b Since $\sum \mathrm{P}(X = i) = 1,$

$\therefore \; \dfrac{16k}{81} + \dfrac{8k}{81} + \dfrac{4k}{81} + \dfrac{2k}{81} + \dfrac{k}{81} = 1$

$\therefore \; \dfrac{31k}{81} = 1$

$\therefore \; k = \frac{81}{31}$

$\therefore \; k \approx 2.61$

$\therefore \; \mathrm{P}(X \geqslant 2) = P(2) + P(3) + P(4)$

$= \dfrac{4k}{81} + \dfrac{2k}{81} + \dfrac{k}{81}$

$= \dfrac{7k}{81}$

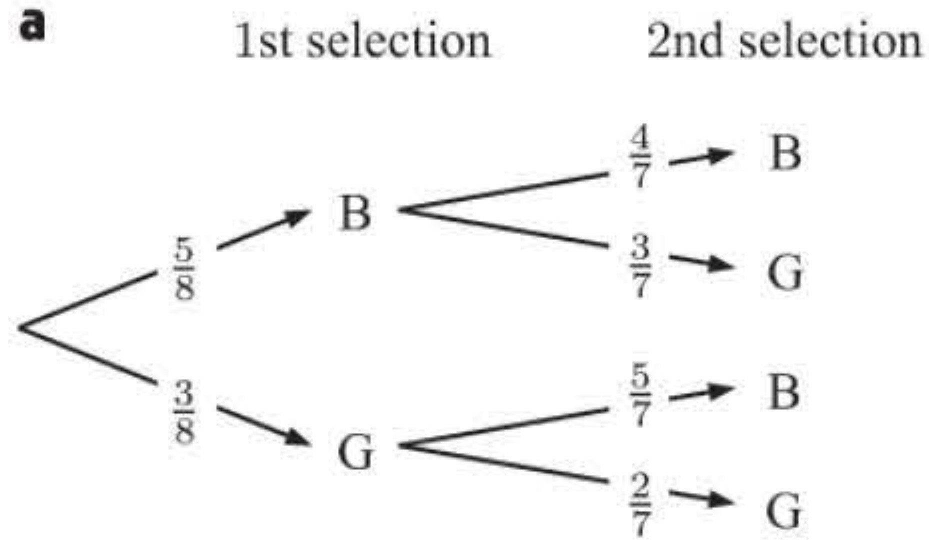
$= \frac{7}{81} \times \frac{81}{31}$

$= \frac{7}{31} \quad (\approx 0.226)$

8 **a** P(no faulty component)
$= \mathrm{P}(X = 0)$
$= \binom{10}{0}(0.04)^0(0.96)^{10-0}$
$= (0.96)^{10}$
≈ 0.665

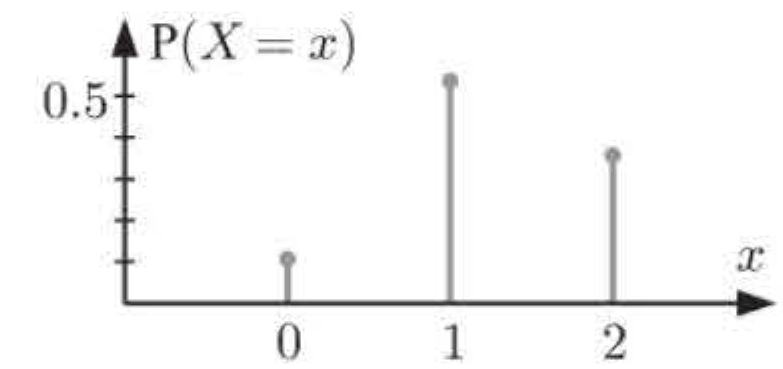
b P(at least one faulty component)
$= \mathrm{P}(X \geqslant 1)$
$= 1 - \mathrm{P}(\text{none are faulty})$
$\approx 1 - (0.96)^{10}$
≈ 0.335

9 **a**

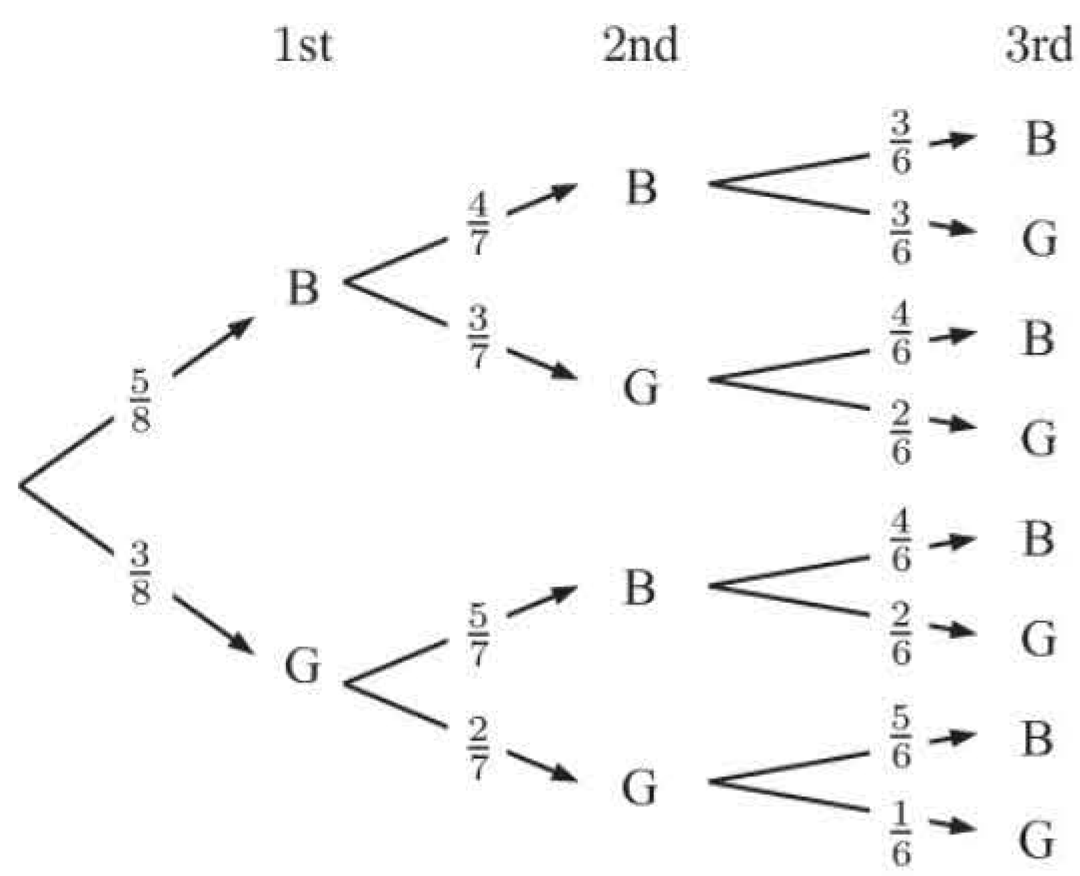


Event	X	Probability
BB	2	$\frac{5}{8} \times \frac{4}{7} = \frac{20}{56}$
BG	1	$\frac{5}{8} \times \frac{3}{7} = \frac{15}{56}$
GB	1	$\frac{3}{8} \times \frac{5}{7} = \frac{15}{56}$
GG	0	$\frac{3}{8} \times \frac{2}{7} = \frac{6}{56}$

x	0	1	2
$\mathrm{P}(X = x)$	$\frac{3}{28}$	$\frac{15}{28}$	$\frac{10}{28}$

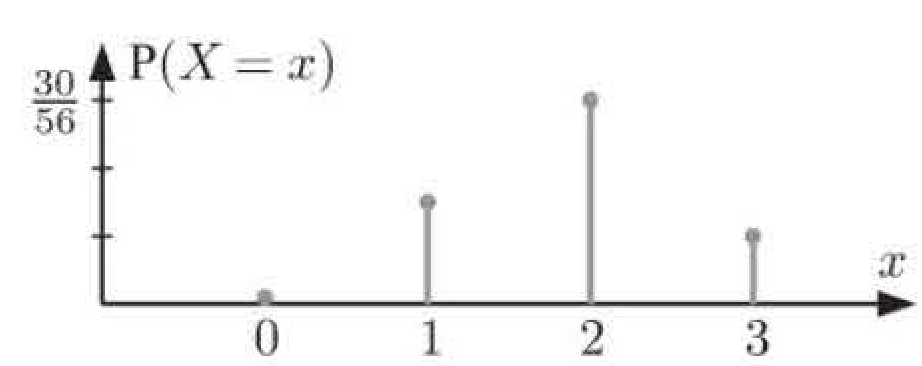


b



Event	X	Probability
BBB	3	$\frac{5}{8} \times \frac{4}{7} \times \frac{3}{6} = \frac{10}{56}$
BBG	2	$\frac{5}{8} \times \frac{4}{7} \times \frac{3}{6} = \frac{10}{56}$
BGB	2	$\frac{5}{8} \times \frac{3}{7} \times \frac{4}{6} = \frac{10}{56}$
BGG	1	$\frac{5}{8} \times \frac{3}{7} \times \frac{2}{6} = \frac{5}{56}$
GBB	2	$\frac{3}{8} \times \frac{5}{7} \times \frac{4}{6} = \frac{10}{56}$
GBG	1	$\frac{3}{8} \times \frac{5}{7} \times \frac{2}{6} = \frac{5}{56}$
GGB	1	$\frac{3}{8} \times \frac{2}{7} \times \frac{5}{6} = \frac{5}{56}$
GGG	0	$\frac{3}{8} \times \frac{2}{7} \times \frac{1}{6} = \frac{1}{56}$

x	0	1	2	3
$\mathrm{P}(X = x)$	$\frac{1}{56}$	$\frac{15}{56}$	$\frac{30}{56}$	$\frac{10}{56}$



10 **a**

Die 2

Die 1		1	2	3	4	5	6
	1	2	3	4	5	6	7
	2	3	4	5	6	7	8
	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

36 possible results

b $\mathrm{P}(D = 7) = \frac{6}{36} = \frac{1}{6}$

c

d	2	3	4	5	6	7	8	9	10	11	12
$\mathrm{P}(D = d)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

d $\mathrm{P}(D \geqslant 8 \mid D \geqslant 6) = \dfrac{\mathrm{P}(D \geqslant 8 \cap D \geqslant 6)}{\mathrm{P}(D \geqslant 6)}$
$= \dfrac{\mathrm{P}(D \geqslant 8)}{\mathrm{P}(D \geqslant 6)}$
$= \frac{15}{36} \div \frac{26}{36}$
$= \frac{15}{26}$

11 **a**

Die 2

Die 1		1	2	3	4	5	6
	1	0	1	2	3	4	5
	2	1	0	1	2	3	4
	3	2	1	0	1	2	3
	4	3	2	1	0	1	2
	5	4	3	2	1	0	1
	6	5	4	3	2	1	0

b

N	0	1	2	3	4	5
$\mathrm{P}(N = n)$	$\frac{6}{36}$	$\frac{10}{36}$	$\frac{8}{36}$	$\frac{6}{36}$	$\frac{4}{36}$	$\frac{2}{36}$

c **i** $\mathrm{P}(N=3) = \frac{6}{36}$
$= \frac{1}{6}$

ii $\mathrm{P}(N \geqslant 3 \mid N \geqslant 1) = \dfrac{\mathrm{P}(N \geqslant 3 \cap N \geqslant 1)}{\mathrm{P}(N \geqslant 1)}$
$= \dfrac{\mathrm{P}(N \geqslant 3)}{\mathrm{P}(N \geqslant 1)}$
$= \frac{12}{36} \div \frac{30}{36}$
$= \frac{2}{5}$

12 **a** $e^x = \sum_{n=0}^{\infty} \dfrac{x^n}{n!} = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + ...$

$$\sum_{n=0}^{\infty} \frac{(0.2)^n e^{-0.2}}{n!} = e^{-0.2} \sum_{n=0}^{\infty} \frac{(0.2)^n}{n!}$$
$$= e^{-0.2} \times e^{0.2} \quad \{\text{definition of } e^x\}$$
$$= e^0$$
$$= 1$$

b **i** $0 \leqslant \dfrac{0.2^x e^{-0.2}}{x!} \leqslant 1$ for all $x = 0, 1, 2, ...$

$\therefore$ $0 \leqslant p_x \leqslant 1$ for all $x = 0, 1, 2, ...$

$$\sum_{x=0}^{\infty} p_x = \sum_{x=0}^{\infty} \frac{0.2^x e^{-0.2}}{x!}$$
$= 1$ {from **a**}

ii $\mathrm{P}(X=0) = \dfrac{(0.2)^0 e^{-0.2}}{0!} = e^{-0.2} \approx 0.819$

$\mathrm{P}(X=1) = \dfrac{(0.2)^1 e^{-0.2}}{1!} = 0.2e^{-0.2} \approx 0.164$

$\mathrm{P}(X=2) = \dfrac{(0.2)^2 e^{-0.2}}{2!} = 0.02e^{-0.2} \approx 0.0164$

iii $\mathrm{P}(\text{at least 3 cars will pass}) = \mathrm{P}(X \geqslant 3)$
$= 1 - \mathrm{P}(X \leqslant 2)$
$= 1 - [\mathrm{P}(X=0) + \mathrm{P}(X=1) + \mathrm{P}(X=2)]$
$= 1 - (e^{-0.2} + 0.2e^{-0.2} + 0.02e^{-0.2})$
$\approx 0.001\,15$

EXERCISE 25C

1 $\mathrm{P}(\text{rain}) = 0.28$ $\therefore$ we would expect rain on $0.28 \times 365.25 \approx 102$ days a year.

2 **a** $\mathrm{P(HHH)} = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$
$= \frac{1}{8}$

b For 200 tosses, we expect $200 \times \frac{1}{8} = 25$ to be '3 heads'.

3 $\mathrm{P}(\text{double}) = \mathrm{P}(1, 1 \text{ or } 2, 2 \text{ or } 3, 3 \text{ or } 4, 4 \text{ or } 5, 5 \text{ or } 6, 6)$
$= \frac{6}{36}$ {6 of the possible 36 outcomes}
$= \frac{1}{6}$

$\therefore$ when rolling the dice 180 times, we expect $180 \times \frac{1}{6} = 30$ doubles.

4

result	*win*
H	\$2
T	−\$1

For playing *once*,
we would expect to win $\frac{1}{2} \times \$2 + \frac{1}{2} \times (-\$1) = \$0.50$
$\therefore$ for 3 games we would expect to win \$1.50.

5 Udo could expect to see snow falling on $\frac{3}{7} \times 5 \times 7 = 15$ days.

6 The goalkeeper would expect to save $\frac{3}{10} \times 90 = 27$ goals.

7 **a** $165 + 87 + 48 = 300$

i $\text{P}(A) \approx \frac{165}{300} = 0.55$

ii $\text{P}(B) \approx \frac{87}{300} = 0.29$

iii $\text{P}(C) \approx \frac{48}{300} = 0.16$

b **i** We expect $7500 \times 0.55 = 4125$ to vote for A.

ii We expect $7500 \times 0.29 = 2175$ to vote for B.

iii We expect $7500 \times 0.16 = 1200$ to vote for C.

8 **a** **i** $\text{P(wins \$10)}$
$= \text{P(rolls a 6)}$
$= \frac{1}{6}$

ii $\text{P(wins \$4)}$
$= \text{P(rolls 4 or 5)}$
$= \frac{2}{6}$ (or $\frac{1}{3}$)

iii $\text{P(wins \$1)}$
$= \text{P(rolls 1, 2, or 3)}$
$= \frac{3}{6}$ (or $\frac{1}{2}$)

b **i** Expectation
$= \frac{2}{6} \times \$4$
$\approx \$1.33$

ii Expectation
$= \frac{3}{6} \times \$1$
$= \$0.50$

iii Expectation
$= \frac{1}{6} \times \$10 + \frac{2}{6} \times \$4 + \frac{3}{6} \times \$1$
$= \frac{1}{6}(\$21) = \$3.50$

c It costs \$4 to play and the expected return is \$3.50.
$\therefore$ you expect to lose \$0.50 per game. (50 cents)

d Over 100 games you expect to lose $100 \times \$0.50 = \$50$.

9 **a** Expect to win $\frac{1}{6} \times €1 + \frac{1}{6} \times €2 + \frac{1}{6} \times €3 + \frac{1}{6} \times €4 + \frac{1}{6} \times €5 + \frac{1}{6} \times €6$
$= \frac{1}{6} \times €21 = €3.50$

b The expected gain is $€3.50 - €4 = -€0.50$
$\therefore$ the player should not play several games, as on each occasion he would expect to lose an average of €0.50.

c **i** The game is fair when the expected gain is 0.
$\therefore\ 3.50 - k = 0$, so $k = 3.50$.

ii $k > 3.50$

10

result	*win*
HH	£10
HT or TH	£3
TT	−£5

a Expectation $= \frac{1}{4} \times £10 + \frac{2}{4} \times £3 + \frac{1}{4} \times (-£5) = £2.75$

b Expected win per game (payout) $= £2.75$
$\therefore$ the organiser would charge $£2.75 + £1.00 = £3.75$ to play each game.

11 **a**

Die 2 (columns), Die 1 (rows)

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

36 possible results

x	2	3	4	5	6	7	8	9	10	11	12
$P(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

So,

$$\text{P}(X \leqslant 3) = \frac{1}{36} + \frac{2}{36} = \frac{1}{12}$$

$$\text{P}(4 \leqslant X \leqslant 6) = \frac{3}{36} + \frac{4}{36} + \frac{5}{36} = \frac{1}{3}$$

$$\text{P}(7 \leqslant X \leqslant 9) = \frac{6}{36} + \frac{5}{36} + \frac{4}{36} = \frac{5}{12}$$

$$\text{P}(X \geqslant 10) = \frac{3}{36} + \frac{2}{36} + \frac{1}{36} = \frac{1}{6}$$

b The expected gain is

$$\left(\frac{1}{12} + \frac{5}{12}\right)\left(-\frac{a}{3}\right) + \frac{1}{3}(7) + \frac{1}{6}(21) - a$$

$$= -\frac{a}{6} + \frac{7}{3} + \frac{21}{6} - a$$

$$= -\frac{7a}{6} + \frac{35}{6}$$

$= \frac{1}{6}(35 - 7a)$ dollars, as required.

c The game is fair when the expected gain is 0.

$$\therefore\ \frac{1}{6}(35 - 7a) = 0$$

$$\therefore\ 35 - 7a = 0$$

$$\therefore\ a = 5$$

d If $a = 4$, expected gain $= \frac{1}{6}(35 - 7(4))$
$= \frac{7}{6}$ dollars

So, the people playing would expect to win about $1.17 per game, which means the organisers expect to lose $1.17 per game.

e If $a = 6$, expected gain $= \frac{1}{6}(35 - 7(6))$
$= -\frac{7}{6}$ dollars

Expectation from 2406 games is $-\frac{7}{6} \times 2406$
$= -2807$

$\therefore$ the organisers would expect to gain $2807.

EXERCISE 25D

1

x	0	1	2	3	4	5	> 5
$\mathrm{P}(X = x)$	0.54	0.26	0.15	k	0.01	0.01	0.00

a $0.54 + 0.26 + 0.15 + k + 0.01 + 0.01 = 1$
$\therefore \quad k + 0.97 = 1$
$\therefore \quad k = 0.03$

b $\mu = \sum x_i p_i$
$= 0 \times 0.54 + 1 \times 0.26 + 2 \times 0.15 + 3 \times 0.03 + 4 \times 0.01 + 5 \times 0.01$
$= 0.26 + 0.30 + 0.09 + 0.04 + 0.05$
$= 0.74$ So, over a long period the mean number of deaths per dozen crayfish is 0.74.

c $\sigma = \sqrt{\sum (x_i - \mu)^2 p_i}$
$= \sqrt{(0-0.74)^2 \times 0.54 + (1-0.74)^2 \times 0.26 + (2-0.74)^2 \times 0.15 + ... + (5-0.74)^2 \times 0.01}$
≈ 0.996

2 $\mathrm{P}(X = x) = \dfrac{x^2 + x}{20}$ for $x = 1, 2, 3$

x	1	2	3
$\mathrm{P}(X = x)$	$\frac{2}{20} = 0.1$	$\frac{6}{20} = 0.3$	$\frac{12}{20} = 0.6$

a The most likely value of X is 3, so this is the mode of the distribution.

b
$p_1 = 0.1$
$p_1 + p_2 = 0.1 + 0.3 = 0.4$
$p_1 + p_2 + p_3 = 0.4 + 0.6 = 1.0$
Since $p_1 + p_2 + p_3 \geqslant 0.5$, the median is 3.

c $\mu = \sum x_i p_i$
$= 1 \times 0.1 + 2 \times 0.3 + 3 \times 0.6$
$= 2.5$

d $\sigma = \sqrt{\sum (x_i - \mu)^2 p_i}$
$= \sqrt{(1 - 2.5)^2 \times 0.1 + (2 - 2.5)^2 \times 0.3 + (3 - 2.5)^2 \times 0.6}$
≈ 0.671

3 **a** $P(x) = \binom{3}{x}(0.4)^x(0.6)^{3-x}$ for $x = 0, 1, 2, 3$

$\therefore \quad P(0) = \binom{3}{0}(0.4)^0(0.6)^3$
$= (0.6)^3$
$= 0.216$

$P(1) = \binom{3}{1}(0.4)^1(0.6)^2$
$= 3(0.4)(0.6)^2$
$= 0.432$

$P(2) = \binom{3}{2}(0.4)^2(0.6)^1$
$= 3(0.16)(0.6)$
$= 0.288$

$P(3) = \binom{3}{3}(0.4)^3(0.6)^0$
$= 1(0.4)^3$
$= 0.064$

x_i	0	1	2	3
$P(x_i)$	0.216	0.432	0.288	0.064

b $\mu = \sum x_i p_i = 0(0.216) + 1(0.432) + 2(0.288) + 3(0.064) = 1.2$
$\sigma = \sqrt{\sum (x_i - \mu)^2 p_i}$
$= \sqrt{(0 - 1.2)^2(0.216) + (1 - 1.2)^2(0.432) + (2 - 1.2)^2 \times 0.288 + (3 - 1.2)^2 \times 0.064}$
≈ 0.849

4
$$\sigma = \sqrt{\sum (x_i - \mu)^2 p_i}$$
$$\begin{aligned}
\therefore \quad \sigma^2 &= \sum (x_i - \mu)^2 p_i \\
&= (x_1 - \mu)^2 p_1 + (x_2 - \mu)^2 p_2 + \ldots\ldots + (x_n - \mu)^2 p_n \\
&= ({x_1}^2 - 2x_1\mu + \mu^2)p_1 + ({x_2}^2 - 2x_2\mu + \mu^2)p_2 + \ldots\ldots + ({x_n}^2 - 2x_n\mu + \mu^2)p_n \\
&= ({x_1}^2 p_1 + {x_2}^2 p_2 + {x_3}^2 p_3 + \ldots\ldots + {x_n}^2 p_n) - 2\mu(x_1p_1 + x_2p_2 + \ldots\ldots + x_np_n) \\
&\quad + \mu^2(p_1 + p_2 + p_3 + \ldots\ldots + p_n)
\end{aligned}$$

Now $p_1 + p_2 + \ldots\ldots + p_n = 1$

$$\begin{aligned}
\therefore \quad \sigma^2 &= \sum {x_i}^2 p_i - 2\mu(\textstyle\sum x_ip_i) + \mu^2(1) \\
&= \sum {x_i}^2 p_i - 2\mu(\mu) + \mu^2 \qquad \{\text{since } \textstyle\sum x_ip_i = \mu\} \\
&= \sum {x_i}^2 p_i - \mu^2
\end{aligned}$$

5 **a**

x_i	1	2	3	4	5
$P(x_i)$	0.1	0.2	0.4	0.2	0.1

b
$$\begin{aligned}
\mu &= \sum x_ip_i \\
&= 1(0.1) + 2(0.2) + \ldots\ldots + 5(0.1) \\
&= 0.1 + 0.4 + 1.2 + 0.8 + 0.5 \\
&= 3
\end{aligned}$$

$$\begin{aligned}
\sigma &= \sqrt{\sum (x_i - \mu)^2 p_i} \\
&= \sqrt{\sum x_i^2 p_i - \mu^2} \\
&= \sqrt{1^2(0.1) + 2^2(0.2) + \ldots\ldots + 5^2(0.1) - (3.0)^2} \\
&= \sqrt{0.1 + 0.8 + 3.6 + 3.2 + 2.5 - 9} \\
&= \sqrt{1.2} \\
&\approx 1.10
\end{aligned}$$

c **i**
$$\begin{aligned}
&\mathrm{P}(\mu - \sigma < X < \mu + \sigma) \\
&= \mathrm{P}(3 - 1.095 < X < 3 + 1.095) \\
&= \mathrm{P}(1.905 < X < 4.095) \\
&= \mathrm{P}(X = 2, 3, 4) \\
&= 0.2 + 0.4 + 0.2 \\
&= 0.8
\end{aligned}$$

ii
$$\begin{aligned}
&\mathrm{P}(\mu - 2\sigma < X < \mu + 2\sigma) \\
&= \mathrm{P}(3 - 2.19 < X < 3 + 2.19) \\
&= \mathrm{P}(0.81 < X < 5.19) \\
&= \mathrm{P}(X = 1, 2, 3, 4 \text{ or } 5) \\
&= 0.1 + 0.2 + 0.4 + 0.2 + 0.1 \\
&= 1
\end{aligned}$$

6 Let X be the payout, so $x = \$20\,000$, $\$8000$, or $\$0$.

$\therefore$ the probability distribution is

x_i	20 000	8000	0
$P(x_i) = p_i$	0.0025	0.03	0.9675

The expectation is $\mu = \sum x_ip_i = 20\,000(0.0025) + 8000(0.03) + 0(0.9675)$
$= \$290$

The company expects to pay out \$290 on average in the long run.

$\therefore$ the company should charge $\$290 + \$100 = \$390$.

7

Die 2

Die 1		1	2	3	4	5	6
	1	1	2	3	4	5	6
	2	2	2	3	4	5	6
	3	3	3	3	4	5	6
	4	4	4	4	4	5	6
	5	5	5	5	5	5	6
	6	6	6	6	6	6	6

a

m_i	1	2	3	4	5	6
$P(m_i)$	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$

b **i** The most likely result is 6, so this is the mode.

ii $p_1 = \frac{1}{36}$ $\qquad p_1+p_2+p_3+p_4 = \frac{9}{36}+\frac{7}{36} = \frac{16}{36}$

$p_1+p_2 = \frac{1}{36}+\frac{3}{36} = \frac{4}{36}$ $\qquad p_1+p_2+p_3+p_4+p_5 = \frac{16}{36}+\frac{9}{36} = \frac{25}{36} \approx 0.694$

$p_1+p_2+p_3 = \frac{4}{36}+\frac{5}{36} = \frac{9}{36}$

Since $p_1+p_2+p_3+p_4+p_5 \approx 0.694 \geqslant 0.5$, the median is 5.

iii $\mu = \sum m_i p_i$

$= 1\left(\frac{1}{36}\right) + 2\left(\frac{3}{36}\right) + 3\left(\frac{5}{36}\right) + ... + 6\left(\frac{11}{36}\right)$

$= \frac{1}{36}+\frac{6}{36}+\frac{15}{36}+\frac{28}{36}+\frac{45}{36}+\frac{66}{36}$

$= \frac{161}{36}$

≈ 4.47

iv $\sigma = \sqrt{\sum m_i^2 p_i - \mu^2}$

$= \sqrt{1^2\left(\frac{1}{36}\right) + 2^2\left(\frac{3}{36}\right) + ... + 6^2\left(\frac{11}{36}\right) - \left(\frac{161}{36}\right)^2}$

$\approx \sqrt{1.971\,45}$

≈ 1.40

8 Examples are:

(1) Tossing one coin, where X is the number of 'heads' resulting. $x = 0$ or 1

x	0	1
$P(x)$	$\frac{1}{2}$	$\frac{1}{2}$

(2) Rolling one die, where X is the number on the uppermost face. $x = 1, 2, 3, 4, 5,$ or 6

x	1	2	3	4	5	6
$P(x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

EXERCISE 25E

1 **a** mean of $X = \mathrm{E}(X)$

$= \sum x_i p_i$

$= 2(0.3)+3(0.3)+4(0.2)+5(0.1)+6(0.1)$

$= 3.4$

b $\mathrm{E}(X^2) = \sum x_i^2 p_i = 4(0.3)+9(0.3)+16(0.2)+25(0.1)+36(0.1) = 13.2$

Now $\mathrm{Var}(X) = \mathrm{E}(X^2) - (\mathrm{E}(X))^2$

$= 13.2 - (3.4)^2$

$= 1.64$

c $\sigma = \sqrt{\mathrm{Var}(X)} \approx 1.28$

2 **a** $\sum p_i = 1$

$\therefore \quad 0.2 + k + 0.4 + 0.1 = 1$

$\therefore \quad k = 0.3$

b $\mathrm{E}(X) = \sum x_i p_i$

$= 5(0.2)+6(0.3)+7(0.4)+8(0.1)$

$= 6.4$

c $\mathrm{Var}(X) = \sum x_i^2 p_i - (\mathrm{E}(X))^2$

$= 25(0.2)+36(0.3)+49(0.4)+64(0.1) - 6.4^2$

$= 0.84$

3 **a** $\mathrm{E}(X)$

$= \sum x_i p_i$

$= 1(0.4)+2(0.3)+3(0.2)+4(0.1)$

$= 2$

b $\mathrm{E}(X^2)$

$= \sum x_i^2 p_i$

$= 1(0.4)+4(0.3)+9(0.2)+16(0.1)$

$= 5$

c $\text{Var}(X)$
$= \text{E}(X^2) - (\text{E}(X))^2$
$= 5 - 2^2$
$= 1$

d $\sigma = \sqrt{\text{Var}(X)}$
$= \sqrt{1}$
$= 1$

e $\text{E}(X+1)$
$= \text{E}(X) + \text{E}(1)$
$= 2 + 1$
$= 3$

f $\text{Var}(X+1)$
$= \text{E}\left((X+1)^2\right) - (\text{E}(X+1))^2$
$= \text{E}\left(X^2 + 2X + 1\right) - 3^2$
$= \text{E}(X^2) + 2\,\text{E}(X) + \text{E}(1) - 9$
$= 5 + 2(2) + 1 - 9$
$= 1$

g $\text{E}(2X^2 + 3X - 7)$
$= 2\,\text{E}(X^2) + 3\,\text{E}(X) - \text{E}(7)$
$= 2(5) + 3(2) - 7$
$= 9$

4 **a** $\text{E}(X) = 2.8$
$\therefore\ 1(0.2) + 2a + 3(0.3) + 4b = 2.8$
$\therefore\ 0.2 + 2a + 0.9 + 4b = 2.8$
$\therefore\ 2a + 4b = 1.7$ .... (1)
Also, $0.2 + a + 0.3 + b = 1$
$\therefore\ b = 0.5 - a$ .... (2)

Substituting (2) into (1) gives
$2a + 4(0.5 - a) = 1.7$
$\therefore\ 2a + 2 - 4a = 1.7$
$\therefore\ -2a = -0.3$
$\therefore\ a = 0.15$
and $b = 0.5 - 0.15$
$= 0.35$

b $\text{E}(X^2) = \sum {x_i}^2 p_i$
$= 1(0.2) + 4(0.15) + 9(0.3) + 16(0.35)$
$= 9.1$
$\therefore\ \text{Var}(X) = \text{E}(X)^2 - (\text{E}(X))^2$
$= 9.1 - 2.8^2$
$= 1.26$

5 **a** $\text{P}(X=0) = a(0) = 0$
$\text{P}(X=1) = a(-7) = -7a$
$\text{P}(X=2) = a(-12) = -12a$
$\text{P}(X=3) = a(-15) = -15a$
$\text{P}(X=4) = a(-16) = -16a$
$\text{P}(X=5) = a(-15) = -15a$
$\text{P}(X=6) = a(-12) = -12a$
$\text{P}(X=7) = a(-7) = -7a$
$\text{P}(X=8) = a(0) = 0$

$\therefore\ 2(-7a - 12a - 15a) - 16a = 1$
$\therefore\ a(-84) = 1$
$\therefore\ a = -\frac{1}{84}$

b $\text{E}(X) = \sum x_i p_i$
$= 1\left(\frac{7}{84}\right) + 2\left(\frac{12}{84}\right) + 3\left(\frac{15}{84}\right) + 4\left(\frac{16}{84}\right) + 5\left(\frac{15}{84}\right) + 6\left(\frac{12}{84}\right) + 7\left(\frac{7}{84}\right)$
$= \frac{336}{84} = 4$

c $\text{E}(X^2) = \sum {x_i}^2 p_i$
$= 1\left(\frac{7}{84}\right) + 4\left(\frac{12}{84}\right) + 9\left(\frac{15}{84}\right) + 16\left(\frac{16}{84}\right) + 25\left(\frac{15}{84}\right) + 36\left(\frac{12}{84}\right) + 49\left(\frac{7}{84}\right)$
$= \frac{1596}{84} = 19$
$\therefore\ \text{Var}(X) = \text{E}(X^2) - (\text{E}(X))^2$
$= 19 - 4^2$
$= 3$
$\therefore\ \sigma = \sqrt{\text{Var}(X)} = \sqrt{3}$

6 **a** $\left(\frac{1}{2}+\frac{1}{2}\right)^4 = \left(\frac{1}{2}\right)^4 + 4\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^1 + 6\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4$

$= \frac{1}{16} + \frac{4}{16} + \frac{6}{16} + \frac{4}{16} + \frac{1}{16}$

So, the probability distribution for X, the number of heads occurring, is:

x	0	1	2	3	4
$P(x)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

b **i** $\mathrm{E}(X) = \sum x_i p_i = 0\left(\frac{1}{16}\right) + 1\left(\frac{4}{16}\right) + 2\left(\frac{6}{16}\right) + 3\left(\frac{4}{16}\right) + 4\left(\frac{1}{16}\right) = 2$

$\therefore$ mean $= 2$

ii $\mathrm{E}(X^2) = \sum {x_i}^2 p_i = 0\left(\frac{1}{16}\right) + 1\left(\frac{4}{16}\right) + 4\left(\frac{6}{16}\right) + 9\left(\frac{4}{16}\right) + 16\left(\frac{1}{16}\right) = 5$

$\therefore \quad \sigma = \sqrt{\mathrm{Var}\,(X)} = \sqrt{\mathrm{E}(X^2) - (\mathrm{E}(X))^2} = \sqrt{5-2^2} = 1$

7 **a** $\mathrm{P}(0 \text{ bitter, } 3 \text{ not bitter}) = \dfrac{\binom{2}{0}\binom{8}{3}}{\binom{10}{3}} = \frac{42}{90} = \frac{7}{15}$

$\mathrm{P}(1 \text{ bitter, } 2 \text{ not bitter}) = \dfrac{\binom{2}{1}\binom{8}{2}}{\binom{10}{3}} = \frac{7}{15}$

$\mathrm{P}(2 \text{ bitter, } 1 \text{ not bitter}) = \dfrac{\binom{2}{2}\binom{8}{1}}{\binom{10}{3}} = \frac{6}{90} = \frac{1}{15}$

x	0	1	2
$P(x)$	$\frac{7}{15}$	$\frac{7}{15}$	$\frac{1}{15}$

b **i** $\mathrm{E}(X)$

$= 0\left(\frac{7}{15}\right) + 1\left(\frac{7}{15}\right) + 2\left(\frac{1}{15}\right)$

$= \frac{9}{15}$

$= 0.6$

$\therefore$ mean $= 0.6$ bitter almonds

ii $\mathrm{E}(X^2) = 0^2\left(\frac{7}{15}\right) + 1^2\left(\frac{7}{15}\right) + 2^2\left(\frac{1}{15}\right) = \frac{11}{15}$

$\therefore \quad \mathrm{Var}\,(X) = \mathrm{E}(X^2) - (\mathrm{E}(X))^2$

$= \frac{11}{15} - \left(\frac{3}{5}\right)^2$

$= \frac{11}{15} - \frac{9}{25}$

≈ 0.3733 and so $\sigma \approx 0.611$

8 **a**

$$\begin{aligned} \mathrm{E}(Y) &= 0.9 \\ \therefore \quad -1(0.1) + 0(a) + 1(0.3) + 2b &= 0.9 \\ \therefore \quad 0.2 + 2b &= 0.9 \\ \therefore \quad 2b &= 0.7 \\ \therefore \quad b &= 0.35 \end{aligned}$$

Also, $0.1 + a + 0.3 + b = 1$

$\therefore \quad a = 1 - 0.1 - 0.3 - 0.35$

$\therefore \quad a = 0.25$

b $\mathrm{E}(Y^2) = (-1)^2(0.1) + 0^2(0.25) + 1^2(0.3) + 2^2(0.35) = 1.8$

$\therefore \quad \mathrm{Var}(Y) = \mathrm{E}(Y^2) - (\mathrm{E}(Y))^2 = 1.8 - 0.9^2 = 0.99$

9 **a** $\frac{1}{6} + \frac{1}{3} + \frac{1}{12} + a + \frac{1}{6} = 1$

$\therefore \quad a = \frac{1}{4}$

b **i** $\mathrm{E}(X) = 1\left(\frac{1}{6}\right) + 2\left(\frac{1}{3}\right) + 3\left(\frac{1}{12}\right) + 4\left(\frac{1}{4}\right) + 5\left(\frac{1}{6}\right) = \frac{35}{12} = 2\frac{11}{12}$

$\mathrm{E}(X^2) = 1^2\left(\frac{1}{6}\right) + 2^2\left(\frac{1}{3}\right) + 3^2\left(\frac{1}{12}\right) + 4^2\left(\frac{1}{4}\right) + 5^2\left(\frac{1}{6}\right) = \frac{125}{12}$

$\therefore \quad \mathrm{Var}(X) = \mathrm{E}(X^2) - (\mathrm{E}(X))^2$

$= \frac{125}{12} - \left(\frac{35}{12}\right)^2$

$= \frac{1500}{144} - \frac{1225}{144}$

$= \frac{275}{144} \approx 1.91$

ii $\begin{aligned} &\mathrm{E}(2X) \\ &= 2\,\mathrm{E}(X) \\ &= 2 \times \tfrac{35}{12} \\ &= \tfrac{35}{6} \\ &= 5\tfrac{5}{6} \end{aligned}$ $\quad$ $\begin{aligned} &\mathrm{Var}(2X) \\ &= 2^2\,\mathrm{Var}(X) \\ &= 4 \times \tfrac{275}{144} \\ &= \tfrac{275}{36} \\ &= 7\tfrac{23}{36} \\ &\approx 7.64 \end{aligned}$

iii $\begin{aligned} &\mathrm{E}(X-1) \\ &= \mathrm{E}(X) - \mathrm{E}(1) \\ &= \tfrac{35}{12} - 1 \\ &= \tfrac{23}{12} \\ &= 1\tfrac{11}{12} \end{aligned}$ $\quad$ $\begin{aligned} &\mathrm{Var}(X-1) \\ &= \mathrm{Var}(X) \\ &= \tfrac{275}{144} \\ &\approx 1.91 \end{aligned}$

10 X has mean 6 and standard deviation 2.

$$\begin{aligned} \mathrm{E}(Y) &= \mathrm{E}(2X+5) \\ &= 2\mathrm{E}(X) + \mathrm{E}(5) \\ &= 2 \times 6 + 5 \\ &= 17 \end{aligned} \qquad \begin{aligned} \mathrm{Var}(Y) &= \mathrm{Var}(2X+5) \\ &= 2^2\mathrm{Var}(X) \\ &= 4 \times 2^2 \\ &= 16 \end{aligned}$$

$\therefore$ mean of Y distribution is 17 $\qquad$ $\therefore$ standard deviation of Y distribution is $\sqrt{16} = 4$

11 **a**
$$\begin{aligned} \mathrm{E}(aX+b) &= \mathrm{E}(aX) + \mathrm{E}(b) && \{\text{using } \mathrm{E}(A+B) = \mathrm{E}(A) + \mathrm{E}(B)\} \\ &= a\,\mathrm{E}(X) + \mathrm{E}(b) && \{\text{using } \mathrm{E}(kX) = k\,\mathrm{E}(X)\} \\ &= a\,\mathrm{E}(X) + b && \{\text{using } \mathrm{E}(k) = k, \quad k \text{ a constant}\} \end{aligned}$$

b **i** $\begin{aligned} \mathrm{E}(Y) &= \mathrm{E}(3X+4) \\ &= 3\,\mathrm{E}(X) + 4 \\ &= 3(3) + 4 \\ &= 13 \end{aligned}$ $\quad$ **ii** $\begin{aligned} \mathrm{E}(Y) &= \mathrm{E}(-2X+1) \\ &= -2\,\mathrm{E}(X) + 1 \\ &= -2(3) + 1 \\ &= -5 \end{aligned}$ $\quad$ **iii** $\begin{aligned} \mathrm{E}(Y) &= \mathrm{E}\left(\frac{4X-2}{3}\right) \\ &= \mathrm{E}\left(\tfrac{4}{3}X - \tfrac{2}{3}\right) \\ &= \tfrac{4}{3}\,\mathrm{E}(X) - \tfrac{2}{3} \\ &= \tfrac{4}{3}(3) - \tfrac{2}{3} = 3\tfrac{1}{3} \end{aligned}$

12 X has mean 5 and standard deviation 2.

a $\mathrm{E}(Y) = \mathrm{E}(2X+3) = 2\,\mathrm{E}(X) + 3 = 2 \times 5 + 3 = 13$
$\mathrm{Var}(Y) = \mathrm{Var}(2X+3) = 2^2\ \mathrm{Var}(X) = 4 \times 2^2 = 16$

b $\mathrm{E}(Y) = \mathrm{E}(-2X+3) = -2\,\mathrm{E}(X) + 3 = -2 \times 5 + 3 = -7$
$\mathrm{Var}(Y) = \mathrm{Var}(-2X+3) = (-2)^2\ \mathrm{Var}(X) = 4 \times 2^2 = 16$

c $Y = \dfrac{X-5}{2} = \tfrac{1}{2}X - \tfrac{5}{2}$
$\mathrm{E}(Y) = \mathrm{E}\left(\tfrac{1}{2}X - \tfrac{5}{2}\right) = \tfrac{1}{2}\mathrm{E}(X) - \tfrac{5}{2} = \tfrac{1}{2} \times 5 - \tfrac{5}{2} = 0$
$\mathrm{Var}(Y) = \mathrm{Var}\left(\tfrac{1}{2}X - \tfrac{5}{2}\right) = \left(\tfrac{1}{2}\right)^2\ \mathrm{Var}(X) = \tfrac{1}{4} \times 2^2 = 1$

13 $Y = 2X + 3$

a $\begin{aligned} \mathrm{E}(Y) &= \mathrm{E}(2X+3) \\ &= 2\,\mathrm{E}(X) + 3 \end{aligned}$ $\qquad$ **b** $\begin{aligned} \mathrm{E}(Y^2) &= \mathrm{E}(4X^2 + 12X + 9) \\ &= 4\,\mathrm{E}(X^2) + 12\,\mathrm{E}(X) + 9 \end{aligned}$

c
$$\begin{aligned} \mathrm{Var}(Y) &= \mathrm{E}(Y^2) - (\mathrm{E}(Y))^2 \\ &= [4\mathrm{E}(X^2) + 12\mathrm{E}(X) + 9] - [2\mathrm{E}(X) + 3]^2 \\ &= 4\mathrm{E}(X^2) + 12\mathrm{E}(X) + 9 - [4(\mathrm{E}(X))^2 + 12\mathrm{E}(X) + 9] \\ &= 4\mathrm{E}(X^2) - 4(\mathrm{E}(X))^2 \end{aligned}$$

14
$$\begin{aligned} \mathrm{Var}(aX+b) &= \mathrm{E}\left((aX+b)^2\right) - (\mathrm{E}(aX+b))^2 \\ &= \mathrm{E}\left(a^2X^2 + 2abX + b^2\right) - (a\,\mathrm{E}(X) + b)^2 \\ &= a^2\,\mathrm{E}(X^2) + 2ab\,\mathrm{E}(X) + b^2 - \left[a^2(\mathrm{E}(X))^2 + 2ab\,\mathrm{E}(X) + b^2\right] \\ &= a^2\,\mathrm{E}(X^2) + \cancel{2ab\mathrm{E}(X)} + \cancel{b^2} - a^2(\mathrm{E}(X))^2 - \cancel{2ab\mathrm{E}(X)} - \cancel{b^2} \\ &= a^2\left(\mathrm{E}(X^2) - (\mathrm{E}(X))^2\right) \\ &= a^2\ \mathrm{Var}(X) \end{aligned}$$

EXERCISE 25F.1

1 **a** $(p+q)^4$
$= p^4 + 4p^3q + 6p^2q^2 + 4pq^3 + q^4$

b P(3 heads) $= 4p^3q$
$= 4\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)$ {as $p = q = \frac{1}{2}$}
$= \frac{1}{4}$

2 **a** $(p+q)^5 = p^5 + 5p^4q + 10p^3q^2 + 10p^2q^3 + 5pq^4 + q^5$

b **i** P(4H and 1T)
$= 5p^4q$
$= 5\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)$
$= \frac{5}{32}$

ii P(2H and 3T)
$= 10p^2q^3$
$= 10\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^3$
$= \frac{10}{32}$
$= \frac{5}{16}$

iii P(HHHHT)
$= \left(\frac{1}{2}\right)^4 \times \frac{1}{2}$
$= \frac{1}{32}$

3 **a** $\left(\frac{2}{3}+\frac{1}{3}\right)^4 = \left(\frac{2}{3}\right)^4 + 4\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right) + 6\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^2 + 4\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^3 + \left(\frac{1}{3}\right)^4$

b $\text{P(S)} = \frac{2}{3}$, $\text{P(S}') = \frac{1}{3}$ S$'$ represents an almond centre

i P(all S)
$= \left(\frac{2}{3}\right)^4$
$= \frac{16}{81}$

ii P(two of each)
$= 6\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^2$
$= \frac{8}{27}$

iii P(at least 2 strawberry creams)
= P(all S or 3S, 1S$'$ or 2S, 2S$'$)
$= \left(\frac{2}{3}\right)^4 + 4\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right) + 6\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^2$
$= \frac{16}{81} + \frac{32}{81} + \frac{24}{81}$
$= \frac{72}{81}$
$= \frac{8}{9}$

4 **a** $\left(\frac{3}{4}+\frac{1}{4}\right)^5 = \left(\frac{3}{4}\right)^5 + 5\left(\frac{3}{4}\right)^4\left(\frac{1}{4}\right) + 10\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^2 + 10\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^3 + 5\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^4 + \left(\frac{1}{4}\right)^5$

b P('normal' kiwi) $= \frac{3}{4}$, P('flat back') $= \frac{1}{4}$

i P(2 'flat backs')
= P(3F$'$, 2F)
$= 10 \times \left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^2$
$= \frac{135}{512}$

ii P(at least 3 'flat backs')
= P(2F$'$, 3F or 1F$'$, 4F or 5F)
$= 10\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^3 + 5\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^4 + \left(\frac{1}{4}\right)^5$
$= \frac{53}{512}$ on simplifying

iii P(at most 3 'normal' kiwis) = 1 − P(4 or 5 normal kiwis)
= 1 − P(4F$'$, 1F or 5F$'$)
$= 1 - \left(5\left(\frac{3}{4}\right)^4\left(\frac{1}{4}\right) + \left(\frac{3}{4}\right)^5\right)$
$= \frac{47}{128}$

5 Let X be the number of Huy's hits.

a Using the binomial expansion,
$P(X=2) = 6\left(\frac{4}{5}\right)^2\left(\frac{1}{5}\right)^2 \approx 0.154$

b $P(X \geqslant 2)$
$= 1 - P(X \leqslant 1)$
$\approx 1 - \left(4\left(\frac{4}{5}\right)\left(\frac{1}{5}\right)^3 + \left(\frac{1}{5}\right)^4\right)$
≈ 0.973

EXERCISE 25F.2

1 **a** The binomial distribution applies, as tossing a coin has two possible outcomes (a head or a tail) and each toss is independent of every other toss.

b The binomial distribution applies, as this is equivalent to tossing one coin 100 times.

c The binomial distribution applies as we can draw out a red or a blue marble with the same chances each time.

d The binomial distribution does not apply as the result of each draw is dependent upon the results of previous draws.

e The binomial distribution does not apply, assuming that ten bolts are drawn without replacement, as we do not have a repetition of independent trials.

2 Let X be the number of defective light bulbs.

a $\text{P}(X = 2) \approx 0.0305$ {using technology}

b $\text{P}(X \geqslant 1) \approx 0.265$

3 If X is the number of questions Raj answers correctly, then X is binomial. There are $n = 10$ independent trials with probability $p = \frac{1}{5}$ of a corrrect answer for each.

$$\begin{aligned}\text{P(Raj passes)} &= \text{P}(X \geqslant 7)\\ &\approx 0.000\,864 \quad \text{\{or about 9 in 10 000\}}\end{aligned}$$

4 X is the random variable for the number working night-shift.

$\therefore$ $X = 0, 1, 2, 3, 4, 5, 6, 7$ and $X \sim \text{B}(7, 0.35)$.

a
$$\begin{aligned}&\text{P}(X = 3)\\ &= \binom{7}{3}(0.35)^3(0.65)^4\\ &\approx 0.268\end{aligned}$$

b
$$\begin{aligned}&\text{P}(X < 4)\\ &= \text{P}(X \leqslant 3)\\ &\approx 0.800\end{aligned}$$

c
$$\begin{aligned}&\text{P(at least 4 work night-shift)}\\ &= \text{P}(X \geqslant 4)\\ &\approx 0.200\end{aligned}$$

5 X is the number of faulty items.

$\therefore$ $X = 0, 1, 2, 3, ..., 12$ and $X \sim \text{B}(12, 0.06)$.

a $\text{P}(X = 0) = \binom{12}{0}(0.06)^0(0.94)^{12}$
≈ 0.476

b $\text{P(at most one is faulty)} = \text{P}(X \leqslant 1)$
≈ 0.840

c $\text{P(at least two are faulty)} = \text{P}(X \geqslant 2)$
≈ 0.160

d $\text{P(less than four are faulty)} = \text{P}(X < 4)$
$= \text{P}(X \leqslant 3)$
≈ 0.996

6 X is the random variable for the number of apples with a blemish.

$\therefore$ $X = 0, 1, 2, 3, ..., 25$ and $X \sim \text{B}(25, 0.05)$.

a
$$\begin{aligned}&\text{P}(X = 2)\\ &= \binom{25}{2}(0.05)^2(0.95)^{23}\\ &\approx 0.231\end{aligned}$$

b
$$\begin{aligned}&\text{P}(X \geqslant 1)\\ &\approx 0.723\end{aligned}$$

c
$$\begin{aligned}\text{E}(X) &= np\\ &= 25 \times 0.05\\ &= 1.25 \text{ apples}\end{aligned}$$

7 X is the random variable for the number of times in a week that the bus is on time.
Since it is late 2 in every 5 days, and on time 3 in every 5 days,
$X = 0, 1, 2, 3, 4, 5, 6,$ or 7 and $X \sim \text{B}(7, 0.6)$.

a $\text{P}(X = 7) = \binom{7}{7}(0.6)^7(0.4)^0$
≈ 0.0280

b $\text{P(on time only on Monday)} = 0.6 \times (0.4)^6$
$\approx 0.002\,46$

c $\text{P}(X = 6) = \binom{7}{6}(0.6)^6(0.4)$
≈ 0.131

d $\text{P}(X \geqslant 4) \approx 0.710$

8 X is the random variable for the number of students with the flu.

$\therefore$ $X = 0, 1, 2, 3, ..., 25$ and $X \sim \text{B}(25, 0.3)$.

a **i** $\text{P}(X \geqslant 2) \approx 0.998$

ii $\text{P(test cancelled)} = \text{P}(X \geqslant 6)$ $\{20\% \text{ of } 25 = 5\}$
≈ 0.807

b Expected absentees from 350 students $= 0.3 \times 350$
$= 105$ students

9 X is the random variable for the number of successful shots from the free throw line.
$\therefore$ $X = 0, 1, 2, 3, ..., 20$ and $X \sim \text{B}(20, 0.94)$.

a **i** $\text{P}(X = 20) = \binom{20}{20}(0.94)^{20}(0.06)^0$
≈ 0.290

ii $\text{P}(X \geqslant 18) \approx 0.885$

b $\text{E}(X) = np = 20 \times 0.94$
$= 18.8$ successful throws

10 P(M wins a game against J) $= \frac{2}{3}$ $\therefore$ P(M wins) $= \frac{2}{3}$ P(J wins) $= \frac{1}{3}$

P(J wins a set 6 games to 4) $=$ P($\underbrace{\text{J wins 5 of the first 9 games}}_{\text{this is binomial with } n=9 \text{ trials of probability } p=\frac{1}{3}}$ **and** J wins the 10th game)

$\approx 0.1024 \times \frac{1}{3}$
≈ 0.0341

11 If there are n dice thrown, $\text{P(no sixes)} = \left(\frac{5}{6}\right)^n$
$\therefore$ $\text{P(at least 1 six)} = 1 - \left(\frac{5}{6}\right)^n$

$\therefore$ need to find the smallest integer n such that $1 - \left(\frac{5}{6}\right)^n > 0.5$

$$\therefore \left(\frac{5}{6}\right)^n < 0.5$$
$$\therefore n \log\left(\frac{5}{6}\right) < \log(0.5)$$
$$\therefore n > \frac{\log(0.5)}{\log\left(\frac{5}{6}\right)} \qquad \{\log\left(\tfrac{5}{6}\right) < 0\}$$
$$\therefore n > 3.80$$

$\therefore$ at least 4 dice are needed.

12 If a fair coin is tossed 200 times, then $n = 200$ and $p = \frac{1}{2}$.

a $\text{P}(90 \leqslant X \leqslant 110)$
≈ 0.863

b $\text{P}(95 < X < 105)$
$= \text{P}(96 \leqslant X \leqslant 104)$
≈ 0.475

13 $n = 38$, $p = 0.75$
$\text{P}(24 \leqslant X \leqslant 31) \approx 0.837$

14 **a**
$$\text{P}(x) = \binom{n}{x}p^{n-x}(1-p)^x = \left(\frac{n!}{x!(n-x)!}\right)p^{n-x}(1-p)^x$$

$$\begin{aligned}\therefore \text{P}(x+1) &= \binom{n}{x+1}p^{n-(x+1)}(1-p)^{x+1} \\ &= \frac{n!}{(x+1)![n-(x+1)]!}\,p^{n-x-1}(1-p)^{x+1} \\ &= \frac{n!}{(x+1)x!(n-x-1)!}\left(\frac{p^{n-x}}{p}\right)(1-p)^x \times (1-p) \\ &= \frac{n!(n-x)}{(x+1)x!(n-x)!}\,p^{n-x}(1-p)^x\left(\frac{1-p}{p}\right) \\ &= \left(\frac{n-x}{x+1}\right)\left(\frac{1-p}{p}\right)\left(\frac{n!}{x!(n-x)!}\right)p^{n-x}(1-p)^x \\ &= \left(\frac{n-x}{x+1}\right)\left(\frac{1-p}{p}\right)\text{P}(x), \quad \text{where } \text{P}(0) = \binom{n}{0}p^{n-0}(1-p)^0 = p^n\end{aligned}$$

b $\mathrm{P}(0) = p^n = \left(\frac{1}{2}\right)^5 = \frac{1}{32}$

$\mathrm{P}(1) = \left(\dfrac{n-x}{x+1}\right)\left(\dfrac{1-p}{p}\right) \mathrm{P}(0)$

$= \left(\frac{5-0}{0+1}\right)(1)(\frac{1}{32}) \qquad \{p = \frac{1}{2} \text{ and } 1-p = \frac{1}{2} \quad \therefore \quad \dfrac{1-p}{p} = 1\}$

$= \frac{5}{32}$

$\mathrm{P}(2) = \left(\frac{5-1}{1+1}\right)(1)(\frac{5}{32}) = \frac{10}{32}$

$\mathrm{P}(3) = \left(\frac{5-2}{2+1}\right)(1)(\frac{10}{32}) = \frac{10}{32}$

$\mathrm{P}(4) = \left(\frac{5-3}{3+1}\right)(1)(\frac{10}{32}) = \frac{5}{32}$

$\mathrm{P}(5) = \left(\frac{5-4}{4+1}\right)(1)(\frac{5}{32}) = \frac{1}{32}$

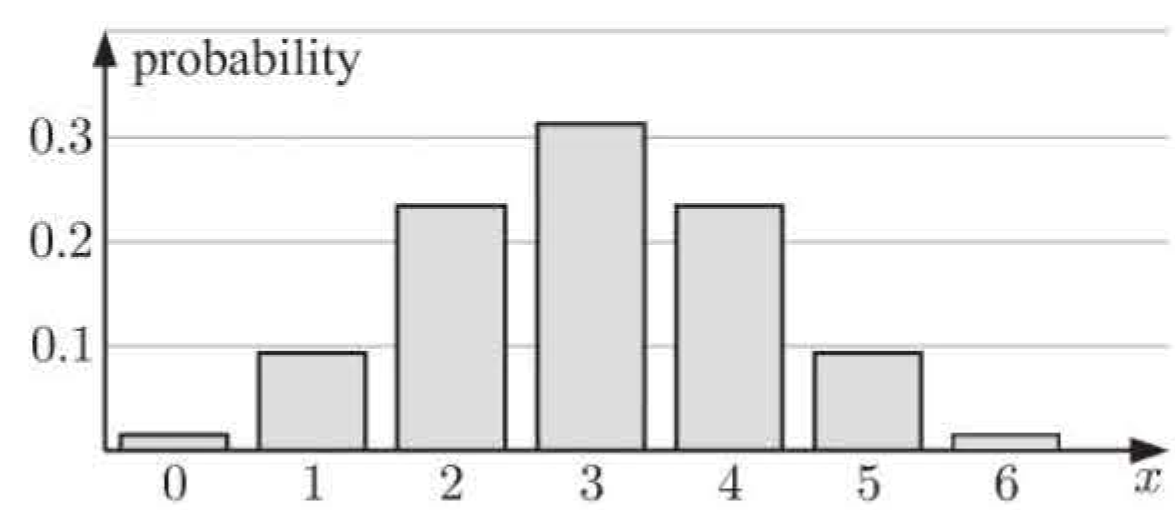
EXERCISE 25F.3

1 **a** **i** $\mu = np$
$= 6 \times 0.5$
$= 3$

$\sigma = \sqrt{np(1-p)}$
$= \sqrt{6 \times 0.5 \times 0.5}$
≈ 1.22

ii

x	0	1	2	3	4	5	6
$P(x)$	0.0156	0.0938	0.2344	0.3125	0.2344	0.0938	0.0156



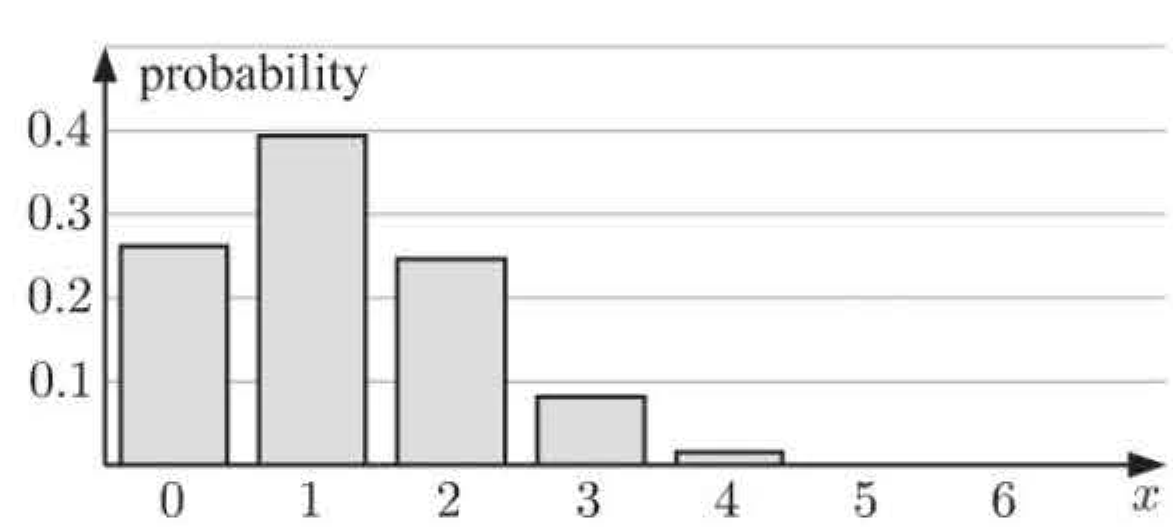
iii The distribution is bell-shaped.

b **i** $\mu = np$
$= 6 \times 0.2$
$= 1.2$

$\sigma = \sqrt{np(1-p)}$
$= \sqrt{6 \times 0.2 \times 0.8}$
≈ 0.980

ii

x	0	1	2	3	4	5	6
$P(x)$	0.2621	0.3932	0.2458	0.0819	0.0154	0.0015	0.0001



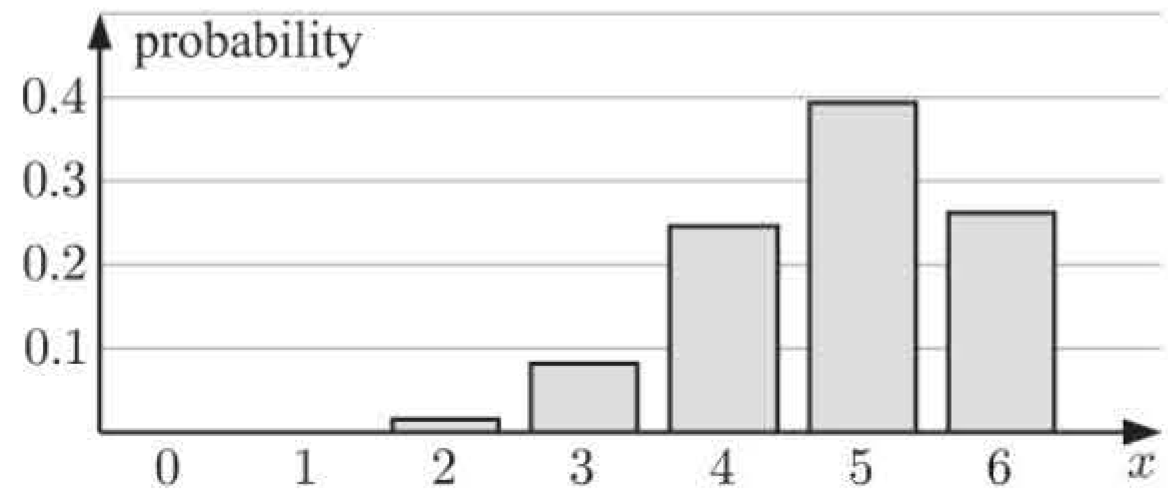
iii The distribution is positively skewed.

c **i** $\mu = np$
$= 6 \times 0.8$
$= 4.8$

$\sigma = \sqrt{np(1-p)}$
$= \sqrt{6 \times 0.8 \times 0.2}$
≈ 0.980

ii

x	0	1	2	3	4	5	6
$P(x)$	0.0001	0.0015	0.0154	0.0819	0.2458	0.3932	0.2621



iii The distribution is negatively skewed, and is the exact reflection of the distribution in **b**.

2 $n = 10,\quad p = \frac{1}{2},$ mean $\mu = np = 10 \times \frac{1}{2} = 5$ and variance $\sigma^2 = np(1-p) = 10 \times \frac{1}{2} \times \frac{1}{2} = 2.5$

3 **a** $X \sim \mathrm{B}(3, p)$

$$\mathrm{P}(X=0) = \binom{3}{0}p^0(1-p)^3 = (1-p)^3$$

$$\mathrm{P}(X=1) = \binom{3}{1}p^1(1-p)^2 = 3p(1-p)^2$$

$$\mathrm{P}(X=2) = \binom{3}{2}p^2(1-p)^1 = 3p^2(1-p)$$

$$\mathrm{P}(X=3) = \binom{3}{3}p^3(1-p)^0 = p^3$$

x_i	0	1	2	3
$P(x_i)$	$(1-p)^3$	$3p(1-p)^2$	$3p^2(1-p)$	p^3

b
$$\begin{aligned}\mu &= \sum x_i p_i\\ &= 0(1-p)^3 + 1 \times 3p(1-p)^2 + 2 \times 3p^2(1-p) + 3p^3\\ &= 3p(1-p)^2 + 6p^2(1-p) + 3p^3\\ &= 3p(1-2p+p^2) + 6p^2 - 6p^3 + 3p^3\\ &= 3p - 6p^2 + 3p^3 + 6p^2 - 6p^3 + 3p^3\\ &= 3p \quad \text{as required}\end{aligned}$$

c
$$\begin{aligned}\sigma^2 &= \sum x_i^2 p_i - \mu^2\\ &= 0^2 \times (1-p)^3 + 1^2 \times 3p(1-p)^2 + 2^2 \times 3p^2(1-p) + 3^2p^3 - (3p)^2\\ &= 3p(1-p)^2 + 12p^2(1-p) + 9p^2(p-1)\\ &= (1-p)\left[3p(1-p) + 12p^2 - 9p^2\right]\\ &= (1-p)\left[3p - 3p^2 + 3p^2\right]\\ &= 3p(1-p)\end{aligned}$$

$\therefore\ \sigma = \sqrt{3p(1-p)}$ as required

4 **a** $n = 30,\quad p = 0.04$

$$\mu = np = 30 \times 0.04 = 1.2$$

$$\sigma = \sqrt{np(1-p)} = \sqrt{30 \times 0.04 \times 0.96} \approx 1.07$$

b $n = 30,\quad p = 0.96$

$$\mu = np = 30 \times 0.96 = 28.8$$

$$\sigma = \sqrt{np(1-p)} = \sqrt{30 \times 0.96 \times 0.04} \approx 1.07$$

5 $n = 30,\quad p = 0.13$

$\therefore$ mean $\mu = np = 30 \times 0.13 = 3.9$ and standard deviation $\sigma = \sqrt{np(1-p)} = \sqrt{30 \times 0.13 \times 0.87} \approx 1.84$

EXERCISE 25G

1 **a** mean $= \dfrac{\sum f_i x_i}{\sum f_i} = \dfrac{0+18+24+18+12+0+6}{52} = \dfrac{78}{52} = 1.5$

b Using $m = 1.5$, we find $p_x = \dfrac{(1.5)^x e^{-1.5}}{x!}$, $x = 0, 1, 2, 3, ...$

So, we can obtain:

$$p_0 = \frac{(1.5)^0 e^{-1.5}}{0!} \approx 0.2231 \quad \therefore \quad 52p_0 \approx 11.6$$

$$p_1 = \frac{(1.5)^1 e^{-1.5}}{1!} \approx 0.3347 \quad \therefore \quad 52p_1 \approx 17.4$$

$$p_2 = \frac{(1.5)^2 e^{-1.5}}{2!} \approx 0.2510 \quad \therefore \quad 52p_2 \approx 13.1$$

$$p_3 = \frac{(1.5)^3 e^{-1.5}}{3!} \approx 0.1255 \quad \therefore \quad 52p_3 \approx 6.5$$

$$p_4 = \frac{(1.5)^4 e^{-1.5}}{4!} \approx 0.0471 \quad \therefore \quad 52p_4 \approx 2.4$$

$$p_5 = \frac{(1.5)^5 e^{-1.5}}{5!} \approx 0.0141 \quad \therefore \quad 52p_5 \approx 0.7$$

$$p_6 = \frac{(1.5)^6 e^{-1.5}}{6!} \approx 0.0035 \quad \therefore \quad 52p_6 \approx 0.2$$

Comparison:

x	0	1	2	3	4	5	6
f	12	18	12	6	3	0	1
$52p_x$	11.6	17.4	13.1	6.5	2.4	0.7	0.2

The fit is excellent.

2 Standard deviation $= 2.67$

a mean $= \sigma^2$
$= 2.67^2$
$= 7.1289$
≈ 7.13

b $m = 7.1289$

$\therefore \; p_x \approx \dfrac{(7.1289)^x e^{-7.1289}}{x!}$ where $x = 0, 1, 2, 3, 4, 5, ...$

c **i** $\text{P}(X = 2)$
$= \dfrac{(7.1289)^2 e^{-7.1289}}{2!}$
≈ 0.0204

ii $\text{P}(X \leqslant 3)$
≈ 0.0753

iii $\text{P}(X \geqslant 5)$
$= 1 - \text{P}(X \leqslant 4)$
$\approx 1 - 0.162$
≈ 0.838

iv $\text{P}(X \geqslant 3 \mid X \geqslant 1)$
$= \dfrac{\text{P}(X \geqslant 3 \cap X \geqslant 1)}{\text{P}(X \geqslant 1)}$
$= \dfrac{\text{P}(X \geqslant 3)}{\text{P}(X \geqslant 1)}$
$= \dfrac{1 - \text{P}(X \leqslant 2)}{1 - \text{P}(X = 0)}$
$\approx \dfrac{1 - 0.0269}{1 - 0.0008}$
≈ 0.974

3 **a** $\mu = \dfrac{1 \times 156 + 2 \times 132 + 3 \times 75 + 4 \times 33 + 5 \times 9 + 6 \times 3 + 7 \times 1}{91 + 156 + 132 + 75 + 33 + 9 + 3 + 1}$
$= \dfrac{847}{500}$
$= 1.694$

b Using $m = 1.694$, we find $p_x = \dfrac{(1.694)^x e^{-1.694}}{x!}$ where $x = 0, 1, 2, 3, 4, ...$

So, we can obtain:

$$\begin{aligned}
500p_0 &= 500 \times 1.694^0 \times e^{-1.694} \times \tfrac{1}{0!} \approx 91.9\\
500p_1 &= 500 \times 1.694^1 \times e^{-1.694} \times \tfrac{1}{1!} \approx 155.7\\
500p_2 &= 500 \times 1.694^2 \times e^{-1.694} \times \tfrac{1}{2!} \approx 131.8\\
500p_3 &= 500 \times 1.694^3 \times e^{-1.694} \times \tfrac{1}{3!} \approx 74.4\\
500p_4 &= 500 \times 1.694^4 \times e^{-1.694} \times \tfrac{1}{4!} \approx 31.5\\
500p_5 &= 500 \times 1.694^5 \times e^{-1.694} \times \tfrac{1}{5!} \approx 10.7\\
500p_6 &= 500 \times 1.694^6 \times e^{-1.694} \times \tfrac{1}{6!} \approx 3.0\\
500p_7 &= 500 \times 1.694^7 \times e^{-1.694} \times \tfrac{1}{7!} \approx 0.7
\end{aligned}$$

Comparison:

x	0	1	2	3	4	5	6	7
f	91	156	132	75	33	9	3	1
$500p_x$	92	156	132	74	32	11	3	1

The fit is excellent.

c

$$\begin{aligned}
&\text{Var}(X)\\
&= \text{E}(X^2) - (\text{E}(X))^2\\
&= \sum {x_i}^2 p_i - (1.694)^2\\
&= 1 \times \tfrac{156}{500} + 4 \times \tfrac{132}{500} + 9 \times \tfrac{75}{500} + 16 \times \tfrac{33}{500} + 25 \times \tfrac{9}{500} + 36 \times \tfrac{3}{500} + 49 \times \tfrac{1}{500} - (1.694)^2\\
&\approx 1.6683
\end{aligned}$$

$\therefore$ $\sigma \approx 1.29$ and $\sqrt{m} = \sqrt{1.694} \approx 1.30$

$\therefore$ σ is very close to the square root of the mean.

4 $p_x = \dfrac{3^x e^{-3}}{x!}$ where $x = 0, 1, 2, 3, 4, 5, ...$

a

$$\begin{aligned}
&\text{P}(X = 0)\\
&= \frac{3^0 e^{-3}}{0!}\\
&\approx 0.0498
\end{aligned}$$

b

$$\begin{aligned}
&\text{P}(X \geqslant 3)\\
&= 1 - \text{P}(X \leqslant 2)\\
&\approx 1 - 0.423\\
&\approx 0.577
\end{aligned}$$

c

$$\begin{aligned}
&\text{P(some requests are refused)}\\
&= \text{P}(X \geqslant 5)\\
&= 1 - \text{P}(X \leqslant 4)\\
&\approx 1 - 0.815\\
&\approx 0.185
\end{aligned}$$

d

$$\begin{aligned}
&\text{P}(X \geqslant 4 \mid X \geqslant 2)\\
&= \frac{\text{P}(X \geqslant 4 \cap X \geqslant 2)}{\text{P}(X \geqslant 2)}\\
&= \frac{\text{P}(X \geqslant 4)}{\text{P}(X \geqslant 2)}\\
&= \frac{1 - \text{P}(X \leqslant 3)}{1 - \text{P}(X \leqslant 1)}\\
&\approx \frac{1 - 0.647\,23}{1 - 0.199\,14}\\
&\approx 0.440
\end{aligned}$$

5 $\mathrm{P}(X=x)=\dfrac{m^x e^{-m}}{x!}$ where $x=0, 1, 2, 3, 4, ...$

a If $\mathrm{P}(X=1)+\mathrm{P}(X=2)=\mathrm{P}(X=3)$,

then $\dfrac{me^{-m}}{1!}+\dfrac{m^2e^{-m}}{2!}=\dfrac{m^3e^{-m}}{3!}$

$\therefore \quad m+\dfrac{m^2}{2}=\dfrac{m^3}{6} \quad \{\div e^{-m}\}$

$\therefore \quad 6m+3m^2=m^3$

$\therefore \quad m(m^2-3m-6)=0$ where $m\neq 0$

$\therefore \quad m^2-3m-6=0$

$\therefore \quad m=\dfrac{3\pm\sqrt{9-4(1)(-6)}}{2}=\dfrac{3\pm\sqrt{33}}{2}$

But $m>0$, so $m=\dfrac{3+\sqrt{33}}{2}$

b **i** $\mathrm{P}(X\geqslant 3)$

$=1-\mathrm{P}(X\leqslant 2)$

$\approx 1-0.494$

≈ 0.506

ii $\mathrm{P}(X\leqslant 4 \mid X\geqslant 2)$

$=\dfrac{\mathrm{P}(X\leqslant 4 \cap X\geqslant 2)}{\mathrm{P}(X\geqslant 2)}$

$=\dfrac{\mathrm{P}(X=2, 3 \text{ or } 4)}{\mathrm{P}(X\geqslant 2)}$

$=\dfrac{\mathrm{P}(X\leqslant 4) - \mathrm{P}(X\leqslant 1)}{1 - \mathrm{P}(X\leqslant 1)}$

$\approx \dfrac{0.8629-0.248\,66}{1-0.248\,66}$

≈ 0.818

6 Let X be the number of aerofoils which disintegrate from a sample of 100.
Each aerofoil has a 2% chance of disintegrating, so $m=\mathrm{E}(X)=0.02\times 100=2$

$\therefore \quad \mathrm{P}(X=x)=\dfrac{2^x e^{-2}}{x!}$ where $x=0, 1, 2, 3, ...$

a $\mathrm{P}(X=1)=\dfrac{2^1e^{-2}}{1!}=\dfrac{2}{e^2}\approx 0.271$

b $\mathrm{P}(X=2)=\dfrac{2^2e^{-2}}{2!}=\dfrac{4}{2e^2}\approx 0.271$

c $\mathrm{P}(X\leqslant 2)=\mathrm{P}(X=0)+\mathrm{P}(X=1)+\mathrm{P}(X=2)$

$=\dfrac{2^0e^{-2}}{0!}+\dfrac{2^1e^{-2}}{1!}+\dfrac{2^2e^{-2}}{2!}$

$=\dfrac{1}{e^2}+\dfrac{2}{e^2}+\dfrac{2}{e^2}$

$=\dfrac{5}{e^2}\approx 0.677$

7 **a** A person who drives 10 times per week will drive $10\times 52=520$ times in one year.
Let X be the number of fatalities from driving 520 times.

$m=\mathrm{E}(X)=0.0002\times 520=0.104$

$\therefore \quad \mathrm{P}(X=x)=\dfrac{0.104^x e^{-0.104}}{x!}$

$\therefore \quad \mathrm{P}(X=0)=\dfrac{0.104^0 e^{-0.104}}{0!}\approx 0.901 \qquad \therefore$ the probability of surviving is 0.901.

b P(driving for n years and surviving) $= (0.901)^n$

$\therefore$ we need to find n such that $(0.901)^n = 0.5$

$$\therefore \quad n \log 0.901 = \log 0.5$$

$$\therefore \quad n = \frac{\log 0.5}{\log 0.901} \approx 6.66$$

$\therefore$ you can drive for 6 years and still have a better than even chance of surviving.

8 Let X be the number of flaws in 1 metre of material.

$m = 1.7 \quad \therefore \ \mathrm{P}(X = x) = \dfrac{1.7^x e^{-1.7}}{x!}, \quad x = 0, 1, 2, 3, ...$

a $\mathrm{P}(X = 3) = \dfrac{1.7^3 e^{-1.7}}{3!} \approx 0.150$

b P(at least one flaw in 2 metres)

$= 1 - \mathrm{P}(\text{no flaws in 2 metres})$

$= 1 - (\mathrm{P}(X = 0))^2$

$\approx 1 - (0.1827)^2$

≈ 0.967

c $\mathrm{P}(X = 0) \approx 0.183$

$\mathrm{P}(X = 1) \approx 0.311$

$\mathrm{P}(X = 2) \approx 0.264$

$\mathrm{P}(X = 3) \approx 0.150$

$\mathrm{P}(X = 4) \approx 0.064$

Finding the highest of the probabilities, the mode is 1 flaw per metre.

9 **a** $\mathrm{P}(Y = y) = \dfrac{m^y e^{-m}}{y!}, \ y = 0, 1, 2, 3, ...$

$\mathrm{P}(Y = 3) = \mathrm{P}(Y = 1) + 2\mathrm{P}(Y = 2)$

$$\therefore \quad \frac{m^3 e^{-m}}{3!} = \frac{m^1 e^{-m}}{1!} + 2\frac{m^2 e^{-m}}{2!}$$

$$\therefore \quad \frac{m^3}{6} = m + m^2 \qquad \{\times e^m\}$$

$$\therefore \quad m^3 = 6m + 6m^2$$

$$\therefore \quad m(m^2 - 6m - 6) = 0 \ \text{ where } \ m \neq 0$$

$$\therefore \quad m^2 - 6m - 6 = 0$$

$$\therefore \quad m = \frac{6 \pm \sqrt{36 - 4(1)(-6)}}{2} = 3 \pm \sqrt{15}$$

But $m > 0$, so $m = 3 + \sqrt{15} \approx 6.8730$

b **i** $\mathrm{P}(1 < Y < 5)$

$= \mathrm{P}(Y \leqslant 4) - \mathrm{P}(Y \leqslant 1)$

≈ 0.177

ii $\mathrm{P}(2 \leqslant Y \leqslant 6 \mid Y \geqslant 4)$

$$= \frac{\mathrm{P}(2 \leqslant Y \leqslant 6 \ \cap \ Y \geqslant 4)}{\mathrm{P}(Y \geqslant 4)}$$

$$= \frac{\mathrm{P}(4 \leqslant Y \leqslant 6)}{\mathrm{P}(Y \geqslant 4)}$$

$$= \frac{\mathrm{P}(Y \leqslant 6) - \mathrm{P}(Y \leqslant 3)}{1 - \mathrm{P}(Y \leqslant 3)}$$

≈ 0.417

10 $\mathrm{P}(U = u) = \dfrac{x^u e^{-x}}{u!}$ where $u = 0, 1, 2, 3, ...$

a $y = \mathrm{P}(U = 0, 1 \text{ or } 2)$

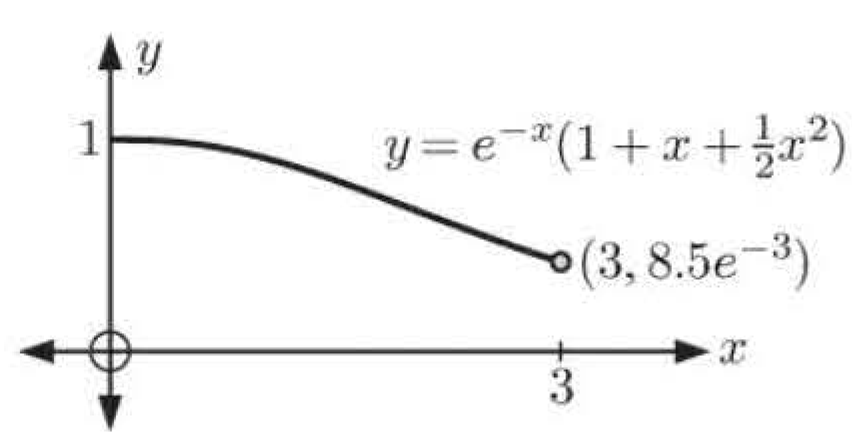
$= \mathrm{P}(U = 0) + \mathrm{P}(U = 1) + \mathrm{P}(U = 2)$

$$= \frac{x^0 e^{-x}}{0!} + \frac{x^1 e^{-x}}{1!} + \frac{x^2 e^{-x}}{2!}$$

$= e^{-x} + xe^{-x} + \frac{1}{2}x^2 e^{-x}$

$\therefore \quad y = e^{-x}(1 + x + \frac{1}{2}x^2)$

b



c
$$y = e^{-x}(1 + x + \tfrac{1}{2}x^2)$$
$$\therefore \quad \frac{dy}{dx} = -e^{-x}(1 + x + \tfrac{1}{2}x^2) + e^{-x}(1 + x)$$
$$= -e^{-x} - xe^{-x} - \tfrac{1}{2}x^2e^{-x} + e^{-x} + xe^{-x}$$
$$= -\tfrac{1}{2}x^2e^{-x}$$

Since $x^2e^{-x} > 0$, $\dfrac{dy}{dx} < 0$ for all $x > 0$.

$\therefore$ as the mean x increases, $y = \text{P}(U \leqslant 2)$ decreases.

REVIEW SET 25A

1 **a** $\text{P}(X = x) = \dfrac{a}{x^2 + 1}$ for $x = 0, 1, 2, 3$

x	0	1	2	3
$\text{P}(X = x)$	a	$\frac{a}{2}$	$\frac{a}{5}$	$\frac{a}{10}$

Now $a + \dfrac{a}{2} + \dfrac{a}{5} + \dfrac{a}{10} = 1$ $\quad \{\text{as } \sum_{x=0}^{3} \text{P}(X = x) = 1\}$

$\therefore \quad 10a + 5a + 2a + a = 10$

$\therefore \quad 18a = 10$

$\therefore \quad a = \frac{5}{9}$

b $\text{P}(X \geqslant 1) = \text{P}(X = 1, 2, \text{ or } 3)$
$= \text{P}(X = 1) + \text{P}(X = 2) + \text{P}(X = 3)$
$= \frac{5}{18} + \frac{1}{9} + \frac{5}{90}$
$= \frac{4}{9}$

or $\text{P}(X \geqslant 1) = 1 - \text{P}(X < 1)$
$= 1 - \text{P}(X = 0)$
$= 1 - \frac{5}{9}$
$= \frac{4}{9}$

2 Let X be the number of defective toothbrushes.

$\therefore \quad X \sim \text{B}(120, 0.04)$

$\mu = np$
$= 120 \times 0.04$
$= 4.8$ defectives

$\sigma = \sqrt{np(1-p)}$
$= \sqrt{120 \times 0.04 \times 0.96}$
≈ 2.15

3

x	0	1	2	3	4
$\text{P}(X = x)$	0.10	0.30	0.45	0.10	k

a If this is a probability distribution then $\sum P(x_i) = 1$

$\therefore \quad 0.1 + 0.3 + 0.45 + 0.1 + k = 1$

$\therefore \quad 0.95 + k = 1$

$\therefore \quad k = 0.05$

b $\text{P}(X \geqslant 3)$
$= \text{P}(X = 3) + \text{P}(X = 4)$
$= 0.10 + 0.05$
$= 0.15$

c $\text{E}(X) = \sum x_i p_i$
$= 0(0.10) + 1(0.30) + 2(0.45) + 3(0.10) + 4(0.05)$
$= 0 + 0.3 + 0.9 + 0.3 + 0.2$
$= 1.7$

d $\text{E}(X^2) = \sum x_i^2 p_i$
$= 0(0.10) + 1(0.30) + 4(0.45) + 9(0.10) + 16(0.05)$
$= 3.8$

$\sigma = \sqrt{\text{Var}(X)} = \sqrt{\text{E}(X^2) - (\text{E}(X))^2}$
$= \sqrt{3.8 - 1.7^2}$
≈ 0.954

4 **a** $\left(\frac{3}{5}+\frac{2}{5}\right)^4 = \underbrace{\left(\frac{3}{5}\right)^4}_{4B} + \underbrace{4\left(\frac{3}{5}\right)^3\left(\frac{2}{5}\right)}_{3B\ 1B'} + \underbrace{6\left(\frac{3}{5}\right)^2\left(\frac{2}{5}\right)^2}_{2B\ 2B'} + \underbrace{4\left(\frac{3}{5}\right)\left(\frac{2}{5}\right)^3}_{1B\ 3B'} + \underbrace{\left(\frac{2}{5}\right)^4}_{4B'}$

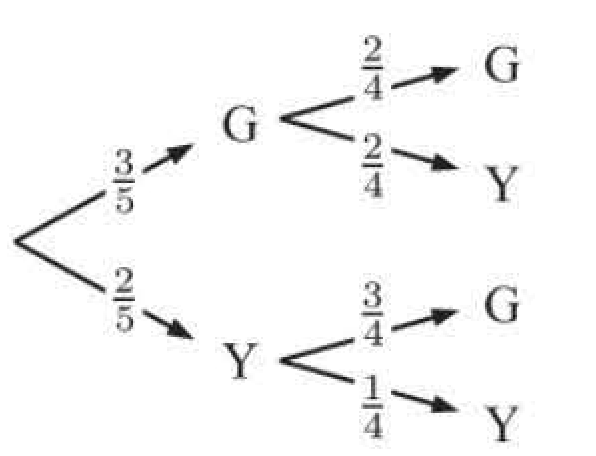
$P(B) = \frac{12}{20}$
$= \frac{3}{5}$
$\therefore\ P(B') = \frac{2}{5}$

b **i** P(2 Blue inks)
$= P(2B \text{ and } 2B')$
$= 6\left(\frac{3}{5}\right)^2\left(\frac{2}{5}\right)^2$
$= \dfrac{6\times 9\times 4}{5^4}$
$= \frac{216}{625}$

ii P(at most 2 Blue inks)
$= P(2B \text{ and } 2B' \text{ or } 1B \text{ and } 3B' \text{ or } 4B')$
$= 6\left(\frac{3}{5}\right)^2\left(\frac{2}{5}\right)^2 + 4\left(\frac{3}{5}\right)\left(\frac{2}{5}\right)^3 + \left(\frac{2}{5}\right)^4$
$= \dfrac{6\times 9\times 4 + 4\times 3\times 8 + 16}{625}$
$= \frac{328}{625}$

5

1st draw 2nd draw



Event	X	Probability
GG	2	$\frac{3}{5}\times\frac{2}{4}=\frac{3}{10}$
GY	1	$\frac{3}{5}\times\frac{2}{4}=\frac{3}{10}$
YG	1	$\frac{2}{5}\times\frac{3}{4}=\frac{3}{10}$
YY	0	$\frac{2}{5}\times\frac{1}{4}=\frac{1}{10}$

	a	**b**	**c**
x	0	1	2
$P(X=x)$	$\frac{1}{10}$	$\frac{3}{5}$	$\frac{3}{10}$

d $E(X) = 0\times\frac{1}{10} + 1\times\frac{3}{5} + 2\times\frac{3}{10} = \frac{6}{5} \quad (=1\frac{1}{5})$

6 **a**

Result	*Pays*
1	£2
2	£4
3	£6
4	£8
5	£10
6	£12

Expectation
$= \frac{1}{6}\times £2 + \frac{1}{6}\times £4 + \frac{1}{6}\times £6 + \frac{1}{6}\times £8 + \frac{1}{6}\times £10 + \frac{1}{6}\times £12$
$= \frac{1}{6}\times £42$
$= £7$

b Expected gain is $£7 - £8 = -£1$.
$\therefore$ advise Lakshmi against playing several games, as £1 is expected to be lost per game in the long run.

7 **a** $n = 7$ and $r = \{0, 1, 2, 3, ..., 7\}$
$= x$

$\therefore\ k = \binom{7}{x}$

b $n = 7,\ p = \frac{1}{3}$

$\therefore\ \mu = np$
$= \frac{7}{3} \quad (\approx 2.33)$

and $\sigma^2 = np(1-p)$
$= 7\times\frac{1}{3}\times\frac{2}{3}$
$= \frac{14}{9} \quad (\approx 1.56)$

8 **a** $\left(\frac{4}{5}+\frac{1}{5}\right)^5 = \left(\frac{4}{5}\right)^5 + 5\left(\frac{4}{5}\right)^4\left(\frac{1}{5}\right)^1 + 10\left(\frac{4}{5}\right)^3\left(\frac{1}{5}\right)^2 + 10\left(\frac{4}{5}\right)^2\left(\frac{1}{5}\right)^3 + 5\left(\frac{4}{5}\right)^1\left(\frac{1}{5}\right)^4 + \left(\frac{1}{5}\right)^5$

b Let X = the number of goals scored

i P(3 goals **then** 2 misses)
$= P(GGGG'G')$
$= (\frac{4}{5})^3\times(\frac{1}{5})^2$
$= \frac{64}{3125}$
≈ 0.0205

ii P(3 goals and 2 misses)
$= P(X=3)$
$= 10\left(\frac{4}{5}\right)^3\left(\frac{1}{5}\right)^2$
$= \frac{128}{625}$
≈ 0.205

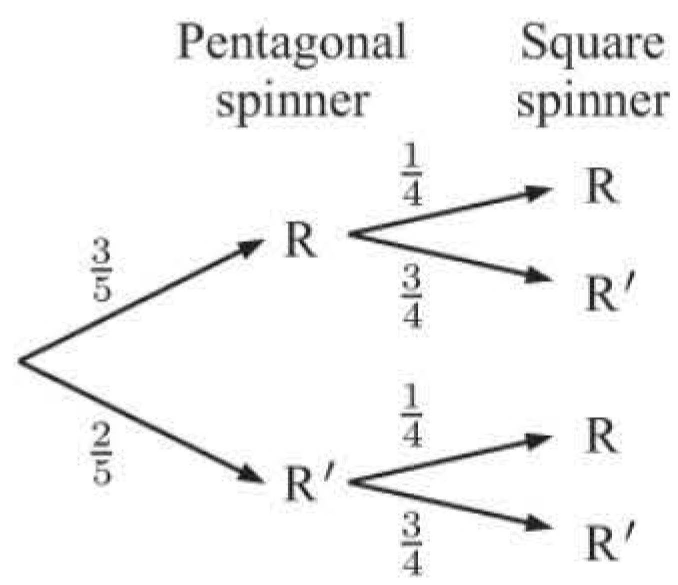
9

Result	*Pays*
1, 3, 5	\$2
2	\$3
4	\$6
6	\$9

a $\text{Expected return} = \frac{3}{6} \times \$2 + \frac{1}{6} \times \$3 + \frac{1}{6} \times \$6 + \frac{1}{6} \times \$9$
$= \frac{1}{6}(\$24)$
$= \$4$

b For a \$5 amount to play the game, the club expects a \$1 return per game
$\therefore$ for 75 people, the return expected is \$75.

10 **a**



b $\text{P(exactly one red)} = \text{P}(RR') + \text{P}(R'R)$
$= \frac{3}{5} \times \frac{3}{4} + \frac{2}{5} \times \frac{1}{4}$
$= \frac{9}{20} + \frac{1}{10}$
$= \frac{11}{20}$

c **i** $n = 10, \quad p = \frac{11}{20}$

$\text{P}(X = 1) = \binom{10}{1}\left(\frac{11}{20}\right)^1\left(\frac{9}{20}\right)^9$

$\text{P}(X = 9) = \binom{10}{9}\left(\frac{11}{20}\right)^9\left(\frac{9}{20}\right)^1$

ii $\binom{10}{1} = \binom{10}{9} = 10$ so we need only consider the parts $\left(\frac{11}{20}\right)^1\left(\frac{9}{20}\right)^9$ and $\left(\frac{11}{20}\right)^9\left(\frac{9}{20}\right)^1$.
Now, $11 \times 9^9 < 11^9 \times 9$, and the denominators are the same in each case.
$\therefore$ it is more likely that exactly one red will occur 9 times.

11 **a** $\frac{1}{3} + \frac{1}{6} + \frac{1}{4} + y = 1$
$\therefore \quad y = \frac{1}{4}$
$\therefore$ the probability of obtaining the number 24 is $y = \frac{1}{4}$.

b $\text{E}(X) = 6\left(\frac{1}{3}\right) + 12\left(\frac{1}{6}\right) + x\left(\frac{1}{4}\right) + 24\left(\frac{1}{4}\right) = 14$
$\therefore \quad 2 + 2 + \frac{x}{4} + 6 = 14$
$\therefore \quad \frac{x}{4} = 4$
$\therefore \quad x = 16$
So, the fourth number is 16.

c $p_1 = \frac{1}{3}$
$p_1 + p_2 = \frac{1}{3} + \frac{1}{6} = \frac{1}{2}$
Since $p_1 + p_2 = 0.5$, the median is $\dfrac{12 + 16}{2} = 14$.
The most likely result is 6, so this is the mode.

12 X has mean μ and standard deviation σ.
$\therefore \quad \text{E}(X) = \mu$ and $\sigma^2 = \text{Var}(X) = \text{E}(X^2) - (\text{E}(X))^2$
Now $Y = aX + b \quad \therefore$ mean of $Y = \text{E}(Y)$
$= \text{E}(aX + b)$
$= \text{E}(aX) + \text{E}(b)$
$= a\,\text{E}(X) + b$
$= a\mu + b$

Also, $\text{Var}(aX + b) = \text{E}\left((aX + b)^2\right) - (\text{E}(aX + b))^2$
$= \text{E}\left(a^2X^2 + 2abX + b^2\right) - (a\,\text{E}(X) + b)^2$
$= a^2\,\text{E}(X^2) + 2ab\,\text{E}(X) + b^2 - \left[a^2(\text{E}(X))^2 + 2ab\,\text{E}(X) + b^2\right]$
$= a^2\,\text{E}(X^2) + \cancel{2ab\,\text{E}(X)} + \cancel{b^2} - a^2(\text{E}(X))^2 - \cancel{2ab\,\text{E}(X)} - \cancel{b^2}$
$= a^2\left(\text{E}(X^2) - (\text{E}(X))^2\right)$
$= a^2\sigma^2$

$\therefore$ standard deviation of $Y = \sqrt{a^2\sigma^2}$
$= \sqrt{a^2}\,\sigma$ {since $\sigma > 0$}
$= |a|\sigma$

REVIEW SET 25B

1 **a** $P(x) = k\left(\frac{3}{4}\right)^x\left(\frac{1}{4}\right)^{3-x}$ for $x = 0, 1, 2, 3$

$P(0) = k\left(\frac{3}{4}\right)^0\left(\frac{1}{4}\right)^3 = \dfrac{k}{64}$ $\quad P(2) = k\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^1 = \dfrac{9k}{64}$

$P(1) = k\left(\frac{3}{4}\right)^1\left(\frac{1}{4}\right)^2 = \dfrac{3k}{64}$ $\quad P(3) = k\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^0 = \dfrac{27k}{64}$

x	0	1	2	3
$P(x)$	$\dfrac{k}{64}$	$\dfrac{3k}{64}$	$\dfrac{9k}{64}$	$\dfrac{27k}{64}$

Now $\dfrac{k}{64} + \dfrac{3k}{64} + \dfrac{9k}{64} + \dfrac{27k}{64} = 1$ {as $\sum P(x_i) = 1$}

$\therefore \dfrac{40k}{64} = 1$

$\therefore k = \frac{8}{5}$ $(= 1.6)$

b $P(X \geqslant 1) = 1 - P(X = 0)$
$= 1 - \dfrac{k}{64}$
$= 1 - \dfrac{1.6}{64}$
$= 0.975$

c $E(X) = \sum x_i p_i$
$= 0 \times \frac{1.6}{64} + 1 \times \frac{3\times1.6}{64} + 2 \times \frac{9\times1.6}{64} + 3 \times \frac{27\times1.6}{64}$
$= 2.55$

d $\sigma = \sqrt{\sum(x_i - \mu)^2 p_i}$

$= \sqrt{(0-2.55)^2 \times \frac{1.6}{64} + (1-2.55)^2 \times \frac{3\times1.6}{64} + (2-2.55)^2 \times \frac{9\times1.6}{64} + (3-2.55)^2 \times \frac{27\times1.6}{64}}$

≈ 0.740

2 X is the number of defectives. Then $X \sim B(10, 0.18)$. $X = 0, 1, 2, 3, ..., 10$.

a $P(X = 1)$
$= \binom{10}{1}(0.18)^1(0.82)^9$
≈ 0.302

b $P(X = 2)$
$= \binom{10}{2}(0.18)^2(0.82)^8$
≈ 0.298

c $P(X \geqslant 2)$
≈ 0.561

3 Expected number of major knee surgeries $= np$
$= 487 \times 0.0132$
≈ 6.43

4 X is the number of visitors who make a voluntary donation upon entry.
Then $X = 0, 1, 2, 3, ..., 175$ and $X \sim B(175, 0.24)$.

a $E(X) = np$
$= 175 \times 0.24$
$= 42$

b $P(X < 40) = P(X \leqslant 39)$
≈ 0.334

5 If X is the number of X-rays which show the fracture, then $X = 0, 1, 2, 3, 4$ and $X \sim B(4, 0.96)$.

a $P(X = 4) = \binom{4}{4}(0.96)^4(0.04)^0$
≈ 0.849

b $P(X = 0) = \binom{4}{0}(0.96)^0(0.04)^4$
$\approx 2.56 \times 10^{-6}$

c $P(X \geqslant 3) \approx 0.991$

d $P(X = 1) = \binom{4}{1}(0.96)^1(0.04)^3$
$\approx 0.000\,246$

6 X is the number of players who turn up to a game.
Then $X = 0, 1, 2, 3, ..., 8$ and $X \sim \text{B}(8, 0.75)$.

a **i** $\text{P}(X = 8) = \binom{8}{8}(0.75)^8(0.25)^0$
≈ 0.100

ii $\text{P}(\text{team has to forfeit}) = \text{P}(X \leqslant 4)$
≈ 0.114

b Expected number of games forfeited in $30 = np$
$\approx 30 \times 0.1138$ {from **a ii**}
≈ 3.41

7 **a** If the mean is 30 then $np = 30$ .... (1)
If the variance is 22.5 then $np(1-p) = 22.5$ .... (2)
Substituting (1) into (2), we get $30(1-p) = 22.5$

$$\therefore \ 1 - p = \frac{22.5}{30}$$

$$\therefore \ p = 1 - \frac{22.5}{30}$$

$$\therefore \ p = 0.25$$

and so $n \times 0.25 = 30$

$$\therefore \ n = 120$$

So, $n = 120$ and $p = 0.25$ (or $\frac{1}{4}$).

b **i** $\text{P}(X = 25)$
≈ 0.0501

ii $\text{P}(X \geqslant 25)$
≈ 0.878

iii $\text{P}(15 \leqslant X \leqslant 25)$
≈ 0.172

8 $\text{P}(X = 0) + \text{P}(X = 1) + \text{P}(X = 2) + ... = 1$

$$\therefore \ a\left(\tfrac{5}{6}\right)^0 + a\left(\tfrac{5}{6}\right)^1 + a\left(\tfrac{5}{6}\right)^2 + ... = 1$$

$$\therefore \ a\underbrace{\left(1 + \tfrac{5}{6} + \left(\tfrac{5}{6}\right)^2 + ...\right)}_{\text{infinite geometric series with } u_1 = 1,\ r = \frac{5}{6}} = 1$$

$$\therefore \ a\left(\frac{1}{1 - \frac{5}{6}}\right) = 1$$

$$\therefore \ a(6) = 1$$

$$\therefore \ a = \tfrac{1}{6}$$

9 **a** $\text{P(X wins)} = \frac{3}{5} = 0.6$ $\quad \text{P(Y wins)} = \frac{2}{5} = 0.4$
Probability generator is $(0.6 + 0.4)^6$

$$= \underbrace{(0.6)^6}_{\substack{\text{X wins 6}}} + \underbrace{6(0.6)^5(0.4)}_{\substack{\text{X wins 5} \\ \text{Y wins 1}}} + \underbrace{15(0.6)^4(0.4)^2}_{\substack{\text{X wins 4} \\ \text{Y wins 2}}} + \underbrace{20(0.6)^3(0.4)^3}_{\substack{\text{X wins 3} \\ \text{Y wins 3}}} + \underbrace{15(0.6)^2(0.4)^4}_{\substack{\text{X wins 2} \\ \text{Y wins 4}}} + \underbrace{6(0.6)(0.4)^5}_{\substack{\text{X wins 1} \\ \text{Y wins 5}}} + \underbrace{(0.4)^6}_{\substack{\text{X wins 0} \\ \text{Y wins 6}}}$$

b **i** P(Y wins 3)
$= 20(0.6)^3(0.4)^3$
≈ 0.276

ii $\text{P(Y wins at least 5)}$
$= 6(0.6)^1(0.4)^5 + (0.4)^6$
≈ 0.0410

10 Let X be the number of objects broken by the first glass blower.
$\therefore$ X has mean $m = 20 \times \frac{1}{200} = 0.1$
$\therefore$ $X \sim \text{Po}(0.1)$
Let Y be the number of objects broken by the second glass blower.
$\therefore$ Y has mean $m = 40 \times \frac{3}{200} = 0.6$
$\therefore$ $Y \sim \text{Po}(0.6)$

$\therefore$ P(glass blowers break 2 or more objects between them)
$= 1 -$ P(glass blowers break 0 or 1 objects between them)
$= 1 - \text{P}(X = 0, Y = 0 \quad \text{or} \quad X = 1, Y = 0 \quad \text{or} \quad X = 0, Y = 1)$

$$= 1 - \left[\frac{0.1^0 e^{-0.1}}{0!} \times \frac{0.6^0 e^{-0.6}}{0!} + \frac{0.1^1 e^{-0.1}}{1!} \times \frac{0.6^0 e^{-0.6}}{0!} + \frac{0.1^0 e^{-0.1}}{0!} \times \frac{0.6^1 e^{-0.6}}{1!}\right]$$

≈ 0.156

11 **a** $\text{P}(X = 3) = 0.226\,89$

$\therefore \binom{7}{3} p^3 (1-p)^{7-3} = 0.226\,89$

$\therefore 35p^3(1-p)^4 = 0.226\,89$

$\therefore p \approx 0.300$ or 0.564 {using technology}

$\therefore p \approx 0.300$ {smallest p}

b $\text{P}(X \leqslant 4) \approx 0.850$

12 $\text{P}(X = x) = k\left(x + \dfrac{1}{x}\right)$

a $\text{P}(X = 1) = k(1 + \frac{1}{1}) = 2k$

$\text{P}(X = 2) = k(2 + \frac{1}{2}) = \frac{5}{2}k$

$\text{P}(X = 3) = k(3 + \frac{1}{3}) = \frac{10}{3}k$

$\text{P}(X = 4) = k(4 + \frac{1}{4}) = \frac{17}{4}k$

Now $\sum P(x_i) = 1$

$\therefore 2k + \frac{5}{2}k + \frac{10}{3}k + \frac{17}{4}k = 1$

$\therefore \frac{145}{12}k = 1$

$\therefore k = \frac{12}{145}$

b Using $k = \frac{12}{145}$ we obtain:

x	1	2	3	4
$P(x)$	$\frac{24}{145}$	$\frac{30}{145}$	$\frac{40}{145}$	$\frac{51}{145}$

$\therefore \text{E}(X) = 1\left(\frac{24}{145}\right) + 2\left(\frac{30}{145}\right) + 3\left(\frac{40}{145}\right) + 4\left(\frac{51}{145}\right) = \frac{408}{145} \approx 2.81$

$\text{E}(X^2) = 1^2\left(\frac{24}{145}\right) + 2^2\left(\frac{30}{145}\right) + 3^2\left(\frac{40}{145}\right) + 4^2\left(\frac{51}{145}\right) = \frac{264}{29}$

$\therefore \text{Var}(X) = \text{E}(X^2) - (\text{E}(X))^2$

$= \frac{264}{29} - \left(\frac{408}{145}\right)^2$

≈ 1.19

c $p_1 = \frac{24}{145}$

$p_1 + p_2 = \frac{24}{145} + \frac{30}{145} = \frac{54}{145}$

$p_1 + p_2 + p_3 = \frac{54}{145} + \frac{40}{145} = \frac{94}{145} \approx 0.648$

Since $p_1 + p_2 + p_3 \geqslant 0.5$, the median is 3.

The most likely value of X is 4, so this is the mode.

13 $\text{P}(Y > 3) \approx 0.033\,768\,97$

$\therefore \text{P}(Y \leqslant 3) \approx 1 - 0.033\,768\,97$

$\therefore \text{P}(Y = 0, 1, 2, \text{ or } 3) \approx 0.966\,231\,03$

$$\therefore \frac{m^0 e^{-m}}{0!} + \frac{m^1 e^{-m}}{1!} + \frac{m^2 e^{-m}}{2!} + \frac{m^3 e^{-m}}{3!} \approx 0.966\,231\,03$$

$$\therefore e^{-m}\left(1 + m + \frac{m^2}{2} + \frac{m^3}{6}\right) \approx 0.966\,231\,03$$

Using technology on the domain $m > 0$, we find $m = 1.2$

$\therefore \text{P}(Y < 3) = \text{P}(Y \leqslant 2)$

≈ 0.879

REVIEW SET 25C

1 **a** $\sum P(X = x_i) = 1$

$\therefore \dfrac{k}{2 \times 1} + \dfrac{k}{2 \times 2} + \dfrac{k}{2 \times 3} = 1$

$\therefore \dfrac{k}{2} + \dfrac{k}{4} + \dfrac{k}{6} = 1$

$\therefore 6k + 3k + 2k = 12$

$\therefore 11k = 12$

$\therefore k = \frac{12}{11}$

b $\sum P(x_i) = 1$

$\therefore \dfrac{k}{2} + 0.2 + k^2 + 0.3 = 1$

$\therefore 2k^2 + k + 1 = 2$

$\therefore 2k^2 + k - 1 = 0$

$\therefore (2k - 1)(k + 1) = 0$

$\therefore k = -1, \frac{1}{2}$

If $k = -1$, then $P(0) = \frac{-1}{2} < 0$, so $P(x)$ would not be a valid probability distribution function.

$\therefore k = \frac{1}{2}$

2 **a** $P(X = x) = \binom{4}{x}\left(\frac{1}{2}\right)^x\left(\frac{1}{2}\right)^{4-x}$

$\therefore P(X = 0) = \binom{4}{0}\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^4 = 0.0625$

$P(X = 1) = \binom{4}{1}\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^3 = 0.25$

$P(X = 2) = \binom{4}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 = 0.375$

$P(X = 3) = \binom{4}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^1 = 0.25$

$P(X = 4) = \binom{4}{4}\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^0 = 0.0625$

x	0	1	2	3	4
$P(X = x)$	0.0625	0.25	0.375	0.25	0.0625

b $\mu = \sum x_i P(X = x_i)$

$= 0 \times 0.0625 + 1 \times 0.25 + 2 \times 0.375 + 3 \times 0.25 + 4 \times 0.0625$

$= 2$

c $n = 4$, $p = \frac{1}{2}$ $\quad \therefore \sigma = \sqrt{np(1-p)} = \sqrt{4 \times \frac{1}{2} \times \frac{1}{2}}$

$= \sqrt{1}$

$= 1$

3 $X \sim B(1200, 0.4)$

So, mean of $X = np$

$= 1200 \times 0.4$

$\therefore \mu = 480$

and standard deviation of $X = \sqrt{np(1-p)}$

$= \sqrt{1200 \times 0.4 \times 0.6}$

$\therefore \sigma \approx 17.0$

4 X is the number of trees that survive the first year.

$\therefore X = 0, 1, 2, 3, 4, 5$ and $X \sim B(5, 0.4)$

a $P(X = 1) = \binom{5}{1}(0.4)^1(0.6)^4$

≈ 0.259

b $P(X \leqslant 1) \approx 0.337$

c $P(X \geqslant 1) \approx 0.922$

5 **a** **i** In the numbers 1 to 20, there are 10 even numbers.
However, '4' and '16' are square numbers, so 8 of the numbers in the bag win \$3.

$\therefore$ P(player wins \$3) $= \frac{8}{20} = \frac{2}{5}$

ii 1, 4, 9, and 16 are the only square numbers in the bag.
But '4' and '16' are even, so 2 of the numbers in the bag win \$6.

$\therefore$ P(player wins \$6) $= \frac{2}{20} = \frac{1}{10}$

iii 2 numbers are both even and square (4 and 16).

$\therefore$ P(player wins \$9) $= \frac{2}{20} = \frac{1}{10}$

b Expected winnings $= \frac{2}{5} \times \$3 + \frac{1}{10} \times \$6 + \frac{1}{10} \times \$9$

$= \$2.70$

$\therefore$ for the game to be fair, players should be charged \$2.70 per game.

6 Suppose X is the event that a 6 is rolled. $\therefore\ n = 360,\quad p = \frac{1}{6}$

a $P(X < 50) = P(X \leqslant 49)$
≈ 0.0660

b $P(55 \leqslant X \leqslant 65) \approx 0.563$

7 With n tosses, P(getting at least 2 heads) $= 1 -$ P(getting at most 1 head)

We need to find n such that P(getting at least 2 heads) > 0.99

So, $1 - \left(\binom{n}{0}\left(\frac{1}{2}\right)^n + \binom{n}{1}\left(\frac{1}{2}\right)^n\right) > 0.99$

Using technology, $n \geqslant 11\quad \{n \in \mathbb{Z}\}$

$\therefore\ n = 11$ is the smallest value of n.

8 **a** P(hot water unit fails within one year) $=$ P(all 20 components fail)
$= (0.85)^{20}$
≈ 0.0388

b P(hot water unit with n components fails within one year) $= 0.85^n$

$\therefore$ P(hot water unit with n components is operating after one year) $= 1 - 0.85^n$

$\therefore$ we need to find the smallest integer n such that $1 - 0.85^n \geqslant 0.98$

$$\therefore\ 0.85^n \leqslant 0.02$$
$$\therefore\ n \log 0.85 \leqslant \log 0.02$$
$$\therefore\ n \geqslant \frac{\log 0.02}{\log 0.85}\quad \{\log 0.85 < 0\}$$
$$\therefore\ n \geqslant 24.1$$

$\therefore$ at least 25 solar components are needed.

9 **a** $P(\text{hit}) = \frac{1}{3},\quad P(\text{miss}) = \frac{2}{3}$

Probability generator is

$$\left(\tfrac{1}{3} + \tfrac{2}{3}\right)^5 = \underset{X=5}{\underset{\uparrow}{\left(\tfrac{1}{3}\right)^5}} + \underset{X=4}{\underset{\uparrow}{5\left(\tfrac{1}{3}\right)^4\left(\tfrac{2}{3}\right)}} + \underset{X=3}{\underset{\uparrow}{10\left(\tfrac{1}{3}\right)^3\left(\tfrac{2}{3}\right)^2}} + \underset{X=2}{\underset{\uparrow}{10\left(\tfrac{1}{3}\right)^2\left(\tfrac{2}{3}\right)^3}} + \underset{X=1}{\underset{\uparrow}{5\left(\tfrac{1}{3}\right)\left(\tfrac{2}{3}\right)^4}} + \underset{X=0}{\underset{\uparrow}{\left(\tfrac{2}{3}\right)^5}}$$

b
$$P(X \text{ odd} \mid X \geqslant 2) = \frac{P(X \text{ odd} \cap X \geqslant 2)}{P(X \geqslant 2)}$$
$$= \frac{P(X = 3 \text{ or } 5)}{1 - P(X \leqslant 1)}$$
$$= \frac{10\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^5}{1 - \left(5\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^4 + \left(\frac{2}{3}\right)^5\right)}$$
$$\approx 0.313$$

10 **a**
$$P(X = x) = \frac{m^x e^{-m}}{x!},\quad x = 0, 1, 2, \ldots..$$

Now $P(X = 1) = P(2 \leqslant x \leqslant 4)$

$\therefore\ P(X = 1) = P(X = 2) + P(X = 3) + P(X = 4)$

$$\therefore\ \frac{me^{-m}}{1!} = \frac{m^2e^{-m}}{2!} + \frac{m^3e^{-m}}{3!} + \frac{m^4e^{-m}}{4!}$$
$$\therefore\ m = \frac{m^2}{2} + \frac{m^3}{6} + \frac{m^4}{24}\qquad \{\times\, e^m\}$$
$$\therefore\ 24m = 12m^2 + 4m^3 + m^4$$
$$\therefore\ m(m^3 + 4m^2 + 12m - 24) = 0\quad \text{where}\quad m \neq 0$$
$$\therefore\ m \approx 1.28\quad \{\text{using technology}\}$$

i mean of $X = m \approx 1.28$

standard deviation $= \sqrt{m} \approx 1.13$

ii $Y = \dfrac{X+1}{2} = \frac{1}{2}X + \frac{1}{2}$

$$\begin{aligned}\therefore \text{ mean of } Y &= \mathrm{E}(Y) \\ &= \mathrm{E}\left(\tfrac{1}{2}X + \tfrac{1}{2}\right) \\ &= \tfrac{1}{2}\mathrm{E}(X) + \mathrm{E}(\tfrac{1}{2}) \\ &\approx 1.14\end{aligned}$$

Now, $\mathrm{Var}(X) \approx 1.28$

$$\begin{aligned}\text{and } \mathrm{Var}(Y) &= \mathrm{Var}(\tfrac{1}{2}X) \\ &\approx (\tfrac{1}{2})^2 \times 1.28 \\ &\approx 0.320 \\ \therefore \sigma_Y &\approx \sqrt{0.320} \\ &\approx 0.566\end{aligned}$$

b $$\begin{aligned}\mathrm{P}(X \geqslant 2) &= 1 - \mathrm{P}(X \leqslant 1) \\ &\approx 1 - 0.634 \\ &\approx 0.366\end{aligned}$$

11 **a** For a Poisson random variable X,

$\mathrm{E}(X) = \mathrm{Var}(X) = m$

$$\begin{aligned}\therefore \quad 5m &= 2m^2 - 12 \\ \therefore \quad 2m^2 - 5m - 12 &= 0 \\ \therefore \quad (2m+3)(m-4) &= 0 \\ \therefore \quad m &= 4 \quad \{\text{as } m > 0\}\end{aligned}$$

$\therefore$ the mean of X is 4.

b $\mathrm{P}(X = x) = \dfrac{4^x e^{-4}}{x!}, \quad x = 0, 1, 2, 3, ...$

$$\begin{aligned}\therefore \mathrm{P}(X < 3) &= \frac{4^0 e^{-4}}{0!} + \frac{4^1 e^{-4}}{1!} + \frac{4^2 e^{-4}}{2!} \\ &= e^{-4}(1 + 4 + 8) \\ &= \frac{13}{e^4} \approx 0.238\end{aligned}$$

12 $$\begin{aligned}\mathrm{P}(X > 2) &\approx 0.070\,198 \\ \therefore \mathrm{P}(X \leqslant 2) &\approx 1 - 0.070\,198 \\ \therefore \mathrm{P}(X = 0, 1 \text{ or } 2) &\approx 0.929\,802\end{aligned}$$

$$\therefore \binom{10}{0} p^0(1-p)^{10} + \binom{10}{1} p^1(1-p)^9 + \binom{10}{2} p^2(1-p)^8 \approx 0.929\,802$$

$$\therefore (1-p)^{10} + 10p(1-p)^9 + 45p^2(1-p)^8 \approx 0.929\,802$$

Using technology and the domain $0 \leqslant p \leqslant 1$, we find $p \approx 0.100$

$$\begin{aligned}\therefore \mathrm{P}(X < 2) &= \mathrm{P}(X \leqslant 1) \\ &\approx 0.736\end{aligned}$$

13 **a** Let X be the number of customers arriving at the shop in a 15 minute period.

$X \sim \mathrm{Po}(20) \quad \therefore \mathrm{P}(X = x) = \dfrac{20^x e^{-20}}{x!}$, where $x = 0, 1, 2, 3, ...$

$$\mathrm{P}(X = 15) = \frac{20^{15} e^{-20}}{15!} \approx 0.0516$$

b Let Y be the number of customers arriving at the shop in a 10 minute period.

$\therefore$ Y has mean $m = \frac{10}{15} \times 20 = \frac{40}{3}$

$\therefore$ $Y \sim \mathrm{Po}\left(\frac{40}{3}\right)$

$$\begin{aligned}\mathrm{P}(Y > 10) &= 1 - \mathrm{P}(Y \leqslant 10) \\ &\approx 1 - 0.224 \\ &\approx 0.776 < 0.8\end{aligned}$$

$\therefore$ the probability that more than 10 customers will arrive at the shop in a 10 minute period is *not* greater than 80%.

$\therefore$ the manager will not hire an extra shop assistant.

Chapter 26

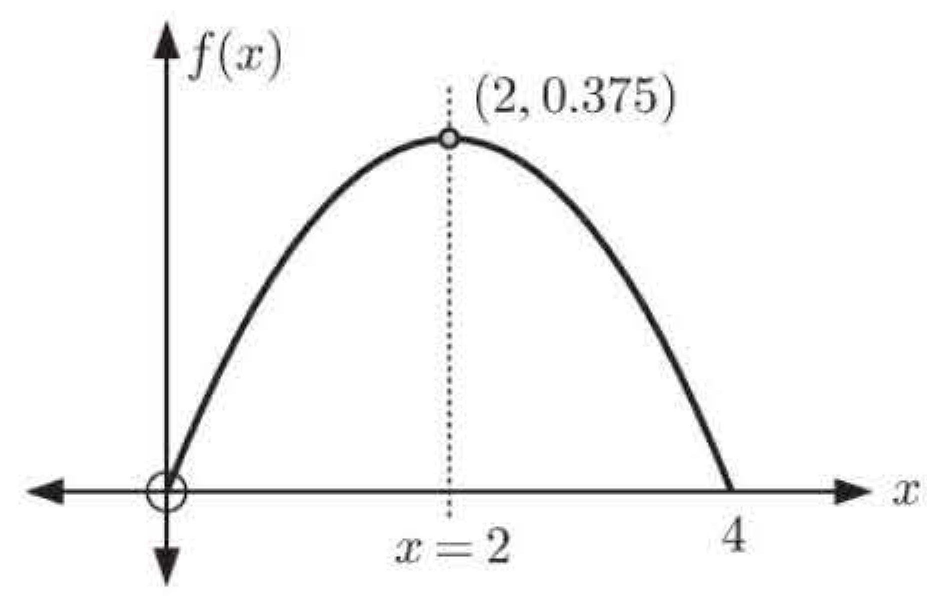
CONTINUOUS RANDOM VARIABLES

EXERCISE 26A

1 **a**

$$\int_0^4 ax(x-4)\,dx = 1$$
$$\therefore\ a\int_0^4 (x^2-4x)\,dx = 1$$
$$\therefore\ a\left[\frac{x^3}{3}-\frac{4x^2}{2}\right]_0^4 = 1$$
$$\therefore\ a\left(\tfrac{64}{3}-32\right) = 1$$
$$\therefore\ a\left(\tfrac{-32}{3}\right) = 1$$
$$\therefore\ a = -\tfrac{3}{32}$$

b $f(x) = -\frac{3}{32}x(x-4),\quad 0 \leqslant x \leqslant 4$



c **i**

$$\begin{aligned}\mu &= \int_0^4 x\,f(x)\,dx\\ &= \int_0^4 -\tfrac{3}{32}x^2(x-4)\,dx\\ &= -\tfrac{3}{32}\int_0^4 (x^3-4x^2)\,dx\\ &= -\tfrac{3}{32}\left[\tfrac{1}{4}x^4-\tfrac{4}{3}x^3\right]_0^4\\ &= -\tfrac{3}{32}\left(\tfrac{1}{4}(4)^4-\tfrac{4}{3}(4)^3\right)\\ &= -\tfrac{3}{32}\left(4^3-\tfrac{4}{3}\times 4^3\right)\\ &= -\tfrac{3}{32}\left(-\tfrac{64}{3}\right)\\ &= 2\end{aligned}$$

ii mode $= 2$ {symmetry of graph}

iii If $\int_0^m -\frac{3}{32}x(x-4)\,dx = \frac{1}{2}$

then $\int_0^m (x^2-4x)\,dx = -\frac{16}{3}$

$$\therefore\ \left[\frac{x^3}{3}-\frac{4x^2}{2}\right]_0^m = -\tfrac{16}{3}$$
$$\therefore\ \frac{m^3}{3}-2m^2-0 = -\tfrac{16}{3}$$
$$\therefore\ m^3-6m^2 = -16$$
$$\therefore\ m^3-6m^2+16 = 0$$
$$\therefore\ (m-2)(m^2-4m-8) = 0$$
$$\therefore\ m = 2 \text{ or } \frac{4\pm\sqrt{16+32}}{2}$$
$$\therefore\ m = 2 \text{ or } 2\pm 2\sqrt{3}$$
$$\therefore\ m = 2 \quad \{\text{as } 0 < m < 4\}$$

$\therefore$ the median is 2

iv

$$\begin{aligned}&\int_0^4 x^2\,f(x)\,dx\\ &= \int_0^4 -\tfrac{3}{32}x^3(x-4)\,dx\\ &= -\tfrac{3}{32}\int_0^4 (x^4-4x^3)\,dx\\ &= -\tfrac{3}{32}\left[\tfrac{1}{5}x^5-x^4\right]_0^4\\ &= -\tfrac{3}{32}\left(\tfrac{4}{5}(4)^4-4^4\right)\\ &= -\tfrac{3}{32}\left(-\tfrac{256}{5}\right) = \tfrac{24}{5}\end{aligned}$$

$$\therefore\ \operatorname{Var}(X) = \tfrac{24}{5}-2^2 = \tfrac{4}{5} = 0.8$$

2 **a**

$$\int_0^b -0.2x(x-b)\,dx = 1$$
$$\therefore\ -0.2\int_0^b (x^2-bx)\,dx = 1$$
$$\therefore\ \left[\tfrac{1}{3}x^3-\tfrac{1}{2}bx^2\right]_0^b = -5$$
$$\therefore\ \tfrac{1}{3}b^3-\tfrac{1}{2}b^3-0 = -5$$
$$\therefore\ 2b^3-3b^3 = -30$$
$$\therefore\ -b^3 = -30$$
$$\therefore\ b^3 = 30$$
$$\therefore\ b = \sqrt[3]{30}$$

b **i**

$$\begin{aligned}\mu &= \int_0^{\sqrt[3]{30}} -0.2x^2(x-\sqrt[3]{30})\,dx\\ &\approx 1.5536 \quad \{\text{using technology}\}\\ &\approx 1.55\end{aligned}$$

ii

$$\begin{aligned}&\int_0^{\sqrt[3]{30}} x^2\,f(x)\,dx\\ &= \int_0^{\sqrt[3]{30}} -0.2x^3(x-\sqrt[3]{30})\,dx\\ &\approx 2.8965 \quad \{\text{using technology}\}\end{aligned}$$

$$\therefore\ \operatorname{Var}(X) \approx 2.8965-\mu^2 \approx 0.483$$

3 **a**

$$\int_0^3 ke^{-x}\,dx = 1$$
$$\therefore\ k\int_0^3 e^{-x}\,dx = 1$$
$$\therefore\ k\left[\frac{e^{-x}}{-1}\right]_0^3 = 1$$
$$\therefore\ k(-e^{-3}-(-1)) = 1$$
$$\therefore\ k(1-e^{-3}) = 1$$
$$\therefore\ k \approx 1.0524$$

b If m is the median then

$$\int_0^m ke^{-x}\,dx = \tfrac{1}{2}$$
$$\therefore\ \int_0^m e^{-x}\,dx = \frac{1}{2k}$$
$$\therefore\ \left[\frac{e^{-x}}{-1}\right]_0^m = \frac{1}{2k}$$
$$\therefore\ -e^{-m}-(-1) = \frac{1}{2k}$$
$$\therefore\ e^{-m} \approx 1-\frac{1}{2(1.0524)}$$
$$\therefore\ e^{-m} \approx 0.524\,89$$
$$\therefore\ -m \approx \ln(0.524\,89)$$
$$\therefore\ m \approx 0.645$$

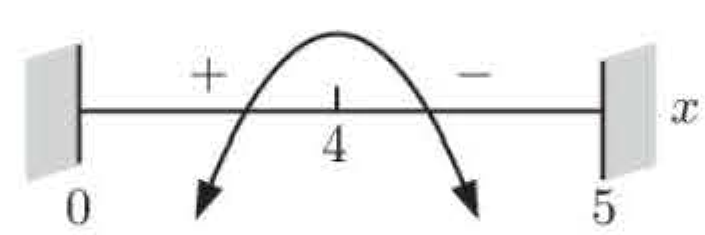
4 **a**

$$\int_0^5 kx^2(x-6)\,dx = 1$$
$$\therefore\ k\int_0^5 (x^3-6x^2)\,dx = 1$$
$$\therefore\ k\left[\tfrac{1}{4}x^4-\tfrac{6}{3}x^3\right]_0^5 = 1$$
$$\therefore\ k\left(\tfrac{625}{4}-250\right) = 1$$
$$\therefore\ k\left(\tfrac{-375}{4}\right) = 1$$
$$\therefore\ k = -\tfrac{4}{375}$$

b

$$f(x) = -\tfrac{4}{375}x^2(x-6)$$
$$= -\tfrac{4}{375}(x^3-6x^2)$$
$$\therefore\ f'(x) = -\tfrac{4}{375}(3x^2-12x)$$
$$\therefore\ f'(x) = 0 \quad \text{when} \quad 3x(x-4) = 0$$
$$\therefore\ x = 0 \text{ or } 4$$

$f'(x)$ has sign diagram:



There is a maximum when $x = 4$, so the mode is 4.

c If m is the median,

then $\int_0^m -\tfrac{4}{375}\,x^2(x-6)\,dx = \tfrac{1}{2}$

$$\therefore\ \int_0^m (x^3-6x^2)\,dx = -\tfrac{375}{8}$$
$$\therefore\ \left[\tfrac{1}{4}x^4-\tfrac{6}{3}x^3\right]_0^m = -\tfrac{375}{8}$$
$$\therefore\ \tfrac{1}{4}m^4-2m^3 = -\tfrac{375}{8}$$
$$\therefore\ 2m^4-16m^3+375 = 0$$

Using technology, $m \approx 3.46$

d $\mu = \int_0^5 x\,f(x)\,dx$

$$= \int_0^5 -\tfrac{4}{375}\,x^3(x-6)\,dx$$
$$= 3\tfrac{1}{3} \quad \{\text{using technology}\}$$

e

$$\mathrm{E}(X^2) = \int_0^5 x^2\,f(x)\,dx$$
$$= \int_0^5 -\tfrac{4}{375}\,x^4(x-6)\,dx$$
$$= 12\tfrac{2}{9} \quad \{\text{using technology}\}$$
$$\therefore\ \mathrm{Var}(X) = 12\tfrac{2}{9}-\left(3\tfrac{1}{3}\right)^2 = 1\tfrac{1}{9}$$

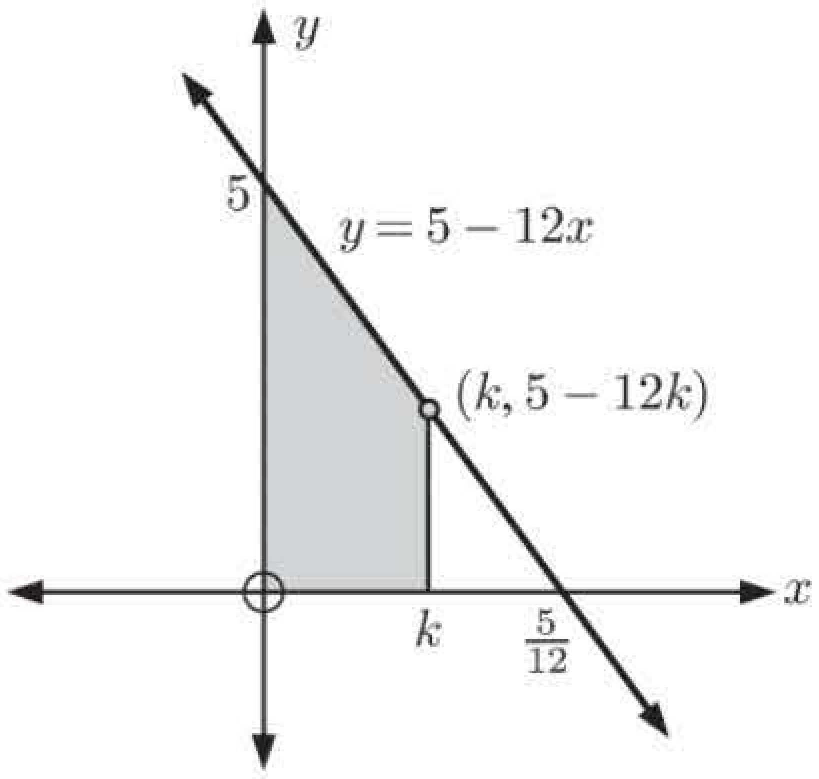
5 **a** Y is a continuous random variable if $5-12y \geqslant 0$ for all $0 \leqslant y \leqslant k$ and $\int_0^k (5-12y)\,dy = 1$.

Since $f(y) = 5-12y$ is a decreasing function, $f(k) = 5-12k$ is the smallest value of $f(y)$ on $0 \leqslant y \leqslant k$.

$$\therefore\ 5-12k \geqslant 0$$
$$\therefore\ 12k \leqslant 5$$
$$\therefore\ k \leqslant \tfrac{5}{12}$$

So, $k \leqslant \tfrac{5}{12}$ and $\int_0^k (5-12y)\,dy = 1$

b



$\int_0^k (5-12y)\,dy = \text{shaded area} = 1$

$\therefore\ k \times \left(\dfrac{5+(5-12k)}{2}\right) = 1$

$\therefore\ k(5-6k) = 1$

$\therefore\ 5k - 6k^2 = 1$

$\therefore\ 6k^2 - 5k + 1 = 0$

$\therefore\ (3k-1)(2k-1) = 0$

$\therefore\ k = \frac{1}{3}$ or $\frac{1}{2}$

But $k \leqslant \frac{5}{12}$, so $k = \frac{1}{3}$

c If $k = \frac{1}{2}$, the graph $f(y) = 5 - 12y$ falls below the horizontal axis.

d $\mu = \int_0^{\frac{1}{3}} y\,f(y)\,dy$

$= \int_0^{\frac{1}{3}} (5y - 12y^2)\,dy$

$= \left[\frac{5}{2}y^2 - 4y^3\right]_0^{\frac{1}{3}}$

$= \frac{5}{2}\left(\frac{1}{9}\right) - 4\left(\frac{1}{27}\right)$

$= \frac{7}{54}$

If $\int_0^m (5-12y)\,dy = \frac{1}{2}$

then $\left[5y - 6y^2\right]_0^m = \frac{1}{2}$

$\therefore\ 5m - 6m^2 = \frac{1}{2}$

$\therefore\ 12m^2 - 10m + 1 = 0$

$\therefore\ m = \frac{5 \pm \sqrt{13}}{12}$

But $m < \frac{5}{12}$, so $m = \frac{5-\sqrt{13}}{12} \approx 0.116$

$\therefore$ the median ≈ 0.116

6 **a** $\int_a^b k\,dx = 1$

$\therefore\ [kx]_a^b = 1$

$\therefore\ bk - ak = 1$

$\therefore\ k = \dfrac{1}{b-a}$

b $\mu = \int_a^b kx\,dx$

$= \dfrac{1}{b-a}\left[\frac{1}{2}x^2\right]_a^b$

$= \dfrac{1}{b-a}\left(\frac{1}{2}b^2 - \frac{1}{2}a^2\right)$

$= \frac{1}{2}\dfrac{(b-a)(b+a)}{(b-a)}$

$\therefore$ mean $= \dfrac{a+b}{2}$

If $\int_a^m k\,dx = \frac{1}{2}$ then $[kx]_a^m = \frac{1}{2}$

$\therefore\ \dfrac{m}{b-a} - \dfrac{a}{b-a} = \frac{1}{2}$

$\therefore\ m - a = \dfrac{b-a}{2}$

$\therefore\ m = a + \dfrac{b-a}{2}$

$= \dfrac{a+b}{2}$

$\therefore$ median $= \dfrac{a+b}{2}$

The mode is undefined as the function is constant for all $a \leqslant x \leqslant b$.

c $\int_a^b kx^2\,dx = \dfrac{1}{b-a}\left[\dfrac{x^3}{3}\right]_a^b$

$= \dfrac{1}{b-a}\left(\dfrac{b^3}{3} - \dfrac{a^3}{3}\right)$

$= \frac{1}{3}\,\dfrac{b^3 - a^3}{b-a}$

$= \frac{1}{3}\,\dfrac{(b-a)(b^2+ab+a^2)}{(b-a)}$

$= \dfrac{a^2+ab+b^2}{3}$

$\therefore\ \text{Var}(X) = \dfrac{a^2+ab+b^2}{3} - \left(\dfrac{a+b}{2}\right)^2$

$= \dfrac{4(a^2+ab+b^2) - 3(a+b)^2}{12}$

$= \dfrac{4a^2 + 4ab + 4b^2 - 3a^2 - 6ab - 3b^2}{12}$

$= \dfrac{a^2 - 2ab + b^2}{12}$

$= \dfrac{(a-b)^2}{12}$

$\therefore\ \sigma_X = \sqrt{\dfrac{(a-b)^2}{12}} = \dfrac{b-a}{\sqrt{12}}$ {as $b > a$}

7 **a** If $\int_0^m 2e^{-2x}\,dx = \frac{1}{2}$

then $\left[-e^{-2x}\right]_0^m = \frac{1}{2}$

$\therefore\ -e^{-2m} - (-e^0) = \frac{1}{2}$

$\therefore\ \frac{1}{2} = e^{-2m}$

$\therefore\ -2m = \ln\frac{1}{2}$

$\therefore\ m = -\frac{1}{2}\ln\frac{1}{2} \approx 0.347$

b $f(x) = 2e^{-2x}$

$\therefore\ f'(x) = -4e^{-2x}$

$\therefore\ f'(x) < 0$ for all $x \geqslant 0$ $\{e^{-2x} > 0\}$

$\therefore\ f(x)$ is always decreasing for $x \geqslant 0$

$\therefore$ the mode $= 0$

8 **a** $\int_0^a 6\cos 3x\,dx = 1$

$\therefore\ [2\sin 3x]_0^a = 1$

$\therefore\ 2\sin 3a - 2\sin 0 = 1$

$\therefore\ \sin 3a = \frac{1}{2}$

$\therefore\ 3a = \frac{\pi}{6}$

$\therefore\ a = \frac{\pi}{18}$

b $\mu = \int_0^{\frac{\pi}{18}} 6x\cos 3x\,dx$

We integrate by parts with $u = 6x$ $v' = \cos 3x$

$u' = 6$ $v = \frac{1}{3}\sin 3x$

$\therefore\ \int 6x\cos 3x\,dx = 2x\sin 3x - \int 2\sin 3x\,dx$

$= 2x\sin 3x + \frac{2}{3}\cos 3x + c$

$\therefore\ \mu = \int_0^{\frac{\pi}{18}} 6x\cos 3x\,dx$

$= \left[2x\sin 3x + \frac{2}{3}\cos 3x\right]_0^{\frac{\pi}{18}}$

$= \left(\frac{\pi}{9}\sin\frac{\pi}{6} + \frac{2}{3}\cos\frac{\pi}{6}\right) - \left(0 + \frac{2}{3}\cos 0\right)$

$= \frac{\pi}{9}\left(\frac{1}{2}\right) + \frac{2}{3}\left(\frac{\sqrt{3}}{2}\right) - \frac{2}{3}$

$= \frac{\pi}{18} + \frac{\sqrt{3}-2}{3}$

≈ 0.0852

c If k is the 20th percentile of X,

then $\int_0^k 6\cos 3x\,dx = 0.2$

$\therefore\ [2\sin 3x]_0^k = 0.2$

$\therefore\ 2\sin 3k - 2\sin 0 = 0.2$

$\therefore\ \sin 3k = 0.1$

$\therefore\ 3k \approx 0.100$

$\therefore\ k \approx 0.0334$

So, the 20th percentile of $X \approx 0.0334$

d $\mathrm{E}(X^2) = \int_0^{\frac{\pi}{18}} 6x^2\cos 3x\,dx$

$\approx 0.009\,773$ {using technology}

$\therefore\ \mathrm{Var}(X) \approx 0.009\,773 - (0.0852)^2$

$\approx 0.002\,511$

$\therefore\ \sigma_X \approx \sqrt{0.002\,511}$

≈ 0.0501

9 $\mathrm{P}\left(X \leqslant \frac{2}{3}\right) = \frac{1}{243}$

$\therefore\ \int_0^{\frac{2}{3}} ax^4\,dx = \frac{1}{243}$

$\therefore\ \left[\frac{1}{5}ax^5\right]_0^{\frac{2}{3}} = \frac{1}{243}$

$\therefore\ \frac{1}{5}a \times \left(\frac{2}{3}\right)^5 = \frac{1}{243}$

$\therefore\ \frac{1}{5}a \times \frac{32}{243} = \frac{1}{243}$

$\therefore\ a = \frac{5}{32}$

So, $\int_0^k \frac{5}{32}x^4\,dx = 1$

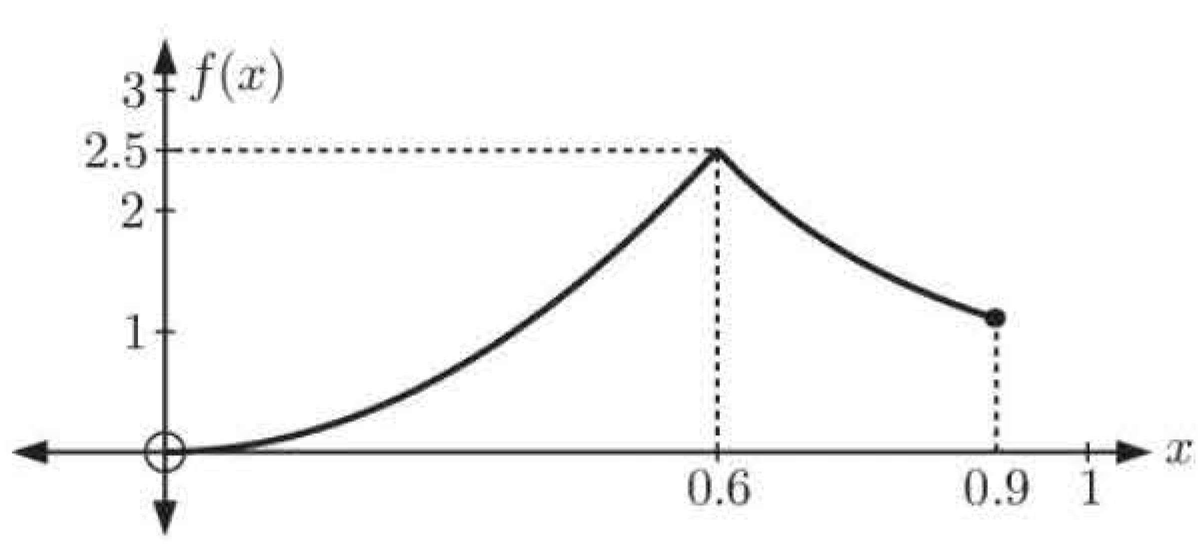
$\therefore\ \left[\frac{1}{32}x^5\right]_0^k = 1$

$\therefore\ \frac{1}{32}k^5 = 1$

$\therefore\ k^5 = 32$

$\therefore\ k = 2$

10 **a**



b From the graph in **a**, $f(x) \geqslant 0$ for all $x \in [0, 0.9]$.

Also, the area under the curve $= \displaystyle\int_0^{0.6} \tfrac{125}{18}x^2\,dx + \int_{0.6}^{0.9} \frac{9}{10x^2}\,dx$

$$= \left[\tfrac{125}{54}x^3\right]_0^{0.6} + \left[-\frac{9}{10x}\right]_{0.6}^{0.9}$$
$$= \tfrac{125}{54}\left(\tfrac{3}{5}\right)^3 + \left(-\tfrac{9}{9}\right) - \left(-\tfrac{9}{6}\right)$$
$$= \tfrac{1}{2} - 1 + \tfrac{3}{2}$$
$$= 1 \quad \text{as required}$$

c $\mu = \int_0^{0.9} x\,f(x)\,dx$
$$= \int_0^{0.6} \tfrac{125}{18}\,x^3\,dx + \int_{0.6}^{0.9} \frac{9}{10x}\,dx$$
$$= \left[\tfrac{125}{72}x^4\right]_0^{0.6} + \left[\tfrac{9}{10}\ln x\right]_{0.6}^{0.9}$$
$$= \tfrac{125}{72}(0.6)^4 + \tfrac{9}{10}\ln(0.9) - \tfrac{9}{10}\ln(0.6)$$
$$\approx 0.590$$

From the calculations in **b**,

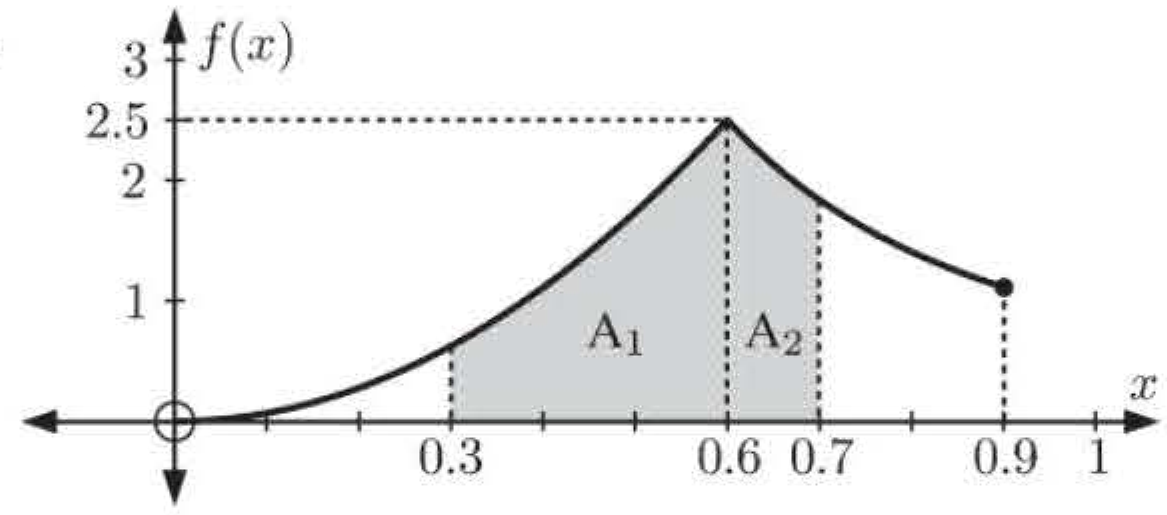
$$\int_0^{0.6} f(x)\,dx = \int_{0.6}^{0.9} f(x)\,dx = \tfrac{1}{2}$$
$\therefore$ median $= 0.6$

From the graph in **a**, the highest value of $f(x)$ occurs at $x = 0.6$

$\therefore$ mode $= 0.6$

d
$$\mathrm{E}(X^2) = \int_0^{0.9} x^2\,f(x)\,dx$$
$$= \int_0^{0.6} \tfrac{125}{18}x^4\,dx + \int_{0.6}^{0.9} \tfrac{9}{10}\,dx$$
$$= \left[\tfrac{25}{18}x^5\right]_0^{0.6} + \left[\tfrac{9}{10}x\right]_{0.6}^{0.9}$$
$$= \tfrac{25}{18}\left(\tfrac{3}{5}\right)^5 + \tfrac{9}{10}\left(\tfrac{9}{10}\right) - \tfrac{9}{10}\left(\tfrac{3}{5}\right)$$
$$= \tfrac{189}{500}$$
$$\therefore\ \mathrm{Var}(X) \approx \tfrac{189}{500} - 0.589\,92^2$$
$$\approx 0.0300$$
$$\therefore\ \sigma_X \approx \sqrt{0.029\,994}$$
$$\approx 0.173$$

e

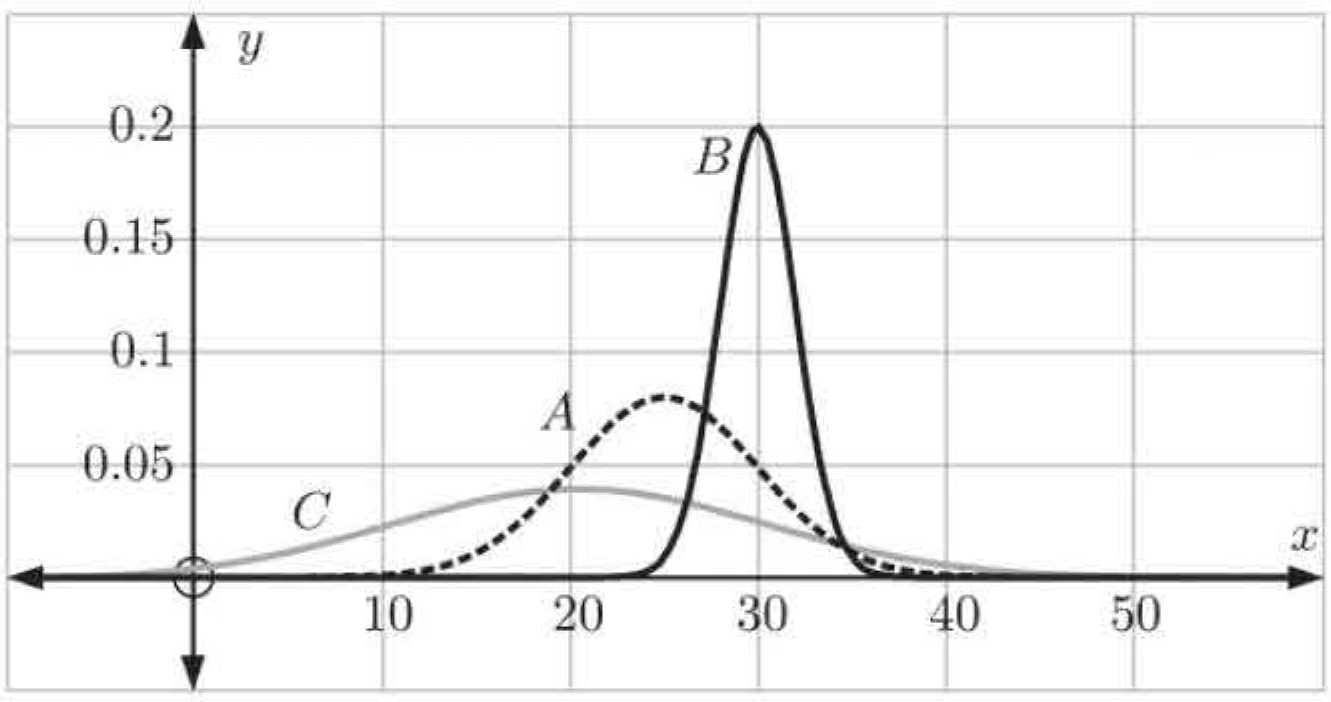


$\mathrm{P}(0.3 < X < 0.7)$
$$= \text{shaded area}$$
$$= \mathrm{A}_1 + \mathrm{A}_2$$
$$= \int_{0.3}^{0.6} \tfrac{125}{18}x^2\,dx + \int_{0.6}^{0.7} \frac{9}{10x^2}\,dx$$
$$= \left[\tfrac{125}{54}x^3\right]_{0.3}^{0.6} + \left[-\frac{9}{10x}\right]_{0.6}^{0.7}$$
$$= \tfrac{125}{54}(0.6^3 - 0.3^3) + \left(-\tfrac{9}{7}\right) - \left(-\tfrac{9}{6}\right)$$
$$\approx 0.652$$

$\therefore$ it takes between 0.3 hours ($=$ 18 minutes) and 0.7 hours ($=$ 42 minutes) to perform the task 65.2% of the time.

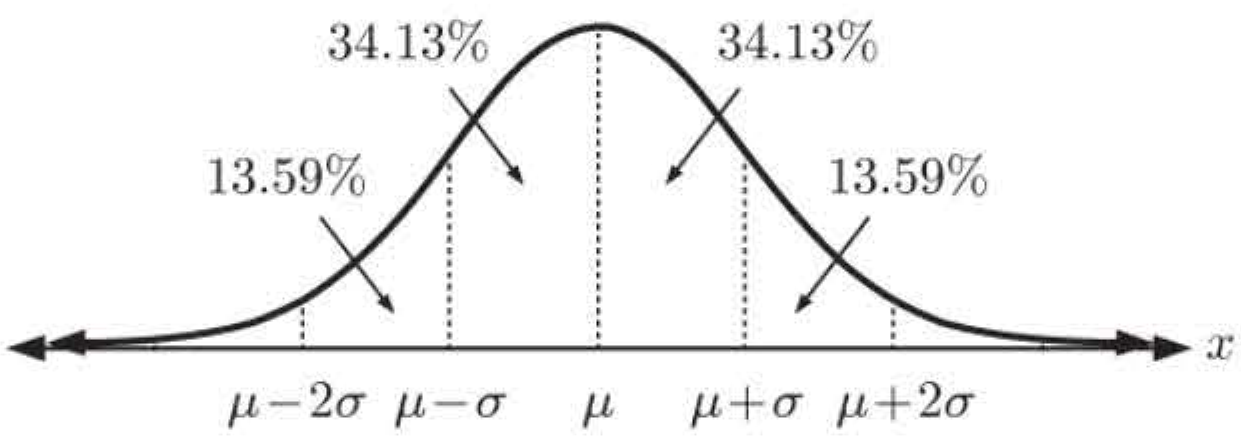
EXERCISE 26B

1



2 a, b The mean volume (or diameter) is likely to occur most often with variations around the mean occurring symmetrically as a result of random variations in the production process.

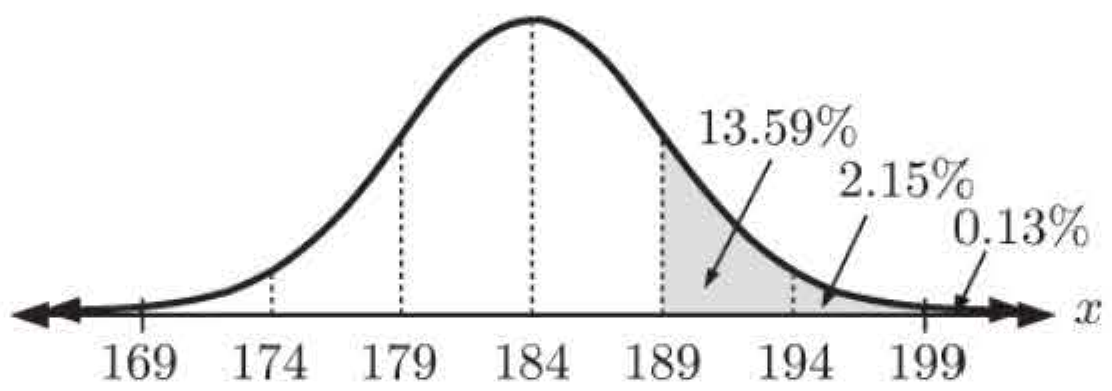
3



a P(value between $\mu - \sigma$ and $\mu + \sigma$)
$\approx 34.13\% + 34.13\%$
≈ 0.683

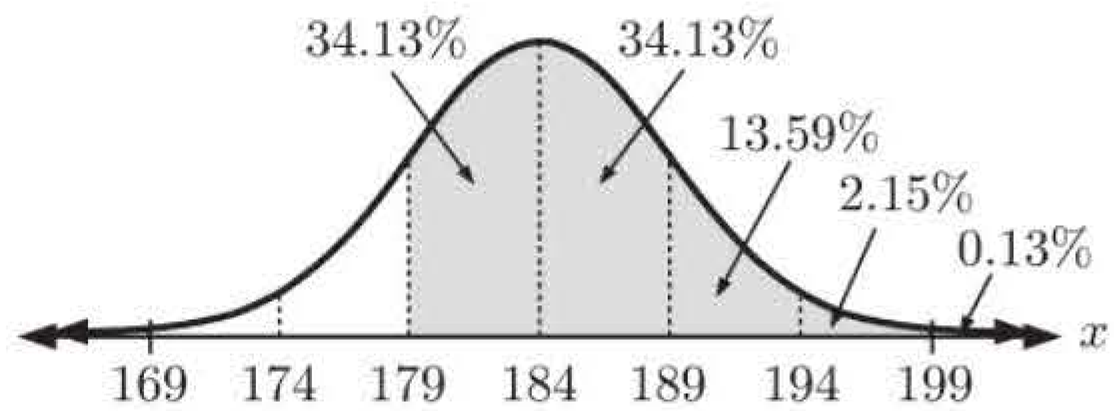
b P(value between μ and $\mu + 2\sigma$)
$\approx 34.13\% + 13.59\%$
≈ 0.477

4 a



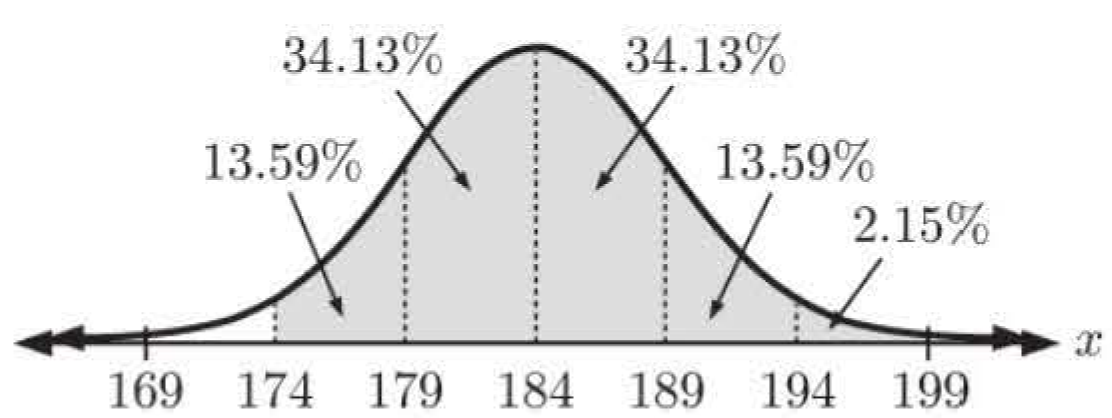
We need the percentage greater than 189 cm.
This is $13.59\% + 2.15\% + 0.13\%$
$\approx 15.9\%$

b



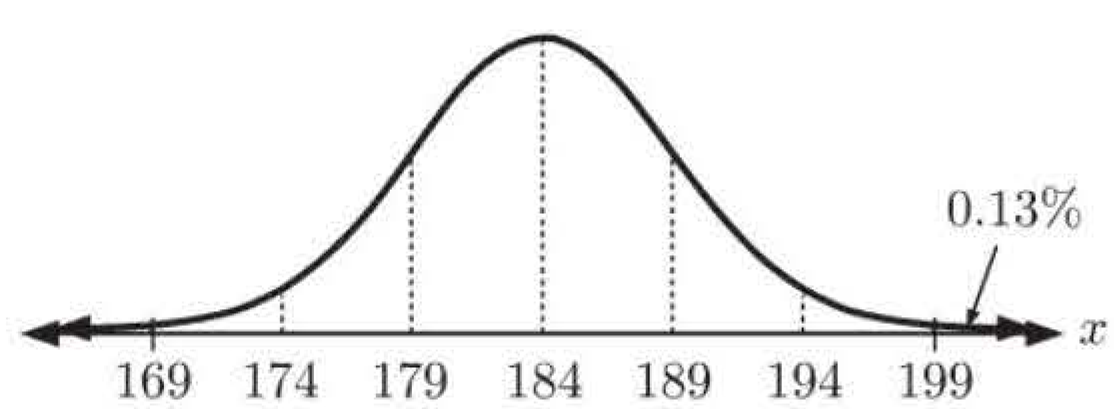
We need the percentage greater than 179 cm.
This is $34.13\% + 34.13\% + 13.59\%$
$+ 2.15\% + 0.13\%$
$\approx 84.1\%$

c



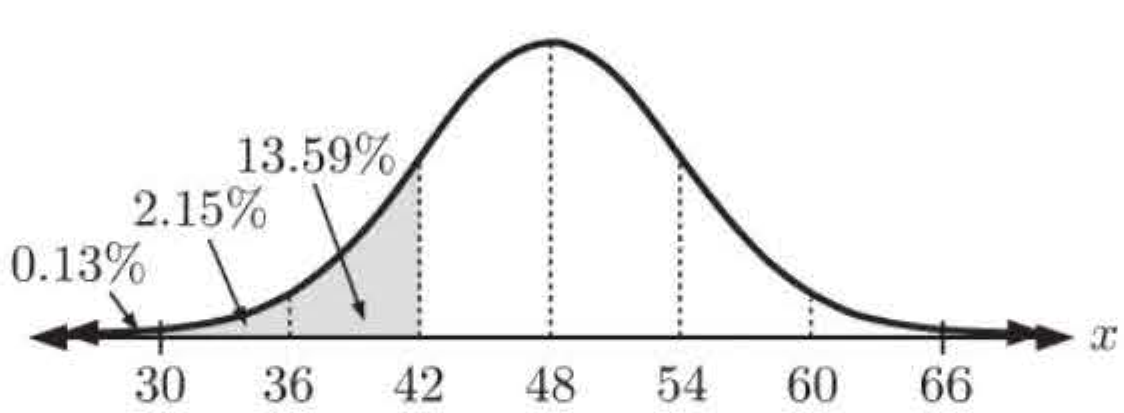
We need the percentage between 174 cm and 199 cm.
This is $13.59\% + 34.13\% + 34.13\%$
$+ 13.59\% + 2.15\%$
$\approx 97.6\%$

d



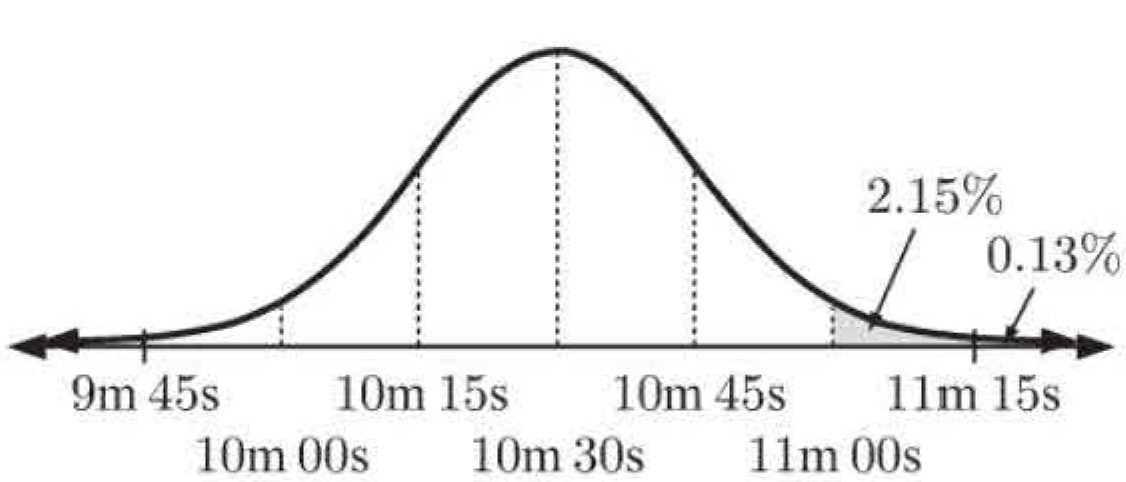
We need the percentage greater than 199 cm.
This is 0.13%.

5



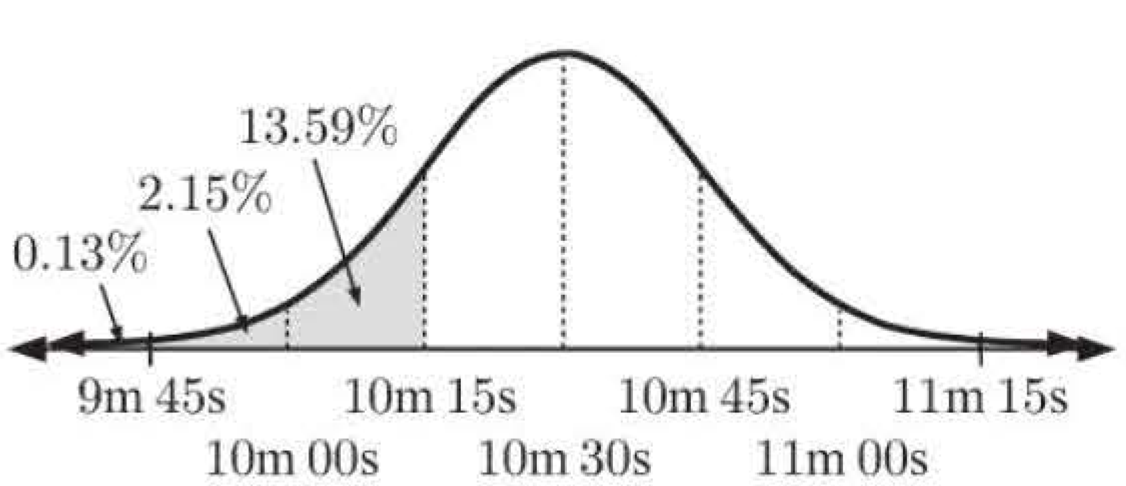
The chance of there being less than 42 mm of rain during August is
$0.13\% + 2.15\% + 13.59\% = 15.87\%$
and 15.87% of $20 = 3.174$
So, over a 20 year period, there would be less than 42 mm of rain during August three times.

6 a



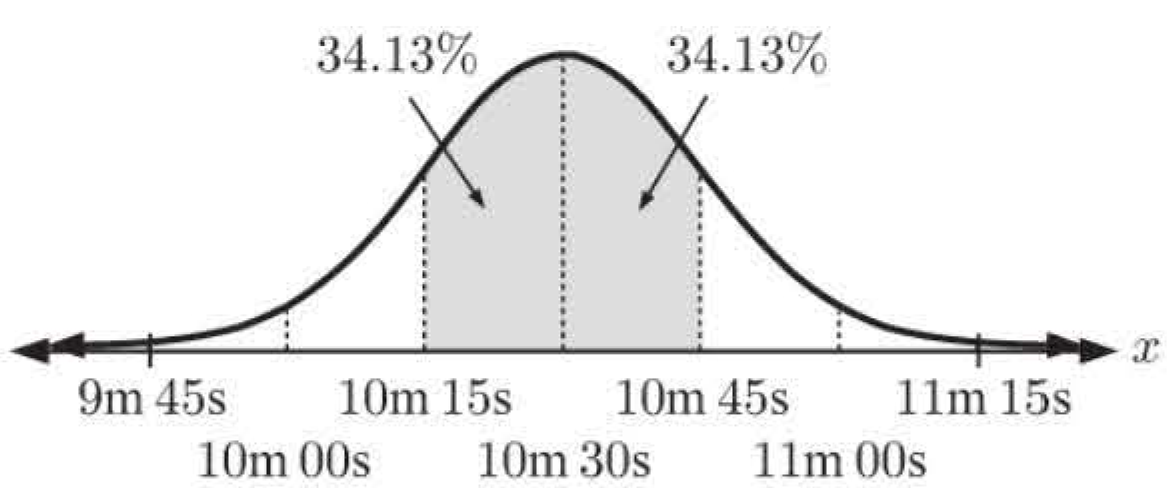
$2.15\% + 0.13\% = 2.28\%$ of competitors took over 11 minutes,
and 2.28% of $200 = 4.56$
So, 5 competitors took longer than 11 minutes.

b



$0.13\% + 2.15\% + 13.59\% = 15.87\%$ of competitors took less than 10 minutes 15 seconds,
and 15.87% of $200 = 31.74$
So, 32 competitors took less than 10 minutes 15 seconds.

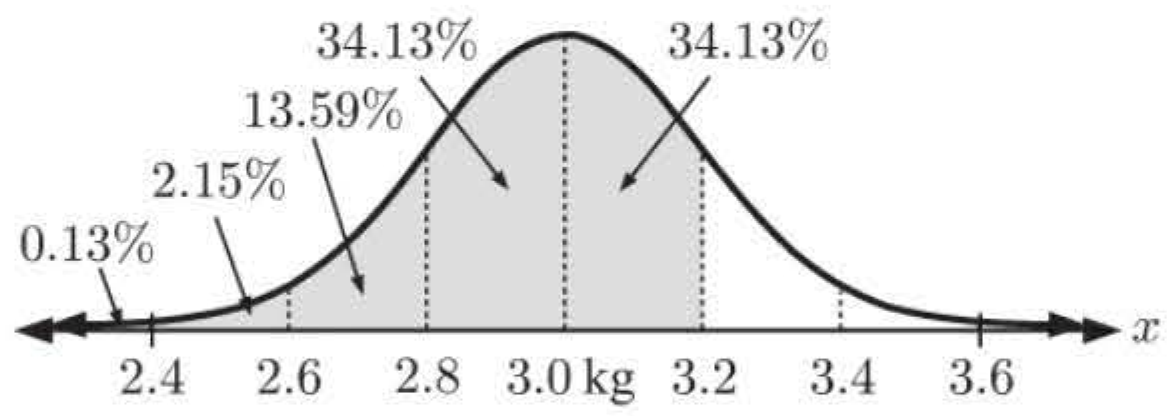
c



$34.13\% + 34.13\% = 68.26\%$ of competitors took between 10 minutes 15 seconds and 10 minutes 45 seconds, and 68.26% of $200 = 136.52$

So, 137 competitors took between 10 minutes 15 seconds and 10 minutes 45 seconds.

7 **a**

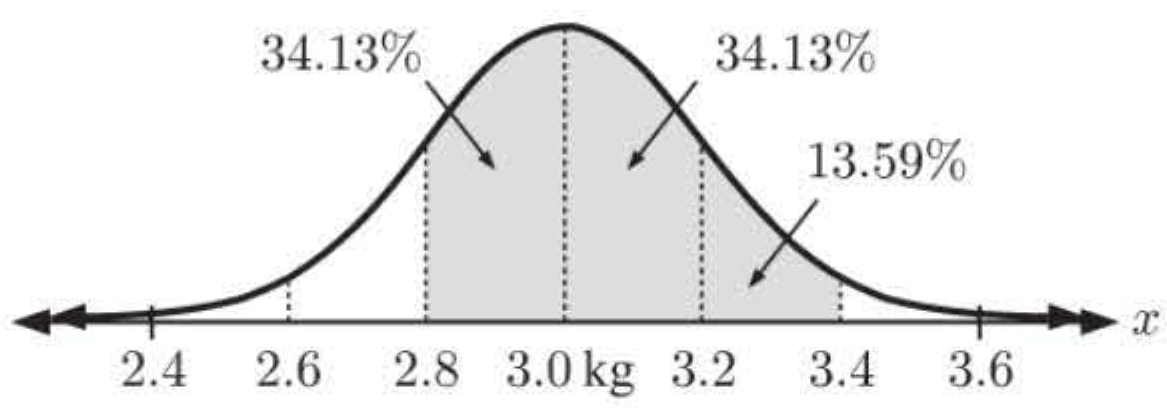


$0.13\% + 2.15\% + 13.59\% + 34.13\% + 34.13\% = 84.13\%$ of babies born weighed less than 3.2 kg, and

84.13% of $545 = 458.5085$

So, 459 babies born weighed less than 3.2 kg.

b

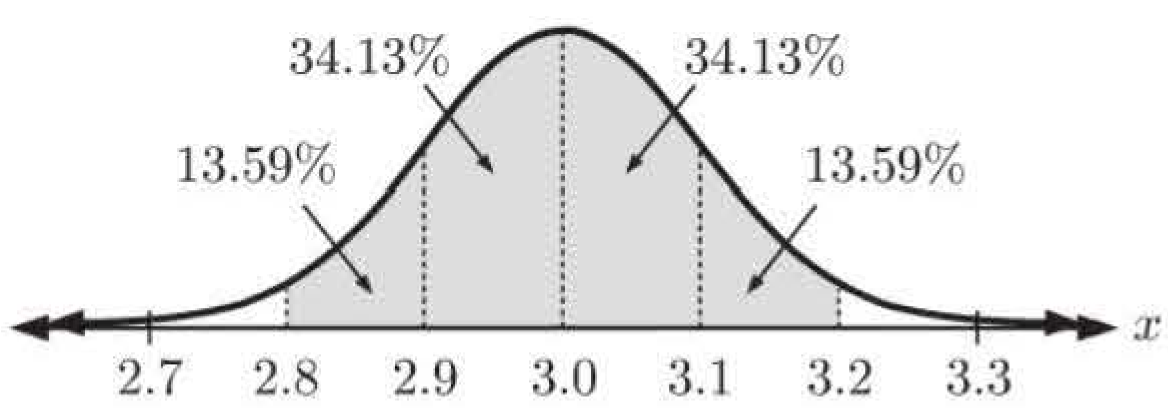


$34.13\% + 34.13\% + 13.59\% = 81.85\%$ of babies born weighed between 2.8 kg and 3.4 kg, and

81.85% of $545 = 446.0825$

So, 446 babies born weighed between 2.8 kg and 3.4 kg.

8



a P(value is within 2 standard deviations of the mean)

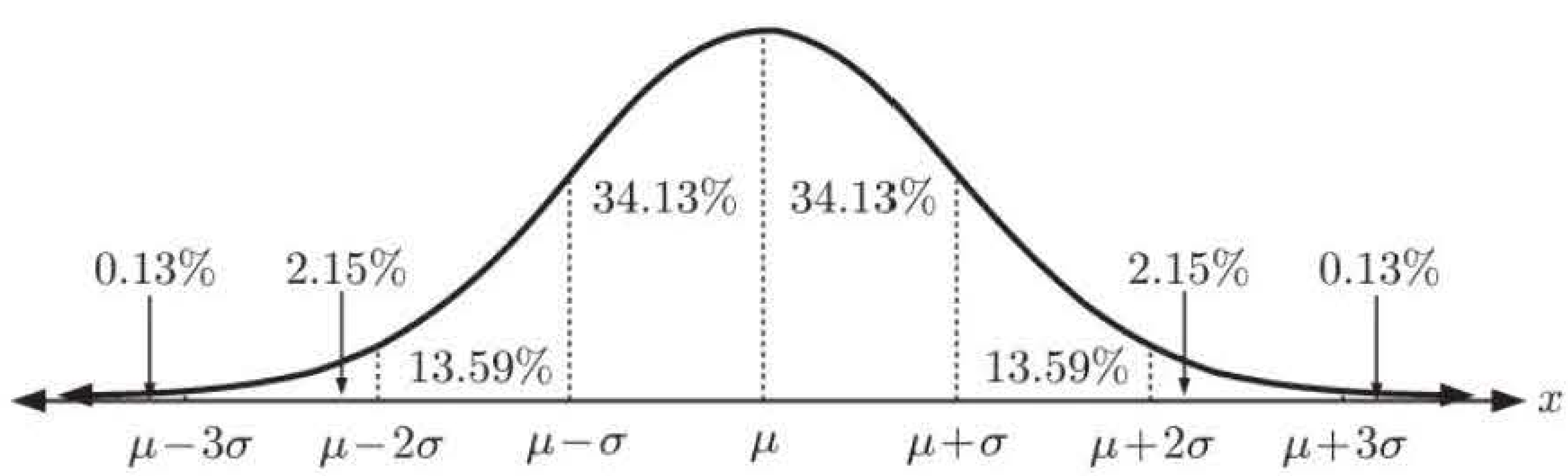
$= \text{P}(2.8 \leqslant X \leqslant 3.2)$

$\approx 13.59\% + 34.13\% + 34.13\% + 13.59\%$

≈ 0.954

b The value 1 standard deviation below the mean is $X = 3 - 0.1 = 2.9$

9 **a**



84% of the crop weigh more than 152 g $\quad \therefore \quad \mu - \sigma = 152$

16% of the crop weigh more than 200 g $\quad \therefore \quad \mu + \sigma = 200$ .... (1)

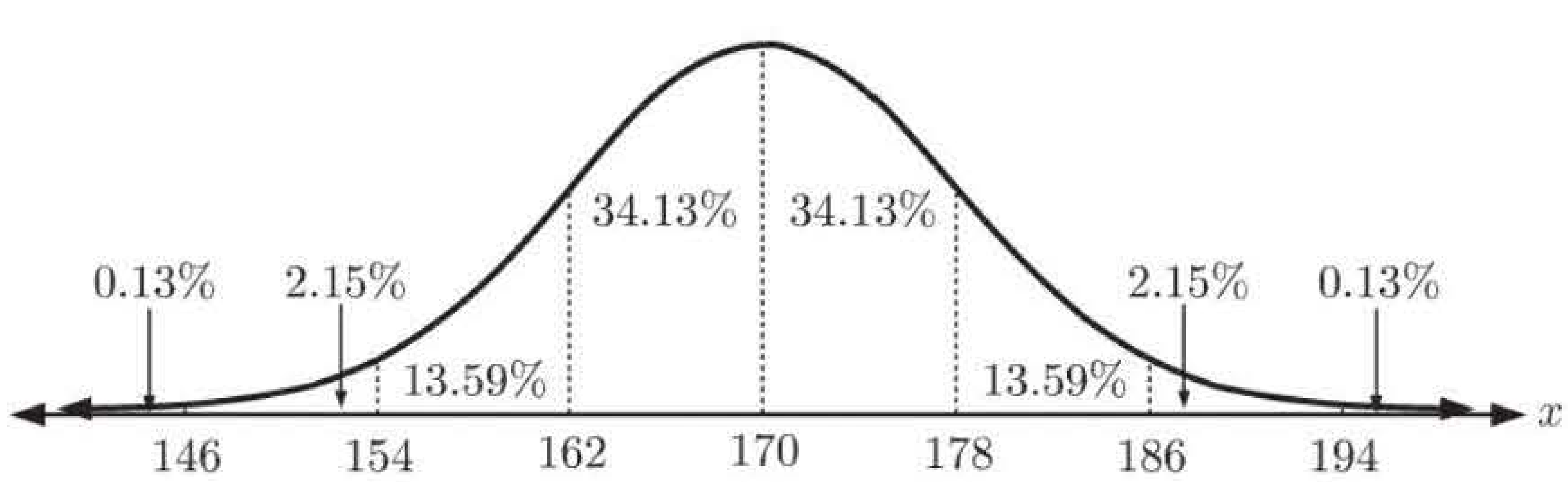
Adding: $\quad 2\mu \quad = 352$, and so $\mu = 176$ g

Substituting $\mu = 176$ into (1) gives $\sigma = 200 - \mu = 24$ g.

b For $\mu = 176$ g and $\sigma = 24$ g, $152 \text{ g} = \mu - \sigma$, and $224 \text{ g} = \mu + 2\sigma$.

$\therefore$ between 152 g and 224 g, the percentage is $34.13\% + 34.13\% + 13.59\% \approx 81.9\%$

10



a **i** $P(162 < X < 170) \approx 34.1\%$

ii $P(170 < X < 186) \approx 34.13\% + 13.59\%$
$\approx 47.7\%$

b **i** $P(178 < X < 186)$
$\approx 13.59\%$
≈ 0.136

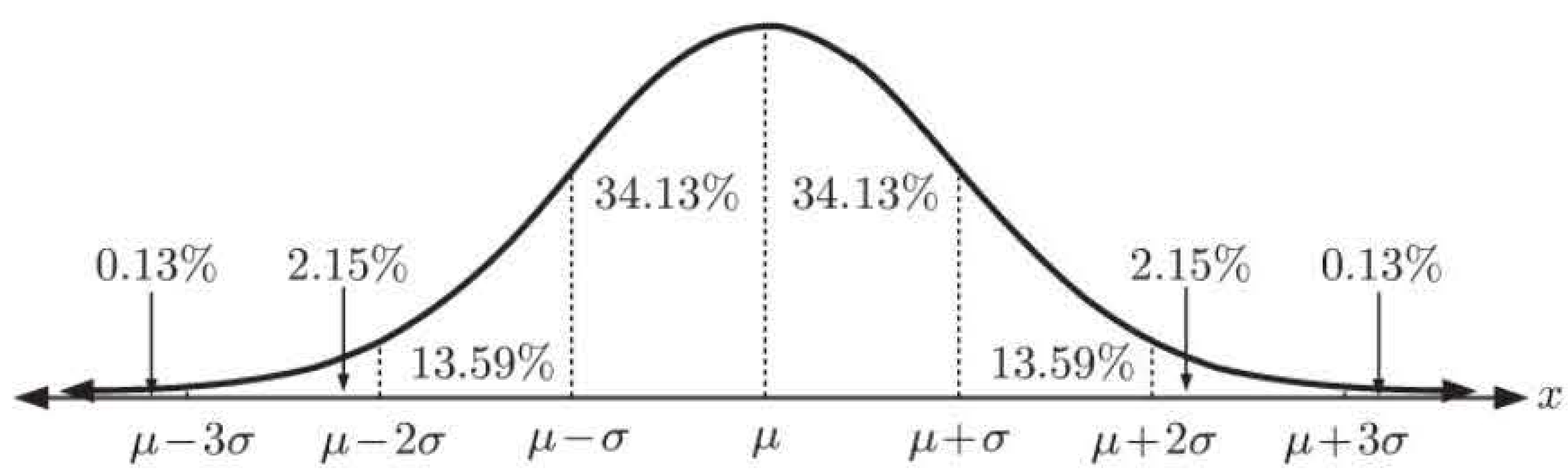
ii $P(X < 162)$
$\approx 1 - (0.5 + 0.3413)$
≈ 0.159

iii $P(X < 154)$
$\approx 0.0215 + 0.0013$
≈ 0.0228

iv $P(X > 162)$
$\approx 1 - 0.159$ {using **b ii**}
≈ 0.841

c 16% of students are taller than 178 cm $\{13.59\% + 2.15\% + 0.13\% \approx 16\%\}$
$\therefore\ k = 178$

11

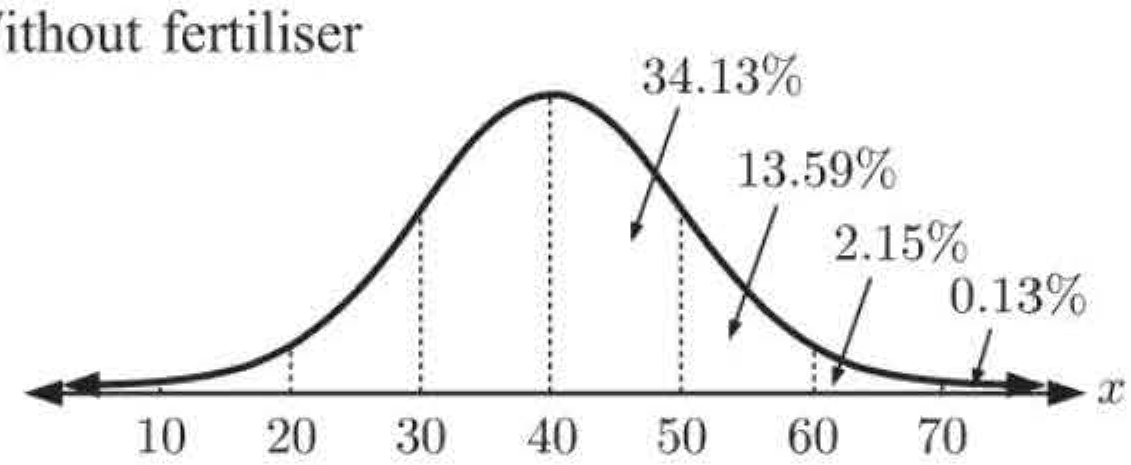


a 97.72% of 13 year old boys are taller than 131 cm $\therefore\ \mu - 2\sigma = 131$
2.28% of 13 year old boys are taller than 179 cm $\therefore\ \mu + 2\sigma = 179$ .... (1)
Adding: $2\mu = 310$, and so $\mu = 155$ cm

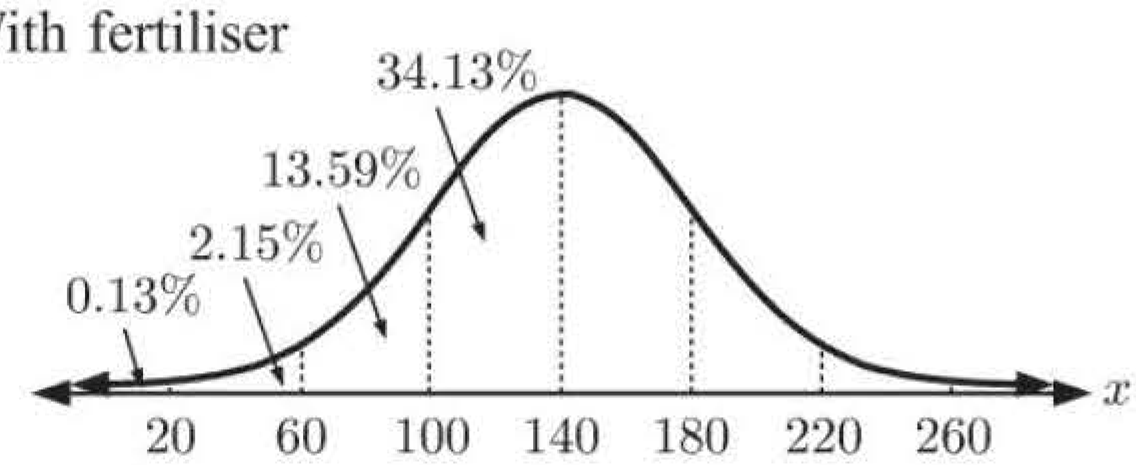
Substituting $\mu = 155$ cm into (1) gives $\sigma = \dfrac{179 - \mu}{2} = \dfrac{24}{2} = 12$ cm

b For $\mu = 155$ cm and $\sigma = 12$ cm, $143 \text{ cm} = \mu - \sigma$, and $191 \text{ cm} = \mu + 3\sigma$
$\therefore$ between 143 cm and 191 cm, the percentage is $34.13\% + 34.13\% + 13.59\% + 2.15\% \approx 84.0\%$
So the probability is 0.84.

12 Without fertiliser



With fertiliser



a P(without and < 50)
$\approx 50\% + 34.13\%$
$\approx 84.1\%$

b P(with and < 60)
$\approx 0.13\% + 2.15\%$
$\approx 2.28\%$

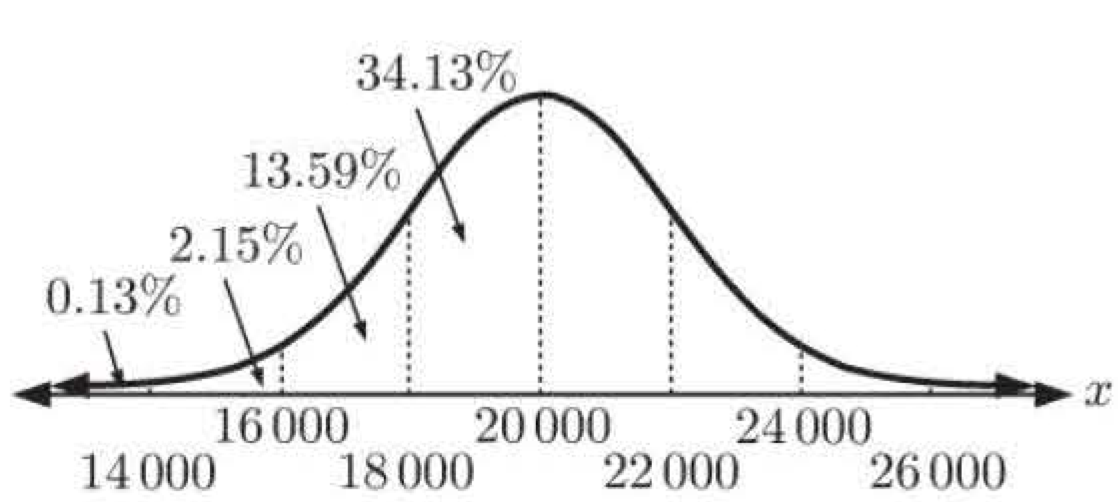
c **i** P(with and $20 \leqslant X \leqslant 60$)
$\approx 2.15\%$

ii P(without and $20 \leqslant X \leqslant 60$)
$\approx 2(34.13\% + 13.59\%)$
$\approx 95.4\%$

d **i** P(with and $X \geqslant 60$)
$\approx 13.59\% + 34.13\% + 50\%$
$\approx 97.7\%$

ii P(without and $X \geqslant 60$)
$\approx 2.15\% + 0.13\%$
$\approx 2.28\%$

13



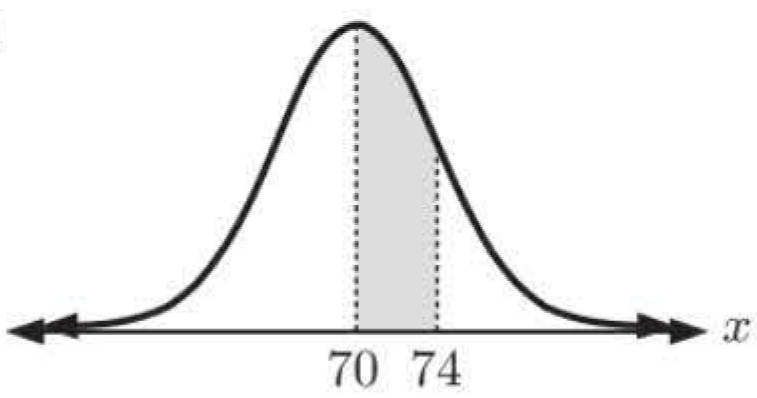
a $P(X < 18\,000)$
$\approx 1 - 0.5 - 0.3413$
≈ 0.1587
$\therefore$ we expect that less than 18 000 bottles are filled on $260 \times 0.1587 \approx 41$ days.

b $\text{P}(X > 16\,000)$
$\approx 0.1359 + 0.3413 + 0.5$
≈ 0.9772
$\therefore$ we expect that over 16 000 bottles are filled on $260 \times 0.9772 \approx 254$ days.

c $\text{P}(18\,000 \leqslant X \leqslant 24\,000)$
$\approx 0.3413 \times 2 + 0.1359$
≈ 0.8185
$\therefore$ we expect that between 18 000 and 24 000 bottles are filled on 260×0.8185
≈ 213 days.

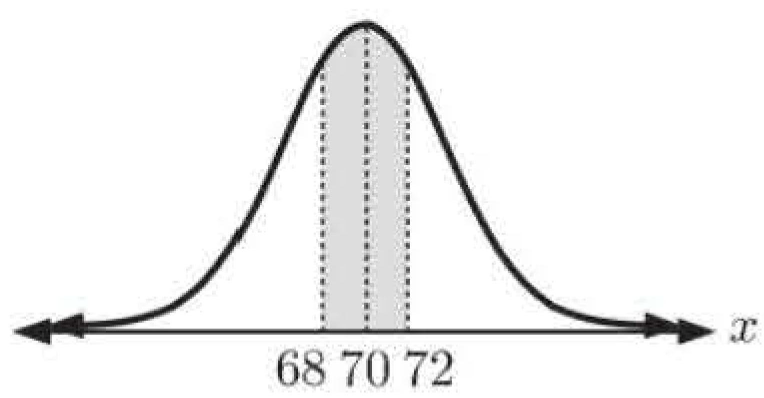
EXERCISE 26C

1 **a**



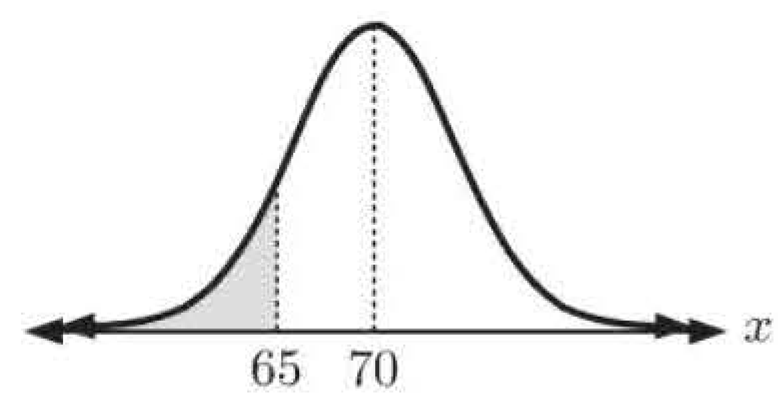
$\text{P}(70 \leqslant X \leqslant 74) \approx 0.341$

b



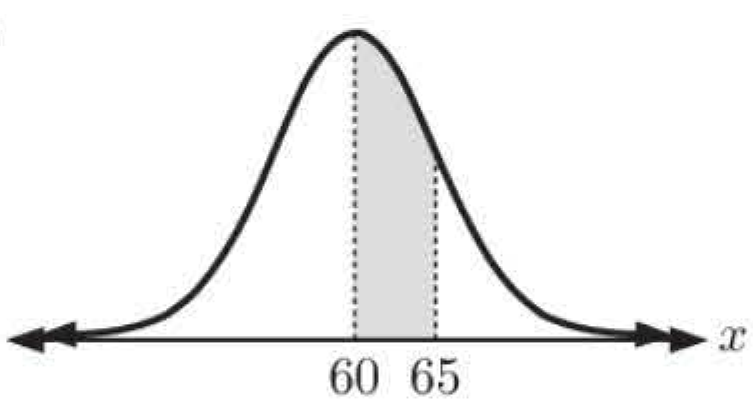
$\text{P}(68 \leqslant X \leqslant 72) \approx 0.383$

c



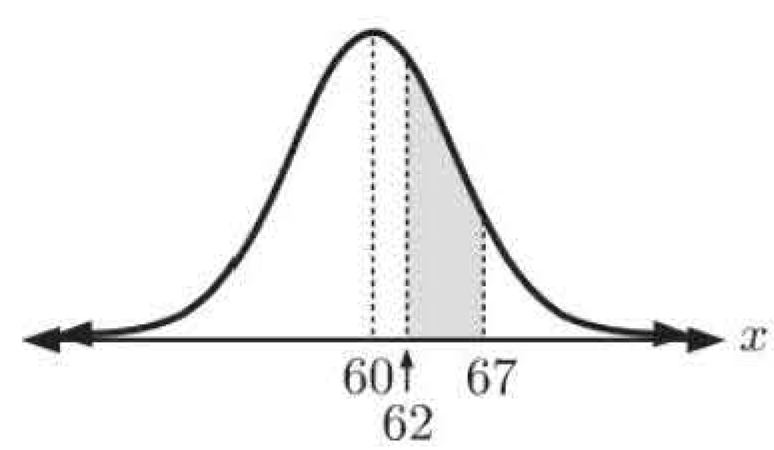
$\text{P}(X \leqslant 65) \approx 0.106$

2 **a**



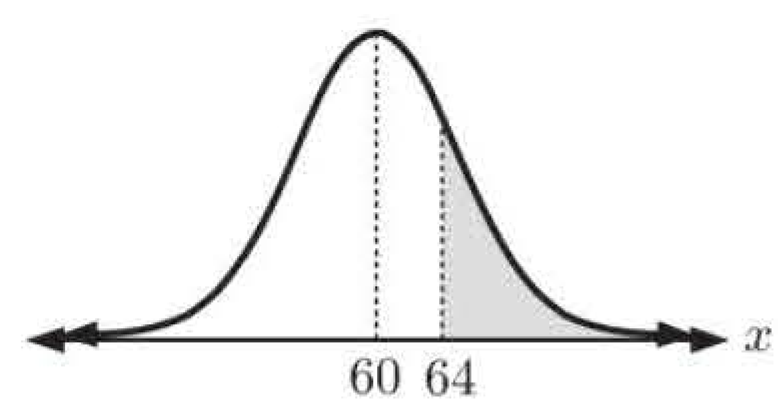
$\text{P}(60 \leqslant X \leqslant 65) \approx 0.341$

b



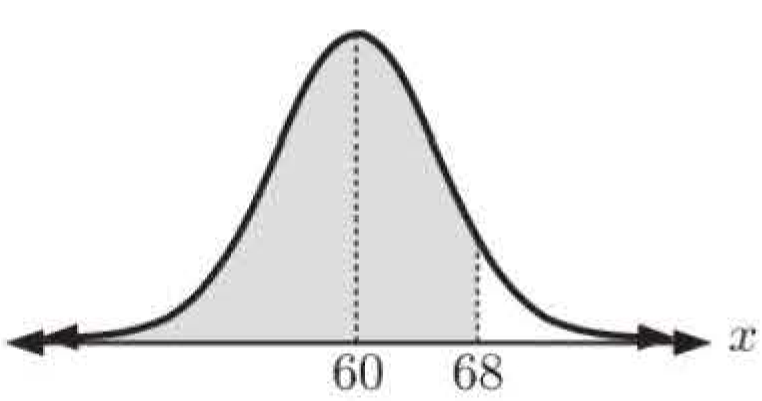
$\text{P}(62 \leqslant X \leqslant 67) \approx 0.264$

c



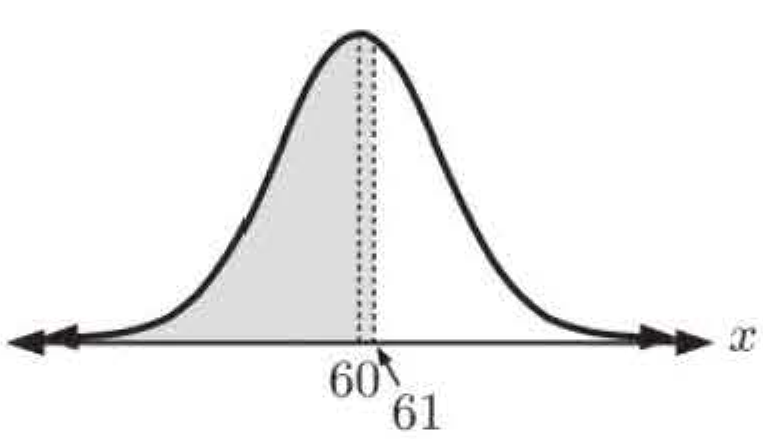
$\text{P}(X \geqslant 64) \approx 0.212$

d



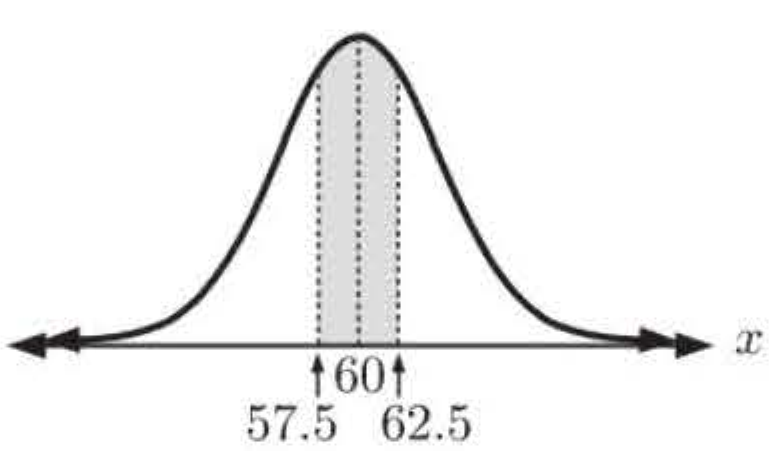
$\text{P}(X \leqslant 68) \approx 0.945$

e



$\text{P}(X \leqslant 61) \approx 0.579$

f



$\text{P}(57.5 \leqslant X \leqslant 62.5) \approx 0.383$

3 If X is the length of a bolt in cm, then X is normally distributed with $\mu = 19.8$ and $\sigma = 0.3$.
$\therefore$ $\text{P}(19.7 < X < 20) \approx 0.378$

4 If X is the money collected in dollars, then X is normally distributed with $\mu = 40$ and $\sigma = 6$.

a $\text{P}(30.00 < X < 50.00) \approx 0.904$
$\approx 90.4\%$

b $\text{P}(X \geqslant 50) \approx 0.0478$
$\approx 4.78\%$

5 If X is the length of an eel in cm, then X is normally distributed with $\mu = 41$ and $\sigma = \sqrt{11}$.

a $\text{P}(X \geqslant 50) \approx 0.003\,33$

b $\text{P}(40 \leqslant X \leqslant 50) \approx 0.615$
$\approx 61.5\%$

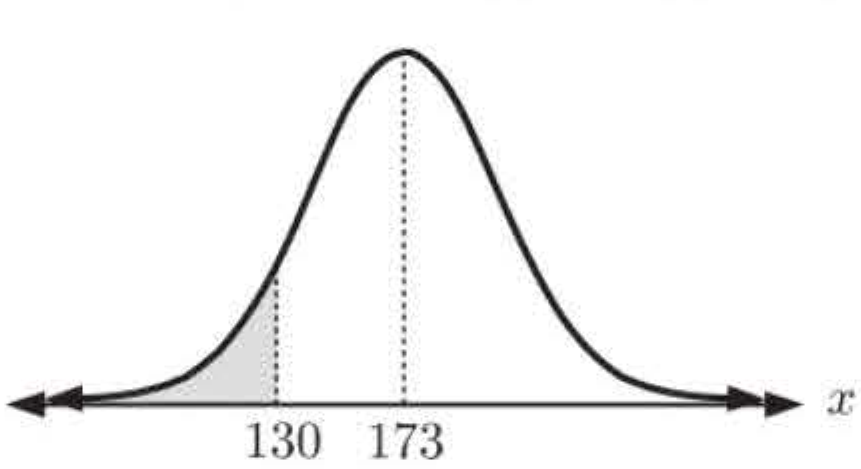
c $\text{P}(X \geqslant 45) \approx 0.114$
So, we would expect $200 \times 0.114 \approx 23$ eels to be at least 45 cm long.

6 If X is the speed of a car in km h^{-1} then X is normally distributed with $\mu = 56.3$ and $\sigma = 7.4$.

a $\text{P}(60 < X < 75) \approx 0.303$ **b** $\text{P}(X \leqslant 70) \approx 0.968$ **c** $\text{P}(X \geqslant 60) \approx 0.309$

7 If X is the weight of an apple in grams, then X is normally distributed with $\mu = 173$ and $\sigma = 34$.

a



$\text{P}(X < 130) \approx 0.102\,988\,39$
≈ 0.103

So, 10.3% of the apples from this crop were too small to sell.

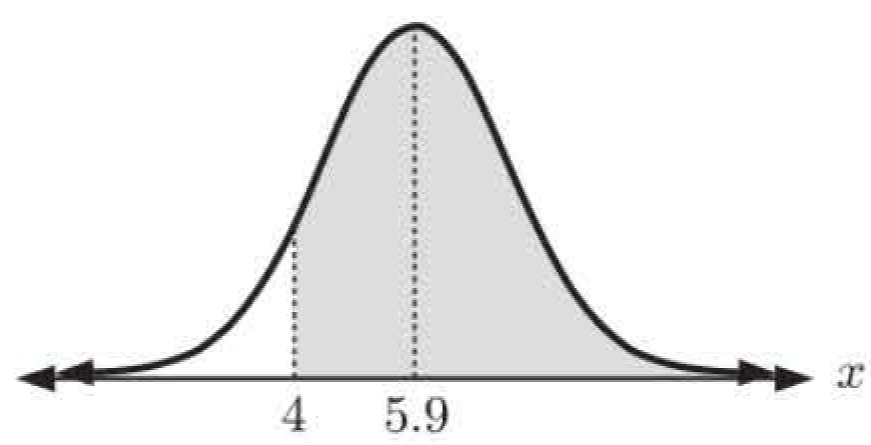
b The chance of one apple being too small to sell is 0.102 988 39.

$\therefore$ the distribution is B(100, 0.102 988 39)

$\therefore$ $P(X \leqslant 10) \approx 0.544$

So, the probability that up to 10 apples were too small to sell is 0.544.

8



If X is the drop in blood pressure (in units) then X is normally distributed with $\mu = 5.9$ and $\sigma = 1.9$

a $P(X \geqslant 4) \approx 0.841\,344\,74$

So, 84.1% of people show a drop of more than 4 units.

b The chance of one person showing a drop of more than 4 units is 0.841 344 74.

$\therefore$ the distribution is B(8, 0.841 344 74)

$$\therefore\ P(X > 5) = P(X \geqslant 6) \approx 0.880$$

So, the probability that more than 5 people show a drop of more than 4 units is 0.880.

EXERCISE 26D

1 **a** For English, $z\text{-score} = \dfrac{48-40}{4.4} \approx 1.82$

For Mandarin, $z\text{-score} = \dfrac{81-60}{9} \approx 2.33$

For Geography, $z\text{-score} = \dfrac{84-55}{18} \approx 1.61$

For Biology, $z\text{-score} = \dfrac{68-50}{20} = 0.9$

For Maths, $z\text{-score} = \dfrac{84-50}{15} \approx 2.27$

b Mandarin, Maths, English, Geography, Biology

2 **a** For Physics, $Z = \dfrac{73-78}{10.8} \approx -0.463$

For Chemistry, $Z = \dfrac{77-72}{11.6} \approx 0.431$

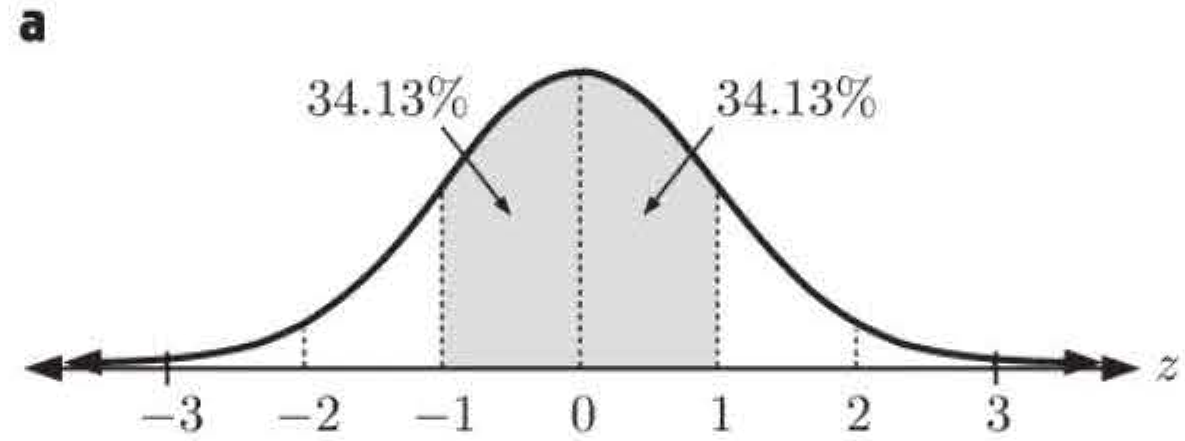
For Maths, $Z = \dfrac{76-74}{10.1} \approx 0.198$

For German, $Z = \dfrac{91-86}{9.6} \approx 0.521$

For Biology, $Z = \dfrac{58-62}{5.2} \approx -0.769$

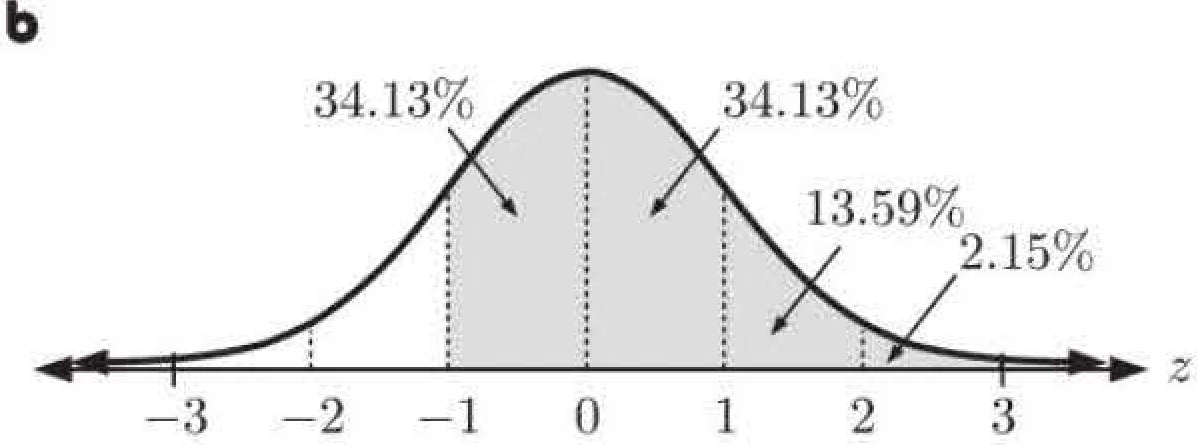
b German, Chemistry, Maths, Physics, Biology

3 **a**



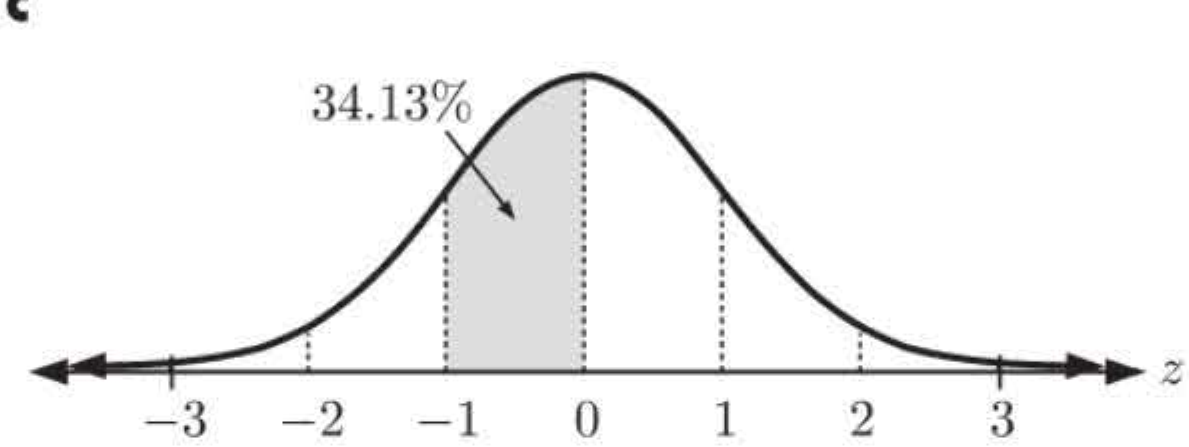
$$\therefore\ P(-1 < Z < 1) \approx 34.13\% + 34.13\% \approx 0.683$$

b



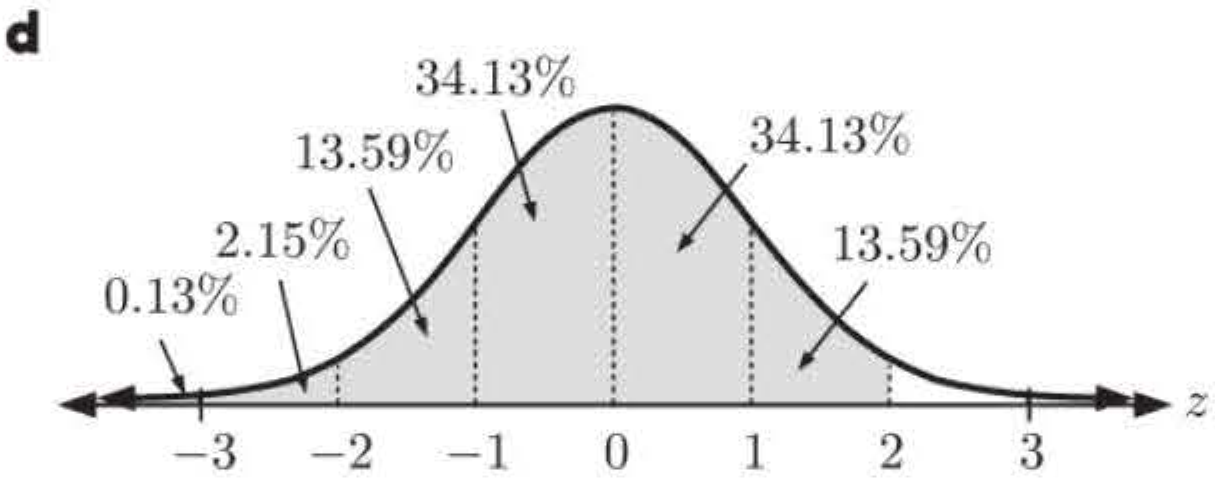
$$\therefore\ P(-1 \leqslant Z \leqslant 3) \approx 34.13\% + 34.13\% + 13.59\% + 2.15\% \approx 0.84$$

c



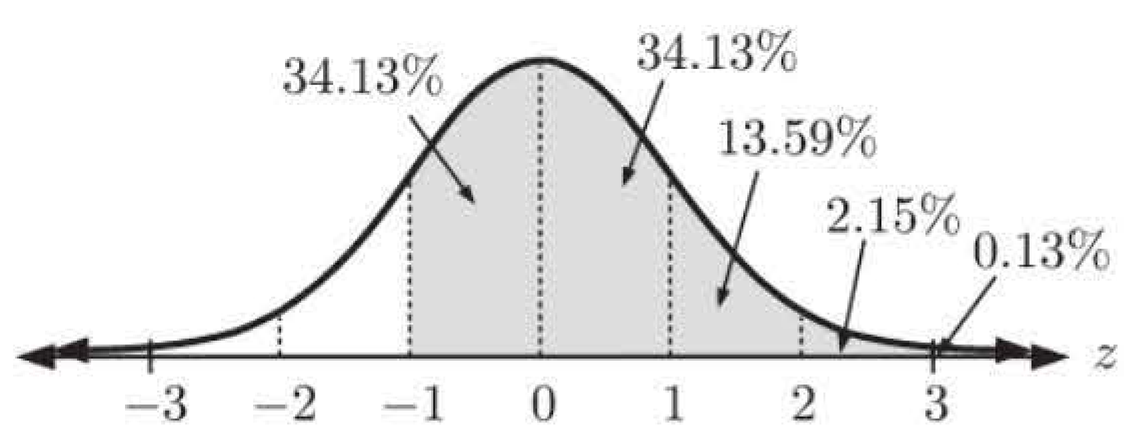
$$\therefore \quad \mathrm{P}(-1 < Z < 0) \approx 34.13\% \approx 0.341$$

d



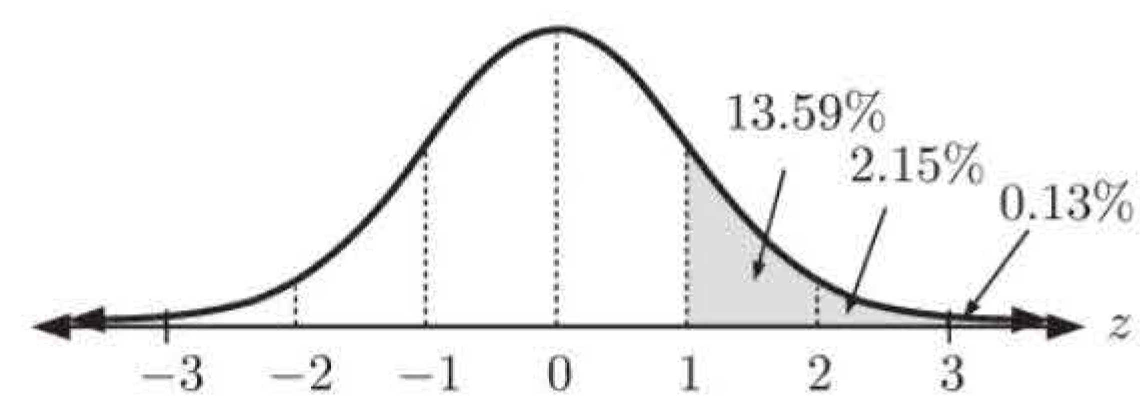
$$\begin{aligned}\therefore \quad & \mathrm{P}(Z < 2) \\ & \approx 0.13\% + 2.15\% + 13.59\% + 34.13\% \\ & \quad + 34.13\% + 13.59\% \\ & \approx 97.72\% \\ & \approx 0.977\end{aligned}$$

e



$$\begin{aligned}\therefore \quad & \mathrm{P}(-1 < Z) \\ & = \mathrm{P}(Z > -1) \\ & \approx 34.13\% + 34.13\% + 13.59\% + 2.15\% \\ & \quad + 0.13\% \\ & \approx 0.841\end{aligned}$$

f



$$\begin{aligned}\therefore \quad & \mathrm{P}(Z \geqslant 1) \\ & \approx 13.59\% + 2.15\% + 0.13\% \\ & \approx 0.159\end{aligned}$$

4 **a**

$$\begin{aligned}\mathrm{E}\left(\frac{X-\mu}{\sigma}\right) &= \mathrm{E}\left(\frac{1}{\sigma}X - \frac{\mu}{\sigma}\right) \\ &= \frac{1}{\sigma}\,\mathrm{E}(X) - \frac{\mu}{\sigma} \\ &= \frac{1}{\sigma}\mu - \frac{\mu}{\sigma} \\ &= 0\end{aligned}$$

b

$$\begin{aligned}\mathrm{Var}\left(\frac{X-\mu}{\sigma}\right) &= \mathrm{Var}\left(\frac{1}{\sigma}X - \frac{\mu}{\sigma}\right) \\ &= \left(\frac{1}{\sigma}\right)^2 \mathrm{Var}(X) \\ &= \frac{1}{\sigma^2} \times \sigma^2 \\ &= 1\end{aligned}$$

5 **a** If $\mathrm{P}(\mu - \sigma < X < \mu + 2\sigma) = \mathrm{P}(a < Z < b)$

then $\dfrac{(\mu - \sigma) - \mu}{\sigma} = a$ and $\dfrac{(\mu + 2\sigma) - \mu}{\sigma} = b$

$$\therefore \quad a = \frac{-\sigma}{\sigma} = -1 \qquad \therefore \quad b = \frac{2\sigma}{\sigma} = 2$$

$\therefore \quad a = -1, \quad b = 2$

b If $\mathrm{P}(\mu - 0.5\sigma < X < \mu) = \mathrm{P}(a < Z < b)$

then $\dfrac{(\mu - 0.5\sigma) - \mu}{\sigma} = a$ and $\dfrac{\mu - \mu}{\sigma} = b$

$$\therefore \quad a = \frac{-0.5\sigma}{\sigma} = -0.5 \qquad \therefore \quad b = 0$$

$\therefore \quad a = -0.5, \quad b = 0$

c If $\mathrm{P}(0 \leqslant Z \leqslant 3) = \mathrm{P}(\mu - a\sigma \leqslant X \leqslant \mu + b\sigma)$

then $\dfrac{(\mu - a\sigma) - \mu}{\sigma} = 0$ and $\dfrac{(\mu + b\sigma) - \mu}{\sigma} = 3$

$\therefore\ \mu - a\sigma - \mu = 0$ $\quad\therefore\ \mu + b\sigma - \mu = 3\sigma$

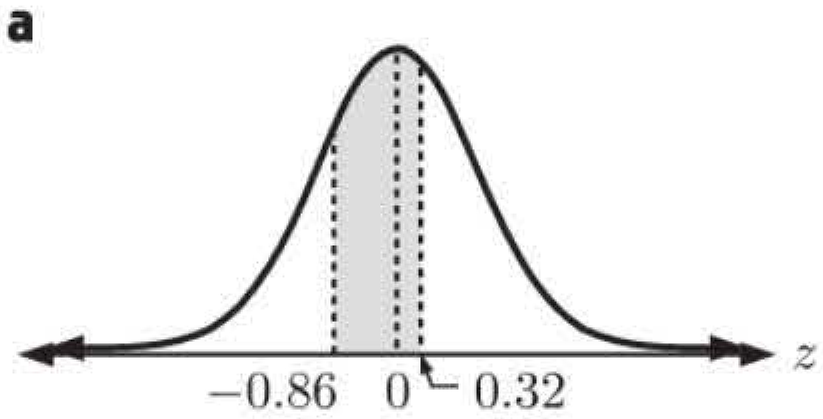
$\therefore\ -a\sigma = 0$ $\quad\therefore\ b\sigma = 3\sigma$

$\therefore\ a = 0$ $\quad\therefore\ b = 3$

$\therefore\ a = 0, \quad b = 3$

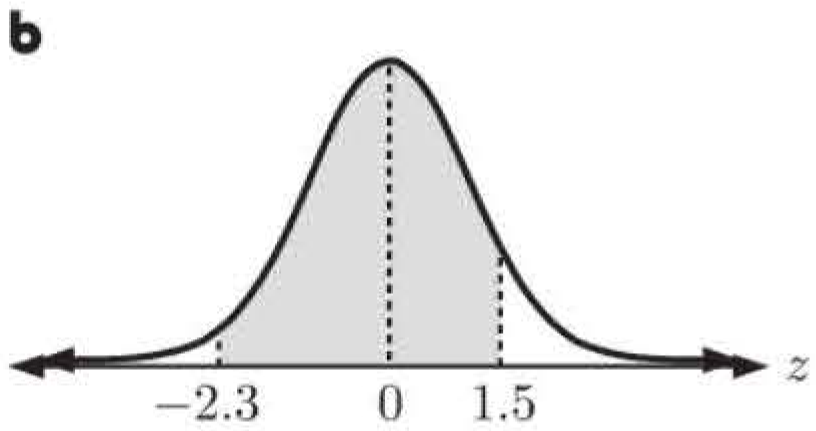
6 $Z \sim \mathrm{N}(0, 1)$

a



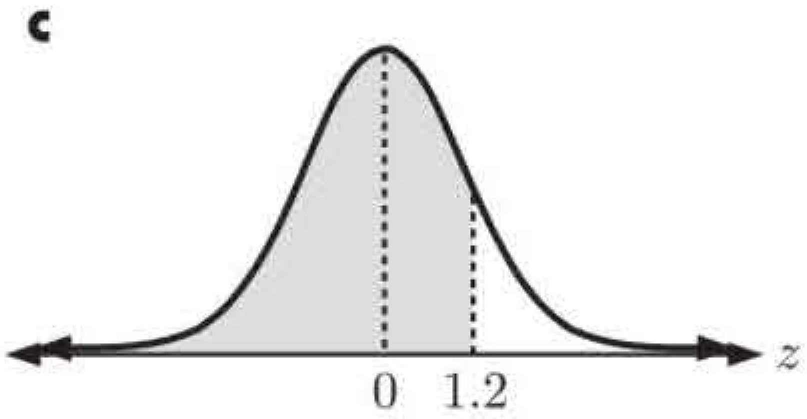
$\therefore\ \mathrm{P}(-0.86 \leqslant Z \leqslant 0.32)$
≈ 0.431

b



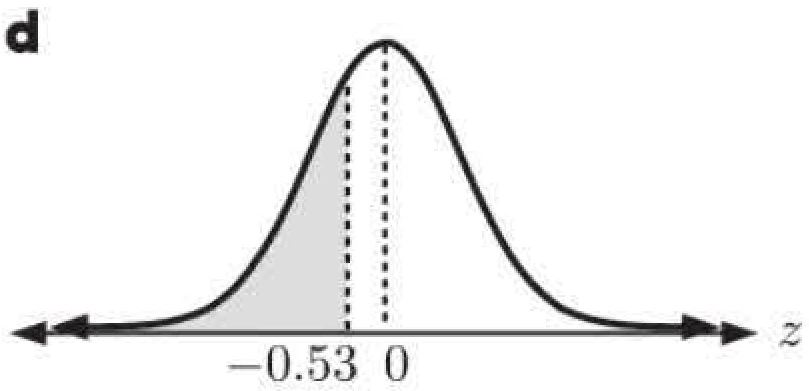
$\therefore\ \mathrm{P}(-2.3 \leqslant Z \leqslant 1.5)$
≈ 0.922

c



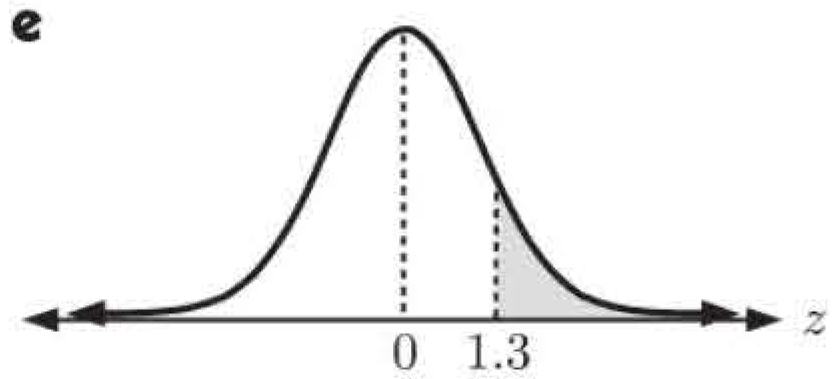
$\therefore\ \mathrm{P}(Z \leqslant 1.2)$
≈ 0.885

d



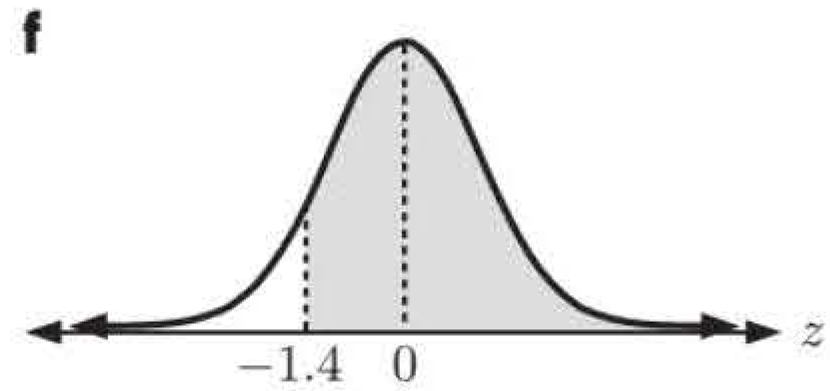
$\therefore\ \mathrm{P}(Z \leqslant -0.53)$
≈ 0.298

e



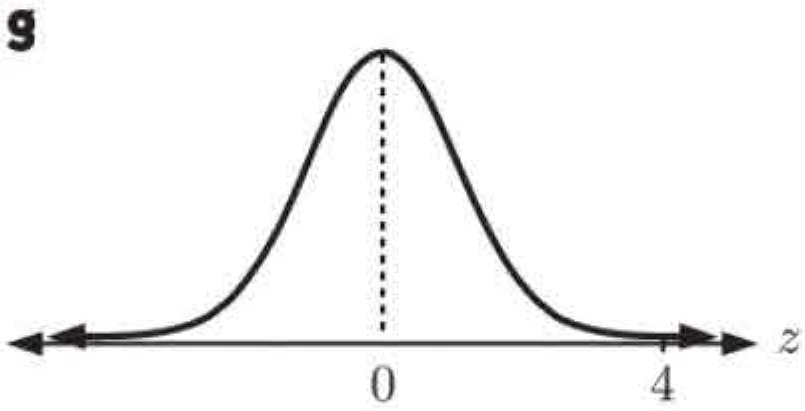
$\therefore\ \mathrm{P}(Z \geqslant 1.3)$
≈ 0.0968

f



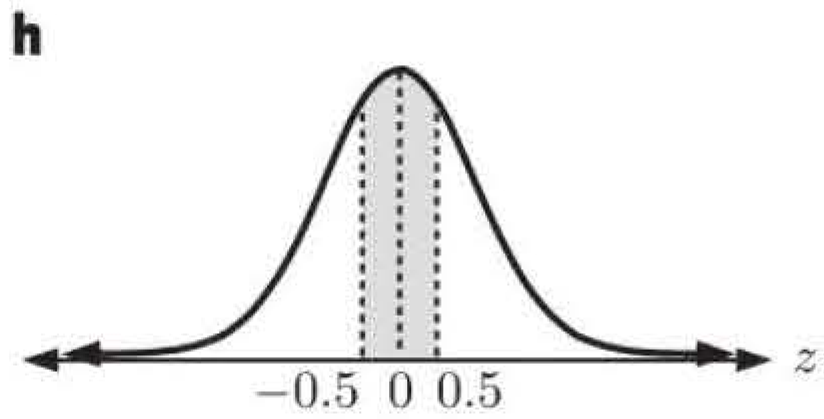
$\therefore\ \mathrm{P}(Z \geqslant -1.4)$
≈ 0.919

g



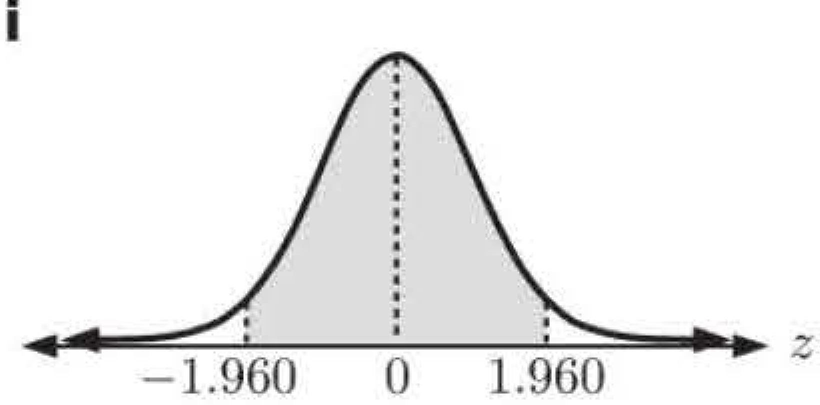
$\therefore\ \mathrm{P}(Z > 4)$
$\approx 0.000\,031\,7$
(3.17×10^{-5})

h



$\therefore\ \mathrm{P}(-0.5 < Z < 0.5)$
≈ 0.383

i



$\therefore\ \mathrm{P}(-1.960 \leqslant Z \leqslant 1.960)$
≈ 0.950

7 **a** **i** $z_1 = \dfrac{50.6 - 58.3}{8.96}$

$\therefore\ z_1 \approx -0.859\,375$
≈ -0.859

$z_2 = \dfrac{68.9 - 58.3}{8.96}$
$\approx 1.183\,035\,714$
≈ 1.18

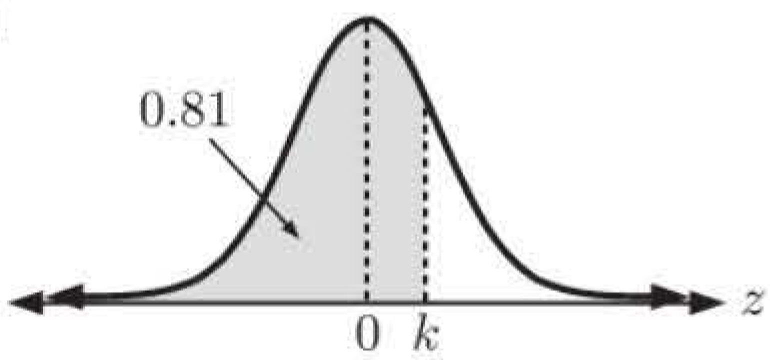
ii $Z \sim \mathrm{N}(0, 1)$
$\mathrm{P}(-0.859\,375 \leqslant Z \leqslant 1.183\,035\,714)$
$\approx 0.686\,535\,67$
≈ 0.687

b $X \sim \mathrm{N}(58.3, 8.96^2)$

$\therefore\ \mathrm{P}(50.6 \leqslant X \leqslant 68.9) \approx 0.687$ ✓

EXERCISE 26E.1

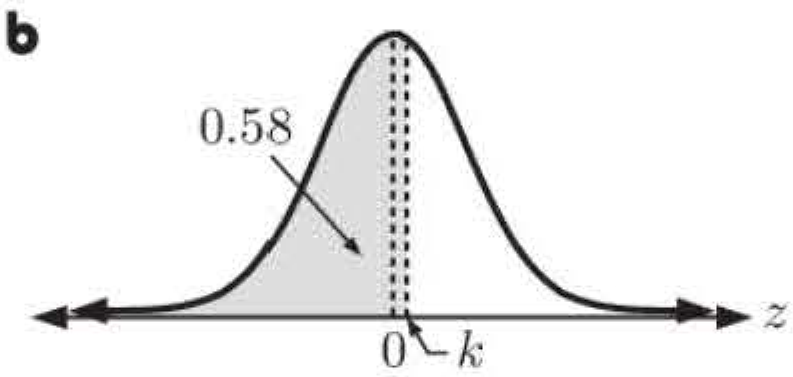
1 **a**



$\text{P}(Z \leqslant k) = 0.81$

$\therefore \quad k \approx 0.878$

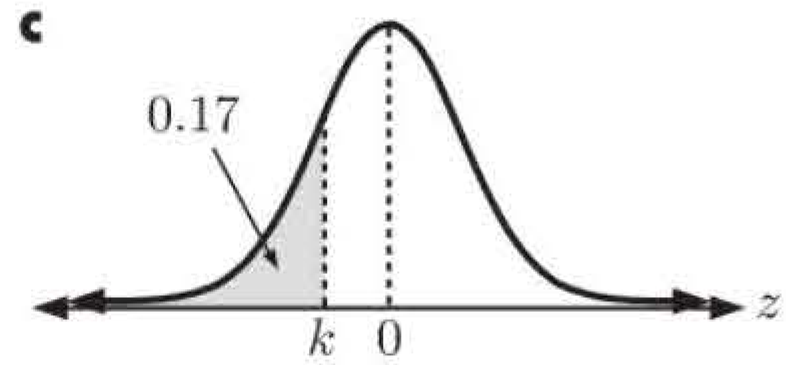
b



$\text{P}(Z \leqslant k) = 0.58$

$\therefore \quad k \approx 0.202$

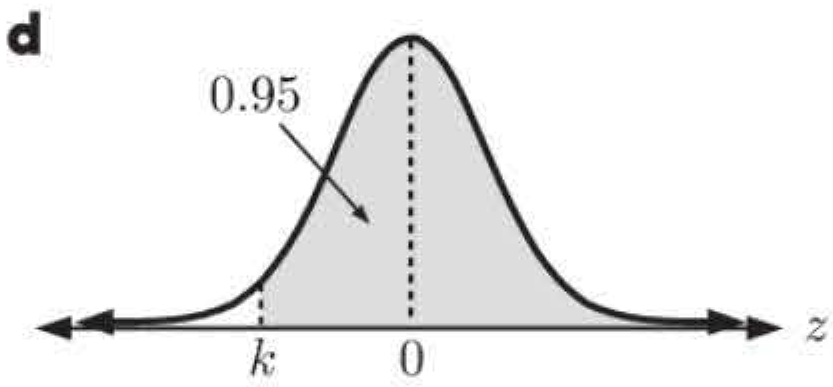
c



$\text{P}(Z \leqslant k) = 0.17$

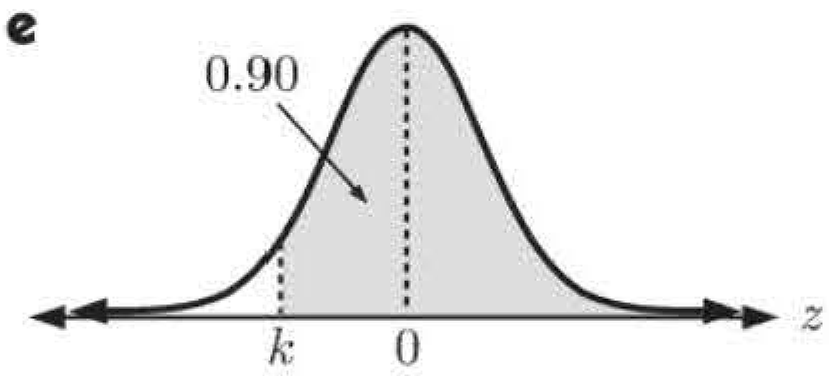
$\therefore \quad k \approx -0.954$

d



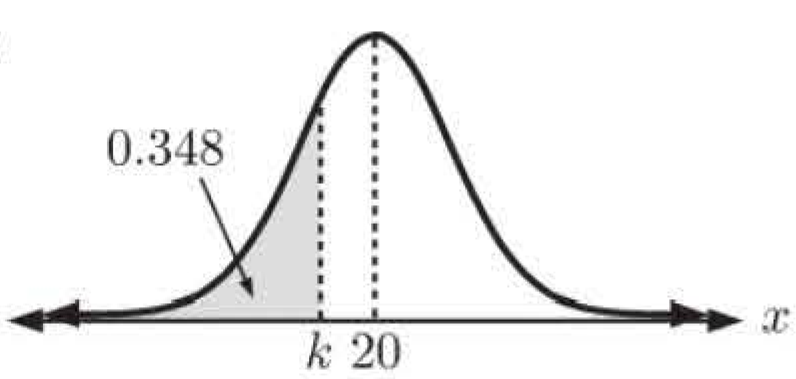
$$\begin{aligned} \text{P}(Z \geqslant k) &= 0.95 \\ \therefore \quad \text{P}(Z \leqslant k) &= 1 - 0.95 \\ &= 0.05 \\ \therefore \quad k &\approx -1.64 \end{aligned}$$

e



$$\begin{aligned} \text{P}(Z \geqslant k) &= 0.90 \\ \therefore \quad \text{P}(Z \leqslant k) &= 1 - 0.90 \\ &= 0.1 \\ \therefore \quad k &\approx -1.28 \end{aligned}$$

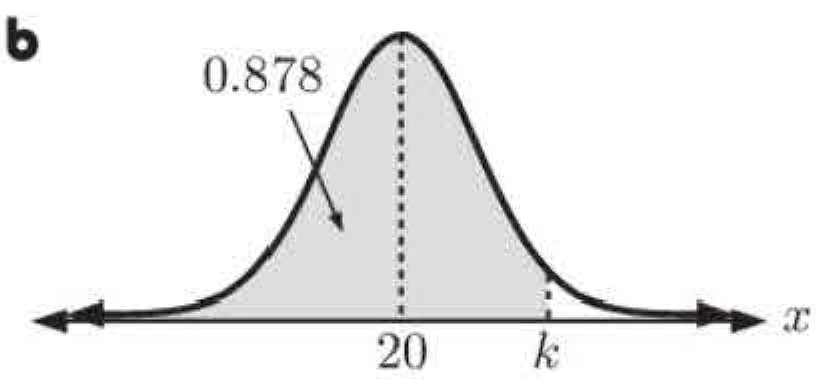
2 **a**



$\text{P}(X \leqslant k) = 0.348$

$\therefore \quad k \approx 18.8$

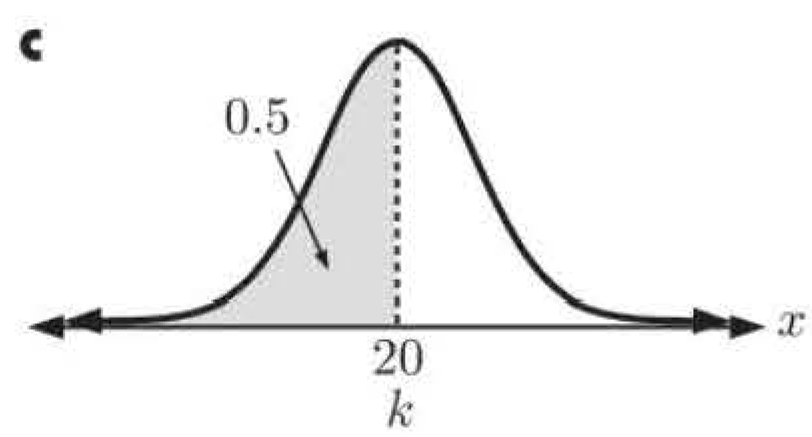
b



$\text{P}(X \leqslant k) = 0.878$

$\therefore \quad k \approx 23.5$

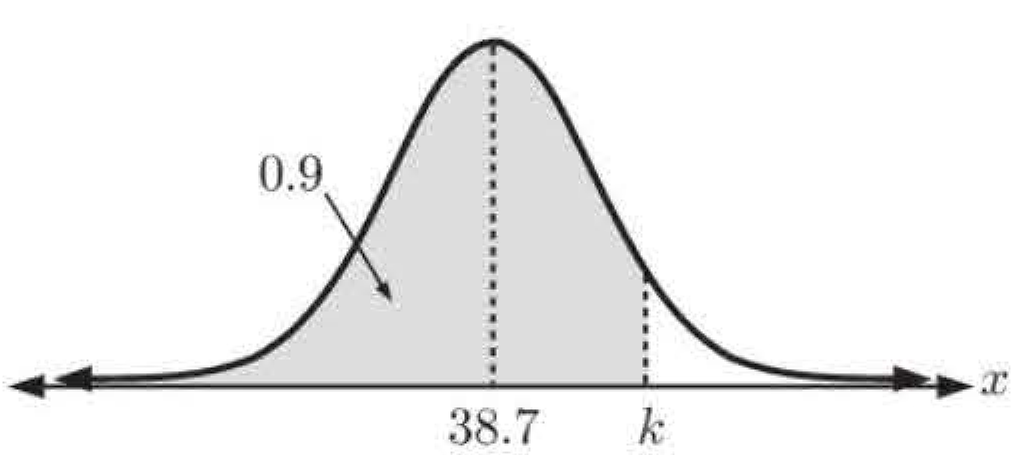
c



$\text{P}(X \leqslant k) = 0.5$

$\therefore \quad k = 20$

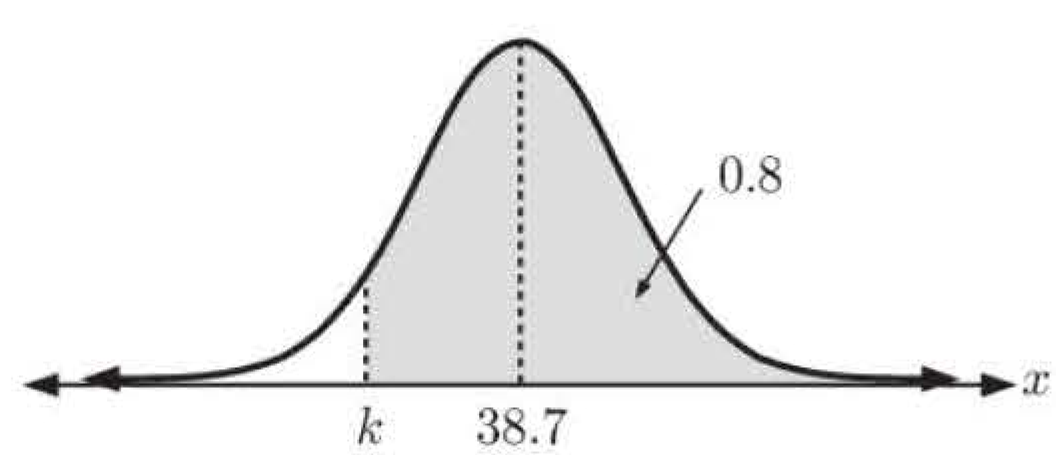
3 **a**



$\text{P}(X \leqslant k) = 0.9$

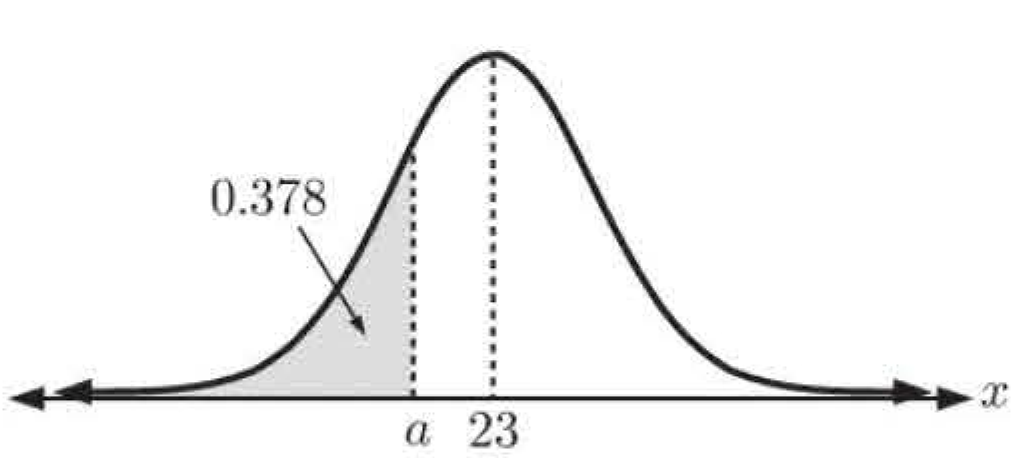
$\therefore \quad k \approx 49.2$

b



$$\begin{aligned} \text{P}(X \geqslant k) &= 0.8 \\ \therefore \quad \text{P}(X \leqslant k) &= 1 - 0.8 = 0.2 \\ \therefore \quad k &\approx 31.8 \end{aligned}$$

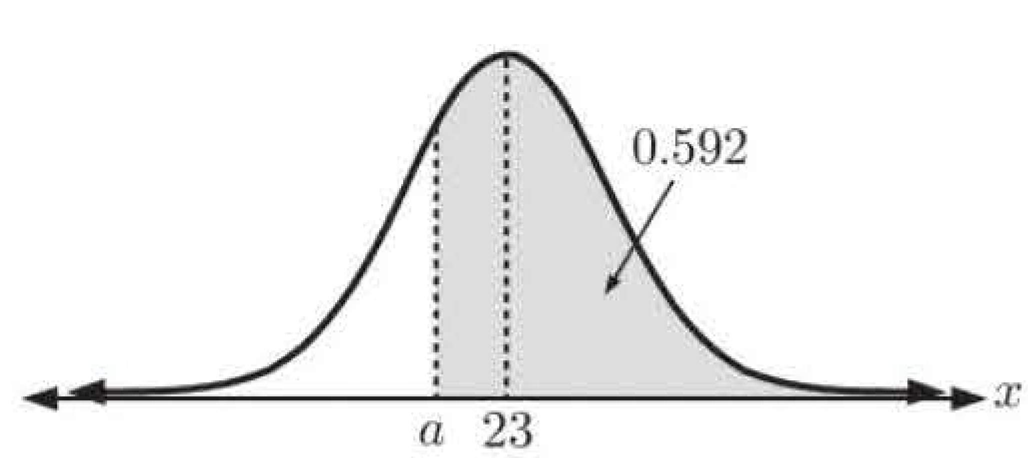
4 **a**



$\text{P}(X < a) = 0.378$

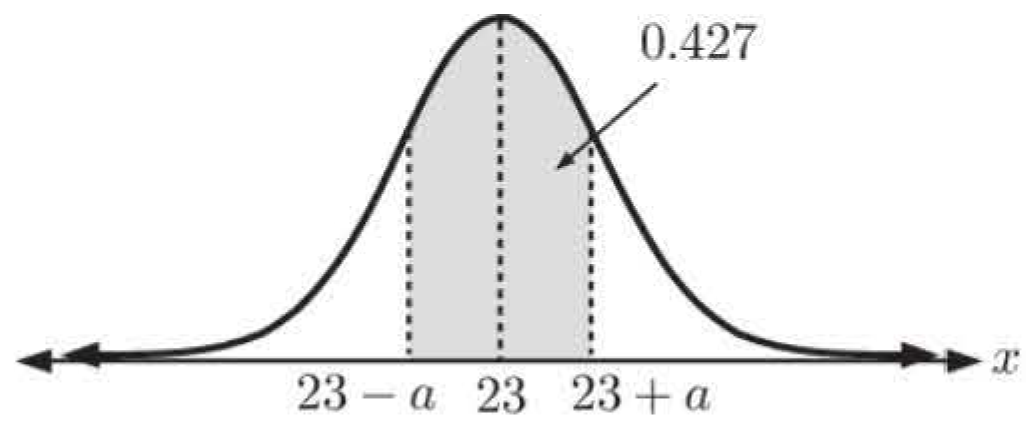
$\therefore \quad a \approx 21.4$

b



$$\begin{aligned} \text{P}(X \geqslant a) &= 0.592 \\ \therefore \quad \text{P}(X \leqslant a) &= 1 - 0.592 \\ &= 0.408 \\ \therefore \quad a &\approx 21.8 \end{aligned}$$

c



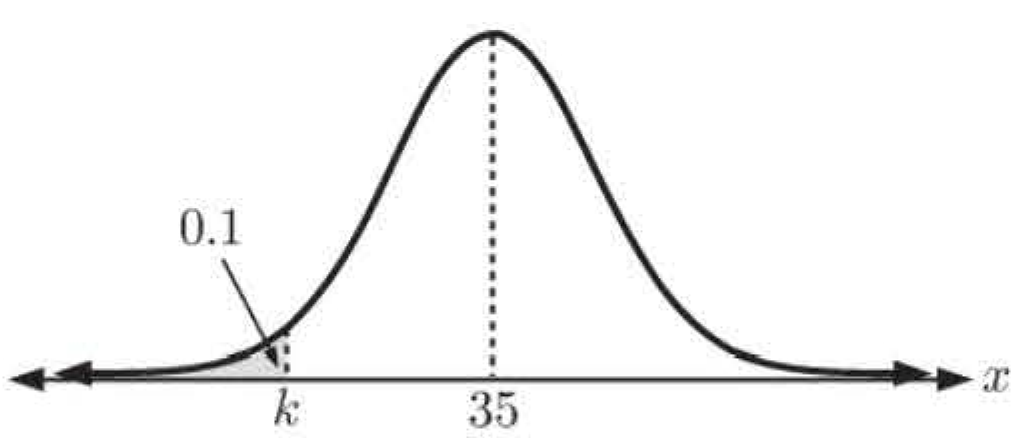
$$\begin{aligned} \mathrm{P}(23-a<X<23+a) &= 0.427 \\ \therefore\ 1-2\times \mathrm{P}(X\leqslant 23-a) &= 0.427 \\ \therefore\ -2\times \mathrm{P}(X\leqslant 23-a) &= -0.573 \\ \therefore\ \mathrm{P}(X\leqslant 23-a) &= 0.2865 \\ \therefore\ 23-a &= 20.181\,806\,2 \\ \therefore\ a &\approx 23-20.181\,806\,2 \\ \therefore\ a &\approx 2.82 \end{aligned}$$

5 Let X be the result of the Physics test, so $X \sim \mathrm{N}(46,\, 25^2)$.

We need to find k such that $\mathrm{P}(X\geqslant k)=0.07$

$$\begin{aligned} \therefore\ 1-\mathrm{P}(X<k) &= 0.07 \\ \therefore\ \mathrm{P}(X<k) &= 0.93 \\ \therefore\ k &\approx 82.894 \\ \therefore\ k &\approx 82.9 \end{aligned}$$

6



$X \sim \mathrm{N}(35,\, 8^2)$

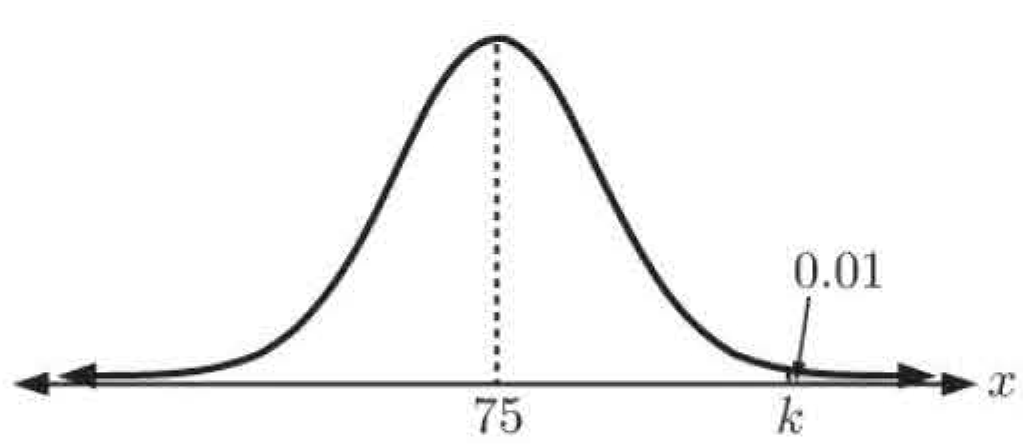
We need to find k such that

$\mathrm{P}(X\leqslant k)=0.1$

$\therefore\ k\approx 24.747\,587\,5$

So, the length of the smallest fish to be harvested is 24.7 cm.

7



$X \sim \mathrm{N}(75,\, 0.1^2)$

We need to find k such that

$\mathrm{P}(X\geqslant k)=0.01$

$\therefore\ \mathrm{P}(X\leqslant k)=0.99$

$\therefore\ k\approx 75.232\,634\,8$

So, the length of the smallest screw to be rejected is 75.2 mm.

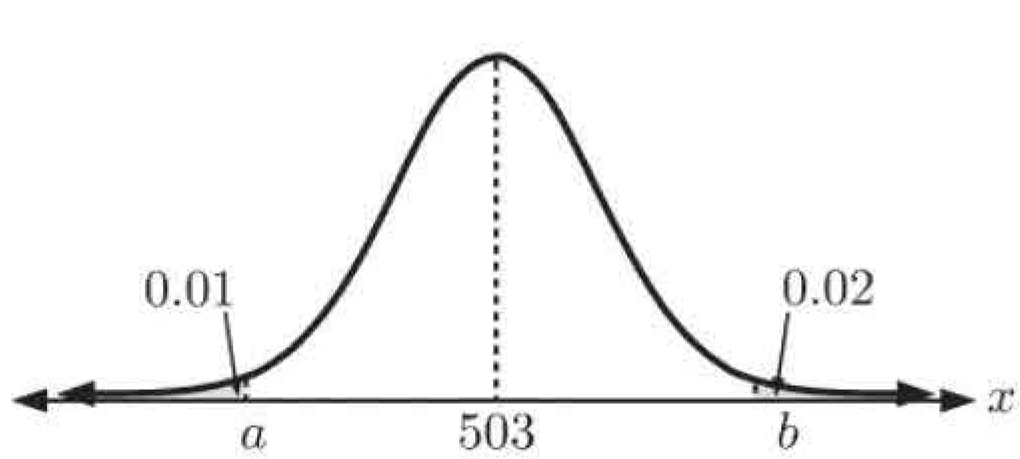
8 Z-score for algebra $=\dfrac{56-50.2}{15.8}\approx 0.3671$ $\qquad$ Z-score for geometry $=\dfrac{x-58.7}{18.7}$

$\therefore$ we need to solve $\dfrac{x-58.7}{18.7}=0.3671$

$$\begin{aligned} \therefore\ x-58.7 &\approx 6.86 \\ \therefore\ x &\approx 65.6 \end{aligned}$$

So, Pedro needs a result of 65.6%.

9



$X \sim \mathrm{N}(503,\, 0.5^2)$

We need to find a such that

$\mathrm{P}(X\leqslant a)=0.01$

$\therefore\ a\approx 502$

We also need to find b such that $\mathrm{P}(X\geqslant b)=0.02$

$\therefore\ \mathrm{P}(X\leqslant b)=0.98$

$\therefore\ b\approx 504$

So, the range of volumes in the bottles that are kept is 502 mL to 504 mL.

EXERCISE 26E.2

1 Let the mean IQ of a student at school be μ.
If X is the IQ of a student at the school, then $X \sim \mathrm{N}(\mu,\, 15^2)$.

$$\text{Now,}\quad \mathrm{P}(X \geqslant 125) = 0.2$$

$$\therefore\quad \mathrm{P}\left(\frac{X-\mu}{15} \geqslant \frac{125-\mu}{15}\right) = 0.2$$

$$\therefore\quad \mathrm{P}\left(Z \geqslant \frac{125-\mu}{15}\right) = 0.2$$

$$\therefore\quad \mathrm{P}\left(Z < \frac{125-\mu}{15}\right) = 0.8$$

$$\therefore\quad \frac{125-\mu}{15} \approx 0.8416$$

$$\therefore\quad \mu \approx 112.4$$

The mean IQ at the school is 112.4.

2 Let the standard deviation of the distances jumped be σ m.
If X is the distance jumped by the athlete, then $X \sim \mathrm{N}(5.2,\, \sigma^2)$.

$$\text{Now,}\quad \mathrm{P}(X < 5) = 0.15$$

$$\therefore\quad \mathrm{P}\left(\frac{X-5.2}{\sigma} < \frac{5-5.2}{\sigma}\right) = 0.15$$

$$\therefore\quad \mathrm{P}\left(Z < -\frac{0.2}{\sigma}\right) = 0.15$$

$$\therefore\quad -\frac{0.2}{\sigma} \approx -1.036$$

$$\therefore\quad \sigma \approx 0.193$$

So, the standard deviation of the distances jumped is 0.193 m.

3 Let the standard deviation of the weekly income be $\$\sigma$.
If X denotes the weekly income of the bakery, then $X \sim \mathrm{N}(6100,\, \sigma^2)$.

$$\text{Now,}\quad \mathrm{P}(X \geqslant 6000) = 0.85$$

$$\therefore\quad \mathrm{P}\left(Z \geqslant \frac{6000-6100}{\sigma}\right) = 0.85$$

$$\therefore\quad \mathrm{P}\left(Z < \frac{6000-6100}{\sigma}\right) = 0.15$$

Using invNorm for $\mathrm{N}(0,\, 1^2)$,

$$\frac{-100}{\sigma} \approx -1.036\,433\,4$$

$$\therefore\quad \sigma \approx \frac{-100}{-1.036\,433\,4}$$

$$\therefore\quad \sigma \approx 96.5$$

So, the standard deviation is \$96.50.

4 Let the mean arrival time be μ minutes after midday.
If X denotes the arrival time of a bus, then $X \sim \mathrm{N}(\mu,\, 5^2)$.

Now, $\mathrm{P}(X \leqslant 235) = 0.1$
$\{$3:55 pm $= 3 \times 60 + 55 = 235$ minutes after midday$\}$

$$\therefore\quad \mathrm{P}\left(Z \leqslant \frac{235-\mu}{5}\right) = 0.1$$

Using invNorm for $\mathrm{N}(0,\, 1^2)$,

$$\frac{235-\mu}{5} \approx -1.281\,551\,6$$

$$\therefore\quad 235-\mu \approx -6.407\,758$$

$$\therefore\quad \mu \approx 235 + 6.407\,758$$

$$\therefore\quad \mu \approx 241.407\,758 \text{ minutes after midday}$$

and $241.407\,758$ minutes $= 4$ h 1 m 24 s

So, the mean arrival time of buses at the depot is 4:01:24 pm.

5 $X \sim \mathrm{N}(\mu,\, \sigma^2)$ where we have to find μ and σ.
We start by finding z_1 and z_2 which correspond to $x_1 = 35$ and $x_2 = 8$.

$$\text{Now}\quad \mathrm{P}(X \geqslant 35) = 0.32$$

$$\therefore\quad \mathrm{P}(X < 35) = 0.68$$

$$\therefore\quad \mathrm{P}\left(\frac{X-\mu}{\sigma} < \frac{35-\mu}{\sigma}\right) = 0.68$$

$$\therefore\quad \mathrm{P}\left(Z < \frac{35-\mu}{\sigma}\right) = 0.68$$

$$\therefore\quad z_1 = \frac{35-\mu}{\sigma} \approx 0.4677$$

$$\therefore\quad 35-\mu \approx 0.4677\sigma \quad .... \ (1)$$

and

$$\mathrm{P}(X \leqslant 8) = 0.26$$

$$\therefore\quad \mathrm{P}\left(\frac{X-\mu}{\sigma} \leqslant \frac{8-\mu}{\sigma}\right) = 0.26$$

$$\therefore\quad \mathrm{P}\left(Z \leqslant \frac{8-\mu}{\sigma}\right) = 0.26$$

$$\therefore\quad z_2 = \frac{8-\mu}{\sigma} \approx -0.6433$$

$$\therefore\quad 8-\mu \approx -0.6433\sigma \quad .... \ (2)$$

Solving (1) and (2) simultaneously we get $\mu \approx 23.6$ and $\sigma \approx 24.3$.

6 **a** $X \sim N(\mu, \sigma^2)$ where we have to find μ and σ.

We start by finding z_1 and z_2 which correspond to $x_1 = 30$ and $x_2 = 80$.

Now $P(X \leqslant 30) = 0.15$

$\therefore P\left(\frac{X-\mu}{\sigma} \leqslant \frac{30-\mu}{\sigma}\right) = 0.15$

$\therefore P\left(Z \leqslant \frac{30-\mu}{\sigma}\right) = 0.15$

$\therefore z_1 = \frac{30-\mu}{\sigma} \approx -1.0364$

$\therefore 30 - \mu \approx -1.0364\sigma$ .... (1)

and $P(X \geqslant 80) = 0.1$

$\therefore P(X < 80) = 0.9$

$\therefore P\left(\frac{X-\mu}{\sigma} < \frac{80-\mu}{\sigma}\right) = 0.9$

$\therefore P\left(Z < \frac{80-\mu}{\sigma}\right) = 0.9$

$\therefore z_2 = \frac{80-\mu}{\sigma} \approx 1.2816$

$\therefore 80 - \mu \approx 1.2816\sigma$ .... (2)

Solving (1) and (2) simultaneously, $\mu \approx 52.36 \approx 52.4$ and $\sigma \approx 21.57 \approx 21.6$.

b Let X be the result of the mathematics exam.

X is normally distributed with mean μ and standard deviation σ.

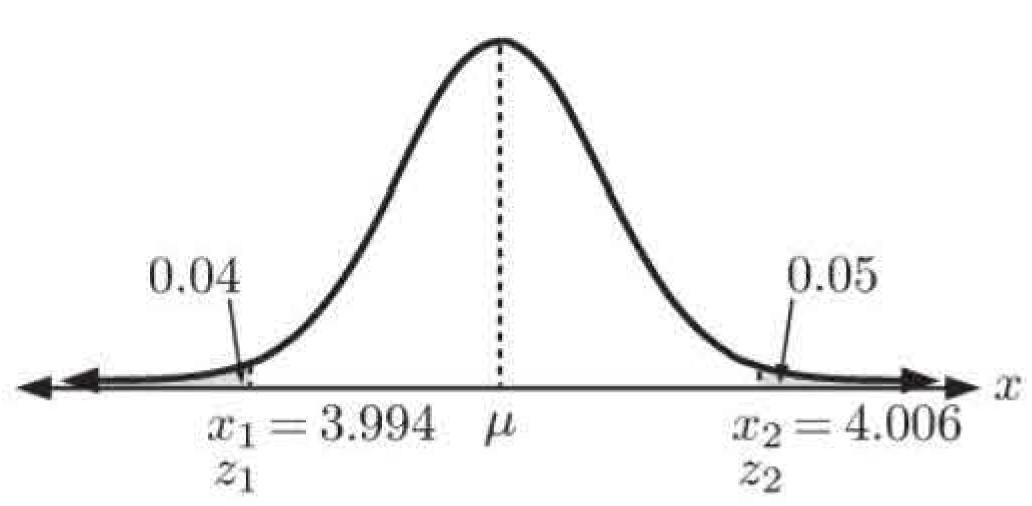
We know that $P(X \geqslant 80) = 0.1$ and $P(X \leqslant 30) = 0.15$.

So, from **a**, $\mu \approx 52.4$ and $\sigma \approx 21.6$.

If part marks can be given, $P(X > 50) \approx 0.544$
$\approx 54.4\%$

So, 54.4% of students scored more than 50.

7 **a**



$X \sim N(\mu, \sigma^2)$ where we have to find μ and σ.

We find z_1 and z_2 which correspond to $x_1 = 3.994$ and $x_2 = 4.006$

Now $P(X \leqslant x_1) = 0.04$

$\therefore P\left(Z \leqslant \frac{3.994-\mu}{\sigma}\right) = 0.04$

$\therefore \frac{3.994-\mu}{\sigma} = -1.750\,686\,1$

$\therefore 3.994 - \mu = -1.750\,686\,1\sigma$ .... (1)

and $P(X \geqslant x_2) = 0.05$

$\therefore P\left(Z \leqslant \frac{4.006-\mu}{\sigma}\right) = 0.95$

$\therefore \frac{4.006-\mu}{\sigma} = 1.644\,853\,63$

$\therefore 4.006 - \mu = 1.644\,853\,63\sigma$ .... (2)

Solving simultaneously, $\mu \approx 4.000\,187\,009$ and $\sigma \approx 0.003\,534\,047\,88$

$\therefore \mu \approx 4.00$ cm and $\sigma \approx 0.003\,53$ cm

b From **a**, $\mu \approx 4.000$ and $\sigma \approx 0.003\,534$

$\therefore X \sim N(4.000, 0.003\,534^2)$

$\therefore P(3.997 \leqslant X \leqslant 4.003) \approx 0.604$

So, the probability that a randomly chosen piston has diameter between 3.997 cm and 4.003 cm is 0.604.

8 **a** $X \sim \mathrm{N}(\mu, \sigma^2)$ where we have to find μ and σ.

We start by finding z_1 and z_2 which correspond to $x_1 = 1.94$ and $x_2 = 2.06$.

Now $\mathrm{P}(X < 1.94) = 0.02$

$\therefore \mathrm{P}\left(\dfrac{X-\mu}{\sigma} < \dfrac{1.94-\mu}{\sigma}\right) = 0.02$

$\therefore \mathrm{P}\left(Z < \dfrac{1.94-\mu}{\sigma}\right) = 0.02$

$\therefore z_1 = \dfrac{1.94-\mu}{\sigma} \approx -2.054$

$\therefore 1.94 - \mu \approx -2.054\sigma$ .... (1)

and $\mathrm{P}(X > 2.06) = 0.03$

$\therefore \mathrm{P}\left(\dfrac{X-\mu}{\sigma} > \dfrac{2.06-\mu}{\sigma}\right) = 0.03$

$\therefore \mathrm{P}\left(Z > \dfrac{2.06-\mu}{\sigma}\right) = 0.03$

$\therefore \mathrm{P}\left(Z \leqslant \dfrac{2.06-\mu}{\sigma}\right) = 0.97$

$\therefore z_2 = \dfrac{2.06-\mu}{\sigma} \approx 1.881$

$2.06 - \mu \approx 1.881\sigma$ .... (2)

Solving (1) and (2) simultaneously, we get $\mu \approx 2.00$ cm and $\sigma \approx 0.0305$ cm.

b Let Y be the number of tokens which will not operate the machine. This is a binomial situation with the probability $p = 0.02 + 0.03 = 0.05$ of failure to operate and $n = 20$. So, $Y \sim \mathrm{B}(20, 0.05)$.

$\therefore$ P(at most one will not operate) $= \mathrm{P}(Y \leqslant 1)$

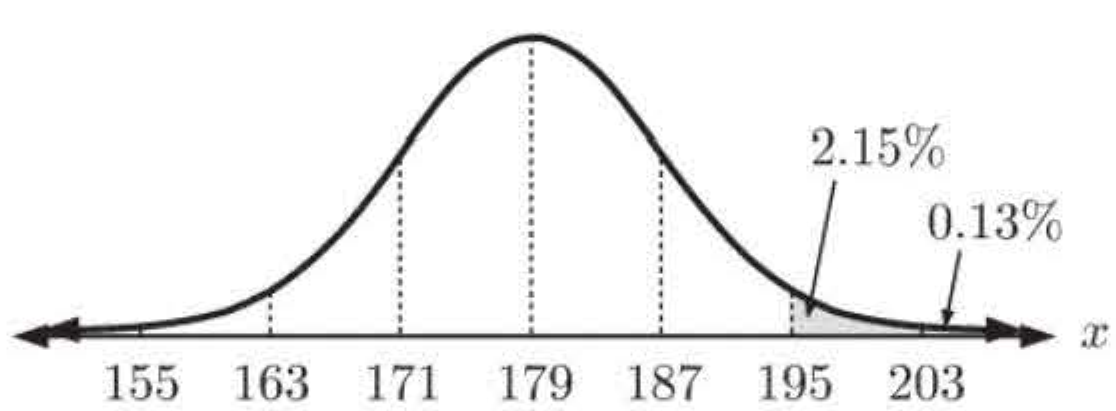
≈ 0.736

REVIEW SET 26A

1 X is the height of a 17 year old boy.

X is normally distributed with $\mu = 179$ cm and $\sigma = 8$ cm.

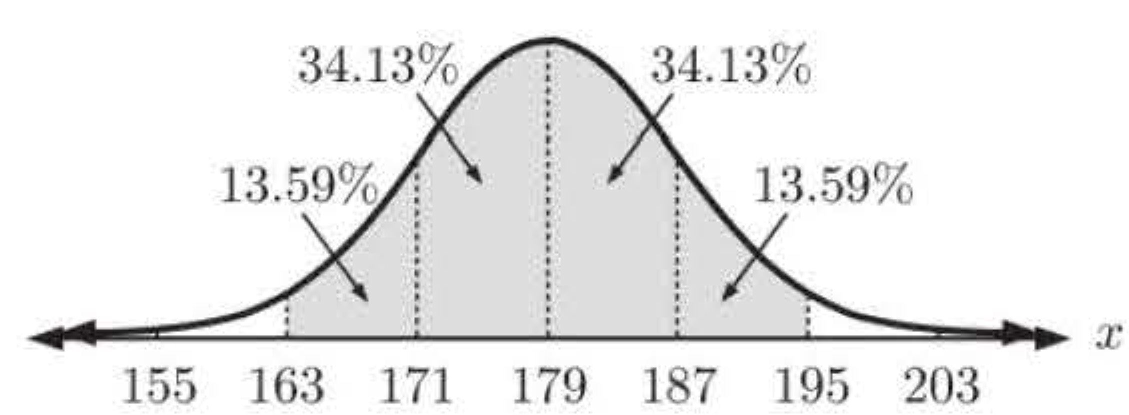
a



$\mathrm{P}(X \geqslant 195) \approx 2.15\% + 0.13\%$

$\approx 2.28\%$

b



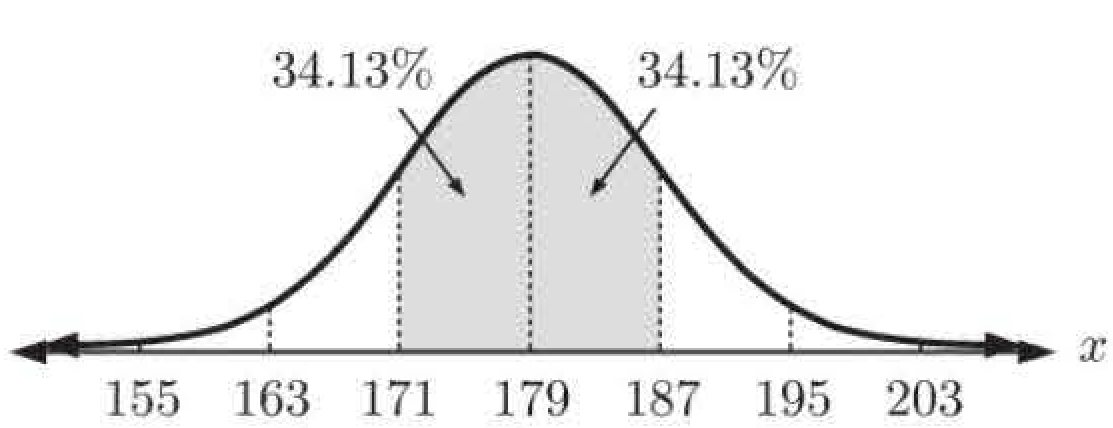
$\mathrm{P}(163 \leqslant X \leqslant 195)$

$\approx 13.59\% + 34.13\% + 34.13\% + 13.59\%$

$\approx 95.44\%$

$\approx 95.4\%$

c



$\mathrm{P}(171 \leqslant X \leqslant 187) \approx 34.13\% + 34.13\%$

$\approx 68.26\%$

$\approx 68.3\%$

2 If X is the contents of the container in mL, then $X \sim \mathrm{N}(377, 4.2^2)$.

a **i** $\mathrm{P}(X < 368.6)$

$\approx 2.15\% + 0.13\%$

$\approx 2.28\%$

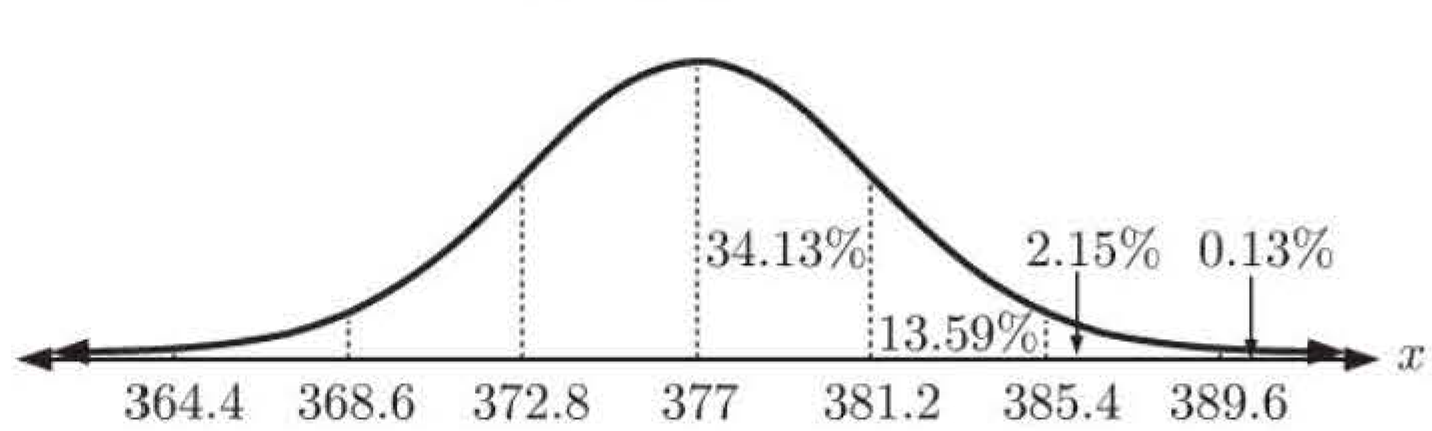
ii $\mathrm{P}(372.8 < X < 389.6)$

$\approx 2 \times 34.13\% + 13.59\% + 2.15\%$

$\approx 84.0\%$

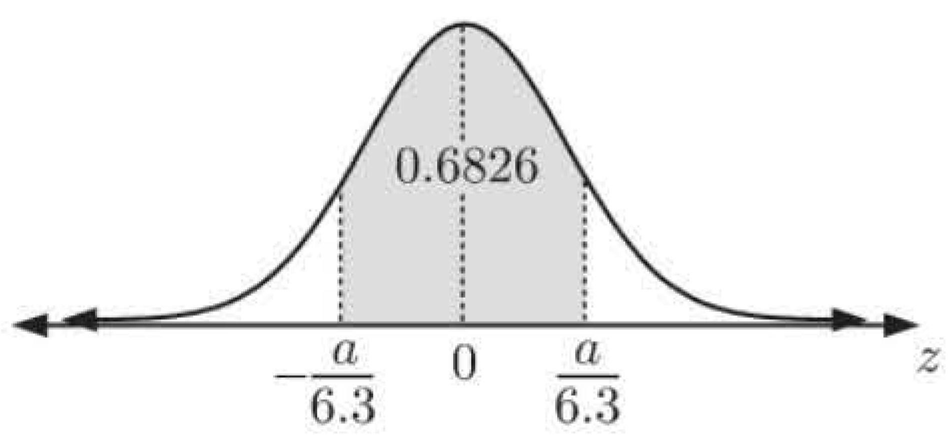
b $\mathrm{P}(377 < X < 381.2)$

≈ 0.341



3 If X is the mass of a Coffin Bay Oyster, then $X \sim \text{N}(38.6, 6.3^2)$.

a
$$\text{P}(38.6 - a \leqslant X \leqslant 38.6 + a) = 0.6826$$
$$\therefore \quad \text{P}\left(\frac{38.6 - a - 38.6}{6.3} \leqslant \frac{X - 38.6}{6.3} \leqslant \frac{38.6 + a - 38.6}{6.3}\right) = 0.6826$$
$$\therefore \quad \text{P}\left(-\frac{a}{6.3} \leqslant Z \leqslant \frac{a}{6.3}\right) = 0.6826$$



$$\therefore \quad \text{by symmetry,} \quad \text{P}\left(Z \leqslant -\frac{a}{6.3}\right) = \frac{1 - 0.6826}{2}$$
$$\therefore \quad \text{P}\left(Z \leqslant -\frac{a}{6.3}\right) = 0.1587 \quad .... \ (*)$$
$$\therefore \quad -\frac{a}{6.3} \approx -1.00$$
$$\therefore \quad a \approx 6.30 \text{ g}$$

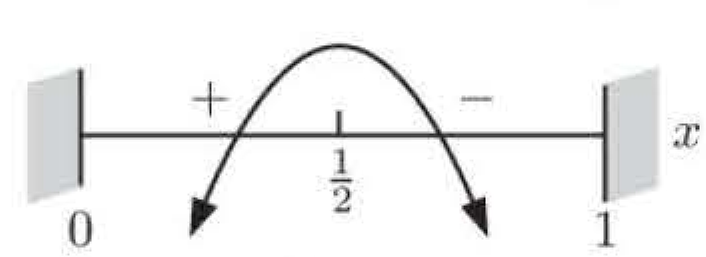
b
$$\text{P}(X \geqslant b) = 0.8413$$
$$\therefore \quad \text{P}(X < b) = 0.1587$$
$$\therefore \quad \text{P}\left(\frac{X - 38.6}{6.3} < \frac{b - 38.6}{6.3}\right) = 0.1587$$
$$\therefore \quad \text{P}\left(Z < \frac{b - 38.6}{6.3}\right) = 0.1587$$

Comparing with $(*)$, $\dfrac{b - 38.6}{6.3} = -\dfrac{a}{6.3}$
$$\therefore \quad b - 38.6 \approx -6.30$$
$$\therefore \quad b \approx 32.3 \text{ g}$$

4
$$\begin{aligned} f(x) &= a(x+1)x(x-1)(x-2), \quad 0 < x < 1 \\ &= a(x^2 + x)(x^2 - 3x + 2) \\ &= a(x^4 - 2x^3 - x^2 + 2x) \end{aligned}$$

a
$$\int_0^1 f(x)\, dx = 1$$
$$\therefore \quad a \int_0^1 x^4 - 2x^3 - x^2 + 2x \, dx = 1$$
$$\therefore \quad a\left[\frac{x^5}{5} - \frac{x^4}{2} - \frac{x^3}{3} + x^2\right]_0^1 = 1$$
$$\therefore \quad a\left[\tfrac{1}{5} - \tfrac{1}{2} - \tfrac{1}{3} + 1\right] = 1$$
$$\therefore \quad a\left(\tfrac{11}{30}\right) = 1$$
$$\therefore \quad a = \tfrac{30}{11}$$

b The mode is the value of x when $f(x)$ is a maximum.
$$\begin{aligned} f(x) &= a(x^4 - 2x^3 - x^2 + 2x) \\ \therefore \quad f'(x) &= a(4x^3 - 6x^2 - 2x + 2) \\ &= 2a(2x^3 - 3x^2 - x + 1) \\ &= 2a(2x - 1)(x^2 - x - 1) \end{aligned}$$
$$\therefore \quad f'(x) = 0 \quad \text{when} \quad x = \tfrac{1}{2} \quad \{0 < x < 1\}$$



$\therefore$ the mode is $\frac{1}{2}$.

c
$$\begin{aligned} f(\tfrac{1}{2} + x) &= a(\tfrac{1}{2} + x + 1)(\tfrac{1}{2} + x)(\tfrac{1}{2} + x - 1)(\tfrac{1}{2} + x - 2) \\ &= a(\tfrac{3}{2} + x)(\tfrac{1}{2} + x)(-\tfrac{1}{2} + x)(-\tfrac{3}{2} + x) \\ f(\tfrac{1}{2} - x) &= a(\tfrac{1}{2} - x + 1)(\tfrac{1}{2} - x)(\tfrac{1}{2} - x - 1)(\tfrac{1}{2} - x - 2) \\ &= a(\tfrac{3}{2} - x)(\tfrac{1}{2} - x)(-\tfrac{1}{2} - x)(-\tfrac{3}{2} - x) \\ &= a(-1)(-\tfrac{3}{2} + x)(-1)(-\tfrac{1}{2} + x)(-1)(\tfrac{1}{2} + x)(-1)(\tfrac{3}{2} + x) \\ &= a(-\tfrac{3}{2} + x)(-\tfrac{1}{2} + x)(\tfrac{1}{2} + x)(\tfrac{3}{2} + x) \end{aligned}$$
$$\therefore \quad f(\tfrac{1}{2} - x) = f(\tfrac{1}{2} + x)$$

d From **c**, $f(\frac{1}{2} - x) = f(\frac{1}{2} + x)$ for all $0 < x < 1$.

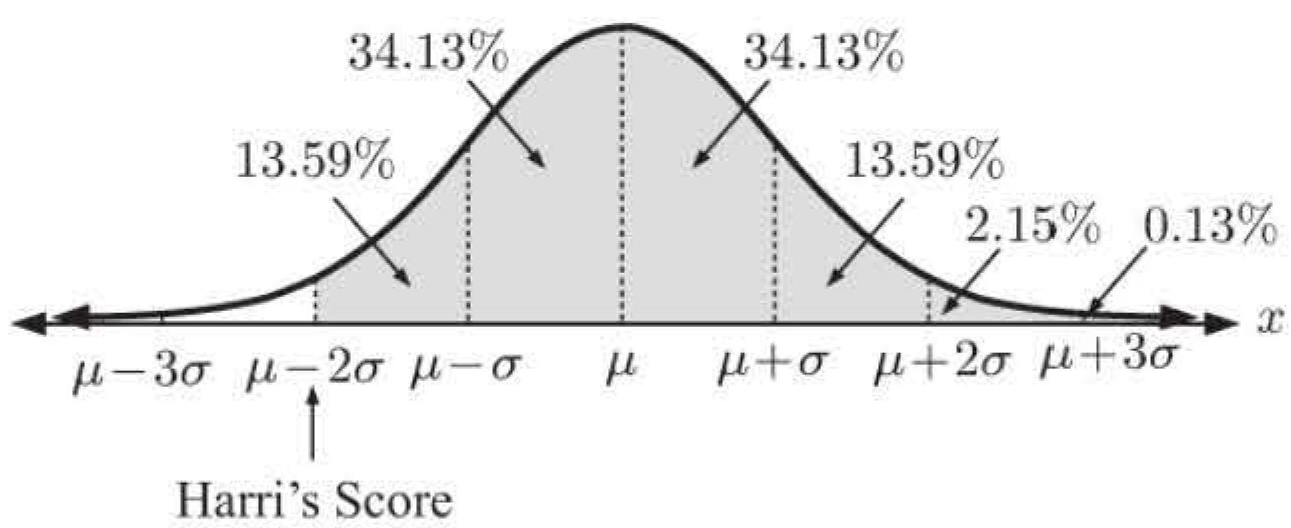
$\therefore$ $f(x)$ is symmetric about $x = \frac{1}{2}$.

$\therefore$ half the data values lie below $x = \frac{1}{2}$ and half lie above.

$\therefore$ median $= \frac{1}{2}$

5 **a** Harri's score is 2 standard deviations below the mean.

b



Proportion of students who scored better than Harri

$\approx 13.59\% + 34.13\% + 34.13\% + 13.59\% + 2.15\% + 0.13\%$

$\approx 97.72\%$

$\approx 97.7\%$

c $\mu = 151$ and $\mu - 2\sigma = 117$

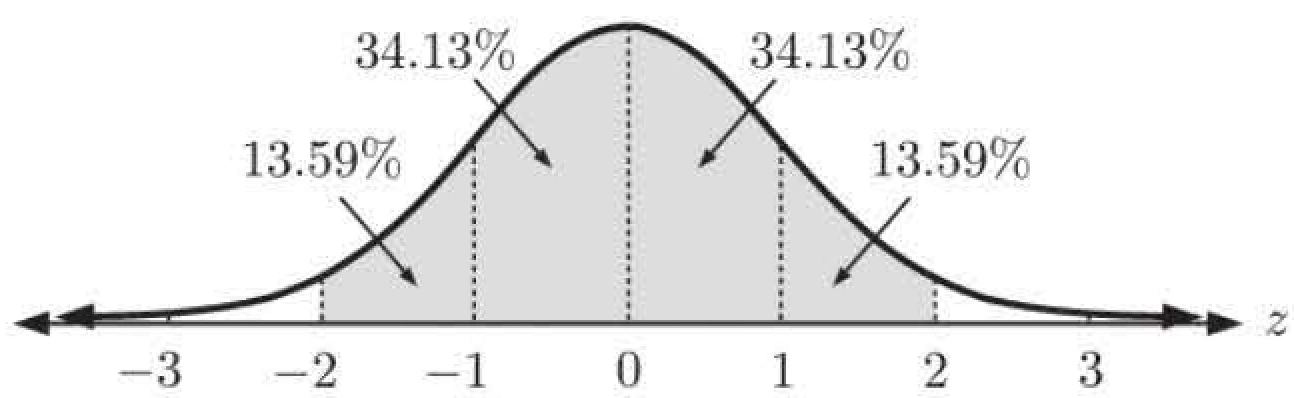
$\therefore\ 151 - 2\sigma = 117$

$\therefore\ -2\sigma = -34$

$\therefore\ \sigma = 17$

The standard deviation was 17.

6



The shaded part of the diagram has an area of approximately 0.95 .

$\therefore\ \mathrm{P}(-2 \leqslant Z \leqslant 2) \approx 0.95$

$\therefore\ k \approx 2$

7 **a**

$\int_0^2 ax(4-x^2)\,dx = 1$

$\therefore\ a\int_0^2 (4x - x^3)\,dx = 1$

$\therefore\ a\left[2x^2 - \dfrac{x^4}{4}\right]_0^2 = 1$

$\therefore\ a(8-4) = 1$

$\therefore\ a = \frac{1}{4}$

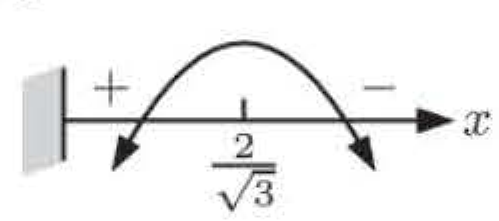
b The mode is the value of x when $f(x)$ is a maximum.

$f(x) = \frac{1}{4}x(4-x^2) = x - \frac{1}{4}x^3$

$\therefore\ f'(x) = 1 - \frac{3}{4}x^2$

$\therefore\ f'(x) = 0$ when $x^2 = \frac{4}{3}$

$\therefore\ x = \frac{2}{\sqrt{3}}$ $\{0 \leqslant x \leqslant 2\}$



$\therefore$ the mode is $\frac{2}{\sqrt{3}}$

c If the median is m, then

$\int_0^m (x - \frac{1}{4}x^3)\,dx = \frac{1}{2}$

$\therefore\ \left[\dfrac{x^2}{2} - \dfrac{x^4}{16}\right]_0^m = \frac{1}{2}$

$\therefore\ \dfrac{m^2}{2} - \dfrac{m^4}{16} = \frac{1}{2}$

$\therefore\ m^4 - 8m^2 + 8 = 0$

$\therefore\ m^2 = \dfrac{8 \pm \sqrt{64-32}}{2}$

$= \dfrac{8 \pm 4\sqrt{2}}{2}$

$= 4 \pm 2\sqrt{2}$

$\therefore\ m^2 = 4 - 2\sqrt{2}$ {as $0 < m < 2$}

$\therefore\ m = \pm\sqrt{4 - 2\sqrt{2}}$

$\therefore\ m = \sqrt{4 - 2\sqrt{2}}$ {as $0 < m < 2$}

$\therefore$ the median is $\sqrt{4 - 2\sqrt{2}}$.

d $\mu = \int_0^2 (x^2 - \frac{1}{4}x^4)\,dx$

$= \left[\dfrac{x^3}{3} - \dfrac{x^5}{20}\right]_0^2$

$= \frac{8}{3} - \frac{32}{20}$

$= \frac{16}{15}$

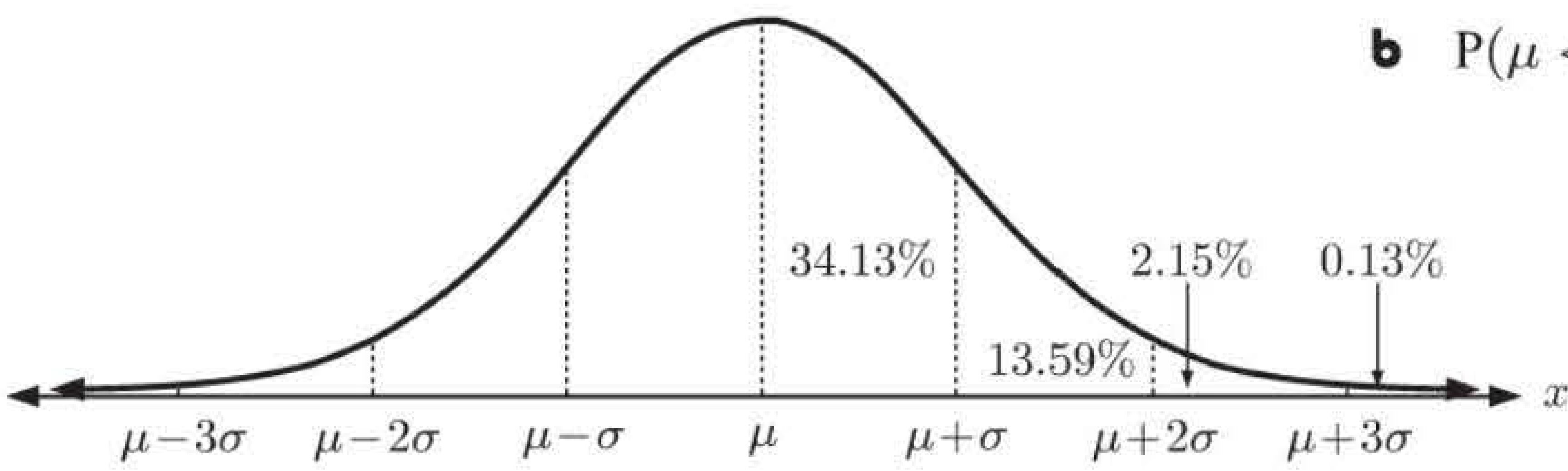
8 Jarrod's z-score is $\dfrac{41-35}{4} = 1.5$

$\therefore$ Paul needs x such that $\dfrac{x-25}{3} = 1.5$

$\therefore \; x = 25 + 4.5 = 29.5$

Paul needs to throw a tennis ball 29.5 m to perform as well as Jarrod.

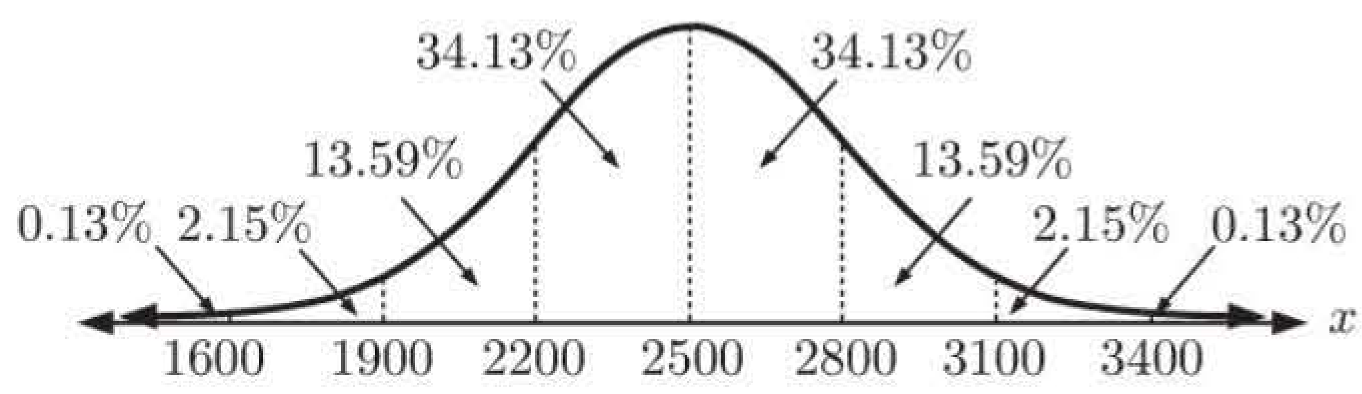
9



a $\text{P}(\mu + \sigma < X < \mu + 2\sigma) \approx 13.59\%$
≈ 0.136

b $\text{P}(\mu < X < \mu + \sigma) \approx 34.13\%$
≈ 0.341

10 If X is the number of bottles sold per day, then $X \sim \text{N}(2500, 300^2)$.



a $\text{P}(X < 1900) \approx 0.13\% + 2.15\%$
$\approx 2.28\%$

b $\text{P}(X > 2200)$
$\approx 2 \times 34.13\% + 13.59\% + 2.15\% + 0.13\%$
$\approx 84.13\%$
$\approx 84.1\%$

c $\text{P}(2200 \leqslant X \leqslant 3100) \approx 34.13\% + 34.13\% + 13.59\%$
$\approx 81.85\%$
$\approx 81.9\%$

11 $f(x) = 2e^{-x}, \quad 0 \leqslant x \leqslant k$

a
$$\begin{aligned}
\int_0^k f(x)\,dx &= 1\\
\therefore \int_0^k 2e^{-x}\,dx &= 1\\
\therefore \left[-2e^{-x}\right]_0^k &= 1\\
\therefore -2e^{-k} + 2e^0 &= 1\\
\therefore 2e^{-k} &= 1\\
\therefore -k &= \ln\tfrac{1}{2}\\
\therefore k &= \ln 2
\end{aligned}$$

b
$$\begin{aligned}
&\text{P}(\ln\tfrac{4}{3} < X < \ln\tfrac{5}{3})\\
&= \int_{\ln\frac{4}{3}}^{\ln\frac{5}{3}} 2e^{-x}\,dx\\
&= \left[-2e^{-x}\right]_{\ln\frac{4}{3}}^{\ln\frac{5}{3}}\\
&= -2e^{-\ln\frac{5}{3}} + 2e^{-\ln\frac{4}{3}}\\
&= -2e^{\ln\frac{3}{5}} + 2e^{\ln\frac{3}{4}}\\
&= -2 \times \tfrac{3}{5} + 2 \times \tfrac{3}{4}\\
&= \tfrac{3}{10}\\
&= 0.3
\end{aligned}$$

c $\mu = \int_0^{\ln 2} x f(x)\, dx$

$= \int_0^{\ln 2} 2xe^{-x}\, dx$

We integrate by parts with

$u = 2x \qquad v' = e^{-x}$

$u' = 2 \qquad v = -e^{-x}$

$\therefore \quad \int 2xe^{-x}\, dx$

$= -2xe^{-x} - \int -2e^{-x}\, dx$

$= -2xe^{-x} - 2e^{-x} + c$

$= -2e^{-x}(x+1) + c \quad .... \ (*)$

So, $\mu = \int_0^{\ln 2} 2xe^{-x}\, dx$

$= \left[-2e^{-x}(x+1)\right]_0^{\ln 2}$

$= -2e^{-\ln 2}(\ln 2 + 1) + 2e^0(1)$

$= -2(\frac{1}{2})(\ln 2 + 1) + 2$

$= -\ln 2 - 1 + 2$

$= 1 - \ln 2$

d Now consider $\mathrm{E}(X^2) = \int_0^{\ln 2} 2x^2 e^{-x}\, dx$

We integrate by parts with

$u = 2x^2 \qquad v' = e^{-x}$

$u' = 4x \qquad v = -e^{-x}$

$\therefore \quad \int 2x^2e^{-x}\, dx$

$= -2x^2e^{-x} - \int -4xe^{-x}\, dx$

$= -2x^2e^{-x} + 2\int 2xe^{-x}\, dx$

$= -2x^2e^{-x} + 2(-2e^{-x}(x+1)) + c$

{using $*$}

$= -2e^{-x}(x^2 + 2x + 2) + c$

$\therefore \quad \mathrm{E}(X^2)$

$= \int_0^{\ln 2} 2x^2e^{-x}\, dx$

$= \left[-2e^{-x}(x^2 + 2x + 2)\right]_0^{\ln 2}$

$= -2e^{-\ln 2}((\ln 2)^2 + 2\ln 2 + 2) + 2e^0(2)$

$= -2(\frac{1}{2})((\ln 2)^2 + 2\ln 2 + 2) + 4$

$= 2 - 2\ln 2 - (\ln 2)^2$

$\therefore \quad \mathrm{Var}(X)$

$= \mathrm{E}(X^2) - (\mathrm{E}(X))^2$

$= \mathrm{E}(X^2) - \mu^2$

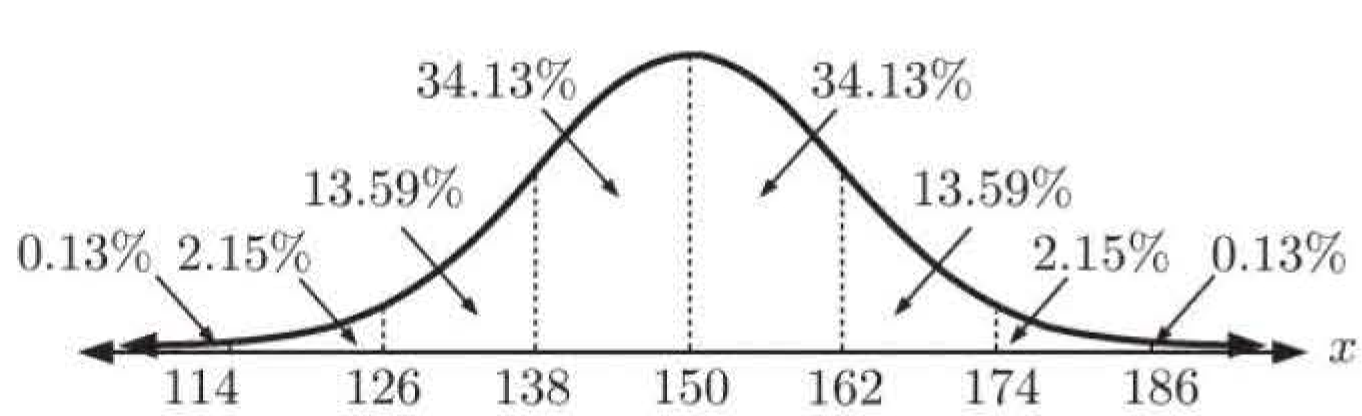
$= 2 - 2\ln 2 - (\ln 2)^2 - (1 - \ln 2)^2$

$= 2 - \cancel{2\ln 2} - (\ln 2)^2 - (1 - \cancel{2\ln 2} + (\ln 2)^2)$

$= 1 - 2(\ln 2)^2$

REVIEW SET 26B

1 $X \sim \mathrm{N}(150,\ 12^2)$



a $\mathrm{P}(138 \leqslant X \leqslant 162)$

$\approx 34.13\% + 34.13\%$

$\approx 68.26\%$

$\approx 68.3\%$

b $\mathrm{P}(126 \leqslant X \leqslant 174)$

$\approx 13.59\% + 34.13\% + 34.13\% + 13.59\%$

$\approx 95.44\%$

$\approx 95.4\%$

c $\mathrm{P}(126 \leqslant X \leqslant 162)$

$\approx 13.59\% + 34.13\% + 34.13\%$

$\approx 81.85\%$

$\approx 81.9\%$

d $\mathrm{P}(162 \leqslant X \leqslant 174)$

$\approx 13.59\%$

$\approx 13.6\%$

2 If random variable X is the arm length in cm then $X \sim \mathrm{N}(64,\ 4^2)$.

a **i** $\mathrm{P}(60 < X < 72)$

$\approx 2 \times 34.13\% + 13.59\%$

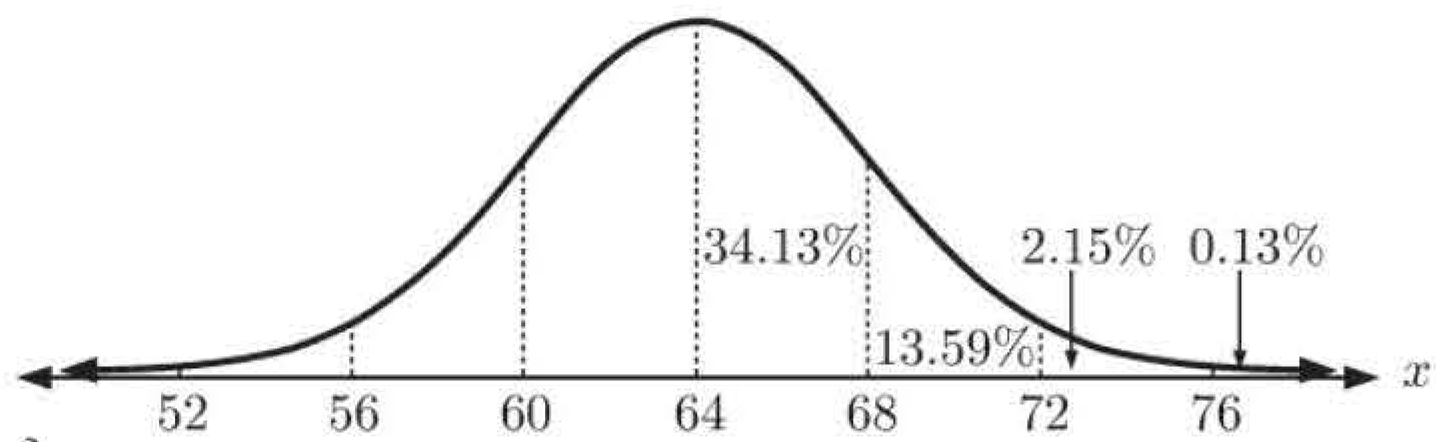
$\approx 81.9\%$

ii $\mathrm{P}(X > 60)$

$\approx 50\% + 34.13\%$

$\approx 84.1\%$

b $\text{P}(56 < X < 64) \approx 0.3413 + 0.1359$
≈ 0.477



c $\text{P}(X > x) = 0.7$
$\therefore \text{P}(X \leqslant x) = 0.3$
$\therefore x \approx 61.9$ {using technology}

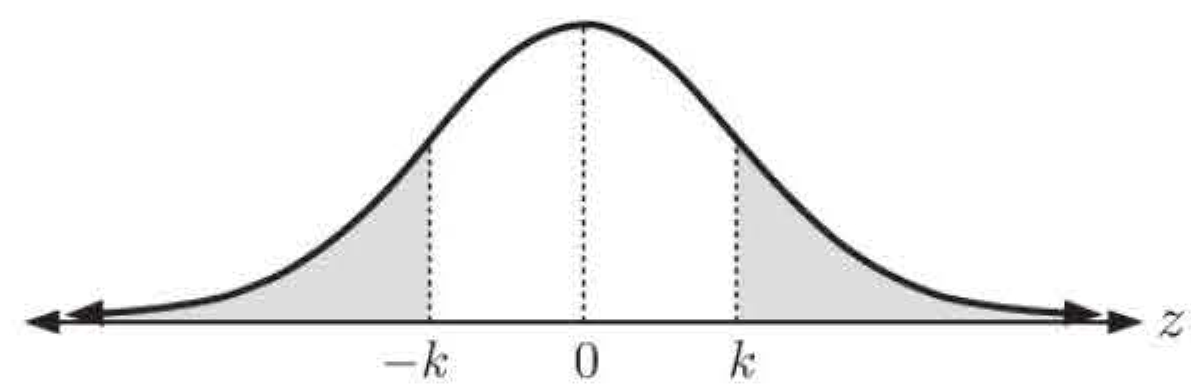
3 If X is the rod length in mm, then $X \sim \text{N}(\mu, 3^2)$.

Now $\text{P}(X < 25) = 0.02$

$$\therefore \text{P}\left(\frac{X-\mu}{3} < \frac{25-\mu}{3}\right) = 0.02$$
$$\therefore \text{P}\left(Z < \frac{25-\mu}{3}\right) = 0.02$$
$$\therefore \frac{25-\mu}{3} \approx -2.0537$$
$$\therefore 25 - \mu \approx -6.161$$
$$\therefore \mu \approx 31.2$$

$\therefore$ the mean rod length is 31.2 mm.

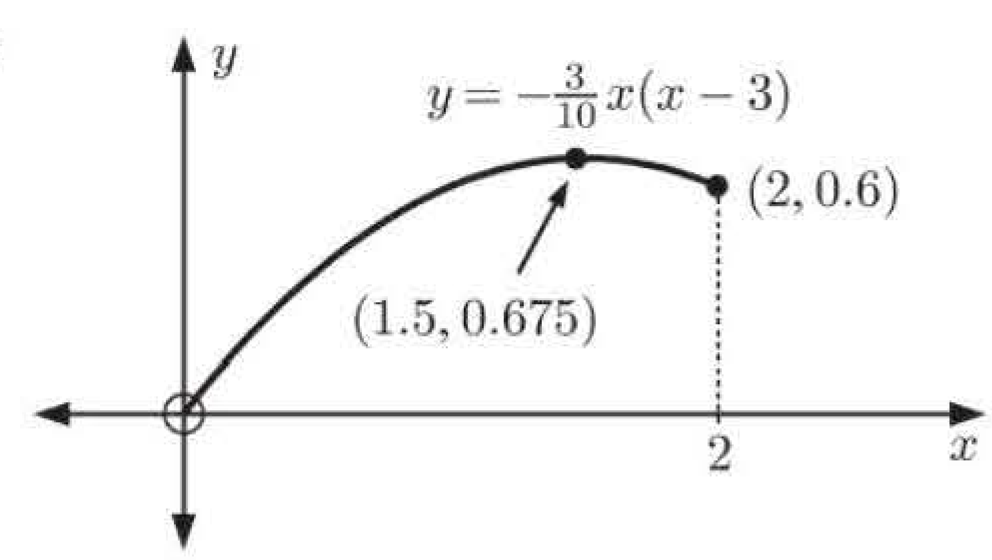
4 $\text{P}(|Z| > k) = 0.376$
$\therefore \text{P}(Z > k \text{ or } Z < -k) = 0.376$



$\therefore \text{P}(Z < -k) = \frac{1}{2}(0.376) = 0.188$
$\therefore -k \approx -0.885$
$\therefore k \approx 0.885$

5 **a**
$$\int_0^2 ax(x-3)\,dx = 1$$
$$\therefore a\int_0^2 (x^2 - 3x)\,dx = 1$$
$$\therefore a\left[\tfrac{1}{3}x^3 - \tfrac{3}{2}x^2\right]_0^2 = 1$$
$$\therefore a\left[\tfrac{8}{3} - 6\right] = 1$$
$$\therefore a\left(-\tfrac{10}{3}\right) = 1$$
$$\therefore a = -\tfrac{3}{10}$$

b



c **i**
$$\begin{aligned}\mu &= \int_0^2 x\,f(x)\,dx\\ &= \int_0^2 -\tfrac{3}{10}x^2(x-3)\,dx\\ &= -\tfrac{3}{10}\int_0^2 (x^3 - 3x^2)\,dx\\ &= -\tfrac{3}{10}\left[\tfrac{1}{4}x^4 - x^3\right]_0^2\,dx\\ &= -\tfrac{3}{10}\left(\tfrac{1}{4}(16) - 8\right)dx\\ &= -\tfrac{3}{10}(-4)\\ &= \tfrac{6}{5} = 1.2\end{aligned}$$

ii $f(x)$ has maximum value when $x = 1.5$
$\therefore$ the mode $= 1.5$

iii If the median is m, then
$$\int_0^m f(x)\,dx = \tfrac{1}{2}$$
$$\int_0^m -\tfrac{3}{10}x(x-3)\,dx = \tfrac{1}{2}$$
$$\therefore \int_0^m (x^2 - 3x)\,dx = -\tfrac{5}{3}$$
$$\left[\tfrac{1}{3}x^3 - \tfrac{3}{2}x^2\right]_0^m = -\tfrac{5}{3}$$
$$\therefore \tfrac{1}{3}m^3 - \tfrac{3}{2}m^2 + \tfrac{5}{3} = 0$$
$$\therefore 2m^3 - 9m^2 + 10 = 0$$
$\therefore m \approx -0.957, 1.24, 4.22$
{using technology}

But $0 \leqslant m \leqslant 2$, so $m \approx 1.24$

iv
$$\begin{aligned}\text{E}(X^2) &= \int_0^2 x^2\,f(x)\,dx\\ &= \int_0^2 -\tfrac{3}{10}x^3(x-3)\,dx\\ &= 1.68 \quad \text{\{using technology\}}\end{aligned}$$
$$\begin{aligned}\therefore \text{Var}(X) &= \text{E}(X^2) - \{\text{E}(X)\}^2\\ &= 1.68 - (1.2)^2\\ &= 0.24\end{aligned}$$

d $\quad \text{P}(1 \leqslant x \leqslant 2)$

$= \int_1^2 -\frac{3}{10}x(x-3)\,dx$

$= 0.65 \quad$ {using technology}

$= \frac{13}{20}$

6 **a** Since Area A = Area B, 20 and 38 must be equal distances away from the mean μ, because of the symmetry of the normal distribution.

$\therefore$ μ is halfway between 20 and 38, so $\mu = \dfrac{20+38}{2} = 29$

Now $\text{P}(X \leqslant 20) = 0.2$

$$\therefore\ \text{P}\left(\frac{X-29}{\sigma} \leqslant \frac{20-29}{\sigma}\right) = 0.2$$

$$\therefore\ \text{P}\left(Z \leqslant -\frac{9}{\sigma}\right) = 0.2$$

$$\therefore\ -\frac{9}{\sigma} \approx -0.8416$$

$$\therefore\ \sigma \approx 10.69$$

$$\therefore\ \mu = 29,\quad \sigma \approx 10.7$$

b Using the values obtained for μ and σ in **a** and technology:

i $\text{P}(X \leqslant 35) \approx 0.713$ **ii** $\text{P}(23 \leqslant X \leqslant 30) \approx 0.250$

7 $X \sim \text{N}(503,\, 2^2)$

a $\quad \text{P}(X < 500)$

$\approx 0.066\,807\,2$

≈ 0.0668

So, approximately 6.68% of the bags are underweight.

b This is a binomial distribution where X is the number of underweight bags,

$n = 20$ and $p = 0.066\,807\,2$

$\therefore$ $\text{P}(X \leqslant 2) \approx 0.854$

{using technology}

8 If X is the marks in the examination, then $X \sim \text{N}(49,\, 15^2)$.

a $\text{P}(X \geqslant 45) \approx 0.6051$

So, $2376 \times 0.6051 \approx 1438$ candidates passed the examination.

b Let k be the minimum mark required for a '7'.

$\therefore$ $\text{P}(X \geqslant k) = 0.07$

$\therefore$ $\text{P}(X < k) = 1 - 0.07 = 0.93$

$\therefore$ $k \approx 71.1$

So the minimum mark required to obtain a '7' is 71.1 marks.

c Let L and U be the lower and upper quartiles of the distribution.

$\therefore$ $\text{P}(X \leqslant L) = 0.25$ and $\text{P}(X \leqslant U) = 0.75$

$\therefore$ $L \approx 38.88$ $\qquad$ $\therefore$ $U \approx 59.12$

$\therefore$ the interquartile range $= U - L \approx 59.12 - 38.88 \approx 20.2$ marks

9 X is the life of a battery in weeks.

X is normally distributed with $\mu = 33.2$ and $\sigma = 2.8$.

a $\text{P}(X \geqslant 35) \approx 0.260$

b We need to find k such that $\text{P}(X \leqslant k) = 0.08$

$\therefore$ $k \approx 29.3$

So, the manufacturer can expect the batteries to last 29.3 weeks before 8% of them fail.

10 **a** $\mathrm{P}(X \leqslant 30) = 0.0832$

$\therefore \mathrm{P}\left(\frac{X-\mu}{\sigma} \leqslant \frac{30-\mu}{\sigma}\right) = 0.0832$

$\therefore \mathrm{P}\left(Z \leqslant \frac{30-\mu}{\sigma}\right) = 0.0832$

$\therefore \frac{30-\mu}{\sigma} \approx -1.383\,864$

$\therefore 30-\mu \approx -1.383\,864\sigma$ .... (1)

and $\mathrm{P}(X \geqslant 90) = 0.101$

$\therefore \mathrm{P}(X < 90) = 0.899$

$\therefore \mathrm{P}\left(\frac{X-\mu}{\sigma} < \frac{90-\mu}{\sigma}\right) = 0.899$

$\therefore \mathrm{P}\left(Z < \frac{90-\mu}{\sigma}\right) = 0.899$

$\therefore \frac{90-\mu}{\sigma} \approx 1.275\,874$

$\therefore 90-\mu \approx 1.275\,874\sigma$ .... (2)

Solving (1) and (2) simultaneously, we get $\mu \approx 61.218 \approx 61.2$ and $\sigma \approx 22.559 \approx 22.6$.

b $\mathrm{P}(-7 \leqslant X-\mu \leqslant 7) \approx \mathrm{P}(-7 \leqslant X - 61.218 \leqslant 7)$

$\approx \mathrm{P}(54.218 \leqslant X \leqslant 68.218)$

≈ 0.244

11 **a** The relative difficulty of each test is not known. We would need the mean mark and standard deviation for each test.

b Kerry's English z-score $= \frac{26-22}{4}$

$= \frac{4}{4}$

$= 1$

Kerry's Chemistry z-score $= \frac{82-75}{7}$

$= \frac{7}{7}$

$= 1$

Since the z-scores are the same, Kerry's performance relative to the rest of the class is the same in both tests.

REVIEW SET 26C

1 **a** The middle 68% of the distribution lies between 16.2 and 21.4, and the middle 68% of data lies between one standard deviation of the mean.

$\therefore \mu \approx \frac{16.2+21.4}{2}$

$\therefore \mu \approx \frac{37.6}{2}$

$\therefore \mu \approx 18.8$

and $\sigma \approx 18.8 - 16.2$

$\therefore \sigma \approx 2.6$

b The middle 95% of the data lies between 2 standard deviations of the mean.

$\mu - 2\sigma \approx 18.8 - 2 \times 2.6$

≈ 13.6

and $\mu + 2\sigma \approx 18.8 + 2 \times 2.6$

≈ 24.0

$\therefore$ the middle 95% of the data lies between 13.6 and 24.0.

2 Using technology:

a $\mathrm{P}(X \geqslant 22) \approx 0.364$

b $\mathrm{P}(18 \leqslant X \leqslant 22) \approx 0.356$

c $\mathrm{P}(X \leqslant k) = 0.3$

$\therefore k \approx 18.2$

3 $\mathrm{P}(-0.524 < X-\mu < 0.524) = \mathrm{P}\left(\frac{-0.524}{2} < \frac{X-\mu}{2} < \frac{0.524}{2}\right)$

$= \mathrm{P}(-0.262 < Z < 0.262)$

≈ 0.207 {using technology}

4 If X is the length of a rod, then $X \sim \mathrm{N}(\mu,\, 6^2)$.

$$\begin{aligned}
\text{Now} \quad \mathrm{P}(X \geqslant 89.52) &= 0.0563 \\
\therefore \quad \mathrm{P}(X < 89.52) &= 1 - 0.0563 \\
\therefore \quad \mathrm{P}\left(\frac{X-\mu}{6} < \frac{89.52-\mu}{6}\right) &= 0.9437 \\
\therefore \quad \mathrm{P}\left(Z < \frac{89.52-\mu}{6}\right) &= 0.9437 \\
\therefore \quad \frac{89.52-\mu}{6} &\approx 1.5866 \\
\therefore \quad 89.52 - \mu &\approx 9.52 \\
\therefore \quad \mu &\approx 80.0
\end{aligned}$$

So, the mean is 80.0 cm.

Since the normal distribution is symmetrical and bell-shaped, the median and modal lengths are also 80.0 cm.

5 **a** T is the lifetime in years of a solar cell component.

$$\begin{aligned}
\therefore \quad \mathrm{P}(T \leqslant 1) &= \int_0^1 0.4e^{-0.4t}\, dt \\
&= \left[-e^{-0.4t}\right]_0^1 \\
&= -e^{-0.4} - (-e^0) \\
&= 1 - e^{-0.4} \\
&\approx 0.329\,68 \\
&\approx 0.330
\end{aligned}$$

b Let X be the number of components not working after one year.

Then $X \sim \mathrm{B}(5,\, 0.329\,68)$

$\therefore$ P(solar cell still operates)

$= \mathrm{P}(X \leqslant 2)$ {at least 3 work}

≈ 0.796

6

$$\begin{aligned}
\mathrm{P}(X < 90) &\approx 0.975 \\
\therefore \quad \mathrm{P}\left(\frac{X-50}{\sigma} < \frac{90-50}{\sigma}\right) &\approx 0.975 \\
\therefore \quad \mathrm{P}\left(Z < \frac{40}{\sigma}\right) &\approx 0.975 \\
\therefore \quad \frac{40}{\sigma} &\approx 1.959\,96 \\
\therefore \quad \sigma &\approx 20.409
\end{aligned}$$

So, $X \sim \mathrm{N}(50,\, 20.409^2)$

Now, the shaded area $= \mathrm{P}(X \geqslant 80)$

≈ 0.0708 units^2

7 If X is the weight of an apple, then $X \sim \mathrm{N}(300,\, 50^2)$.

a $\mathrm{P}(250 \leqslant X \leqslant 350)$

$\approx 0.682\,689\,49$

$\approx 68.3\%$

b This is a binomial distribution where X is the number of apples that are fit for sale.

$n = 100$ and $p \approx 0.682\,689\,49$

$$\begin{aligned}
\mathrm{P}(X \geqslant 75) &= 1 - \mathrm{P}(X \leqslant 74) \\
&\approx 1 - 0.911\,645\,43 \\
&\approx 0.0884
\end{aligned}$$

8 **a** Consider the integral $\displaystyle\int_0^1 \frac{4}{1+x^2}\, dx$.

Let $x = \tan\theta$, $\dfrac{dx}{d\theta} = \sec^2\theta$

When $x = 0$, $\theta = 0$, and when $x = 1$, $\theta = \frac{\pi}{4}$

$$\therefore \int_0^1 \frac{4}{1+x^2}\,dx = \int_0^{\frac{\pi}{4}} \frac{4}{1+\tan^2\theta}\sec^2\theta\,d\theta$$
$$= \int_0^{\frac{\pi}{4}} 4\,d\theta \qquad \{1+\tan^2\theta = \sec^2\theta\}$$
$$= \left[4\theta\right]_0^{\frac{\pi}{4}} = \pi$$

$\therefore \int_0^1 \frac{4}{1+x^2}\,dx \neq 1$, and so $f(x)$ cannot be a probability density function.

b $k\,f(x) = \begin{cases} \dfrac{4k}{1+x^2} & \text{for } 0 \leqslant x \leqslant 1 \\ 0 & \text{otherwise} \end{cases}$

For a probability density function, $\int_0^1 \frac{4k}{1+x^2}\,dx = 1$

$$\therefore k\int_0^1 \frac{4}{1+x^2}\,dx = 1$$
$$\therefore k(\pi) = 1$$
$$\therefore k = \tfrac{1}{\pi}$$

c $\mu = \frac{1}{\pi}\int_0^1 \frac{4x}{1+x^2}\,dx$

$= \frac{2}{\pi}\int_0^1 \frac{2x}{1+x^2}\,dx$

$= \frac{2}{\pi}\left[\ln(1+x^2)\right]_0^1 \quad \{\text{as } 1+x^2>0\}$

$= \frac{2}{\pi}(\ln 2 - \ln 1)$

$= \frac{2}{\pi}\ln 2$

$\mathrm{E}(X^2) = \frac{4}{\pi}\int_0^1 \frac{x^2}{1+x^2}\,dx$

$= \frac{4}{\pi}\int_0^1 \left(1 - \frac{1}{1+x^2}\right)dx$

$= \frac{4}{\pi}\int_0^1 1\,dx - \frac{4}{\pi}\int_0^1 \frac{1}{1+x^2}\,dx$

$= \frac{4}{\pi}[\,x\,]_0^1 - \frac{4}{\pi}[\arctan x\,]_0^1$

$= \frac{4}{\pi} - \frac{4}{\pi}\times\frac{\pi}{4}$

$= \frac{4}{\pi} - 1$

$\therefore \mathrm{Var}(X) = \mathrm{E}(X^2) - \mu^2$

$= \frac{4}{\pi} - 1 - \left(\frac{2}{\pi}\ln 2\right)^2$

$= \frac{4}{\pi} - 1 - \left(\frac{2\ln 2}{\pi}\right)^2$

9 If X is the volume of drink in mL, then $X \sim \mathrm{N}(376, \sigma^2)$.

Now $\mathrm{P}(X < 375) = 0.023$

$\therefore \mathrm{P}\left(\frac{X-376}{\sigma} < \frac{375-376}{\sigma}\right) = 0.023$

$\therefore \mathrm{P}\left(Z < \frac{-1}{\sigma}\right) = 0.023$

$\therefore -\frac{1}{\sigma} \approx -1.995$

$\therefore \sigma \approx 0.501$

$\therefore$ the standard deviation is 0.501 mL.

10 If X is the height of an 18 year old boy, then $X \sim \mathrm{N}(187, \sigma^2)$.

Now $\mathrm{P}(X > 193) = 0.15$

$\therefore \mathrm{P}(X \leqslant 193) = 0.85$

$\therefore \mathrm{P}\left(\frac{X-187}{\sigma} \leqslant \frac{193-187}{\sigma}\right) = 0.85$

$\therefore \mathrm{P}\left(Z \leqslant \frac{6}{\sigma}\right) = 0.85$

$\therefore \frac{6}{\sigma} \approx 1.0364$

$\therefore \sigma \approx 5.789$

So, $\mathrm{P}(X > 185) \approx 0.635$

$\therefore$ the probability that two 18 year old boys are taller than 185 cm $\approx 0.635^2$

≈ 0.403

11 **a**

$$\int_0^k f(x)\,dx = 1$$

$$\therefore\ \int_0^2 \frac{x}{5}\,dx + \int_2^k \frac{8}{5x^2}\,dx = 1$$

$$\therefore\ \left[\frac{x^2}{10}\right]_0^2 + \left[-\frac{8}{5x}\right]_2^k = 1$$

$$\therefore\ \tfrac{4}{10} + \left(-\frac{8}{5k}\right) - \left(-\tfrac{8}{10}\right) = 1$$

$$\therefore\ -\frac{8}{5k} = -\tfrac{2}{10}$$

$$\therefore\ 10k = 80$$

$$\therefore\ k = 8$$

b If m is the median of X, then $\int_0^m f(x)\,dx = \frac{1}{2}$

$\therefore$ since $\int_0^2 \frac{x}{5}\,dx < \frac{1}{2}$,

$$\int_0^2 \frac{x}{5}\,dx + \int_2^m \frac{8}{5x^2}\,dx = \tfrac{1}{2}$$

$$\therefore\ \tfrac{4}{10} + \left[-\frac{8}{5x}\right]_2^m = \tfrac{1}{2}$$

$$\therefore\ \tfrac{4}{10} + \left(-\frac{8}{5m}\right) - \left(-\tfrac{8}{10}\right) = \tfrac{1}{2}$$

$$\therefore\ -\frac{8}{5m} = -\tfrac{7}{10}$$

$$\therefore\ 35m = 80$$

$$\therefore\ m = \tfrac{16}{7}$$

$\therefore$ the median is $2\frac{2}{7}$

c

$$\begin{aligned}\mu &= \int_0^8 x\,f(x)\,dx \\ &= \int_0^2 \frac{x^2}{5}\,dx + \int_2^8 \frac{8}{5x}\,dx \\ &= \left[\frac{x^3}{15}\right]_0^2 + \left[\tfrac{8}{5}\ln|x|\right]_2^8 \\ &= \tfrac{8}{15} + \tfrac{8}{5}\ln 8 - \tfrac{8}{5}\ln 2 \\ &\approx 2.75\end{aligned}$$

$$\begin{aligned}\mathrm{E}(X^2) &= \int_0^8 x^2\,f(x)\,dx \\ &= \int_0^2 \frac{x^3}{5}\,dx + \int_2^8 \tfrac{8}{5}\,dx \\ &= \left[\frac{x^4}{20}\right]_0^2 + \left[\tfrac{8}{5}x\right]_2^8 \\ &= \tfrac{16}{20} + \tfrac{64}{5} - \tfrac{16}{5} \\ &= \tfrac{52}{5}\end{aligned}$$

$$\begin{aligned}\therefore\ \mathrm{Var}(X) &= \mathrm{E}(X^2) - \mu^2 \\ &\approx \tfrac{52}{5} - 2.751^2 \\ &\approx 2.83\end{aligned}$$

Chapter 27

MISCELLANEOUS QUESTIONS

EXERCISE 27A

1 **a** $\quad (1-i)^2 = 1 - 2i + i^2 = -2i$

$\therefore\ (1-i)^{4n} = \left[(1-i)^2\right]^{2n} = (-2i)^{2n} = \left[(-2i)^2\right]^n = (-4)^n$

b $(1-i)^{16} = (1-i)^{4\times 4} = (-4)^4 = 256 \quad \{\text{using } (1-i)^{4n} = (-4)^n\}$

c If $z^{16} = 256$ we have a polynomial with real coefficients.
From **b**, $1-i$ is one solution and so $1+i$ must also be a solution {Theorem of real polynomials}
Thus $z = 1 \pm i$ are two solutions of $z^{16} = 256$.

2 $\quad S_n = n^3 + 2n - 1$

Now $u_n = S_n - S_{n-1}, \quad n > 1$

$$\begin{aligned}
&= n^3 + 2n - 1 - \left[(n-1)^3 + 2(n-1) - 1\right] \\
&= n^3 + 2n - 1 - \left[n^3 - 3n^2 + 3n - 1\right] - 2n + 2 + 1 \\
&= n^3 + 2n - 1 - n^3 + 3n^2 - 3n + 1 - 2n + 3 \\
&= 3n^2 - 3n + 3, \quad n > 1
\end{aligned}$$

and $u_1 = S_1 = 2$

$\therefore\ u_1 = 2,\ u_n = 3n^2 - 3n + 3,\ n > 1$

3 Consider $\dfrac{3x-1}{|x+1|} > 2.$

If $x = -1$, LHS is undefined, so $x = -1$ is not a solution.
If $x \neq -1$, $|x+1| > 0$ and so $3x - 1 > 2\,|x+1|$.
So, if $x > -1$, $\quad 3x - 1 > 2x + 2 \quad \therefore\ x > 3$
if $x < -1$, $\quad 3x - 1 > -2x - 2 \quad \therefore\ 5x > -1$ and so $x > -\frac{1}{5}$, which is impossible.
Thus, $x > 3$ is the solution.

4 **a** $z = \dfrac{-1 + i\sqrt{3}}{4}$

$= \frac{1}{2}\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$

$= \frac{1}{2}\operatorname{cis}\left(\frac{2\pi}{3}\right)$

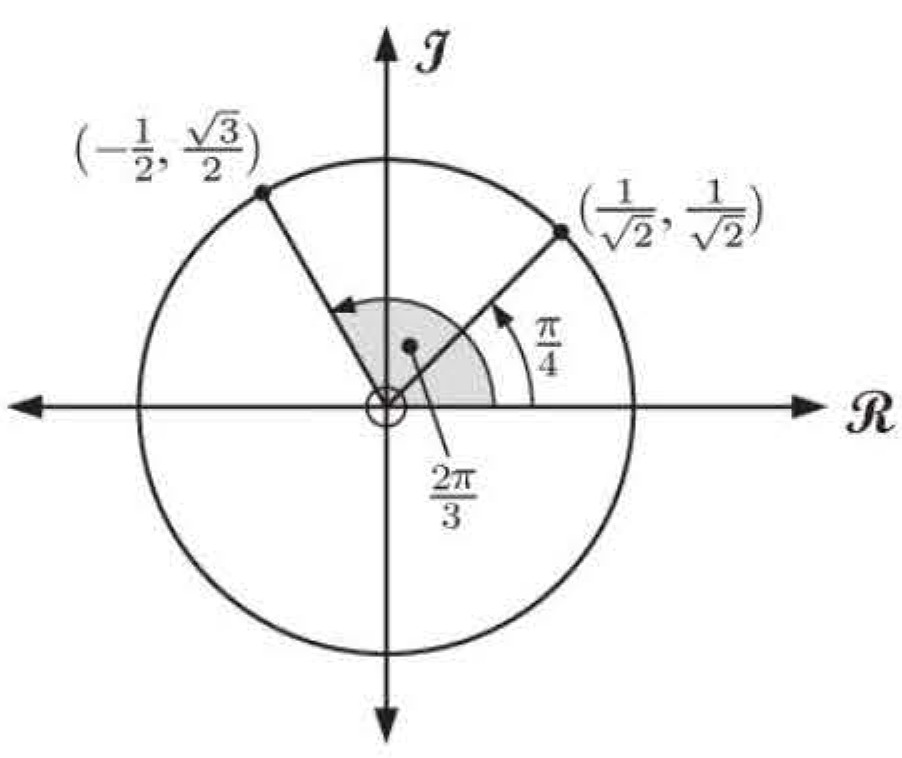
$= \frac{1}{2}\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

$w = \dfrac{\sqrt{2} + i\sqrt{2}}{4}$

$= \frac{1}{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$

$= \frac{1}{2}\operatorname{cis}\left(\frac{\pi}{4}\right)$

$= \frac{1}{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$



b $zw = \frac{1}{2}\operatorname{cis}\left(\frac{2\pi}{3}\right) \times \frac{1}{2}\operatorname{cis}\left(\frac{\pi}{4}\right)$

$= \frac{1}{4}\operatorname{cis}\left(\frac{2\pi}{3} + \frac{\pi}{4}\right)$

$= \frac{1}{4}\operatorname{cis}\left(\frac{11\pi}{12}\right)$

$= \frac{1}{4}\left(\cos\left(\frac{11\pi}{12}\right) + i\sin\left(\frac{11\pi}{12}\right)\right)$

c $zw = \frac{1}{16}(-1 + i\sqrt{3})(\sqrt{2} + i\sqrt{2}) = \frac{1}{16}\left(\left[-\sqrt{2} - \sqrt{6}\right] + i\left[\sqrt{6} - \sqrt{2}\right]\right)$

Equating real and imaginary parts of zw gives $\cos\left(\frac{11\pi}{12}\right) = \frac{-\sqrt{2}-\sqrt{6}}{4}$ and $\sin\left(\frac{11\pi}{12}\right) = \frac{\sqrt{6}-\sqrt{2}}{4}$.

5 As $y^2 = 4x$, $2y\,\dfrac{dy}{dx} = 4$ and so $\dfrac{dy}{dx} = \dfrac{2}{y}$.

But the tangent has gradient m, so at the point of contact, $m = \dfrac{2}{y}$ or $y = \dfrac{2}{m}$.

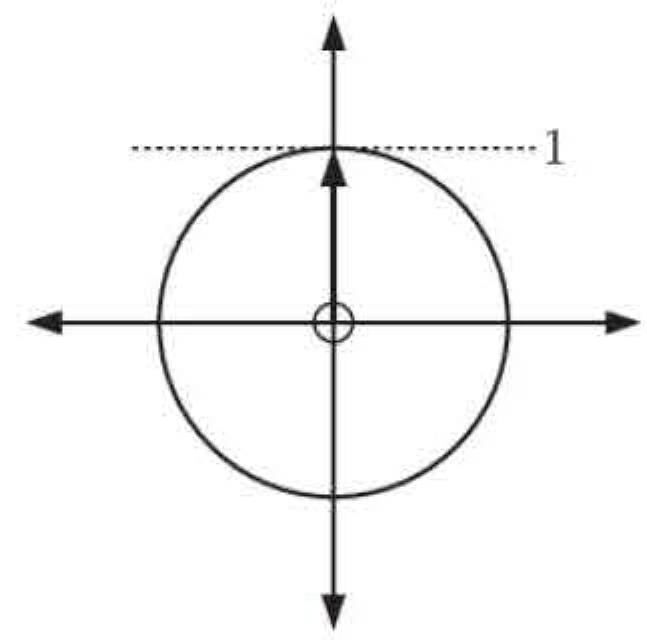
So, the y-coordinate of the point of contact is $\dfrac{2}{m}$.

The x-coordinate of the point of contact is $\dfrac{y^2}{4} = \dfrac{4}{m^2} \div 4 = \dfrac{1}{m^2}$

$\therefore$ the point of contact is $\left(\dfrac{1}{m^2}, \dfrac{2}{m}\right)$.

This point lies on $y = mx + c$, so
$$\begin{aligned} \frac{2}{m} &= m\left(\frac{1}{m^2}\right) + c \\ \therefore\ \frac{2}{m} &= \frac{1}{m} + c \\ \therefore\ c &= \frac{1}{m} \end{aligned}$$

6
$$\begin{aligned} \sin^2 x + \sin x - 2 &= 0, \quad -2\pi \leqslant x \leqslant 2\pi \\ \therefore\ (\sin x + 2)(\sin x - 1) &= 0 \\ \therefore\ \sin x &= -2 \text{ or } 1 \\ \therefore\ \sin x &= 1 \qquad \{\text{as } -1 \leqslant \sin x \leqslant 1\} \\ \therefore\ x &= \tfrac{\pi}{2} + k2\pi, \ k \in \mathbb{Z} \\ \therefore\ x &= -\tfrac{3\pi}{2} \text{ or } \tfrac{\pi}{2} \end{aligned}$$



7 $f(x) = \ln x$ has inverse $f^{-1}(x) = e^x$.

$g(x) = 3 + x$ has inverse given by $x = 3 + y$

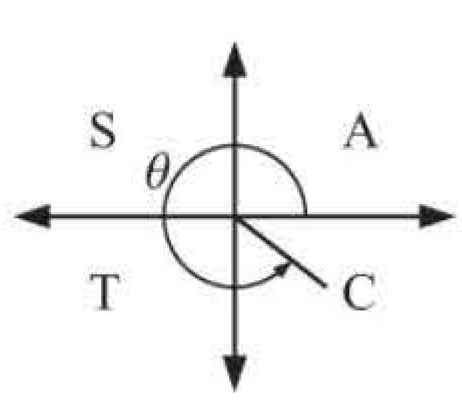
$\therefore\ y = x - 3$ so $g^{-1}(x) = x - 3$.

a
$$\begin{aligned} &f^{-1}(2) \times g^{-1}(2) \\ &= e^2 \times -1 \\ &= -e^2 \end{aligned}$$

b
$$\begin{aligned} (f \circ g)(x) = f(g(x)) &= f(3 + x) \\ &= \ln(3 + x) \end{aligned}$$
$\therefore$ the inverse of $(f \circ g)(x)$ is $x = \ln(3 + y)$
$$\begin{aligned} \therefore\ 3 + y &= e^x \\ \therefore\ y &= e^x - 3 \end{aligned}$$
So, $(f \circ g)^{-1}(x) = e^x - 3$

and $(f \circ g)^{-1}(2) = e^2 - 3$

8 $\sin\theta = -\frac{5}{13}$



a
$$\begin{aligned} &\cos\theta \\ &= \sqrt{1 - \sin^2\theta} \\ &= \sqrt{1 - \tfrac{25}{169}} \\ &= \sqrt{\tfrac{144}{169}} \\ &= \tfrac{12}{13} \end{aligned}$$

b
$$\begin{aligned} &\tan\theta \\ &= \frac{\sin\theta}{\cos\theta} \\ &= -\tfrac{5}{13} \div \tfrac{12}{13} \\ &= -\tfrac{5}{12} \end{aligned}$$

c
$$\begin{aligned} &\sin 2\theta \\ &= 2\sin\theta\cos\theta \\ &= 2\left(\tfrac{-5}{13}\right)\left(\tfrac{12}{13}\right) \\ &= -\tfrac{120}{169} \end{aligned}$$

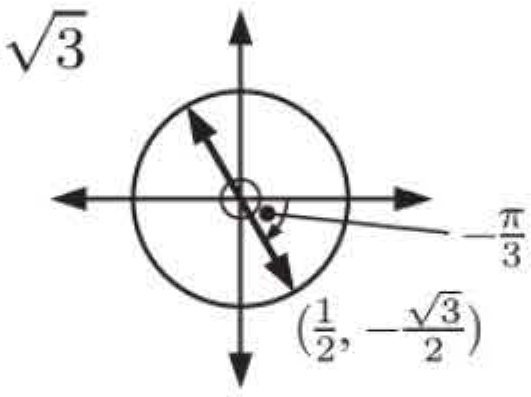
d
$$\begin{aligned} &\sec 2\theta \\ &= \frac{1}{\cos 2\theta} \\ &= \frac{1}{2\cos^2\theta - 1} \\ &= \frac{1}{2\left(\frac{12}{13}\right)^2 - 1} \\ &= \frac{1}{2\left(\frac{144}{169}\right) - 1} \times \frac{169}{169} \\ &= \tfrac{169}{288 - 169} \\ &= \tfrac{169}{119} \end{aligned}$$

9 $\sqrt{3}\cos x \csc x + 1 = 0, \quad 0 \leqslant x \leqslant 2\pi$

$\therefore \sqrt{3}\cos x\left(\frac{1}{\sin x}\right) = -1$

$\therefore \frac{\cos x}{\sin x} = -\frac{1}{\sqrt{3}}$

$\therefore \tan x = -\sqrt{3}$



$\therefore x = \frac{2\pi}{3}$ or $\frac{5\pi}{3}$

10 Let X be the number of snails.

$\mu = \sigma^2 = m$ and $\sigma = d \quad \therefore m = d^2$

Now $\mathrm{P}(X = 8) = \frac{1}{2}\mathrm{P}(X = 7)$

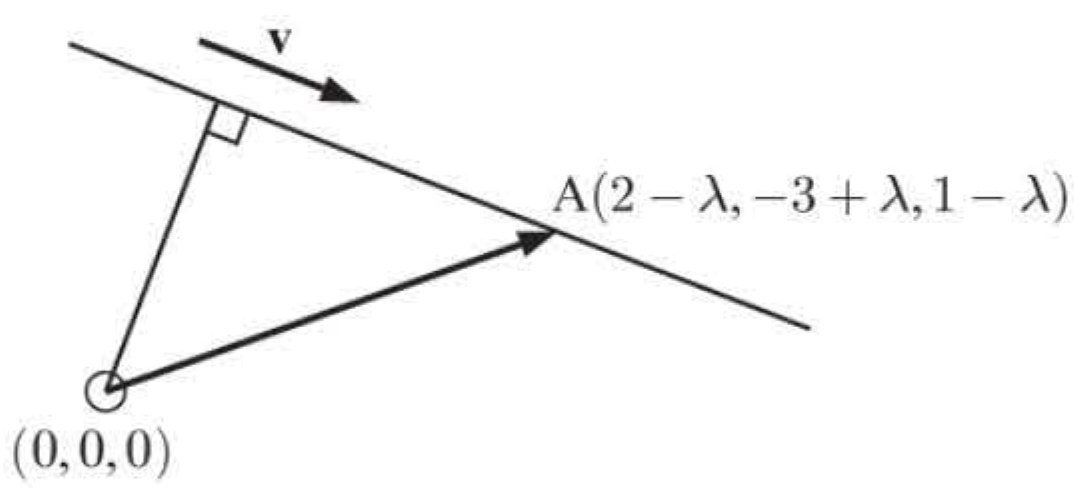
where $\mathrm{P}(X = x) = \frac{m^x e^{-m}}{x!} = \frac{d^{2x}e^{-d^2}}{x!}$

$\therefore \frac{d^{16}e^{-d^2}}{8!} = \frac{1}{2}\frac{d^{14}e^{-d^2}}{7!}$

$\frac{d^2}{8} = \frac{1}{2}$

$\therefore d = 2 \ \{\text{as } d > 0\}$

11



$\overrightarrow{\mathrm{OA}} = \begin{pmatrix} 2-\lambda \\ \lambda-3 \\ 1-\lambda \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$

The shortest distance occurs when $\overrightarrow{\mathrm{OA}} \bullet \mathbf{v} = 0$

$\therefore -(2-\lambda) + \lambda - 3 + (-1)(1-\lambda) = 0$

$\therefore \lambda - 2 + \lambda - 3 - 1 + \lambda = 0$

$\therefore 3\lambda = 6 \quad \therefore \lambda = 2$

So, the point on L that is nearest the origin is $(0, -1, -1)$.

12 $f'(x) > 0$ and $f''(x) < 0$ for all x

$\therefore f(x)$ is increasing and concave downwards for all x.

a $f(2) = 1$ and $f'(2) = 2$

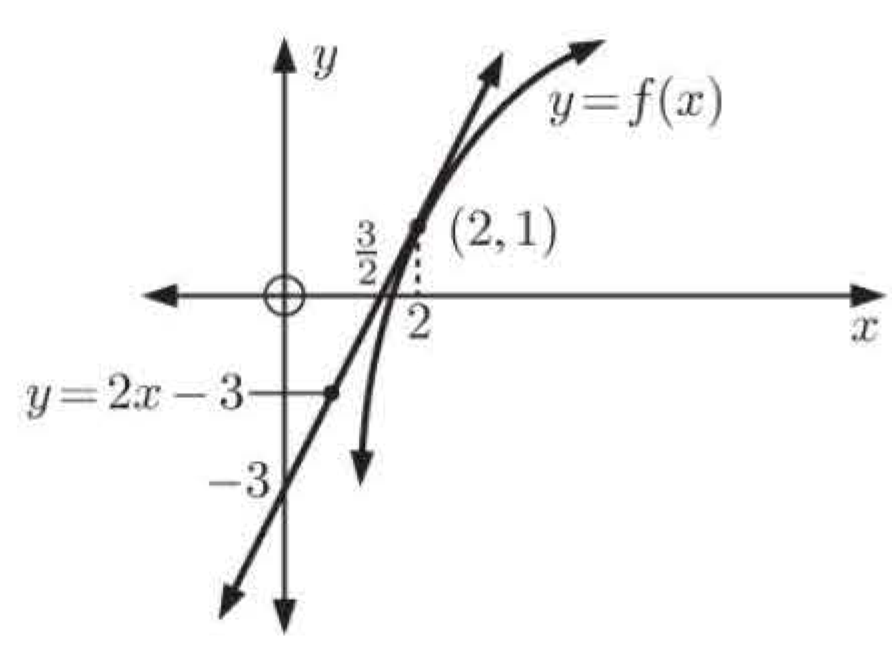
$\therefore$ $(2, 1)$ lies on the curve and the tangent at this point has gradient 2

$\therefore$ the equation of the tangent is $y = 2x + c$

and $1 = 2(2) + c$, so $c = -3$

$\therefore$ the tangent has equation $y = 2x - 3$.

b



c As $f(x)$ is increasing it has *at most one* zero. But $f(x)$ is also concave downwards for all x, so it always lies below the tangent shown. So, for $x < \frac{3}{2}$, the tangent's y-values are negative and so $f(x)$ is also negative. Thus $f(x)$ has *exactly one* zero.

d From the graph, the x-intercept of $y = f(x)$ lies inside $]\frac{3}{2}, 2[$.

13 P_n is "$2n^3 - 3n^2 + n + 31 \geqslant 0$" for $n \in \mathbb{Z}$, $n \geqslant -2$.

Proof: (By the principle of mathematical induction)

(1) If $n = -2$, $2(-2)^3 - 3(-2)^2 + (-2) + 31 = -16 - 12 - 2 + 31 = 1$ which is $\geqslant 0$

$\therefore P_{-2}$ is true.

(2) If P_k is assumed true then $2k^3 - 3k^2 + k + 31 \geqslant 0$

Thus $2(k+1)^3 - 3(k+1)^2 + (k+1) + 31$

$$= 2(k^3 + 3k^2 + 3k + 1) - 3(k^2 + 2k + 1) + k + 32$$

$$= \left[2k^3 - 3k^2 + k + 31\right] + 6k^2 + 6k + 2 - 6k - 3 + 1$$

$$= \underbrace{\left[2k^3 - 3k^2 + k + 31\right]}_{\geqslant 0 \ \{\text{using } P_k\}} + \underbrace{6k^2}_{\geqslant 0 \text{ as } k^2 \geqslant 0 \text{ for all } k}$$

$$\geqslant 0$$

Thus P_{k+1} is true whenever P_k is true.

$\therefore$ since P_{-2} is true, P_n is true for all $n \in \mathbb{Z}$, $n \geqslant -2$ {Principle of mathematical induction}

14 We integrate by parts with $u = \arctan x$ $\quad v' = x$

$$u' = \frac{1}{1+x^2} \qquad v = \frac{x^2}{2}$$

$\therefore \quad \int x \arctan x \, dx$

$$= \arctan x \left(\frac{x^2}{2}\right) - \int \frac{x^2}{2(1+x^2)}\, dx$$

$$= \tfrac{1}{2}x^2 \arctan x - \tfrac{1}{2}\int \frac{1+x^2-1}{1+x^2}\, dx$$

$$= \tfrac{1}{2}x^2 \arctan x - \tfrac{1}{2}\int \left(1 - \frac{1}{1+x^2}\right) dx$$

$$= \tfrac{1}{2}x^2 \arctan x - \frac{x}{2} + \tfrac{1}{2}\arctan x + c$$

Check:

$$\frac{d}{dx}\left(\tfrac{1}{2}x^2 \arctan x - \tfrac{1}{2}x + \tfrac{1}{2}\arctan x + c\right)$$

$$= x \arctan x + \tfrac{1}{2}x^2\left(\frac{1}{1+x^2}\right) - \tfrac{1}{2} + \tfrac{1}{2}\left(\frac{1}{1+x^2}\right) + 0$$

$$= x \arctan x + \frac{\tfrac{1}{2}x^2 - \tfrac{1}{2}(1+x^2) + \tfrac{1}{2}}{1+x^2}$$

$$= x \arctan x + \frac{0}{1+x^2}$$

$$= x \arctan x \quad \checkmark$$

15 **a** $\log_2(x^2 - 2x + 1) = 1 + \log_2(x-1)$

$\therefore \log_2(x-1)^2 - \log_2(x-1) = 1$

$\therefore 2\log_2(x-1) - \log_2(x-1) = 1$

$\therefore \log_2(x-1) = 1$

$\therefore x - 1 = 2^1$

$\therefore x = 3$

b $3^{2x+1} = 5(3^x) + 2$

$\therefore 3(3^x)^2 - 5(3^x) - 2 = 0$

$\therefore 3m^2 - 5m - 2 = 0 \quad \{m = 3^x\}$

$\therefore (3m+1)(m-2) = 0$

$\therefore m = -\tfrac{1}{3}$ or 2

$\therefore 3^x = -\tfrac{1}{3}$ or $3^x = 2$

The first equation is impossible as $3^x > 0$ for all x.

$\therefore 3^x = 2$

$\therefore x = \dfrac{\ln 2}{\ln 3}$

(or $x = \log_3 2$)

16 P_n is "$\sum_{r=1}^{n} r3^r = \tfrac{3}{4}\left[(2n-1)3^n + 1\right]$" for $n \in \mathbb{Z}^+$.

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, $\text{LHS} = \sum_{r=1}^{1} r3^r = 1(3)^1 = 3$

$\text{RHS} = \tfrac{3}{4}\left[(1)3^1 + 1\right] = \tfrac{3}{4} \times 4 = 3 \qquad \therefore P_1$ is true.

(2) If P_k is assumed true, then $\sum_{r=1}^{k} (r3^r) = \tfrac{3}{4}\left[(2k-1)3^k + 1\right]$

Thus $\sum_{r=1}^{k+1} r3^r = \sum_{r=1}^{k} r3^r + (k+1)3^{k+1}$

$$= \tfrac{3}{4}\left[(2k-1)3^k + 1\right] + (k+1)3^{k+1} \qquad \{\text{using } P_k\}$$

$$= \tfrac{3}{4}\left[(2k-1)3^k + 1 + \tfrac{4}{3}(k+1)3^{k+1}\right]$$

$$= \tfrac{3}{4}\left[(2k-1)3^k + 1 + (4k+4)3^k\right]$$

$$= \tfrac{3}{4}\left[(6k+3)3^k + 1\right]$$

$$= \tfrac{3}{4}\left[(2k+1)3^{k+1} + 1\right]$$

$$= \tfrac{3}{4}\left[(2(k+1)-1)3^{k+1} + 1\right]$$

Thus P_{k+1} is true whenever P_k is true.

$\therefore$ since P_1 is true, P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

17 Since $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1, \quad b^2x^2+a^2y^2=a^2b^2$ .... $(*)$

$$\therefore\ b^2(2x)+a^2(2y)\,\frac{dy}{dx}=0$$

$$\therefore\ \text{at the point } (x_1, y_1),\quad \frac{dy}{dx}=-\frac{b^2x}{a^2y}=-\frac{b^2x_1}{a^2y_1}$$

$$\therefore\ \text{the equation of the tangent is } \frac{y-y_1}{x-x_1}=\frac{-b^2x_1}{a^2y_1}$$

$$\therefore\ a^2y_1y-a^2{y_1}^2=-b^2x_1x+b^2{x_1}^2$$

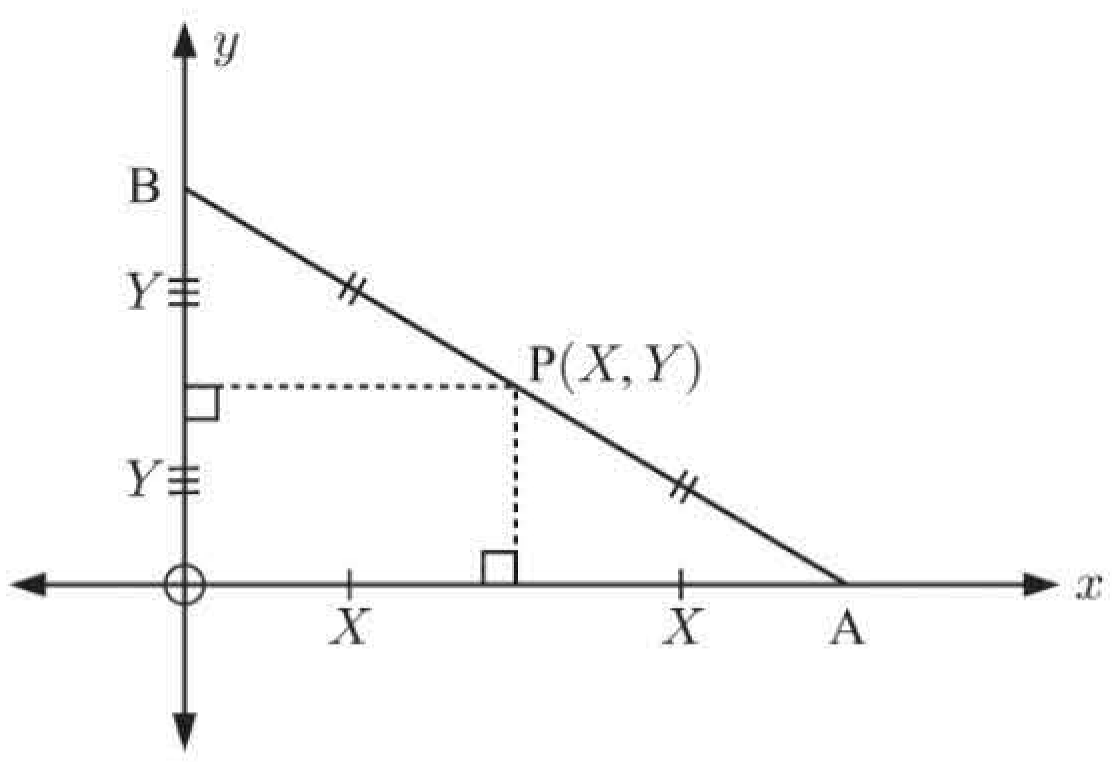
$$\therefore\ a^2y_1y+b^2x_1x=b^2{x_1}^2+a^2{y_1}^2$$

As (x_1, y_1) lies on the curve, $b^2{x_1}^2+a^2{y_1}^2=a^2b^2$ {from $(*)$}

$\therefore$ the equation of the tangent is $b^2x_1x+a^2y_1y=a^2b^2$

or $\left(\dfrac{x_1}{a^2}\right)x+\left(\dfrac{y_1}{b^2}\right)y=1$ {dividing throughout by a^2b^2}

18



Let $\mathrm{P}(X, Y)$ be the midpoint of [AB].

$\therefore$ A is at $(2X, 0)$ and B is at $(0, 2Y)$.

Let [AB] have fixed length l units.

$$\therefore\ (2X)^2+(2Y)^2=l^2$$

$$\therefore\ X^2+Y^2=\left(\frac{l}{2}\right)^2$$

which is the equation of a circle, centre $(0, 0)$ and radius $\dfrac{l}{2}$ units.

19 **a**

$$\frac{Ax+B}{x^2+5}+\frac{C}{x-1}=\frac{Ax+B}{x^2+5}-\frac{C}{1-x}$$
$$=\frac{(Ax+B)(1-x)-C(x^2+5)}{(x^2+5)(1-x)}$$
$$=\frac{-(A+C)x^2+(A-B)x+(B-5C)}{(x^2+5)(1-x)}$$

So, if $\dfrac{x+5}{(x^2+5)(1-x)}=\dfrac{Ax+B}{x^2+5}+\dfrac{C}{x-1}$ for all x then

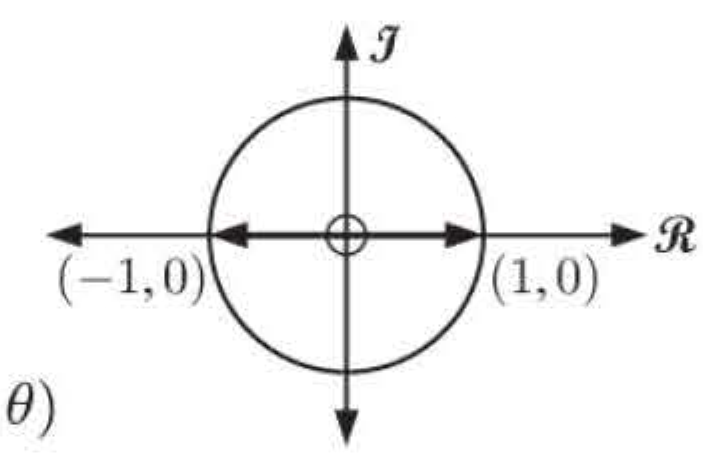
$$-(A+C)=0,\ A-B=1,\ \text{and}\ B-5C=5$$

Solving these simultaneously gives $A=1,\ B=0,\ C=-1$

b

$$\int_2^4 \frac{x+5}{(x^2+5)(1-x)}\,dx=\int_2^4\left(\frac{x}{x^2+5}-\frac{1}{x-1}\right)dx$$
$$=\tfrac{1}{2}\int_2^4 \frac{2x}{x^2+5}\,dx-\int_2^4\frac{1}{x-1}\,dx$$
$$=\tfrac{1}{2}\left[\ln\left|x^2+5\right|\right]_2^4-[\ln|x-1|]_2^4$$
$$=\tfrac{1}{2}(\ln 21-\ln 9)-(\ln 3-\ln 1)$$
$$=\tfrac{1}{2}\ln\left(\tfrac{7}{3}\right)-\ln 3$$
$$=\tfrac{1}{2}(\ln 7-\ln 3)-\ln 3$$
$$=\tfrac{1}{2}\ln 7-\tfrac{3}{2}\ln 3$$

20 **a** Let $z = |z| \operatorname{cis}\theta$, $\quad \therefore \quad \dfrac{1}{z} = \dfrac{\operatorname{cis} 0}{|z|\operatorname{cis}\theta} = \dfrac{1}{|z|}\operatorname{cis}(-\theta)$



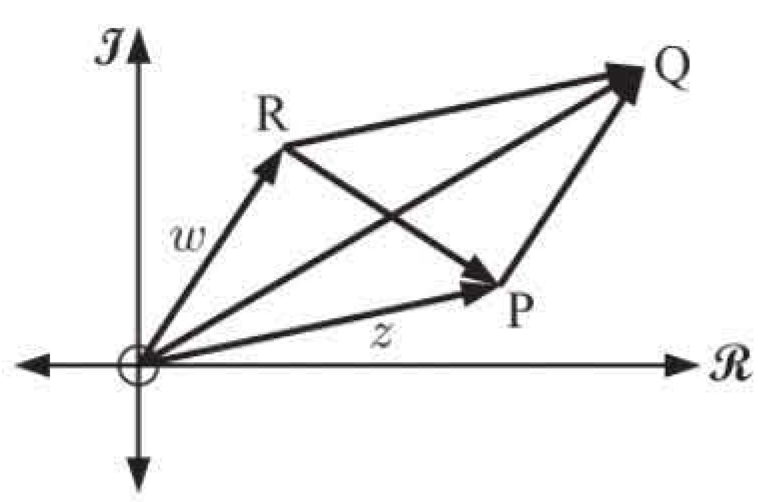
$$\begin{aligned}
\therefore \quad z + \frac{1}{z} &= |z|\operatorname{cis}\theta + \frac{1}{|z|}\operatorname{cis}(-\theta) \\
&= |z|(\cos\theta + i\sin\theta) + \frac{1}{|z|}(\cos\theta - i\sin\theta) \\
&= \cos\theta\left(|z| + \frac{1}{|z|}\right) + i\sin\theta\left(|z| - \frac{1}{|z|}\right)
\end{aligned}$$

Now $z + \dfrac{1}{z}$ is real, $\quad \therefore \quad \sin\theta = 0$ or $|z| - \dfrac{1}{|z|} = 0$

$\therefore \quad \theta = k\pi$ or $|z| = 1$

$\therefore \quad z$ is real or $|z| = 1$

b



If $z = \overrightarrow{OP}$ and $w = \overrightarrow{OR}$ then $\overrightarrow{RP} = z - w$ and $\overrightarrow{OQ} = z + w$ in parallelogram OPQR.

So, if $|z + w| = |z - w|$ the diagonals are equal in length, which is only possible when the parallelogram is a rectangle.

Thus, $\widehat{POR} = \frac{\pi}{2}$, so $\arg w - \arg z = \frac{\pi}{2}$

If P and R were interchanged then $\arg z - \arg w = \frac{\pi}{2}$

Thus, the arguments differ by $\frac{\pi}{2}$.

c If $z = r\operatorname{cis}\theta$

$$\begin{aligned}
z^4 &= [r\operatorname{cis}\theta]^4 \\
&= r^4\operatorname{cis}4\theta \\
&\quad \{\text{De Moivre}\}
\end{aligned}$$

$$\begin{aligned}
\frac{1}{z} &= \frac{1}{r\operatorname{cis}\theta} \\
&= \frac{1}{r\operatorname{cis}\theta}\left(\frac{\operatorname{cis}(-\theta)}{\operatorname{cis}(-\theta)}\right) \\
&= \frac{1}{r}\,\frac{\operatorname{cis}(-\theta)}{\operatorname{cis}0} \\
&= \frac{1}{r}\operatorname{cis}(-\theta)
\end{aligned}$$

$$\begin{aligned}
iz^* &= \operatorname{cis}\left(\tfrac{\pi}{2}\right)(r\operatorname{cis}(-\theta)) \\
&= r\operatorname{cis}\left(\tfrac{\pi}{2} - \theta\right)
\end{aligned}$$

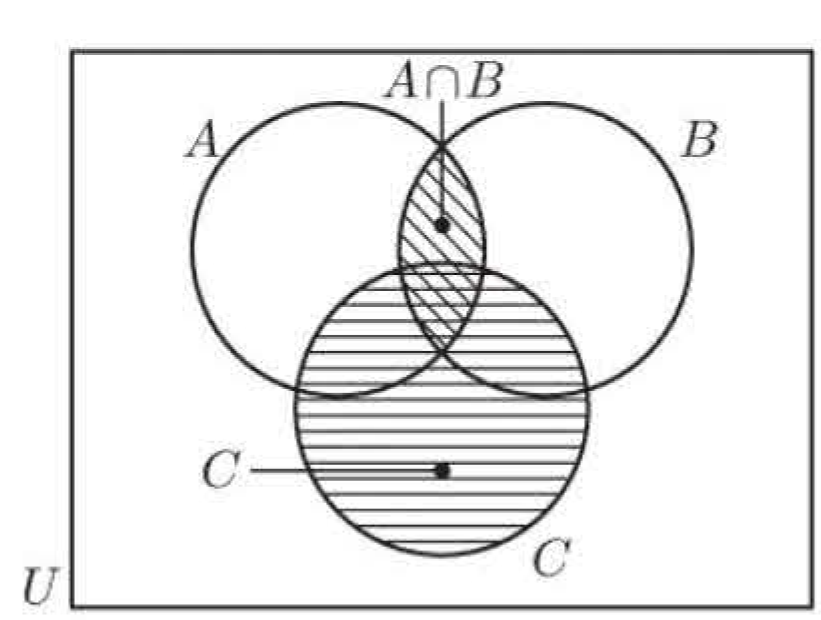
21 **a** **i**

$$\begin{aligned}
&(A \cup B) \cap A' \\
&= A' \cap (A \cup B) \\
&= (A' \cap A) \cup (A' \cap B) \\
&= \varnothing \cup (A' \cap B) \\
&= A' \cap B
\end{aligned}$$

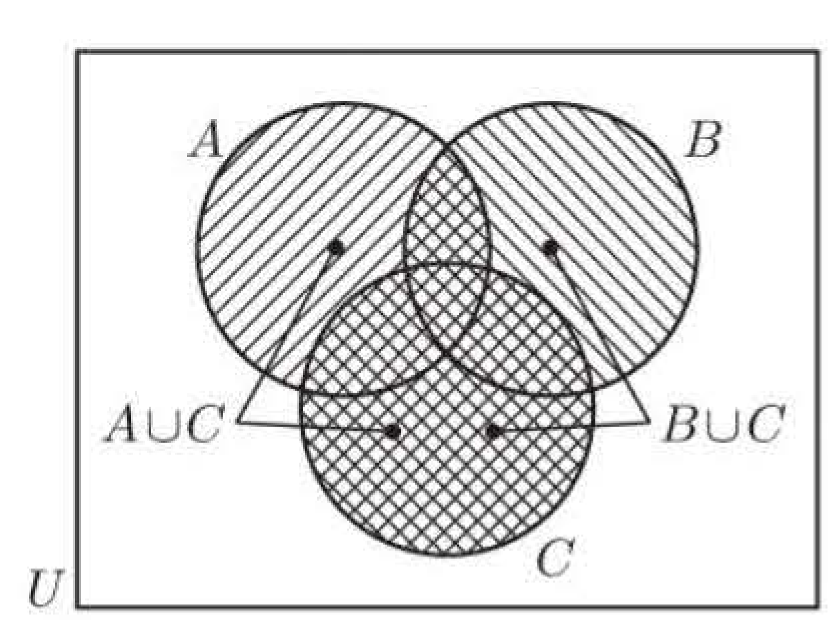
ii

$$\begin{aligned}
&(A \cap B) \cup (A' \cap B) \\
&= (A \cup A') \cap B \\
&= U \cap B \\
&= B
\end{aligned}$$

b



$(A \cap B) \cup C$ includes all that is shaded.



$(A \cup C) \cap (B \cup C)$ includes all parts double shaded.

As the shadings match, the identity is verified.

c **i**

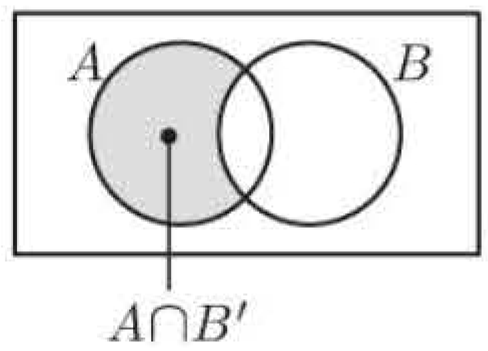
$$\begin{aligned}&\mathrm{P}(A' \cap B')\\ &= \mathrm{P}((A \cup B)')\\ &= 1 - \mathrm{P}(A \cup B)\\ &= 1 - [\mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)]\\ &= 1 - \mathrm{P}(A) - \mathrm{P}(B) + \mathrm{P}(A)\mathrm{P}(B)\\ &= [1 - \mathrm{P}(A)]\,[1 - \mathrm{P}(B)]\\ &= \mathrm{P}(A')\,\mathrm{P}(B')\end{aligned}$$

$\therefore$ A' and B' are independent.

ii

$$\begin{aligned}&\mathrm{P}(A \cap B')\\ &= \mathrm{P}(A) - \mathrm{P}(A \cap B)\\ &= \mathrm{P}(A) - \mathrm{P}(A)\mathrm{P}(B)\\ &= \mathrm{P}(A)\,[1 - \mathrm{P}(B)]\\ &= \mathrm{P}(A)\,\mathrm{P}(B')\end{aligned}$$

$\therefore$ A and B' are independent.



22 **a** Let $x = \sin\theta$, $\dfrac{dx}{d\theta} = \cos\theta$

$$\begin{aligned}\therefore \int \frac{x}{\sqrt{1-x^2}}\,dx &= \int \frac{\sin\theta}{\cos\theta}\cos\theta\,d\theta\\ &= \int \sin\theta\,d\theta\\ &= -\cos\theta + c\\ &= -\sqrt{1-\sin^2\theta} + c\\ &= -\sqrt{1-x^2} + c\end{aligned}$$

b

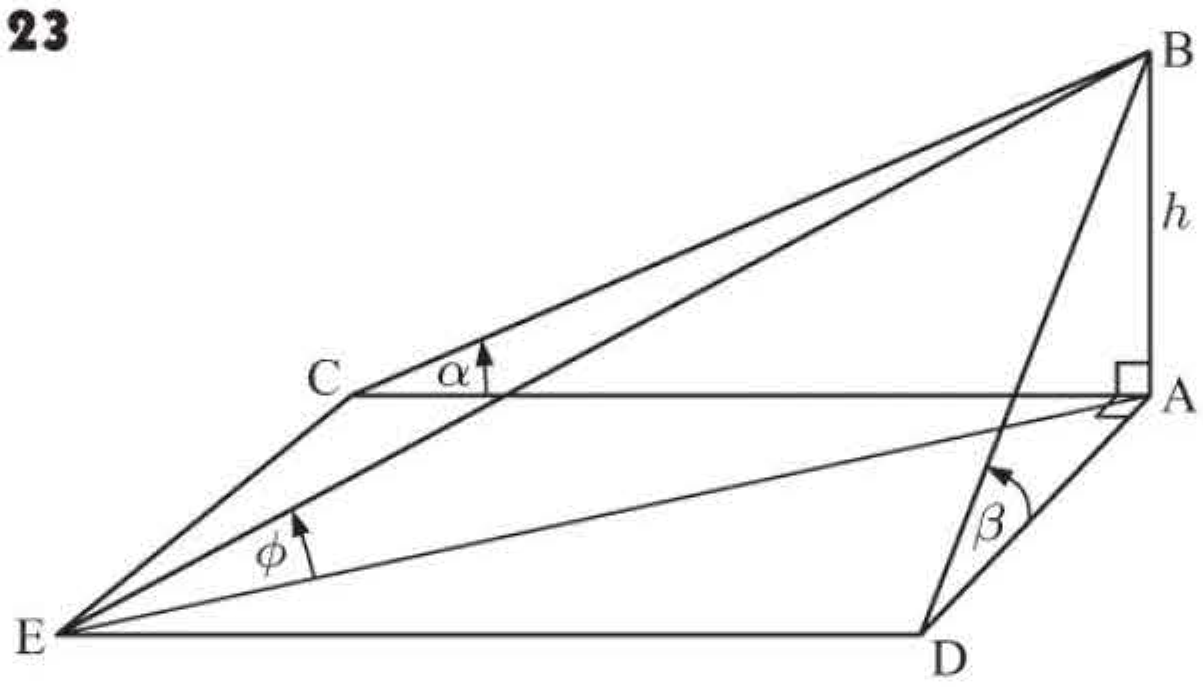
$$\begin{aligned}&\int \frac{1+x}{1+x^2}\,dx\\ &= \int \frac{1}{1+x^2}\,dx + \int \frac{x}{1+x^2}\,dx\\ &= \int \frac{1}{1+x^2}\,dx + \tfrac{1}{2}\int \frac{2x}{1+x^2}\,dx\\ &= \arctan x + \tfrac{1}{2}\ln\left|1+x^2\right| + c\\ &= \arctan x + \tfrac{1}{2}\ln\left(1+x^2\right) + c\end{aligned}$$

$\{x^2 + 1 > 0\}$

c Let $x = \sin\theta$, $\dfrac{dx}{d\theta} = \cos\theta$

$$\begin{aligned}\therefore \int \frac{1}{\sqrt{1-x^2}}\,dx &= \int \frac{1}{\sqrt{1-\sin^2\theta}}\cos\theta\,d\theta\\ &= \int \frac{1}{\cos\theta}\cos\theta\,d\theta\\ &= \int 1\,d\theta = \theta + c\\ &= \arcsin x + c\end{aligned}$$

23



Suppose the pole has height h and the angle of elevation of B from E is ϕ.

$\tan\alpha = \dfrac{h}{\mathrm{AC}}$, $\tan\beta = \dfrac{h}{\mathrm{AD}}$, $\tan\phi = \dfrac{h}{\mathrm{AE}}$

$\therefore$ $\mathrm{AC} = h\cot\alpha$, $\mathrm{AD} = h\cot\beta$, $\mathrm{AE} = h\cot\phi$

But $\mathrm{AE}^2 = \mathrm{AC}^2 + \mathrm{AD}^2$ {Pythagoras}

$$\begin{aligned}&\therefore\ h^2\cot^2\phi = h^2\cot^2\alpha + h^2\cot^2\beta\\ &\therefore\ \cot^2\phi = \cot^2\alpha + \cot^2\beta\\ &\therefore\ \cot\phi = \sqrt{\cot^2\alpha + \cot^2\beta}\quad \{\cot\phi > 0\}\\ &\therefore\ \phi = \operatorname{arccot}\left(\sqrt{\cot^2\alpha + \cot^2\beta}\right)\end{aligned}$$

24 **a**

$$\begin{aligned}&\frac{1}{1+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} + \dots + \frac{1}{\sqrt{99}+\sqrt{100}}\\ &= \frac{1}{1+\sqrt{2}}\left(\frac{1-\sqrt{2}}{1-\sqrt{2}}\right) + \frac{1}{\sqrt{2}+\sqrt{3}}\left(\frac{\sqrt{2}-\sqrt{3}}{\sqrt{2}-\sqrt{3}}\right) + \frac{1}{\sqrt{3}+\sqrt{4}}\left(\frac{\sqrt{3}-\sqrt{4}}{\sqrt{3}-\sqrt{4}}\right) + \dots\\ &\qquad + \frac{1}{\sqrt{99}+\sqrt{100}}\left(\frac{\sqrt{99}-\sqrt{100}}{\sqrt{99}-\sqrt{100}}\right)\\ &= \frac{1-\sqrt{2}}{1-2} + \frac{\sqrt{2}-\sqrt{3}}{2-3} + \frac{\sqrt{3}-\sqrt{4}}{3-4} + \dots + \frac{\sqrt{99}-\sqrt{100}}{99-100}\\ &= -\left(1-\sqrt{2}+\sqrt{2}-\sqrt{3}+\sqrt{3}-\sqrt{4}+\dots+\sqrt{99}-\sqrt{100}\right)\\ &= -(1-10)\\ &= 9\end{aligned}$$

b Using the same technique, $\dfrac{1}{1+\sqrt{2}}+\dfrac{1}{\sqrt{2}+\sqrt{3}}+\dfrac{1}{\sqrt{3}+\sqrt{4}}+\ldots.+\dfrac{1}{\sqrt{n}+\sqrt{n+1}}$

$$= -\left(1-\sqrt{n+1}\right)$$
$$= \sqrt{n+1}-1$$

25 Since $x > y > z > 0$, $\dfrac{1}{x} < \dfrac{1}{y} < \dfrac{1}{z}$

$\therefore\ \dfrac{1}{z}-\dfrac{1}{y}=\dfrac{1}{y}-\dfrac{1}{x}$ and so $\dfrac{1}{x}+\dfrac{1}{z}=\dfrac{2}{y}$ $\left\{\dfrac{1}{x}, \dfrac{1}{y}, \dfrac{1}{z} \text{ in arithmetic progression}\right\}$

$\therefore\ \dfrac{x+z}{xz}=\dfrac{2}{y}$

$\therefore\ xy+yz=2xz$ .... $(*)$

Now $(x-y+z)^2 = x^2+y^2+z^2-2xy+2xz-2yz$

$= x^2+y^2+z^2-2(xy+yz)+2xz$

$\therefore\ (x-y+z)^2 = x^2+y^2+z^2-4xz+2xz$ {using $(*)$}

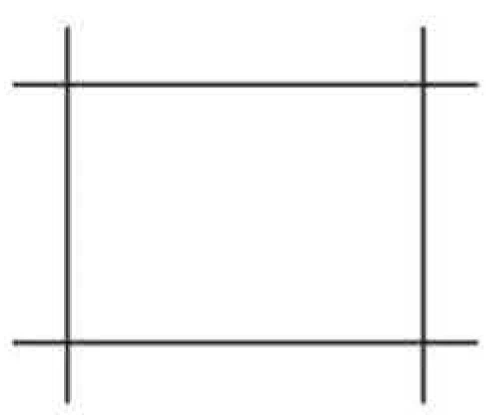
$= x^2-2xz+z^2+y^2$

$= (x-z)^2+y^2$

Hence $x-z$, y, and $x-y+z$ form the sides of a right angled triangle.

26 Each rectangle is determined by choosing the two pairs of opposite sides.

This can be done in $\binom{m+2}{2}\times\binom{n+2}{2}=\dfrac{(m+2)(m+1)(n+2)(n+1)}{4}$ ways.



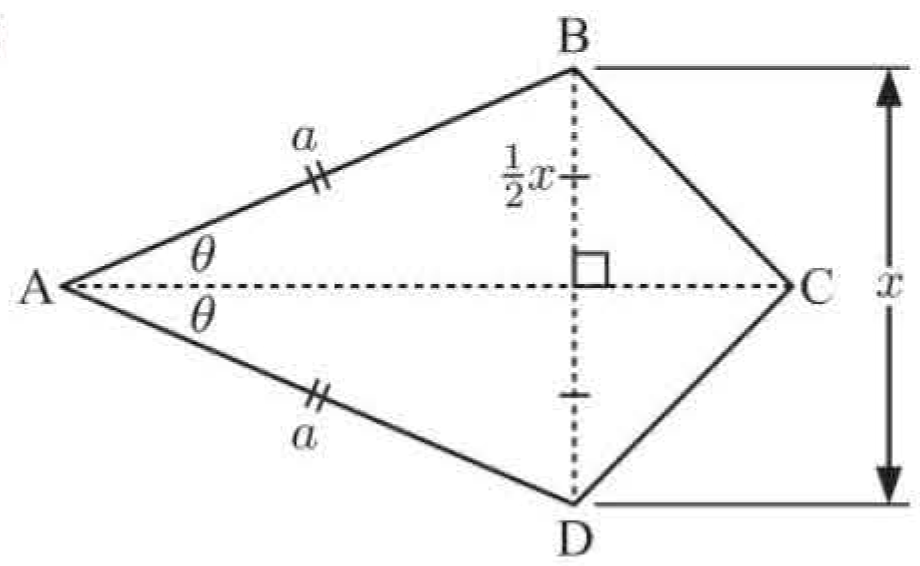
27 The two numbers selected are different and there are $\binom{n}{2}$ different selections.

Of these $(1, 4)$, $(2, 8)$, $(3, 12)$, ..., $(\frac{n}{4}, n)$ are the different outcomes where one is 4 times the other.

There are $\frac{n}{4}$ of these.

$\therefore$ P(one is 4 times the other) $= \dfrac{\frac{n}{4}}{\binom{n}{2}} = \dfrac{\frac{n}{4}}{\frac{n(n-1)}{2}} = \dfrac{1}{2(n-1)}$

28



The diagonals [AC] and [BD] intersect at right angles.

Let BD $= x$. Using the cosine rule,

$x^2 = a^2+a^2-2aa\cos 2\theta$

$\therefore\ x^2 = 2a^2(1-\cos 2\theta)$ .... (1)

But $\sin\theta = \dfrac{\frac{x}{2}}{a} = \dfrac{x}{2a}$

$\therefore\ x^2 = 4a^2\sin^2\theta$ .... (2)

From (1) and (2), $2a^2(1-\cos 2\theta) = 4a^2\sin^2\theta$

$\therefore\ 1-\cos 2\theta = 2\sin^2\theta$

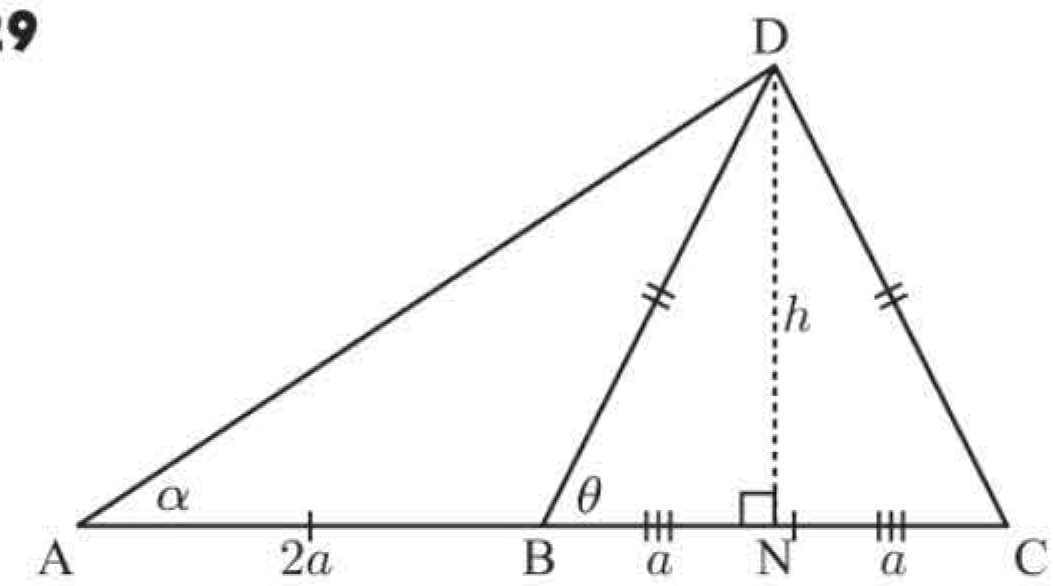
$\therefore\ \sin^2\theta = \frac{1}{2}-\frac{1}{2}\cos 2\theta$

and $\cos^2\theta = 1-\sin^2\theta$

$= 1-\left(\frac{1}{2}-\frac{1}{2}\cos 2\theta\right)$

$\therefore\ \cos^2\theta = \frac{1}{2}+\frac{1}{2}\cos 2\theta$

29



Draw [DN] $\perp$ [BC] in isosceles $\triangle$BCD.

$\therefore$ BN = NC $= a$ say and so AB = BC $= 2a$

If DN $= h$, then $\tan\alpha = \dfrac{\text{DN}}{\text{AN}} = \dfrac{h}{3a}$

and $\tan\theta = \dfrac{h}{a}$

$\therefore\ 3\tan\alpha = \tan\theta$

30 **a**

$$8^{2x+3} = 4\sqrt[3]{2}$$
$$\therefore\ 2^{3(2x+3)} = 2^2 \times 2^{\frac{1}{3}}$$
$$\therefore\ 6x + 9 = \tfrac{7}{3}$$
$$\therefore\ 18x + 27 = 7$$
$$\therefore\ 18x = -20$$
$$\therefore\ x = -1\tfrac{1}{9}$$

b

$$3^{2x+1} + 8(3^x) = 3$$
$$\therefore\ 3(3^x)^2 + 8(3^x) - 3 = 0$$
$$\therefore\ 3m^2 + 8m - 3 = 0 \quad \text{where } m = 3^x$$
$$\therefore\ (3m-1)(m+3) = 0 \quad \text{where } m = 3^x$$
$$\therefore\ m = \tfrac{1}{3} \text{ or } -3$$

But $m = 3^x > 0$, so $3^x = \frac{1}{3} = 3^{-1}$

$$\therefore\ x = -1$$

c

$$\ln(\ln x) = 1$$
$$\therefore\ \ln x = e^1 = e$$
$$\therefore\ x = e^e$$

d

$$\log_{\frac{1}{9}} x = \log_9 5$$
$$\therefore\ \frac{\log x}{\log\left(\frac{1}{9}\right)} = \frac{\log 5}{\log 9}$$
$$\therefore\ \frac{\log x}{-\log 9} = \frac{\log 5}{\log 9}$$
$$\therefore\ \log x = -\log 5 = \log(5^{-1})$$
$$\therefore\ x = 5^{-1} = \tfrac{1}{5}$$

31

$$\frac{d}{dx}(\tan^3 x) = 3(\tan x)^2 \times \sec^2 x$$
$$= 3\,\frac{\sin^2 x}{\cos^2 x}\,\frac{1}{\cos^2 x}$$
$$= 3(1 - \cos^2 x)\sec^4 x$$
$$= 3\sec^4 x - 3\sec^2 x$$

$$\therefore\ \int (3\sec^4 x - 3\sec^2 x)\,dx = \tan^3 x + c$$
$$\therefore\ 3\int \sec^4 x\,dx - 3\tan x + c = \tan^3 x + c$$
$$\therefore\ 3\int \sec^4 x\,dx = 3\tan x + \tan^3 x + c$$
$$\therefore\ \int \sec^4 x\,dx = \tan x + \tfrac{1}{3}\tan^3 x + c$$

32 If X and Y are independent events then $\text{P}(X \cap Y) = \text{P}(X)\,\text{P}(Y)$.

Thus $\text{P}((A \cap B) \cap (A \cup B)) = \text{P}(A \cap B)\,\text{P}(A \cup B)$

$\therefore\ \text{P}(A \cap B) = \text{P}(A \cap B)\,\text{P}(A \cup B)$ {since $A \cap B \subseteq A \cup B$}

$\therefore\ \text{P}(A \cap B) = 0$ or $\text{P}(A \cup B) = 1$

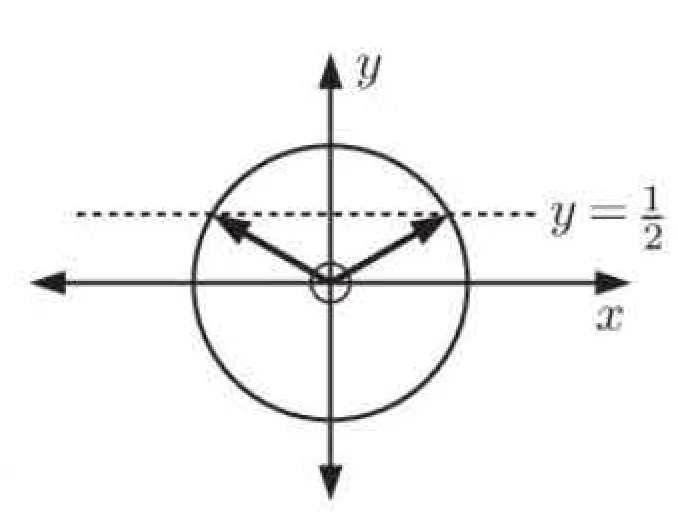
$\therefore$ A and B are disjoint or either A or B must occur.

33

$$(3 - i\sqrt{2})^4 = 3^4 + 4(3^3)(-i\sqrt{2}) + 6(3^2)(-i\sqrt{2})^2 + 4(3)(-i\sqrt{2})^3 + (-i\sqrt{2})^4$$
$$= 81 - 108\sqrt{2}i - 108 + 24\sqrt{2}i + 4$$
$$= -23 - 84\sqrt{2}i$$

34

$$\sin\theta\cos\theta = \tfrac{1}{4}, \quad \theta \in [-\pi, \pi]$$
$$\therefore\ \tfrac{1}{2}\sin 2\theta = \tfrac{1}{4}$$
$$\therefore\ \sin 2\theta = \tfrac{1}{2}$$
$$\therefore\ 2\theta = \tfrac{\pi}{6} + k2\pi \text{ or } \tfrac{5\pi}{6} + k2\pi$$
$$\therefore\ 2\theta = \tfrac{\pi}{6}, -\tfrac{11\pi}{6}, \tfrac{5\pi}{6}, -\tfrac{7\pi}{6} \qquad \{\text{as } -2\pi \leqslant 2\theta \leqslant 2\pi\}$$
$$\therefore\ \theta = -\tfrac{11\pi}{12}, -\tfrac{7\pi}{12}, \tfrac{\pi}{12}, \tfrac{5\pi}{12}$$



35 As the cubic has real coefficients, both $2+i$ and $2-i$ are roots.

These have sum 4 and product $4+1=5$, and so come from the quadratic factor z^2-4z+5.

Thus $z^3+az^2+bz+15=(z^2-4z+5)(z+3)$

$\therefore\ z^3+az^2+bz+15=z^3-z^2-7z+15$

$\therefore\ a=-1$ and $b=-7$ {equating coefficients}

36 **a**

$(0.5)^{x+1}>0.125$

$\therefore\ (0.5)^{x+1}>(0.5)^3$

But $y=(0.5)^{x+1}$ is decreasing,

so $x+1<3$

$\therefore\ x<2$

b

$\left(\frac{2}{3}\right)^x>\left(\frac{3}{2}\right)^{x-1}$

$\therefore\ \left(\frac{2}{3}\right)^x>\left(\frac{2}{3}\right)^{1-x}$

But $y=\left(\frac{2}{3}\right)^x$ is decreasing,

$\therefore\ x<1-x$

$\therefore\ 2x<1$

$\therefore\ x<\frac{1}{2}$

c

$4^x+2^{x+3}<48$

$\therefore\ (2^x)^2+8(2^x)-48<0$

$(2^x+12)(2^x-4)<0$

where $2^x+12>0$ for all x

$\therefore\ 2^x-4<0$

$\therefore\ 2^x<2^2$

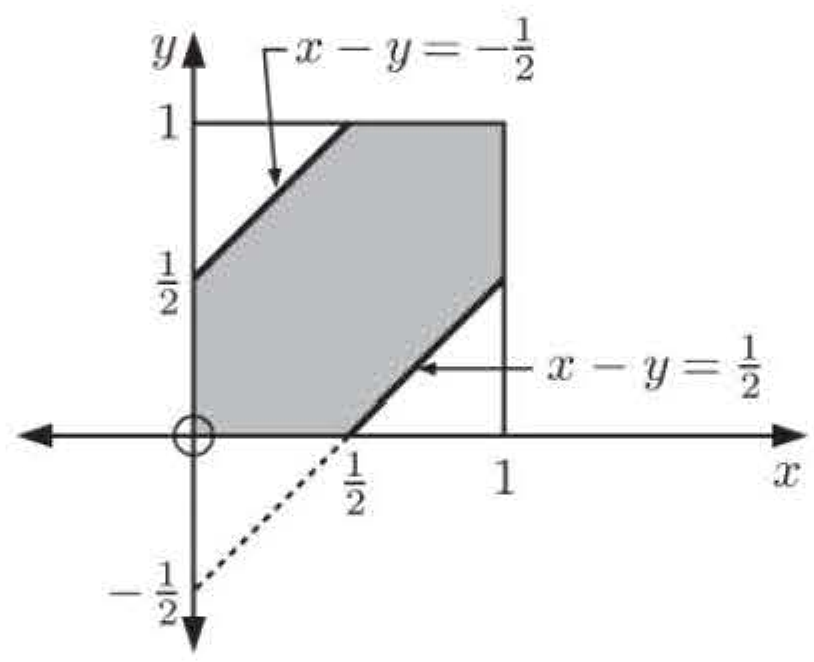
But $y=2^x$ is increasing,

$\therefore\ x<2$

37 Let X arrive x hours after 1 pm, $0\leqslant x\leqslant 1$ and Y arrive y hours after 1 pm, $0\leqslant y\leqslant 1$.

They meet provided $-\frac{1}{2}<x-y<\frac{1}{2}$ as the difference between their arrival times is not more than half an hour.

$\therefore$ P(they meet) $=\dfrac{\text{shaded area}}{\text{area of square}}=\dfrac{\frac{3}{4}}{1}=\frac{3}{4}$.



38 We draw the smaller circle in another position, and deliberately put P close to the x-axis, but not on it.

We let $X\widehat{O}P=\alpha$ and M be the centre of the smaller circle.

We join [MP] and let $M\widehat{O}P=\theta$

$\therefore\ Q\widehat{M}P=2\theta$ {angle at the centre theorem}

Now arc QX = arc QP, since both represent the distance the smaller circle has been rolled.

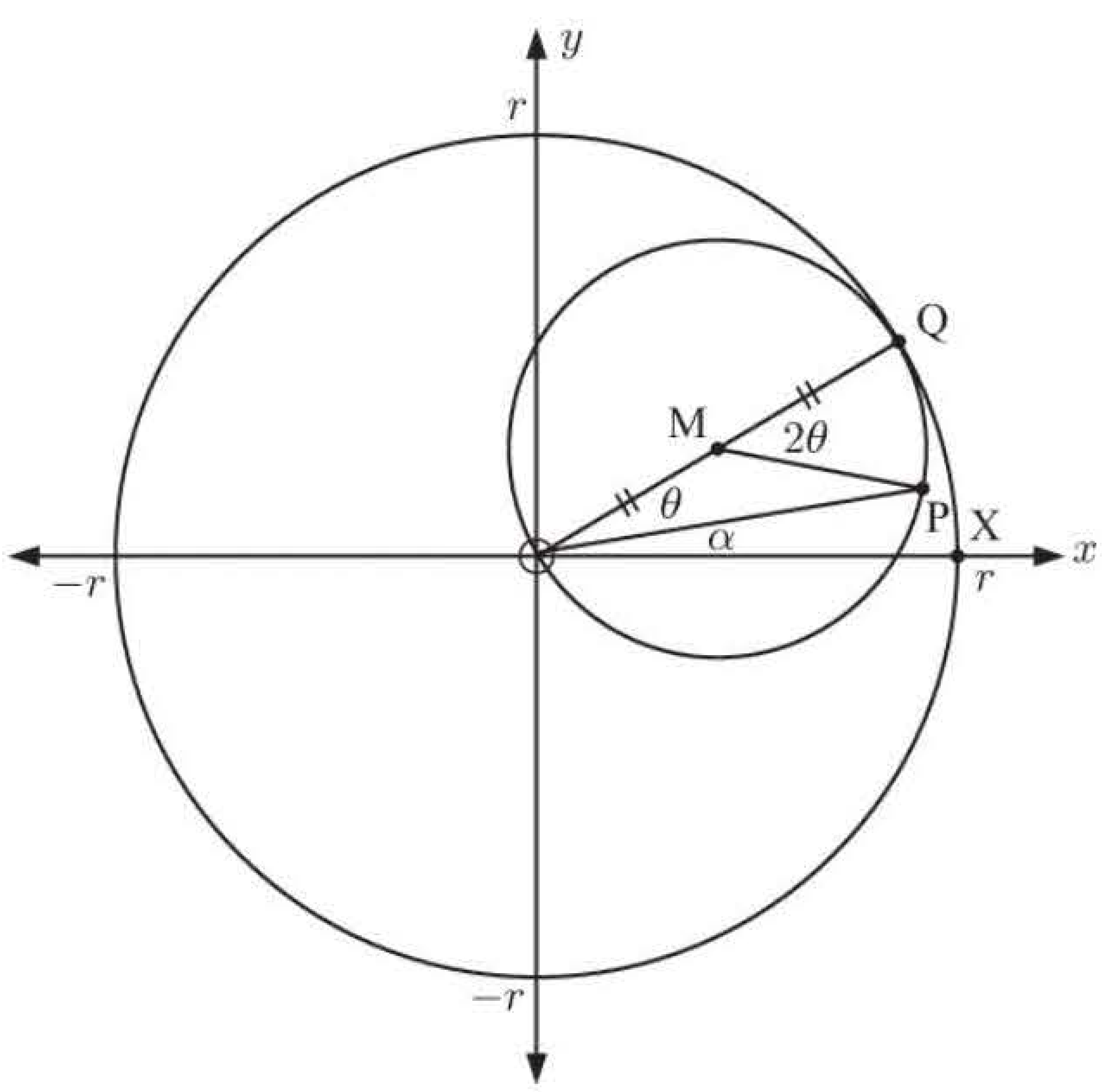
$\therefore\ r(\theta+\alpha)=\dfrac{r}{2}(2\theta)$

$\therefore\ r\theta+r\alpha=r\theta$

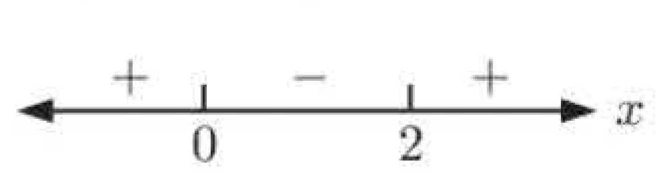
$\therefore\ r\alpha=0$

$\therefore\ \alpha=0$ as $r\neq 0$

$\therefore$ P lies on the x-axis.



39 **a** $f(x)=\ln(x(x-2))$ is defined when $x(x-2)>0$



$\therefore\ x<0$ or $x>2$

So the domain is $x\in\,]-\infty,0[\,\cup\,]2,\infty[$

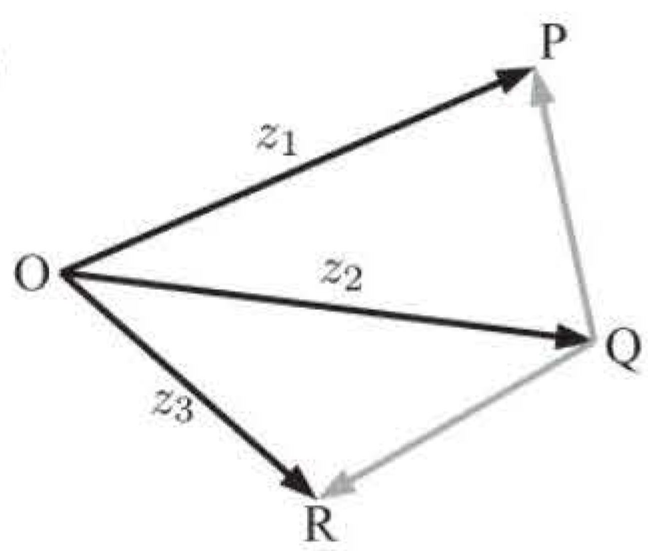
b $$f(x) = \ln x + \ln(x-2) \quad \{\text{log law}\}$$
$$\therefore\ f'(x) = \frac{1}{x} + \frac{1}{x-2} = \frac{2x-2}{x(x-2)}$$

c $f'(3) = \frac{1}{3} + 1 = \frac{4}{3}$ at $(3, \ln 3)$

$\therefore$ the tangent has equation
$$\frac{y - \ln 3}{x-3} = \frac{4}{3}$$
$$\therefore\ 4x - 12 = 3y - 3\ln 3$$
$$\therefore\ 4x - 3y = 12 - 3\ln 3$$

40 **a** Let $z = R\operatorname{cis}\theta$
$$\therefore\ iz = \operatorname{cis}\tfrac{\pi}{2} \times R\operatorname{cis}\theta$$
$$\therefore\ iz = R\operatorname{cis}\left(\tfrac{\pi}{2} + \theta\right)$$
$$\therefore\ \arg(iz) = \tfrac{\pi}{2} + \theta$$

b



$$z_3 - z_2 = \overrightarrow{QR}$$
$$z_1 - z_2 = \overrightarrow{QP}$$
Now $i(z_3 - z_2) = z_1 - z_2$
$$\therefore\ i\,\overrightarrow{QR} = \overrightarrow{QP}$$
$$\therefore\ \overrightarrow{QR} \perp \overrightarrow{QP} \quad \{\text{from } \mathbf{a}\}$$

$\therefore$ $\widehat{PQR}$ is a right angle.

Also $|\overrightarrow{QP}| = |i\,\overrightarrow{QR}| = |i|\,|\overrightarrow{QR}|$

$\therefore$ QP $= 1 \times$ QR $=$ QR

$\therefore$ $\triangle$PQR is right angled and isosceles, right angled at Q.

41 $(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \dots + \binom{n}{n}x^n$ .... (1)

$(1+x)^2(1+x)^{n-2} = (1 + 2x + 1x^2)\left[1 + \binom{n-2}{1}x + \binom{n-2}{2}x^2 + \dots + \binom{n-2}{n-2}x^{n-2}\right]$ .... (2)

In (1), the coefficient of x^r is $\binom{n}{r}$

In (2), the coefficient of x^r is $1 \times \binom{n-2}{r} + 2\binom{n-2}{r-1} + 1\binom{n-2}{r-2}$

Equating these gives $\binom{n}{r} = \binom{n-2}{r} + 2\binom{n-2}{r-1} + \binom{n-2}{r-2}$

42
$$x^2 + y^2 = 52xy$$
$$\therefore\ x^2 - 2xy + y^2 = 50xy$$
$$\therefore\ (x-y)^2 = 50xy$$
$$\therefore\ \frac{(x-y)^2}{25} = 2xy$$
$$\therefore\ \left(\frac{x-y}{5}\right)^2 = 2xy$$
$$\therefore\ \log\left(\frac{x-y}{5}\right)^2 = \log(2xy)$$
$$\therefore\ 2\log\left(\frac{x-y}{5}\right) = \log x + \log 2y$$
$$\therefore\ \log\left(\frac{x-y}{5}\right) = \tfrac{1}{2}(\log x + \log 2y)$$

43
$$\sin(xy) + y^2 = x$$
$$\therefore\ \cos(xy)\left[1y + x\frac{dy}{dx}\right] + 2y\frac{dy}{dx} = 1$$
$$\therefore\ \frac{dy}{dx}(x\cos(xy) + 2y) = 1 - y\cos(xy)$$
$$\therefore\ \frac{dy}{dx} = \frac{1 - y\cos(xy)}{x\cos(xy) + 2y}$$

44 After labelling the triangles,

$\triangle$ABC and $\triangle$ABD are isosceles {equal base angles}

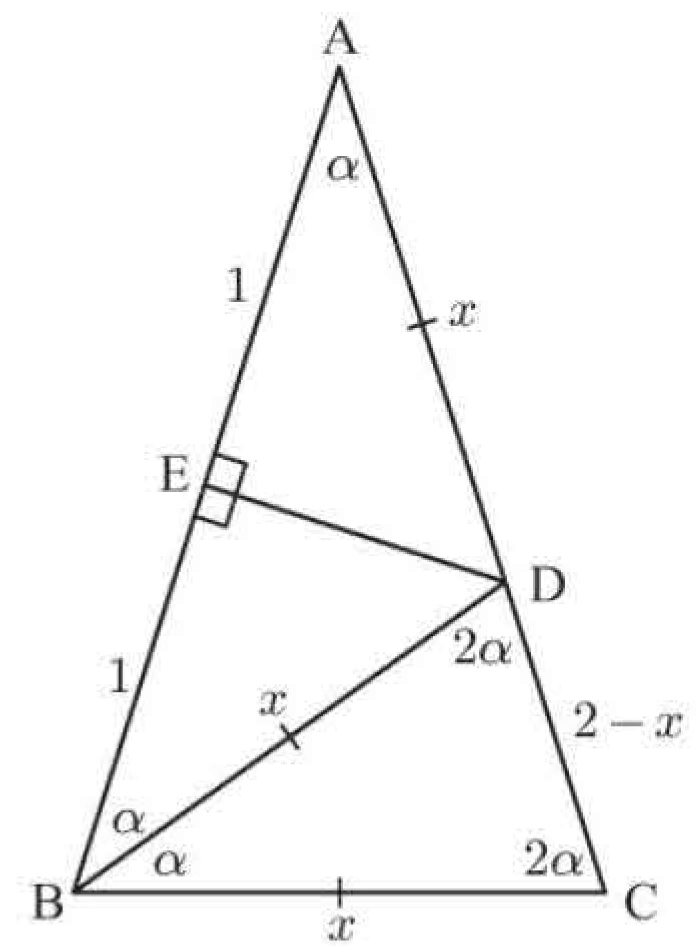
and $\widehat{BDC} = 2\alpha$ {exterior angle of $\triangle$ABD}

$\therefore$ $\triangle$BDC is isosceles {equal base angles}

Thus AD = BD = BC = x, say.

If we let AE = EB = 1 unit, then AC = 2 and DC = $2 - x$.

Now $\triangle$ABC and $\triangle$BCD are similar {equiangular}



$$\therefore \quad \frac{AB}{BC} = \frac{BC}{CD} \quad \text{and so} \quad \frac{2}{x} = \frac{x}{2-x}$$

$$\therefore \quad x^2 + 2x - 4 = 0$$

$$\therefore \quad x = -1 \pm \sqrt{5}$$

$$\therefore \quad x = \sqrt{5} - 1 \qquad \{\text{as } x \text{ must be} > 0\}$$

$$\text{But } 5\alpha = 180^\circ \qquad \{\text{angle sum of a triangle}\}$$

$$\therefore \quad \alpha = 36^\circ$$

and in $\triangle$BED, $\cos 36^\circ = \dfrac{1}{\sqrt{5}-1} = \dfrac{1}{\sqrt{5}-1}\left(\dfrac{\sqrt{5}+1}{\sqrt{5}+1}\right) = \dfrac{1+\sqrt{5}}{4}$.

45 By the Remainder theorem, $P(1) = -3$ and $P(-3) = -15$ where $P(x) = x^n + ax^2 - 6$

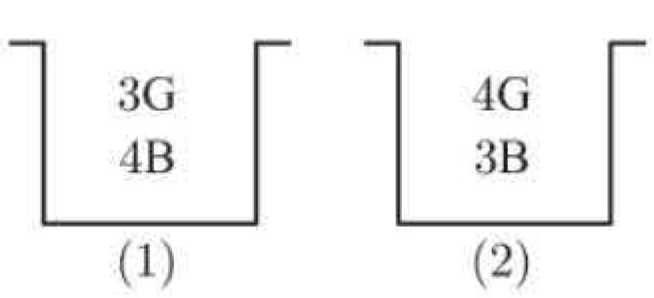
$\therefore$ $(1)^n + a(1)^2 - 6 = -3$ and $(-3)^n + a(-3)^2 - 6 = -15$

$\therefore$ $1 + a - 6 = -3$ and $(-3)^n + 9a = -9$

$\therefore$ $a = 2$ and $(-3)^n = -27$

$\therefore$ $n = 3$

46 **a**



P(both same colour)

$= $ P(GG or BB)

$= \frac{3}{7} \times \frac{4}{7} + \frac{4}{7} \times \frac{3}{7}$

$= \frac{24}{49}$

b P(G from (2) | both different)

$= \dfrac{\text{P(G from (2) } \cap \text{ both different)}}{\text{P(both different)}}$

$= \dfrac{\text{P(G from (2) } \textbf{and} \text{ B from (1))}}{1 - \frac{24}{49}}$

$= \dfrac{\frac{4}{7} \times \frac{4}{7}}{\frac{25}{49}}$

$= \frac{16}{25}$

47 **a** $y = f(x-2) + 1$ is a translation of $y = f(x)$ through $\binom{2}{1}$. $\therefore$ A$(-2, 3) \to$ A$'(0, 4)$.

b $y = 2f(x-2)$ is obtained from $y = f(x)$ by a translation through $\binom{2}{0}$ followed by a vertical stretch with scale factor $k = 2$. $\therefore$ A$(-2, 3) \to$ A$'(0, 3) \to$ A$''(0, 6)$.

c Consider $y = -|f(x)| - 2$. When $x = -2$, $y = -|f(-2)| - 2 = -3 - 2 = -5$

$\therefore$ A$(-2, 3) \to$ A$'(-2, -5)$.

d $y = f(2x - 3) = f\left(2(x - \frac{3}{2})\right)$.

It is obtained from $y = f(x)$ by a horizontal stretch with scale factor $k = \frac{1}{2}$ followed by a translation through $\begin{pmatrix} \frac{3}{2} \\ 0 \end{pmatrix}$. $\therefore$ A$(-2, 3) \to$ A$'(-1, 3) \to$ A$''(\frac{1}{2}, 3)$.

e Consider $y = \dfrac{1}{f(x)}$. When $x = -2$, $y = \dfrac{1}{f(-2)} = \frac{1}{3}$. $\therefore$ A$(-2, 3) \to$ A$'(-2, \frac{1}{3})$.

f Consider $y = f^{-1}(x)$. For an inverse function, the point is reflected in the line $y = x$.

$\therefore$ A$(-2, 3) \to$ A$'(3, -2)$.

48 **a** $z = re^{i\theta} = r\operatorname{cis}\theta$

$$\therefore\ z + \frac{1}{z} = r\operatorname{cis}\theta + \frac{1}{r\operatorname{cis}\theta}$$
$$= r\operatorname{cis}\theta + \frac{1}{r}\operatorname{cis}(-\theta)$$
$$= r(\cos\theta + i\sin\theta) + \frac{1}{r}(\cos\theta - i\sin\theta)$$
$$= \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta$$

Thus $a = \left(r + \frac{1}{r}\right)\cos\theta$ and $b = \left(r - \frac{1}{r}\right)\sin\theta$

b If $z + \frac{1}{z}$ is real then

$$\left(r - \frac{1}{r}\right)\sin\theta = 0$$
$$\therefore\ r - \frac{1}{r} = 0 \text{ or } \sin\theta = 0$$
$$\therefore\ r^2 = 1 \text{ or } \sin\theta = 0$$
$$\therefore\ r = 1 \text{ or } \theta = k\pi,\ k \in \mathbb{Z}\ \{r > 0\}$$
$$\therefore\ r = 1 \text{ or } z \text{ is real and non-zero}$$

49 P_n is "If u_n is defined by $u_1 = u_2 = 1$ and $u_{n+2} = u_{n+1} + u_n$ then $u_n \leqslant 2^n$" for $n \in \mathbb{Z}^+$.

Proof: (By the principle of mathematical induction)

(1) If $n = 1$ or 2, $u_1 = u_2 = 1$ and so $u_1, u_2 \leqslant 2^1$ $\therefore$ P_1 and P_2 are true.

(2) If P_k is assumed true then $u_k \leqslant 2^k$

Now $u_{k+1} = u_k + u_{k-1},\quad k \geqslant 2$

$\therefore\ u_{k+1} \leqslant u_k + u_k$ {the sequence $\{u_n\}$ is increasing, so $u_{k-1} \leqslant u_k$ for all $k \in \mathbb{Z}$}

$\therefore\ u_{k+1} \leqslant 2u_k$

$\therefore\ u_{k+1} \leqslant 2(2^k)$ {by P_k}

$\therefore\ u_{k+1} \leqslant 2^{k+1}$

Thus P_{k+1} is true whenever P_k is true, for $k \geqslant 2$.

$\therefore$ since P_1 and P_2 are true, P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

50 P_n is "$\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right)\ldots\left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n}$" for $n \in \mathbb{Z}^+,\ n \geqslant 2$.

Proof: (By the principle of mathematical induction)

(1) If $n = 2$, LHS $= 1 - \frac{1}{2^2} = \frac{3}{4}$ and RHS $= \frac{2+1}{2(2)} = \frac{3}{4}$ $\therefore$ P_2 is true.

(2) If P_k is assumed true then

$$\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\ldots\left(1 - \frac{1}{k^2}\right) = \frac{k+1}{2k}$$

$$\therefore\ \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\ldots\left(1 - \frac{1}{k^2}\right)\left(1 - \frac{1}{(k+1)^2}\right) = \frac{k+1}{2k}\left(1 - \frac{1}{(k+1)^2}\right)$$
$$= \frac{k+1}{2k}\left(\frac{(k+1)^2 - 1}{(k+1)^2}\right)$$
$$= \frac{k+1}{2k} \times \frac{k^2 + 2k}{(k+1)^2}$$
$$= \frac{k(k+2)}{2k(k+1)}$$
$$= \frac{(k+1)+1}{2(k+1)}$$

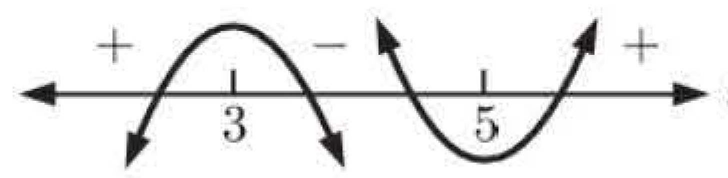
Thus P_{k+1} is true whenever P_k is true.

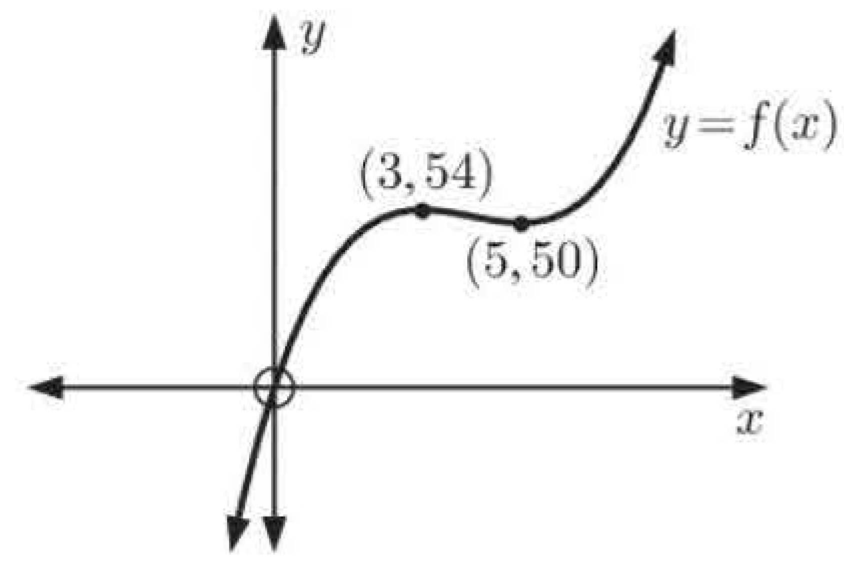
$\therefore$ since P_2 is true, P_n is true for all $n \geqslant 2$ {Principle of mathematical induction}

51 **a** $y = x^3 - 12x^2 + 45x$

$$\therefore \frac{dy}{dx} = 3x^2 - 24x + 45$$
$$= 3(x^2 - 8x + 15)$$
$$= 3(x-3)(x-5)$$

which is 0 when $x = 3$ or 5

The sign diagram is: 



local maximum at $(3, 54)$

local minimum at $(5, 50)$

b $y = x^3 - 12x^2 + 45x$ meets $y = k$ where $x^3 - 12x^2 + 45x = k$.

Now $y = k$ is a horizontal line, so for 3 real roots we need to observe where $y = k$ meets the curve in 3 places.

Thus $50 < k < 54$ or $k \in \,]50, 54[$.

52 **a** Using the cosine rule,

$$y^2 = x^2 + 8^2 - 2(x)(8)\cos\theta$$
$$\therefore y^2 = x^2 + 64 - 16x\cos\theta$$

But $x + y + 8 = 20$

$$\therefore y = 12 - x$$

Hence, $(12-x)^2 = x^2 + 64 - 16x\cos\theta$

$$\therefore 144 - 24x + x^2 = x^2 + 64 - 16x\cos\theta$$
$$\therefore 16x\cos\theta = 24x - 80$$
$$\therefore \cos\theta = \frac{3x-10}{2x}$$

b The area, $A = \frac{1}{2}(8x)\sin\theta = 4x\sin\theta$

$$\therefore A^2 = 16x^2\sin^2\theta$$
$$= 16x^2(1 - \cos^2\theta)$$
$$= 16x^2\left(1 - \left(\frac{3x-10}{2x}\right)^2\right)$$
$$= 16x^2\left(1 - \frac{9x^2 - 60x + 100}{4x^2}\right)$$
$$= 16x^2 - 36x^2 + 240x - 400$$
$$= -20x^2 + 240x - 400$$
$$= -20(x^2 - 12x + 20)$$

c A^2 is a quadratic in x with $a = -20$ and so the shape is 

Thus it has a maximum value when $x = \dfrac{-b}{2a} = \frac{-240}{-40} = 6$

$$\therefore A^2_{\max} = -20(36 - 72 + 20) = 320$$

So, $A_{\max} = \sqrt{320} = 8\sqrt{5}\text{ cm}^2$ when $x = y = 6$

$\therefore$ the triangle is isosceles.

53 **a** $u_3 = u_1 + 2d = \dfrac{1}{k}$ .... (1)

$u_4 = u_1 + 3d = k$

$u_6 = u_1 + 5d = k^2 + 1$

So, $u_1 + 3d - u_1 - 2d = k - \dfrac{1}{k}$

$$\therefore d = k - \frac{1}{k} \quad .... (2)$$

and $u_1 + 5d - u_1 - 3d = k^2 + 1 - k$

$$\therefore 2d = k^2 + 1 - k \quad .... (3)$$

From (2) and (3), $k^2 - k + 1 = 2k - \dfrac{2}{k}$

$$\therefore k^3 - 3k^2 + k + 2 = 0$$
$$\therefore (k-2)(k^2 - k - 1) = 0$$
$$\therefore k = 2 \text{ or } \frac{1 \pm \sqrt{1 - 4(1)(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

But $k \in \mathbb{Q}$, so $k = 2$

b Using (2), $d = 2 - \frac{1}{2} = \frac{3}{2}$

$\therefore$ using (1), $u_1 + 2\left(\frac{3}{2}\right) = \frac{1}{2}$

$\therefore \; u_1 = -\frac{5}{2}$

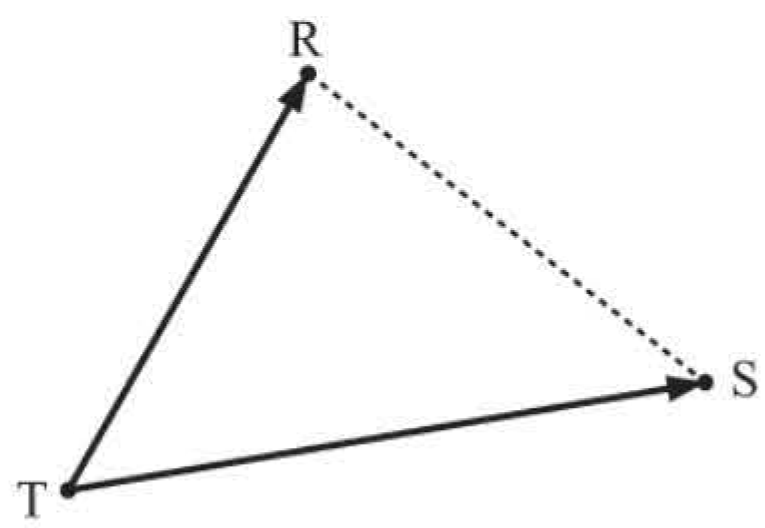
Now $u_n = u_1 + (n-1)d$

$\therefore \; u_n = -\frac{5}{2} + (n-1)\frac{3}{2}$

$\therefore \; u_n = \frac{3}{2}n - 4$

$\therefore \; u_n = \dfrac{3n-8}{2}$ for all $n \in \mathbb{Z}^+$

54



$$\begin{aligned}\overrightarrow{\text{TR}} &= \mathbf{r} - \mathbf{t} \\ &= 2\mathbf{i} - 2\mathbf{j} + \mathbf{k} - \mathbf{i} - 2\mathbf{j} + \mathbf{k} \\ &= \mathbf{i} - 4\mathbf{j} + 2\mathbf{k}\end{aligned}$$

$$\begin{aligned}\overrightarrow{\text{TS}} &= \mathbf{s} - \mathbf{t} \\ &= 3\mathbf{i} + \mathbf{j} + 2\mathbf{k} - \mathbf{i} - 2\mathbf{j} + \mathbf{k} \\ &= 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}\end{aligned}$$

$\therefore$ area $\triangle$RST

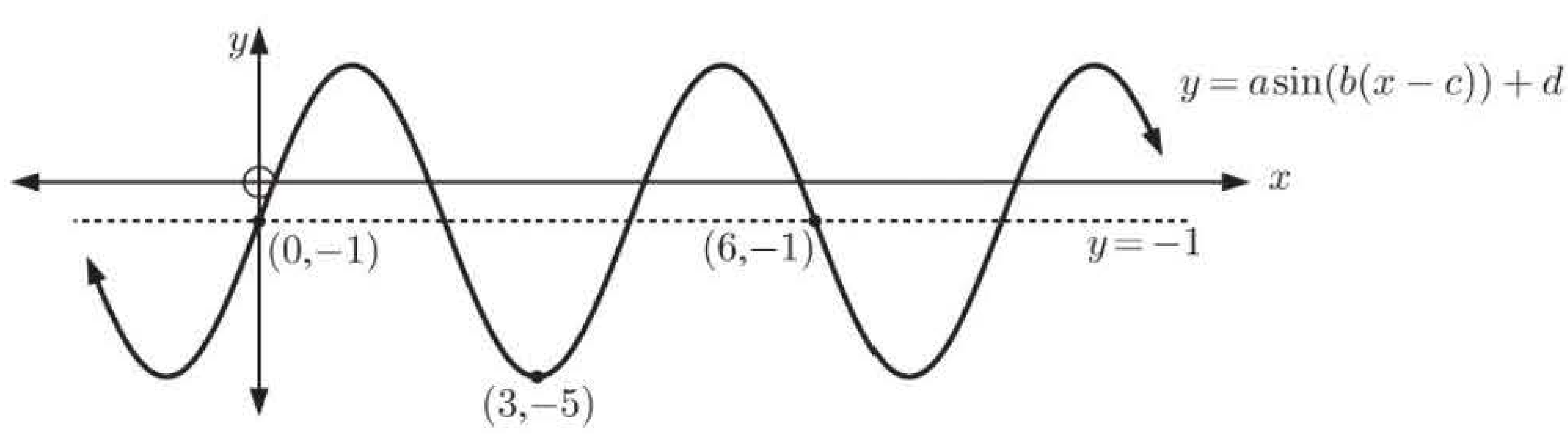
$$\begin{aligned}&= \tfrac{1}{2} \,|\, \overrightarrow{\text{TR}} \times \overrightarrow{\text{TS}} \,| \\ &= \tfrac{1}{2} \left\| \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -4 & 2 \\ 2 & -1 & 3 \end{vmatrix} \right\| \\ &= \tfrac{1}{2} \,|-10\mathbf{i} + \mathbf{j} + 7\mathbf{k}| \\ &= \tfrac{1}{2}\sqrt{100 + 1 + 49} \\ &= \tfrac{1}{2}\sqrt{150} \\ &= \tfrac{5}{2}\sqrt{6} \text{ units}^2\end{aligned}$$

55 $x = \log_3 y^2 \quad \therefore \; y^2 = 3^x = (81^{\frac{1}{4}})^x = 81^{\frac{x}{4}}$

$\therefore \; y = (81^{\frac{x}{4}})^{\frac{1}{2}} = 81^{\frac{x}{8}}$

$\therefore \; 81 = y^{\frac{8}{x}}$ and so $\log_y 81 = \dfrac{8}{x}$

56



The amplitude $= a = 4$. The period $= 4 = \dfrac{2\pi}{b} \quad \therefore \; b = \frac{\pi}{2}$

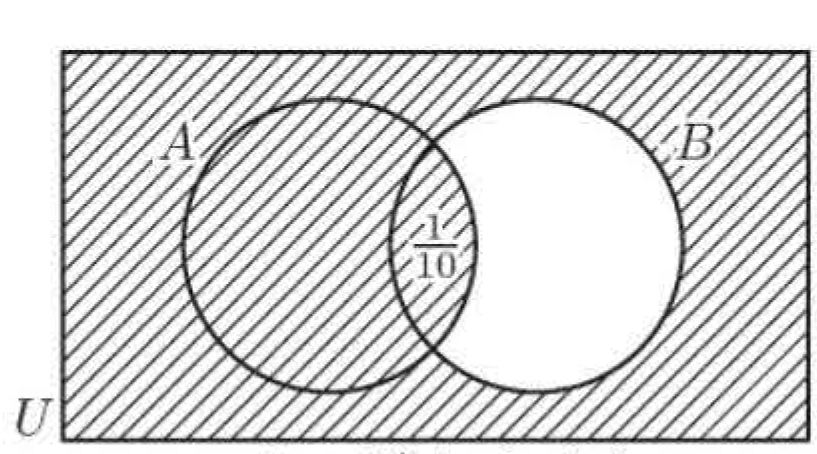
The basic sine curve has been translated through $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$. $\quad \therefore \; c = 0, \; d = -1$

Thus $y = 4\sin\left(\frac{\pi}{2}x\right) - 1$

Check: $y(3) = 4\sin\left(\frac{3\pi}{2}\right) - 1 = 4(-1) - 1 = -5$ ✓

$y(6) = 4\sin(3\pi) - 1 = 4(0) - 1 = -1$ ✓

57 As A and B are independent, $\text{P}(A \mid B) = \text{P}(A)$ and $\text{P}(B \mid A) = \text{P}(B)$



$A \cup B'$ is shaded

$\therefore \; \text{P}(A) = \frac{1}{4}$ and $\text{P}(B) = \frac{2}{5}$

and $\text{P}(A \cap B) = \text{P}(A)\,\text{P}(B) = \frac{1}{4} \times \frac{2}{5} = \frac{1}{10}$

$$\begin{aligned}\text{So, } \text{P}(A \cup B') &= 1 - \text{P}(B) + \text{P}(A \cap B) \\ &= 1 - \tfrac{2}{5} + \tfrac{1}{10} \\ &= \tfrac{7}{10}\end{aligned}$$

58 $$\frac{58}{9(3-7i)} = \frac{58}{9(3-7i)}\left(\frac{3+7i}{3+7i}\right)$$
$$= \frac{58(3+7i)}{9(9+49)}$$
$$= \frac{3+7i}{9}$$

Thus, $z^2 = 1 + i + \dfrac{3+7i}{9} = \dfrac{12+16i}{9}$

$\therefore\ z = \pm\dfrac{\sqrt{12+16i}}{3}$

We now let $\sqrt{12+16i} = c + di,\ c > 0$

$\therefore\ 12 + 16i = c^2 - d^2 + 2cdi$

$\therefore\ c^2 - d^2 = 12$ and $cd = 8$

$\therefore\ c = 4,\ d = 2$ or $\cancel{c = -4,\ d = -2}$ {as $c > 0$}

$\therefore\ z = \pm\dfrac{4+2i}{3}$

$\therefore\ z = \frac{4}{3} + \frac{2}{3}i$ or $-\frac{4}{3} - \frac{2}{3}i$

59 $4^x = 8^y$ and $9^y = \dfrac{243}{3^x}$

$\therefore\ (2^2)^x = (2^3)^y$ and $(3^2)^y \times 3^x = 3^5$

$\therefore\ 2x = 3y$ and $2y + x = 5$

{equating indices}

Solving these simultaneously gives

$2(5 - 2y) = 3y$

$\therefore\ 10 = 7y$

$\therefore\ y = \frac{10}{7}$

So, $2x = 3\left(\frac{10}{7}\right)$

$\therefore\ x = \frac{15}{7}$

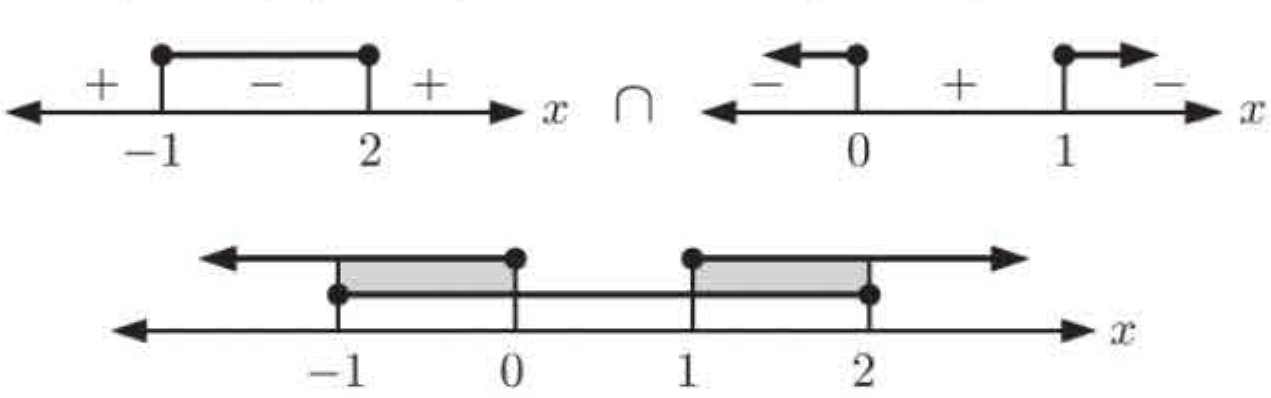
So, $x = 2\frac{1}{7},\ y = 1\frac{3}{7}$

60 For $f(x)$ to be defined we require that

$-1 \leqslant 1 + x - x^2 \leqslant 1$

$\therefore\ x^2 - x - 2 \leqslant 0$ and $x - x^2 \leqslant 0$

$\therefore\ (x-2)(x+1) \leqslant 0$ and $x(1-x) \leqslant 0$



$\therefore\ x \in [-1, 0] \cup [1, 2]$

$f(x) = \arccos u$ where $u = 1 + x - x^2$

$$f'(x) = f'(u)\,\frac{du}{dx}$$
$$= \frac{-1}{\sqrt{1-u^2}}\,(1-2x)$$
$$= \frac{2x-1}{\sqrt{1-(1+x-x^2)^2}}$$

61 $\displaystyle\sum_{n=1}^{m} f(n) = m^3 + 3m$

$\therefore\ f(1) + f(2) + f(3) + \ldots. + f(m) = m^3 + 3m$

$\therefore\ f(1) + f(2) + f(3) + \ldots. + f(m-1) = (m-1)^3 + 3(m-1)$

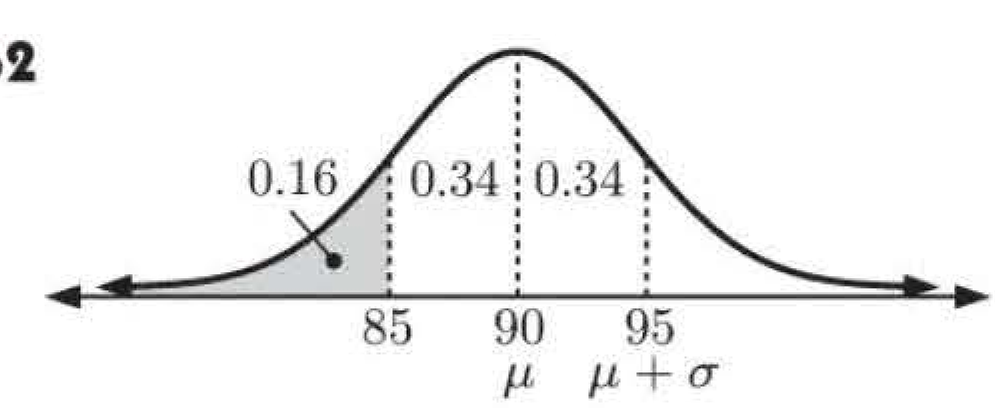
Thus $(m-1)^3 + 3(m-1) + f(m) = m^3 + 3m$

Hence $\cancel{m^3} - 3m^2 + \cancel{3m} - 1 + 3m - 3 + f(m) = \cancel{m^3} + \cancel{3m}$

$\therefore\ f(m) = 3m^2 - 3m + 4$

Hence $f(n) = 3n^2 - 3n + 4$ {replacing m by n}

62



a $\mathrm{P}(X < 85) \approx 0.16$
From the diagram,
$\mathrm{P}(90 < X < 95) \approx 0.34$

b As roughly 34% of scores lie between μ and $\mu + \sigma$ for the normal distribution then $\sigma \approx 5$.

63 $y = \dfrac{\tan x}{\sin(2x) + 1} = \dfrac{\sin x}{\cos x(\sin 2x + 1)}$ is undefined if $\cos x = 0$ or $\sin 2x = -1$

$\therefore\ x = \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}$ or $2x = -\frac{\pi}{2} + k2\pi,\ k \in \mathbb{Z}$

$\therefore\ x = -\frac{\pi}{2}, -\frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$ for $x \in [-\pi, \pi]$

$\therefore$ the vertical asymptotes are: $x = -\frac{\pi}{2},\ x = -\frac{\pi}{4}, x = \frac{\pi}{2},\ x = \frac{3\pi}{4}$.

There are no horizontal asymptotes.

64 **a** $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -3 \\ 0 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & -3 \\ 1 & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & -3 \\ 0 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} \mathbf{k}$

$= 5\mathbf{i} - 2\mathbf{j} + \mathbf{k}$

b $|\mathbf{a} \times \mathbf{b}| = \sqrt{25+4+1} = \sqrt{30}$ units

$\therefore$ $\frac{1}{\sqrt{30}}(5\mathbf{i} - 2\mathbf{j} + \mathbf{k})$ is a unit vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

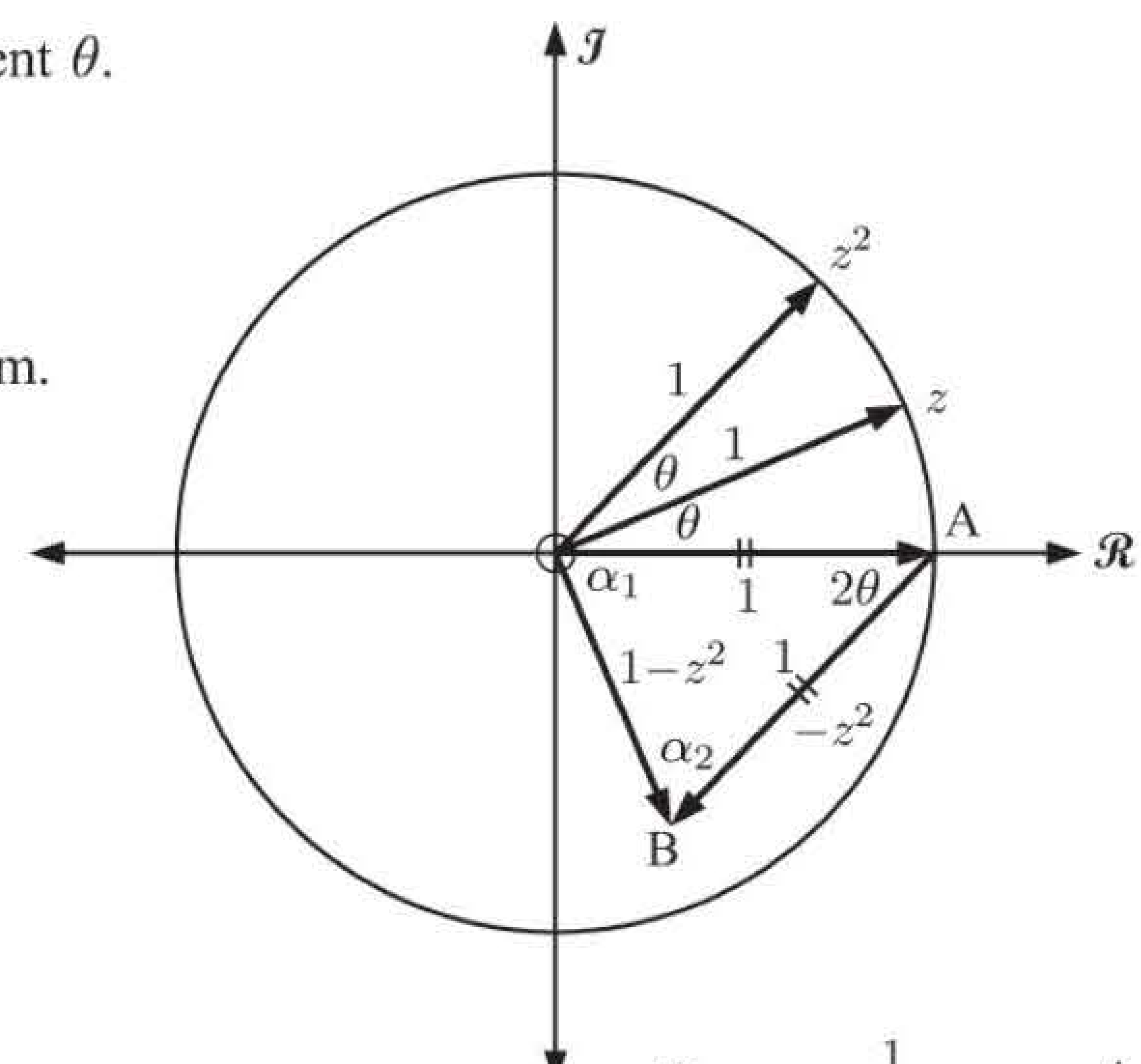
$\therefore$ the required vector is $\frac{5}{\sqrt{30}}(5\mathbf{i} - 2\mathbf{j} + \mathbf{k}) = \frac{\sqrt{30}}{6}(5\mathbf{i} - 2\mathbf{j} + \mathbf{k})$

65 $z = \cos\theta + i\sin\theta = \operatorname{cis}\theta$ has modulus 1 and argument θ.

$z^2 = (\operatorname{cis}\theta)^2 = \operatorname{cis} 2\theta$ {De Moivre}

$\therefore$ z^2 has modulus 1 and argument 2θ

$1 - z^2$ is found using vector subtraction on the diagram.



$\triangle$OAB is isosceles as OA = BA = 1

Hence $\alpha_1 = \alpha_2$ {isosceles $\triangle$ theorem}

As z^2 and $-z^2$ are parallel,

$\widehat{OAB} = 2\theta$ {equal alternate angles}

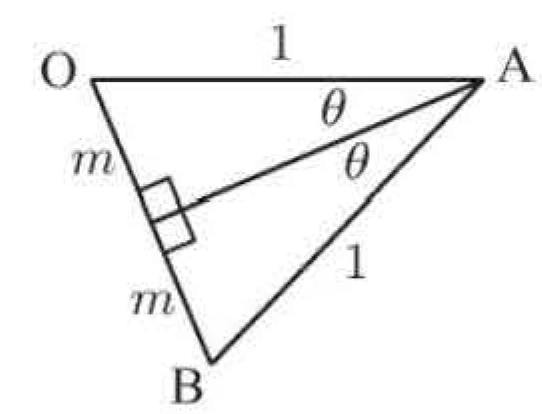
Thus $2\alpha + 2\theta = \pi$ and so $\alpha = \frac{\pi}{2} - \theta$

Now $\arg(1 - z^2) = -\alpha_1 = \theta - \frac{\pi}{2}$

and $\sin\theta = \dfrac{m}{1} = m$

$\therefore$ $|1 - z^2| = 2m = 2\sin\theta$

{$\sin\theta > 0$, as $0 < \theta < \frac{\pi}{4}$}



So, $1 - z^2$ has argument $\theta - \frac{\pi}{2}$ and modulus $2\sin\theta$.

66 We integrate by parts with $u = x^2$ $\quad v' = \sin x$

$u' = 2x$ $\quad v = -\cos x$

$\therefore$ $\int x^2 \sin x \, dx = x^2(-\cos x) - \int -\cos x \,(2x)\, dx$

$= -x^2 \cos x + 2\int x\cos x \, dx$

We again integrate by parts, this time with $u = x$ $\quad v' = \cos x$

$u' = 1$ $\quad v = \sin x$

$\therefore$ $\int x^2 \sin x \, dx = -x^2\cos x + 2\left[x\sin x - \int \sin x \, dx\right]$

$= -x^2\cos x + 2x\sin x - 2(-\cos x) + c$

$= -x^2\cos x + 2x\sin x + 2\cos x + c$

67 Let $f(x) = ax + b$

$\therefore$ $f(2x+3) = a(2x+3) + b$

$= 2ax + [3a + b]$

So, $2a = 5$ and $3a + b = -7$

$\therefore$ $a = \frac{5}{2}$ and $\frac{15}{2} + b = -7$

$\therefore$ $b = -\frac{29}{2}$

$\therefore$ $f(x) = \frac{5}{2}x - \frac{29}{2}$ or $\dfrac{5x - 29}{2}$

To obtain $f^{-1}(x)$ we use $x = \dfrac{5y - 29}{2}$

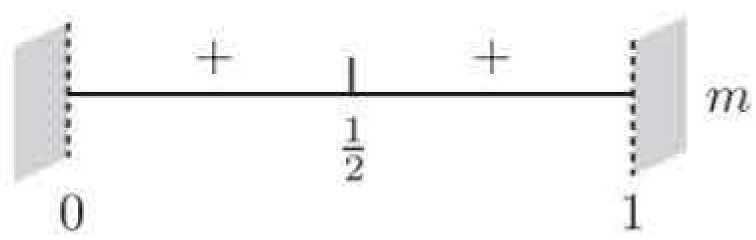
$\therefore$ $2x = 5y - 29$

$y = \dfrac{2x + 29}{5}$

So, $f^{-1}(x) = \dfrac{2x + 29}{5}$

68
$$\log_x 4 + \log_2 x = 3$$
$$\therefore \frac{\log 4}{\log x} + \frac{\log x}{\log 2} - 3 = 0$$
$$\therefore \log 4 \log 2 + (\log x)^2 - 3\log 2 \log x = 0$$
$$\therefore (\log x)^2 - 3\log 2(\log x) + \log 4 \log 2 = 0$$
$$\therefore (\log x)^2 - 3\log 2(\log x) + 2(\log 2)^2 = 0$$
$$\therefore (\log x - 2\log 2)(\log x - \log 2) = 0$$
$$\therefore \log x = 2\log 2 \text{ or } \log 2$$
$$\therefore x = 4 \text{ or } 2$$

69 The discriminant, $\Delta = 1^2 - 4(m-1)(-m)$
$$= 1 + 4m(m-1)$$
$$= 4m^2 - 4m + 1$$
$$= (2m-1)^2$$
$$\geqslant 0 \text{ for all } 0 < m < 1$$



So, the roots are always real.

$$\text{sum of roots} = \frac{-b}{a} = \frac{-1}{m-1} \quad \text{which is positive since } m - 1 < 0 \text{ for all } 0 < m < 1$$
$$\text{product of roots} = \frac{c}{a} = \frac{-m}{m-1} \quad \text{which is positive since } -m < 0 \text{ for all } 0 < m < 1.$$

As the sum and product of the roots are both positive, both roots are positive.

70 **a**
$$\sin 15^\circ$$
$$= \sin(45^\circ - 30^\circ)$$
$$= \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ$$
$$= \left(\tfrac{1}{\sqrt{2}}\right)\left(\tfrac{\sqrt{3}}{2}\right) - \left(\tfrac{1}{\sqrt{2}}\right)\left(\tfrac{1}{2}\right)$$
$$= \left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)\frac{\sqrt{2}}{\sqrt{2}}$$
$$= \frac{\sqrt{6}-\sqrt{2}}{4}$$

b
$$\cos^2 165^\circ + \cos^2 285^\circ$$
$$= [\cos(180^\circ - 15^\circ)]^2 + [\cos(270^\circ + 15^\circ)]^2$$
$$= (-\cos 15^\circ)^2 + (\sin 15^\circ)^2$$
$$\{\cos(\pi - \theta) = -\cos\theta \text{ and } \cos\left(\tfrac{3\pi}{2} + \theta\right) = \sin\theta\}$$
$$= \cos^2 15^\circ + \sin^2 15^\circ$$
$$= 1$$

71 **a**
$$x^2 - 3xy + y^2 = 7$$
$$\therefore 2x - \left[3y + 3x\frac{dy}{dx}\right] + 2y\frac{dy}{dx} = 0$$
$$\therefore 2x - 3y + (2y - 3x)\frac{dy}{dx} = 0$$
$$\therefore \frac{dy}{dx} = \frac{3y - 2x}{2y - 3x}$$

b We need to find where $\dfrac{3y-2x}{2y-3x} = \frac{2}{3}$
$$\therefore 9y - 6x = 4y - 6x$$
$$\therefore 5y = 0$$
$$\therefore y = 0$$
$$\therefore x^2 = 7 \text{ and so } x = \pm\sqrt{7}$$
$\therefore$ the points are $(\sqrt{7}, 0)$ and $(-\sqrt{7}, 0)$.

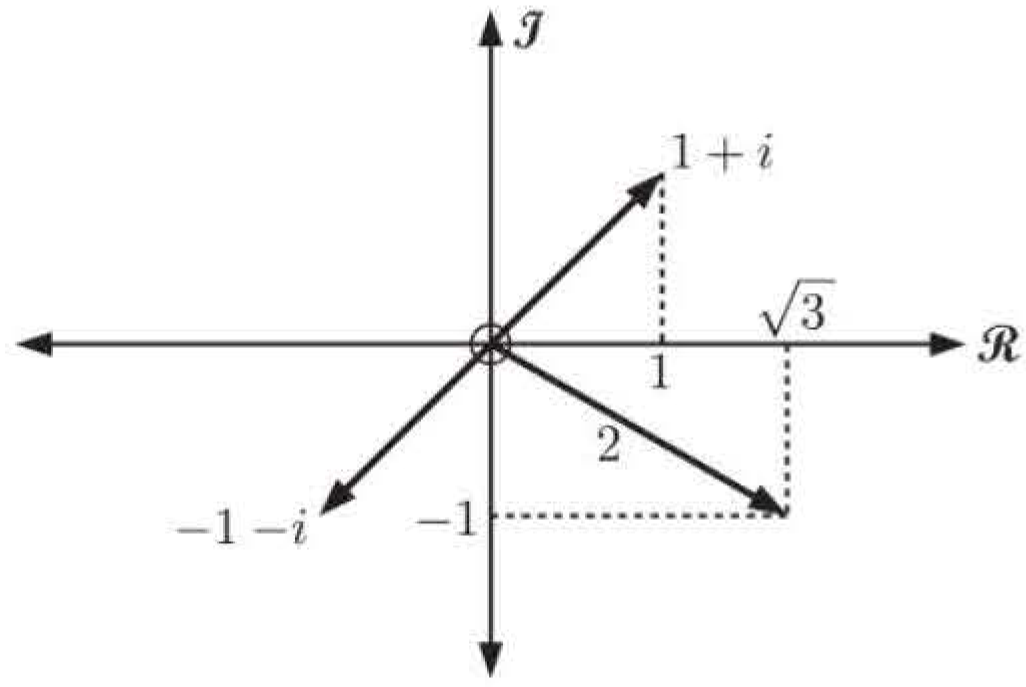
72 **a** **i** If A and B are mutually exclusive then
$$P(A \cup B) = P(A) + P(B) = \tfrac{1}{3} + \tfrac{2}{7} = \tfrac{13}{21}$$

ii If A and B are independent then
$$P(A \cup B) = P(A) + P(B) - P(A)P(B)$$
$$= \tfrac{1}{3} + \tfrac{2}{7} - \tfrac{1}{3} \times \tfrac{2}{7}$$
$$= \tfrac{11}{21}$$

b
$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A) + P(B) - P(A \cup B)}{P(B)} = \left(\frac{\frac{13}{21} - \frac{3}{7}}{\frac{2}{7}}\right)\frac{21}{21} = \frac{4}{6} = \frac{2}{3}$$

73 **a**

$$1+i=\sqrt{2}\operatorname{cis}\tfrac{\pi}{4}=\sqrt{2}e^{i\frac{\pi}{4}}$$
$$\sqrt{3}-i=2\operatorname{cis}\left(-\tfrac{\pi}{6}\right)=2e^{i\left(-\frac{\pi}{6}\right)}$$
$$\therefore\ \frac{-1-i}{\sqrt{3}-i}=\frac{\sqrt{2}e^{i\left(-\frac{3\pi}{4}\right)}}{2e^{i\left(-\frac{\pi}{6}\right)}}$$
$$=\tfrac{1}{\sqrt{2}}e^{i\left(-\frac{3\pi}{4}+\frac{\pi}{6}\right)}$$
$$=\tfrac{1}{\sqrt{2}}e^{i\left(-\frac{7\pi}{12}\right)}$$



b $z^n=\left(\frac{1}{\sqrt{2}}\right)^n e^{i\left(-\frac{7\pi n}{12}\right)}$ which is real when $-\dfrac{7\pi n}{12}=0+k\pi$

$$\therefore\ n=0-\frac{12k}{7},\ k\in\mathbb{Z}$$

$\therefore$ the smallest positive integer is $n=12$ when $k=-7$.

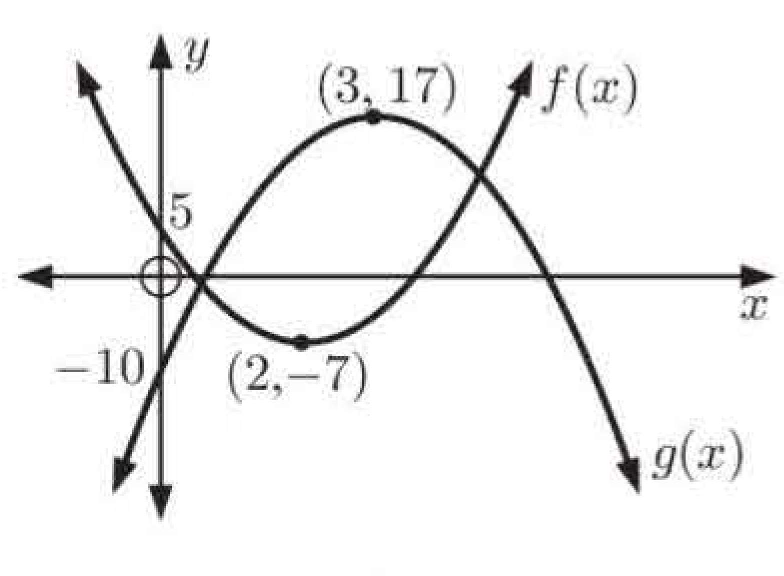
74 **a** Since $\mathbf{p}$ and $\mathbf{q}$ perpendicular,

$$\mathbf{p}\bullet\mathbf{q}=0$$
$$\therefore\ -t+2+2t-4t=0$$
$$\therefore\ 3t=2$$
$$\therefore\ t=\tfrac{2}{3}$$

b Since $\mathbf{p}$ and $\mathbf{q}$ parallel,

$$\mathbf{q}=k\mathbf{p}\ \text{ for some scalar } k$$
$$\therefore\ \begin{pmatrix}-t\\1+t\\2t\end{pmatrix}=\begin{pmatrix}k\\2k\\-2k\end{pmatrix}$$
$$\therefore\ k=-t\ \text{ and }\ 2k=1+t=-2t$$
$$\therefore\ 3t=-1$$
$$\therefore\ t=-\tfrac{1}{3}$$

75
$$f(x)=3x^2-12x+5$$
$$=3(x^2-4x+4)+5-12$$
$$=3(x-2)^2-7$$
$$g(x)=-3x^2+18x-10$$
$$=-3(x^2-6x+9)-10+27$$
$$=-3(x-3)^2+17$$



$$g(x)=-3(x-3)^2+17$$
$$=-(3(x-1-2)^2-7-10)$$
$$=-(f(x-1)^2-10)$$

So, we translate $y=f(x)$ through $\binom{1}{-10}$ and then reflect the result in the x-axis.

76
$$\text{Area}=\int_0^{\frac{\pi}{4}}\tan^2 x+2\sin^2 x\,dx$$
$$=\int_0^{\frac{\pi}{4}}\sec^2 x-1+2\left(\tfrac{1}{2}-\tfrac{1}{2}\cos 2x\right)dx$$
$$=\int_0^{\frac{\pi}{4}}\sec^2 x-\cos 2x\,dx$$
$$=\left[\tan x-\tfrac{1}{2}\sin 2x\right]_0^{\frac{\pi}{4}}$$
$$=\left(1-\tfrac{1}{2}\right)-(0-0)$$
$$=\tfrac{1}{2}\text{ unit}^2$$

77 The x-intercepts -3 and $-\frac{1}{4}$ indicate that $(x+3)$ and $(4x+1)$ are factors of $f(x)$.

$f(x)$ touches the x-axis at $\frac{3}{2}$, so $(2x-3)^2$ is also a factor of $f(x)$.

Thus, the quartic has the form $f(x)=a(x+3)(4x+1)(2x-3)^2$, where $a\neq 0$.

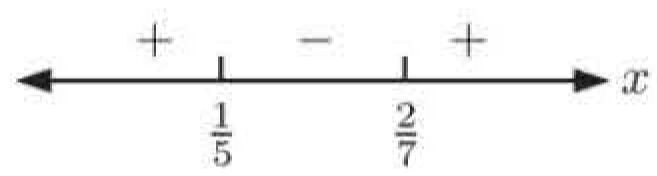
But $f(0)=9$, so $a(3)(1)(-3)^2=9$

$$\therefore\ a=\tfrac{1}{3}$$
$$\therefore\ f(x)=\tfrac{1}{3}(x+3)(4x+1)(2x-3)^2$$

78 **a**

$$|1-4x| > \tfrac{1}{3}|2x-1|$$
$$\therefore\ 3|1-4x| > |2x-1|$$
$$\therefore\ 9(1-4x)^2 > (2x-1)^2$$
$$\therefore\ 9(1-4x)^2 - (2x-1)^2 > 0$$
$$\therefore\ [3(1-4x)+(2x-1)]\,[3(1-4x)-(2x-1)] > 0$$
$$\therefore\ (-10x+2)(-14x+4) > 0$$

Sign diagram:

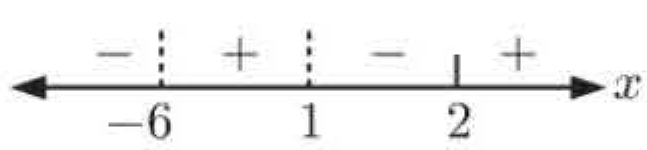


$$\therefore\ x < \tfrac{1}{5} \text{ or } x > \tfrac{2}{7}$$

b

$$\frac{x-2}{6-5x-x^2} \leqslant 0$$
$$\therefore\ \frac{x-2}{x^2+5x-6} \geqslant 0$$
$$\therefore\ \frac{x-2}{(x-1)(x+6)} \geqslant 0$$

Sign diagram:



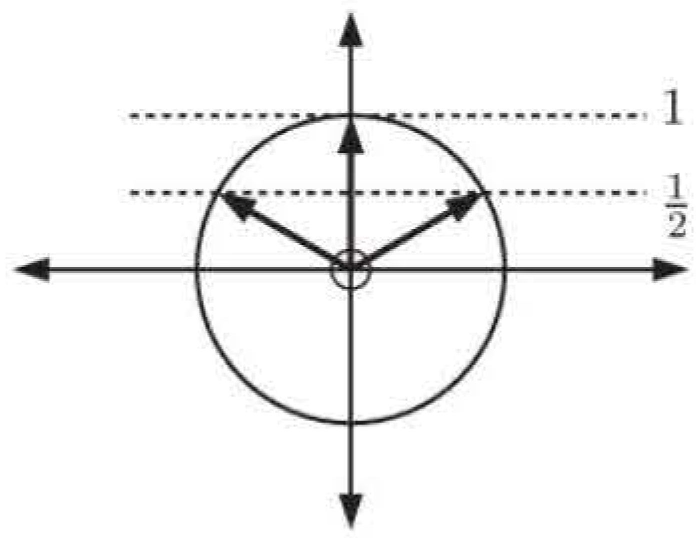
$$\therefore\ -6 < x < 1 \text{ or } x \geqslant 2$$

79 $3\sec 2x = \cot 2x + 3\tan 2x, \quad -\pi \leqslant x \leqslant \pi$

$$\therefore\ \frac{3}{\cos 2x} = \frac{\cos 2x}{\sin 2x} + 3\,\frac{\sin 2x}{\cos 2x}, \quad -2\pi \leqslant 2x \leqslant 2\pi$$

Multiplying all terms by $\sin 2x \cos 2x$ gives:

$$3\sin 2x = \cos^2 2x + 3\sin^2 2x$$
$$\therefore\ 3\sin 2x = 1 - \sin^2 2x + 3\sin^2 2x$$
$$\therefore\ 2\sin^2 2x - 3\sin 2x + 1 = 0$$
$$\therefore\ (2\sin 2x - 1)(\sin 2x - 1) = 0$$
$$\therefore\ \sin 2x = \tfrac{1}{2} \text{ or } 1$$
$$\therefore\ 2x = \tfrac{-11\pi}{6}, \tfrac{-3\pi}{2}, \tfrac{-7\pi}{6}, \tfrac{\pi}{6}, \tfrac{\pi}{2}, \text{ or } \tfrac{5\pi}{6}$$
$$\therefore\ x = \tfrac{-11\pi}{12}, \tfrac{-3\pi}{4}, \tfrac{-7\pi}{12}, \tfrac{\pi}{12}, \tfrac{\pi}{4}, \text{ or } \tfrac{5\pi}{12}$$



80

$$e^{xy} + xy^2 - \sin y = 2$$
$$e^{xy}\left(1y + x\,\frac{dy}{dx}\right) + 1y^2 + x\left(2y\,\frac{dy}{dx}\right) - \cos y\,\frac{dy}{dx} = 0$$
$$\frac{dy}{dx}\left(xe^{xy} + 2xy - \cos y\right) = -y^2 - ye^{xy}$$
$$\therefore\ \frac{dy}{dx} = \frac{-y^2 - ye^{xy}}{xe^{xy} + 2xy - \cos y}$$

81

$$\frac{z+u}{z-u} = \frac{x+2i+3+iy}{x+2i-3-iy}$$
$$= \frac{(x+3)+i(y+2)}{(x-3)-i(y-2)} \times \frac{(x-3)+i(y-2)}{(x-3)+i(y-2)}$$
$$= \frac{\left[(x^2-9)-(y^2-4)\right] + i\left[(x+3)(y-2)+(y+2)(x-3)\right]}{(x-3)^2+(y-2)^2}$$

This is purely imaginary when

$$x^2 - 9 - y^2 + 4 = 0 \quad \text{and} \quad (x+3)(y-2)+(y+2)(x-3) \neq 0$$
$$\therefore\ x^2 - y^2 = 5 \quad \text{and} \quad xy - 2x + 3y - 6 + xy - 3y + 2x - 6 \neq 0$$
$$\therefore\ 2xy \neq 12$$
$$\therefore\ xy \neq 6$$

Since $x^2 = 5 + y^2$ where $y^2 \geqslant 0$, $x^2 \geqslant 5$

So, $x \geqslant \sqrt{5}$ or $x \leqslant -\sqrt{5}$, and the smallest positive x is $\sqrt{5}$.

82 Area $= \int_0^{\frac{\pi}{3}} \frac{\tan x}{\cos 2x + 1}\,dx$

$= \int_0^{\frac{\pi}{3}} \frac{\tan x}{2\cos^2 x - 1 + 1}\,dx$

$= \frac{1}{2}\int_0^{\frac{\pi}{3}} \tan x \sec^2 x\,dx$

$= \frac{1}{2}\left[\frac{(\tan x)^2}{2}\right]_0^{\frac{\pi}{3}} \quad \{\frac{d}{dx}\tan x = \sec^2 x\}$

$= \frac{1}{4}\left((\sqrt{3})^2 - 0^2\right)$

$= \frac{3}{4}$ unit^2

83 $\frac{a+b+26}{5} = 8$ {as the mean is 8}

$\therefore\ a + b = 14$ .... (1)

But $\frac{\sum(x_i - \overline{x})^2}{n} = 8$ also,

so $\frac{(a-8)^2 + (b-8)^2 + 4 + 25 + 1}{5} = 8$

Using (1), $(a-8)^2 + (6-a)^2 + 30 = 40$

$\therefore\ a^2 - 16a + 64 + 36 - 12a + a^2 - 10 = 0$

$\therefore\ 2a^2 - 28a + 90 = 0$

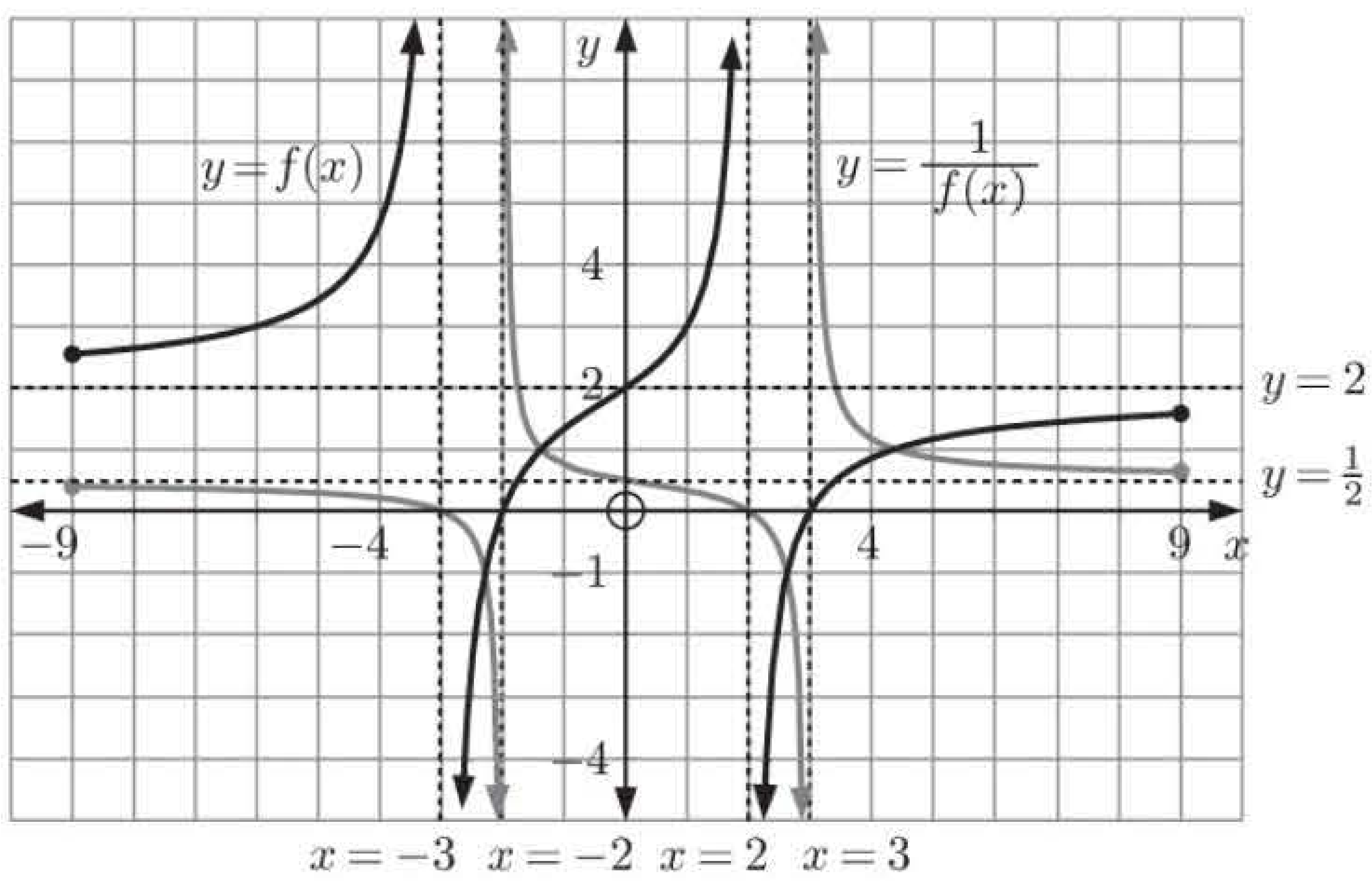
$\therefore\ a^2 - 14a + 45 = 0$

$\therefore\ (a-9)(a-5) = 0$

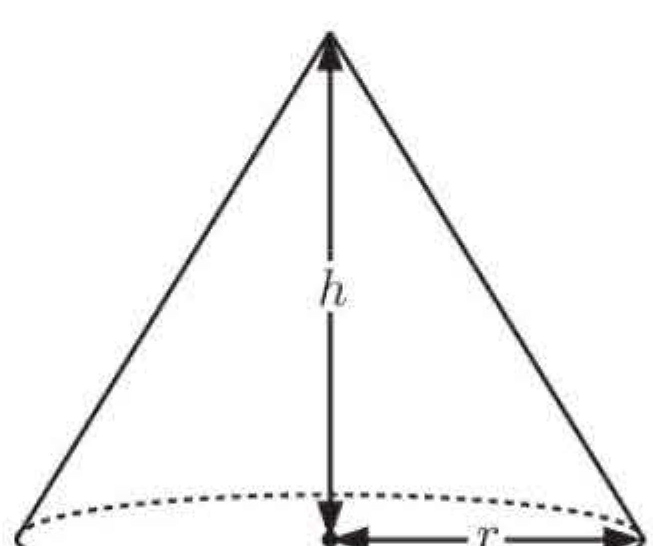
$\therefore\ a = 5$ or 9

When $a = 5,\ b = 9,$ and when $a = 9,\ b = 5.$
As $b > a,\ a = 5,\ b = 9$

84



85



$h = 2r$

$V = \frac{1}{3}\pi r^2 h$

$\therefore\ V = \frac{\pi}{3} r^2(2r)$

$\therefore\ V = \frac{2\pi}{3} r^3$

So, $\frac{dV}{dt} = 2\pi r^2 \frac{dr}{dt}$

Particular case:

When $h = 20$ cm, $r = 10$ cm

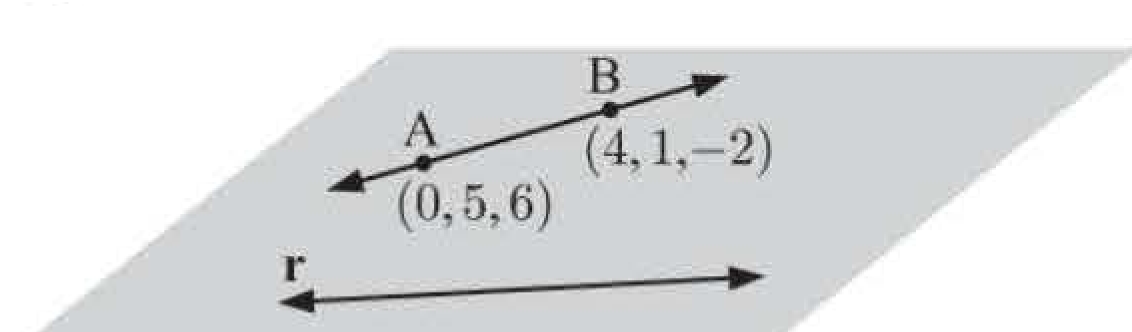
and $\frac{dV}{dt} = 5\ \text{cm}^3\ \text{s}^{-1}$

$\therefore\ 5 = 2\pi(10^2)\frac{dr}{dt}$

$\therefore\ 5 = 200\pi \frac{dr}{dt}$

$\therefore\ \frac{dr}{dt} = \frac{1}{40\pi}$ cm s^{-1}

86



$\overrightarrow{AB} = \begin{pmatrix} 4 \\ -4 \\ -8 \end{pmatrix} = 4\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$

$\therefore$ the first line has equation $x = t,\ y = 5 - t,\ z = 6 - 2t$

The lines are not parallel as the direction vectors of the lines are not multiples of each other.
For the lines to be coplanar, they must intersect.

The first line meets the second line where $\begin{pmatrix}0\\5\\6\end{pmatrix}+t\begin{pmatrix}1\\-1\\-2\end{pmatrix}=\begin{pmatrix}a\\3\\2\end{pmatrix}+s\begin{pmatrix}2\\-1\\1\end{pmatrix}$

$$\therefore \quad t=a+2s, \qquad \underbrace{5-t=3-s}_{\therefore\ s=t-2}, \qquad \text{and} \qquad \underbrace{6-2t=2+s}_{\therefore\ s+2t=4}$$

$$\begin{aligned} \text{Thus }\ t-2+2t&=4\\ \therefore\ 3t&=6\\ \therefore\ t&=2 \ \text{ and } \ s=0 \end{aligned}$$

$\therefore$ the lines are coplanar if $2=a+2(0)$

$$\therefore\ a=2$$

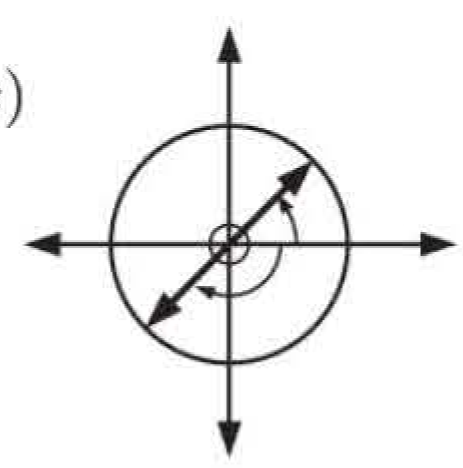
87 $y=\dfrac{\sin x}{\tan x+1}, \quad -\pi\leqslant x\leqslant\frac{\pi}{2}$

$$\therefore\ \frac{dy}{dx}=\frac{\cos x(\tan x+1)-\sin x\sec^2 x}{(\tan x+1)^2}$$

$$\therefore\ \frac{dy}{dx}=\frac{\sin x+\cos x-\dfrac{\sin x}{\cos^2 x}}{(\tan x+1)^2}$$

which is 0 when $\sin x+\cos x=\dfrac{\sin x}{\cos^2 x}$

$$\begin{aligned} \therefore\ \sin x\cos^2 x+\cos^3 x&=\sin x\\ \therefore\ \cos^3 x&=\sin x(1-\cos^2 x)\\ \therefore\ \cos^3 x&=\sin^3 x\\ \therefore\ \tan^3 x&=1\\ \therefore\ \tan x&=1\\ \therefore\ x&=\tfrac{\pi}{4},\ -\tfrac{3\pi}{4} \end{aligned}$$



$\therefore$ the stationary points are at $\left(\frac{\pi}{4}, \frac{\sqrt{2}}{4}\right)$ and $\left(-\frac{3\pi}{4}, -\frac{\sqrt{2}}{4}\right)$.

88 $P(X=x)=a\left(\frac{2}{5}\right)^x$ where $x=0, 1, 2, 3, 4, 5, ...$

$$\therefore\ a\left(\tfrac{2}{5}\right)^0+a\left(\tfrac{2}{5}\right)^1+a\left(\tfrac{2}{5}\right)^2+....=1 \qquad \{\textstyle\sum P(x)=1\}$$

$$\therefore\ a\left(1+\tfrac{2}{5}+\left(\tfrac{2}{5}\right)^2+....\right)=1$$

$$\therefore\ a\left(\frac{1}{1-\frac{2}{5}}\right)=1 \qquad \{\text{sum of an infinite geometric series}\}$$

$$\therefore\ \frac{a}{\frac{3}{5}}=1$$

$$\therefore\ a=\tfrac{3}{5}$$

89 $4\sin x=\sqrt{3}\csc x+2-2\sqrt{3}$ where $0\leqslant x\leqslant 2\pi$

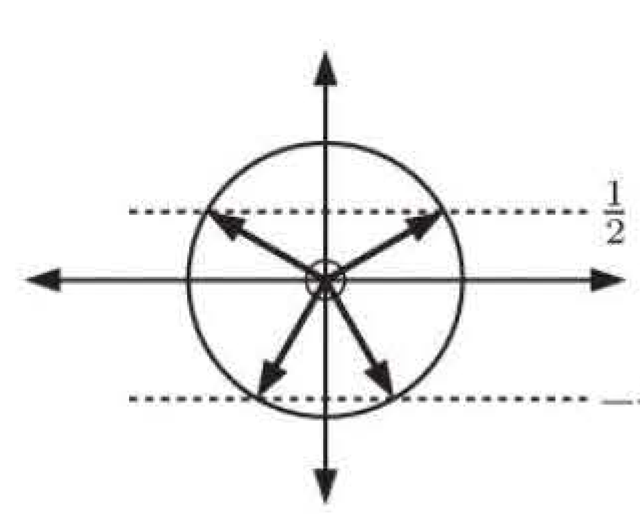
$$\therefore\ 4\sin x=\frac{\sqrt{3}}{\sin x}+2-2\sqrt{3}$$

$$\therefore\ 4\sin^2 x+(2\sqrt{3}-2)\sin x-\sqrt{3}=0$$

$$\therefore\ (2\sin x+\sqrt{3})(2\sin x-1)=0$$

$$\therefore\ \sin x=-\tfrac{\sqrt{3}}{2} \text{ or } \tfrac{1}{2}$$

$$\therefore\ x=\tfrac{\pi}{6}, \tfrac{5\pi}{6}, \tfrac{4\pi}{3}, \tfrac{5\pi}{3}$$



90 **a** $\left(\begin{array}{ccc|c} 1 & -2 & 3 & 1\\ 1 & p & 2 & 0\\ -2 & p^2 & -4 & q \end{array}\right)$

b $$\sim\left(\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & p+2 & -1 & -1 \\ 0 & p^2-4 & 2 & q+2 \end{array}\right) \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 + 2R_1 \end{array}$$

$$\sim\left(\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & p+2 & -1 & -1 \\ 0 & 0 & p & p+q \end{array}\right) \; R_3 \to R_3 - (p-2)R_2$$

$$\begin{array}{cccc} 0 & p^2-4 & 2 & q+2 \\ 0 & -(p^2-4) & p-2 & p-2 \\ \hline 0 & 0 & p & p+q \\ \hline \end{array}$$

c **i** For a unique solution, $p \neq 0$. The planes meet at one point only.

ii There are no solutions if $p = 0,\ q \neq 0$. The three planes have no common point of intersection as planes 2 and 3 are parallel.

iii There are infinitely many solutions when $p = q = 0$. Planes 2 and 3 are coincident.

d When $p = q = 0$ the augmented matrix is $\left(\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$ ← this equation is $2y - z = -1$.

Letting $y = t,\quad z = 1 + 2t$.

Using $x - 2y + 3z = 1$, we find $x - 2(t) + 3(1+2t) = 1$

$$\therefore\ x - 2t + 3 + 6t = 1$$

$$\therefore\ x = -2 - 4t$$

So, $x = -2 - 4t,\ y = t,\ z = 1 + 2t,\ t \in \mathbb{R}$.

91 Since the polynomial is real, $1 \pm 2i$ and $\pm ai$ are zeros, $a \neq 0$.

$1 \pm 2i$ have sum 2 and product $1 + 4 = 5$ and so come from $z^2 - 2z + 5$

$\pm ai$ have sum 0 and product a^2 and so come from $z^2 + a^2$

$\therefore\ P(z) = k(z^2 - 2z + 5)(z^2 + a^2)$

But $k = 1$ and $P(0) = 1(5)(a^2) = 10$ so $a^2 = 2$

$$\therefore\ P(z) = (z^2 - 2z + 5)(z^2 + 2)$$

92
$$\tan 2A = \tfrac{3}{2}$$
$$\therefore\ \frac{2\tan A}{1 - \tan^2 A} = \tfrac{3}{2}$$
$$\therefore\ 4\tan A = 3 - 3\tan^2 A$$
$$\therefore\ 3\tan^2 A + 4\tan A - 3 = 0$$
$$\therefore\ \tan A = \frac{-4 \pm \sqrt{16 - 4(3)(-3)}}{6}$$
$$= -\frac{2}{3} \pm \frac{\sqrt{13}}{3}$$
But A is acute, so $\tan A > 0$
$$\therefore\ \tan A = \frac{\sqrt{13} - 2}{3}$$

93 **a** $$\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -1 \\ 2 & -1 & 1 \end{vmatrix}$$
$$= \mathbf{i}(0) - \mathbf{j}(3) + \mathbf{k}(-3) = -3\mathbf{j} - 3\mathbf{k}$$

b $$\text{LHS} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & -1 \\ 0 & -3 & -3 \end{vmatrix}$$
$$= \mathbf{i}(-9) - \mathbf{j}(-9) + \mathbf{k}(-9)$$
$$= -9\mathbf{i} + 9\mathbf{j} - 9\mathbf{k}$$

$$\text{RHS} = \mathbf{b}(\mathbf{a} \bullet \mathbf{c}) - \mathbf{c}(\mathbf{a} \bullet \mathbf{b})$$
$$= 3\mathbf{b} - 6\mathbf{c}$$
$$= 3\mathbf{i} + 3\mathbf{j} - 3\mathbf{k} - 12\mathbf{i} + 6\mathbf{j} - 6\mathbf{k}$$
$$= -9\mathbf{i} + 9\mathbf{j} - 9\mathbf{k}$$

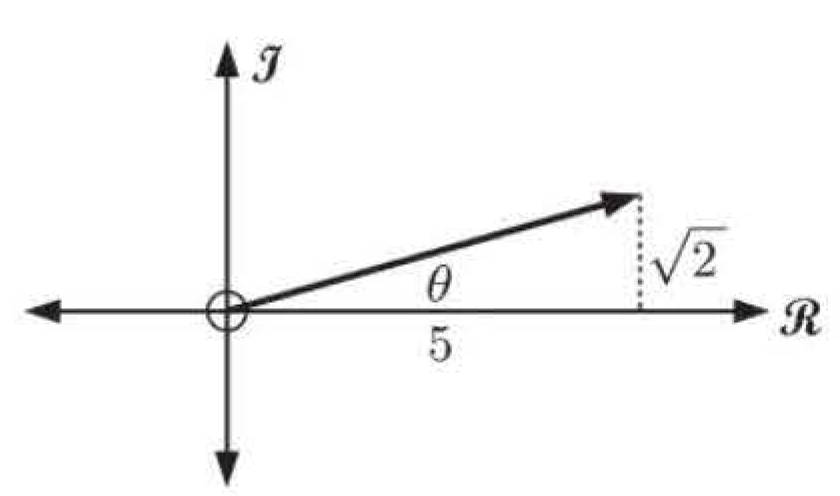
94 **a** $$(-1 + i\sqrt{2})^3 = (-1)^3 + 3(-1)^2 i\sqrt{2} + 3(-1)(i\sqrt{2})^2 + (i\sqrt{2})^3$$
$$= -1 + 3\sqrt{2}i + 6 - 2\sqrt{2}i$$
$$= 5 + i\sqrt{2}$$

b $$\left|5 + i\sqrt{2}\right| = \sqrt{25 + 2} = \sqrt{27} = (\sqrt{3})^3$$

$$\arg(5 + i\sqrt{2}) = \theta = \arctan\left(\tfrac{\sqrt{2}}{5}\right)$$

$$\therefore\ 5 + i\sqrt{2} = \left(\sqrt{3}\right)^3 \operatorname{cis}\left[\arctan\left(\tfrac{\sqrt{2}}{5}\right)\right]$$

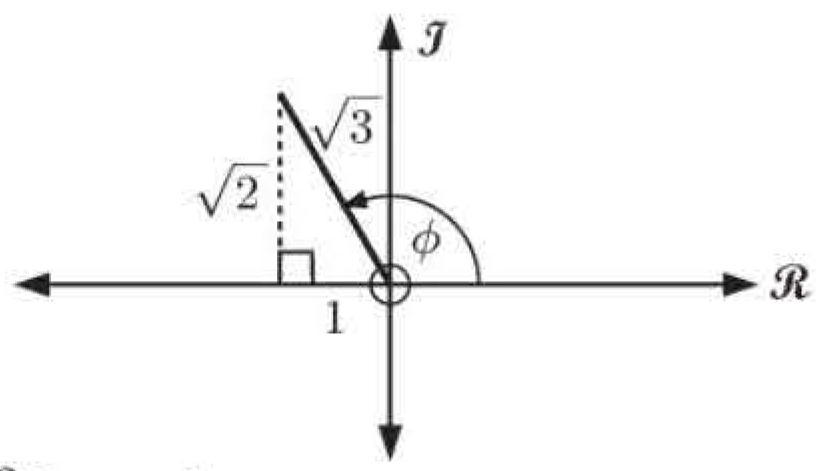
$$\therefore\ a = \sqrt{3},\ \theta = \arctan\left(\tfrac{\sqrt{2}}{5}\right)$$



c $z^3 = 5 + i\sqrt{2} = \left(\sqrt{3}\right)^3 \text{cis}\left[\arctan\left(\frac{\sqrt{2}}{5}\right)\right]$

$\therefore\ z = \sqrt{3}\ \text{cis}\left[\dfrac{\arctan\left(\frac{\sqrt{2}}{5}\right) + k2\pi}{3}\right]$ where $k = 0, 1, 2$ {De Moivre}

d From **a**, one of the solutions to $z^3 = 5 + i\sqrt{2}$ is



$z = -1 + i\sqrt{2}$

$\therefore\ z = \sqrt{3}\ \text{cis}\ \phi$

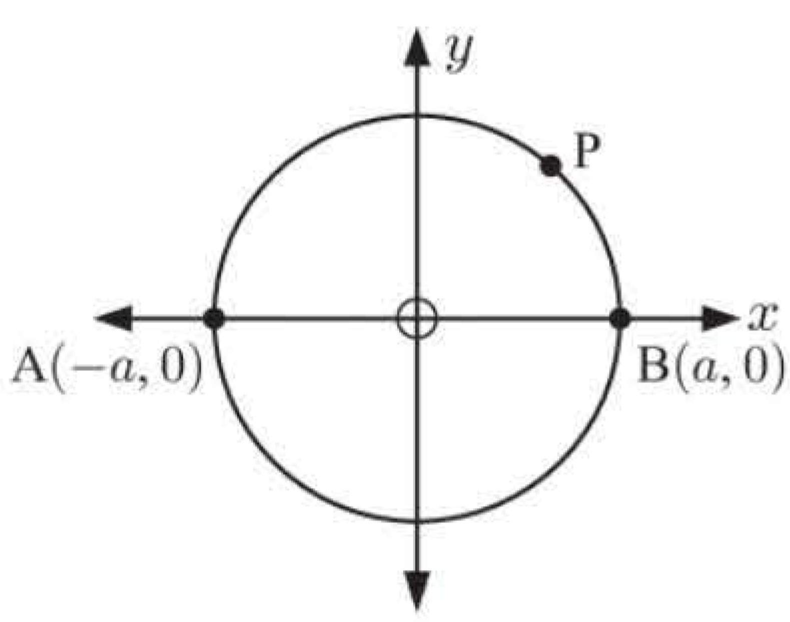
$\therefore\ z = \sqrt{3}\ \text{cis}\left[\arccos\left(\frac{-1}{\sqrt{3}}\right)\right]$

This corresponds to the solution in **c** where $k = 1$ $\{\frac{\theta}{3} + \frac{2\pi}{3} = \phi\}$

Equating arguments gives: $\dfrac{\arctan\left(\frac{\sqrt{2}}{5}\right) + 2\pi}{3} = \arccos\left(\frac{-1}{\sqrt{3}}\right)$

$\therefore\ \arctan\left(\frac{\sqrt{2}}{5}\right) + 2\pi = 3\arccos\left(\frac{-1}{\sqrt{3}}\right)$

95 **a**



As $x^2 + y^2 = a^2$,

$y^2 = a^2 - x^2$

$\therefore\ y = \sqrt{a^2 - x^2}$ as y is > 0

$\therefore$ P is $(x, \sqrt{a^2 - x^2})$

b **i** $\overrightarrow{\text{AP}} = \begin{pmatrix} x+a \\ \sqrt{a^2-x^2} \end{pmatrix}$, $\overrightarrow{\text{AB}} = \begin{pmatrix} 2a \\ 0 \end{pmatrix}$, $\overrightarrow{\text{OP}} = \begin{pmatrix} x \\ \sqrt{a^2-x^2} \end{pmatrix}$

ii $\cos(\text{P}\widehat{\text{A}}\text{B})$

$= \dfrac{\overrightarrow{\text{AP}} \bullet \overrightarrow{\text{AB}}}{|\overrightarrow{\text{AP}}||\overrightarrow{\text{AB}}|}$

$= \dfrac{2a(x+a) + 0}{\sqrt{(x+a)^2 + a^2 - x^2} \times 2a}$

$= \dfrac{x+a}{\sqrt{x^2 + 2ax + a^2 + a^2 - x^2}}$

$= \dfrac{x+a}{\sqrt{2a(x+a)}}$

$= \sqrt{\dfrac{x+a}{2a}}$

$\cos(\text{P}\widehat{\text{O}}\text{B})$

$= \dfrac{\overrightarrow{\text{OP}} \bullet \overrightarrow{\text{OB}}}{|\overrightarrow{\text{OP}}||\overrightarrow{\text{OB}}|}$ where $\overrightarrow{\text{OB}} = \begin{pmatrix} a \\ 0 \end{pmatrix}$

$= \dfrac{ax}{\sqrt{x^2 + a^2 - x^2} \times a}$

$= \dfrac{ax}{a^2}$

$= \dfrac{x}{a}$

c Thus, $2\cos^2(\text{P}\widehat{\text{A}}\text{B}) - 1 = \dfrac{2(x+a)}{2a} - 1$

$= \dfrac{x}{a} + 1 - 1$

$= \dfrac{x}{a}$

$= \cos(\text{P}\widehat{\text{O}}\text{B})$

Hence, $\text{P}\widehat{\text{O}}\text{B} = 2 \times \text{P}\widehat{\text{A}}\text{B}$

The angle at the centre of a circle is twice the angle on the circle subtended by the same arc.

96 P_n is "$x^n - y^n$ has factor $x - y$" for $n \in \mathbb{Z}^+$.

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, $x^1 - y^1$ has factor $x - y$ ✓ $\therefore$ P_1 is true.

(2) If P_k is assumed true then $x^k - y^k = (x-y)f_k(x, y)$, where $f_k(x, y)$ is another factor.

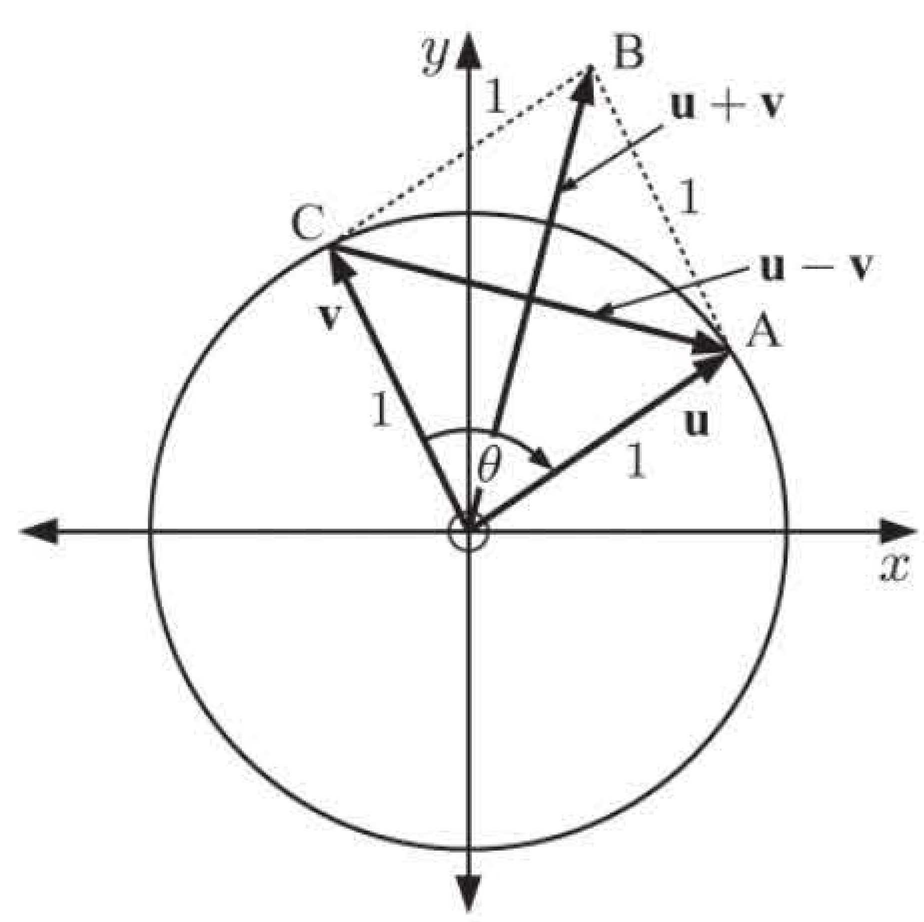
Now
$$\begin{aligned} x^{k+1} - y^{k+1} &= x(x^k - y^k) + xy^k - y^{k+1} \\ &= x(x-y)f_k(x, y) + y^k(x-y) \\ &= (x-y)\left[x f_k(x, y) + y^k\right] \\ &\equiv (x-y)\, f_{k+1}(x, y) \end{aligned}$$

$\therefore$ $x^{k+1} - y^{k+1}$ has factor $x - y$.

Thus P_{k+1} is true whenever P_k is true.

$\therefore$ since P_1 is true, P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

97 **a**



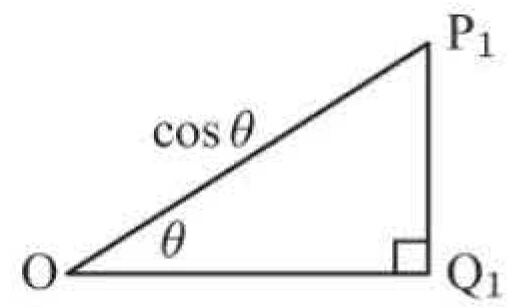
We complete the rhombus OABC (each side has length 1).
Using the cosine rule in triangle OAC,

$$|\mathbf{u} - \mathbf{v}|^2 = 1^2 + 1^2 - 2(1)(1)\cos\theta$$
$$\therefore \quad |\mathbf{u} - \mathbf{v}|^2 = 2 - 2\cos\theta$$
$$\therefore \quad |\mathbf{u} - \mathbf{v}| = \sqrt{2 - 2\cos\theta}$$

Now in $\triangle$OAB, $\widehat{\text{OAB}} = 180^\circ - \theta$, and using the cosine rule again,
$$\begin{aligned} |\mathbf{u} + \mathbf{v}|^2 &= 1^2 + 1^2 - 2(1)(1)\cos(180^\circ - \theta) \\ &= 2 - 2(-\cos\theta) \\ &= 2 + 2\cos\theta \end{aligned}$$
$$\therefore \quad |\mathbf{u} + \mathbf{v}| = \sqrt{2 + 2\cos\theta}$$

b Now, if $|\mathbf{u} + \mathbf{v}| = 5\,|\mathbf{u} - \mathbf{v}|$ then $\sqrt{2 + 2\cos\theta} = 5\sqrt{2 - 2\cos\theta}$
$$\therefore \quad 2 + 2\cos\theta = 25(2 - 2\cos\theta)$$
$$\therefore \quad 2 + 2\cos\theta = 50 - 50\cos\theta$$
$$\therefore \quad 52\cos\theta = 48$$
$$\therefore \quad \cos\theta = \tfrac{12}{13}$$

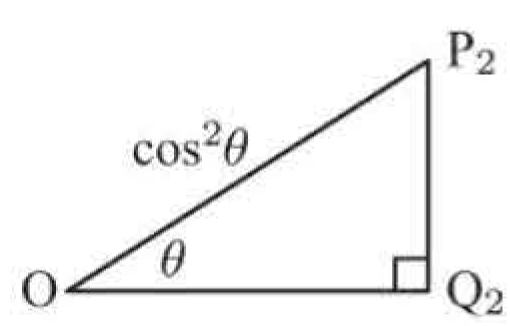
98 P lies on the unit circle, so $\text{OQ} = \cos\theta$ and $\text{PQ} = \sin\theta$



In $\triangle\text{OP}_1\text{Q}_1$, $\text{OP}_1 = \text{OQ} = \cos\theta$

So, $\sin\theta = \dfrac{\text{P}_1\text{Q}_1}{\cos\theta}$ and $\cos\theta = \dfrac{\text{OQ}_1}{\cos\theta}$

$\therefore$ $\text{OQ}_1 = \cos^2\theta$ and $\text{P}_1\text{Q}_1 = \sin\theta\cos\theta$



Likewise in $\triangle\text{OP}_2\text{Q}_2$, $\sin\theta = \dfrac{\text{P}_2\text{Q}_2}{\cos^2\theta}$ and $\cos\theta = \dfrac{\text{OQ}_2}{\cos^2\theta}$

$\therefore$ $\text{OQ}_2 = \cos^3\theta$ and $\text{P}_2\text{Q}_2 = \sin\theta\cos^2\theta$

Thus $\text{PQ} + \text{P}_1\text{Q}_1 + \text{P}_2\text{Q}_2 + \text{P}_3\text{Q}_3 + ...$
$$\begin{aligned} &= \sin\theta + \sin\theta\cos\theta + \sin\theta\cos^2\theta + \sin\theta\cos^3\theta + ... \\ &= \sin\theta(1 + \cos\theta + \cos^2\theta + \cos^3\theta + ...) \\ &= \sin\theta\left(\frac{1}{1 - \cos\theta}\right) \quad \{\text{sum of an infinite geometric series with } |r| = |\cos\theta| \leqslant 1\} \\ &= \frac{\sin\theta}{1 - \cos\theta} \\ &= \frac{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}{2\sin^2\left(\frac{\theta}{2}\right)} \\ &= \cot\left(\frac{\theta}{2}\right) \end{aligned}$$

99 Suppose the common root is α and the other roots are β and γ.

$\therefore\ x^2+ax+bc=(x-\alpha)(x-\beta)$ and $x^2+bx+ca=(x-\alpha)(x-\gamma)$

Thus $\alpha+\beta=-a$ and $\alpha+\gamma=-b$ so $(\alpha+\beta)(\alpha+\gamma)=ab$ .... (1)

Also, $\alpha\beta=bc$ and $\alpha\gamma=ca$ so $\alpha^2\beta\gamma=abc^2$ .... (2)

Now α is a common root of both equations $\therefore\ \alpha^2+a\alpha+bc=0$ .... (3) and

$\alpha^2+b\alpha+ca=0$ .... (4)

Subtracting (4) from (3), $(a-b)\alpha-(a-b)c=0$

$\therefore\ (a-b)(\alpha-c)=0$

$\therefore\ a=b$ or $\alpha=c$

But if $a=b$, both equations are the same, and so the equations would have two common roots

$\therefore\ \alpha=c$ .... (5)

Using (2), $\alpha^2\beta\gamma=abc^2$

$\therefore\ c^2\beta\gamma=abc^2$

$\therefore\ \beta\gamma=ab$ .... (6)

Using (1), $\alpha^2+\alpha(\beta+\gamma)+\beta\gamma=ab$

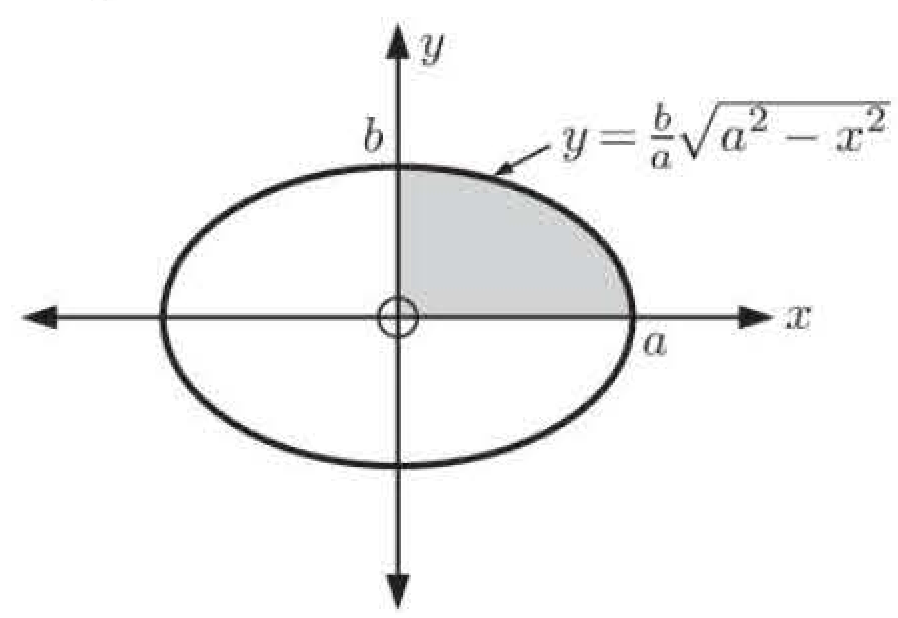
$\therefore\ c^2+c(\beta+\gamma)=0$ {using (5), (6)}

$\therefore\ \beta+\gamma=-c$ .... (7)

From (6) and (7), β and γ are the roots of $x^2+cx+ab=0$.

100 **a** Since $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, $y^2=b^2\left(1-\dfrac{x^2}{a^2}\right)=\dfrac{b^2}{a^2}(a^2-x^2)$

$$\therefore\ y=\pm\frac{b}{a}\sqrt{a^2-x^2}$$



$y=\dfrac{b}{a}\sqrt{a^2-x^2}$ is the top half of the ellipse

$$\therefore\ \text{shaded area}=\int_0^a y\,dx=\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx$$

$$=\frac{b}{a}\int_0^a \sqrt{a^2-x^2}\,dx$$

b Let $x=a\sin\theta$, $\dfrac{dx}{d\theta}=a\cos\theta$

when $x=0$, $\sin\theta=0$, $\theta=0$

when $x=a$, $\sin\theta=1$, $\theta=\frac{\pi}{2}$

$$\therefore\ \text{shaded area}=\frac{b}{a}\int_0^{\frac{\pi}{2}}\sqrt{a^2-a^2\sin^2\theta}\,a\cos\theta\,d\theta$$

$$=b\int_0^{\frac{\pi}{2}} a\sqrt{1-\sin^2\theta}\,\cos\theta\,d\theta$$

$$=ab\int_0^{\frac{\pi}{2}}\cos^2\theta\,d\theta$$

$$=ab\int_0^{\frac{\pi}{2}}\left(\tfrac{1}{2}+\tfrac{1}{2}\cos 2\theta\right)d\theta$$

$$=ab\left[\tfrac{1}{2}\theta+\tfrac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{2}}$$

$$=ab\left(\tfrac{\pi}{4}+\tfrac{1}{4}\sin\pi-0-0\right)$$

$$=\frac{\pi ab}{4}$$

$\therefore$ area of ellipse $=4\times$ shaded area $=\pi ab$

c Volume $=2\pi\int_0^a y^2\,dx$

$$=2\pi\frac{b^2}{a^2}\int_0^a(a^2-x^2)\,dx$$

$$=2\pi\frac{b^2}{a^2}\left[a^2x-\frac{x^3}{3}\right]_0^a$$

$$=2\pi\frac{b^2}{a^2}\left(a^3-\frac{a^3}{3}-0\right)$$

$$=2\pi\times\frac{b^2}{a^2}\times\frac{2a^3}{3}$$

$$=\tfrac{4}{3}\pi ab^2$$

(**Note:** When $a=b$, $V=\frac{4}{3}\pi a^3$, which is the volume of a sphere of radius a.)

101 Now $f(x) = (a_1x - b_1)^2 + (a_2x - b_2)^2 + ... + (a_nx - b_n)^2$
is the sum of squares, so $f(x) \geqslant 0$ for all x.

But $f(x) = \sum_{i=1}^{n}(a_ix - b_i)^2$

$$= \left(\sum_{i=1}^{n} a_i^{\ 2}\right)x^2 - 2\left(\sum_{i=1}^{n} a_ib_i\right)x + \left(\sum_{i=1}^{n} b_i^{\ 2}\right)$$

Since $f(x) \geqslant 0$ for all x, the discriminant must be non-positive.

Hence $\left(2\sum_{i=1}^{n} a_ib_i\right)^2 - 4\left(\sum_{i=1}^{n} a_i^{\ 2}\right)\left(\sum_{i=1}^{n} b_i^{\ 2}\right) \leqslant 0$

$$\therefore\ 4\left(\sum_{i=1}^{n} a_i^{\ 2}\right)\left(\sum_{i=1}^{n} b_i^{\ 2}\right) \geqslant 4\left(\sum_{i=1}^{n} a_ib_i\right)^2$$

$$\therefore\ \left(\sum_{i=1}^{n} a_i^{\ 2}\right)\left(\sum_{i=1}^{n} b_i^{\ 2}\right) \geqslant \left(\sum_{i=1}^{n} a_ib_i\right)^2$$

102 **a** P_n is: "$(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + ... + \binom{n}{n}x^n$" for $n \in \mathbb{Z}^+$.

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, LHS $= (1+x)^1 = 1 + x$

RHS $= 1 + \binom{1}{1}x = 1 + x$ $\quad\therefore\ P_1$ is true.

(2) If P_k is assumed true, then

$$(1+x)^k = 1 + \binom{k}{1}x + \binom{k}{2}x^2 + ... + \binom{k}{k}x^k$$

Thus
$$\begin{aligned}(1+x)^{k+1} &= \left[1 + \binom{k}{1}x + \binom{k}{2}x^2 + ... + \binom{k}{k}x^k\right][1+x]\\
&= 1 + \binom{k}{1}x + \binom{k}{2}x^2 + \binom{k}{3}x^3 + ... + \binom{k}{k}x^k\\
&\qquad + x + \binom{k}{1}x^2 + \binom{k}{2}x^3 + ... + \binom{k}{k-1}x^k + \binom{k}{k}x^{k+1}\\
&= 1 + \left[\binom{k}{1} + \binom{k}{0}\right]x + \left[\binom{k}{2} + \binom{k}{1}\right]x^2 + ...\\
&\qquad + \left[\binom{k}{k} + \binom{k}{k-1}\right]x^k + \binom{k}{k}x^{k+1}\\
&= 1 + \binom{k+1}{1}x + \binom{k+1}{2}x^2 + ... + \binom{k+1}{k}x^k + \binom{k+1}{k+1}x^{k+1}\end{aligned}$$

{Using Pascal's Rule and $\binom{k+1}{k+1} = \binom{k}{k} = 1$}

Thus P_{k+1} is true whenever P_k is true.

$\therefore$ since P_1 is true, P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

b Letting $x = \frac{b}{a}$, $\quad \left(1 + \frac{b}{a}\right)^n = 1 + \binom{n}{1}\frac{b}{a} + \binom{n}{2}\frac{b^2}{a^2} + ... + \binom{n}{n}\frac{b^n}{a^n}$, $\ n \in \mathbb{Z}^+$

$$\therefore\ a^n\left(1 + \frac{b}{a}\right)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + ... + \binom{n}{n}b^n$$

$$\therefore\ \left[a\left(1 + \frac{b}{a}\right)\right]^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + ... + \binom{n}{n}b^n$$

$$\therefore\ (a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + ... + \binom{n}{n}b^n$$

103 $\log_y 16y - \log_{16y} y = \frac{8}{3}$ $\quad$ {as $x = 16y$}

$\therefore\ \dfrac{\log 16y}{\log y} - \dfrac{\log y}{\log 16y} - \frac{8}{3} = 0$

$\therefore\ m - \dfrac{1}{m} - \frac{8}{3} = 0$ $\quad$ {letting $\dfrac{\log 16y}{\log y} = m$}

$\therefore\ 3m^2 - 8m - 3 = 0$ $\quad$ {$\times 3m$}

$\therefore\ (3m+1)(m-3) = 0$

$$\therefore\ m = \frac{\log 16y}{\log y} = -\tfrac{1}{3} \text{ or } 3$$

$$\therefore\ \log 16y = -\tfrac{1}{3}\log y \quad \text{or} \quad \log 16y = 3\log y$$

$$\therefore\ 16y = y^{-\frac{1}{3}} \quad \text{or} \quad 16y = y^3$$

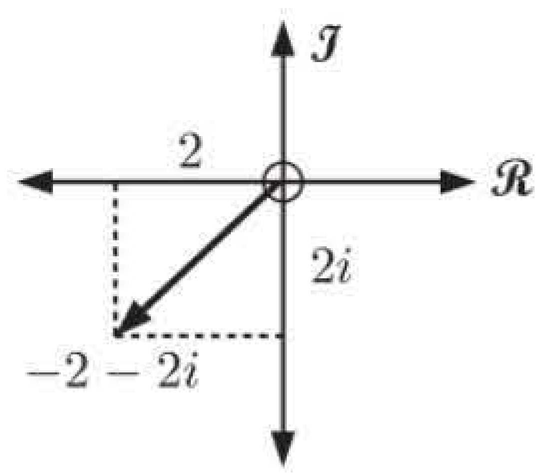
$$\therefore\ y^{\frac{4}{3}} = \tfrac{1}{16} \quad \text{or} \quad y(y^2-16)=0$$

$$\therefore\ y = \left(\pm\tfrac{1}{2}\right)^3 \quad \text{or} \quad y = 0 \text{ or } \pm 4$$

$$\therefore\ y = \tfrac{1}{8} \text{ or } 4 \quad \{\text{as } y \text{ is a base, } y>0\}$$

$$\therefore\ y = \tfrac{1}{8},\ x = 2 \quad \text{or} \quad y = 4,\ x = 64$$

104 **a**



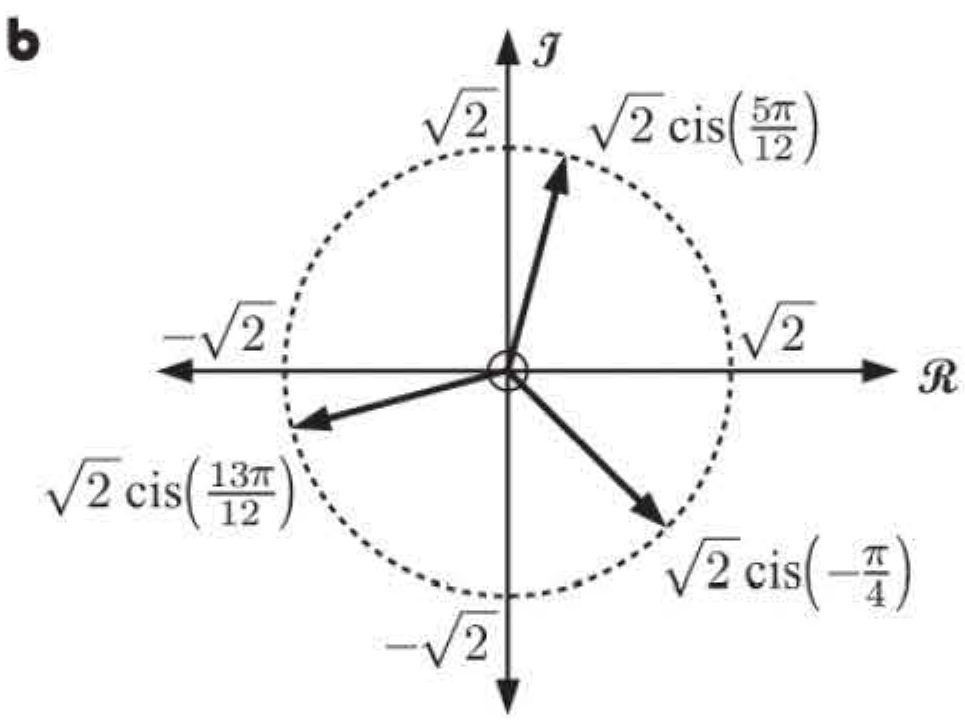
$|-2-2i| = \sqrt{4+4} = \sqrt{8} = (\sqrt{2})^3$ and $\arg(-2-2i) = \frac{-3\pi}{4}$

So, when $z^3 = -2-2i$, $z^3 = (\sqrt{2})^3 \operatorname{cis}\left(\frac{-3\pi}{4} + k2\pi\right)$

$$\therefore\ z = \sqrt{2}\operatorname{cis}\left(\tfrac{-\pi}{4} + k\tfrac{2\pi}{3}\right),\quad k = 0, 1, 2 \quad \{\text{De Moivre}\}$$

$\therefore$ the cube roots are: $\sqrt{2}\operatorname{cis}\left(-\frac{\pi}{4}\right)$, $\sqrt{2}\operatorname{cis}\left(\frac{5\pi}{12}\right)$, $\sqrt{2}\operatorname{cis}\left(\frac{13\pi}{12}\right)$

b



c

$$\begin{aligned}
&\alpha_1 + \alpha_2 + \alpha_3 \\
&= \sqrt{2}\left[\operatorname{cis}\left(\tfrac{-\pi}{4}\right) + \operatorname{cis}\left(\tfrac{5\pi}{12}\right) + \operatorname{cis}\left(\tfrac{13\pi}{12}\right)\right] \\
&= \sqrt{2}\left[a^{-3} + a^5 + a^{13}\right] \quad \text{where } a = \operatorname{cis}\left(\tfrac{\pi}{12}\right) \\
&= \sqrt{2}a^{-3}\left[1 + a^8 + a^{16}\right] \\
&= \sqrt{2}a^{-3}\left[\frac{(a^8)^3 - 1}{a^8 - 1}\right] \quad \{\text{sum of a geometric series}\} \\
&= \frac{\sqrt{2}}{a^3}\left[\frac{a^{24}-1}{a^8-1}\right] \\
&= \frac{\sqrt{2}}{a^3}\left[\frac{\left[\operatorname{cis}\left(\frac{\pi}{12}\right)\right]^{24} - 1}{a^8-1}\right] \\
&= \frac{\sqrt{2}}{a^3}\left[\frac{\operatorname{cis}(2\pi) - 1}{a^8-1}\right] \\
&= 0
\end{aligned}$$

d Let $z^n = \beta\operatorname{cis}(0 + k2\pi)$ $\{\text{as } \operatorname{cis}(0+k2\pi) = 1\}$

$$\therefore\ z = \beta^{\frac{1}{n}}\left[\operatorname{cis}\left(\frac{k2\pi}{n}\right)\right],\quad k = 0, 1, 2, 3, ..., n-1 \quad \{\text{De Moivre}\}$$

$$\therefore\ z = \beta^{\frac{1}{n}}\operatorname{cis} 0,\ \beta^{\frac{1}{n}}\operatorname{cis}\left(\frac{2\pi}{n}\right),\ \beta^{\frac{1}{n}}\operatorname{cis}\left(\frac{4\pi}{n}\right),\ ....,\ \beta^{\frac{1}{n}}\operatorname{cis}\left(\frac{(n-1)2\pi}{n}\right)$$

$$\therefore\ z = \beta^{\frac{1}{n}},\ \beta^{\frac{1}{n}}\alpha,\ \beta^{\frac{1}{n}}\alpha^2,\ \beta^{\frac{1}{n}}\alpha^3,\ ....,\ \beta^{\frac{1}{n}}\alpha^{n-1} \quad \text{where } \alpha = \operatorname{cis}\left(\frac{2\pi}{n}\right)$$

The sum of these zeros is

$$\begin{aligned}
&\beta^{\frac{1}{n}}\left[1 + \alpha + \alpha^2 + \alpha^3 + ... + \alpha^{n-1}\right] \\
&= \beta^{\frac{1}{n}}\left[\tfrac{1-\alpha^n}{1-\alpha}\right] \quad \{\text{sum of a geometric series with } u_1 = 1,\ r = \alpha,\ \text{“}n\text{”} = n\} \\
&= \beta^{\frac{1}{n}}\left[\tfrac{1-\operatorname{cis} 2\pi}{1-\alpha}\right] \quad \left\{\text{since } \alpha^n = \left[\operatorname{cis}\left(\frac{2\pi}{n}\right)\right]^n = \operatorname{cis} 2\pi\right\} \\
&= \beta^{\frac{1}{n}}(0) \quad \{\text{as } \operatorname{cis} 2\pi = 1\} \\
&= 0
\end{aligned}$$

105 Let the roots be $a-3b,\ a-b,\ a+b,\ a+3b$

$a-b$ and $a+b$ have sum $2a$ and product a^2-b^2

$a-3b$ and $a+3b$ have sum $2a$ and product a^2-9b^2

$$\therefore\ x^4-(3m+2)x^2+m^3=\left[x^2-2ax+(a^2-b^2)\right]\left[x^2-2ax+(a^2-9b^2)\right]$$

Equating coefficients of x^3 gives $0=-2a-2a=-4a \qquad \therefore\ a=0$

$$\begin{aligned}\text{Thus }\ x^4-(3m+2)x^2+m^2&=(x^2-b^2)(x^2-9b^2)\\&=x^4-10b^2x^2+9b^4\end{aligned}$$

Equating coefficients, $3m+2=10b^2$ and $m^2=9b^4$

$$\therefore\ m=\pm 3b^2$$

$$\therefore\ 3m+2=10\left(\pm\frac{m}{3}\right)$$

$$\therefore\ 9m+6=\pm 10m$$

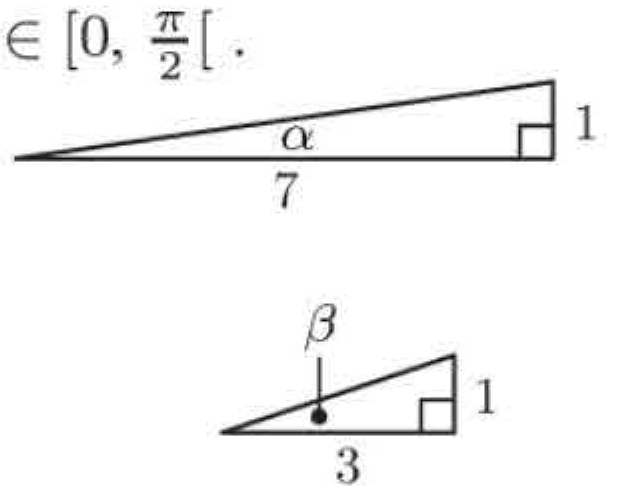
$$\therefore\ m=6 \text{ or } -\tfrac{6}{19}$$

106 Let $\arctan\left(\frac{1}{7}\right)=\alpha$ and $\arctan\left(\frac{1}{3}\right)=\beta$ where $\alpha,\beta\in\left]-\frac{\pi}{2},\frac{\pi}{2}\right[$

$\therefore\ \tan\alpha=\frac{1}{7}$ and $\tan\beta=\frac{1}{3}$

Since $\tan\alpha,\ \tan\beta>0$ and $\alpha,\beta\in\left]-\frac{\pi}{2},\frac{\pi}{2}\right[$, we know that actually $\alpha,\beta\in\left[0,\frac{\pi}{2}\right[$.

From the diagrams we see that $\alpha,\beta\in\left]0,\frac{\pi}{4}\right[$.



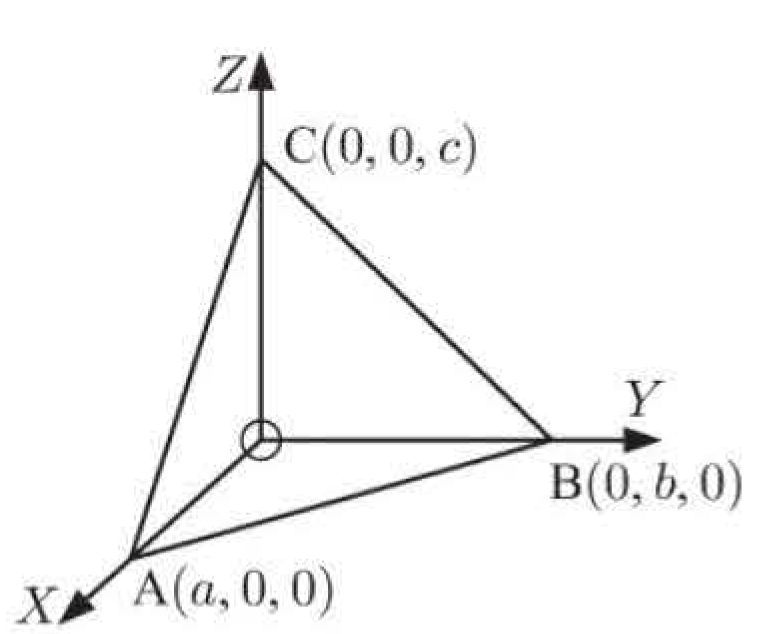
We need to find $\tan(\alpha+2\beta)$ where $\tan 2\beta=\dfrac{2\tan\beta}{1-\tan^2\beta}=\dfrac{\frac{2}{3}}{1-\frac{1}{9}}=\frac{3}{4}$

Now $\tan(\alpha+2\beta)=\dfrac{\tan\alpha+\tan 2\beta}{1-\tan\alpha\tan 2\beta}=\dfrac{\frac{1}{7}+\frac{3}{4}}{1-\frac{3}{28}}=1$

$\therefore\ \alpha+2\beta=\frac{\pi}{4}+k\pi,\ k\in\mathbb{Z}$

$\therefore\ \alpha+2\beta=\frac{\pi}{4}$, the only solution satisfying $\alpha,\beta\in\left]0,\frac{\pi}{4}\right[$.

107



$$\overrightarrow{AB}=\begin{pmatrix}-a\\b\\0\end{pmatrix} \text{ and } \overrightarrow{BC}=\begin{pmatrix}0\\-b\\c\end{pmatrix}$$

$$\begin{aligned}\therefore\ \overrightarrow{AB}\times\overrightarrow{BC}&=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-a&b&0\\0&-b&c\end{vmatrix}\\&=\mathbf{i}\begin{vmatrix}b&0\\-b&c\end{vmatrix}-\mathbf{j}\begin{vmatrix}-a&0\\0&c\end{vmatrix}+\mathbf{k}\begin{vmatrix}-a&b\\0&-b\end{vmatrix}\\&=bc\mathbf{i}+ac\mathbf{j}+ab\mathbf{k}\end{aligned}$$

$\therefore$ the plane has equation $bcx+acy+abz=bc(a)+0+0$ {using point A}

Dividing through by abc gives $\dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c}=1.$

108 Given: $(1+x)^n=\binom{n}{0}+\binom{n}{1}x+\binom{n}{2}x^2+\binom{n}{3}x^3+\ldots.+\binom{n}{n}x^n,\ n\in\mathbb{Z}^+ \quad \ldots.\ (*)$

a Differentiating both sides gives:

$$n(1+x)^{n-1}=\binom{n}{1}+2\binom{n}{2}x+3\binom{n}{3}x^2+\ldots.+n\binom{n}{n}x^{n-1}$$

Letting $x=1$, $\binom{n}{1}+2\binom{n}{2}+3\binom{n}{3}+\ldots.+n\binom{n}{n}=n\,2^{n-1}$

b
$$\begin{aligned}&\binom{n}{0}+2\binom{n}{1}+3\binom{n}{2}+\ldots.+(n+1)\binom{n}{n}\\&=\left[\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\ldots.+\binom{n}{n}\right]+\left[\binom{n}{1}+2\binom{n}{2}+\ldots.+n\binom{n}{n}\right]\\&=2^n+n2^{n-1} \quad \{\text{letting } x=1 \text{ in } (*)\text{, and using } \mathbf{a}\}\\&=(n+2)2^{n-1}\end{aligned}$$

c We notice that $\dfrac{1}{r+1}\dbinom{n}{r} = \dfrac{1}{r+1}\left(\dfrac{n!}{r!(n-r!)}\right) = \dfrac{n!}{(r+1)!(n-r)!}$

$$= \left[\frac{n!}{(r+1)!(n-r)!}\right]\frac{n+1}{n+1}$$

$$= \frac{1}{n+1}\left[\frac{(n+1)!}{(r+1)!(n-r)!}\right]$$

$$= \frac{1}{n+1}\binom{n+1}{r+1}$$

$\therefore \quad \frac{1}{1}\binom{n}{0} + \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + ... + \frac{1}{n+1}\binom{n}{n}$

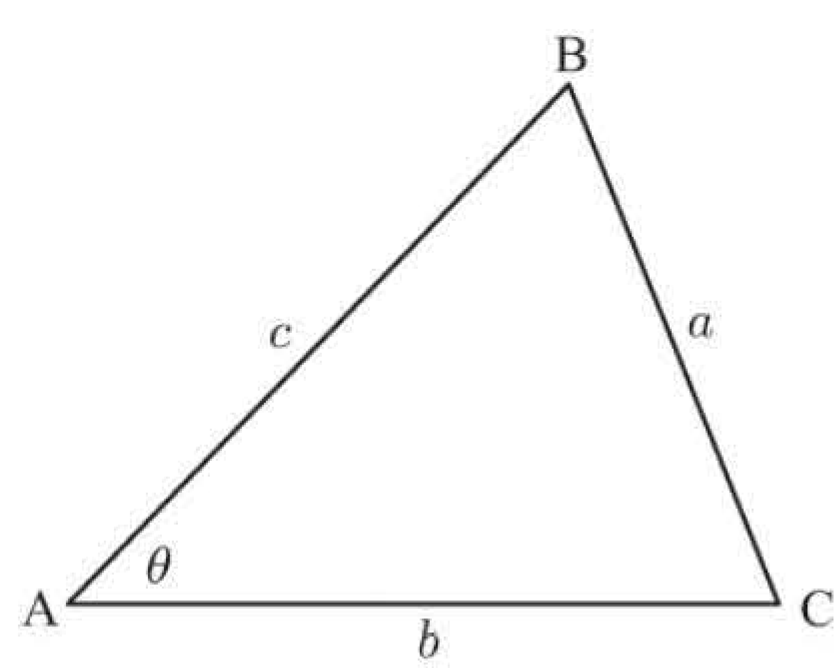
$= \frac{1}{n+1}\binom{n+1}{1} + \frac{1}{n+1}\binom{n+1}{2} + \frac{1}{n+1}\binom{n+1}{3} + ... + \frac{1}{n+1}\binom{n+1}{n+1}$

$= \frac{1}{n+1}\left[\binom{n+1}{1} + \binom{n+1}{2} + \binom{n+1}{3} + ... + \binom{n+1}{n+1}\right]$

$= \frac{1}{n+1}\left[(1+1)^{n+1} - \binom{n+1}{0}\right]$ {using $(*)$ with n replaced by $n+1$, $x=1$}

$= \dfrac{2^{n+1}-1}{n+1}$

109



Area of $\triangle$ABC, $A = \frac{1}{2}bc\sin A$

$\therefore \quad A^2 = \frac{1}{4}b^2c^2\sin^2\theta$

$$= \tfrac{1}{4}b^2c^2\left(1-\cos^2\theta\right)$$

$$= \frac{b^2c^2}{4}\left(1-\left[\frac{b^2+c^2-a^2}{2bc}\right]^2\right)$$

$$= \frac{b^2c^2}{4}\left(1+\frac{b^2+c^2-a^2}{2bc}\right)\left(1-\frac{b^2+c^2-a^2}{2bc}\right)$$

$$= \frac{b^2c^2}{4}\left(\frac{2bc+b^2+c^2-a^2}{2bc}\right)\left(\frac{2bc-b^2-c^2+a^2}{2bc}\right)$$

$$= \tfrac{1}{16}\left((b+c)^2-a^2\right)\left(a^2-(b-c)^2\right)$$

$$= \tfrac{1}{16}(b+c-a)(b+c+a)(a-b+c)(a+b-c)$$

$$= \left(\frac{a+b+c}{2}\right)\left(\frac{b+c-a}{2}\right)\left(\frac{a+c-b}{2}\right)\left(\frac{a+b-c}{2}\right)$$

$$= s(s-a)(s-b)(s-c) \quad \text{where} \quad s = \frac{a+b+c}{2}$$

Thus $A = \sqrt{s(s-a)(s-b)(s-c)}$ where $s = \dfrac{a+b+c}{2}$.

110 $\text{P}(X=x) = \dfrac{m^x e^{-m}}{x!}$ for $x = 0, 1, 2, 3, 4, ...$ where m = mean = variance = σ^2

Thus $\text{P}(X=x) = \dfrac{\sigma^{2x}e^{-\sigma^2}}{x!}$

But $\text{P}(X=2) - \text{P}(X=1) = 3\text{P}(X=0)$

$$\therefore \quad \frac{\sigma^4 e^{-\sigma^2}}{2!} - \frac{\sigma^2 e^{-\sigma^2}}{1!} = \frac{3\sigma^0 e^{-\sigma^2}}{0!}$$

$$\therefore \quad \frac{\sigma^4}{2} - \sigma^2 = 3$$

$$\therefore \quad \sigma^4 - 2\sigma^2 - 6 = 0$$

$$\therefore \quad \sigma^2 = \frac{2 \pm \sqrt{4-4(1)(-6)}}{2} = \frac{2\pm\sqrt{28}}{2} = 1 \pm \sqrt{7}$$

But $\sigma^2 > 0$, so $\sigma^2 = 1+\sqrt{7}$ and $\sigma > 0$ so $\sigma = \sqrt{1+\sqrt{7}}$

111 **a** Let $\dfrac{1}{n(n+2)} = \dfrac{A}{n} + \dfrac{B}{n+2} = \dfrac{A(n+2)+Bn}{n(n+2)}$ for all n

$\therefore \dfrac{1}{n(n+2)} = \dfrac{(A+B)n+2A}{n(n+2)}$ for all n

$\therefore A+B=0$ and $2A=1$ $\therefore A=\frac{1}{2},\ B=-\frac{1}{2}$

b $\dfrac{1}{1\times 3}+\dfrac{1}{2\times 4}+\dfrac{1}{3\times 5}+ .... +\dfrac{1}{n(n+2)}$

$= \dfrac{\frac{1}{2}}{1}-\dfrac{\frac{1}{2}}{3}+\dfrac{\frac{1}{2}}{2}-\dfrac{\frac{1}{2}}{4}+\dfrac{\frac{1}{2}}{3}-\dfrac{\frac{1}{2}}{5}+\dfrac{\frac{1}{2}}{4}-\dfrac{\frac{1}{2}}{6}+ .... +\dfrac{\frac{1}{2}}{n-1}-\dfrac{\frac{1}{2}}{n+1}+\dfrac{\frac{1}{2}}{n}-\dfrac{\frac{1}{2}}{n+2}$

$= \dfrac{1}{2}+\dfrac{1}{4}-\dfrac{1}{2n+2}-\dfrac{1}{2n+4}$ {as all other terms cancel}

$= \dfrac{3}{4}-\dfrac{1}{2n+2}-\dfrac{1}{2n+4}$

c $\displaystyle\sum_{r=1}^{\infty}\frac{1}{r(r+2)} = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{r(r+2)} = \lim_{n\to\infty}\left(\tfrac{3}{4}-\tfrac{1}{2n+2}-\tfrac{1}{2n+4}\right)=\tfrac{3}{4}$

d P_n is "$\dfrac{1}{1\times 3}+\dfrac{1}{2\times 4}+\dfrac{1}{3\times 5}+ .... +\dfrac{1}{n(n+2)} = \dfrac{3}{4}-\dfrac{1}{2n+2}-\dfrac{1}{2n+4}$" for $n\in\mathbb{Z}^+$.

Proof: (By the principle of mathematical induction)

(1) If $n=1$, LHS $=\frac{1}{1\times 3}=\frac{1}{3}$, RHS $=\frac{3}{4}-\frac{1}{4}-\frac{1}{6}=\frac{1}{3}$ ✓

(2) If P_k is assumed true then

$$\frac{1}{1\times 3}+\frac{1}{2\times 4}+\frac{1}{3\times 5}+ .... +\frac{1}{k(k+2)} = \frac{3}{4}-\frac{1}{2k+2}-\frac{1}{2k+4}$$

So, $$\frac{1}{1\times 3}+\frac{1}{2\times 4}+\frac{1}{3\times 5}+ .... +\frac{1}{k(k+2)}+\frac{1}{(k+1)(k+3)}$$

$$= \frac{3}{4}-\frac{1}{2k+2}-\frac{1}{2k+4}+\frac{1}{(k+1)(k+3)}$$

$$= \frac{3}{4}-\frac{1}{2k+4}+\frac{1}{(k+1)(k+3)}-\frac{1}{2(k+1)}$$

$$= \frac{3}{4}-\frac{1}{2k+4}+\frac{2-(k+3)}{2(k+1)(k+3)}$$

$$= \frac{3}{4}-\frac{1}{2k+4}+\frac{{}^{-1}\cancel{-(k+1)}}{2\cancel{(k+1)}(k+3)}$$

$$= \frac{3}{4}-\frac{1}{2k+4}-\frac{1}{2k+6}$$

$$= \frac{3}{4}-\frac{1}{2(k+1)+2}-\frac{1}{2(k+1)+4}$$

Thus P_{k+1} is true whenever P_k is true.

Since P_1 is true, P_n is true for all $n\in\mathbb{Z}^+$ {Principle of mathematical induction}

112 **a** $y=\ln(\tan x)$

$\therefore \dfrac{dy}{dx} = \dfrac{\sec^2 x}{\tan x}$

$= \dfrac{1}{\cos^2 x}\times\dfrac{\cos x}{\sin x}$

$= \dfrac{1}{\sin x\cos x}\left(\dfrac{2}{2}\right)$

$= \dfrac{2}{\sin 2x} = 2\csc(2x)$

$\therefore k=2$

b As $\dfrac{d[\ln(\tan x)]}{dx} = 2\csc(2x)$,

$\int \csc(2x)\,dx = \frac{1}{2}\ln(\tan x)+c$

$\therefore$ area $= \displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\csc(2x)\,dx = \left[\tfrac{1}{2}\ln(\tan x)\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$

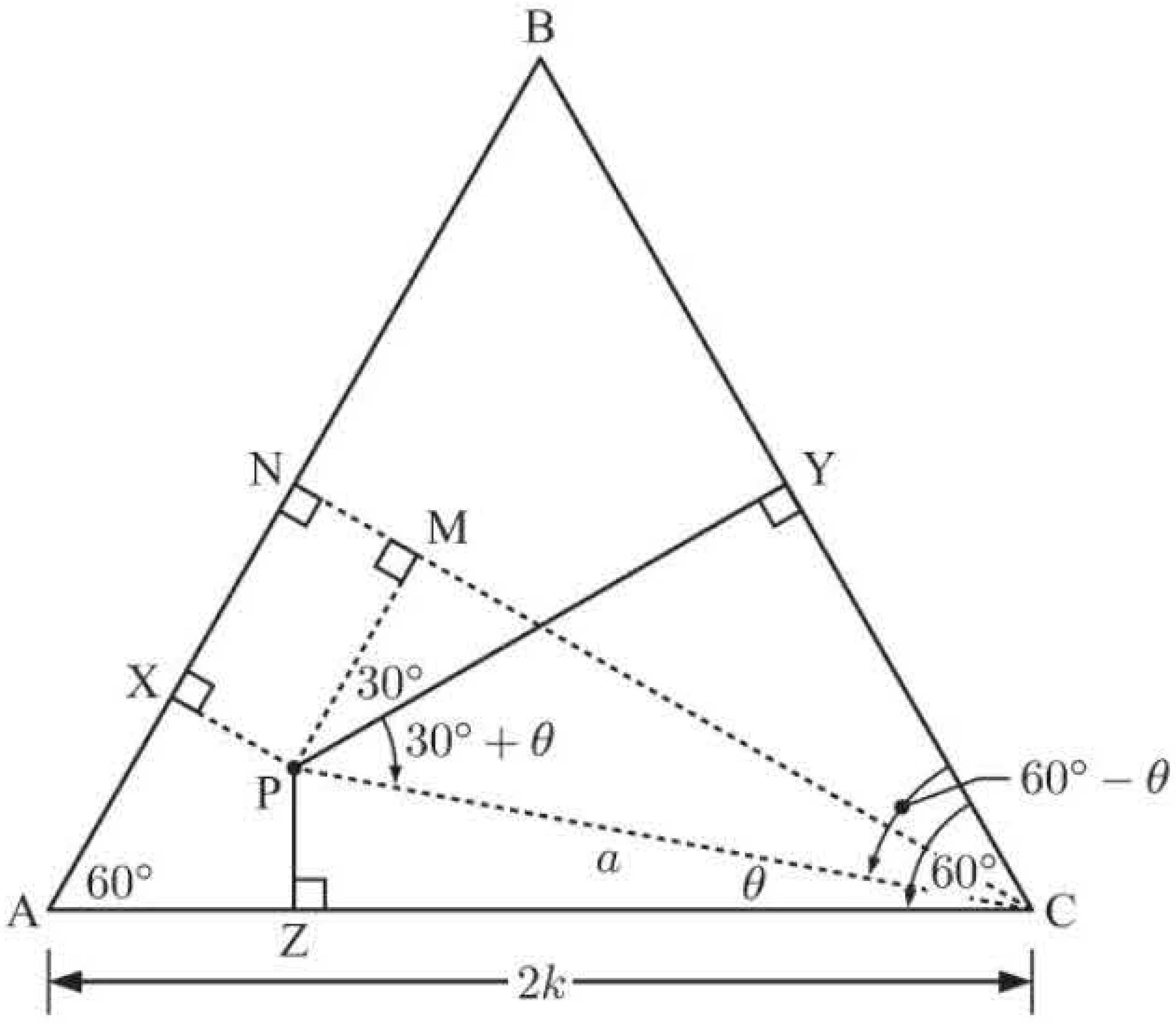
$= \frac{1}{2}\ln(\sqrt{3})-\frac{1}{2}\ln\left(\frac{1}{\sqrt{3}}\right)$

$= \frac{1}{2}\ln(3^{\frac{1}{2}})-\frac{1}{2}\ln(3^{-\frac{1}{2}})$

$= \frac{1}{4}\ln 3+\frac{1}{4}\ln 3$

$= \frac{1}{2}\ln 3$ units^2

113 **a** Let $\text{PC} = a$ units



So, $\sin\theta = \dfrac{\text{PZ}}{a}$

$\therefore \quad \text{PZ} = a\sin\theta \quad .... (1)$

and $\sin(60° - \theta) = \dfrac{\text{PY}}{a}$

$\therefore \quad \text{PY} = a\sin(60° - \theta) \quad .... (2)$

Now $\widehat{\text{MPC}} = 30° + 30° + \theta$
$= 60° + \theta$

so in $\triangle$CMP, $\sin(60° + \theta) = \dfrac{\text{CM}}{a}$

$\therefore \quad \text{CM} = a\sin(60° + \theta) \quad .... (3)$

Also, in $\triangle$ACN, $\sin 60° = \dfrac{\text{CN}}{2k}$

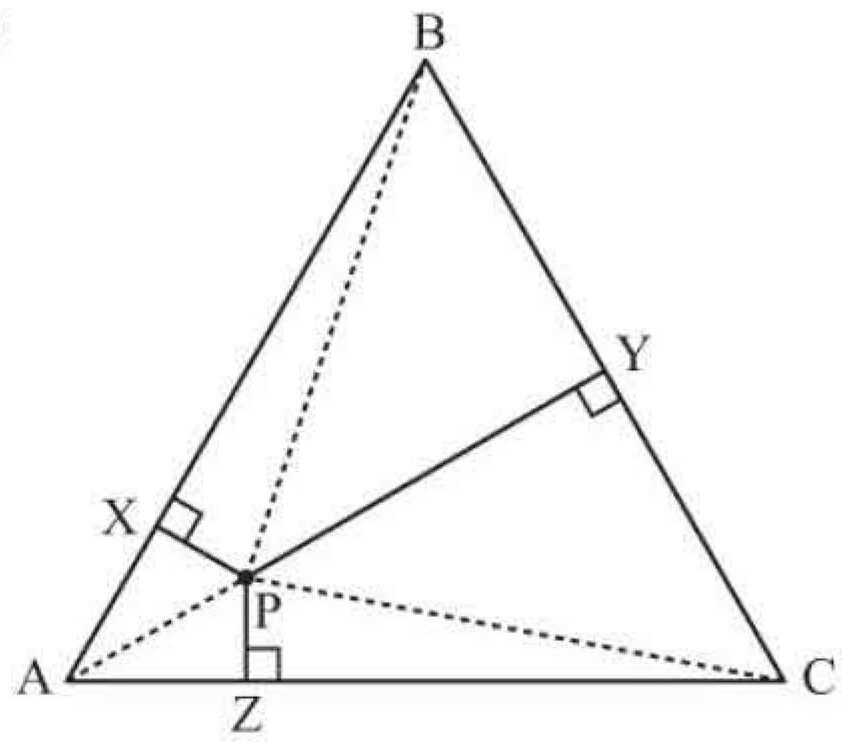
$\therefore \quad \text{CN} = 2k\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}k \quad .... (4)$

$\therefore \quad \text{PX} = \text{NM} = \text{CN} - \text{CM}$
$= k\sqrt{3} - a\sin(60° + \theta) \qquad$ {using (3) and (4)}

Thus $\text{PX} + \text{PY} + \text{PZ} = k\sqrt{3} - a\sin(60° + \theta) + a\sin(60° - \theta) + a\sin\theta \qquad$ {using (1) and (2)}
$= k\sqrt{3} + a[\sin\theta + \sin(60° - \theta) - \sin(60° + \theta)]$
$= k\sqrt{3} + a[\sin\theta - \sin\theta]$
$= k\sqrt{3}$ for all θ, which is a constant.

b If P is at A, $\text{PX} + \text{PY} + \text{PZ} =$ altitude from A to $[\text{BC}] + 0 + 0$
$= \text{AC}\sin 60° = k\sqrt{3}$ ✓

c



Area $\triangle$ABC = area $\triangle$ABP + area $\triangle$BCP + area $\triangle$ACP
$= \frac{1}{2}(2k\text{PX}) + \frac{1}{2}(2k\text{PY}) + \frac{1}{2}(2k\text{PZ})$
$= k(\text{PX} + \text{PY} + \text{PZ}) \quad .... (5)$

But area $\triangle$ABC$= \frac{1}{2}(2k)(\sqrt{3}k) = k^2\sqrt{3} \quad .... (6)$

From (5) and (6), $\text{PX} + \text{PY} + \text{PZ} = k\sqrt{3}$.

114 **a** Consider $1 + a\operatorname{cis}\theta + a^2\operatorname{cis}2\theta + a^3\operatorname{cis}3\theta + ... + a^n\operatorname{cis}n\theta \quad .... (1)$
$= 1 + a[\operatorname{cis}\theta]^1 + a^2[\operatorname{cis}\theta]^2 + a^3[\operatorname{cis}\theta]^3 + ... + a^n[\operatorname{cis}\theta]^n$

which is a geometric series with $u_1 = 1$, $r = a\operatorname{cis}\theta$ and has $n+1$ terms. So, its sum is:

$$1\left[\frac{(a\operatorname{cis}\theta)^{n+1} - 1}{a\operatorname{cis}\theta - 1}\right]$$

$$= \frac{a^{n+1}\operatorname{cis}(n+1)\theta - 1}{a\operatorname{cis}\theta - 1}$$

$$= \frac{a^{n+1}[\cos(n+1)\theta + i\sin(n+1)\theta] - 1}{a\cos\theta + ai\sin\theta - 1}$$

$$= \left[\frac{a^{n+1}\cos(n+1)\theta - 1 + ia^{n+1}\sin(n+1)\theta}{a\cos\theta - 1 + ia\sin\theta}\right]\left[\frac{a\cos\theta - 1 - ia\sin\theta}{a\cos\theta - 1 - ia\sin\theta}\right]$$

$$= \frac{(a^{n+1}\cos(n+1)\theta - 1 + ia^{n+1}\sin(n+1)\theta)(a\cos\theta - 1 - ia\sin\theta)}{(a\cos\theta - 1)^2 + a^2\sin^2\theta} \quad .... (2)$$

Equating the real parts of (1) and (2) gives

$$\begin{aligned}
&1 + a\cos\theta + a^2\cos 2\theta + a^3\cos 3\theta + ... + a^n(\cos n\theta)\\
&= \frac{(a^{n+1}\cos(n+1)\theta - 1)(a\cos\theta - 1) + a^{n+2}\sin(n+1)\theta\sin\theta}{a^2\cos^2\theta - 2a\cos\theta + 1 + a^2\sin^2\theta}\\
&= \frac{a^{n+2}\cos(n+1)\theta\cos\theta - a\cos\theta - a^{n+1}\cos(n+1)\theta + 1 + a^{n+2}\sin(n+1)\theta\sin\theta}{a^2 - 2a\cos\theta + 1}\\
&= \frac{a^{n+2}[\cos(n+1)\theta\cos\theta + \sin(n+1)\theta\sin\theta] - a\cos\theta - a^{n+1}\cos(n+1)\theta + 1}{a^2 - 2a\cos\theta + 1}\\
&= \frac{a^{n+2}\cos n\theta - a\cos\theta - a^{n+1}\cos(n+1)\theta + 1}{a^2 - 2a\cos\theta + 1}\\
&= \frac{a^{n+1}(a\cos n\theta - \cos(n+1)\theta) - a\cos\theta + 1}{a^2 - 2a\cos\theta + 1}
\end{aligned}$$

b Equating the imaginary parts of (1) and (2) gives

$$\begin{aligned}
&a\sin\theta + a^2\sin 2\theta + a^3\sin 3\theta + ... + a^n\sin n\theta\\
&= \frac{(a^{n+1}\sin(n+1)\theta)(a\cos\theta - 1) - a\sin\theta(a^{n+1}\cos(n+1)\theta - 1)}{a^2 - 2a\cos\theta + 1}\\
&= \frac{a^{n+2}\sin(n+1)\theta\cos\theta - a^{n+1}\sin(n+1)\theta - a^{n+2}\cos(n+1)\theta\sin\theta + a\sin\theta}{a^2 - 2a\cos\theta + 1}\\
&= \frac{a^{n+2}(\sin(n+1)\theta\cos\theta - \cos(n+1)\theta\sin\theta) - a^{n+1}\sin(n+1)\theta + a\sin\theta}{a^2 - 2a\cos\theta + 1}\\
&= \frac{a^{n+2}\sin n\theta - a^{n+1}\sin(n+1)\theta + a\sin\theta}{a^2 - 2a\cos\theta + 1}\\
&= \frac{a^{n+1}(a\sin n\theta - \sin(n+1)\theta) + a\sin\theta}{a^2 - 2a\cos\theta + 1}
\end{aligned}$$

115 Let $u = \sqrt{x+2}$, so $u^2 = x + 2$

$$\therefore\ 2u\frac{du}{dx} = 1$$

$$\begin{aligned}
\therefore \int \frac{x}{1+\sqrt{x+2}}\,dx &= \int \left(\frac{u^2-2}{1+u}\right) 2u\,du\\
&= \int \frac{2u^3 - 4u}{u+1}\,du\\
&= \int \left(2u^2 - 2u - 2 + \frac{2}{u+1}\right) du\\
&= \frac{2u^3}{3} - \frac{2u^2}{2} - 2u + 2\ln|u+1| + c\\
&= \tfrac{2}{3}(x+2)^{\frac{3}{2}} - x - 2 - 2\sqrt{x+2} + 2\ln(\sqrt{x+2}+1) + c
\end{aligned}$$

$$\begin{array}{r|r}
 & 2u^2 - 2u - 2\\
\hline
u+1 & 2u^3 + 0u^2 - 4u + 0\\
 & -(2u^3 + 2u^2)\\
\hline
 & -2u^2 - 4u + 0\\
 & -(-2u^2 - 2u)\\
\hline
 & -2u + 0\\
 & -(-2u - 2)\\
\hline
 & 2
\end{array}$$

116 **a** P(stopped at least once)

$= 1 -$ P(never stopped in n intersections)

$= 1 - (1-p)^n$

b
$$\begin{aligned}
\mathrm{P}(A_k \mid B_k) &= \frac{\mathrm{P}(A_k \cap B_k)}{\mathrm{P}(B_k)}\\
&= \frac{\mathrm{P}(A_k)}{\mathrm{P}(B_k)}\\
&= \frac{\binom{n}{k} p^k (1-p)^{n-k}}{\sum\limits_{r=k}^{n} \binom{n}{r} p^r (1-p)^{n-r}}
\end{aligned}$$

c **i** If A_1 and B_1 are independent then

$$P(A_1 \mid B_1) = P(A_1)$$

$$\therefore \quad \frac{\binom{n}{1} p(1-p)^{n-1}}{1-(1-p)^n} = \binom{n}{1} p(1-p)^{n-1}$$

$$\therefore \quad 1-(1-p)^n = 1$$

$$\therefore \quad (1-p)^n = 0$$

$$\therefore \quad p = 1$$

ii If $P(A_2 \mid B_2) = P(A_1)$ and $n = 2$,

$$\frac{\binom{2}{2} p^2}{\binom{2}{2} p^2} = \binom{2}{1} p(1-p)$$

$$\therefore \quad 1 = 2p(1-p)$$

$$\therefore \quad 2p^2 - 2p + 1 = 0$$

where $\Delta = 4 - 4(2)(1) < 0$

$\therefore$ there are no real solutions

117 **a** $\cos 4\theta + i \sin 4\theta = \text{cis}\, 4\theta$

$$= (\text{cis}\,\theta)^4 \quad \{\text{De Moivre}\}$$

$$= (C + iS)^4 \quad \text{where } C = \cos\theta \text{ and } S = \sin\theta$$

$$= C^4 + 4C^3(iS) + 6C^2(iS)^2 + 4C(iS)^3 + (iS)^4$$

$$= (C^4 - 6C^2S^2 + S^4) + i(4C^3S - 4CS^3)$$

Equating real and imaginary parts, $\sin 4\theta = 4C^3S - 4CS^3$ and $\cos 4\theta = C^4 - 6C^2S^2 + S^4$.

Hence, $\tan 4\theta = \dfrac{\sin 4\theta}{\cos 4\theta} = \dfrac{4C^3S - 4CS^3}{C^4 - 6C^2S^2 + S^4}$

$$= \frac{4\left(\frac{S}{C}\right) - 4\left(\frac{S}{C}\right)^3}{1 - 6\left(\frac{S}{C}\right)^2 + \left(\frac{S}{C}\right)^4} \quad \{\text{dividing all terms by } C^4\}$$

$$= \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$$

b $x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$

If we let $x = \tan\theta$ then $\tan^4\theta + 4\tan^3\theta - 6\tan^2\theta - 4\tan\theta + 1 = 0$

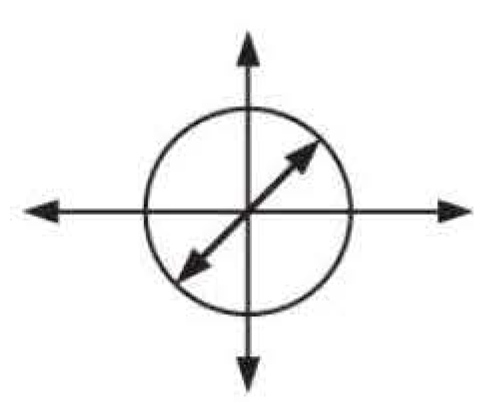
$\therefore \quad 1 - 6\tan^2\theta + \tan^4\theta = 4\tan\theta - 4\tan^3\theta$

$$\therefore \quad \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta} = 1$$

$$\therefore \quad \tan 4\theta = 1 \quad \{\text{by } \mathbf{a}\}$$

$$\therefore \quad 4\theta = \tfrac{\pi}{4} + k\pi, \quad k \in \mathbb{Z}$$

$$\therefore \quad \theta = \tfrac{\pi}{16} + \tfrac{k\pi}{4}$$



Thus $x = \tan\left(\frac{\pi}{16}\right), \ \tan\left(\frac{5\pi}{16}\right), \ \tan\left(\frac{9\pi}{16}\right), \ \tan\left(\frac{13\pi}{16}\right)$

118 **a** The system has augmented matrix

$$\left(\begin{array}{ccc|c} 2 & -1 & 3 & 4 \\ 2 & 1 & a+3 & 10-a \\ 4 & 6 & a^2+6 & a^2 \end{array}\right) = \left(\begin{array}{ccc|c} 2 & -1 & 3 & 4 \\ 0 & 2 & a & 6-a \\ 0 & 8 & a^2 & a^2-8 \end{array}\right) \begin{array}{l} \\ R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{array}$$

$$= \left(\begin{array}{ccc|c} 2 & -1 & 3 & 4 \\ 0 & 2 & a & 6-a \\ 0 & 0 & a^2-4a & a^2+4a-32 \end{array}\right) \begin{array}{l} \\ \\ R_3 \to R_3 - 4R_2 \end{array}$$

b The last equation is $a(a-4)z = (a+8)(a-4)$

i The system has no solutions if the LHS $= 0$ and the RHS $\neq .0$.
This occurs when $a = 0$, since we get $0z = -32$.
The line of intersection of any two planes is parallel to the third.

ii The system has infinitely many solutions if the last equation has the form $0z = 0$, which is true for any real number z. This occurs when $a = 4$.

The augmented matrix becomes $\left(\begin{array}{ccc|c} 2 & -1 & 3 & 4 \\ 0 & 2 & 4 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$$\text{Letting } z = t, \ 2y + 4t = 2$$
$$\therefore \ y + 2t = 1$$
$$\therefore \ y = 1 - 2t$$
$$\text{and so } 2x - (1 - 2t) + 3t = 4$$
$$\therefore \ 2x - 1 + 2t + 3t = 4$$
$$\therefore \ x = \frac{5 - 5t}{2}$$

$\therefore$ when $a = 4$, there are infinitely many solutions of the form

$$x = \frac{5 - 5t}{2}, \ y = 1 - 2t, \ z = t, \ t \in \mathbb{R}.$$

The planes meet in a common line of intersection.

iii The system has a unique solution for all other values of a. So, $a \neq 0$ or 4.

In this case $z = \dfrac{a + 8}{a}$

$$\therefore \ 2y + a\left(\frac{a+8}{a}\right) = 6 - a \qquad \text{and} \qquad 2x + a + 1 + 3\left(\frac{a+8}{a}\right) = 4$$
$$\therefore \ 2y = 6 - a - a - 8 \qquad\qquad \therefore \ 2x = 4 - a - 1 - 3 - \frac{24}{a}$$
$$= -2a - 2 \qquad\qquad = -a - \frac{24}{a}$$
$$\therefore \ y = -a - 1 \qquad\qquad \therefore \ x = -\frac{a}{2} - \frac{12}{a}$$

$\therefore$ when $a \neq 0$ or 4, there is a unique solution of the form

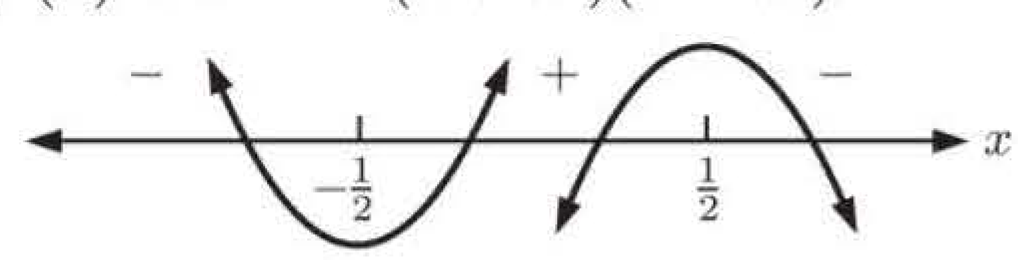
$$x = -\frac{a}{2} - \frac{12}{a}, \quad y = -a - 1, \quad z = \frac{a+8}{a}$$

When $a = 2$, the solution is $x = -7$, $y = -3$, $z = 5$.
This means that the 3 planes meet at one point only.

119 $f(x) = xe^{1-2x^2}$

a
$$f'(x) = 1e^{1-2x^2} + xe^{1-2x^2}(-4x)$$
$$= e^{1-2x^2}(1 - 4x^2)$$
$$f''(x) = -4xe^{1-2x^2}(1 - 4x^2) + e^{1-2x^2}(-8x)$$
$$= e^{1-2x^2}(-4x + 16x^3 - 8x)$$
$$= e^{1-2x^2}(16x^3 - 12x)$$

b $f'(x) = e^{1-2x^2}(1 + 2x)(1 - 2x)$



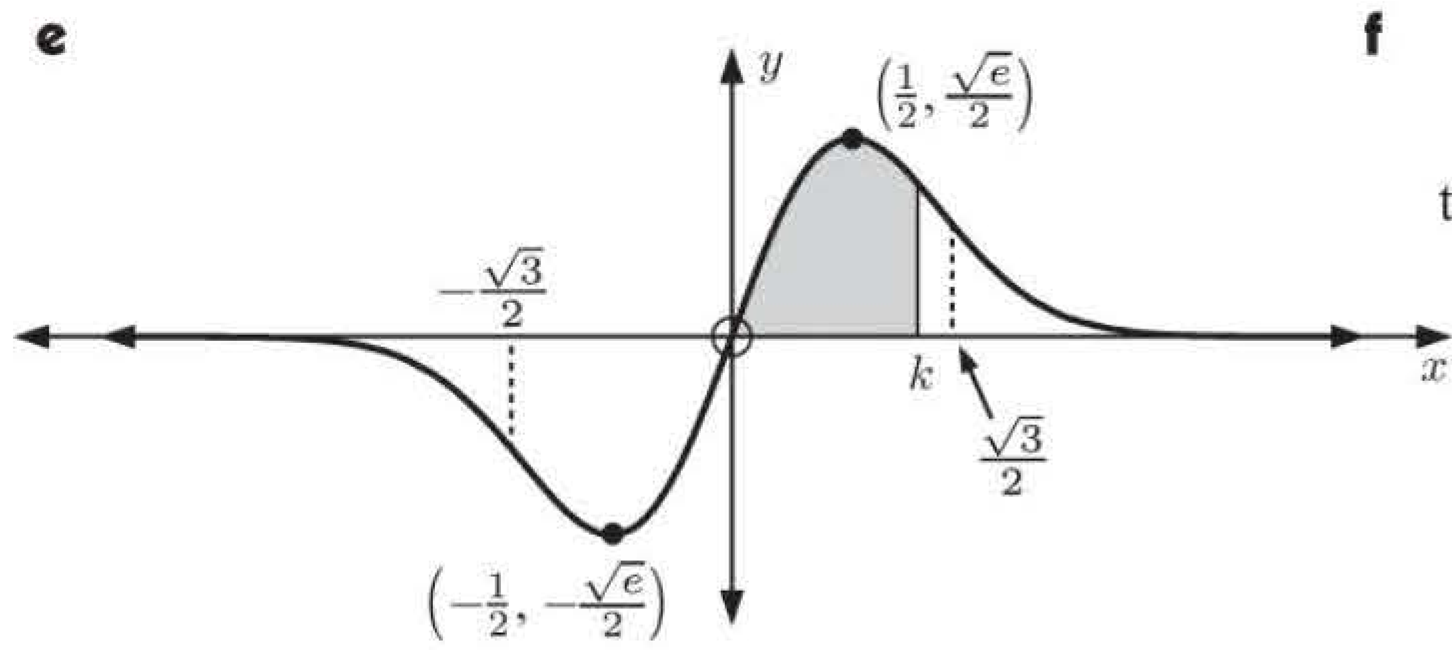
$\therefore$ there is a local minimum at $(-\frac{1}{2}, -\frac{\sqrt{e}}{2})$

and a local maximum at $(\frac{1}{2}, \frac{\sqrt{e}}{2})$.

c $f''(x) = 0$ when $16x^3 - 12x = 0$
$$\therefore \ 4x(4x^2 - 3) = 0$$
$$\therefore \ x = 0 \text{ or } \pm\tfrac{\sqrt{3}}{2}$$

d As $x \to \infty$, $f(x) \to 0$ (above).
As $x \to -\infty$, $f(x) \to 0$ (below).

e



f
$$\text{If } \int_0^k xe^{1-2x^2}\,dx = \frac{e-1}{4}$$
$$\text{then } \tfrac{1}{-4}\int_0^k e^{1-2x^2}(-4x)\,dx = \frac{e-1}{4}$$
$$\therefore \ \left[e^{1-2x^2}\right]_0^k = 1 - e$$
$$\therefore \ e^{1-2k^2} - e = 1 - e$$
$$\therefore \ e^{1-2k^2} = 1$$
$$\therefore \ 1 - 2k^2 = 0$$
$$\therefore \ k = \pm\tfrac{1}{\sqrt{2}}$$

But $k > 0$, so $k = \frac{1}{\sqrt{2}}$

120 **a** $\operatorname{cis}\theta + \operatorname{cis}\phi = \cos\theta + i\sin\theta + \cos\phi + i\sin\phi$

$$= [\cos\theta + \cos\phi] + i\,[\sin\theta + \sin\phi]$$

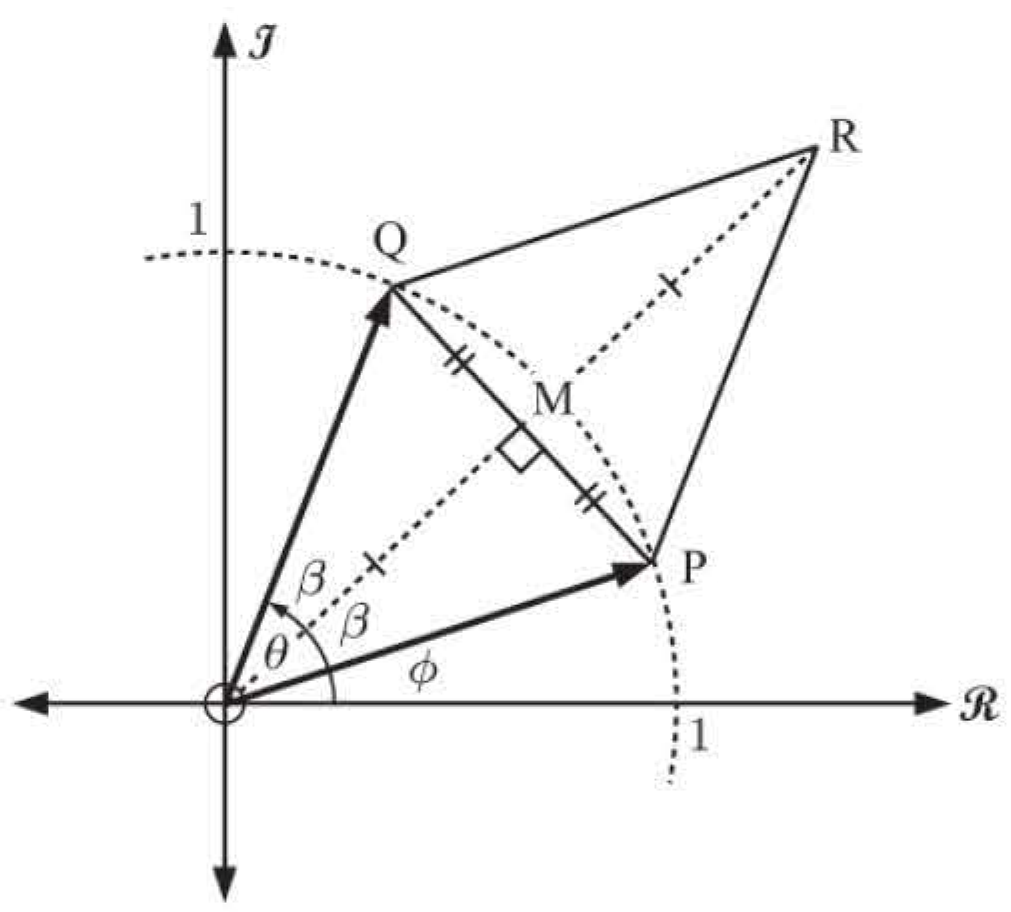
$$= 2\cos\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right) + i2\sin\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right)$$

$$= 2\cos\left(\frac{\theta-\phi}{2}\right)\left[\cos\left(\frac{\theta+\phi}{2}\right) + i\sin\left(\frac{\theta+\phi}{2}\right)\right]$$

$$= 2\cos\left(\frac{\theta-\phi}{2}\right)\operatorname{cis}\left(\frac{\theta+\phi}{2}\right)$$

b $|\operatorname{cis}\theta + \operatorname{cis}\phi| = 2\left|\cos\left(\frac{\theta-\phi}{2}\right)\right|$ and $\arg(\operatorname{cis}\theta + \operatorname{cis}\phi) = \frac{\theta+\phi}{2}$

c



Let $\overrightarrow{\mathrm{OP}} = \operatorname{cis}\phi$ and $\overrightarrow{\mathrm{OQ}} = \operatorname{cis}\theta,\ \theta > \phi$

We complete rhombus OPRQ, where the diagonals bisect each other at right angles.

$\overrightarrow{\mathrm{OR}} = \overrightarrow{\mathrm{OP}} + \overrightarrow{\mathrm{PR}} = \overrightarrow{\mathrm{OP}} + \overrightarrow{\mathrm{OQ}} = \operatorname{cis}\phi + \operatorname{cis}\theta$

Since $\theta = \phi + 2\beta,\ \beta = \dfrac{\theta-\phi}{2}$

Now $\arg\overrightarrow{\mathrm{OR}} = \phi + \beta = \dfrac{\theta+\phi}{2}$

and $|\overrightarrow{\mathrm{OR}}| = 2(\mathrm{OM})$ where $\cos\beta = \dfrac{\mathrm{OM}}{\mathrm{OP}} = \mathrm{OM}$

$\therefore\ |\overrightarrow{\mathrm{OR}}| = 2\cos\left(\dfrac{\theta-\phi}{2}\right)$

Note: If θ and ϕ are interchanged, $|\overrightarrow{\mathrm{OR}}| = 2\cos\left(\dfrac{\phi-\theta}{2}\right)$.

Thus $|\overrightarrow{\mathrm{OR}}|$ is actually $2\left|\cos\left(\dfrac{\theta-\phi}{2}\right)\right|$.

d $\left(\dfrac{z+1}{z-1}\right)^5 = 1 \quad \therefore\ \dfrac{z+1}{z-1} = 1,\ \alpha,\ \alpha^2,\ \alpha^3$, and α^4 where $\alpha = \operatorname{cis}\left(\frac{2\pi}{5}\right)$

$$\therefore\ \frac{z+1}{z-1} = \alpha^k \text{ where } k = 0, 1, 2, 3, 4 \text{ and } \alpha = \operatorname{cis}\left(\tfrac{2\pi}{5}\right)$$

$$\therefore\ z = \frac{1+\alpha^k}{\alpha^k - 1} \quad \{\text{making } z \text{ the subject}\}$$

$$\therefore\ z = \frac{\operatorname{cis}0 + \left[\operatorname{cis}\left(\frac{2\pi}{5}\right)\right]^k}{\left[\operatorname{cis}\left(\frac{2\pi}{5}\right)\right]^k + \operatorname{cis}\pi} \qquad \{\text{as } \operatorname{cis}\pi = -1\}$$

$$\therefore\ z = \frac{\operatorname{cis}\left(\frac{2\pi k}{5}\right) + \operatorname{cis}0}{\operatorname{cis}\left(\frac{2\pi k}{5}\right) + \operatorname{cis}\pi}$$

$$\therefore\ z = \frac{\cancel{2}\cos\left(\frac{\pi k}{5}\right)\operatorname{cis}\left(\frac{\pi k}{5}\right)}{\cancel{2}\cos\left(\frac{\left(\frac{2\pi k}{5}\right)-\pi}{2}\right)\operatorname{cis}\left(\frac{\left(\frac{2\pi k}{5}\right)+\pi}{2}\right)}$$

$$\therefore\ z = \frac{\cos\left(\frac{\pi k}{5}\right)}{\cos\left(\frac{\pi k}{5} - \frac{\pi}{2}\right)}\operatorname{cis}\left(\tfrac{\pi k}{5} - \tfrac{\pi k}{5} - \tfrac{\pi}{2}\right)$$

$$\therefore\ z = \frac{\cos\left(\frac{\pi k}{5}\right)}{\sin\left(\frac{\pi k}{5}\right)}\operatorname{cis}\left(-\tfrac{\pi}{2}\right) \qquad \{\cos(\theta - \tfrac{\pi}{2}) = \sin\theta\}$$

$$\therefore\ z = \cot\left(\tfrac{\pi k}{5}\right) \times -i$$

$$\therefore\ z = -i\cot\left(\tfrac{\pi k}{5}\right) \quad \text{for } k = 0, 1, 2, 3, 4$$

However $\cot 0$ is undefined, so we reject the solution when $k = 0$

$\therefore\ z = -i\cot\left(\frac{k\pi}{5}\right)$ where $k = 1, 2, 3, 4$.

121 If $u = \ln x$, $\dfrac{du}{dx} = \dfrac{1}{x}$ and $\therefore$ $\int \sin(\ln x)\,dx = \int \sin u(x\,du)$

$$= \int \sin u e^u\,du$$
$$= \int e^u \sin u\,du \quad .... (*)$$

Now consider $\int e^x \sin x\,dx$ using integration by parts

$$\therefore \int e^x \sin x\,dx = e^x \sin x - \int e^x \cos x\,dx$$
$$= e^x \sin x - \left[e^x \cos x - \int e^x(-\sin x)\,dx\right]$$

Thus $\int e^x \sin x\,dx = e^x(\sin x - \cos x) - \int e^x \sin x\,dx$

$$\therefore 2\int e^x \sin x\,dx = e^x(\sin x - \cos x)$$
$$\therefore \int e^x \sin x\,dx = \frac{e^x}{2}(\sin x - \cos x)$$

Hence, in $(*)$, $\int \sin(\ln x)\,dx = \dfrac{e^u}{2}(\sin u - \cos u)$

$$= \frac{x}{2}(\sin(\ln x) - \cos(\ln x))$$

122 **a** Since $P(x)$ is a real polynomial, both $1 + ki$ and $1 - ki$ are zeros, $k \in \mathbb{Z}$.
These have sum 2 and product $1 + k^2$ and therefore come from the quadratic factor $x^2 - 2x + 1 + k^2$.

b By comparison with $P(x)$, $1 + k^2$ is a factor of -10.

$\therefore$ $1 + k^2 = \pm1, \pm2, \pm5, \pm10$ where $k \in \mathbb{Z}$

$\therefore$ $k^2 = 0, 1, 4, 9$

$\therefore$ $k = 0, \pm1, \pm2, \pm3$

c As p and q are integer zeros, they come from
$(x - p)(x - q) = x^2 - (p + q)x + pq$
$\therefore$ pq is a factor of -10
The possibilities are as shown in the table:

$1 + k^2$	pq
1	-10
2	-5
5	-2
10	-1

d Without loss of generality we assume $p > q$. Then as $p + q = -1$, the only possibility is $p = 1$, $q = -2$, and $1 + k^2 = 5$ $\therefore$ $k = \pm2$

$\therefore$ $P(x) = (x - 1)(x + 2)(x - 1 - 2i)(x - 1 + 2i)$

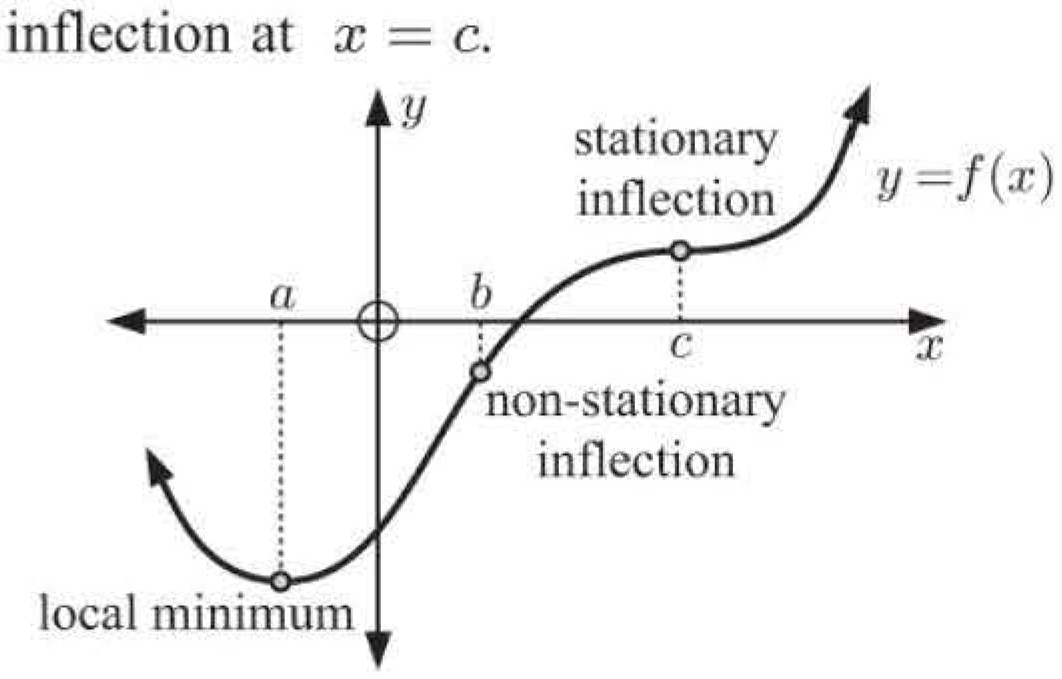
So, the zeros of $P(x)$ are 1, -2, $1 + 2i$, and $1 - 2i$.

123 **a** $f'(x)$ near a has sign diagram

$f'(x)$ near b has sign diagram , but $f'(b) \neq 0$

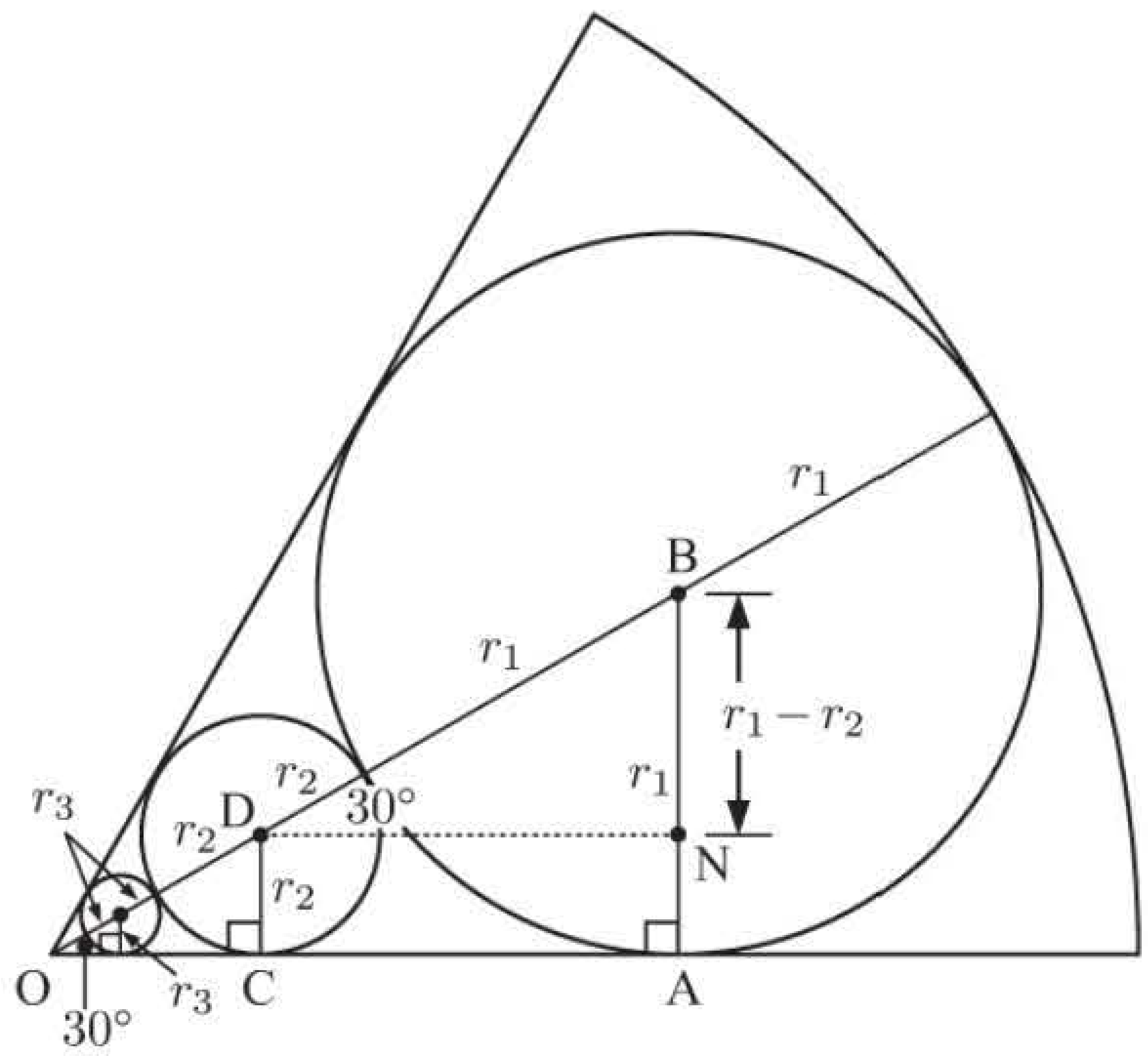
$f'(x)$ near c has sign diagram , but $f'(c) = 0$

$\therefore$ there is a local minimum at $x = a$, a non-stationary inflection at $x = b$, and a stationary inflection at $x = c$.



b $f(x) = f_1(x) + k$, k a constant

124 **a**



$$\sin 30° = \frac{r_1}{10 - r_1} \quad \{\text{in } \triangle\text{OAB}\}$$

$$\therefore \ \tfrac{1}{2} = \frac{r_1}{10 - r_1} \quad \therefore \ r_1 = \tfrac{10}{3}$$

In $\triangle$DBN, $\sin 30° = \dfrac{r_1 - r_2}{r_1 + r_2} = \tfrac{1}{2}$

$$\therefore \ 2r_1 - 2r_2 = r_1 + r_2$$

$$\therefore \ r_1 = 3r_2$$

$$\therefore \ r_2 = \tfrac{1}{3} r_1$$

So, in successive circles, radii are reduced by a factor of 3.

$\therefore \ r_2 = \frac{10}{9}, \ r_3 = \frac{10}{27}, \ r_4 = \frac{10}{81}$, and so on.

Thus the total area of the circles $= \pi {r_1}^2 + \pi {r_2}^2 + \pi {r_3}^2 + \pi {r_4}^2 + ...$

$$= \pi \left(\left(\tfrac{10}{3}\right)^2 + \left(\tfrac{10}{9}\right)^2 + \left(\tfrac{10}{27}\right)^2 + \left(\tfrac{10}{81}\right)^2 + ... \right)$$

$$= \pi \times \left(\tfrac{10}{3}\right)^2 \left[1 + \left(\tfrac{1}{3}\right)^2 + \left(\tfrac{1}{3}\right)^4 + \left(\tfrac{1}{3}\right)^6 + ... \right]$$

$$= \pi \times \tfrac{100}{9} \times \left(\frac{1}{1 - \frac{1}{9}} \right)$$

$$= \tfrac{25\pi}{2} \text{ units}^2$$

b We let $\frac{\alpha}{2}$ replace $30°$ in the calculations of **a**, and let $a = \sin\left(\frac{\alpha}{2}\right)$.

In this case $a = \dfrac{r_1}{10 - r_1}$, and so $r_1 = \dfrac{10a}{1 + a}$

Now $a = \dfrac{r_1 - r_2}{r_1 + r_2}$ $\{$using $\triangle$DBN$\}$

$$\therefore \ a(r_1 + r_2) = r_1 - r_2$$

$$\therefore \ r_2(a + 1) = r_1(1 - a)$$

$$\therefore \ r_2 = r_1 \left(\frac{1 - a}{a + 1} \right) = \frac{10a}{1 + a} \left(\frac{1 - a}{a + 1} \right) = \frac{10a(1 - a)}{(a + 1)^2}$$

Thus $r_3 = r_2 \left(\dfrac{1 - a}{a + 1} \right) = \dfrac{10a(1 - a)^2}{(a + 1)^3}$, and so on.

$$\therefore \ \text{total area} = \pi({r_1}^2 + {r_2}^2 + {r_3}^2 + {r_4}^2 + ...)$$

$$= \pi \left[\left(\frac{10a}{1 + a} \right)^2 + \left(\frac{10a(1 - a)}{(1 + a)^2} \right)^2 + \left(\frac{10a(1 - a)^2}{(1 + a)^3} \right)^2 + ... \right]$$

$$= \pi \frac{100a^2}{(1 + a)^2} \left[1 + \left(\frac{1 - a}{1 + a} \right)^2 + \left(\frac{1 - a}{1 + a} \right)^4 + ... \right]$$

This is the sum of an infinite geometric series which converges since $\left| \dfrac{1 - \sin\left(\frac{\alpha}{2}\right)}{1 + \sin\left(\frac{\alpha}{2}\right)} \right| < 1$.

$$\therefore \ \text{total area} = \pi \times \frac{100a^2}{(1 + a)^2} \times \frac{1}{1 - \left(\frac{1 - a}{1 + a} \right)^2}$$

$$= \pi \times \frac{100a^2}{(1 + a)^2 - (1 - a)^2}$$

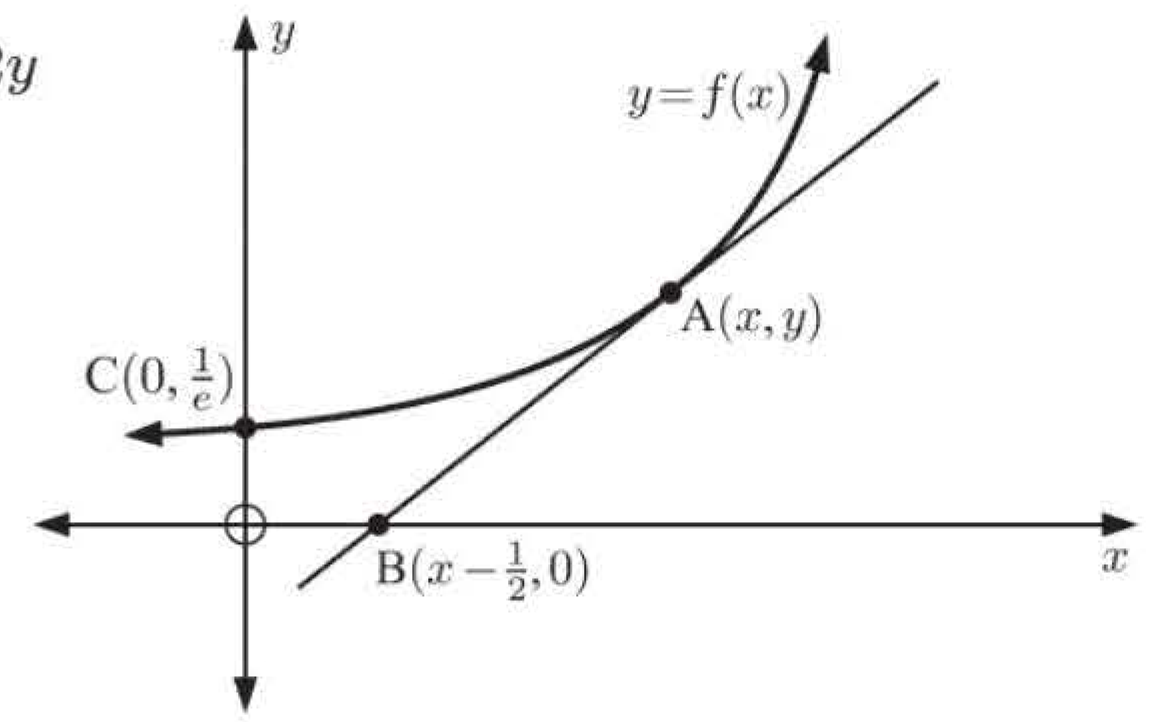
$$= \pi \times \frac{100a^2}{4a}$$

$$= 25\pi a$$

$$= 25\pi \sin\left(\tfrac{\alpha}{2}\right) \text{ units}^2$$

125 **a** At any point $A(x, y)$, $\dfrac{dy}{dx} = \dfrac{y-0}{x - \left(x - \frac{1}{2}\right)} = 2y$

$$\therefore \ \frac{dy}{dx} = \frac{1}{2y}$$



b

$$\therefore \ x = \int \frac{1}{2y}\, dy$$

$$\therefore \ x = \tfrac{1}{2}\ln|y| + c$$

But when $x = 0$, $y = e^{-1}$

$$\therefore \ 0 = \tfrac{1}{2}(-1) + c$$

$$\therefore \ c = \tfrac{1}{2}$$

$$\therefore \ x = \tfrac{1}{2}\ln|y| + \tfrac{1}{2}$$

$$\therefore \ 2x = \ln|y| + 1$$

$$\therefore \ \ln|y| = 2x - 1$$

$$|y| = e^{2x-1}$$

$$\therefore \ y = \pm e^{2x-1}$$

$$\therefore \ y = e^{2x-1} \qquad (\text{If } y = -e^{2x-1} \text{ and } x = 0,\ y = -\frac{1}{e} \neq \frac{1}{e})$$

126 **a** **i**

$$\begin{aligned}
&\sin 2A + \sin 2B + \sin 2C \\
&= \sin 2A + \sin 2B + \sin(2\pi - 2A - 2B) && \{\text{as } A + B + C = \pi\} \\
&= \sin 2A + \sin 2B - \sin(2A + 2B) && \{\sin(2\pi - \theta) = -\sin\theta\} \\
&= \sin 2A + \sin 2B - [\sin 2A \cos 2B + \cos 2A \sin 2B] \\
&= \sin 2A(1 - \cos 2B) + \sin 2B(1 - \cos 2A) \\
&= 2\sin A\cos A(2\sin^2 B) + 2\sin B\cos B\left(2\sin^2 A\right) && \{\cos 2\theta = 1 - 2\sin^2\theta\} \\
&= 4\sin A\cos A\sin^2 B + 4\sin^2 A\sin B\cos B \\
&= 4\sin A\sin B\,[\sin B\cos A + \cos B\sin A] \\
&= 4\sin A\sin B\sin(A + B) \\
&= 4\sin A\sin B\sin(\pi - C) \\
&= 4\sin A\sin B\sin C && \{\sin(\pi - \theta) = \sin\theta\}
\end{aligned}$$

ii

$$\begin{aligned}
&\tan A + \tan B + \tan C \\
&= \tan A + \tan B + \tan(\pi - (A + B)) \\
&= \tan A + \tan B - \tan(A + B) && \{\tan(\pi - \theta) = -\tan\theta\} \\
&= \frac{\tan A + \tan B}{1} - \frac{\tan A + \tan B}{1 - \tan A\tan B} \\
&= \frac{(\tan A + \tan B)(1 - \tan A\tan B) - (\tan A + \tan B)}{1 - \tan A\tan B} \\
&= \frac{\cancel{\tan A} + \cancel{\tan B} - \tan^2 A\tan B - \tan A\tan^2 B - \cancel{\tan A} - \cancel{\tan B}}{1 - \tan A\tan B} \\
&= -\tan A\tan B\,\frac{(\tan A + \tan B)}{1 - \tan A\tan B} \\
&= -\tan A\tan B\tan(A + B) \\
&= \tan A\tan B\tan(\pi - (A + B)) && \{\tan(\pi - \theta) = -\tan\theta\} \\
&= \tan A\tan B\tan C
\end{aligned}$$

b $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

$\therefore\ \tan A + \tan B = \tan C(\tan A \tan B - 1)$ .... (1)

Suppose $\tan A \tan B = 1$ .... (2)

Then $\tan A + \tan B + \tan C = \tan C$

$\therefore\ \tan A + \tan B = 0$

$\therefore\ \tan A = -\tan B$

$\therefore\ \tan A = -\dfrac{1}{\tan A}$ {using (2)}

$\therefore\ \tan^2 A = -1$ which is impossible

$\therefore\ \tan A \tan B \neq 1$, and the supposition (2) is false

So, using (1), $\dfrac{\tan A + \tan B}{1 - \tan A \tan B} = -\tan C$

$\therefore\ \tan(A+B) = \tan(-C)$

$\therefore\ A + B = -C + k\pi,\ k \in \mathbb{Z}$ {equal $\tan$s are π apart}

$\therefore\ A + B + C = k\pi,\quad k \in \mathbb{Z}$

127 **a** Suppose $\sqrt{14 - 4\sqrt{6}} = a + b\sqrt{6}$ where $a, b \in \mathbb{Z}$ .... $(*)$

$\therefore\ a^2 + 2ab\sqrt{6} + 6b^2 = 14 - 4\sqrt{6}$ {squaring both sides}

$\therefore\ a^2 + 6b^2 = 14$ and $ab = -2$

$\therefore\ a^2 + 6\left(\dfrac{-2}{a}\right)^2 = 14$

$\therefore\ a^2 + \dfrac{24}{a^2} - 14 = 0$

$\therefore\ a^4 - 14a^2 + 24 = 0$

$\therefore\ (a^2 - 2)(a^2 - 12) = 0$

$\therefore\ a^2 = 2$ or 12

$\therefore\ a = \pm\sqrt{2}$ or $\pm 2\sqrt{3}$ which is a contradiction to $(*)$

Hence the supposition is false, and so $\sqrt{14 - 4\sqrt{6}}$ cannot be written in the form $a + b\sqrt{6}$ with $a, b \in \mathbb{Z}$.

b As $\sqrt{6} = \sqrt{2}\sqrt{3}$ we try $\sqrt{14 - 4\sqrt{6}} = a\sqrt{3} + b\sqrt{2}$ where $a, b \in \mathbb{Z}$.

$\therefore\ 3a^2 + 2b^2 + 2ab\sqrt{6} = 14 - 4\sqrt{6}$

$\therefore\ 3a^2 + 2b^2 = 14$ and $ab = -2$

$\therefore\ 3a^2 + 2\left(\dfrac{-2}{a}\right)^2 - 14 = 0$

$\therefore\ 3a^2 + \dfrac{8}{a^2} - 14 = 0$

$\therefore\ 3a^4 - 14a^2 + 8 = 0$

$\therefore\ (a^2 - 4)(3a^2 - 2) = 0$

$\therefore\ a^2 = 4$ or $\frac{2}{3}$

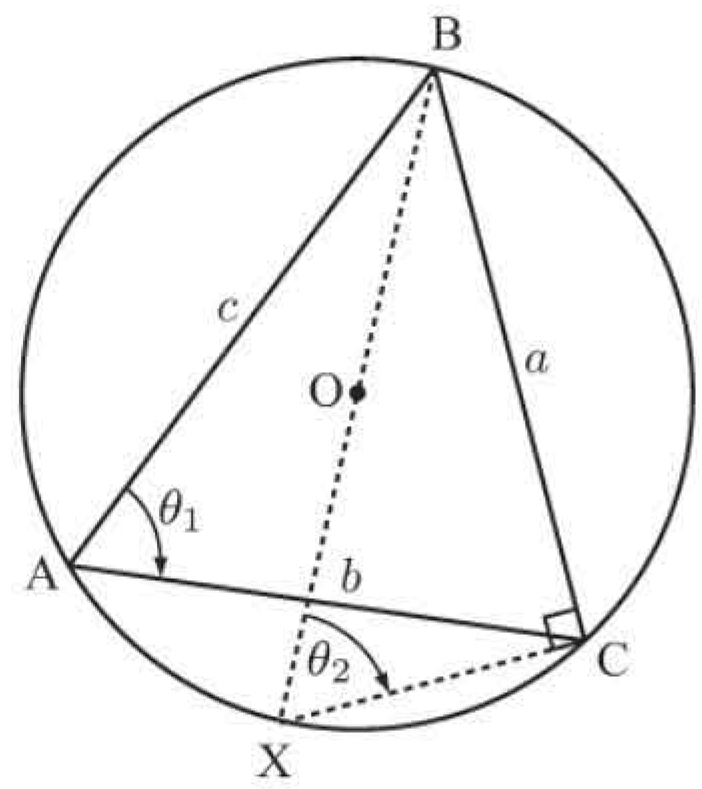
Since $a \in \mathbb{Z}$, $a = \pm 2$ and so $b = -\dfrac{2}{a} = \mp 1$

$\therefore\ \sqrt{14 - 4\sqrt{6}} = 2\sqrt{3} - \sqrt{2}$ or $-2\sqrt{3} + \sqrt{2}$

We reject the second one as it is negative.

So, $\sqrt{14 - 4\sqrt{6}} = 2\sqrt{3} - \sqrt{2}$.

128 **a**



Draw diameter BOX and join [CX].

Now $\theta_1 = \theta_2$ {angles in the same segment theorem}

and $\widehat{BCX} = 90^\circ$ {angle in a semi-circle theorem}

Now the area is $\frac{1}{2}bc\sin\theta_1 = \frac{1}{2}bc\sin\theta_2$

$$= \tfrac{1}{2}bc\,\frac{a}{\text{BX}}$$
$$= \tfrac{1}{2}bc \times \frac{a}{2r}$$
$$= \frac{abc}{4r}$$

b

$$\sin A = \cos B + \cos C$$
$$\therefore \sin(\pi - [B+C]) = \cos B + \cos C$$
$$\therefore \sin[B+C] = \cos B + \cos C$$
$$\therefore 2\sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B+C}{2}\right) = 2\cos\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)$$
$$\therefore 2\cos\left(\frac{B+C}{2}\right)\left[\sin\left(\frac{B+C}{2}\right) - \cos\left(\frac{B-C}{2}\right)\right] = 0$$
$$\therefore \cos\left(\frac{B+C}{2}\right) = 0 \quad .... (1) \qquad \text{or} \qquad \sin\left(\frac{B+C}{2}\right) = \cos\left(\frac{B-C}{2}\right) \quad .... (2)$$

In (1), $\dfrac{B+C}{2} = \dfrac{\pi}{2} + k\pi$

$\therefore B + C = \pi + k2\pi,\ k \in \mathbb{Z}$

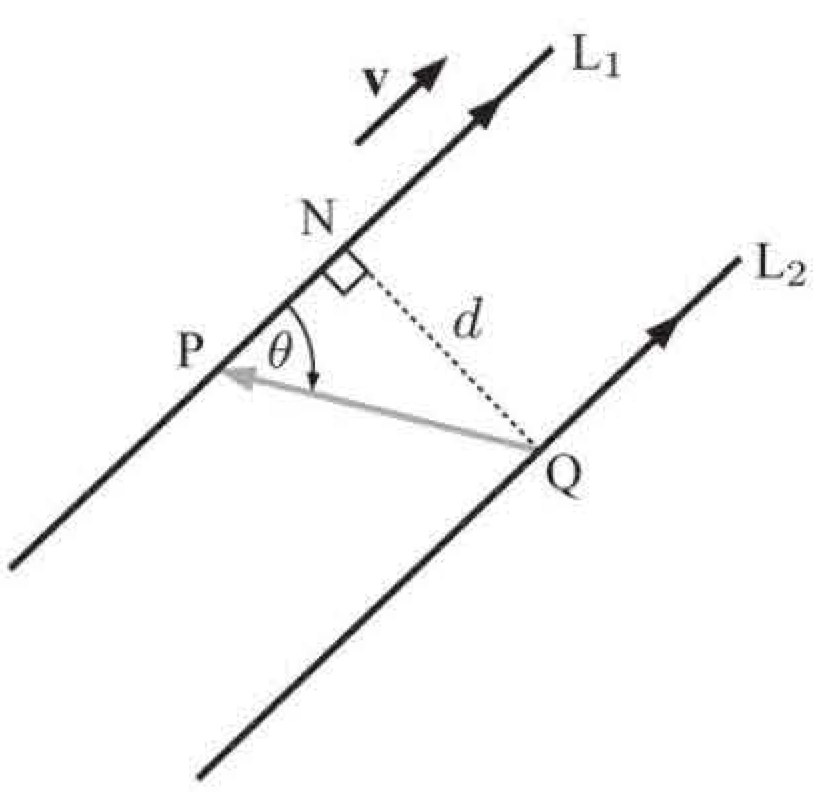
$\therefore B + C = \pi$, 3π, $-\pi$, and so on, all of which are impossible.

In (2), $\sin\left(\dfrac{B+C}{2}\right) = \sin\left(\frac{\pi}{2} - \dfrac{B-C}{2}\right)$ $\quad \{\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)\}$

$$\therefore \frac{B+C}{2} = \frac{\pi - B + C}{2} \quad \text{or} \quad \frac{B+C}{2} = \pi - \left(\frac{\pi - B + C}{2}\right)$$
$$\therefore B + \cancel{C} = \pi - B + \cancel{C} \quad \text{or} \quad \cancel{B} + C = 2\pi - \pi + \cancel{B} - C$$
$$\therefore B = \tfrac{\pi}{2} \quad \text{or} \quad C = \tfrac{\pi}{2}$$

$\therefore$ the triangle is right angled at B or C.

129



Let θ be the angle between L_1 and $\overrightarrow{QP}$.

Now in $\triangle$PQN, $\sin\theta = \dfrac{d}{|\overrightarrow{QP}|}$

$$\therefore d = |\overrightarrow{QP}|\sin\theta$$
$$= \frac{|\overrightarrow{QP}|\,|\mathbf{v}|\sin\theta}{|\mathbf{v}|}$$
$$= \frac{|\overrightarrow{QP} \times \mathbf{v}|}{|\mathbf{v}|}$$

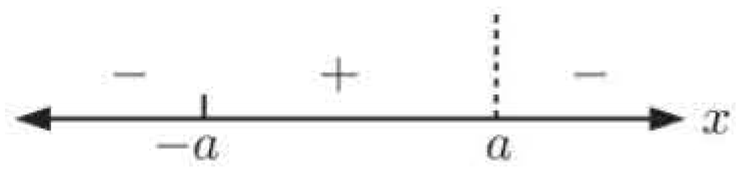
130 **a** $\dfrac{P}{a-x} + \dfrac{Q}{a+x} = \dfrac{P(a+x) + Q(a-x)}{(a-x)(a+x)} = \dfrac{[P-Q]x + [P+Q]a}{a^2 - x^2}$

So, if $\dfrac{1}{a^2 - x^2} = \dfrac{P}{a-x} + \dfrac{Q}{a+x}$ for all x, then $P - Q = 0$ and $[P+Q]a = 1$

Thus $P = Q$ and so $P = Q = \dfrac{1}{2a}$.

b $$\int \frac{1}{a^2-x^2}\,dx = \int \left(\frac{\frac{1}{2a}}{a-x} + \frac{\frac{1}{2a}}{a+x} \right) dx = \frac{1}{2a} \int \left(\frac{1}{a-x} + \frac{1}{a+x} \right) dx$$
$$= \frac{1}{2a}\left(-\ln|a-x| + \ln|a+x|\right) + c$$
$$= \frac{1}{2a} \ln\left|\frac{a+x}{a-x}\right| + c, \quad \text{provided } x \neq a \text{ or } -a$$

c $\dfrac{a+x}{a-x}$ has sign diagram



So, $$\frac{d}{dx}\left[\frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + c\right] = \begin{cases} \dfrac{d}{dx}\left[\dfrac{1}{2a}\ln\left(\dfrac{a+x}{a-x}\right)\right] & \text{if } -a < x < a \\ \dfrac{d}{dx}\left[\dfrac{1}{2a}\ln\left(-\dfrac{a+x}{a-x}\right)\right] & \text{if } x < -a \text{ or } x > a \end{cases}$$

$$= \begin{cases} \dfrac{1}{2a}\dfrac{d}{dx}\left[\ln(a+x) - \ln(a-x)\right] & \text{if } -a < x < a \\ \dfrac{1}{2a}\dfrac{d}{dx}\left[\ln(a+x) - \ln(x-a)\right] & \text{if } x > a \\ \dfrac{1}{2a}\dfrac{d}{dx}\left[\ln(-a-x) - \ln(a-x)\right] & \text{if } x < -a \end{cases}$$

$$= \begin{cases} \dfrac{1}{2a}\left[\dfrac{1}{a+x} - \dfrac{-1}{a-x}\right] & \text{if } -a < x < a \\ \dfrac{1}{2a}\left[\dfrac{1}{a+x} - \dfrac{1}{x-a}\right] & \text{if } x > a \\ \dfrac{1}{2a}\left[\dfrac{-1}{-a-x} - \dfrac{-1}{a-x}\right] & \text{if } x < -a \end{cases}$$

$$= \frac{1}{2a}\left(\frac{1}{a+x} + \frac{1}{a-x}\right)$$
$$= \frac{1}{2a} \times \frac{a-x+a+x}{(a+x)(a-x)}$$
$$= \frac{1}{2a} \times \frac{2a}{a^2-x^2}$$
$$= \frac{1}{a^2-x^2} \quad \text{as required}$$

131 If $(2-x)\mathbf{a} + y\mathbf{b} = y\mathbf{a} + (x-3)\mathbf{b}$

then $(2-x-y)\mathbf{a} = (x-3-y)\mathbf{b}$

which are parallel unless $2-x-y=0$ and $x-3-y=0$

$\therefore$ $x+y=2$ and $x-y=3$

Solving simultaneously, $x = \frac{5}{2}$, $y = -\frac{1}{2}$.

132 **a** $\theta_1 = \theta_2$ and $\phi_1 = \phi_2$ {equal alternate angles}

Now $\tan\theta = \dfrac{\text{PN}}{b}$ and $\tan\phi = \dfrac{\text{PM}}{a}$

$\therefore$ $\text{PN} = b\tan\theta$ and $\text{PM} = a\tan\phi$

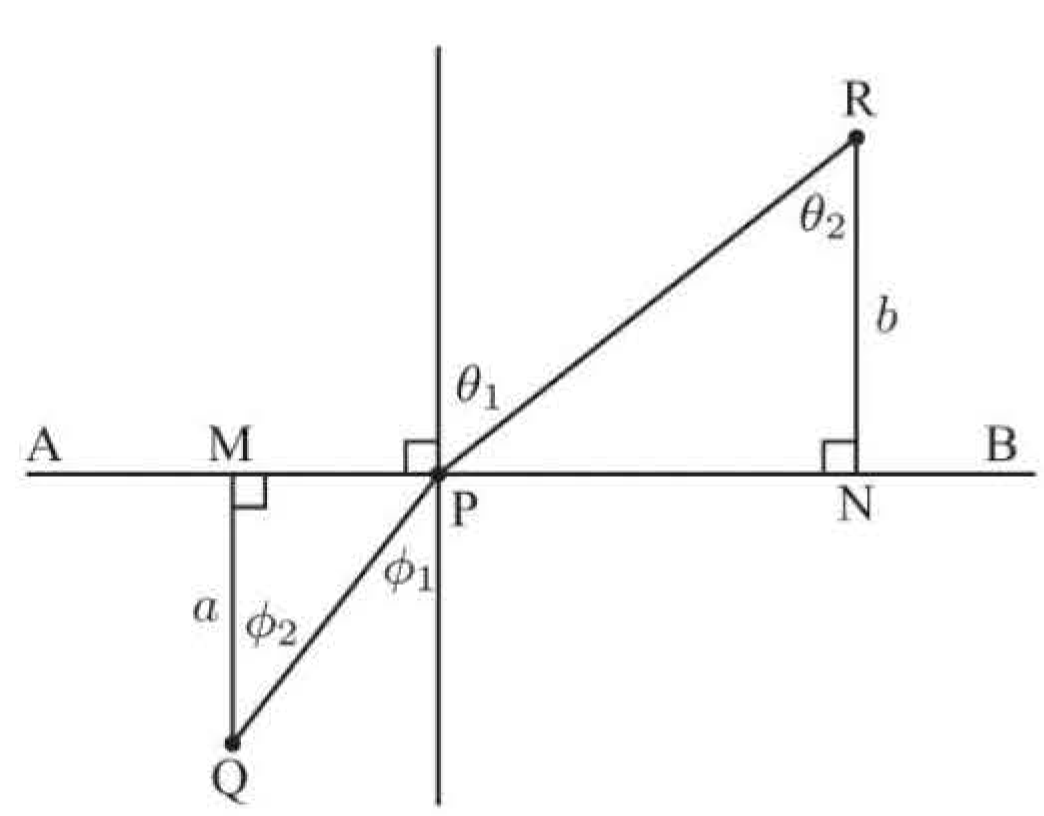
But MN is constant, so we let

$b\tan\theta + a\tan\phi = k$

Differentiating with respect to θ gives

$$b\sec^2\theta + a\sec^2\phi\frac{d\phi}{d\theta} = 0$$

$$\therefore \quad \frac{d\phi}{d\theta} = \frac{-b\sec^2\theta}{a\sec^2\phi} = \frac{-b\cos^2\phi}{a\cos^2\theta}$$



b Now speed $= \dfrac{\text{distance}}{\text{time}}$ $\quad\therefore\;$ time $= \dfrac{\text{distance}}{\text{speed}}$

$\therefore$ the total time T to travel from R to P to Q is given by

$$\begin{aligned}
T &= \frac{\text{PR}}{v_1} + \frac{\text{PQ}}{v_2} \\
&= \frac{b}{v_1 \cos\theta} + \frac{a}{v_2 \cos\phi} \qquad \left\{\text{since } \cos\theta = \frac{b}{\text{PR}},\ \cos\phi = \frac{a}{\text{PQ}}\right\} \\
&= \frac{b}{v_1}[\cos\theta]^{-1} + \frac{a}{v_2}[\cos\phi]^{-1}
\end{aligned}$$

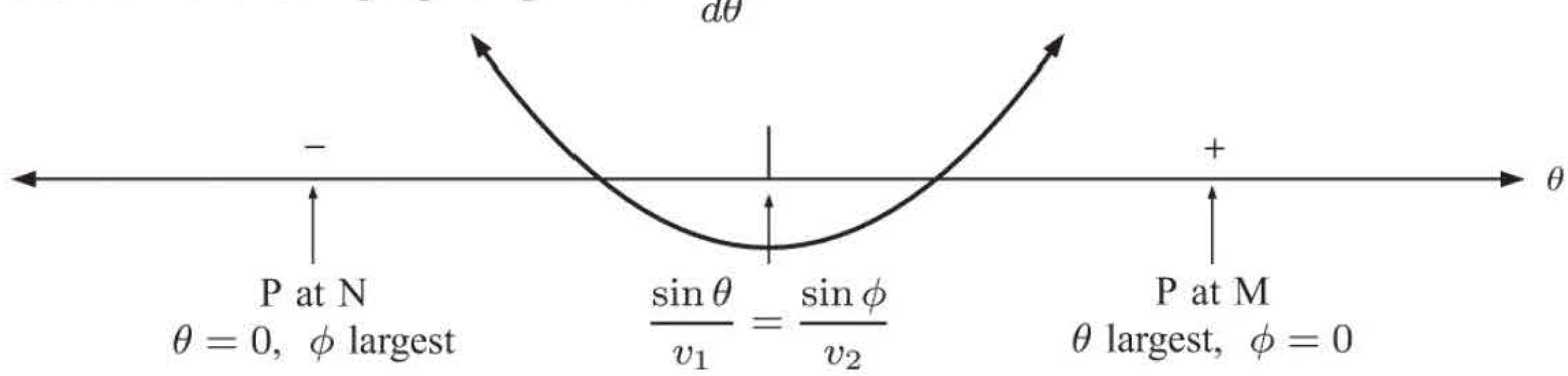
$$\begin{aligned}
\therefore\ \frac{dT}{d\theta} &= -\frac{b}{v_1}[\cos\theta]^{-2} \times -\sin\theta - \frac{a}{v_2}[\cos\phi]^{-2} \times -\sin\phi\,\frac{d\phi}{d\theta} \\
&= \frac{b\sin\theta}{v_1\cos^2\theta} + \frac{a\sin\phi}{v_2\cos^2\phi}\left(\frac{-b\cos^2\phi}{a\cos^2\theta}\right) \qquad \{\text{from } \mathbf{a}\} \\
&= \frac{b\sin\theta}{v_1\cos^2\theta} - \frac{b\sin\phi}{v_2\cos^2\theta} \\
&= \frac{b}{\cos^2\theta}\left[\frac{\sin\theta}{v_1} - \frac{\sin\phi}{v_2}\right]
\end{aligned}$$

$\therefore\ \dfrac{dT}{d\theta} = 0$ when $\dfrac{\sin\theta}{v_1} = \dfrac{\sin\phi}{v_2}$, or when $\dfrac{\sin\theta}{\sin\phi} = \dfrac{v_1}{v_2}$.

So, there is only one case when $\dfrac{dT}{d\theta} = 0$.

The question is whether we have a minimum or a maximum T in this case.

We construct the following sign diagram for $\dfrac{dT}{d\theta}$:



$\dfrac{dT}{d\theta} = \dfrac{b}{\cos^2\theta}\left[-\dfrac{\sin\phi}{v_2}\right]$

$\therefore\ \dfrac{dT}{d\theta} < 0$

$\dfrac{dT}{d\theta} = \dfrac{b}{\cos^2\theta}\left(\dfrac{\sin\theta}{v_1} - 0\right)$

$\therefore\ \dfrac{dT}{d\theta} > 0$ as $b, v_1, \sin\theta, \cos\theta > 0$

$\therefore$ the minimum value of T occurs when $\dfrac{dT}{d\theta} = 0$.

133 Let X be the outcome of a spin.

x	1	2	4
$P(X = x)$	$\frac{1}{2}$	$\frac{3}{8}$	$\frac{1}{8}$

Since spins are independent,

P(player spins x_1 and operator spins x_2)
$= P(x_1) \times P(x_2)$

For example, P(player spins 2 and operator spins 4) $= \frac{3}{8} \times \frac{1}{8} = \frac{3}{64}$

Results table:

		Operator		
		1	2	4
Player	1	$\frac{1}{4}$	$\frac{3}{16}$	$\frac{1}{16}$
	2	$\frac{3}{16}$	$\frac{9}{64}$	$\frac{3}{64}$
	4	$\frac{1}{16}$	$\frac{3}{64}$	$\frac{1}{64}$

In these 3 the player wins $\therefore$ gaining $a - k$ dollars

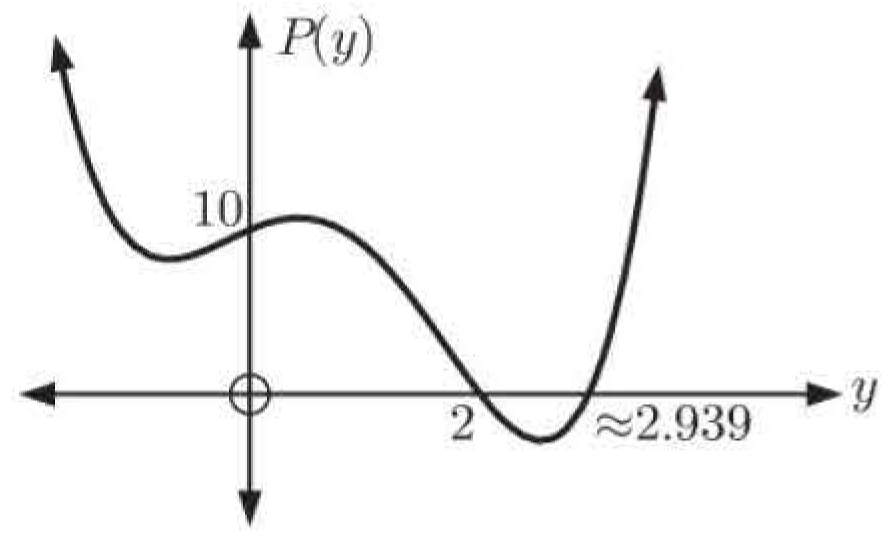
In these 6 the player loses $\therefore$ gaining $-k$ dollars

$$\therefore \text{ expected gain} = (a-k)\left[\tfrac{3}{16}+\tfrac{1}{16}+\tfrac{3}{64}\right] + (-k)\left[\tfrac{1}{4}+\tfrac{3}{16}+\tfrac{1}{16}+\tfrac{9}{64}+\tfrac{3}{64}+\tfrac{1}{64}\right]$$
$$= (a-k)\left(\tfrac{19}{64}\right) - k\left(\tfrac{45}{64}\right)$$
$$= \tfrac{19}{64}a - k \quad \text{which is zero in a 'fair' game}$$
$$\therefore \ k = \frac{19a}{64}$$

EXERCISE 27B

1 Since $x = 1 - y^2$, $\quad (1-y^2)^2 + 3(1-y^2)y + 9 = 0$

$$\therefore \ 1 - 2y^2 + y^4 + 3y - 3y^3 + 9 = 0$$
$$\therefore \ y^4 - 3y^3 - 2y^2 + 3y + 10 = 0$$



Using technology there are two real solutions,
$y = 2$ and $y \approx 2.939$
When $y = 2$, $x = -3$ and when $y \approx 2.939$, $x \approx -7.64$.
So, $x = -3$ and $x \approx -7.64$ are the solutions.

2 Let n be the number of years after winter 1969 and let u_n be the number of trees at time n.
Each summer, 10% die out and 100 new ones are planted, so $u_{n+1} = 0.9u_n + 100$.
We also know that $u_{11} = 1200$, since there were 1200 trees in 1980.
We hence have a sequence of the form $u_{n+1} = au_n + b$, $n = 1, 2, 3, 4, 5, ...$

Now $u_2 = au_1 + b$
$$u_3 = au_2 + b = a(au_1 + b) + b = a^2u_1 + ab + b$$
$$u_4 = au_3 + b = a(a^2u_1 + ab + b) + b = a^3u_1 + a^2b + ab + b \quad \text{and so on}$$

This suggests: $u_{n+1} = a^n u_1 + b(1 + a + a^2 + ... + a^{n-1})$

$$\therefore \ u_{n+1} = a^n u_1 + b\left(\frac{1-a^n}{1-a}\right) \qquad \{\text{sum of a geometric series}\}$$

a In this case $a = 0.9$ and $b = 100$

$$\therefore \ u_{n+1} = (0.9)^n u_1 + 100\left(\frac{1-(0.9)^n}{1-0.9}\right)$$
$$\therefore \ u_{n+1} = (0.9)^n u_1 + 1000\,(1-(0.9)^n)$$

But $u_{11} = (0.9)^{10}u_1 + 1000(1-(0.9)^{10}) = 1200$

$$\therefore \ 0.348\,68u_1 + 651.32 \approx 1200$$
$$\therefore \ u_1 \approx \frac{548.68}{0.348\,68} \approx 1574 \text{ trees}$$

$\therefore$ there were about 1574 trees at the end of winter in 1970.

b As n gets large,
$$(0.9)^n \to 0$$
and $1-(0.9)^n \to 1$
$$\therefore \ u_{n+1} \to 1000$$
This indicates a stable number of trees at 1000.

3 Let $\text{P}(A \text{ hits}) = 2p$ and $\text{P}(B \text{ hits}) = p$

Now $\text{P(at least one hits)} = \frac{1}{2}$, so $\text{P(both miss)} = \frac{1}{2}$

$$\therefore \ (1-2p)(1-p) = \tfrac{1}{2}$$
$$\therefore \ 4p^2 - 6p + 1 = 0$$
$$\therefore \ p = \frac{3 \pm \sqrt{5}}{4}$$
$$\therefore \ p = \frac{3 - \sqrt{5}}{4} \quad \left\{\text{since } \frac{3+\sqrt{5}}{4} > 1\right\}$$
$$\therefore \ 2p = \frac{3-\sqrt{5}}{2} \approx 0.382$$
$$\therefore \ \text{P}(A \text{ hits}) \approx 0.382$$

4 P_n is "$\frac{1}{a(a+1)}+\frac{1}{(a+1)(a+2)}+\frac{1}{(a+2)(a+3)}+....+\frac{1}{(a+n-1)(a+n)}=\frac{n}{a(a+n)}$" for $n \in \mathbb{Z}^+$.

Proof: (By the principle of mathematical induction)

(1) If $n=1$, LHS $=\frac{1}{a(a+1)}$ and RHS $=\frac{1}{a(a+1)}$ $\therefore$ P_1 is true.

(2) If P_k is assumed true then

$$\frac{1}{a(a+1)}+\frac{1}{(a+1)(a+2)}+....+\frac{1}{(a+k-1)(a+k)}=\frac{k}{a(a+k)}$$

Thus $\frac{1}{a(a+1)}+\frac{1}{(a+1)(a+2)}+....+\frac{1}{(a+k-1)(a+k)}+\frac{1}{(a+k)(a+k+1)}$

$$=\frac{k}{a(a+k)}+\frac{1}{(a+k)(a+k+1)} \quad \{\text{using } P_k\}$$

$$=\frac{k}{a(a+k)}\left(\frac{a+k+1}{a+k+1}\right)+\frac{1}{(a+k)(a+k+1)}\left(\frac{a}{a}\right)$$

$$=\frac{ak+k^2+k+a}{a(a+k)(a+k+1)}$$

$$=\frac{\cancel{(a+k)}(k+1)}{a\cancel{(a+k)}(a+k+1)}$$

$$=\frac{k+1}{a(a+[k+1])}$$

Thus P_{k+1} is true whenever P_k is true.

$\therefore$ since P_1 is true, P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

5 Now $2z+w=i$,

so $6z+3w=3i$

Also, $z-3w=7-10i$

$\therefore$ $7z \quad =7-7i$

$\therefore$ $z=1-i$

$\therefore$ $w=i-2z$

$=i-2+2i$

$=-2+3i$

Thus $z+w=-1+2i$

6 **Proof:** (By contradiction)

Suppose neither equation has real roots

$\therefore$ $b_1^2-4c_1<0$ and $b_2^2-4c_2<0$

$\therefore$ $b_1^2+b_2^2<4c_1+4c_2$

$\therefore$ $b_1^2+b_2^2<4(c_1+c_2)$

$\therefore$ $b_1^2+b_2^2<2b_1b_2$ {given $b_1b_2=2(c_1+c_2)$}

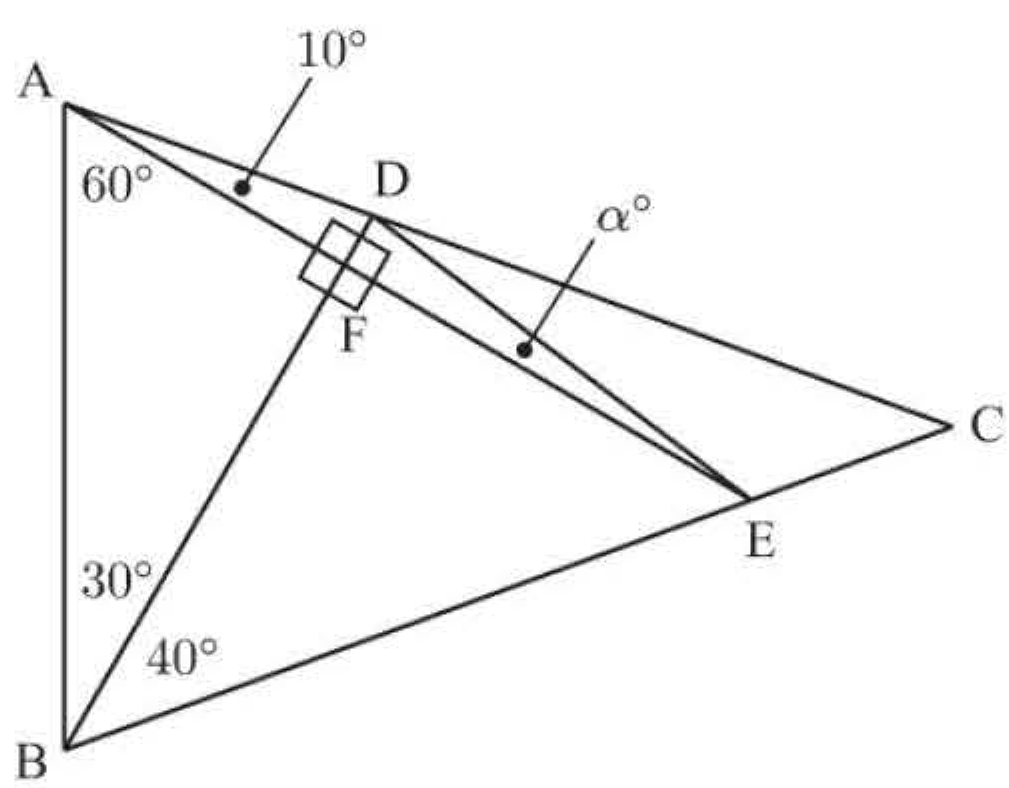
$\therefore$ $b_1^2-2b_1b_2+b_2^2<0$

$\therefore$ $(b_1-b_2)^2<0$

which is a contradiction as no perfect square of real numbers can be negative.

Thus the supposition is false and so at least one of the equations has real roots.

7



We notice that $\widehat{AFB}=90°$ {angles in a $\triangle$}

$\therefore$ $\tan 30°=\frac{AF}{BF}$, $\tan 40°=\frac{EF}{BF}$, $\tan 10°=\frac{DF}{AF}$

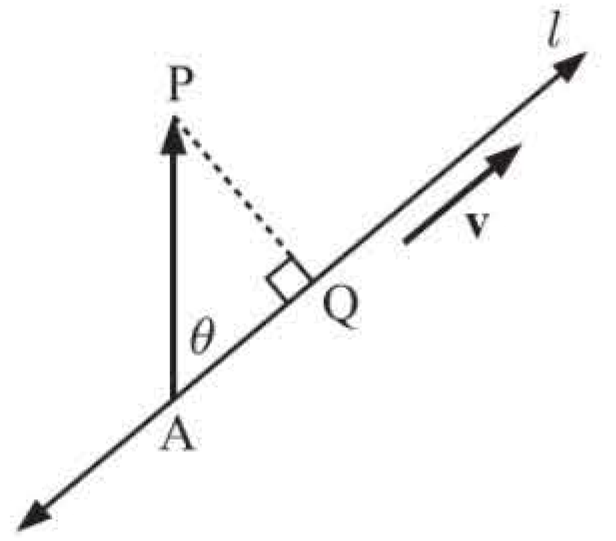
Now $\tan\alpha=\frac{DF}{EF}=\frac{DF}{AF}\times\frac{AF}{BF}\times\frac{BF}{EF}$

$$=\frac{\tan 10°\times\tan 30°}{\tan 40°}$$

$\therefore$ $\alpha=\arctan\left(\frac{\tan 10°\times\tan 30°}{\tan 40°}\right)$

$\therefore$ $\alpha\approx 6.92°$

8 **a**



Let angle PAQ $= \theta$

$$\therefore \quad \sin\theta = \frac{\text{PQ}}{|\overrightarrow{\text{AP}}|}$$

$$\therefore \quad \text{PQ} = |\overrightarrow{\text{AP}}|\sin\theta$$

$$= \frac{|\overrightarrow{\text{AP}}|\,|\mathbf{v}|\sin\theta}{|\mathbf{v}|}$$

$$= \frac{|\overrightarrow{\text{AP}} \times \mathbf{v}|}{|\mathbf{v}|}$$

$\{$as $|\mathbf{a}|\,|\mathbf{b}|\sin\theta = |\mathbf{a} \times \mathbf{b}|\,\}$

b P is at $(2,\ -1,\ 3)$.

A is at $(-1,\ 1,\ 2)$ and $\mathbf{v} = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$

$$\overrightarrow{\text{AP}} \times \mathbf{v} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 1 \\ 3 & -1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} -2 & 1 \\ -1 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & 1 \\ 3 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & -2 \\ 3 & -1 \end{vmatrix}\mathbf{k}$$

$$= -\mathbf{i} + 3\mathbf{k}$$

$$\therefore \quad |\overrightarrow{\text{AP}} \times \mathbf{v}| = \sqrt{1+0+9} = \sqrt{10} \text{ units}$$

$$\therefore \quad \text{PQ} = \frac{\sqrt{10}}{\sqrt{9+1+1}}$$

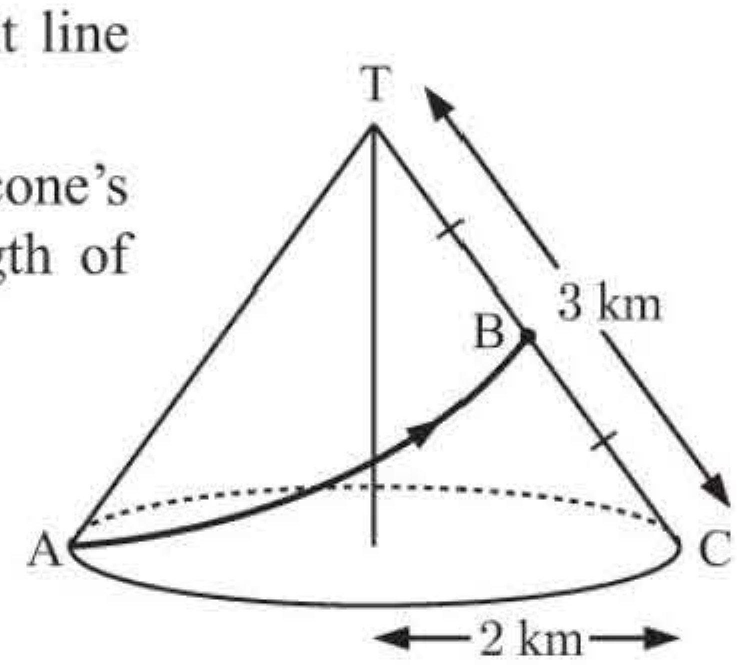
$$\approx 0.953 \text{ units}$$

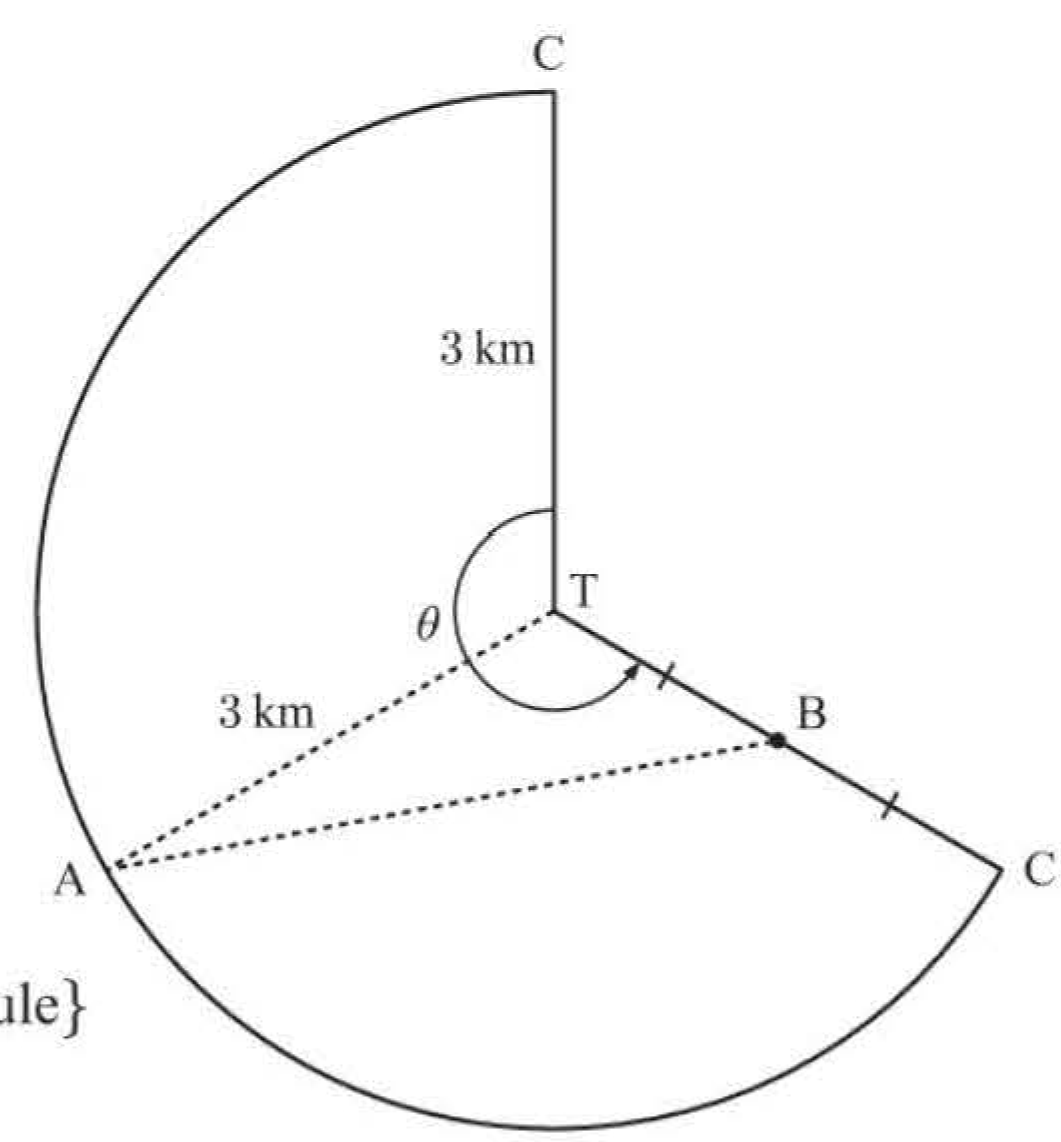
9 Consider a model of the mountain. We cut the model along [CT] and flatten it out. To make AB as short as possible, [AB] is a straight line on a sector of a circle.

The circumference of the cone's base is equal to the arc length of the sector, so

$$2\pi \times 2 = \theta \times 3$$

$$\therefore \quad \theta = \tfrac{4\pi}{3}$$





Thus $\widehat{\text{ATB}} = \frac{1}{2}\theta = \frac{2\pi}{3}$

and $\text{AB}^2 = 3^2 + \left(\frac{3}{2}\right)^2 - 2 \times 3 \times \frac{3}{2}\cos\left(\frac{2\pi}{3}\right)$ $\{$cosine rule$\}$

$$\therefore \quad \text{AB} = \sqrt{3^2 + (1.5)^2 - 9\cos\left(\tfrac{2\pi}{3}\right)}$$

$$\therefore \quad \text{AB} \approx 3.97 \text{ km}$$

10 $\log_3(x-k) + \log_3(x+2) = 1$

$$\therefore \quad \log_3(x-k)(x+2) = 1$$

$$\therefore \quad (x-k)(x+2) = 3^1 = 3$$

$$\therefore \quad x^2 + [2-k]x - 2k - 3 = 0$$

This quadratic in x has $\Delta = (2-k)^2 - 4(1)(-2k-3)$

$$= 4 - 4k + k^2 + 8k + 12$$

$$= k^2 + 4k + 16$$

$$= (k+2)^2 + 12$$

Since $(k+2)^2 \geqslant 0$, $\Delta \geqslant 0$ for all k

$\therefore$ the original equation has a real solution for all real k.

11 **a** $u_1 = \dfrac{1}{\sin\theta} - \sin\theta = \dfrac{1-\sin^2\theta}{\sin\theta} = \dfrac{\cos^2\theta}{\sin\theta}$

$$u_4 = \frac{1}{\cos\theta} - \cos\theta = \frac{1-\cos^2\theta}{\cos\theta} = \frac{\sin^2\theta}{\cos\theta}$$

$$\therefore \quad \frac{u_2}{u_1} = \frac{\cos\theta}{\left(\dfrac{\cos^2\theta}{\sin\theta}\right)} = \tan\theta, \quad \frac{u_3}{u_2} = \frac{\sin\theta}{\cos\theta} = \tan\theta, \quad \text{and} \quad \frac{u_4}{u_3} = \frac{\left(\dfrac{\sin^2\theta}{\cos\theta}\right)}{\sin\theta} = \tan\theta$$

So, the sequence is geometric with $u_1 = \dfrac{\cos^2\theta}{\sin\theta}$ and $r = \tan\theta$

$\therefore\ u_n = u_1 r^{n-1} = \dfrac{\cos^2\theta}{\sin\theta} \times (\tan\theta)^{n-1} = \dfrac{\cos\theta}{\tan\theta} \times \tan^{n-1}\theta$

$\therefore\ u_n = \cos\theta\tan^{n-2}\theta$

b $u_1 = 1,\ u_2 = \cos^1\theta,\ u_3 = \cos^3\theta,\ u_4 = \cos^7\theta,\ u_5 = \cos^{15}\theta$

We notice that $u_5 = u_4^{\,2}\cos\theta,\ u_4 = u_3^{\,2}\cos\theta,\ u_3 = u_2^{\,2}\cos\theta,\ u_2 = u_1^{\,2}\cos\theta,$

suggesting that $u_1 = 1$ and $u_{n+1} = u_n^{\,2}\cos\theta$ for all $n \in \mathbb{Z}^+$.

12 If X is the number of seedlings in a selected row, then $X \sim \text{B}(10, \frac{1}{2})$.

P(randomly selected row has *at least* 8 seedlings) $= \text{P}(X = 8, 9, \text{ or } 10)$

$$= \frac{\binom{10}{8} + \binom{10}{9} + \binom{10}{10}}{2^{10}}$$

$= \frac{7}{128}$ {technology}

$\therefore$ P(randomly selected row has *less than* 8 seedlings) $= 1 - \frac{7}{128} = \frac{121}{128}$

$\therefore$ P(*all 10 rows* have *less than* 8 seedlings) $= \left(\frac{121}{128}\right)^{10}$

$\therefore$ P(row with maximum germination contains *at least* 8 seedlings) $= 1 - \left(\frac{121}{128}\right)^{10} \approx 0.430$

13 **a** $L = \int_0^1 \sqrt{1 + (2x)^2}\, dx$

≈ 1.48 units {technology}

b $y = \sin x$, so $\dfrac{dy}{dx} = \cos x$

$\therefore\ L = \int_0^{\pi} \sqrt{1 + \cos^2 x}\, dx$

≈ 3.82 units {technology}

14 $\dfrac{P(x)}{(x-a)^2} = Q(x) + \dfrac{bx + c}{(x-a)^2}$ {the division process}

$\therefore\ P(x) = Q(x)(x-a)^2 + bx + c$

$\therefore\ P(a) = Q(a) \times 0 + ab + c = ab + c$ .... (1)

Also, $P'(x) = Q'(x)(x-a)^2 + Q(x)2(x-a) + b$

$\therefore\ P'(a) = 0 + 0 + b = b$ .... (2)

So, the remainder is $bx + c = bx + (ab + c) - ab$

$= P'(a)\,x + P(a) - a\,P'(a)$ {using (1) and (2)}

$= P'(a)(x-a) + P(a)$

15 **a** $-8i = 8(-i) = 8\,\text{cis}\left(-\frac{\pi}{2}\right)$

b As $z^3 = -8i$,

$z^3 = 8\ \text{cis}\left(-\frac{\pi}{2} + k2\pi\right),\ k \in \mathbb{Z}$

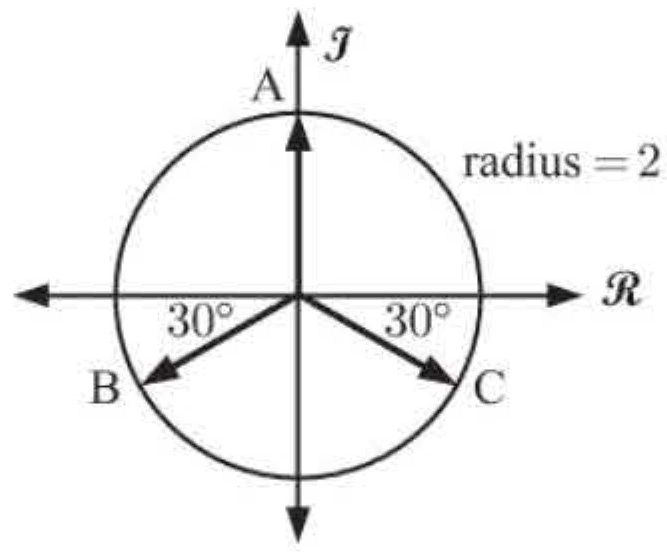
$\therefore\ z = 8^{\frac{1}{3}}\ \text{cis}\left(\dfrac{-\frac{\pi}{2} + k2\pi}{3}\right)$ {De Moivre}

$\therefore\ z = 2\ \text{cis}\left(\dfrac{-\pi + k4\pi}{6}\right)$

$\therefore\ z = 2\ \text{cis}\left(-\frac{5\pi}{6}\right),\ 2\ \text{cis}\left(-\frac{\pi}{6}\right),\ 2\ \text{cis}\left(\frac{\pi}{2}\right)$

$\{k = -1, 0, 1\}$

c



We let A, B, and C be the points corresponding to z_1, z_2, z_3 say.

d If $z_1 = 2 \text{ cis}\left(-\frac{\pi}{6}\right)$,

$$\begin{aligned} {z_1}^2 &= 4 \text{ cis}\left(-\tfrac{\pi}{3}\right) \quad \text{and} \\ z_2 z_3 &= 2 \text{ cis}\left(\tfrac{\pi}{2}\right) \times 2 \text{ cis}\left(-\tfrac{5\pi}{6}\right) \\ &= 4 \text{ cis}\left(\tfrac{\pi}{2} - \tfrac{5\pi}{6}\right) \\ &= 4 \text{ cis}\left(-\tfrac{\pi}{3}\right) \\ &= {z_1}^2 \end{aligned}$$

If $z_1 = 2 \text{ cis}\left(-\frac{5\pi}{6}\right)$,

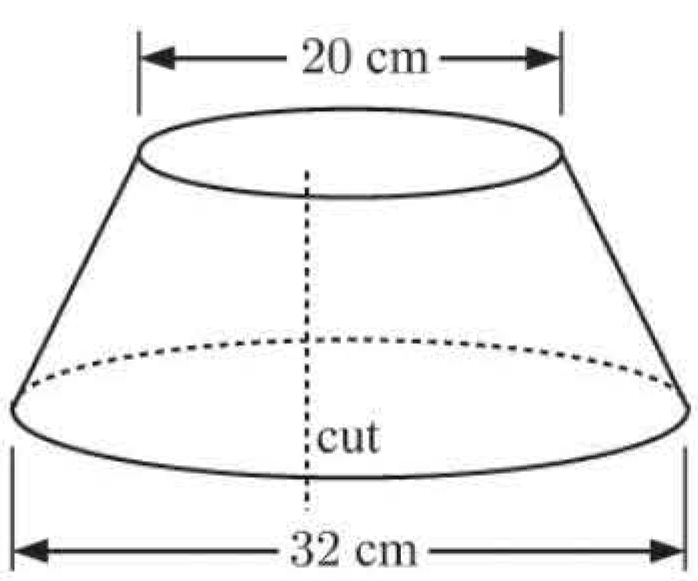
$$\begin{aligned} {z_1}^2 &= 4 \text{ cis}\left(-\tfrac{5\pi}{3}\right) \quad \text{and} \\ z_2 z_3 &= 2 \text{ cis}\left(-\tfrac{\pi}{6}\right) \times 2 \text{ cis}\left(\tfrac{\pi}{2}\right) \\ &= 4 \text{ cis}\left(\tfrac{\pi}{2} - \tfrac{\pi}{6}\right) \\ &= 4 \text{ cis}\, \tfrac{\pi}{3} \\ &= 4 \text{ cis}\left(-\tfrac{5\pi}{3}\right) \\ &= {z_1}^2 \end{aligned}$$

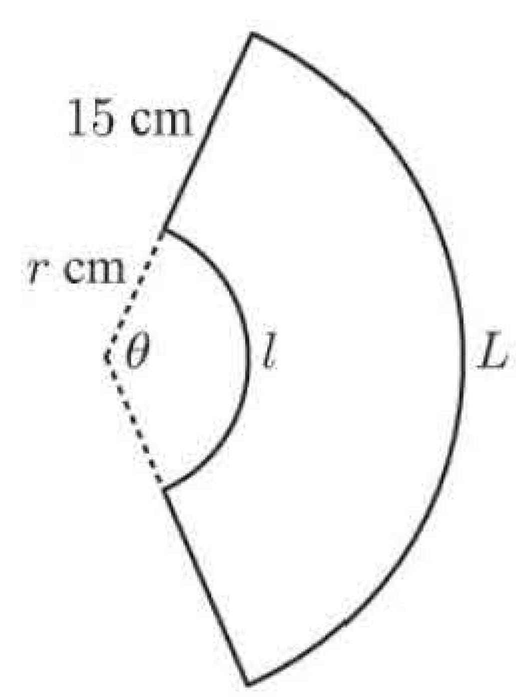
If $z_1 = 2 \text{ cis}\left(\frac{\pi}{2}\right)$,

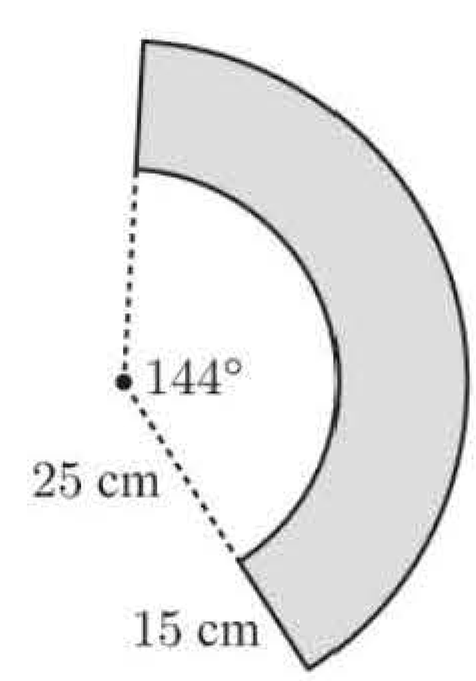
$$\begin{aligned} {z_1}^2 &= (2i)^2 = -4 \quad \text{and} \\ z_2 z_3 &= 2 \text{ cis}\left(\tfrac{-5\pi}{6}\right) \times 2 \text{ cis}\left(\tfrac{-\pi}{6}\right) \\ &= 4 \text{ cis}\left(\tfrac{-5\pi}{6} + \tfrac{-\pi}{6}\right) \\ &= 4 \text{ cis}(-\pi) \\ &= -4 \\ &= {z_1}^2 \end{aligned}$$

e
$$\begin{aligned} z_1 z_2 z_3 &= z_1(z_2 z_3) \\ &= z_1 \times {z_1}^2 \quad \{\text{from } \mathbf{d}\} \\ &= {z_1}^3 \\ &= -8i \quad \{\text{as } z_1 \text{ is a root of } z^3 = -8i\} \end{aligned}$$

16







$$\begin{aligned} l &= 2\pi(10) = 20\pi \\ L &= 2\pi(16) = 32\pi \end{aligned}$$

But $l = r\theta$ and $L = (r+15)\theta$

$\therefore\ r\theta = 20\pi$ and $(r+15)\theta = 32\pi$

Thus $20\pi + 15\theta = 32\pi$

$$\begin{aligned} \therefore\ 15\theta &= 12\pi \\ \therefore\ \theta &= \tfrac{4\pi}{5}^c = 144^\circ \end{aligned}$$

and $r\left(\frac{4\pi}{5}\right) = 20\pi$

$$\begin{aligned} \therefore\ r &= 25 \\ \therefore\ r &= 25 \text{ and } \theta = 144^\circ \end{aligned}$$

17 $X \sim \text{N}(90, \sigma^2)$

a
$$\begin{aligned} \text{P}(X < 88) &\approx 0.289\,25 \\ \therefore\ \text{P}\left(\frac{X-90}{\sigma} < \frac{88-90}{\sigma}\right) &\approx 0.289\,25 \\ \therefore\ \text{P}\left(Z < \frac{-2}{\sigma}\right) &\approx 0.289\,25 \\ \therefore\ \frac{-2}{\sigma} &\approx -0.555\,577 \\ \therefore\ \sigma &\approx 3.599\,86 \end{aligned}$$

b
$$\begin{aligned} &\text{P}(X < 89 \text{ or } X > 91) \\ &= 1 - \text{P}(89 \leqslant X \leqslant 91) \\ &\approx 1 - 0.219 \\ &\approx 0.781 \end{aligned}$$

18 **a** $(f \circ g)(x) = f(g(x))$

$$= f\left(\frac{x+1}{x-2}\right)$$

$$= 2\left(\frac{x+1}{x-2}\right) + 1$$

$$= \frac{2x+2+x-2}{x-2}$$

$$= \frac{3x}{x-2}$$

b $y = \dfrac{x+1}{x-2}$ has inverse $x = \dfrac{y+1}{y-2}$

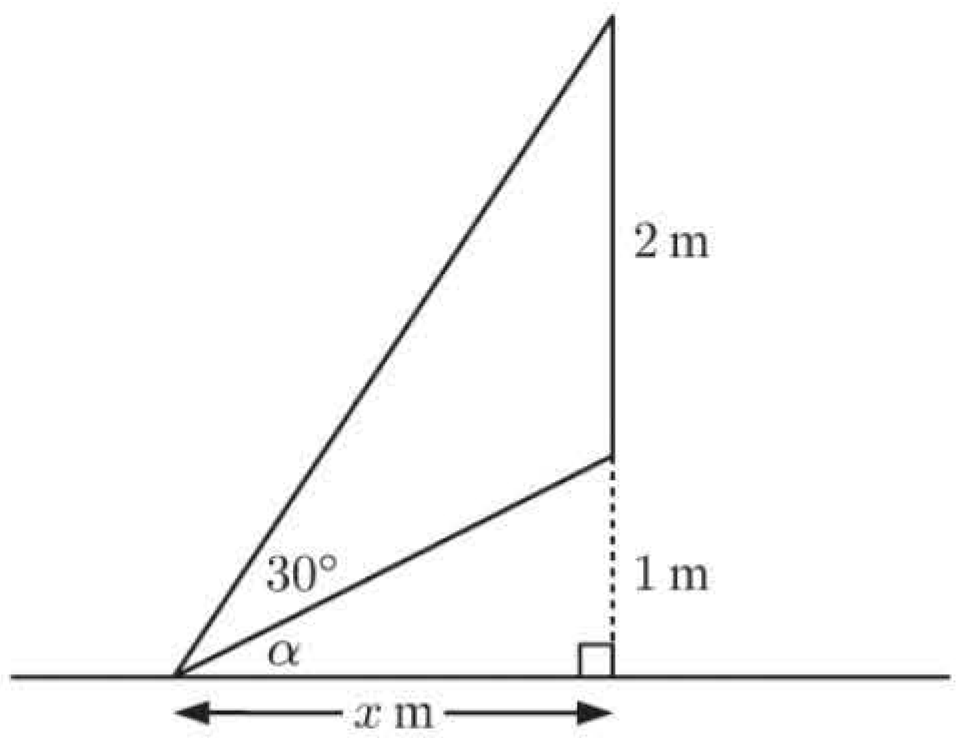
$$\therefore\ xy - 2x = y + 1$$

$$\therefore\ y(x-1) = 2x+1$$

$$\therefore\ y = \frac{2x+1}{x-1}$$

$$\therefore\ f^{-1}(x) = \frac{2x+1}{x-1}$$

19



$\tan\alpha = \dfrac{1}{x}$ and $\tan(\alpha + 30°) = \dfrac{3}{x}$

$$\therefore\ \frac{\tan\alpha + \tan 30°}{1 - \tan\alpha \tan 30°} = \frac{3}{x}$$

$$\therefore\ \frac{1}{x} + \frac{1}{\sqrt{3}} = \frac{3}{x}\left(1 - \frac{1}{x}\frac{1}{\sqrt{3}}\right) = \frac{3}{x} - \frac{\sqrt{3}}{x^2}$$

$$\therefore\ \frac{1}{\sqrt{3}} - \frac{2}{x} + \frac{\sqrt{3}}{x^2} = 0$$

$$\therefore\ x^2 - 2\sqrt{3}x + 3 = 0 \quad \{\times\ \sqrt{3}x^2\}$$

$$\therefore\ (x - \sqrt{3})^2 = 0$$

$$\therefore\ x = \sqrt{3} \approx 1.73$$

So, she is $\sqrt{3}$ m or 1.73 m from the wall.

20

$$\log_a(x+2) = \log_a x + 2$$

$$\therefore\ \log_a(x+2) - \log_a x = 2$$

$$\therefore\ \log_a\left(\frac{x+2}{x}\right) = 2$$

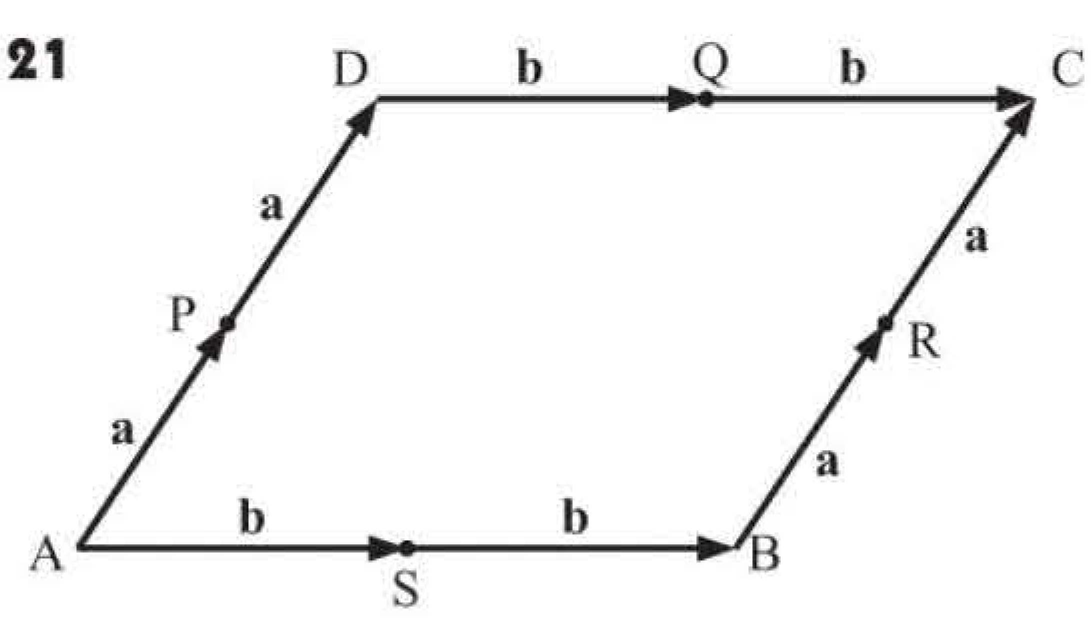
$$\therefore\ \frac{x+2}{x} = a^2$$

$$\therefore\ 1 + \frac{2}{x} = a^2$$

$$\therefore\ \frac{2}{x} = a^2 - 1$$

$$\therefore\ x = \frac{2}{a^2-1}, \quad a > 1$$

21



Using the vectors given, $\overrightarrow{PQ} = \mathbf{a} + \mathbf{b} = \overrightarrow{SR}$

$\overrightarrow{QR} = \mathbf{b} - \mathbf{a} = \overrightarrow{PS}$

$\therefore$ [PQ] $\parallel$ [SR] and [QR] $\parallel$ [PS]

$\therefore$ PQRS is a parallelogram

But $\overrightarrow{PQ} \bullet \overrightarrow{QR} = (\mathbf{a}+\mathbf{b}) \bullet (\mathbf{b}-\mathbf{a})$

$$= \cancel{\mathbf{a} \bullet \mathbf{b}} - \mathbf{a} \bullet \mathbf{a} + \mathbf{b} \bullet \mathbf{b} - \cancel{\mathbf{b} \bullet \mathbf{a}}$$

$$= \mathbf{b} \bullet \mathbf{b} - \mathbf{a} \bullet \mathbf{a} \quad \{\text{as } \mathbf{a} \bullet \mathbf{b} = \mathbf{b} \bullet \mathbf{a}\}$$

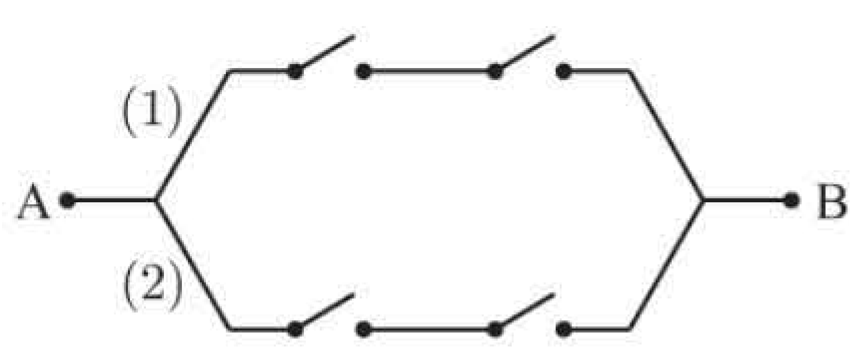
$$= |\mathbf{b}|^2 - |\mathbf{a}|^2 \quad \{\text{as } \mathbf{x} \bullet \mathbf{x} = |\mathbf{x}|^2\}$$

$$= 0 \text{ as } |\mathbf{a}| = |\mathbf{b}|$$

$\therefore$ $\widehat{PQR}$ is a right angle

So, PQRS is a rectangle.

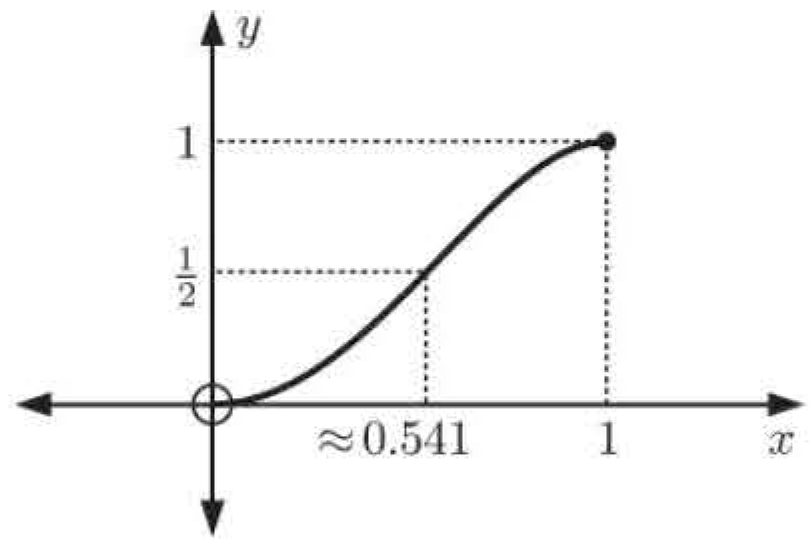
22 **a**



P(current flows)
$= \text{P}((1) \text{ closed} \cup (2) \text{ closed})$
$= \text{P}((1) \text{ closed}) + \text{P}((2) \text{ closed}) - \text{P}((1) \textbf{ and } (2) \text{ closed})$
$= p^2 + p^2 - p^4$
$= 2p^2 - p^4$

b We need to solve $2p^2 - p^4 > \frac{1}{2}$

We graph $y = 2x^2 - x^4$ in $[0, 1]$.



So, for $2p^2 - p^4 > \frac{1}{2}$, $p > 0.541$

$\therefore$ the least value of p is ≈ 0.541

23 $h(x) = x^3 - 6tx^2 + 11t^2x - 6t^3$

a $h(t) = t^3 - 6t^3 + 11t^3 - 6t^3 = 0$

$\therefore$ $x = t$ is a zero of $h(x)$.

b By inspection, $h(x) = (x - t)(x^2 - 5tx + 6t^2)$

$\therefore$ $h(x) = (x - t)(x - 2t)(x - 3t)$

c $y = x^3 + 6x^2$ meets $y = -6 - 11x$ where $x^3 + 6x^2 = -6 - 11x$

$\therefore$ $x^3 + 6x^2 + 11x + 6 = 0$ which is $h(x)$ when $t = -1$

$\therefore$ $(x + 1)(x + 2)(x + 3) = 0$ {using **b**}

$\therefore$ $x = -1, -2,$ or -3

So, the graphs meet at $(-1, 5)$, $(-2, 16)$, and $(-3, 27)$.

24 $\sum\limits_{i=1}^{25} x_i = 1650$ and $\sum\limits_{i=1}^{25} {x_i}^2 = 115\,492$

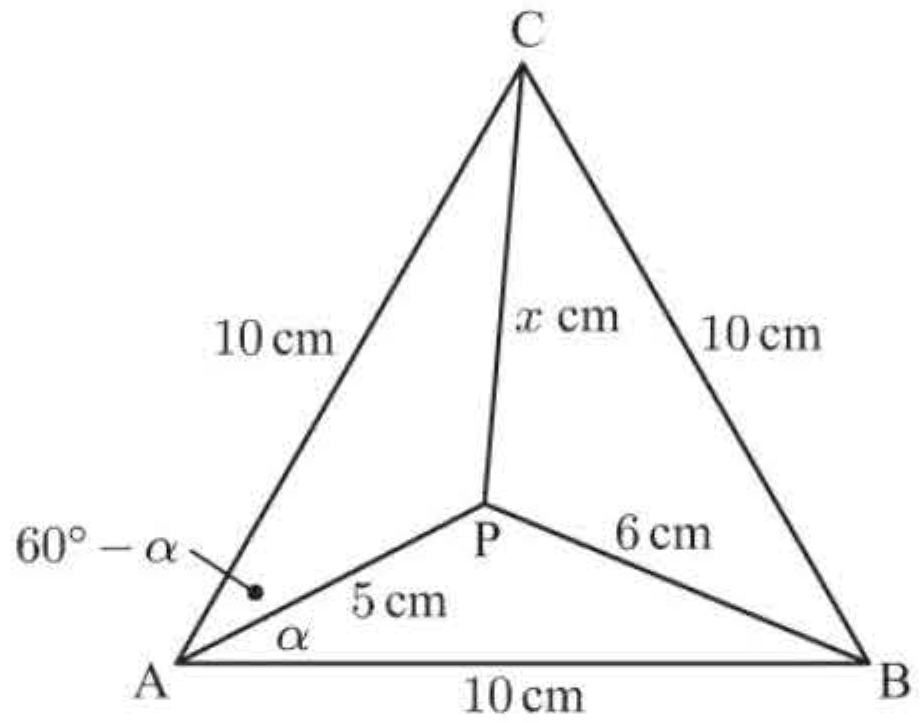
a $\overline{x} = \dfrac{\sum\limits_{i=1}^{25} x_i}{25} = \frac{1650}{25} = 66$

b $\sum\limits_{i=1}^{25} (x_i - \overline{x})^2 = \sum\limits_{i=1}^{25} {x_i}^2 - n\overline{x}^2 = 115\,492 - 25 \times 66^2 = 6592$

Now $s_n^{\ 2} = \dfrac{\sum\limits_{i=1}^{25} (x_i - \overline{x})^2}{25} = \dfrac{6592}{25} \approx 264$

$\therefore$ the variance ≈ 264

25



Using the cosine rule in $\triangle$ABP,

$$\cos\alpha = \frac{5^2 + 10^2 - 6^2}{2(5)(10)} = \frac{89}{100} = 0.89$$

$\therefore$ $\alpha \approx 27.127°$

$\therefore$ $60° - \alpha \approx 32.873°$

So, in $\triangle$APC, $x^2 = 10^2 + 5^2 - 2(10)(5)\cos 32.873°$

$\therefore$ $x^2 \approx 41.0127$

$\therefore$ $x \approx 6.40$

Thus P is about 6.40 cm from C.

26 **a** $w = \dfrac{z - 1}{z^* + 1} = \dfrac{a + bi - 1}{a - bi + 1} = \dfrac{[a-1] + bi}{[a+1] - bi}$

$\therefore$ $w = \left(\dfrac{[a-1] + bi}{[a+1] - bi}\right)\left(\dfrac{[a+1] + bi}{[a+1] + bi}\right)$

$= \dfrac{\left[a^2 - 1 - b^2\right] + i\,[2ab]}{[a+1]^2 + b^2}$

b So, w is purely imaginary when

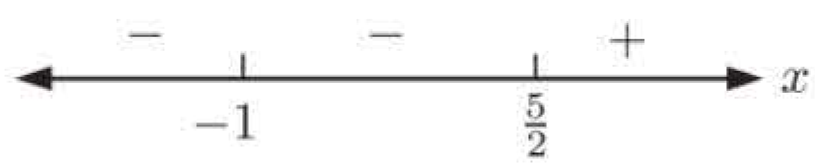
$a^2 - b^2 - 1 = 0$ and $2ab \neq 0$

$\therefore$ $a^2 - b^2 = 1$ and $ab \neq 0$

27 $f(x) = 2x^3 - x^2 - 8x - 5$

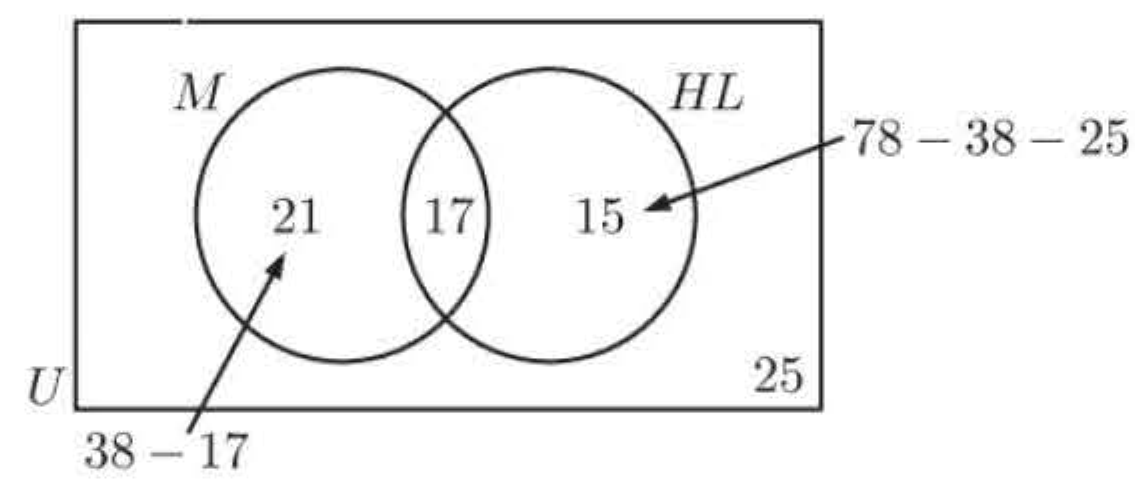
$$= (x+1)(2x^2 - 3x - 5)$$
$$= (x+1)(2x-5)(x+1)$$
$$= (x+1)^2(2x-5)$$

which has sign diagram:



$\therefore\ f(x) \geqslant 0$ for $x = -1$ or $x \geqslant \frac{5}{2}$

28



Of the 32 students studying higher level Mathematics, 17 are male.

$\therefore\ \mathrm{P}(M \mid HL) = \frac{17}{32}$

29 **a** $(2-\sqrt{3})^{n+1} = (2-\sqrt{3})^n(2-\sqrt{3})$

$$= (a_n - b_n\sqrt{3})(2-\sqrt{3})$$
$$= (2a_n + 3b_n) - (a_n + 2b_n)\sqrt{3}$$

$\therefore\ a_{n+1} = 2a_n + 3b_n$ and $b_{n+1} = a_n + 2b_n$

b $(2-\sqrt{3})^1 = 2-\sqrt{3}$ $\quad$ $(2-\sqrt{3})^2 = 7-4\sqrt{3}$ $\quad$ $(2-\sqrt{3})^3 = 26-15\sqrt{3}$

$\therefore\ a_1 = 2,\ b_1 = 1$ $\quad$ $\therefore\ a_2 = 7,\ b_2 = 4$ $\quad$ $\therefore\ a_3 = 26,\ b_3 = 15$

$\therefore\ {a_1}^2 - 3{b_1}^2 = 4 - 3(1) = 1$ $\quad$ $\therefore\ {a_2}^2 - 3{b_2}^2 = 49 - 3(16) = 1$ $\quad$ $\therefore\ {a_3}^2 - 3{b_3}^2 = 676 - 3(225) = 1$

c P_n is: "If $(2-\sqrt{3})^n = a_n - b_n\sqrt{3},\ a_n, b_n \in \mathbb{Z}$, then ${a_n}^2 - 3{b_n}^2 = 1$" for $n \in \mathbb{Z}^+$.

d **Proof:** (By the principle of mathematical induction)

(1) If $n = 1,\ {a_1}^2 - 3{b_1}^2 = 1$ was shown in **b**. $\quad\therefore\ P_1$ is true.

(2) If P_k is assumed true then ${a_k}^2 - 3{b_k}^2 = 1$

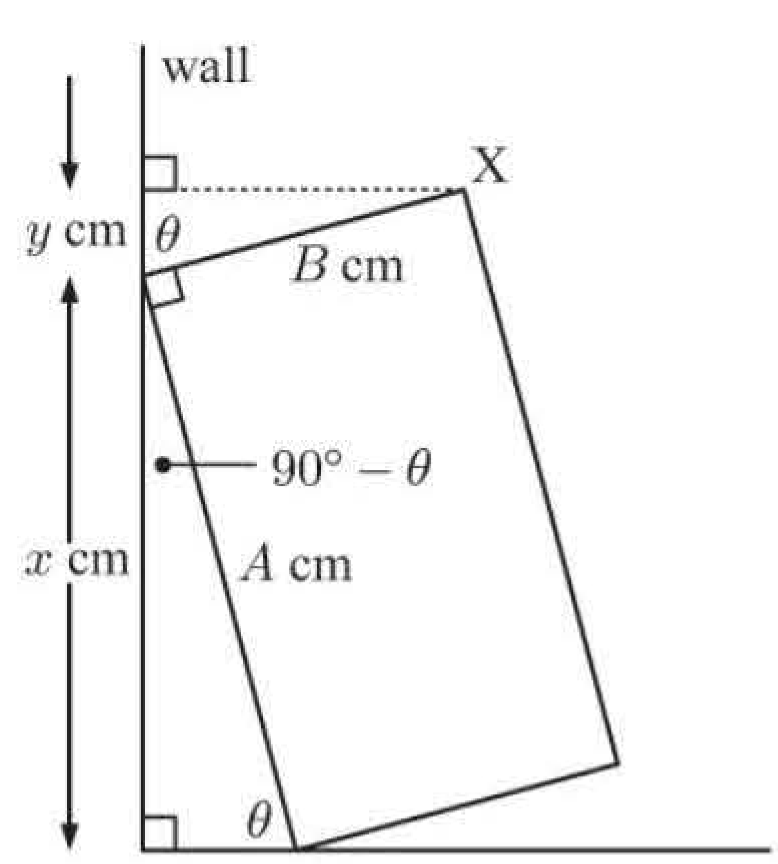
Now ${a_{k+1}}^2 - 3{b_{k+1}}^2 = (2a_k + 3b_k)^2 - 3(a_k + 2b_k)^2$ {from **a**}

$$= 4{a_k}^2 + 12a_kb_k + 9{b_k}^2 - 3\left({a_k}^2 + 4a_kb_k + 4{b_k}^2\right)$$
$$= {a_k}^2 - 3{b_k}^2$$
$$= 1 \quad \{\text{using } P_k\}$$

Thus P_{k+1} is true whenever P_k is true.

$\therefore$ since P_1 is true, P_n is true for all $n \in \mathbb{Z}$ {Principle of mathematical induction}

30 **a**

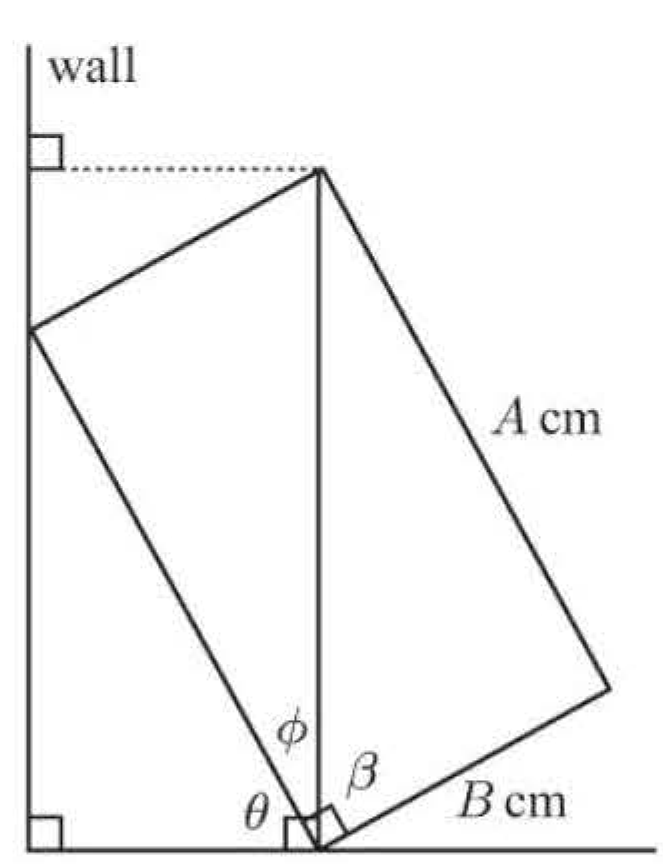


$\sin\theta = \dfrac{x}{A}$ and $\cos\theta = \dfrac{y}{B}$

$\therefore\ x = A\sin\theta$ and $y = B\cos\theta$

$\therefore\ H = x + y = A\sin\theta + B\cos\theta$

b



H must be $\leqslant$ diagonal of refrigerator

$\therefore\ A\sin\theta + B\cos\theta \leqslant \sqrt{A^2 + B^2}$

with equality when H is the diagonal

In this case, $\phi = \frac{\pi}{2} - \theta$

and $\beta = \frac{\pi}{2} - \phi = \theta$

$\therefore\ \tan\theta = \tan\beta = \dfrac{A}{B}$

31 **a**

$$\begin{aligned} T_{r+1} &= \binom{8}{r}(2x^3)^{8-r}\left(\frac{-1}{2x}\right)^r \quad \text{where } r = 0,\,1,\,2,\,3,\,....,\,8 \\ &= \binom{8}{r}2^{8-r}x^{24-3r}\left(\tfrac{-1}{2}\right)^r x^{-r} \\ &= \binom{8}{r}2^{8-r}\left(\tfrac{-1}{2}\right)^r x^{24-4r} \end{aligned}$$

We require $24 - 4r = 12$, so $4r = 12$ or $r = 3$

$$\therefore\ T_4 = \binom{8}{3}2^5\left(\tfrac{-1}{2}\right)^3 x^{12}$$

$\therefore$ the coefficient of x^{12} is $-\binom{8}{3}2^2 = -224$

b $(1+2x)^5(2-x)^6 = \left[1 + \binom{5}{1}2x + \binom{5}{2}(2x)^2 + ...\right]\left[2^6 - \binom{6}{1}2^5x + \binom{6}{2}2^4x^2 - ....\right]$

The coefficient of x^2 is

$$\begin{aligned} &1 \times \binom{6}{2}2^4 + \binom{5}{1}2 \times (-1)\binom{6}{1}2^5 + \binom{5}{2}2^2 \times 2^6 \\ &= 240 - 1920 + 2560 \\ &= 880 \end{aligned}$$

c

$$\begin{aligned} &(1+2x-3x^2)^4 \\ &= \left([1+2x] - 3x^2\right)^4 \\ &= (1+2x)^4 + 4(1+2x)^3(-3x^2)\ \underbrace{+\,6(1+2x)^2(-3x^2)^2 + ...}_{\text{all terms have order higher than } x^3} \\ &= 1 + \binom{4}{1}(2x) + \binom{4}{2}(2x)^2 + \binom{4}{3}(2x)^3 + (2x)^4 - 12x^2\left(1 + \binom{3}{1}(2x) + ...\right) + ... \end{aligned}$$

$\therefore$ the coefficient of x^3 is $4 \times 2^3 - 12 \times 3 \times 2 = -40$

32 Let F be the event of a faulty chip.

$\therefore$ $\mathrm{P}(F) = 0.03$ and $\mathrm{P}(F') = 0.97$

If X is the number which are faulty then $X \sim \mathrm{B}(500,\, 0.03)$

So, $\mathrm{P}(5 \leqslant X \leqslant 10) = \mathrm{P}(X \leqslant 10) - \mathrm{P}(X \leqslant 4)$ $\quad$ {1% is 5, 2% is 10}

$$\begin{aligned} &\approx 0.114\,787 - 0.000\,754 \\ &\approx 0.114 \end{aligned}$$

33 **a** As a is real, $p(x)$ has all real coefficients.

By the theorem on real polynomials, $-2+i$ and $-2-i$ are both zeros.

These have sum -4 and product $4+1=5$, so come from the quadratic factor x^2+4x+5.

b Hence, $p(x) = x^3 + (5+4a)x + 5a = (x^2+4x+5)(x+a)$ $\quad$ {comparing constant terms}

$$\begin{aligned} \therefore\ x^3 + (5+4a)x + 5a &= x^3 + [a+4]x^2 + [4a+5]x + 5a \\ \therefore\ a+4 &= 0 \qquad \{\text{equating coefficients of } x^2\} \\ \therefore\ a &= -4 \qquad \text{and the real zero is } -a \text{ which is 4.} \end{aligned}$$

34

$$\begin{aligned} \binom{n}{3} &= 3\binom{n-1}{2} - \binom{n-1}{1} \\ \therefore\ \frac{n(n-1)(n-2)}{6} - \frac{3(n-1)(n-2)}{2} + (n-1) &= 0 \\ \therefore\ \frac{n-1}{6}\left(n(n-2) - 9(n-2) + 6\right) &= 0 \\ \therefore\ (n-1)(n^2 - 2n - 9n + 18 + 6) &= 0 \\ \therefore\ (n-1)(n^2 - 11n + 24) &= 0 \\ \therefore\ (n-1)(n-3)(n-8) &= 0 \\ \therefore\ n &= 1,\,3,\text{ or } 8 \end{aligned}$$

But $n \geqslant 3$, $n-1 \geqslant 2$ and $n-1 \geqslant 1$, so $n \geqslant 3$

$$\therefore\ n = 3 \text{ or } 8$$

35 **a** As the sum of the probabilities must be 1,

$$\tfrac{1}{12} + k + \tfrac{1}{4} + \tfrac{1}{3} = 1$$

$$\therefore\ k + \frac{1+3+4}{12} = 1$$

$$\therefore\ k = 1 - \tfrac{2}{3}$$

$$\therefore\ k = \tfrac{1}{3}$$

b Let X be the number of 2s when the die is rolled 2400 times.

So, $X \sim \text{B}(2400,\ \frac{1}{3})$

mean

$$= np$$
$$= 2400 \times \tfrac{1}{3}$$
$$= 800$$

standard deviation

$$= \sqrt{np(1-p)}$$
$$= \sqrt{2400 \times \tfrac{1}{3} \times \tfrac{2}{3}}$$
$$= \tfrac{\sqrt{4800}}{3}$$
$$= \tfrac{40\sqrt{3}}{3}$$

36 Let the terms of the geometric series be x, rx, and r^2x, where r is the common ratio.

$\therefore\ x + rx + r^2x = 39$ .... (1)

But $x,\ \frac{5}{3}rx,$ and r^2x are arithmetic, so $\frac{5}{3}rx - x = r^2x - \frac{5}{3}rx$

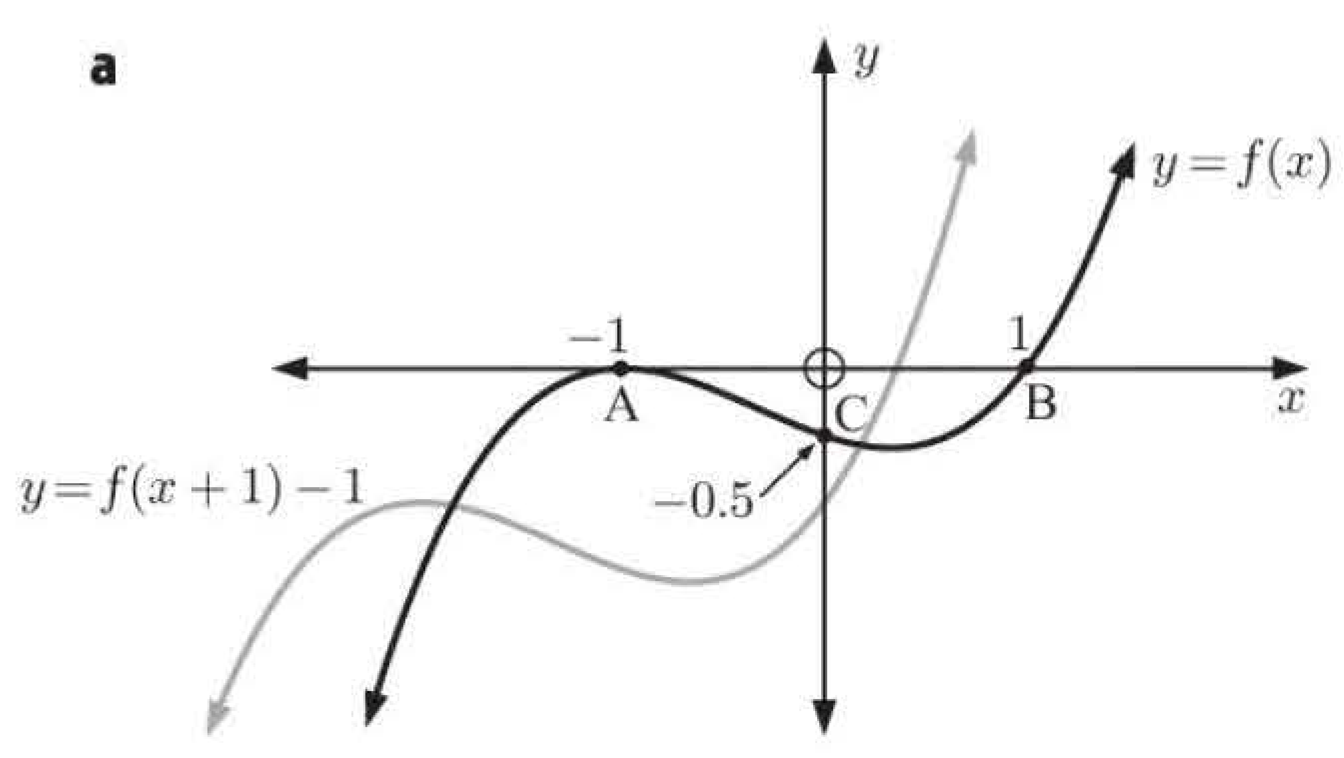
$$\therefore\ \tfrac{10}{3}rx - x = r^2x$$
$$\therefore\ \tfrac{10}{3}r - 1 = r^2$$
$$\therefore\ 3r^2 - 10r + 3 = 0$$
$$\therefore\ (3r-1)(r-3) = 0$$
$$\therefore\ r = \tfrac{1}{3} \text{ or } 3$$

Now from (1), $x = \dfrac{39}{1+r+r^2}$

$\therefore$ when $r = \frac{1}{3}$, $x = \dfrac{39}{1+\frac{1}{3}+\frac{1}{9}} = 27$, and when $r = 3$, $x = \dfrac{39}{1+3+9} = 3$

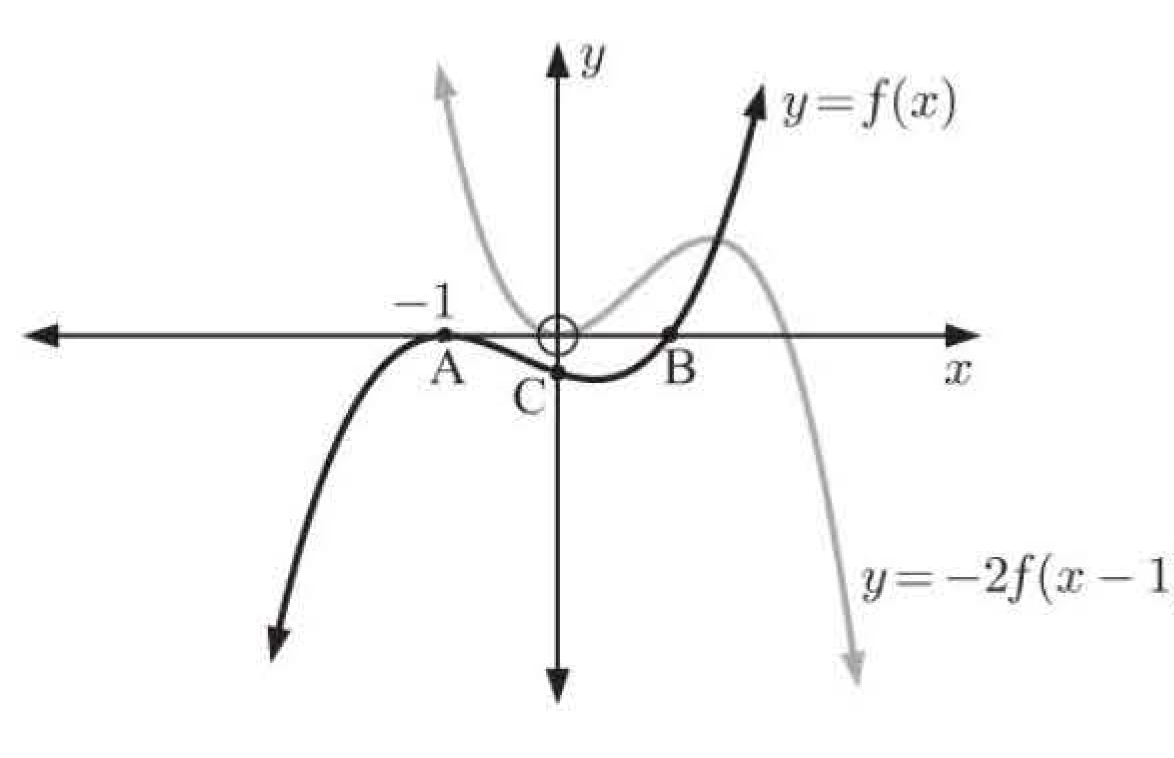
$\therefore$ the smallest possible value of the first term is 3.

37 **a**



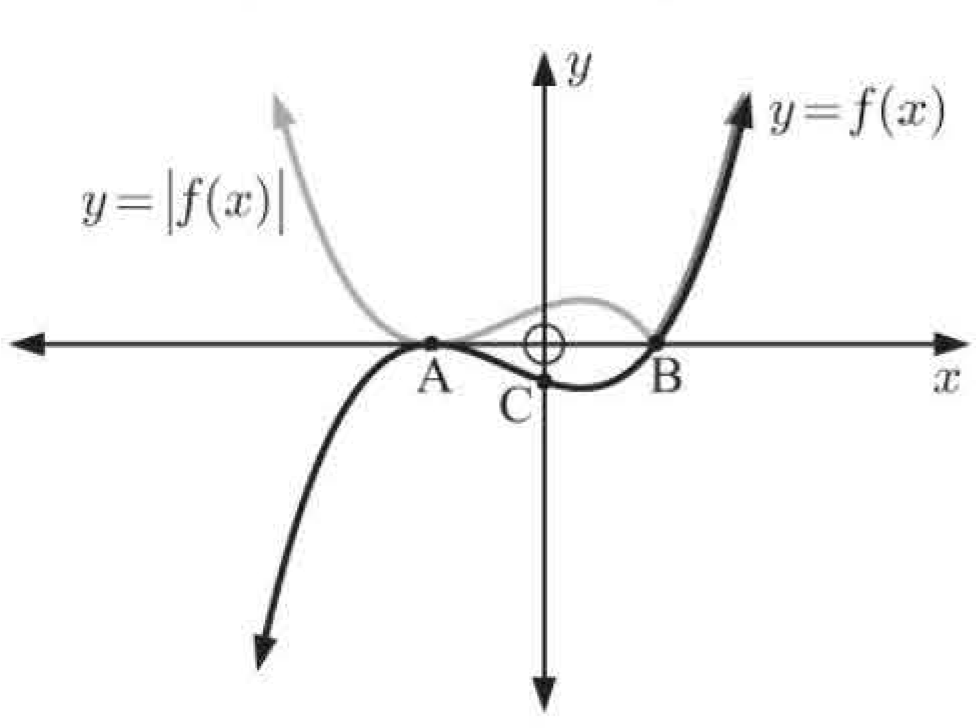
$\text{A}(-1,\ 0) \to \text{A}'(-2,\ -1)$
$\text{B}(1,\ 0) \to \text{B}'(0,\ -1)$
$\text{C}(0,\ -\frac{1}{2}) \to \text{C}'(-1,\ -\frac{3}{2})$

b



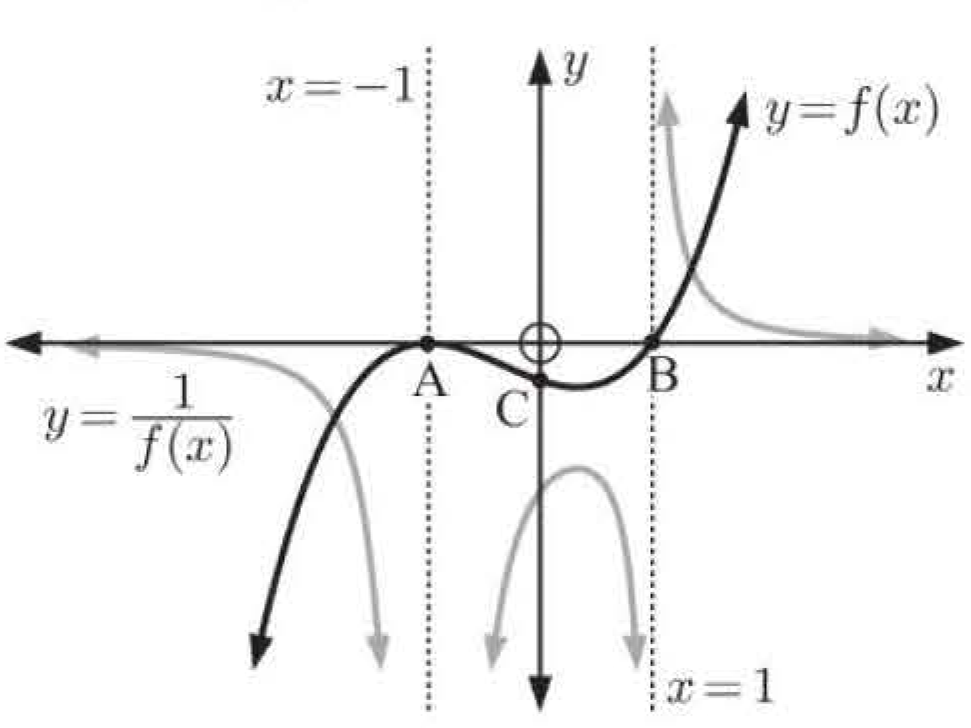
$\text{A}(-1,\ 0) \to \text{A}'(0,\ 0)$
$\text{B}(1,\ 0) \to \text{B}'(2,\ 0)$
$\text{C}(0,\ -\frac{1}{2}) \to \text{C}'(1,\ 1)$

c



$\text{A}(-1,\ 0) \to \text{A}'(-1,\ 0)$
$\text{B}(1,\ 0) \to \text{B}'(1,\ 0)$
$\text{C}(0,\ -\frac{1}{2}) \to \text{C}'(0,\ \frac{1}{2})$

d



$\text{A}(-1,\ 0) \to \text{VA},\ x = -1$
$\text{B}(1,\ 0) \to \text{VA},\ x = 1$
$\text{C}(0,\ -\frac{1}{2}) \to \text{C}'(0,\ -2)$

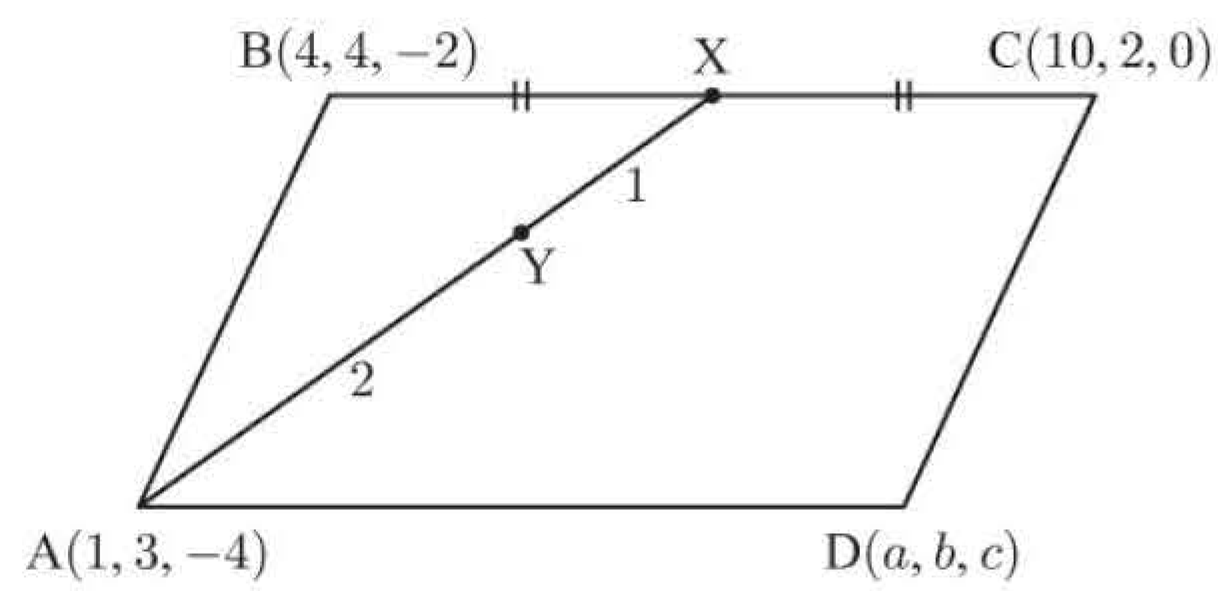
38 **a** For 200 metres of rope, $m = 1.4$

$$\text{P(2 flaws)} = \frac{1.4^2 e^{-1.4}}{2!} \approx 0.242$$

b For 400 metres of rope, $m = 2.8$

$$\begin{aligned}\text{P(at least 2 flaws)} &= 1 - \text{P}(X \leqslant 1)\\ &\approx 1 - 0.231\\ &\approx 0.769\end{aligned}$$

39 **a**



$$\overrightarrow{\text{AD}} = \overrightarrow{\text{BC}}$$

$$\therefore \begin{pmatrix} a-1 \\ b-3 \\ c+4 \end{pmatrix} = \begin{pmatrix} 6 \\ -2 \\ 2 \end{pmatrix}$$

$\therefore$ $a = 7, \ b = 1, \ c = -2$

$\therefore$ D is at $(7, 1, -2)$, X is at $(7, 3, -1)$

Now $\overrightarrow{\text{OY}} = \overrightarrow{\text{OA}} + \frac{2}{3}\overrightarrow{\text{AX}}$

$$= \begin{pmatrix} 1 \\ 3 \\ -4 \end{pmatrix} + \frac{2}{3}\begin{pmatrix} 6 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ -2 \end{pmatrix}$$

So, Y is at $(5, 3, -2)$.

b $\overrightarrow{\text{BY}} = \begin{pmatrix} 5-4 \\ 3-4 \\ -2+2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and $\overrightarrow{\text{BD}} = \begin{pmatrix} 7-4 \\ 1-4 \\ -2+2 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix}$

$\therefore$ $\overrightarrow{\text{BD}} = 3\overrightarrow{\text{BY}}$

$\therefore$ [BD] $\parallel$ [BY] and BD $= 3$(BY)

Since B is common to both [BD] and [BY], B, D, and Y are collinear.

40

$$\frac{dy}{dx} = \cos^2 x = \tfrac{1}{2} + \tfrac{1}{2}\cos 2x$$

$$\therefore \ y = \int \left(\tfrac{1}{2} + \tfrac{1}{2}\cos 2x\right) dx$$

$$\therefore \ y = \tfrac{1}{2}x + \tfrac{1}{4}\sin 2x + c$$

But $y(0) = 4$, so $4 = 0 + 0 + c$

Thus $y = \frac{1}{2}x + \frac{1}{4}\sin 2x + 4$

41 $X \sim \text{N}(\mu, 2.83^2)$

$$\begin{aligned}\therefore \ & \text{P}(-4 < X - \mu < 4)\\ &= \text{P}\left(\frac{-4}{2.83} < \frac{X-\mu}{2.83} < \frac{4}{2.83}\right)\\ &= \text{P}(-1.4134 < Z < 1.4134)\\ &\approx 0.842\end{aligned}$$

42

$$\text{Area} = \int_a^{a+2} x^2\, dx = \tfrac{31}{6}$$

$$\therefore \ \left[\frac{x^3}{3}\right]_a^{a+2} = \tfrac{31}{6}$$

$$\therefore \ \frac{(a+2)^3}{3} - \frac{a^3}{3} = \tfrac{31}{6}$$

$$\therefore \ \frac{\cancel{a^3} + 6a^2 + 12a + 8 - \cancel{a^3}}{3} = \tfrac{31}{6}$$

$$\therefore \ 12a^2 + 24a + 16 = 31$$

$$\therefore \ 12a^2 + 24a - 15 = 0$$

$$\therefore \ 4a^2 + 8a - 5 = 0$$

$$\therefore \ (2a-1)(2a+5) = 0$$

$$a = \tfrac{1}{2} \text{ or } -\tfrac{5}{2}$$

But $a > 0$, so $a = \frac{1}{2}$

43 As $P(x)$ is a real polynomial, both $1-2i$ and $1+2i$ are zeros. These have sum 2 and product $1+4=5$.

$\therefore$ $P(x)$ has a quadratic factor x^2-2x+5

$\therefore$ $P(x)=(x^2-2x+5)(x^2+ax+10)$

$\therefore$ the coefficient of x^3 is $a-2$

So, $a-2=0$

$\therefore$ $a=2$

Check:

$$\begin{array}{rrrrr} & & 1 & -2 & 5 \\ & \times & 1 & 2 & 10 \\ \hline & & 10 & -20 & 50 \\ & 2 & -4 & 10 & \\ 1 & -2 & 5 & & \\ \hline 1 & 0 & 11 & -10 & 50 \end{array} \checkmark$$

Thus $P(x)=(x^2-2x+5)(x^2+2x+10)$ where

$x^2+2x+10$ has zeros $\dfrac{-2\pm\sqrt{4-4(1)(10)}}{2}=\dfrac{-2\pm 6i}{2}=-1\pm 3i$

So, the other three zeros are $1+2i$, $-1+3i$, and $-1-3i$.

44 **a** The number of ways of selecting 2 females from n is $\binom{n}{2}$

and the number of ways of selecting 1 male from n is $\binom{n}{1}$.

$\therefore$ there are $\binom{n}{2}\binom{n}{1}=n\binom{n}{2}$ ways of selecting 2 females and 1 male.

b The number of ways of selecting 3 females from n is $\binom{n}{3}$.

c The total number of ways of selecting a committee of 3 from $2n$ people is $\binom{2n}{3}$.

As there are equal numbers of male and female members, exactly half of these committees will have more females than males.

$\therefore$ total number of committees with more females than males $=\frac{1}{2}\binom{2n}{3}$.

$\therefore$ using **a** and **b**, $n\binom{n}{2}+\binom{n}{3}=\frac{1}{2}\binom{2n}{3}$.

d **i** If $n=6$,

P(Mrs Jones is on the committee)

$$=\frac{\binom{1}{1}\binom{5}{1}\binom{6}{1}+\binom{1}{1}\binom{5}{2}\binom{6}{0}}{\frac{1}{2}\binom{12}{3}}$$

$$=\frac{30+10}{110}$$

$$=\tfrac{4}{11}$$

ii P(Mr Jones is on | Mrs Jones is on)

$$=\frac{\text{P(Mr Jones is on} \cap \text{Mrs Jones is on)}}{\text{P(Mrs Jones is on)}}$$

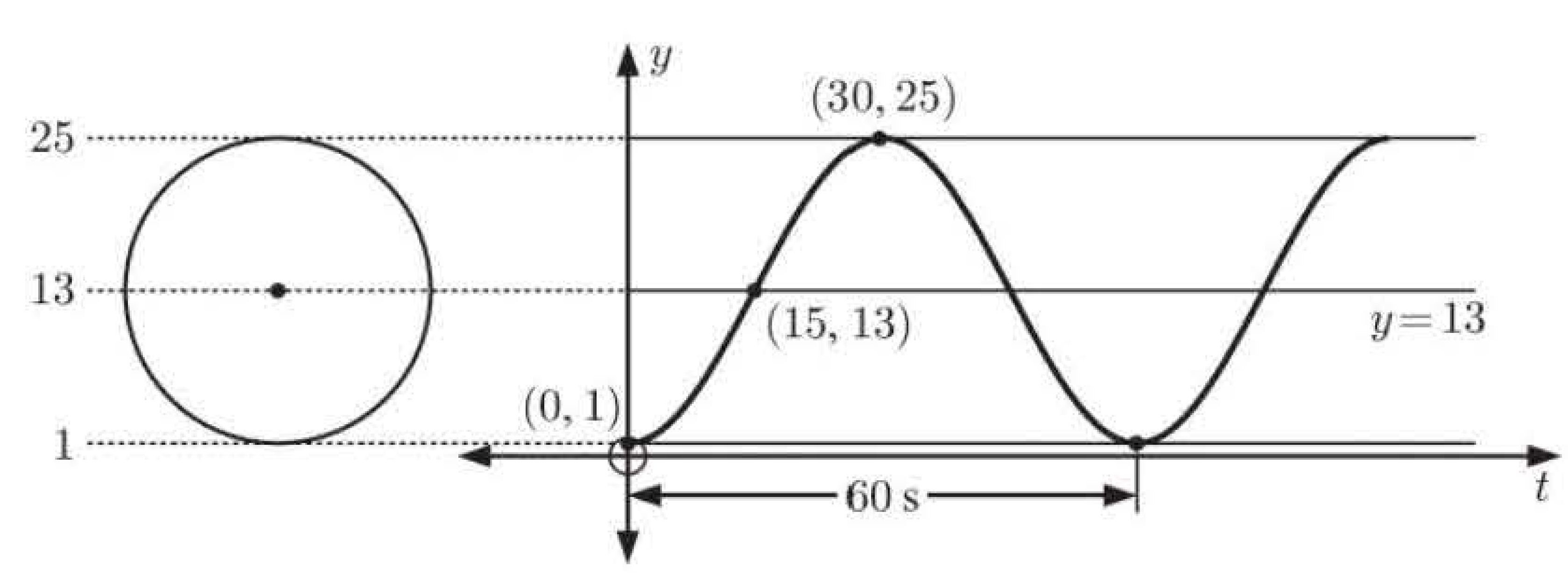
$$=\frac{\text{P(both are on)}}{\text{P(Mrs Jones is on)}}$$

$$=\frac{\binom{2}{2}\binom{5}{1}\binom{5}{0}}{\frac{1}{2}\binom{12}{3}}\div\tfrac{4}{11}$$

$$=\tfrac{5}{110}\times\tfrac{11}{4}$$

$$=\tfrac{1}{8}$$

45 **a**



We model the Ferris wheel using $h(t)=a+b\sin(c(t-d))$.

The amplitude $=b=12$. The period $=\dfrac{2\pi}{c}=60$ $\therefore$ $c=\frac{\pi}{30}$

The basic sine curve has been translated through $\binom{15}{13}$. $\therefore$ $d=15$, $a=13$

Thus $h(t)=12\sin\left(\frac{\pi}{30}(t-15)\right)+13$

Check: $h(0)=12\sin\left(\frac{-\pi}{2}\right)+13=12(-1)+13=1$ ✓

$h(30)=12\sin\left(\frac{\pi}{2}\right)+13=12(1)+13=25$ ✓

b $h(91) = 12\sin\left(\dfrac{\pi \times 76}{30}\right) + 13 \approx 24.9$ m

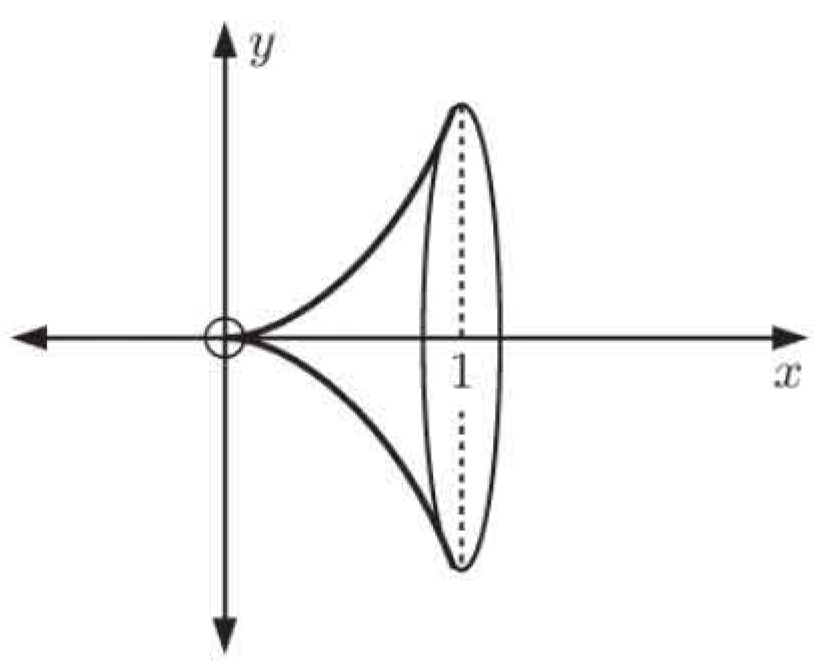
46 Let $x^2 + ax + b$ have zeros α and 2α

$\therefore$ the sum of the zeros $= 3\alpha = \dfrac{-a}{1}$ and the product of the zeros $= 2\alpha^2 = \dfrac{b}{1}$

$$\therefore \alpha = \frac{-a}{3} \text{ and } 2\alpha^2 = b$$

$$\therefore 2\left(\frac{a^2}{9}\right) = b \text{ and so } 2a^2 = 9b$$

47



$$\begin{aligned}
\text{Volume} &= \pi \int_0^1 y^2\, dx \\
&= \pi \int_0^1 x^2 e^{2x^3}\, dx \\
&= \tfrac{\pi}{6} \int_0^1 e^{2x^3}(6x^2)\, dx \\
&= \tfrac{\pi}{6}\left[e^{2x^3}\right]_0^1 \\
&= \tfrac{\pi}{6}(e^2 - 1) \text{ units}^3
\end{aligned}$$

48 **a** The line meets the plane where

$$\begin{aligned}
2(-4+3\lambda) + (2+\lambda) - (-1+2\lambda) &= 2 \\
\therefore -8 + 6\lambda + 2 + \lambda + 1 - 2\lambda &= 2 \\
\therefore 5\lambda - 5 &= 2 \\
\therefore \lambda &= \tfrac{7}{5}
\end{aligned}$$

$\therefore$ the point of intersection is $\left(\frac{1}{5}, \frac{17}{5}, \frac{9}{5}\right)$.

b L_1 has $\mathbf{v}_1 = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$.

L_2 has $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$.

Since $\mathbf{v}_2$ is **not** a multiple of $\mathbf{v}_1$, L_1 and L_2 are not parallel.

c The lines meet where

$$-4 + 3\lambda = \frac{2 + \lambda - 5}{2} = \frac{-(-1+2\lambda) - 1}{2}$$

$$\therefore \underbrace{-4 + 3\lambda = \frac{\lambda - 3}{2}} \!\!\! \underbrace{\phantom{\frac{\lambda-3}{2}} = -\lambda}$$

$-8 + 6\lambda = \lambda - 3$
$\therefore 5\lambda = 5$
$\therefore \lambda = 1$

Check:
$\lambda - 3 = -2\lambda$
$\therefore 3\lambda = 3$
$\therefore \lambda = 1$

So, they meet at $(-1, 3, 1)$.

d Normal vector $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2$

$$\begin{aligned}
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & 2 \\ 1 & 2 & -2 \end{vmatrix} \\
&= \mathbf{i}(-6) - \mathbf{j}(-8) + \mathbf{k}(5) \\
&= -6\mathbf{i} + 8\mathbf{j} + 5\mathbf{k}
\end{aligned}$$

$\therefore$ the equation of the plane is

$$\begin{aligned}
-6x + 8y + 5z &= -6(-1) + 8(3) + 5(1) \\
\therefore -6x + 8y + 5z &= 35 \\
\therefore 6x - 8y - 5z &= -35
\end{aligned}$$

49 $f(n) = \begin{cases} 0.6e^{-0.6n}, & n \geqslant 0 \\ 0 & \text{otherwise} \end{cases}$

a P(lasts at least a year)
$= \text{P}(N \geqslant 1)$
$= 1 - \text{P}(0 \leqslant N < 1)$
$= 1 - \int_0^1 0.6e^{-0.6n}\, dn$
≈ 0.549 {technology}

b P(cell fails within one year)
$=$ P(all components fail in one year)
$\approx (1 - 0.548\,81)^8$
$\approx 0.001\,72$ {technology}

50 **a** A and B are mutually exclusive if $A \cap B = \varnothing$.
In this case $\text{P}(A \cap B) = 0$, so $x = 0$.

b If A and B are independent, then $\quad \mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B)$

$$\begin{aligned} \therefore\ x &= (0.3+x)(0.2+x) \\ \therefore\ x &= 0.06 + 0.5x + x^2 \\ \therefore\ x^2 - 0.5x + 0.06 &= 0 \\ \therefore\ (x-0.2)(x-0.3) &= 0 \\ \therefore\ x &= 0.2 \text{ or } 0.3 \end{aligned}$$

51 **a** Zeros are $3 \pm 2i$ with sum 6 and product $9 + 4 = 13$

$\therefore\ f(x) = a(x^2 - 6x + 13),\ a \neq 0$

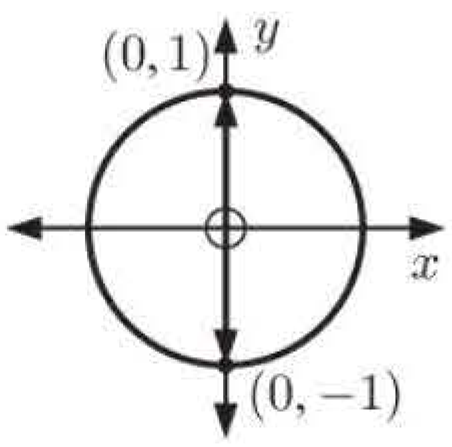
But $f(0) = -13$, so $a = -1$

$\therefore\ f(x) = -x^2 + 6x - 13$

b $$\begin{aligned} f(x) &= -1(x^2 - 6x + 13) \\ &= -1\left([x-3]^2 + 13 - 9\right) \\ &= -(x-3)^2 - 4 \end{aligned}$$

52 $f(x) = 2\tan(3(x-1)) + 4,\quad x \in [-1, 1]$

a $y = \tan nx$ has period $\dfrac{\pi}{n}$, so $f(x)$ has period $= \frac{\pi}{3}$



b Asymptotes are solutions of $\cos(3(x-1)) = 0$

$$\begin{aligned} \therefore\ 3(x-1) &= \tfrac{\pi}{2} + k\pi \\ \therefore\ x - 1 &= \tfrac{\pi}{6} + \frac{k\pi}{3} \\ \therefore\ x &= 1 + \tfrac{\pi}{6} + \frac{k\pi}{3} \end{aligned}$$

For the domain $-1 \leqslant x \leqslant 1$, asymptotes are $x \approx -0.571$ and $x \approx 0.476$.

c $y = \tan x \quad \rightarrow \quad y = \tan(3x) \quad \rightarrow \quad y = 2\tan(3x) \quad \rightarrow \quad y = 2\tan(3(x-1)) + 4$

We have a horizontal stretch with scale factor $\frac{1}{3}$, followed by a vertical stretch with scale factor 2, followed by a translation through $\binom{1}{4}$.

d The domain is $x \in [-1, 1]$ but $x \not\approx -0.571$ or 0.476.

The range is $y \in \mathbb{R}$.

53 $v(t) = \cos\left(\frac{1}{3}t\right)\ \mathrm{cm\,s^{-1}}$

$\therefore$ the total distance travelled in the first 10π seconds $= \displaystyle\int_0^{10\pi} \left|\cos(\tfrac{1}{3}t)\right|\, dt$

≈ 20.6 cm

54 **a** If there are no restrictions, there are $\binom{16}{6} = 8008$ possible choices.

b $\binom{7}{2}\binom{6}{2}\binom{3}{2} + \binom{7}{2}\binom{6}{3}\binom{3}{1} + \binom{7}{2}\binom{6}{4}\binom{3}{0} + \binom{7}{3}\binom{6}{2}\binom{3}{1} + \binom{7}{3}\binom{6}{3}\binom{3}{0} + \binom{7}{4}\binom{6}{2}\binom{3}{0}$

$= 5320$ possible choices

c $\binom{1}{1}\binom{3}{1}\binom{12}{4} + \binom{1}{1}\binom{3}{2}\binom{12}{3} + \binom{1}{1}\binom{3}{3}\binom{12}{2} = 2211$ possible choices

55 **a** Substituting $x = -2t + 2$, $y = t$, and $z = 3t + 1$ into $2x + y + z$, we get:

$$\begin{aligned} &2(-2t+2) + t + 3t + 1 \\ &= -4t + 4 + t + 3t + 1 \\ &= 5\ \checkmark \end{aligned}$$

$\therefore$ the line lies in the plane.

b If $x + ky + z = 3$ contains L_1 then

$(-2t + 2) + k(t) + (3t + 1) = 3$

$$\begin{aligned} \therefore\ -2t + kt + 3t &= 3 - 2 - 1 \\ \therefore\ t(k+1) &= 0 \\ \therefore\ k &= -1 \end{aligned}$$

as t is not necessarily equal to 0.

c From **a** and **b**, both $2x+y+z=5$ and $x-y+z=3$ contain L_1.
The solution of the system of 3 equations is where L_1 meets the third plane.

$$\therefore\ 2(-2t+2)+p(t)+2(3t+1)=q$$
$$\therefore\ -4t+4+pt+6t+2=q$$
$$\therefore\ t(p+2)=q-6$$

Now t can be any real number. So, we have infinitely many solutions when $p+2=0$ and $q-6=0$.
$\therefore\ p=-2,\ q=6$

d With $p=-2$, $q=6$, the system has augmented matrix

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 1 & -1 & 1 & 3 \\ 2 & -2 & 2 & 6 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 1 & -1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_3 \to R_3 - 2R_2$$

The row of all zeros and the fact that the first two rows are not multiples indicates there are infinitely many solutions.

56 $\dfrac{u_1}{1-r}=49$ and $u_1 r=10$

$$\therefore\ \frac{10}{r}=49(1-r)$$
$$\therefore\ 10=49r-49r^2$$
$$\therefore\ 49r^2-49r+10=0$$
$$\therefore\ (7r-2)(7r-5)=0$$
$$\therefore\ r=\tfrac{2}{7} \text{ or } \tfrac{5}{7}$$

When $r=\frac{2}{7}$, $u_1=35$.

When $r=\frac{5}{7}$, $u_1=14$.

$$\text{Thus } S_3=\frac{35\left(1-\left(\frac{2}{7}\right)^3\right)}{1-\frac{2}{7}}=47\tfrac{6}{7}$$

$$\text{or } S_3=\frac{14\left(1-\left(\frac{5}{7}\right)^3\right)}{1-\frac{5}{7}}=31\tfrac{1}{7}$$

57 The average number of amoebas in 10 mL of water, $m=4$. $\therefore\ X \sim \text{Po}(4)$

a $\text{P}(X \leqslant 5) \approx 0.785$

b If Y is the number of days where no more than 5 amoebas are collected then $Y \sim \text{B}(20,\ 0.785\,13)$.

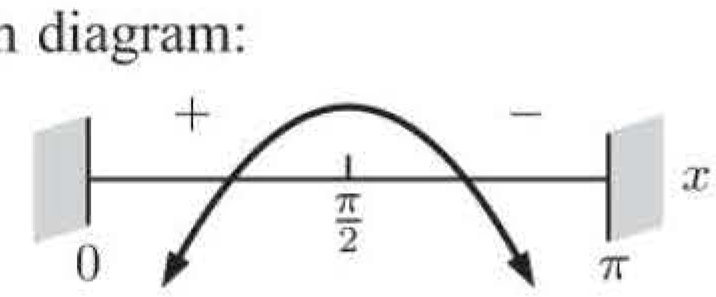
$$\text{P}(Y>10)=1-\text{P}(Y\leqslant 10)$$
$$\approx 1-0.004\,52$$
$$\approx 0.995$$

58 $f(x)=e^{\sin^2 x},\ x\in[0,\pi]$

a $f'(x)=e^{\sin^2 x}\times 2\sin x\cos x$
$\qquad = e^{\sin^2 x}\sin 2x$

which is 0 when $\sin 2x=0$
$$\therefore\ 2x=0+k\pi$$
$$\therefore\ x=0+\frac{k\pi}{2}$$
$$\therefore\ x=0,\ \tfrac{\pi}{2},\ \pi$$

Sign diagram:



$\therefore\ f(x)$ has maximum value when $x=\frac{\pi}{2}$, and this value is e.

b $f''(x)=e^{\sin^2 x}\times\sin 2x\sin 2x+e^{\sin^2 x}2\cos 2x$
$\qquad = e^{\sin^2 x}(\sin^2 2x+2\cos 2x)$

c $\sin^2 2x+2\cos 2x=0$ needs to be solved, as $f''(x)=0$ for points of inflection.
$\sin^2 2x+2\cos 2x=0$
$$\therefore\ x\approx 0.999 \text{ or } 2.14 \quad \{\text{technology}\}$$
$\therefore$ the points of inflection are $(0.999,\ 2.03)$ and $(2.14,\ 2.03)$.

59 $\dfrac{P(x)}{x(2x-3)}=Q(x)+\dfrac{ax+b}{x(2x-3)}$ {division process}

$\therefore\ P(x)=Q(x)\,[x(2x-3)]+ax+b$

a If $Q(x)=ax+b$, $\quad P(x)=(ax+b)[x(2x-3)]+ax+b$
$$\therefore\ P(x)=(ax+b)(2x^2-3x+1)$$

b Factorising the quadratic, $P(x) = (ax+b)(2x-1)(x-1)$

c Now $P(0) = 7$ so $b(-1)(-1) = 7$ $\therefore$ $b = 7$
and $P(2) = 39$ so $(2a+7)(3)(1) = 39$ $\therefore$ $2a+7 = 13$ and so $a = 3$

Thus $P(x) = (3x+7)(2x^2-3x+1)$

60
$$\int_0^a \frac{x}{x^2+1}\,dx = 3$$
$$\therefore\ \tfrac{1}{2}\int_0^a \frac{2x}{x^2+1}\,dx = 3$$
$$\therefore\ \Big[\ln|x^2+1|\Big]_0^a = 6$$
$$\therefore\ \ln|a^2+1| - \ln 1 = 6$$
$$\therefore\ \ln(a^2+1) = 6 \qquad \{a^2+1>0\}$$
$$\therefore\ a^2+1 = e^6$$
$$\therefore\ a^2 = e^6 - 1$$
$\therefore$ since $a > 0$, $a = \sqrt{e^6-1}$

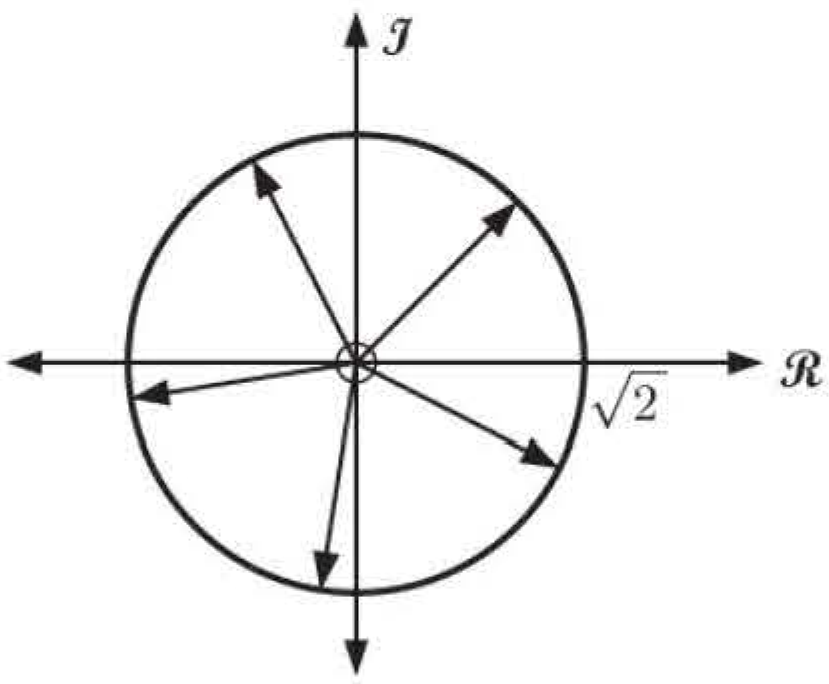
61 $|1+i| = \sqrt{2}$

$\therefore$ the five fifth roots of $a+bi$ are equally spaced around a circle centre O with radius $\sqrt{2}$.

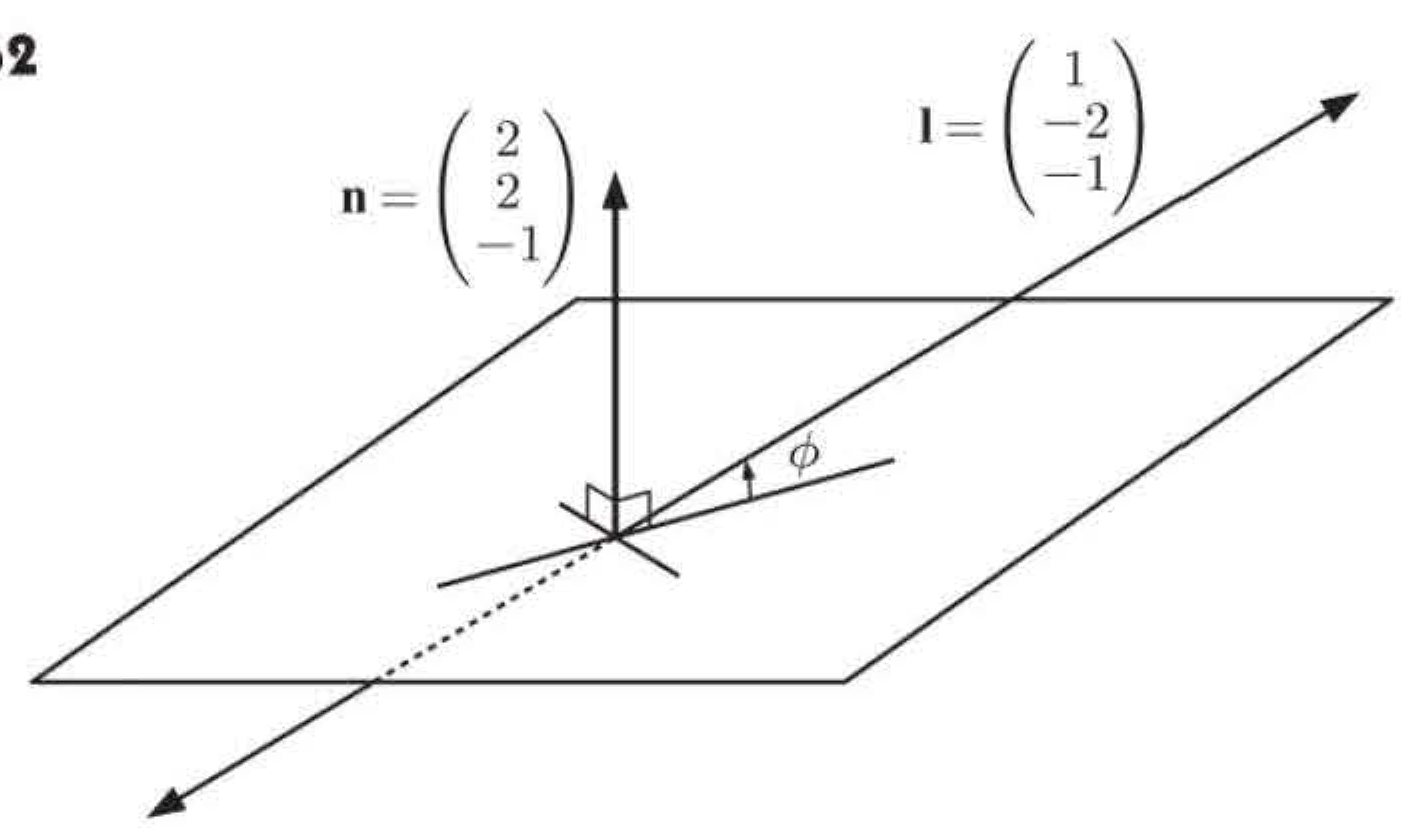
Now $\arg(1+i) = \frac{\pi}{4}$

$$\begin{aligned}\therefore\ \text{the other arguments} &= \tfrac{\pi}{4} + k\,\tfrac{2\pi}{5}, \quad k \in \mathbb{Z}\\ &= \tfrac{5\pi}{20} + \tfrac{k8\pi}{20}, \quad k \in \mathbb{Z}\\ &= -\tfrac{19\pi}{20}, -\tfrac{11\pi}{20}, -\tfrac{3\pi}{20}, \tfrac{13\pi}{20}\\ &\quad \{\text{when } k = -3, -2, -1, 1\}\end{aligned}$$

$\therefore$ the other roots are $\sqrt{2}\operatorname{cis}\left(-\frac{19\pi}{20}\right)$, $\sqrt{2}\operatorname{cis}\left(-\frac{11\pi}{20}\right)$, $\sqrt{2}\operatorname{cis}\left(-\frac{3\pi}{20}\right)$, and $\sqrt{2}\operatorname{cis}\left(\frac{13\pi}{20}\right)$.

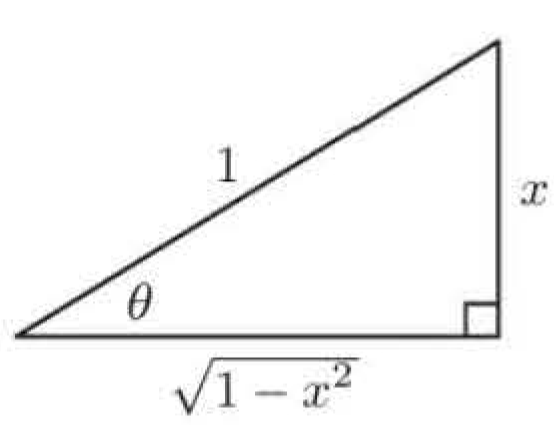


62



$$\begin{aligned}\phi &= \arcsin\left(\frac{|\mathbf{n} \bullet \mathbf{l}|}{|\mathbf{n}|\,|\mathbf{l}|}\right)\\ &= \arcsin\left(\frac{|2-4+1|}{\sqrt{4+4+1}\sqrt{1+4+1}}\right)\\ &= \arcsin\left(\frac{1}{3\sqrt{6}}\right)\\ &\approx 7.82^\circ\end{aligned}$$

63 **a** Let $\theta = \arcsin x$
$$\begin{aligned}&\therefore\ x = \sin\theta\\ &\therefore\ \sin(2\arcsin x)\\ &\quad = \sin 2\theta\\ &\quad = 2\sin\theta\cos\theta\\ &\quad = 2x\sqrt{1-x^2}\end{aligned}$$



b Let $u = 1-x^2$, $\dfrac{du}{dx} = -2x$
$$\begin{aligned}\therefore\ &\int \sin(2\arcsin x)\,dx\\ &= \int 2x\sqrt{1-x^2}\,dx\\ &= \int \sqrt{u}\left(-\frac{du}{dx}\right)dx\\ &= -\int u^{\frac{1}{2}}\,du\\ &= \frac{-u^{\frac{3}{2}}}{\frac{3}{2}} + c\\ &= -\tfrac{2}{3}(1-x^2)^{\frac{3}{2}} + c\end{aligned}$$

c
$$\begin{aligned}&\int_0^1 \sin(2\arcsin x)\,dx\\ &= \left[-\tfrac{2}{3}(1-x^2)^{\frac{3}{2}}\right]_0^1\\ &= (0) - \left(-\tfrac{2}{3}\right)\\ &= \tfrac{2}{3}\end{aligned}$$

64 **a** $x = 3 + a\lambda,\ y = -2 - \lambda,\ z = 2 + 2\lambda$

meets $\dfrac{x-4}{2} = 1 - y = \dfrac{z+2}{3}$ where

$$\frac{3 + a\lambda - 4}{2} = 1 + 2 + \lambda = \frac{2 + 2\lambda + 2}{3}$$

$$\therefore\ \frac{a\lambda - 1}{2} = \underbrace{\lambda + 3 = \frac{2\lambda + 4}{3}}$$

$$\therefore\ 3\lambda + 9 = 2\lambda + 4$$

$$\therefore\ \lambda = -5$$

Hence $\dfrac{-5a - 1}{2} = -2$

$\therefore\ 5a + 1 = 4$

$\therefore\ a = \frac{3}{5}$

and P is $\left(3 + \frac{3}{5}(-5),\ -2 + 5,\ 2 + 2(-5)\right)$

or $(0, 3, -8)$.

b l_1 has direction vector $\begin{pmatrix} 3 \\ -5 \\ 10 \end{pmatrix} = \mathbf{v}_1$

l_2 has direction vector $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \mathbf{v}_2$

$\left\{\text{as } \dfrac{x-4}{2} = \dfrac{y-1}{-1} = \dfrac{z+2}{3}\right\}$

Now $\cos\theta = \dfrac{|\mathbf{v}_1 \bullet \mathbf{v}_2|}{|\mathbf{v}_1|\,|\mathbf{v}_2|}$ {as θ is acute}

$$= \frac{|6 + 5 + 30|}{\sqrt{9 + 25 + 100}\sqrt{4 + 1 + 9}}$$

$$= \frac{41}{\sqrt{134}\sqrt{14}}$$

$\therefore\ \theta \approx 18.8^\circ$

c $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -5 & 10 \\ 2 & -1 & 3 \end{vmatrix}$$

$$= \begin{vmatrix} -5 & 10 \\ -1 & 3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & 10 \\ 2 & 3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & -5 \\ 2 & -1 \end{vmatrix}\mathbf{k}$$

$$= -5\mathbf{i} + 11\mathbf{j} + 7\mathbf{k}$$

$\therefore$ the plane has equation $5x - 11y - 7z = 5(3) - 11(-2) - 7(2)$

$\therefore\ 5x - 11y - 7z = 23$

65 **a** $s(t) = t^3 - 7t^2 + 10t + 14$ m

$\therefore\ v(t) = 3t^2 - 14t + 10$ m s^{-1}

The object is stationary when $v(t) = 0$

$\therefore\ 3t^2 - 14t + 10 = 0$

$\therefore\ t \approx 0.880$ s or 3.79 s {technology}

b Total distance travelled in first 5 seconds

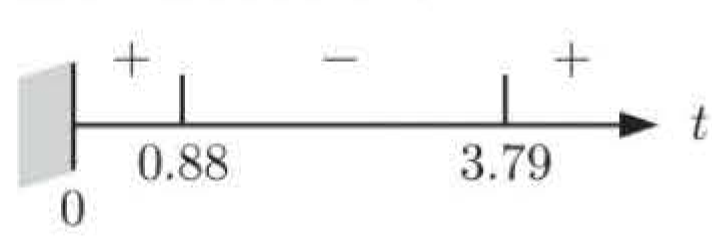
$= \int_0^5 \left|3t^2 - 14t + 10\right| dt$

≈ 24.5 m {technology}

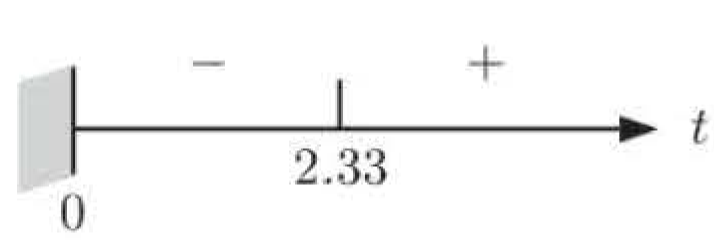
c **i** $a(t) = 6t - 14$ m s^{-2}

$\therefore$ acceleration is zero when $t \approx 2.33$ s

ii $v(t)$ sign diagram is



$a(t)$ sign diagram is

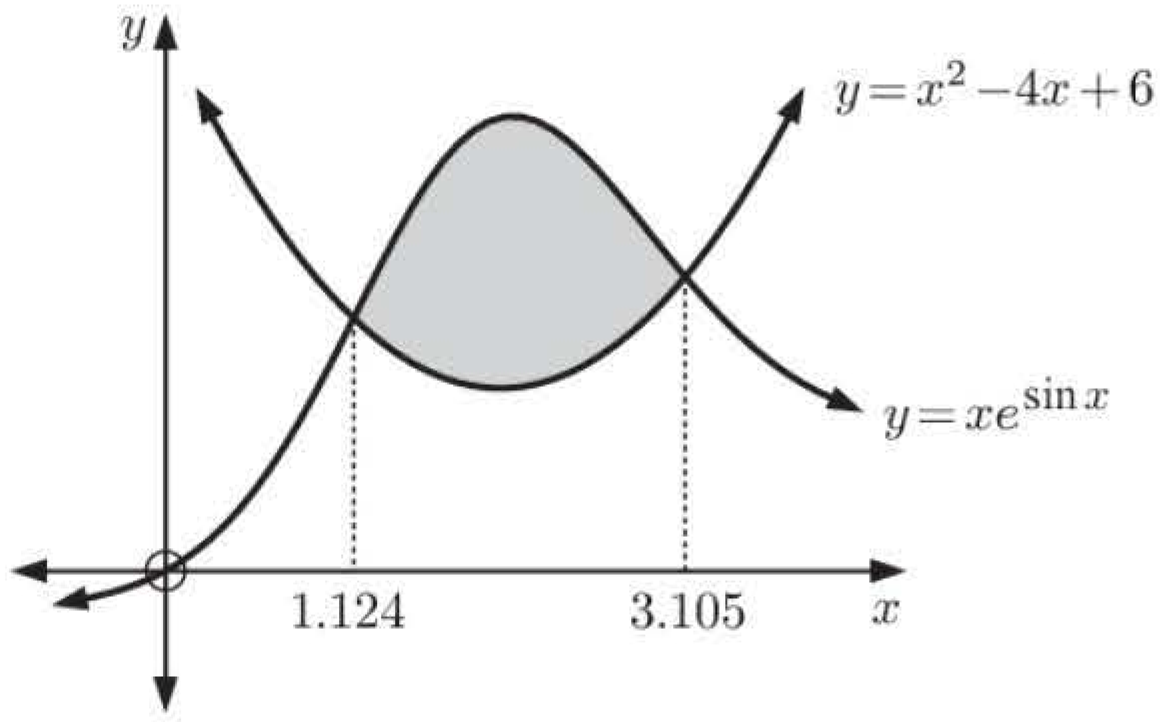


The change of sign about 2.33 on the $a(t)$ sign diagram indicates that velocity is at a minimum at this time.

To the left of 2.33, $v(t)$ and $a(t)$ are the same sign, indicating increasing speed, and to the right they are of opposite sign, indicating decreasing speed.

$\therefore$ speed is at a maximum at $t \approx 2.33$ s.

66



Using technology, the graphs intersect when $x \approx 1.124$ and 3.105

$\therefore$ area $\approx \int_{1.124}^{3.105} (xe^{\sin x} - x^2 + 4x - 6)\,dx$

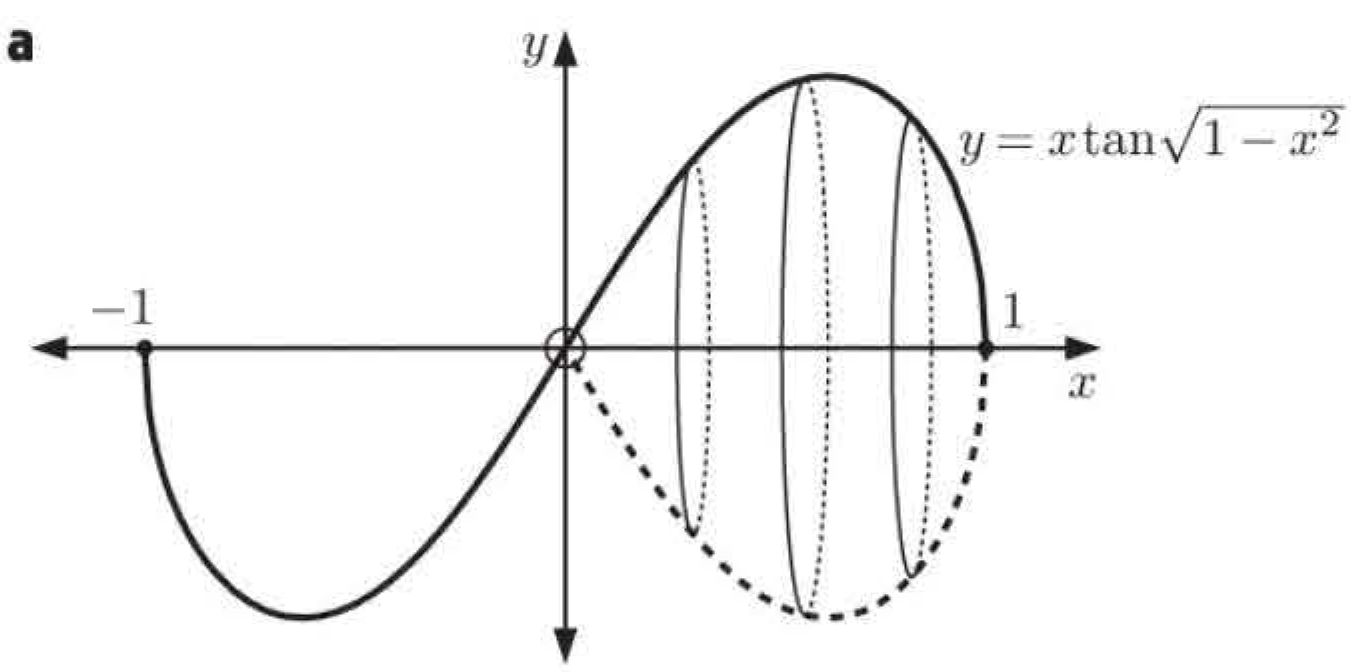
≈ 3.76 units^2 {using technology}

67 **a** As a, b, and c are real, all the coefficients of $P(z)$ are real.
Consequently $-3+2i$ and $-3-2i$ are both zeros.
They have sum -6 and product $9+4=13$.
$\therefore$ $z^2+6z+13$ is a factor of $P(z)$.

Thus $P(z) = (z+2)(z^2+6z+13)$
$= z^3+8z^2+25z+26$
$\therefore$ $a=8$, $b=25$, and $c=26$

b If $P(z) \geqslant 0$ then $(z+2)(z^2+6z+13) \geqslant 0$.
Now $z^2+6z+13>0$ for all z, since its roots are complex and it has shape 
$\therefore$ $P(z) \geqslant 0$ provided $z+2 \geqslant 0$
$\therefore$ $z \geqslant -2$

68 **a**



b $V = \pi \int_0^1 (x\tan\sqrt{1-x^2})^2\,dx$

≈ 0.676 units^3 {technology}

69 P_n is "$3(5^{2n+1}) + 2^{3n+1}$ is divisible by 17" for $n \in \mathbb{Z}^+$.

Proof: (By the principle of mathematical induction)

(1) If $n=1$, $3(5^3)+2^{3+1} = 391 = 17 \times 23$ where $23 \in \mathbb{Z}$ $\therefore$ P_1 is true.

(2) If P_k is assumed true, then $3(5^{2k+1}) + 2^{3k+1} = 17A$ for some $A \in \mathbb{Z}$

Thus $3(5^{2(k+1)+1}) + 2^{3(k+1)+1}$
$= 3(5^{2k+1+2}) + 2^{3k+1+3}$
$= 3 \times 25 \times 5^{2k+1} + 8 \times 2^{3k+1}$
$= 25\left(17A - 2^{3k+1}\right) + 8 \times 2^{3k+1}$ {by rearranging and substituting P_k}
$= 25 \times 17 \times A + (8-25)2^{3k+1}$
$= 25 \times 17 \times A - 17 \times 2^{3k+1}$
$= 17\left(25A - 2^{3k+1}\right)$ where $25A - 2^{3k+1} \in \mathbb{Z}$

Thus P_{k+1} is true whenever P_k is true.

$\therefore$ since P_1 is true, P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

70 **a** If $x = a^{\frac{1}{3}} + b^{\frac{1}{3}}$ then $x^3 = (a^{\frac{1}{3}} + b^{\frac{1}{3}})^3$

$$\therefore\ x^3 = (a^{\frac{1}{3}})^3 + 3(a^{\frac{1}{3}})^2 b^{\frac{1}{3}} + 3(a^{\frac{1}{3}})(b^{\frac{1}{3}})^2 + (b^{\frac{1}{3}})^3$$

$$\therefore\ x^3 = a + b + 3a^{\frac{1}{3}}b^{\frac{1}{3}}(a^{\frac{1}{3}} + b^{\frac{1}{3}})$$

$$\therefore\ x^3 = 3(ab)^{\frac{1}{3}}x + (a+b)$$

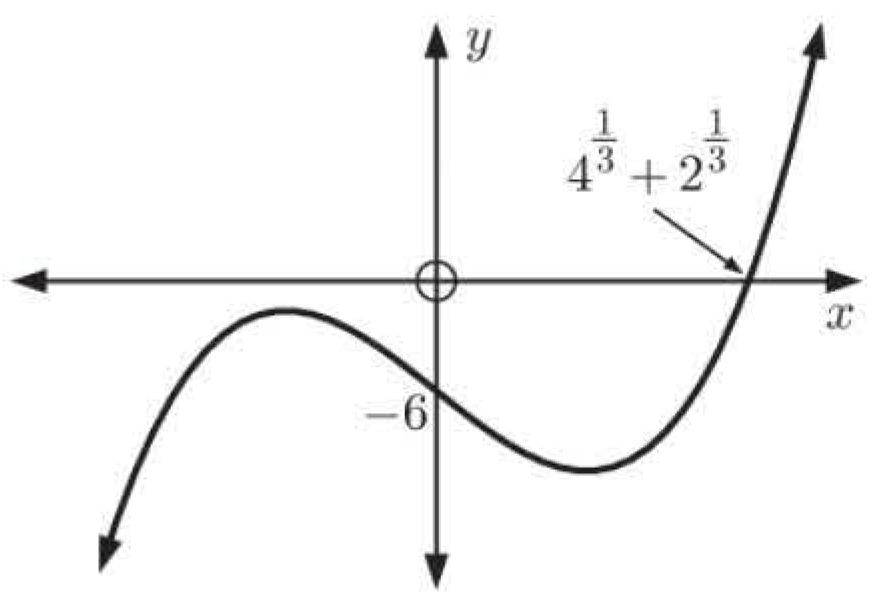
b $x^3 = 6x + 6$ is of the above form where $(ab)^{\frac{1}{3}} = 2$ and $a + b = 6$

$\therefore\ ab = 8$ and $a + b = 6$

$\therefore\ a = 2,\ b = 4$ or $a = 4,\ b = 2$

Thus $x = 4^{\frac{1}{3}} + 2^{\frac{1}{3}}$ is a root of $x^3 = 6x + 6$

The graph of $f(x) = x^3 - 6x - 6$ is:



As a cubic has at most two turning points, it is clear that $x = \sqrt[3]{4} + \sqrt[3]{2}$ is the only real zero of $f(x) = x^3 - 6x - 6$, and hence is the only real solution of $x^3 = 6x + 6$.

71 $y = 3\,\dfrac{dy}{dx}$

$\therefore\ \dfrac{dy}{dx} = \dfrac{y}{3}$

$\therefore\ \dfrac{dx}{dy} = \dfrac{3}{y}$ $\quad\left\{\dfrac{dy}{dx} \times \dfrac{dx}{dy} = 1\right\}$

$\therefore\ x = \displaystyle\int \frac{3}{y}\,dy$

$\therefore\ x = 3\ln|y| + c$

$\therefore\ x = 3\ln y + c$ $\quad\{$as $y > 0$, given$\}$

Now when $x = 0,\ y = 3$

$\therefore\ 0 = 3\ln 3 + c$

$\therefore\ c = -3\ln 3$

Hence, $x = 3\ln y - 3\ln 3$

$\therefore\ \dfrac{x}{3} = \ln y - \ln 3 = \ln\left(\dfrac{y}{3}\right)$

$\therefore\ \dfrac{y}{3} = e^{\frac{x}{3}}$

$\therefore\ y = 3e^{\frac{x}{3}}$

72 **a** Let u_{n+1} be the amount still owing after n quarters and let R be the repayment each quarter. Each quarter, interest is charged at $\frac{12\%}{4} = 3\%$, so $u_{n+1} = 1.03u_n - R$.

From **2**, $u_{n+1} = (1.03)^n u_1 - R\left(\dfrac{1-(1.03)^n}{1-1.03}\right)$

We want $u_{n+1} = 0$, $u_1 = 20\,000$, and $n = 40$

Thus $(1.03)^{40} \times 20\,000 + \dfrac{100R}{3}\left(1 - (1.03)^{40}\right) = 0$

$\therefore\ 65\,240.756 \approx R \times 75.401\,26$

$\therefore\ R \approx 865.25$

So, repayments of \$865.25 each quarter are required.

b From **a**, $0 = \left(1+\dfrac{r}{100m}\right)^{mn} P - R\left(\dfrac{1-\left(1+\dfrac{r}{100m}\right)^{mn}}{-\dfrac{r}{100m}}\right)$

$$\therefore\ R\left(\frac{1-\left(1+\frac{r}{100m}\right)^{mn}}{-\frac{r}{100m}}\right) = P\left(1+\frac{r}{100m}\right)^{mn}$$

$$\therefore\ R = \frac{P\left(1+\frac{r}{100m}\right)^{mn}\left(-\frac{r}{100m}\right)}{1-\left(1+\frac{r}{100m}\right)^{mn}}$$

$$\therefore\ R = \frac{P\left(\frac{r}{100m}\right)\left(1+\frac{r}{100m}\right)^{mn}}{\left(1+\frac{r}{100m}\right)^{mn}-1}$$

73 **a** We select the captain first and then the other 10 from the remaining 21.

This can be done in $\binom{11}{1}\binom{21}{10} = 3\,879\,876$ ways.

b This identity can be established using the general solution of **a**, using n instead of 11. The number of choices is therefore $\binom{n}{1}\binom{2n-1}{n-1}$ .... (1).

We can do this count in a different way.

Suppose we select i members from A and $n-i$ from B. There are i ways of choosing the captain from A.

So, we have $i \times \binom{n}{i}\binom{n}{n-i}$ selections where $i = 1, 2, 3, 4, ..., n$

$= i\binom{n}{i}^2$ {as $\binom{n}{i} = \binom{n}{n-i}$ by Pascal's rule}

Thus the total number of ways is $1\binom{n}{1}^2 + 2\binom{n}{2}^2 + 3\binom{n}{3}^2 + ... + n\binom{n}{n}^2$ .... (2)

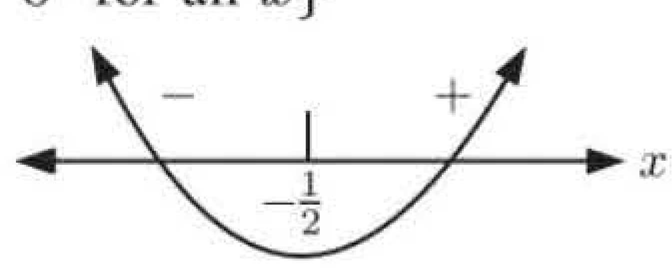
From (1) and (2), $1\binom{n}{1}^2 + 2\binom{n}{2}^2 + 3\binom{n}{3}^2 + ... + n\binom{n}{n}^2 = n\binom{2n-1}{n-1}$.

74 **a** $y = xe^{2x}$

$$\therefore\ \frac{dy}{dx} = (1)e^{2x} + x(2e^{2x})$$
$$= e^{2x}(1+2x)$$

which is 0 when $x = -\frac{1}{2}$ $\{e^{2x} > 0$ for all $x\}$

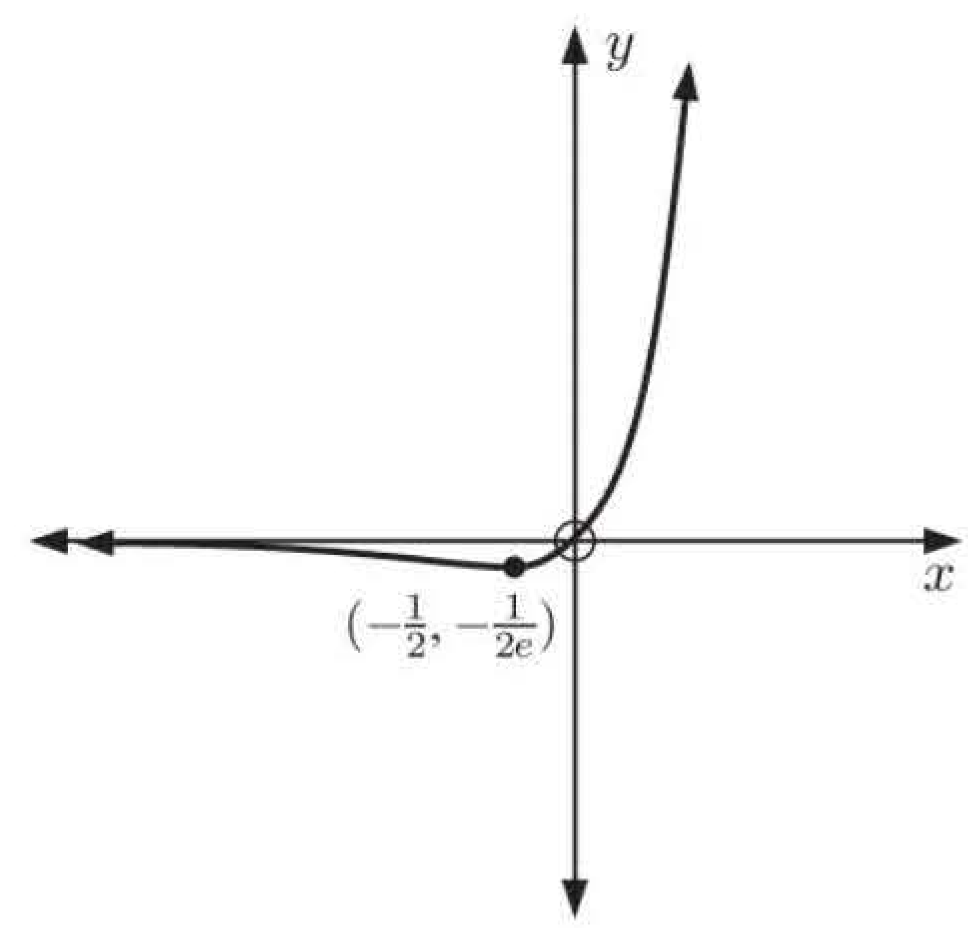
and the sign diagram of $\dfrac{dy}{dx}$ is:



$\therefore$ a local minimum occurs at $\left(-\frac{1}{2}, -\frac{1}{2e}\right)$

$\therefore$ $k = -\frac{1}{2e}$

b The graph of $y = xe^{2x}$ is shown alongside.



i When $k = -\frac{1}{2e}$ or $k \geqslant 0$, $y = k$ cuts the curve once only.

ii When $-\frac{1}{2e} < k < 0$, $y = k$ cuts the curve twice.

iii When $k < -\frac{1}{2e}$, $y = k$ never cuts the graph.

c **i** When $y = xe^{ax}$, $\dfrac{dy}{dx} = e^{ax}(ax+1)$

$y = x$ meets $y = xe^{ax}$ where $x = xe^{ax}$

$$\therefore \ x(1 - e^{ax}) = 0$$
$$\therefore \ x = 0, \quad a \in \mathbb{R}$$

When $x = 0$, $y = 0$ and $\dfrac{dy}{dx} = 1$, which is the gradient of $y = x$.

$\therefore$ $y = x$ is a tangent to $y = xe^{ax}$, and the point of contact is $(0, 0)$.

ii The normal at $(0, 0)$ has slope -1, and equation $y = -x$

$\therefore$ it makes an angle of $45°$ to the x-axis.

75 A is the event of a family having at most one boy in n children.
B is the event of a family having every child the same sex in n children.

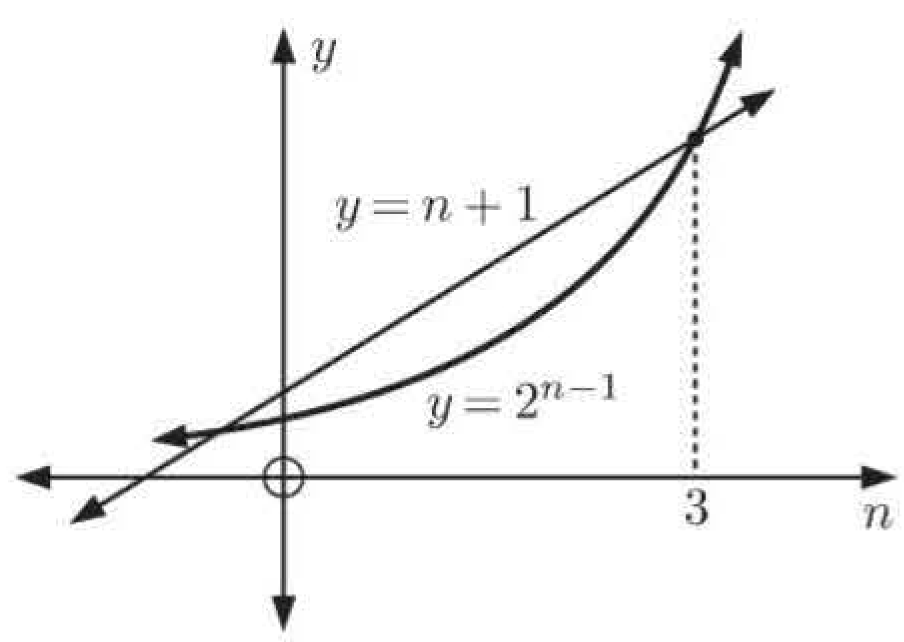
$$\begin{aligned} \text{P}(A) &= \text{P}(0\text{B and } n\text{G or 1B and } (n-1)\text{G}) \\ &= \tbinom{n}{0}\left(\tfrac{1}{2}\right)^n + \tbinom{n}{1}\left(\tfrac{1}{2}\right)^1\left(\tfrac{1}{2}\right)^{n-1} \\ &= \left(\tfrac{1}{2}\right)^n + n\left(\tfrac{1}{2}\right)^n \\ &= \left(\tfrac{1}{2}\right)^n(n+1) \end{aligned}$$

$$\begin{aligned} \text{P}(B) &= \text{P}(0\text{B and } n\text{G or } n\text{B and 0G}) \\ &= \tbinom{n}{0}\left(\tfrac{1}{2}\right)^n + \tbinom{n}{n}\left(\tfrac{1}{2}\right)^n \\ &= 2\left(\tfrac{1}{2}\right)^n \end{aligned}$$

$$\text{P}(A \cap B) = \text{P}(0\text{B and } n\text{G}) = \left(\tfrac{1}{2}\right)^n$$

A and B are independent when

$$\begin{aligned} \text{P}(A \cap B) &= \text{P}(A)\,\text{P}(B) \\ \therefore \ \left(\tfrac{1}{2}\right)^n &= \left(\tfrac{1}{2}\right)^n(n+1) \times 2\left(\tfrac{1}{2}\right)^n \\ \therefore \ 1 &= 2(n+1)\left(\tfrac{1}{2}\right)^n \\ \therefore \ 2^n &= 2(n+1) \\ \therefore \ 2^{n-1} &= n+1 \\ \therefore \ n &= 3 \quad \{n > 1\} \end{aligned}$$



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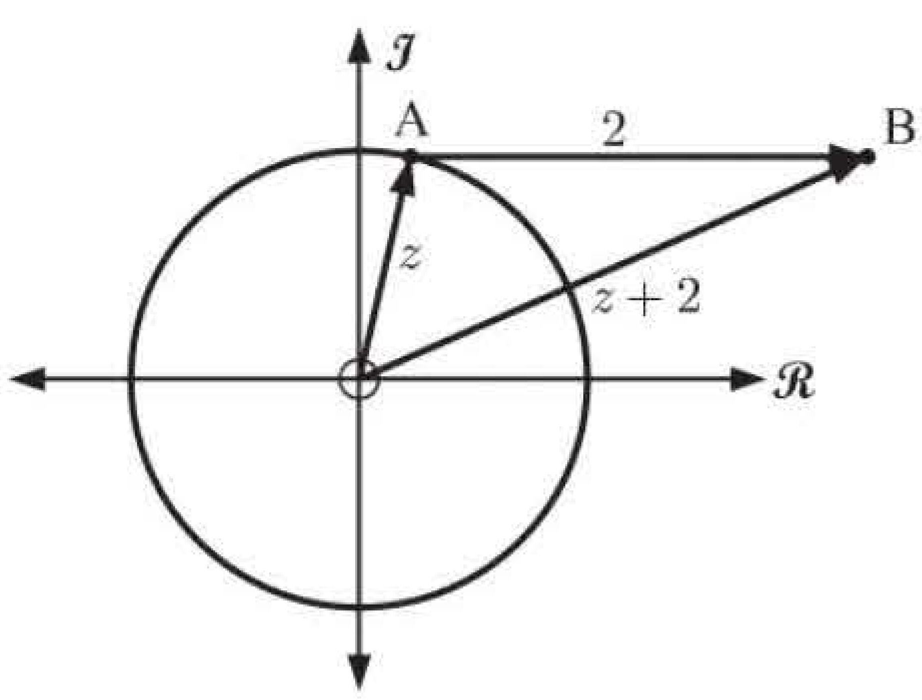
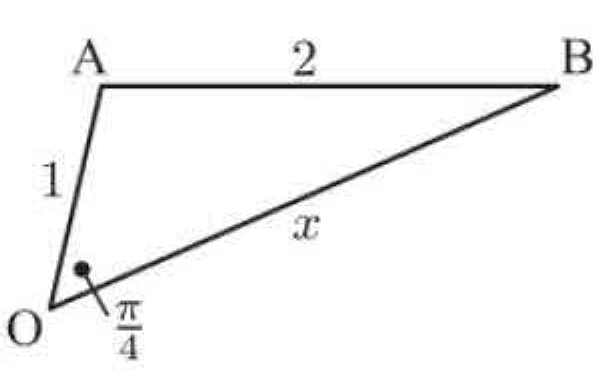




As $|z| = 1$ and $\arg z \in [0, \frac{\pi}{2}]$, z lies on the unit circle in the first quadrant.

$z + 2$ is shown using vector addition.

$$\text{Now} \quad \arg\left(\frac{z}{z+2}\right) = \tfrac{\pi}{4}$$
$$\therefore \ \arg z - \arg(z+2) = \tfrac{\pi}{4}$$
$$\therefore \ \text{A}\widehat{\text{O}}\text{B} = \tfrac{\pi}{4}$$

Using the cosine rule in $\triangle$OAB gives

$$\begin{aligned} 2^2 &= x^2 + 1^2 - 2(x)(1)\cos\tfrac{\pi}{4} \\ \therefore \ 4 &= x^2 + 1 - \sqrt{2}x \\ \therefore \ x^2 - \sqrt{2}x - 3 &= 0 \\ \therefore \ x &\approx 2.58 \quad \{\text{technology, } x > 0\} \\ \therefore \ |z+2| &\approx 2.58 \end{aligned}$$

77 We assume that the typist would know that the coefficients $a = 1$ and $b = 1$ are excluded, so we assume $a \neq 1$ and $b \neq 1$.

Thus a and b are from $\{2, 3, 4, 5, 6, 7, 8, 9\}$, but c is from $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

$\therefore$ the total number of guesses is $8 \times 8 \times 9 = 576$.

For real roots, $b^2 - 4ac \geqslant 0$ and so $ac \leqslant \dfrac{b^2}{4}$.

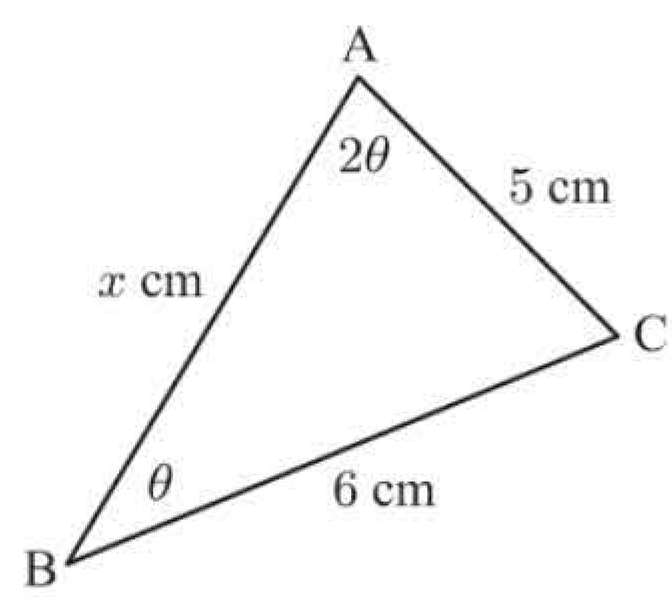
If $b = 2$, $ac \leqslant 1$	which is impossible.	0 solutions
If $b = 3$, $ac \leqslant 2\frac{1}{4}$	$\therefore$ $a = 2$, $c = 1$.	1 solution
If $b = 4$, $ac \leqslant 4$	$\therefore$ $a = 2$, $c = 1$ or 2; $a = 3$, $c = 1$; $a = 4$, $c = 1$.	4 solutions

The number of solutions is best shown in a table:

b values	max. ac	a values: 2	3	4	5	6	7	8	9	Total
2	1	0	0	0	0	0	0	0	0	0
3	$2\frac{1}{4}$	1	0	0	0	0	0	0	0	1
4	4	2	1	1	0	0	0	0	0	4
5	$6\frac{1}{4}$	3	2	1	1	1	0	0	0	8
6	9	4	3	2	1	1	1	1	1	14
7	$12\frac{1}{4}$	6	4	3	2	2	1	1	1	20
8	16	8	5	4	3	2	2	2	1	27
9	$20\frac{1}{4}$	9	6	5	4	3	2	2	2	33
										107

$\therefore$ P(real roots) $= \frac{107}{576}$

78 **a**



Let the angle at B be θ and at A be 2θ

By the sine rule, $\dfrac{\sin 2\theta}{6} = \dfrac{\sin\theta}{5}$

$$\therefore \frac{2\sin\theta\cos\theta}{\sin\theta} = \frac{6}{5}$$

$$\therefore \cos\theta = \tfrac{3}{5} \;....\; (*) \quad \{\text{as } \sin\theta \neq 0\}$$

b Let $\text{AB} = x$ cm.

Using the cosine rule,

$$5^2 = x^2 + 6^2 - 2x(6)\cos\theta$$

$$\therefore 25 = x^2 + 36 - 12x\left(\tfrac{3}{5}\right)$$

$$\therefore x^2 - \tfrac{36}{5}x + 11 = 0$$

$$5x^2 - 36x + 55 = 0$$

$$\therefore (x-5)(5x-11) = 0$$

$$\therefore x = 5 \text{ or } \tfrac{11}{5}$$

$\therefore$ AB $= 5$ cm or 2.2 cm

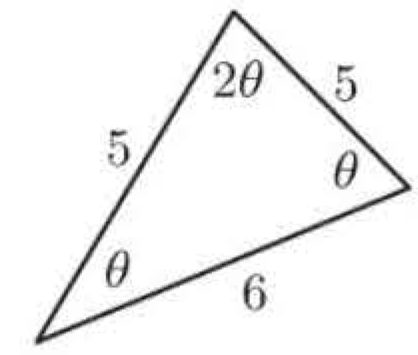
Check: If AB $= 5$ we have an isosceles triangle

$\therefore 4\theta = 180°$

$\therefore \theta = 45°$

which contradicts $(*)$

as $\cos 45° = \frac{1}{\sqrt{2}} \neq \frac{3}{5}$.

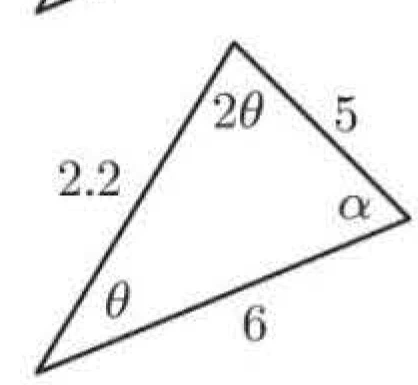


If AB $= 2.2$ we have

$\theta \approx 53.1°$

$\therefore 2\theta \approx 106.3°$

and $\alpha \approx 20.6°$



$\therefore$ AB $= 2.2$ is the only valid solution.

79 P_n is "$\dfrac{1}{\sin 2x} + \dfrac{1}{\sin 4x} + \dfrac{1}{\sin 8x} + ... + \dfrac{1}{\sin(2^n x)} = \cot x - \cot(2^n x)$" for $n \in \mathbb{Z}^+$.

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, LHS $= \dfrac{1}{\sin 2x}$ and RHS $= \cot x - \cot 2x$

$$= \frac{\cos x}{\sin x} - \frac{\cos 2x}{\sin 2x}$$

$$= \frac{\cos x}{\sin x}\left(\frac{2\cos x}{2\cos x}\right) - \frac{\cos 2x}{\sin 2x}$$

$$= \frac{2\cos^2 x - \cos 2x}{\sin 2x}$$

$$= \frac{2\cos^2 x - \left[2\cos^2 x - 1\right]}{\sin 2x}$$

$$= \frac{1}{\sin 2x}$$

$\therefore P_1$ is true.

(2) If P_k is true, $\dfrac{1}{\sin 2x}+\dfrac{1}{\sin 4x}+....+\dfrac{1}{\sin(2^k x)}=\cot x-\cot(2^k x)$

$$\therefore \quad \frac{1}{\sin 2x}+\frac{1}{\sin 4x}+....+\frac{1}{\sin(2^k x)}+\frac{1}{\sin(2^{k+1}x)}$$
$$=\cot x-\cot(2^k x)+\frac{1}{\sin(2^{k+1}x)}$$
$$=\cot x+\frac{1}{\sin(2^{k+1}x)}-\frac{\cos(2^k x)}{\sin(2^k x)}\left(\frac{2\cos(2^k x)}{2\cos(2^k x)}\right)$$
$$=\cot x+\frac{1}{\sin(2^{k+1}x)}-\frac{2\cos^2(2^k x)}{\sin(2^{k+1}x)}$$
$$=\cot x+\frac{1-2\cos^2(2^k x)}{\sin(2^{k+1}x)}$$
$$=\cot x+\frac{-\cos(2^{k+1}x)}{\sin(2^{k+1}x)} \qquad \{\cos 2\theta=2\cos^2\theta-1\}$$
$$=\cot x-\cot(2^{k+1}x)$$

Thus P_{k+1} is true whenever P_k is true.

$\therefore$ since P_1 is true, P_n is true for all $n\in\mathbb{Z}^+$ {Principle of mathematical induction}

80 **a** $I_0=\displaystyle\int_0^{\frac{\pi}{2}}\cos x\,dx$
$$=\Big[\sin x\Big]_0^{\frac{\pi}{2}}$$
$$=1-0$$
$$=1$$

b $I_1=\int x\cos x\,dx \qquad \left\{\begin{array}{ll} u'=\cos x & v=x \\ u=\sin x & v'=1\end{array}\right.$
$$=x\sin x-\int\sin x\,dx$$
$$=x\sin x+\cos x+c$$
$$\therefore \quad \int_0^{\pi} x\cos x\,dx$$
$$=\tfrac{\pi}{2}\sin(\tfrac{\pi}{2})+\cos(\tfrac{\pi}{2})-(0+\cos 0)$$
$$=\tfrac{\pi}{2}-1$$

c $\int x^n\cos x\,dx \qquad \left\{\begin{array}{ll} u'=\cos x & v=x^n \\ u=\sin x & v'=nx^{n-1}\end{array}\right.$

$$=x^n\sin x-n\int x^{n-1}\sin x\,dx \qquad \left\{\begin{array}{l} u'=\sin x \\ u=-\cos x \\ v=x^{n-1} \\ v'=(n-1)x^{n-2}\end{array}\right.$$
$$=x^n\sin x-n\left[-x^{n-1}\cos x+(n-1)\int x^{n-2}\cos x\,dx\right]$$
$$=x^n\sin x+nx^{n-1}\cos x-n(n-1)\int x^{n-2}\cos x\,dx$$

$$\therefore \quad I_n=[x^n\sin x]_0^{\frac{\pi}{2}}+n\left[x^{n-1}\cos x\right]_0^{\frac{\pi}{2}}-n(n-1)I_{n-2}$$
$$=(\tfrac{\pi}{2})^n\sin(\tfrac{\pi}{2})-0+n(0-0)-n(n-1)I_{n-2}$$
$$=(\tfrac{\pi}{2})^n-n(n-1)I_{n-2}$$

d $\displaystyle\int_0^{\frac{\pi}{2}}x^3\cos x\,dx=I_3$
$$=(\tfrac{\pi}{2})^3-(3)(2)I_1$$
$$=(\tfrac{\pi}{2})^3-6(\tfrac{\pi}{2}-1)$$
$$=(\tfrac{\pi}{2})^3-3\pi+6$$

81 **a** If $z = \text{cis}\,\theta$, then $z^n + \dfrac{1}{z^n} = 2\cos n\theta$, $n \in \mathbb{Z}^+$ {see **Ex 16E 12 a i**}

So, with $n = 1$, $z + \dfrac{1}{z} = 2\cos\theta$

$$\therefore \left(z + \frac{1}{z}\right)^3 = 8\cos^3\theta$$

$$\therefore z^3 + 3z + \frac{3}{z} + \frac{1}{z^3} = 8\cos^3\theta$$

$$\begin{aligned}\therefore \cos^3\theta &= \tfrac{1}{8}\left[z^3 + \frac{1}{z^3} + 3\left(z + \frac{1}{z}\right)\right]\\ &= \tfrac{1}{8}\left[2\cos 3\theta + 6\cos\theta\right]\\ &= \tfrac{3}{4}\cos\theta + \tfrac{1}{4}\cos 3\theta\end{aligned}$$

b Letting $x = \frac{1}{m}\cos\theta$, we have

$$\frac{\cos^3\theta}{m^3} - \frac{3\cos\theta}{m} + 1 = 0$$

$$\therefore \cos^3\theta - 3m^2\cos\theta + m^3 = 0$$

$$\therefore \tfrac{3}{4}\cos\theta + \tfrac{1}{4}\cos 3\theta - 3m^2\cos\theta + m^3 = 0$$

$$\therefore \tfrac{1}{4}\cos 3\theta + (\tfrac{3}{4} - 3m^2)\cos\theta + m^3 = 0$$

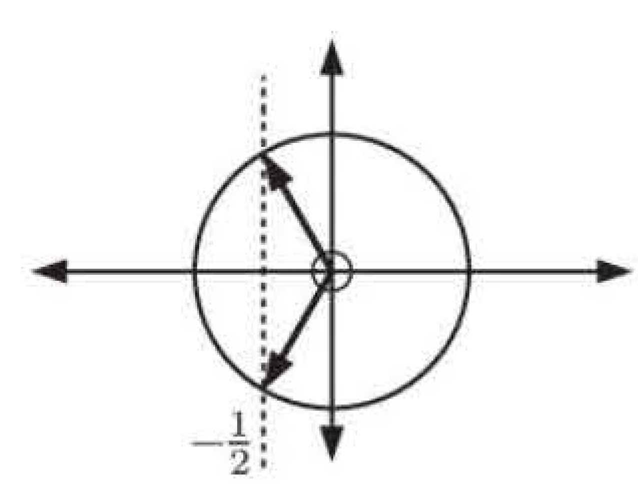
which when choosing $\underline{m = \frac{1}{2}}$ becomes

$\frac{1}{4}\cos 3\theta = -\frac{1}{8}$

$\therefore \cos 3\theta = -\frac{1}{2}$

$\therefore 3\theta = \frac{2\pi}{3} + k2\pi$ or $\frac{-2\pi}{3} + k2\pi$

$\therefore \theta = \frac{2\pi}{9} + \frac{k6\pi}{9}$ or $\frac{-2\pi}{9} + k\frac{6\pi}{9}$



Now $x = \dfrac{\cos\theta}{\frac{1}{2}} = 2\cos\theta$

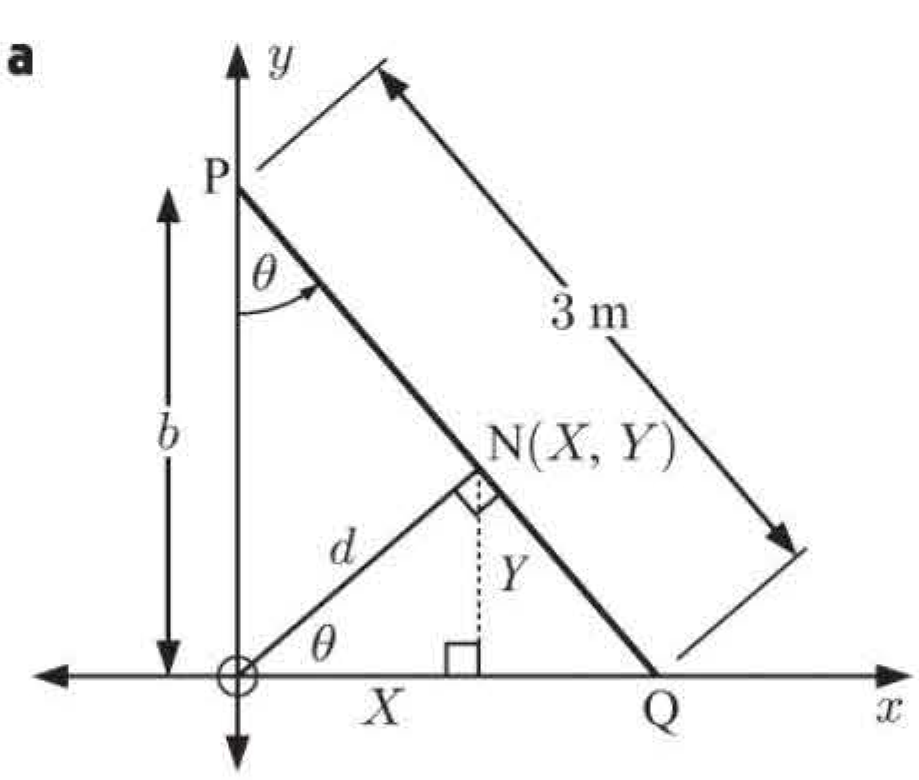
$\therefore x = 2\cos\left(\frac{2\pi}{9}\right),\ 2\cos\left(\frac{8\pi}{9}\right),\ 2\cos\left(\frac{14\pi}{9}\right)$

$\approx 1.53,\ -1.88,\ 0.347$

As these are all different, they are the required roots.

(Any solution of 3 consecutive values of θ will generate these roots when decimalised.)

82 **a**



As $\widehat{OPQ} = \theta$, $\widehat{PON} = 90° - \theta$ and so $\widehat{NOQ} = \theta$

So, $\cos\theta = \dfrac{X}{d}$ and $\sin\theta = \dfrac{Y}{d}$

$\therefore X = d\cos\theta$ and $Y = d\sin\theta$

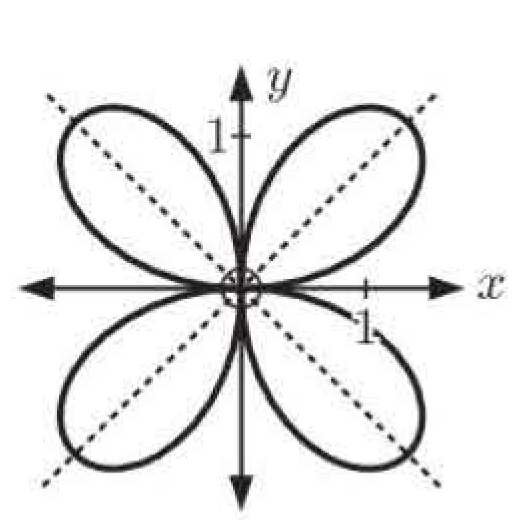
But $\sin\theta = \dfrac{d}{b}$ and $\cos\theta = \dfrac{b}{3}$

$$\begin{aligned}\therefore d &= b\sin\theta\\ &= (3\cos\theta)\sin\theta\end{aligned}$$

Thus $X = 3\sin\theta\cos^2\theta$ and $Y = 3\sin^2\theta\cos\theta$

$\therefore$ N is at $(3\sin\theta\cos^2\theta,\ 3\sin^2\theta\cos\theta)$.

b



a 'four leaf clover'

83 **a** u_{m+1}, u_{n+1}, and u_{p+1} are consecutive terms of a geometric sequence.

$$\therefore \quad \frac{u_{n+1}}{u_{m+1}} = \frac{u_{p+1}}{u_{n+1}}$$

$$\therefore \quad \frac{u_1+nd}{u_1+md} = \frac{u_1+pd}{u_1+nd}$$

$$\therefore \quad (u_1+nd)^2 = (u_1+md)(u_1+pd)$$

$$\therefore \quad \cancel{u_1^2} + 2ndu_1 + n^2d^2 = \cancel{u_1^2} + mdu_1 + pdu_1 + mpd^2$$

Dividing each term by d^2 gives

$$\frac{2nu_1}{d} + n^2 = \frac{mu_1}{d} + \frac{pu_1}{d} + mp$$

$$\therefore \quad \frac{u_1}{d}(2n-m-p) = mp - n^2$$

$$\therefore \quad \frac{d}{u_1} = \frac{2n-m-p}{mp-n^2}$$

b If $\dfrac{2mp}{m+p} = n$,

then $m+p = \dfrac{2mp}{n}$

$$\therefore \quad \frac{d}{u_1} = \frac{2n - \frac{2mp}{n}}{mp-n^2}\left(\frac{n}{n}\right)$$

$$= \frac{2n^2-2mp}{n(mp-n^2)}$$

$$= \frac{2\cancel{(n^2-mp)}}{-n\cancel{(n^2-mp)}}$$

$$= -\frac{2}{n} \quad \text{provided } n^2 \neq mp$$

< 0 for all values as $n > 0$

84 **a** L_1 can be written in the form $x = 8+3\lambda$, $y = -13-5\lambda$, $z = -3-2\lambda$

Substituting into L_2 gives: $\dfrac{8+3\lambda+10}{6} = \dfrac{-13-5\lambda-7}{-5} = \dfrac{-3-2\lambda-11}{-5}$

$$\therefore \quad \frac{3\lambda+18}{6} = \frac{-20-5\lambda}{-5} = \frac{-2\lambda-14}{-5}$$

$$\therefore \quad \frac{\lambda+6}{2} = \lambda+4 = \frac{2\lambda+14}{5}$$

$\therefore \quad \lambda+6 = 2\lambda+8$ and $5\lambda+20 = 2\lambda+14$

$\therefore \quad \lambda = -2$ and $3\lambda = -6$

So, $\lambda = -2$ is a common solution.

Substituting $\lambda = -2$ gives $x = 2$, $y = -3$, $z = 1$, so they meet at A$(2, -3, 1)$.

b L_1 meets $3x+2y-z=-2$ where $3(8+3\lambda) + 2(-13-5\lambda) - (-3-2\lambda) = -2$

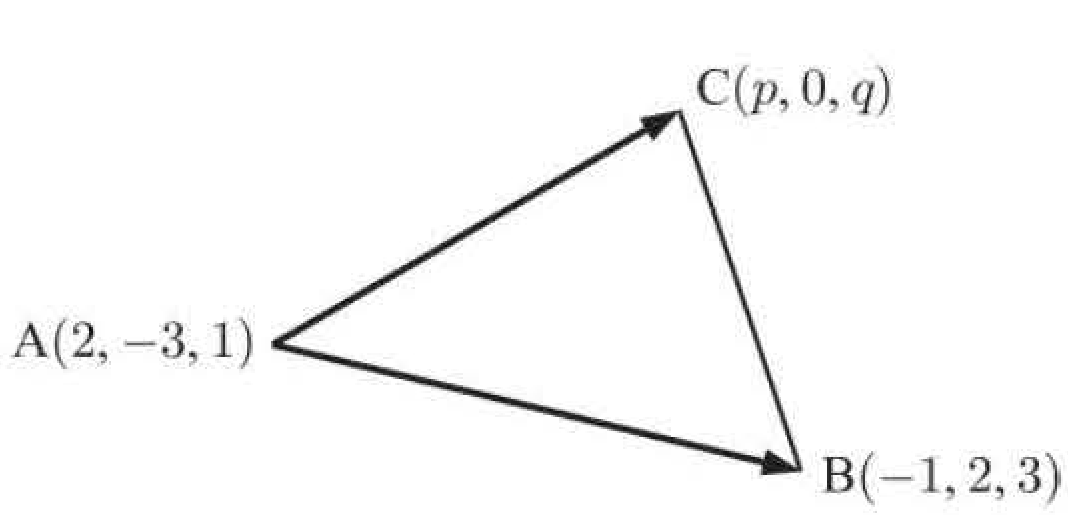
$$\therefore \quad 24 + 9\lambda - 26 - 10\lambda + 3 + 2\lambda = -2$$

$$\therefore \quad \lambda + 1 = -2$$

$$\therefore \quad \lambda = -3$$

So, B is at $(-1, 2, 3)$.

c



$$\overrightarrow{AB} = \begin{pmatrix} -3 \\ 5 \\ 2 \end{pmatrix}, \quad \overrightarrow{AC} = \begin{pmatrix} p-2 \\ 3 \\ q-1 \end{pmatrix}$$

$$\therefore \quad \text{area } \triangle ABC = \tfrac{1}{2}\,|\,\overrightarrow{AB} \times \overrightarrow{AC}\,|$$

$$= \tfrac{1}{2}\left\| \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 5 & 2 \\ p-2 & 3 & q-1 \end{vmatrix} \right\| = \tfrac{\sqrt{3}}{2}$$

But C lies on $3x+2y-z=-2$

$$\therefore \quad 3p - q = -2$$

$$\therefore \quad q = 3p+2$$

$$\text{Thus} \quad \left\| \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 5 & 2 \\ p-2 & 3 & 3p+1 \end{vmatrix} \right\| = \sqrt{3}$$

$$\therefore\ |(15p-1)\mathbf{i}-(1-11p)\mathbf{j}+(1-5p)\mathbf{k}|=\sqrt{3}$$
$$\therefore\ \sqrt{(15p-1)^2+(1-11p)^2+(1-5p)^2}=\sqrt{3}$$
$$\therefore\ 225p^2-30p+1+1-22p+121p^2+1-10p+25p^2=3$$
$$\therefore\ 371p^2-62p=0$$
$$\therefore\ p(371p-62)=0$$
$$\therefore\ p=0 \text{ or } \tfrac{62}{371}$$

85 **a** Suppose $e^x=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+....$ .... (1)

By successive differentiation of both sides, we find:

$e^x=a_1+2a_2x+3a_3x^2+4a_4x^3+5a_5x^4+....$ .... (2)

and $e^x=2a_2+6a_3x+12a_4x^2+20a_5x^3+30a_6x^4+....$ .... (3)

and $e^x=6a_3+24a_4x+60a_5x^2+120a_6x^3+210a_7x^4+....$ .... (4)

and $e^x=24a_4+120a_5x+360a_6x^2+....$ .... (5)

Letting $x=0$ in (1) to (5) gives $1=a_0,\ 1=a_1,\ 1=2a_2,\ 1=6a_3,\ 1=24a_4,\ ....$

Thus $a_0=1,\ a_1=1,\ a_2=\frac{1}{2},\ a_3=\frac{1}{6},\ a_4=\frac{1}{24},\ ....$

$$\therefore\ a_0=\frac{1}{0!},\ a_1=\frac{1}{1!},\ a_2=\frac{1}{2!},\ a_3=\frac{1}{3!},\ a_4=\frac{1}{4!},\ ....$$

b *Conjecture*: $e^x=\sum\limits_{n=0}^{\infty}\dfrac{x^n}{n!}$

c When $x=1,\ e=\sum\limits_{n=0}^{\infty}\dfrac{1}{n!}=1+1+\dfrac{1}{2!}+\dfrac{1}{3!}+\dfrac{1}{4!}+\dfrac{1}{5!}+....$

$\approx 2.718\,281\,828....$ (adding the first 13 terms)

which checks with the calculator result.

86
$$|\mathbf{a}-\mathbf{b}|^2=(\mathbf{a}-\mathbf{b})\bullet(\mathbf{a}-\mathbf{b})$$
$$\therefore\ |\mathbf{a}-\mathbf{b}|^2=\mathbf{a}\bullet\mathbf{a}-2\mathbf{a}\bullet\mathbf{b}+\mathbf{b}\bullet\mathbf{b}$$
$$\therefore\ |\mathbf{a}-\mathbf{b}|^2=|\mathbf{a}|^2-2\mathbf{a}\bullet\mathbf{b}+|\mathbf{b}|^2$$
$$\therefore\ 400=49-2\mathbf{a}\bullet\mathbf{b}+256$$
$$\therefore\ 2\mathbf{a}\bullet\mathbf{b}=-95$$

But $|\mathbf{a}+\mathbf{b}|^2=|\mathbf{a}|^2+2\mathbf{a}\bullet\mathbf{b}+|\mathbf{b}|^2$ {same reasoning as above}

$=49-95+250$

$=210$

$$\therefore\ |\mathbf{a}+\mathbf{b}|=\sqrt{210}\approx 14.5$$

87 If k is the third root then $x^3+ax^2+bx+c=(x-\alpha)(x-\beta)(x-k)$

$=x^3-[\alpha+\beta+k]\,x^2+[\alpha\beta+\alpha k+\beta k]\,x-\alpha\beta k$

Equating coefficients: $a=-(\alpha+\beta+k)$ .... (1)

$b=\alpha\beta+\alpha k+\beta k$ .... (2)

$c=-\alpha\beta k$ and so $k=-\dfrac{c}{\alpha\beta}$ .... (3)

So, $(\alpha\beta)^3-b(\alpha\beta)^2+ac(\alpha\beta)-c^2$

$=(\alpha\beta)^3-[\alpha\beta+\alpha k+\beta k]\,(\alpha\beta)^2-\alpha\beta c(\alpha+\beta+k)-c^2$ {using (1) and (2)}

$=(\alpha\beta)^3-(\alpha\beta)^3-(\alpha+\beta)k(\alpha\beta)^2-\alpha\beta c(\alpha+\beta+k)-c^2$

$=-(\alpha+\beta)(-\dfrac{c}{\alpha\beta})(\alpha\beta)^2-\alpha\beta c(\alpha+\beta-\dfrac{c}{\alpha\beta})-c^2$ {using (3)}

$=-\cancel{(\alpha+\beta)(-c\alpha\beta)}-\cancel{\alpha\beta c(\alpha+\beta)}+\cancel{c}^2-\cancel{c}^2$

$=0$

$\therefore\ \alpha\beta$ is a root of $x^3-bx^2+acx-c^2=0.$

88 **a** It cuts the x-axis when $y = 0$

$$\therefore\ x^4 = x^2$$
$$\therefore\ x^2(x^2-1) = 0$$
$$\therefore\ x^2(x+1)(x-1) = 0$$
$$\therefore\ x = 0, \pm 1$$

$\therefore$ the x-intercepts are -1, 0, and 1.

It cuts the y-axis when $x = 0$

$$\therefore\ y^4 = -y^2$$
$$\therefore\ y^2(y^2+1) = 0$$
$$\therefore\ y = 0$$

$\therefore$ the y-intercept is 0.

b

$$(x^2+y^2)^2 = x^2 - y^2$$
$$\therefore\ 2(x^2+y^2)\left[2x + 2y\,\frac{dy}{dx}\right] = 2x - 2y\,\frac{dy}{dx}$$
$$\therefore\ 4x(x^2+y^2) + 4y(x^2+y^2)\,\frac{dy}{dx} = 2x - 2y\,\frac{dy}{dx}$$
$$\therefore\ \left[4y(x^2+y^2)+2y\right]\frac{dy}{dx} = 2x - 4x(x^2+y^2)$$
$$\therefore\ \frac{dy}{dx} = \frac{x - 2x(x^2+y^2)}{2y(x^2+y^2)+y}$$

which is $0 \Leftrightarrow x\left[1 - 2(x^2+y^2)\right] = 0$

But x is clearly not 0

$\therefore\ x^2+y^2 = \frac{1}{2}$ .... (1)

Thus $(\frac{1}{2})^2 = x^2 - y^2$

$\therefore\ x^2 - y^2 = \frac{1}{4}$ .... (2)

From (1) and (2), $2x^2 = \frac{3}{4}$

$$\therefore\ x = \pm\tfrac{\sqrt{3}}{2\sqrt{2}} = \pm\tfrac{\sqrt{6}}{4}$$

and $y^2 = \frac{1}{8}$

$$\therefore\ y = \pm\tfrac{1}{2\sqrt{2}} = \pm\tfrac{\sqrt{2}}{4}$$

$\therefore$ the four points are $\left(\frac{\sqrt{6}}{4}, \frac{\sqrt{2}}{4}\right)$, $\left(\frac{\sqrt{6}}{4}, -\frac{\sqrt{2}}{4}\right)$, $\left(-\frac{\sqrt{6}}{4}, \frac{\sqrt{2}}{4}\right)$, $\left(-\frac{\sqrt{6}}{4}, -\frac{\sqrt{2}}{4}\right)$.

89 **a** $\int \ln x\, dx = \int 1 \ln x\, dx$

We integrate by parts with $u = \ln x \quad v' = 1$

$$u' = \frac{1}{x} \qquad v = x$$

$$\therefore\ \int \ln x\, dx = x\ln x - \int \left(\frac{1}{x}\right) x\, dx$$
$$= x\ln x - \int 1\, dx$$
$$= x\ln x - x + c$$

Check: $\dfrac{d}{dx}(x\ln x - x + c)$

$$= 1\ln x + x\left(\frac{1}{x}\right) - 1 + 0$$
$$= \ln x + 1 - 1$$
$$= \ln x\ \checkmark$$

b For $f(x)$ to be a pdf,

$$\int_1^k f(x)\, dx = 1$$
$$\therefore\ \int_1^k \ln x\, dx = 1$$
$$\therefore\ [x\ln x - x]_1^k = 1 \quad \{\text{using } \mathbf{a}\}$$
$$\therefore\ (k\ln k - k) - (0 - 1) = 1$$
$$\therefore\ k(\ln k - 1) = 0$$
$$\therefore\ k = 0 \text{ or } \ln k = 1$$
$$\therefore\ k = 0 \text{ or } e$$

But $k \geqslant 1$, so $k = e$

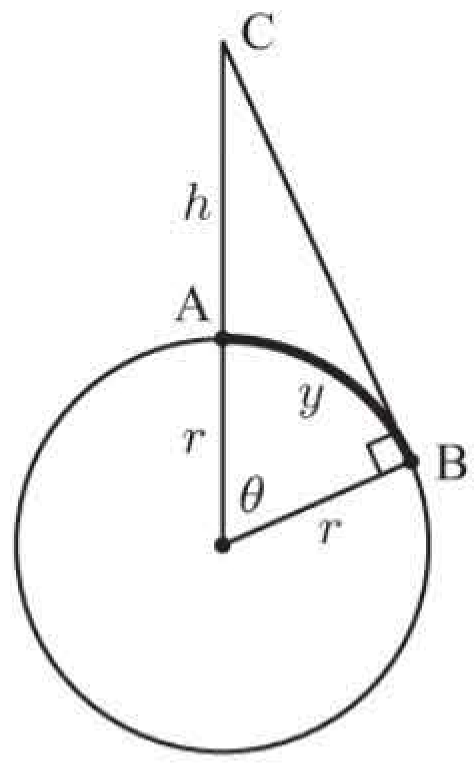
c Let m be the median.

$$\therefore\ \int_1^m \ln x\, dx = \tfrac{1}{2}$$
$$\therefore\ [x\ln x - x]_1^m = \tfrac{1}{2}$$
$$\therefore\ m\ln m - m - (-1) = \tfrac{1}{2}$$
$$\therefore\ m\ln m - m + \tfrac{1}{2} = 0$$
$$\therefore\ m \approx 0.187 \text{ or } 2.16$$

{technology}

But $m \geqslant 1$, so $m \approx 2.16$

90 **a**



Let the arc length AB be y.

$\therefore \quad y = r\theta \quad$ and $\quad \cos\theta = \dfrac{r}{r+h} \quad .... (1)$

Substituting $\theta = \frac{y}{r}$ into (1) gives

$$\cos\left(\tfrac{y}{r}\right) = r(r+h)^{-1}$$

We differentiate with respect to t:

$$-\sin\left(\tfrac{y}{r}\right)\frac{1}{r}\frac{dy}{dt} = -\frac{r}{(r+h)^2}\frac{dh}{dt} \qquad \{r \text{ is constant}\}$$

$$\therefore \quad \sin\theta\,\frac{dy}{dt} = \left(\frac{r}{r+h}\right)^2\frac{dh}{dt}$$

$$\therefore \quad \frac{dy}{dt} = \frac{\cos^2\theta}{\sin\theta}\frac{dh}{dt} \qquad \{\text{using (1)}\}$$

b The rocket has velocity $\dfrac{dh}{dt}$.

$\therefore \quad \dfrac{dh}{dt} = r\sin t \quad$ for $\quad t \in [0, \pi]$

$\therefore \quad h = \int r\sin t\,dt = -r\cos t + c$

But when $t = 0, \quad h = 0$

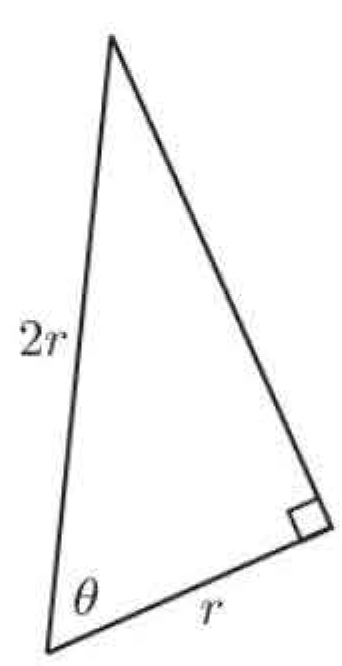
$\therefore \quad c = r$

So, $\quad h = r(1 - \cos t)$

Now when $t = \frac{\pi}{2}$, $\cos t = 0$ and $h = r$.

This means that the rocket is r km above the earth's surface.

c At $t = \frac{\pi}{2}$, we have:



$$\cos\theta = \frac{r}{2r} = \tfrac{1}{2}$$

$$\therefore \quad \theta = \tfrac{\pi}{3} \quad \text{and} \quad \sin\theta = \tfrac{\sqrt{3}}{2}$$

$$\therefore \quad \frac{dy}{dt} \approx \frac{\left(\frac{1}{2}\right)^2}{\frac{\sqrt{3}}{2}} \times 6000 \times \sin\left(\tfrac{\pi}{2}\right) \qquad \left\{\text{using } \mathbf{a},\ \frac{dh}{dt} = r\sin t\right\}$$

$$\approx \tfrac{1}{4} \times \tfrac{2}{\sqrt{3}} \times 6000 \times 1$$

$$\approx 1730 \text{ km h}^{-1}$$

91 **a**

$$\text{arctanh}(z) = \tfrac{1}{2}\left[\ln(1+z) - \ln(1-z)\right]$$

$$\therefore \quad \frac{d}{dz}\left(\text{arctanh}(z)\right) = \tfrac{1}{2}\left[\frac{1}{1+z} - \frac{-1}{1-z}\right]$$

$$= \tfrac{1}{2}\left[\frac{1}{1+z} + \frac{1}{1-z}\right]$$

$$= \tfrac{1}{2}\left[\frac{1-z+1+z}{1-z^2}\right]$$

$$= \frac{1}{1-z^2}$$

b $f(x) = \int x(1-x^2)^{-\frac{3}{2}}\,dx$ Let $u = 1 - x^2$

$= -\frac{1}{2}\int u^{-\frac{3}{2}}(-2x)\,dx$ $\therefore \dfrac{du}{dx} = -2x$

$$= -\tfrac{1}{2}\int u^{-\frac{3}{2}}\,\frac{du}{dx}\,dx$$

$$= -\tfrac{1}{2}\int u^{-\frac{3}{2}}\,du$$

$$= -\tfrac{1}{2}\left(\frac{u^{-\frac{1}{2}}}{-\frac{1}{2}}\right) + c$$

$$= \frac{1}{\sqrt{u}} + c$$

$$= \frac{1}{\sqrt{1-x^2}} + c$$

Now $V = \pi\int_0^{\frac{1}{2}} [f(x)]^2\,dx$

$$= \pi\int_0^{\frac{1}{2}}\left[\frac{1}{1-x^2} + \frac{2c}{\sqrt{1-x^2}} + c^2\right]dx$$

$$= \pi\left[\operatorname{arctanh}(x) + 2c\arcsin x + c^2x\right]_0^{\frac{1}{2}}$$

$$= \pi\left[\left(\operatorname{arctanh}(\tfrac{1}{2}) + 2c\arcsin(\tfrac{1}{2}) + \frac{c^2}{2}\right) - (\operatorname{arctanh}(0) + 2c\arcsin(0) + 0)\right]$$

$$= \pi\left[\tfrac{1}{2}\left(\ln(\tfrac{3}{2}) - \ln(\tfrac{1}{2})\right) + 2c(\tfrac{\pi}{6}) + \frac{c^2}{2} - 0\right]$$

Thus $\pi\left[\frac{1}{2}\ln 3 + \frac{\pi}{3}c + \dfrac{c^2}{2}\right] \approx 14.589$

$$\therefore \frac{c^2}{2} + \tfrac{\pi}{3}c + \tfrac{1}{2}\ln 3 \approx 4.6438$$

$$\therefore c \approx -4.09 \text{ or } 2.00$$

$$\therefore f(x) \approx \frac{1}{\sqrt{1-x^2}} + 2 \quad\text{or}\quad f(x) \approx \frac{1}{\sqrt{1-x^2}} - 4.09$$

92 **a** $f(x) = x - 4\sqrt{2}\sqrt{x}$

i The domain is $\{x \mid x \geqslant 0\}$

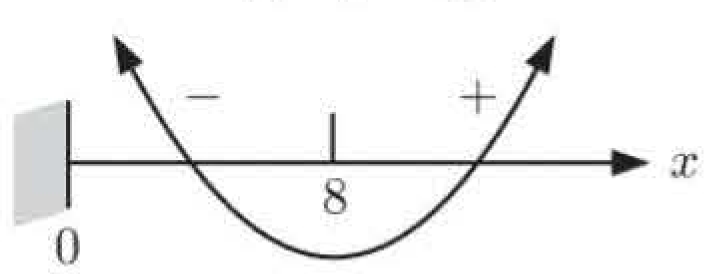
ii $f'(x) = 1 - 4\sqrt{2}\,\frac{1}{2}x^{-\frac{1}{2}}$

$$= 1 - \frac{2\sqrt{2}}{\sqrt{x}}$$

$$= \frac{\sqrt{x} - 2\sqrt{2}}{\sqrt{x}}$$

which is $0 \Leftrightarrow \sqrt{x} = 2\sqrt{2}$

$\therefore x = 8$



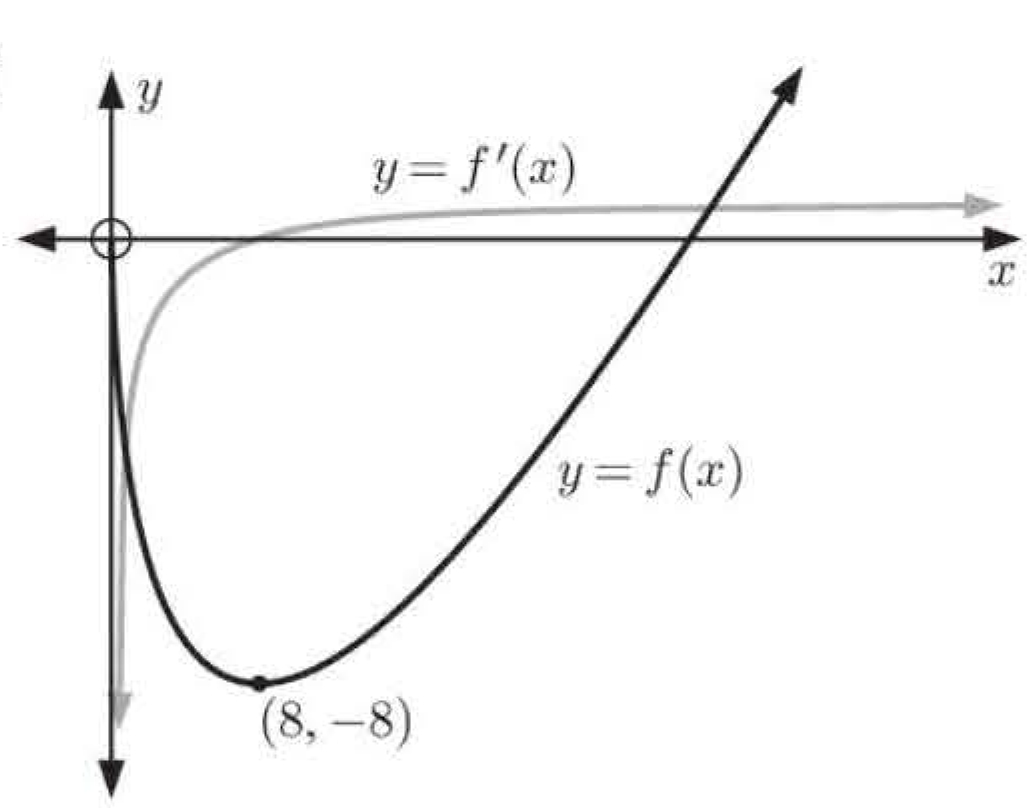
and $f(8) = 8 - 4\sqrt{2}(\sqrt{8})$

$= 8 - 16$

$= -8$

$\therefore$ the minimum value is -8 when $x = 8$.

iii



b **i** $(\sqrt{a}-\sqrt{b})^2 \geqslant 0$ for all $a \geqslant 0,\ b \geqslant 0$

$\therefore\ a - 2\sqrt{ab} + b \geqslant 0$

$\therefore\ a + b \geqslant 2\sqrt{ab}$

$\therefore\ \dfrac{a+b}{2} \geqslant \sqrt{ab}$

Note: Equality occurs when $a = b$.

ii Using $a = x,\ b = 8$:

Given x, 8 are $\geqslant 0$

$\dfrac{x+8}{2} \geqslant \sqrt{8x}$ {AM-GM inequality}

$\therefore\ x + 8 \geqslant 2 \times 2\sqrt{2}\sqrt{x}$

$\therefore\ x + 8 \geqslant 4\sqrt{2}\sqrt{x}$

$\therefore\ x - 4\sqrt{2}\sqrt{x} \geqslant -8$

with equality occuring $\Leftrightarrow\ x = 8$

$\therefore\ -8$ is the minimum value when $x = 8$.

93 **a** If $x = 1+\lambda,\ y = -1 + a\lambda,\ z = 2 - \lambda$ lies on $3x - ky + z = 3$, then

$3(1+\lambda) - k(-1 + a\lambda) + 2 - \lambda = 3$ for all λ

$\therefore\ 3 + 3\lambda + k - ak\lambda + 2 - \lambda = 3$

$\therefore\ (3 + k + 2) + \lambda(3 - ak - 1) = 3$

$\therefore\ k + 5 = 3$ and $2 - ak = 0$

$\therefore\ k = -2$ and $2 + 2a = 0$

$\therefore\ k = -2$ and $a = -1$

b P_2 has normal vector $\mathbf{n}_2 = \begin{pmatrix} 2 \\ -1 \\ -4 \end{pmatrix}$

P_1 has normal vector $\mathbf{n}_1 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$

Now $\mathbf{n}_1 \bullet \mathbf{n}_2 = 6 - 2 - 4 = 0$

$\therefore\ \mathbf{n}_1 \perp \mathbf{n}_2$ and so $P_1 \perp P_2$.

c We need to solve $\begin{cases} 2x - y - 4z = 9 \\ 3x + 2y + z = 3 \end{cases}$

$$\left[\begin{array}{ccc|c} 2 & -1 & -4 & 9 \\ 3 & 2 & 1 & 3 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 2 & -1 & -4 & 9 \\ 0 & 7 & 14 & -21 \end{array}\right] \quad R_2 \to 2R_2 - 3R_1$$

The second equation simplifies to $y + 2z = -3$

If $z = t$ then $y = -3 - 2t$

and $2x + 3 + 2t - 4t = 9$

$\therefore\ 2x = 6 + 2t$

$\therefore\ x = 3 + t$

So, L_2 has equation

$x = 3 + t,\ y = -3 - 2t,\ z = t,\ t \in \mathbb{R}$.

d L_1 and L_2 meet when $1 + \lambda = 3 + t,\quad -1 - \lambda = -3 - 2t,\quad 2 - \lambda = t$

$\therefore\ \lambda = 2 + t,\quad \lambda = 2 + 2t,\quad \lambda = 2 - t$

$\therefore\ 2 + t = 2 + 2t = 2 - t$

The common solution is $t = 0$, so L_1 and L_2 meet at $(3, -3, 0)$.

e $\mathbf{L}_1 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ and $\mathbf{L}_2 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

$$\cos\theta = \frac{|\mathbf{L}_1 \bullet \mathbf{L}_2|}{|\mathbf{L}_1|\,|\mathbf{L}_2|} = \frac{|1 + 2 - 1|}{\sqrt{3}\sqrt{6}} = \tfrac{2}{\sqrt{18}}$$

$$\therefore\ \theta = \arccos\left(\tfrac{2}{\sqrt{18}}\right) \approx 61.9°$$

94 **a** There are $12! = 479\,001\,600$ possible orders.

b **i** There are $\binom{4}{2} \times 2! = 12$ ways to place I and E amongst the last 4, and the other 10 are ordered in $10!$ ways.

$\therefore$ the total number is $12 \times 10! = 43\,545\,600$ ways.

ii The 3 can be together in 2 ways (PIL or LIP) and this group together with the other 9 can be ordered in 10! ways.

$\therefore$ the total number is $2 \times 10! = 7\,257\,600$ ways.

iii Istvan will be between Paul and Laszlo in $\frac{1}{3}$ of all possible cases, since each of them will be the 'middle student' $\frac{1}{3}$ of the time.

$\therefore$ the total number of ways $= \frac{1}{3}$ of $12! = 159\,667\,200$ ways.

iv The students can be arranged in the form

$$\left.\begin{array}{ll} \text{A x x x H x x x x x x x x} & \text{in } 10! \text{ ways} \\ \text{H x x x A x x x x x x x x} & \text{in } 10! \text{ ways} \\ \text{x A x x x H x x x x x x x} & \text{in } 10! \text{ ways} \\ \vdots & \\ \text{x x x x x x x H x x x A} & \text{in } 10! \text{ ways} \end{array}\right\} 8 \times 2 = 16 \text{ of these}$$

$\therefore$ the total number of ways $= 16 \times 10! = 58\,060\,800$ ways.

c **i** There are $\binom{12}{4}$ ways to choose the first group, $\binom{8}{4}$ ways to choose the second group, and $\binom{4}{4}$ ways to choose the third group. The order of groups is not important, so we divide by 3!.

So, there are $\dfrac{1}{3!}\binom{12}{4}\binom{8}{4}\binom{4}{4} = 5775$ ways.

ii There are $\binom{2}{2}\binom{10}{2}$ ways to choose the group with Ben and Marton. There are then $\binom{8}{4}$ ways to choose the second group, and $\binom{4}{4}$ ways to choose the third group. The order of groups 2 and 3 is not important, so we divide by 2!.

So, there are $\dfrac{1}{2!}\binom{2}{2}\binom{10}{2}\binom{8}{4}\binom{4}{4} = 1575$ ways.

95 **a** $a = \frac{1}{2}v^2$

$\therefore \dfrac{dv}{dt} = \dfrac{v^2}{2}$

$\therefore \dfrac{dt}{dv} = \dfrac{2}{v^2}$

b $t = \int 2v^{-2}\,dv$

$= 2\left(\dfrac{v^{-1}}{-1}\right) + c$

$= -\dfrac{2}{v} + c$

But when $t = 0$, $v = -1$

$\therefore 0 = 2 + c$

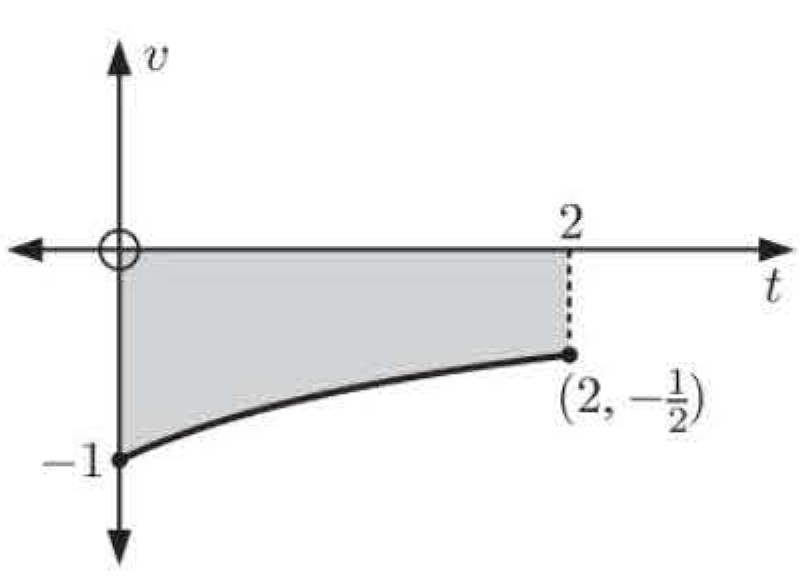
$\therefore c = -2$

Hence, $t = -\dfrac{2}{v} - 2$

$\therefore \dfrac{2}{v} = -t - 2$

$\therefore v = \dfrac{-2}{t+2}$

c



$\therefore$ there is no direction change

total distance travelled = shaded area

$= -\displaystyle\int_0^2 \frac{-2}{t+2}\,dt$

$= 2\left[\ln|t+2|\right]_0^2$

$= 2\ln 4 - 2\ln 2$

$= 2(\ln 4 - \ln 2)$

$= 2\ln 2$

≈ 1.39 units

96 P_n is: "$n^2 \times 1 + (n-1)^2 \times 2 + (n-2)^2 \times 3 + ... + 2^2 \times (n-1) + 1^2 \times n = \dfrac{n(n+1)^2(n+2)}{12}$" for $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, LHS $= 1^2 \times 1 = 1$ and RHS $= \dfrac{1 \times 2^2 \times 3}{12} = 1$ Thus P_1 is true.

(2) If P_k is assumed true then

$$k^2 \times 1 + (k-1)^2 \times 2 + (k-2)^2 \times 3 + ... + 2^2 \times (k-1) + 1^2 \times k = \frac{k(k+1)^2(k+2)}{12}$$

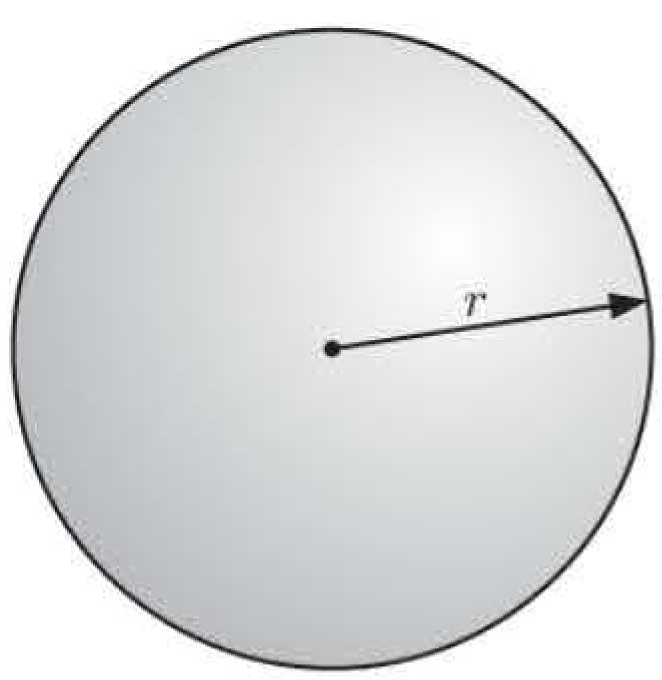
Thus,

$$\begin{aligned}
&(k+1)^2 \times 1 + k^2 \times 2 + (k-1)^2 \times 3 + ... + 3^2(k-1) + 2^2 \times k + 1^2 \times (k+1) \\
&= k^2 \times 1 + (k-1)^2 \times 2 + (k-2)^2 \times 3 + ... + 3^2(k-2) + 2^2 \times (k-1) + 1^2 \times k \\
&\quad + (k+1)^2 + k^2 + (k-1)^2 + (k-2)^2 + ... + 3^2 + 2^2 + 1^2 \\
&= \frac{k(k+1)^2(k+2)}{12} + 1^2 + 2^2 + 3^2 + 4^2 + ... + k^2 + (k+1)^2 \\
&= \frac{k(k+1)^2(k+2)}{12} + \frac{(k+1)(k+2)(2k+3)}{6} \\
&\qquad \left\{1^2 + 2^2 + 3^2 + 4^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6},\ n \in \mathbb{Z}^+\right\} \\
&= \frac{(k+1)(k+2)[k(k+1) + 2(2k+3)]}{12} \\
&= \frac{(k+1)(k+2)(k^2+5k+6)}{12} \\
&= \frac{(k+1)(k+2)(k+2)(k+3)}{12} \\
&= \frac{(k+1)([k+1]+1)^2([k+1]+2)}{12}
\end{aligned}$$

Thus P_{k+1} is true whenever P_k is true

$\therefore$ since P_1 is true, P_n is true for all $n \in \mathbb{Z}^+$. {Principle of mathematical induction}

97 **a**



$$V = \tfrac{4}{3}\pi r^3$$

$$\therefore \quad \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

But $\dfrac{dr}{dt} = -\tfrac{8}{5}$ cm/min {as it is constant}

$$\therefore \quad \frac{dV}{dt} = 4\pi r^2(\tfrac{-8}{5}) = -6.4\pi r^2$$

At time $t = 2.5$, $r = 8 - \tfrac{8}{5} \times 2.5 = 4$ cm

$$\therefore \quad \frac{dV}{dt} = -6.4\pi \times 4^2 \approx -322$$

$\therefore$ the volume is decreasing at 322 cm^3/min.

b The average change in volume $= \dfrac{V(5) - V(1)}{5-1}$

$$= \frac{0 - \frac{4}{3}\pi(6.4)^3}{4} \qquad \{\text{when } t = 1,\ r = 6.4\}$$

$$= -\tfrac{\pi}{3}(6.4)^3$$

$$\approx -275$$

$\therefore$ on average, the volume is decreasing at 275 cm^3/min.

98 **a** Let $A = 3\log_y x$, $\quad B = 3\log_z y$, $\quad C = 7\log_x z$

$\therefore \log_y x = \dfrac{A}{3}$, $\quad \log_z y = \dfrac{B}{3}$, $\quad \log_x z = \dfrac{C}{7}$

$\therefore x = y^{\frac{A}{3}}$, $\quad y = z^{\frac{B}{3}}$, $\quad z = x^{\frac{C}{7}}$

Thus $x = y^{\frac{A}{3}} = (z^{\frac{B}{3}})^{\frac{A}{3}} = z^{\frac{AB}{9}} = (x^{\frac{C}{7}})^{\frac{AB}{9}} = x^{\frac{ABC}{63}}$

Consequently $ABC = 63$ .... (1)

But $3, A, B, C$ are in arithmetic sequence

$\therefore\ A = \dfrac{3+B}{2}$ and $B = \dfrac{A+C}{2}$

$\therefore\ B = 2A - 3$ and $C = 2B - A$

$= 4A - 6 - A$

$= 3A - 6$

Substituting in (1) gives $A(2A-3)(3A-6) = 63$

$\therefore\ A(2A-3)(A-2) = 21$

$\therefore\ 2A^3 - 7A^2 + 6A - 21 = 0$

$\therefore\ A = \frac{7}{2}$, the only real solution {technology}

Hence, $B = 2(\frac{7}{2}) - 3$ and $C = 3(\frac{7}{2}) - 6$

$= 4$ $= \frac{9}{2}$

So, $x^{18} = (y^{\frac{A}{3}})^{18}$ and $y^{21} = (z^{\frac{B}{3}})^{21}$

$= (y^{\frac{7}{6}})^{18}$ $= z^{7B}$

$= y^{21}$ $= z^{28}$

$\therefore\ x^{18} = y^{21} = z^{28}$

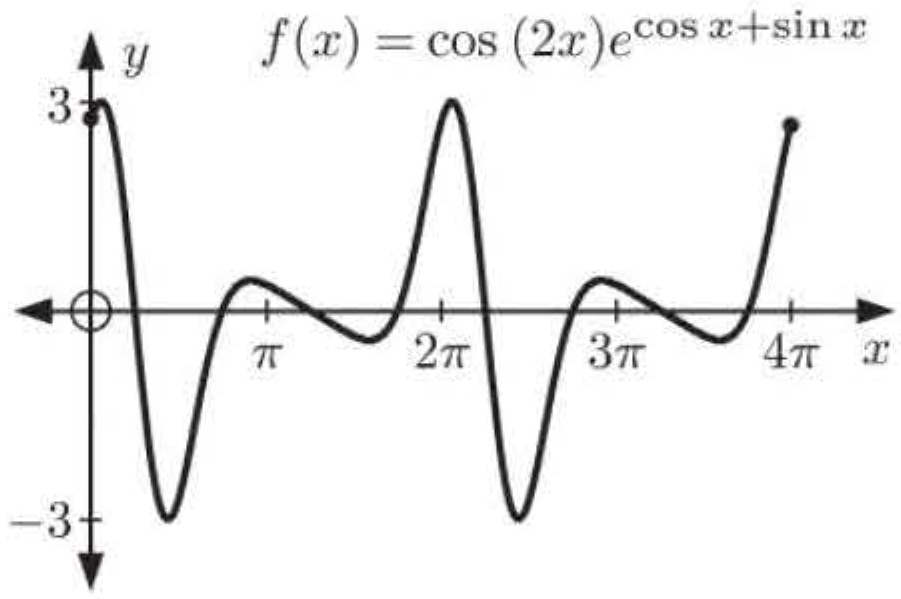
b $x^{18} = y^{21}$ and $x^{18} = z^{28}$

$\therefore\ x = y^{\frac{21}{18}}$ $\therefore\ x^{18} = z^{\frac{28}{18}}$

$\therefore\ x = y^{\frac{7}{6}}$ $\therefore\ x = y^{\frac{14}{9}}$

99 **a** The period is 2π.

b



c **i** If $u = e^{\cos x + \sin x}$

then $\dfrac{du}{dx} = e^{\cos x+\sin x}(-\sin x + \cos x)$ and $\ln u = \cos x + \sin x$

$\therefore\ \int \cos(2x) e^{\cos x+\sin x}\, dx$

$= \int (\cos^2 x - \sin^2 x) e^{\cos x+\sin x}\, dx$

$= \int (\cos x + \sin x)(\cos x - \sin x) e^{\cos x+\sin x}\, dx$

$= \int \ln u \dfrac{du}{dx}\, dx$

$= \int 1 \ln u\, du$ $\left\{ \begin{array}{ll} a' = 1 & b = \ln u \\ a = u & b' = \dfrac{1}{u} \end{array} \right.$

$= u \ln u - \int 1\, du$

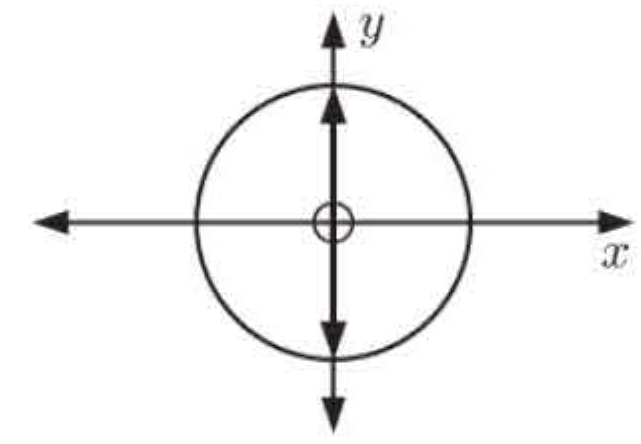
$= u \ln u - u + c$

$= e^{\cos x+\sin x}(\cos x + \sin x) - e^{\cos x+\sin x} + c$

$= e^{\cos x+\sin x}(\cos x + \sin x - 1) + c$

ii $f(x)$ cuts the x-axis when $y = 0$

$$\therefore \cos(2x) = 0$$
$$\therefore 2x = \tfrac{\pi}{2} \quad \{\text{the first positive solution}\}$$
$$\therefore x = \tfrac{\pi}{4}$$

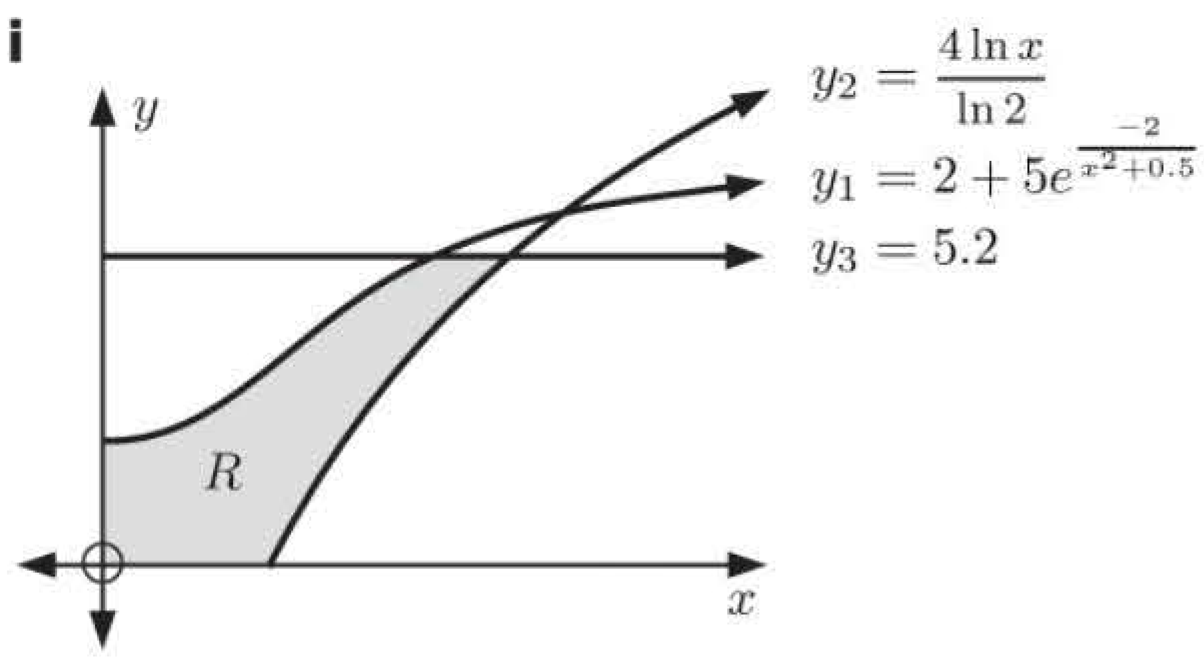


d $\text{Area} = \int_0^{\frac{\pi}{4}} \cos(2x)e^{\cos x+\sin x}\,dx$

$$= \left[e^{\cos x+\sin x}(\cos x + \sin x - 1)\right]_0^{\frac{\pi}{4}}$$
$$= e^{\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}}\left(\tfrac{1}{\sqrt{2}} + \tfrac{1}{\sqrt{2}} - 1\right) - e^1(1 + 0 - 1)$$
$$= e^{\sqrt{2}}(\sqrt{2} - 1) \text{ units}^2$$

(or ≈ 1.70 units^2)

100 **a** **i**



ii y_1 cuts the y-axis when $x = 0$

$$\therefore y_1 = 2 + 5e^{\frac{-2}{0.5}} = 2 + 5e^{-4}$$

$\therefore$ the y-intercept is $2 + 5e^{-4}$.

b **i**

$$y = 2 + 5e^{\frac{-2}{x^2+0.5}}$$
$$\therefore y - 2 = 5e^{\frac{-2}{x^2+0.5}}$$
$$\therefore \frac{y-2}{5} = e^{\frac{-2}{x^2+0.5}}$$
$$\therefore \ln\left(\frac{y-2}{5}\right) = \frac{-2}{x^2+0.5}$$
$$\therefore x^2 + 0.5 = \frac{-2}{\ln\left(\frac{y-2}{5}\right)}$$
$$\therefore x^2 = \frac{-2}{\ln\left(\frac{y-2}{5}\right)} - 0.5$$
$$\therefore x = \sqrt{\frac{-2}{\ln\left(\frac{y-2}{5}\right)} - 0.5} \quad \{x \geqslant 0\}$$

ii If $y = \dfrac{4\ln x}{\ln 2}$, then $4\ln x = y\ln 2$

$$\therefore \ln x^4 = \ln 2^y$$
$$\therefore x^4 = 2^y$$
$$\therefore x = 2^{\frac{y}{4}}$$

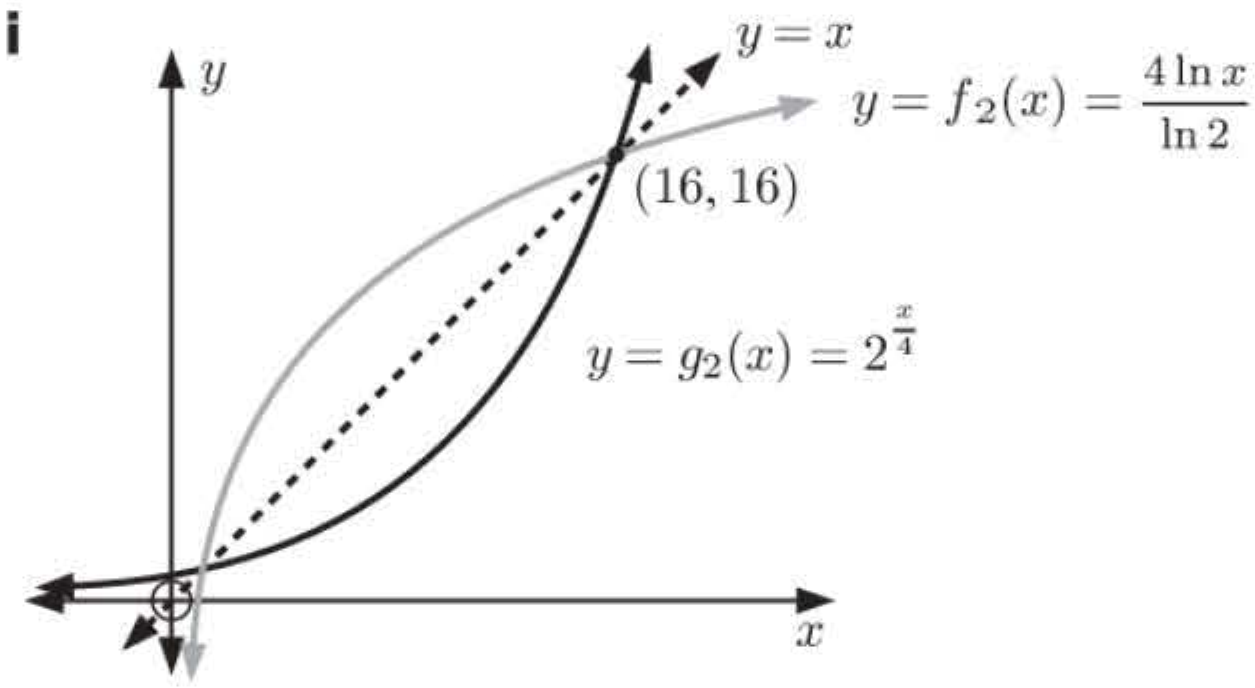
c **i**

$$V = \pi\left[\int_0^{5.2} {x_2}^2\,dy - \int_{2+\frac{5}{e^4}}^{5.2} {x_1}^2\,dy\right]$$
$$\therefore V = \pi\left[\int_0^{5.2} 2^{\frac{y}{2}}\,dy - \int_{2+\frac{5}{e^4}}^{5.2}\left(\frac{-2}{\ln\left(\frac{y-2}{5}\right)} - 0.5\right)dy\right] \text{ cm}^3$$

ii Thus, $V = \pi\left[\displaystyle\int_0^{5.2} 2^{\frac{y}{2}}\,dy + \int_{2+\frac{5}{e^4}}^{5.2} \frac{2}{\ln\left(\frac{y-2}{5}\right)} + 0.5\,dy\right] \text{ cm}^3$

$$\approx 31.1 \text{ cm}^3$$

d **i**



ii
$$\begin{aligned}(f_2 \circ g_2)(x) &= f_2(g_2(x)) \\ &= f_2(2^{\frac{x}{4}}) \\ &= \frac{4\ln 2^{\frac{x}{4}}}{\ln 2} \\ &= \frac{4(\frac{x}{4})\ln 2}{\ln 2} \\ &= x\end{aligned}$$

$$\begin{aligned}(g_2 \circ f_2)(x) &= g_2(f_2(x)) \\ &= g_2\left(\frac{4\ln x}{\ln 2}\right) \\ &= 2^{\frac{\ln x}{\ln 2}} \\ &= 2^{\log_2 x} \\ &= x\end{aligned}$$

iii They are inverse functions.

e **i** $f_1(x) = 2 + 5e^{\frac{-2}{x^2+0.5}}$ has inverse function $x = 2 + 5e^{\frac{-2}{y^2+0.5}}$

From **b i**, the rearrangement making y the subject is $y = \sqrt{\dfrac{-2}{\ln\left(\frac{x-2}{5}\right) - 0.5}}$

$$\therefore \ g_1(x) = \sqrt{\frac{-2}{\ln\left(\frac{x-2}{5}\right) - 0.5}}$$

ii For $f_1(x) = 2 + 5e^{\frac{-2}{x^2+0.5}}$, as $x \to \infty$, $e^{\frac{-2}{x^2+0.5}} \to e^0$

$$\therefore \ f_1(x) \to 7$$
$$\therefore \ x < 7$$

As $f_1(x)$ is increasing for all $x \geqslant 0$, $\min_x = f_1(0) = 2 + 5e^{-4}$

$\therefore$ the range of $f_1(x)$ is $\{y \mid 2 + 5e^{-4} \leqslant y < 7\}$.

$\therefore$ the domain of $g_1(x)$ is $\{x \mid 2 + 5e^{-4} \leqslant x < 7\}$.